Growth gap in hyperbolic groups and amenability

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Abstract

We prove a general version of the amenability conjecture in the unified setting of a Gromov hyperbolic group G acting properly cocompactly either on its Cayley graph, or on a CAT(-1)-space. Namely, for any subgroup H of G, we show that H is co-amenable in G if and only if their exponential growth rates (with respect to the prescribed action) coincide. For this, we prove a quantified, representation-theoretical version of Stadlbauer's amenability criterion for group extensions of a topologically transitive subshift of finite type, in terms of the spectral radii of the classical Ruelle transfer operator and its corresponding extension. As a consequence, we are able to show that, in our enlarged context, there is a gap between the exponential growth rate of a group with Kazhdan's property (T) and the ones of its infinite index subgroups. This also generalizes a well-known theorem of Corlette for lattices of the quaternionic hyperbolic space or the Cayley hyperbolic plane.

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1 Introduction

Amenability has a large number of equivalent formulations. In a seminal work dating back to 1959, Kesten proved that a finitely generated group Q is amenable if and only if 1 is the spectral radius of the Markov operator associated to a symmetric random walk on Q whose support generates Q [28]. Given a finite generating set S of Q, Grigorchuk [24] and Cohen [9] independently related the spectral radius ρ for the random walk with uniform probability measure supported on $S \cup S^{-1}$, to the exponential growth rates of the free group $\mathbf{F}(S)$ and the kernel N of the canonical projection $\mathbf{F}(S) \twoheadrightarrow Q$. Recall that the *exponential growth rate* of a discrete group G of isometries of a proper metric space X, denoted by $\omega(G, X)$, (or simply ω_G if there is no ambiguity) is

$$\omega(G, X) = \limsup_{r\to\infty} \frac{1}{r} \ln |\{g \in G \mid d(gx, x) \leqslant r\}|,$$

where x is any point of X. The Cohen-Grigorchuck formula states that

$$\rho = \frac{\sqrt{e^{\omega_{\mathbf{F}(S)}}}}{1 + e^{\omega_{\mathbf{F}(S)}}} \left(\frac{\sqrt{e^{\omega_{\mathbf{F}(S)}}}}{e^{\omega_N}} + \frac{e^{\omega_N}}{\sqrt{e^{\omega_{\mathbf{F}(S)}}}} \right), \tag{1}$$

where ω_N and $\omega_{\mathbf{F}(S)}$ are the respective growth rates of N and $\mathbf{F}(S)$ acting on the Cayley graph of $\mathbf{F}(S)$ with respect to S. This immediately yielded, by Kesten's criterion, a characterization of amenability for a group Q generated by a subset S, in terms of the growth of the relator subgroup: the quotient $Q = \mathbf{F}(S)/N$ is amenable if and only if $\omega_N = \omega_{\mathbf{F}(S)} = \log(2|S| - 1)$.

Almost at the same time, a geometric version of Kesten's criterion, in terms of the bottom of the L^2-spectrum of the Laplace operator, was discovered in Riemannian geometry by Brooks [5, 6] (see also [7]): for any normal, Riemannian covering $\hat{M} \to M$ of a Riemannian manifold M of *finite topological type*, if the automorphism group of $\hat{M}$ is amenable, then $\lambda_0(\hat{M}) = \lambda_0(M)$; moreover, when $\hat{M} = \tilde{M}$ is the universal covering of a convex-cocompact hyperbolic manifold M (or any Riemannian manifold M of finite topological type satisfying a "Cheeger-type" condition [6, Theorem 2]), then the converse implication also holds. Realizing the base manifold as $M = \tilde{M}/G$ and the cover as $\hat{M} = \tilde{M}/N$ with N normal in G, this precisely says that the $\lambda_0(\hat{M}) = \lambda_0(M)$ whenever the quotient group $Q = G/N$ is amenable. Coupling this with Sullivan's formula relating $\lambda_0(M)$ to ω_G for discrete subgroups of isometries of the hyperbolic space $\mathbf{H}^n$, it yields

$$\lambda_0(M) = \begin{cases} \omega_G(n - 1 - \omega_G) & \text{if } \omega_G \geqslant \dfrac{n-1}{2} \\ \lambda_0(\mathbf{H}^n) & \text{if } \omega_G \leqslant \dfrac{n-1}{2} \end{cases}, \tag{2}$$

(where ω_G is computed, this time, with respect to the action of G on the hyperbolic space). Brooks then derived an analogue of Cohen-Grigorchuk statement for particular, discrete groups of isometries of $\mathbf{H}^n$: for any normal Riemannian covering $\hat{M} = \tilde{M}/N$ of a convex-cocompact hyperbolic manifold $M = \tilde{M}/G$ with $\omega_G > (n-1)/2$, the automorphism group $Q = G/N$ of $\hat{M}$ is amenable if and only if $\omega_G = \omega_N$. This holds, for instance, for uniform lattices. The result was also extended to hyperbolic, non-uniform lattices in [35].

Beyond the evident formal analogies of these results – negatively curved cocompact groups look like free groups at large scale, while the bottom of the Laplacian of M can be related to the spectral radius of the Markov operator associated to particular random walks on $G = \pi_1(M)$, induced by the heat kernel of $\tilde{M}$ [5] – the two statements did not live on a common ground. This opened the door to intensive research for a unifying, generalized setting, and a deeper understanding of the relations between these results in dynamical terms.

The first statement in a general setting was given in 2005 by Roblin [34]. Given a discrete group G of isometries of a CAT(-1) space, he proved, using Patterson-Sullivan theory, that amenability of the quotient G/N always implies the relation $\omega_G = \omega_N$. In this generality, it is worth to remark that the reciprocal is not true. Indeed, there exist Kleinian, geometrically finite groups G – even lattices in pinched, variable negative curvature [19, 20] – admitting a parabolic subgroup P with $\omega_P = \omega_G$. Such groups give easy counterexamples to the converse implication, by taking for N the normal closure of P in G. Indeed in most of these cases G/N contains free subgroups and is not amenable. The most accredited version of the *Amenability Problem* in the last decade can be stated as follows. *Given a group G acting properly on a hyperbolic space X and a normal subgroup N of G, under which circumstances does the equality $\omega_N = \omega_G$ imply that the quotient group $Q = G/N$ is amenable?*

Clearly, for a group G acting on a general space X, an exact formula as (1) or (2) is hopeless. Rather, one should expect that the equality $\omega_N = \omega_G$ reflects the qualitative behavior of the dynamics of G/N on the space X/N. Nevertheless an exact relation, in terms of the asymptotic behavior of the spectral radii of random walks on G/N with probability measure supported by large spheres, resists to this general setting, allowing to show the "easy part" of the implication above even in the generality of cocompact group actions on Gromov-hyperbolic spaces (see for instance Propositions B.5 and B.6 in Appendix B).

A substantial step forward in the solution of the Amenability Problem is due to Stadlbauer [36], who generalized Kesten's amenability criterion in terms of group extensions of topological Markov chains. More precisely, he considered a topologically mixing subshift of finite type (Σ, σ) together with a topologically transitive extension $(\Sigma_\theta, \sigma_\theta)$ of this system by a locally constant evaluation map $\theta : \Sigma \to Q$ into a discrete group Q. He proved that Q is amenable if and only if the *Gurevič pressures* of the two systems (with respect to a weakly symmetric potential with Hölder variations) coincide[1]. As an application, Stadlbauer solved the Amenability Problem for the class of *essentially free* groups G of isometries of $\mathbf{H}^n$, for the first time without assuming that $\omega_G > (n-1)/2$. This result was recently generalized by Dougall and Sharp, using Stadbauer's criterion, to the class of convex-cocompact groups of isometries of pinched, negatively curved Cartan-Hadamard manifolds [22].

[1] Actually Stadlbauer's criterion works in a slightly more general context which allow to consider symbolic dynamical systems over an infinite alphabet. Nevertheless in the context of hyperbolic groups a subshift of finite type is sufficient to conclude.

The first result of this paper solves the Amenability Problem in an enlarged context encompassing two very different cases. The first one, of algebraic nature, concerns the growth of groups with respect to the word metric. The second case, coming from the geometry, focuses on the action of a group on a negatively curved Riemannian manifold or a CAT(-1) space. The aim is two-fold: to give a self-contained proof of all these results in a unified setting, and to make clear the minimum algebraic and geometric structure needed.

Theorem 1.1 (see Theorem 5.1)**.** *Let G be a group acting properly co-compactly by isometries on a Gromov hyperbolic space X. We assume that one of the following holds. Either*

(i) X is the Cayley graph of G with respect to a finite generating set, or

(ii) X is a CAT(-1) *space.*

Let H be a subgroup of G, and let ω_G and ω_H denote the exponential growth rates of G and H acting on X. The subgroup H is co-amenable in G if and only if $\omega_H = \omega_G$.

Recall that the action of a group G on a space X is *amenable* if X admits a G-invariant mean, and that H is called *co-amenable* in G if the action of G on the left coset space $H\backslash G$ is amenable. When the subgroup H is normal in G then H is co-amenable in G if and only if G/H is an amenable group. Notice however that in the above theorem we do not assume that H is a normal subgroup.

Note that the CAT(-1) case in the above theorem extends the Riemannian convex-compact situation studied by Dougall and Sharp [22]. Nevertheless, going from negatively curved manifolds to CAT(-1) spaces is a substantial generalization. Indeed, Dougall and Sharp explicitly use the Riemannian structure to encode the geodesic flow via Markov sections. To the best of our knowledge there is no such coding for the geodesic flow on CAT(-1) spaces. We explain at the end of the introduction our strategy to bypass this difficulty.

The easy part of Theorem 1.2 is the "only if" implication. As we mentioned before, this direction was proved for normal subgroups of discrete groups acting on CAT(-1) spaces by Roblin [34]. In [35] Roblin and Tapie sketched how to extend the argument to the case of groups acting on a Gromov hyperbolic space. Nevertheless, we decided to report in Appendix B a complete proof of this fact via random walks, for general subgroups of Gromov hyperbolic groups, since this also gives an exact formula which is similar to Sullivan's one for the bottom of the Laplacian of hyperbolic quotients (see Theorem B.1).

On the other hand, our proof of the converse implication is strongly inspired by Stadlbauer's work [36] and relies on a variation of his amenability criterion. However, our approach makes an explicit use of representation theory and operator algebra, which was somehow hidden in [36]. We hope that this point of view can enlighten the key conceptual arguments and clarify the exposition. More precisely we take advantage of the Hulanicki–Reiter criterion for amenable actions: the action of a discrete group G on a set Y is amenable if and only if the induced regular representation $\lambda\colon G \to \mathcal{U}(\ell^2(Y))$ admits almost invariant vectors [3, Theorem G.3.2]. Assume now that $(\Sigma_\theta, \sigma_\theta)$ is the extension of a subshift of finite type (Σ, σ) by a locally constant map $\theta\colon \Sigma \to G$. We associate the classical Ruelle transfer operator $\mathcal{L}$ to the original system (Σ, σ). On the other hand, given an action of G on a set Y, we endow the extended system $(\Sigma_\theta, \sigma_\theta)$ with a *twisted transfer operator* $\mathcal{L}_\lambda$ which is naturally related to the induced unitary representation $\lambda\colon G \to \mathcal{U}(\ell^2(Y))$ (see Section A.3). The twisted transfer operator acts on a subspace of the space of continuous functions $C(\Sigma, \ell^2(Y))$ (the appropriate, Hölder regularity will be described in Section A.1). Using the uniform convexity of

Hilbert spaces, we relate the difference between the spectral radii ρ and ρ_λ of $\mathcal{L}$ and $\mathcal{L}_\lambda$ respectively, to the existence of almost invariant vectors for the representation λ.

Theorem 1.2 (see Theorems A.23 and A.25)**.** *Let (Σ, σ) be a topologically transitive subshift of finite type. Let $F\colon \Sigma \to \mathbf{R}_+^*$ be a potential with α-bounded Hölder variations (for some $\alpha \in \mathbf{R}_+^*$), and $\mathcal{L}$ be the Ruelle transfer operator associated with F. Let G be a finitely generated group and $\theta : \Sigma \to G$ a locally constant map. Assume that the extension $(\Sigma_\theta, \sigma_\theta)$ of (Σ, σ) by θ has the visibility property. Then the following holds.*

(i) For every finite subset S of G and every $\varepsilon \in \mathbf{R}_+^*$, there exists $\eta \in \mathbf{R}_+^*$ with the following property: if G acts on a set Y, and $\mathcal{L}_\lambda$ is the corresponding twisted transfer operator, the condition $\rho_\lambda > (1-\eta)\rho$ implies that the representation $\lambda\colon G \to \mathcal{U}(\ell^2(Y))$ admits an (S, ε)-invariant vector.

(ii) In particular, if $\rho_\lambda = \rho$ then the action of G on Y is amenable.

The second statement of this theorem easily follows from the first point, and is very similar to the one obtained by Stadlbauer [36] when (Σ, σ) is a subshift of finite type. Let us highlight a few important differences though. Unlike Stadlbauer's proof, our approach does not really use the Gibbs measure but simply the Perron-Frobenius theorem. Therefore we do not ask the original system (Σ, σ) to be *topologically mixing*, but just *topologically transitive* (this is much weaker, as in many situations one can always reduce to an irreducible component of the system). Secondly, we consider an extension $(\Sigma_\theta, \sigma_\theta)$ of the initial system by the whole group G, and only assume that it has the *visibility property* (which means that the extended flow visits almost the whole group G), whereas Stadlbauer extends the initial system by the quotient Q, and assumes that this extension is *topologically transitive*. This is one of the key points which allows us to consider any subgroup of a hyperbolic group and not only normal subgroups. Moreover, as we state our result in terms of spectral radius instead of pressure, we do not need any kind of symmetry for the potential F (this was already observed by Jaerisch [27]).

More importantly, our approach provides a *quantitative* version[2] of Stadlbauer's statement. In this perspective, the first statement in the above theorem is close to some results of Dougall in [21], which also includes more concrete representation theory (nevertheless she assumes mixing of the initial system, and considers only normal subgroups to ensure the transitivity of the extended system, as well as a condition called *linear visibility with reminders*, a bit stronger than ours, to control the return times of the flow in a fixed cylinder). The quantitative version of the amenability criterion (see Theorem A.28) makes apparent the following consequence for groups satisfying Kazhdan's property (T).

Theorem 1.3 (see Theorem 5.2)**.** *Let G be a group with Kazhdan's property (T) acting properly co-compactly by isometries on a hyperbolic space X. We assume that one of the following holds:*

(i) either X is the Cayley graph of G with respect to a finite generating set,

(ii) or X is a CAT(-1) space.

[2] In the proof of Theorem 1.2 (i), we choose to work with ultra-limit of Banach spaces: this has the advantage of simplifying the arguments involving almost invariant vectors. As a consequence, we do not provide a precise formula for η in terms of S and ε; nevertheless, a careful reader could go through the arguments and make the relation between these quantities explicit.

Then, there exists $\eta > 0$ with the following property. Let H be a subgroup of G, and let ω_G, ω_H denote the exponential growth rates of G and H acting on X. If $\omega_H > \omega_G - \eta$, then H is a finite index subgroup of G.

We stress the fact that also in this statement H is not assumed to be normal. So, this gives the following generalization of Corlette's celebrated growth gap theorem [17] for subgroups of lattices of rank one symmetric spaces of negative curvature possessing Kazhdan's Property (T), i.e. the quaternionic hyperbolic space $\mathbf{H}^n_{\mathbb{H}}$ or the Cayley hyperbolic plane $\mathbf{H}^2_{\mathbb{O}}$. Corlette's theorem was generalized by Dougall [21] for convex-cocompact groups of isometries of pinched, negatively curved Cartan-Hadamard manifolds. Our statement is an even further generalization which unifies the combinatorial and the geometric point of views. Recall that, for a Gromov hyperbolic space, the *visual dimension* $\dim_{vis}(\partial X)$ is defined analogously to the Hausdorff dimension, but with respect to the natural visual measures of ∂X, and it coincides with the exponential growth rate of any cocompact group G of isometries of X – see for instance [32, 33]. We then have:

Corollary 1.4. *Let X be a CAT(-1) metric space admitting uniform lattices, whose isometry group possesses Kazhdan's property (T). Then, there exists $\eta > 0$ such that for any subgroup H of* $\mathrm{Isom}(X)$*, either H is a lattice, or the exponential growth rate of H is at most* $\dim_{vis}(\partial X) - \eta$.

An similar statement holds if X is the Cayley graph of a hyperbolic group. This result shall be added to the list of geometric and dynamical rigidity consequences of property (T), such as Serre's fixed point-edge property for actions on trees [29, 38], or the local C^∞-conjugacy rigidity of isometric actions on compact Riemannian manifolds [23].

Let us now give an overview of the proof of Theorems 1.1 and 1.3. The main idea is to apply our amenability criterion (Theorem 1.2) to the geodesic flow on X. However the criterion requires a coding of this dynamical system, which may not exist for CAT(-1) spaces. To bypass this difficulty we consider a geodesic flow not on the space X but rather on the Cayley graph Γ of the group G with respect to a finite generating set S. More precisely, if $\mathcal{G}\Gamma$ stands for all bi-infinite geodesics $\gamma\colon \mathbf{R} \to \Gamma$, the flow ϕ_t acts on $\mathcal{G}\Gamma$ by shifting the time parameter by t. The issue is that this dynamical system is rather pathological. Generally, any two points in the boundary at infinity $\partial\Gamma$ of Γ are joined by infinitely many orbits of the flow.

As Gromov explained in [25] – see also Coornaert and Papadopoulos [15, 16] – one can restrict our attention to an invariant subset of $\mathcal{G}\Gamma$: roughly speaking, all the bi-infinite geodesics whose labels are minimal for the lexicographic order induced by some fixed, arbitrary order on the symmetric, generating set S of G. Formally the system that we consider is the following. One introduces a space $\mathfrak{H}_0(\Gamma)$ of horofunctions on Γ (which generalize the Busemann functions) on which the group G acts. Any horofunction $h \in \mathfrak{H}_0(\Gamma)$ naturally comes with a *preferred* gradient line starting at 1, i.e. whose labelling is minimal for the fixed lexicographic order. Calling $\theta(h)$ the first letter of our preferred gradient line, the transformation $T\colon \mathfrak{H}_0(\Gamma) \to \mathfrak{H}_0(\Gamma)$ is defined by sending h to $\theta(h)^{-1}h$. Remarkably, the system $(\mathfrak{H}_0(\Gamma), T)$ is conjugated to a subshift of finite type (Σ, σ). Geometrically, the suspension of (Σ, σ) should be thought of as analogue of the geodesic flow on the unit tangent bundle of compact, negatively curved manifold M, while the suspension of the extension $(\Sigma_\theta, \sigma_\theta)$ plays the role of the geodesic flow on the unit tangent bundle of its universal cover $\tilde{M}$. Nevertheless, unlike in the Riemannian setting, this flow is neither mixing nor, a-priori, topologically transitive. This reflects the fact that two points in the boundary at infinity of Γ can still be joined by finitely many orbits of the flow.

Actually, the dynamical properties of $(\mathfrak{H}_0(\Gamma), T)$ are very sensitive to the choice of the order on S. For instance, if G is the direct product of the free group with a finite group, then $\mathfrak{H}_0(\Gamma)$ naturally splits into several disjoint "layers" $\mathfrak{I}_0, \dots, \mathfrak{I}_n$, where $\mathfrak{I}_0$ is invariant under the action of T. Moreover, depending on the choice of the order on S, the other layers $\mathfrak{I}_i$ are either invariant under the action of T, or mapped into $\mathfrak{I}_0$. For more details, we refer the reader to Example 4.9.

To circumvent this difficulty, we are forced to restrict our study to an irreducible component $\mathfrak{I}$ of the system $(\mathfrak{H}_0(\Gamma), T)$. The price to pay though, is that the extension of $(\mathfrak{I}, T)$ by the map $\theta\colon \mathfrak{H}_0(\Gamma) \to G$ may not "visit" the whole Cayley graph Γ of G. If it misses a large portion of Γ, then this system will be useless for counting purposes. However, we show that there exists an irreducible component $\mathfrak{I}$ whose extension has the visibility property (see Definition 2.1). Our strategy to produce such an irreducible component is inspired by an idea of Constantine, Lafont and Thompson [10], and based on a construction of Gromov. Namely, in [25] Gromov build from $(\mathcal{G}\Gamma, \phi_t)$ a new flow $(\widehat{\mathcal{G}}\Gamma, \psi_s)$ with enhanced properties – see also [30, 8]. The space $\widehat{\mathcal{G}}\Gamma$ is quasi-isometric to Γ, hence its boundary at infinity is homeomorphic to $\partial\Gamma$; every two points in $\partial\Gamma$ are joined by a *unique* orbit of the flow ψ_s; there is a natural projection $\mathcal{G}\Gamma \to \widehat{\mathcal{G}}\Gamma$ which send every ϕ_t-orbit homeomorphically onto a ψ_s-orbit. It turns out that the new flow $(\widehat{\mathcal{G}}\Gamma, \psi_s)$ is topologically transitive. In particular, it admits a dense orbit. The irreducible component $\mathfrak{I}$ is, roughly speaking, the closure of a lift of this dense orbit. The transitivity of $(\widehat{\mathcal{G}}\Gamma, \psi_s)$ tells us that the the extension of $(\mathfrak{I}, T)$ passes uniformly near every point of Γ, hence ensuring the visibility property.

In order to apply our criterion (Theorem 1.2) to the system $(\mathfrak{I}, T)$ we finally need to define a potential $F\colon \mathfrak{I} \to \mathbf{R}_+^*$ with bounded Hölder variations. Recall that the dynamical system $(\mathfrak{I}, T)$ was not build directly from the metric space X we are interested in: thus, the role of the potential is to reflect the geometry of X. If X coincides with the Cayley graph of Γ (which corresponds to the first case of Theorem 1.1), we simply take for F the constant map equal to 1. In this situation we prove that the spectral radius of the corresponding Ruelle transfer operator $\mathcal{L}$ is $\rho = e^{\omega_G}$, whereas the one of the twisted transfer operator satisfies $\rho_\lambda \geqslant e^{\omega_H}$. Hence the conclusion of Theorem 1.1 directly follows from our amenability criterion. If X is a CAT(-1) space (which corresponds to the second case of Theorem 1.1) we use a quasi-isometry between Γ and X to define a potential F describing the geometry of X. In this situation the CAT(-1) geometry is crucial to ensure that F has bounded Hölder variations. Once this is done, we provide as before estimates of the spectral radii ρ and ρ_λ in terms of ω_G and ω_H and conclude by the amenability criterion.

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2 Preliminaries

2.1 Subshift of finite type

Vocabulary. Let $\mathcal{A}$ be a finite set. We write $\mathcal{A}^*$ for the set of all finite words over the alphabet $\mathcal{A}$. The length of a word $w \in \mathcal{A}^*$ is denoted by $|w|$. Given $n \in \mathbf{N}$, the set of all words of length n is

denoted by $\mathcal{A}^n$. The set $\mathcal{A}^{\mathbf{N}}$ is is endowed with a distance d defined as follows: given $x, y \in \Sigma$, we let $d(x,y) = e^{-n}$ where n is the length of the longest common prefix of x and y. Let $\sigma\colon \mathcal{A}^{\mathbf{N}} \to \mathcal{A}^{\mathbf{N}}$ be the shift operator, i.e. the map sending the sequence $(x_i)_{i\in\mathbf{N}}$ to $(x_{i+1})_{i\in\mathbf{N}}$. Let Σ be a *subshift* i.e. a closed σ-invariant subset $\mathcal{A}^{\mathbf{N}}$. A word $w \in \mathcal{A}^*$ is *admissible* if it is the prefix of a sequence $x = (x_i)$ in Σ. We denote by $\mathcal{W}$ the set of all finite admissible words. For every $n \in \mathbf{N}$, we write $\mathcal{W}^n = \mathcal{W} \cap \mathcal{A}^n$ for the set of admissible words of length n. Given $w \in \mathcal{W}$, the *cylinder of* w, denoted by $[w]$, is the set of sequences $x \in \Sigma$ such that the prefix of length $|w|$ of x is exactly w. We refer to $|w|$ as the *length* of the cylinder. From now on we assume that Σ is a *subshift of finite type*, i.e. there exists $N \in \mathbf{N}$ with the following property: a sequence $x \in \mathcal{A}^{\mathbf{N}}$ belongs to Σ if and only if every subword of length N of Σ belongs to $\mathcal{W}^N$. We say that (Σ, σ) is *irreducible* or *topologically transitive* if (Σ, σ) admits a dense orbit. As (Σ, σ) is a subshift of finite type it is equivalent to ask that one of the following holds.

(i) For every $x, y \in \Sigma$, for every $\varepsilon \in \mathbf{R}_+^*$, there exists $k \in \mathbf{N}$ and $x' \in \Sigma$ such that $d(x, x') \leqslant \varepsilon$ and $\sigma^k x' = y$

(ii) For every $w, w' \in \mathcal{W}$, there exists $w_0 \in \mathcal{W}$ such that the concatenation ww_0w' is admissible

Irreducible components. We associate to (Σ, σ) an oriented graph $\Gamma = (V, E)$ labeled by $\mathcal{A}$ describing the dynamics of the shift. We still denote by N the integer given by the definition of a subshift of finite type. The vertex set V of Γ is simply $\mathcal{W}^N$. Given two words $w_1, w_2 \in \mathcal{W}^N$ and a letter $a \in \mathcal{A}$, there is an edge from w_1 to w_2 labelled by a if w_1 is the prefix of length N of aw_2. It follows from the definition of N, that the labelling of Γ induces a one-to-one correspondance between Σ and the set of infinite oriented paths in Γ. We now define an equivalence relation on $\mathcal{W}^N$ seen as the vertex set of Γ. Two vertices $w, w' \in \mathcal{W}^N$ are *communicating* and we write $w \sim w'$ if there exists an oriented loop in Γ passing through w and w'. The corresponding equivalence classes $V_1, \dots, V_m$ are called *communicating classes*.

Given $i \in [\![1, m]\!]$, we write Γ_i for the full subgraph of Γ associated to V_i. The subspace $\Sigma_i \subset \Sigma$ is defined as the set of all sequences of $\mathcal{A}^{\mathbf{N}}$ labeling an infinite path in Γ_i. Observe that Σ_i is a closed σ-invariant subset of Σ and (Σ_i, σ) is a subshift of finite type. It follows from the construction that (Σ_i, σ) is irreducible.

We observe that for every sequence $x \in \Sigma$, there exists $n_0 \in \mathbf{N}$, and a unique $i \in [\![0, m]\!]$ such that for all integers $n \geqslant n_0$, the sequence $\sigma^n x$ belongs to Σ_i. Indeed one can derive from the relation $\sim$ a new graph $\Gamma/\!\sim$ defined as follows. Its vertex set is the set of communicating classes $V_1, \dots, V_m$. There is an edge in $\Gamma/\!\sim$ from V_i to V_j, if Γ contains an edge joining a vertex in V_i to a vertex in V_j. The graph $\Gamma/\!\sim$ comes with a natural projection $\Gamma \to \Gamma/\!\sim$. It follows from the definition of $\sim$ that $\Gamma/\sim$ does not contain any oriented loop. Hence if c is an infinite path in Γ, its projection in $\Gamma/\!\sim$ is ultimately constant. This means that there exists $i \in [\![1, m]\!]$ such that the path obtained from c by removing its first edges is contained in Γ_i. Hence if x is the sequence labelling c, there exists $n_0 \in \mathbf{N}$, such that for all integers $n \geqslant n_0$, the sequence $\sigma^n x$ belongs to Σ_i. We say that Σ_i is the *asymptotic irreducible component of* x.

Group extension. Let G be a finitely generated group. Let $\theta\colon \Sigma \to G$ be a continuous map. We denote by Σ_θ the direct product $\Sigma_\theta = \Sigma \times G$ endowed with the product topology. We define a continuous map $\sigma_\theta\colon \Sigma_\theta \to \Sigma_\theta$ by

$$\sigma_\theta(x, g) = (\sigma x, g\theta(x)), \quad \forall (x, g) \in \Sigma_\theta.$$

The dynamical system $(\Sigma_\theta, \sigma_\theta)$ is called the *extension of* (X, σ) *by* θ. For every $n \in \mathbf{N}$, for every $x \in \Sigma$ we define the cocycle $\theta_n(x)$ by

$$\theta_n(x) = \theta(x)\theta(\sigma x) \cdots \theta\left(\sigma^{n-1} x\right). \tag{3}$$

By convention $\theta_0 \colon \Sigma \to G$ is the constant map equal to 1 (the identity of G). Hence we have

$$\sigma_\theta^n(x, g) = \left(\sigma^n x, g\theta_n(x)\right).$$

Definition 2.1. We say that the extension $(\Sigma_\theta, \sigma_\theta)$ has the *visibility property* if there exists a finite subset U of G with the following property: for every $g \in G$, there is a point $x \in \Sigma$, an integer $n \in \mathbf{N}$ and two elements $u_1, u_2 \in U$ such that $g = u_1\theta_n(x)u_2$.

Remark. This definition is somewhat reminiscent of Dougall's *linear visibility with remainder* property [21, Definition 3.1]. Nevertheless our notion is more flexible as we do not ask to control the value of n in terms of the word length of g.

2.2 Hyperbolic geometry

The four point inequality. Let (X, d) be a geodesic metric space. The Gromov product of three points $x, y, z \in X$ is defined by

$$\langle x, y \rangle_z = \frac{1}{2}\left\{d(x, z) + d(y, z) - d(x, y)\right\}.$$

We assume that the space X is δ-*hyperbolic*, i.e. for every $x, y, z, t \in X$,

$$\langle x, z \rangle_t \geqslant \min\left\{\langle x, y \rangle_t, \langle y, z \rangle_t\right\} - \delta, \tag{4}$$

The boundary at infinity. Let x_0 be a base point of X. A sequence (y_n) of points of X *converges to infinity* if $\langle y_n, y_m \rangle_{x_0}$ tends to infinity as n and m approach to infinity. The set $\mathcal{S}$ of such sequences is endowed with a binary relation defined as follows. Two sequences (y_n) and (z_n) are related if

$$\lim_{n \to +\infty} \langle y_n, z_n \rangle_{x_0} = +\infty.$$

If follows from (4) that this relation is actually an equivalence relation. The *boundary at infinity* of X denoted by ∂X is the quotient of $\mathcal{S}$ by this relation. If the sequence (y_n) is an element in the class of $\xi \in \partial X$ we say that (y_n) *converges* to ξ and write

$$\lim_{n \to +\infty} y_n = \xi.$$

Note that the definition of ∂X does not depend on the base point x_0. For more details about the Gromov boundary, we refer the reader to [13, Chapitre 2]. The notation $\partial^2 X$ stands for

$$\partial^2 X = \partial X \times \partial X \setminus \mathrm{diag}(\partial X),$$

where $\mathrm{diag}(\partial X) = \{(\xi, \xi) \mid \xi \in \partial X\}$ is the diagonal.

Quasi-geodesics. One major feature of hyperbolic spaces is the stability of quasi-geodesics also known as Morse's Lemma.

Definition 2.2. Let $\kappa \in \mathbf{R}_+^*$ and $\ell \in \mathbf{R}_+$. Let $f\colon X_1 \to X_2$ be a map between two metric spaces. We say that f is a (κ, ℓ)-*quasi-isometric embedding*, if for every $x, x' \in X_1$ we have

$$\kappa^{-1} d(x, x') - \ell \leqslant d(f(x), f(x')) \leqslant \kappa d(x, x') + \ell.$$

A (κ, ℓ)-*quasi-geodesic* of X, is a (κ, ℓ)-quasi-isometric embedding of an interval of $\mathbf{R}$ into X.

Given a (κ, ℓ)-quasi-geodesic $\gamma\colon \mathbf{R}_+ \to X$, there exists a point $\xi \in \partial X$ such that for every sequence (t_n) diverging to infinity, we have

$$\lim_{n \to +\infty} \gamma(t_n) = \xi,$$

see for instance [13, Chapitre 3, Théorème 2.2]. In this situation we consider ξ as the endpoint at infinity of γ and write $\gamma(\infty) = \xi$. If $\gamma\colon \mathbf{R} \to X$ is a bi-infinite (κ, ℓ)-quasi-geodesic, we define $\gamma(-\infty)$ in the same way.

Proposition 2.3 ([13, Chapitre 3, Théorèmes 1.2 et 3.1])**.** *Let $\kappa \in \mathbf{R}_+^*$ and $\ell \in \mathbf{R}_+$. There exists $D = D(\kappa, \ell, \delta)$ in $\mathbf{R}_+$ such that the Hausdorff distance between any two (κ, ℓ)-quasi-geodesics with the same endpoints (possibly in ∂X) is at most D.*

Group action. Let x_0 be a base point of X. Let G be a group acting properly by isometries on X. The *exponential growth rate of G acting on X* is the quantity

$$\omega(G, X) = \limsup_{r \to \infty} \frac{1}{r} \ln \left| \{ g \in G \mid d(gx_0, x_0) \leqslant r \} \right|. \tag{5}$$

Note that $\omega(G, X)$ does not depend on x_0. It can also be interpreted as the critical exponent of the Poincaré series

$$\mathcal{P}_G(s) = \sum_{g \in G} e^{-s d(gx_0, x_0)}.$$

If there is no ambiguity, we simply write ω_G instead of $\omega(G, X)$.

3 Horofunctions

In this section we recall the definition of horofunctions and gradient lines introduced by Gromov in [25, Section 7.5]. We follow the exposition given by Coornaert and Papadopoulos in [14, 15]. Let (X, x_0) be a pointed geodesic δ-hyperbolic space and G be a group acting by isometries on X.

Horofunctions. Let $\varepsilon \in \mathbf{R}_+$. A map $f\colon X \to \mathbf{R}$ is *ε-quasi-convex* if for every geodesic $\gamma\colon I \to X$, for every $a, b \in I$, for every $t \in [0\,, 1]$ we have

$$f \circ \gamma\left(ta + (1-t)b\right) \leqslant t f \circ \gamma(a) + (1-t) f \circ \gamma(b) + \varepsilon.$$

The map f is *quasi-convex* if there exists $\varepsilon \in \mathbf{R}_+$ such that f is ε-quasi-convex.

Definition 3.1. A *horofunction* is a quasi-convex map $h\colon X \to \mathbf{R}$ satisfying the following distance-like property: for every $x \in X$, for every $r \in \mathbf{R}$, if $r \leqslant h(x)$, then

$$h(x) = r + d\big(x, h^{-1}(r)\big)\,. \tag{6}$$

A *cocycle* is a map $c\colon X \times X \to \mathbf{R}$ of the form $c(x,y) = h(x) - h(y)$ where h is a horofunction. In this situation, h is called a *primitive* of c.

Let $C(X)$ be the set of all continuous function $f\colon X \to \mathbf{R}$ endowed with the topology of uniform convergence on compact subsets. We denote by $C_*(X)$ the quotient of $C(X)$ by the 1-dimensional closed subspace of constant functions. The space $C_*(X)$ is endowed with the quotient topology. It can be identified with the set of all continuous functions $f\colon X \to \mathbf{R}$ vanishing at x_0. This is the point of view that we adopt here. The action of G on X induces an action on $C(X)$ and thus on $C_*(X)$. More precisely $g \in G$, and $f \in C_*(X)$ the map $g \cdot f$ is defined by

$$[g \cdot f]\,(x) = f(g^{-1}x) - f(g^{-1}x_0), \quad \forall x \in X.$$

A horofunction is automatically 1-Lipschitz, hence continuous [15, Proposition 2.2]. Moreover, given a cocycle c, any two primitives h_1 and h_2 of c differ by a constant, i.e. there exists $a \in \mathbf{R}$, such that for every $x \in X$, we have $h_2(x) - h_1(x) = a$. Hence the set of all cocycles, or equivalently all horofunctions vanishing at x_0, embeds in $C_*(X)$. We denote it by $\mathfrak{H}(X)$. It is a compact subspace of $C_*(X)$ [15, Proposition 3.9].

Gradient lines. The gradient lines are an important tool to track the behavior of horofunctions.

Definition 3.2. Let $h \in \mathfrak{H}(X)$ be a horofunction. A *gradient line for* h or an *h-gradient line* is a path $\gamma\colon I \to X$ parametrized by arc length such that for every $t, t' \in I$ we have

$$h(\gamma(t)) - h(\gamma(t')) = t' - t.$$

If the interval I has the form $I = \mathbf{R}_+$ we say that γ a *h-gradient ray.*

Let us recall a few properties of gradient lines. Let $h \in \mathfrak{H}(X)$. Every h-gradient line is a geodesic [15, Proposition 2.10]. If $\gamma\colon I \to X$ be an h-gradient line, then for every $g \in G$, the path $g\gamma$ is a gradient line for gh [15, Proposition 2.14]. For every $x \in X$, there exists an h-gradient ray starting at x [15, Proposition 2.13].

Let $\gamma\colon \mathbf{R}_+ \to X$ be a geodesic ray. The *Busemann function* associated to γ is the map $b_\gamma\colon X \to \mathbf{R}$ defined by

$$b_\gamma(x) = \lim_{t\to\infty} \Big[\, d(x, \gamma(t)) - t \,\Big], \quad \forall x \in X.$$

The map $c_\gamma\colon X \times X \to \mathbf{R}$ defined by $c_\gamma(x,y) = b_\gamma(x) - b_\gamma(y)$ is a cocycle [14, Chapter 3, Proposition 3.6]. Moreover γ is a gradient ray starting at $\gamma(0)$ for this cocycle. However, one has to be careful that c_γ may admit other gradient lines starting at $\gamma(0)$. Similarly, given a horofunction $h \in \mathfrak{H}(X)$, the Busemann function associated to an h-gradient ray starting at x_0 is not necessarily h.

Comparison with the Gromov boundary. Recall that ∂X is the Gromov boundary of X.

Proposition 3.3 (Coornaert-Papadopoulos [15, Proposition 3.1])**.** *Let $h \in \mathfrak{H}(X)$ be a horofunction. Let $\gamma_1 \colon \mathbf{R}_+ \to X$ and $\gamma_2 \colon \mathbf{R}_+ \to X$ be two h-gradient rays. Then $\gamma_1(\infty) = \gamma_2(\infty)$.*

This gives rise to a map $\pi \colon \mathfrak{H}(X) \to \partial X$ which associates to any $h \in \mathfrak{H}(X)$ the endpoint at infinity of any h-gradient ray.

Proposition 3.4 (Coornaert-Papadopoulos [15, Proposition 3.3 and Corollary 3.8])**.** *The map $\pi \colon \mathfrak{H}(X) \to \partial X$ is continuous, G-equivariant and onto. More precisely for every geodesic ray $\gamma \colon \mathbf{R}_+ \to X$ starting at x_0 the corresponding Busemann function b_γ is a preimage of $\gamma(\infty)$ in $\mathfrak{H}(X)$. Two horofunctions $h, h' \in \mathfrak{H}(X)$ have the same image in ∂X if and only if $\|h - h'\|_\infty \leqslant 64\delta$.*

4 Dynamics in a hyperbolic group

In this section we introduce a few dynamical systems to describe the "geodesic flow" of a hyperbolic Cayley graph. Let G be a hyperbolic group and A a finite generating set of G. For simplicity we assume that A is symmetric, i.e. $A^{-1} = A$. We denote by $\Gamma(G, A)$ or simply Γ the Cayley graph of G with respect to A. We identify G with the vertex set of Γ. We consider 1 as a base point in Γ. For every $n \in \mathbf{N}$, we write $S(n)$ for the sphere of radius n in Γ centered at the identity, i.e. the set of all elements $g \in G$ such that $d_\Gamma(1, g) = n$. Similarly we write $B(n)$ for the closed ball of radius n.

4.1 Transitivity of Gromov's geodesic flow

We denote by $\mathcal{G}\Gamma$ the set of all *parametrized* bi-infinite geodesic $\gamma \colon \mathbf{R} \to \Gamma$ of Γ. This set is endowed with a distance defined as follows: given two bi-infinite geodesics $\gamma_1, \gamma_2 \colon \mathbf{R} \to \Gamma$ we let

$$d(\gamma_1, \gamma_2) = \int_{-\infty}^{\infty} e^{-|t|} d(\gamma_1(t), \gamma_2(t))\, dt. \tag{7}$$

The action of G on Γ induces an action by isometries of G on $\mathcal{G}\Gamma$. One checks easily that the map $\mathcal{G}\Gamma \to \Gamma$ sending γ to $\gamma(0)$ is a G-equivariant quasi-isometry. The space $\mathcal{G}\Gamma$ also comes with a flow $\phi = (\phi_s)_{s \in \mathbf{R}}$ defined as follows: for every $\gamma \in \mathcal{G}\Gamma$, for every $s \in \mathbf{R}$, the geodesic $\phi_s(\gamma) \colon \mathbf{R} \to \Gamma$ is given by

$$\phi_s(\gamma)(t) = \gamma(s + t), \quad \forall t \in \mathbf{R}.$$

Starting from $\mathcal{G}\Gamma$, Gromov build a new hyperbolic space $\widehat{\mathcal{G}}\Gamma$ that is quasi-isometric to Γ. In particular it is hyperbolic and its boundary at infinity is homeomorphic to $\partial\Gamma$. Moreover $\widehat{\mathcal{G}}\Gamma$ comes with a flow so that any two distinct points of $\partial\Gamma$ are joined by a *unique* orbit of the flow. The construction is given in [25, Section 8.3], the details can be found in [8, 30]. We recall here the main properties of this space.

(F1) The space $\widehat{\mathcal{G}}\Gamma$ is geodesic and proper. It is endowed with a proper co-compact action by isometries of G as well as a flow $\psi = (\psi_s)_{s \in \mathbf{R}}$. The flow and the action of G commute, i.e. for every $\widehat{\gamma} \in \widehat{\mathcal{G}}\Gamma$, for every $s \in \mathbf{R}$ and $g \in G$ we have $\psi_s(g\widehat{\gamma}) = g\psi_s(\widehat{\gamma})$.

(F2) There exists a continuous G-equivariant quasi-isometric projection $p\colon \mathcal{G}\Gamma \twoheadrightarrow \widehat{\mathcal{G}}\Gamma$. In particular p induces a homeomorphism p_∞ from $\partial\Gamma$ onto the boundary at infinity of $\widehat{\mathcal{G}}\Gamma$. In addition, for every geodesic $\gamma \in \mathcal{G}\Gamma$, the projection p maps the ϕ-orbit of $\gamma \in \mathcal{G}\Gamma$ homeomorphically onto the ψ-orbit of $\widehat{\gamma} = p(\gamma)$.

(F3) For every point $\widehat{\gamma} \in \widehat{\mathcal{G}}\Gamma$, the map $\mathbf{R} \to \widehat{\mathcal{G}}\Gamma$ sending s to $\psi_s(\widehat{\gamma})$ is a quasi-isometric embedding of $\mathbf{R}$ into $\widehat{\mathcal{G}}\Gamma$. Hence for every point $\widehat{\gamma} \in \widehat{\mathcal{G}}\Gamma$ one can associate two distinct points in the boundary at infinity of $\widehat{\mathcal{G}}\Gamma$ defined by

$$\widehat{\gamma}(\infty) = \lim_{s\to\infty} \psi_s(\widehat{\gamma}) \quad \text{and} \quad \widehat{\gamma}(-\infty) = \lim_{s\to-\infty} \psi_s(\widehat{\gamma}).$$

By construction for every geodesic $\gamma \in \mathcal{G}\Gamma$, the homeomorphism p_∞ maps $\gamma(\infty)$ and $\gamma(-\infty)$ to $\widehat{\gamma}(\infty)$ and $\widehat{\gamma}(-\infty)$, where $\widehat{\gamma} = p(\gamma)$.

(F4) The map $\widehat{\mathcal{G}}\Gamma \to \partial^2\Gamma$ sending $\widehat{\gamma}$ to $(\widehat{\gamma}(-\infty), \widehat{\gamma}(\infty))$ induces a homeomorphism from $\widehat{\mathcal{G}}\Gamma/\mathbf{R}$ onto $\partial^2\Gamma$. Actually $\widehat{\mathcal{G}}\Gamma$ is homeomorphic to $\partial^2\Gamma \times \mathbf{R}$.

It is important to notice that in general p does not conjugate the flow, i.e. $p \circ \phi_s \neq \psi_s \circ p$. Since the flow ψ and the action of G commute, the flow ψ induces a flow on $\widehat{\mathcal{G}}\Gamma/G$ that we denote $\overline{\psi} = (\overline{\psi}_s)_{s\in\mathbf{R}}$.

Proposition 4.1. *The flow $\overline{\psi}$ on $\widehat{\mathcal{G}}\Gamma/G$ is topologically transitive, i.e. given any two non-empty open subsets U and V of $\widehat{\mathcal{G}}\Gamma/G$, there exists $s \in \mathbf{R}$ such that $\overline{\psi}_s(U) \cap V$ is non-empty.*

The remainder of the section is dedicated to the proof of this proposition. We follow mostly the strategy used in [2, Chapter III].

Lemma 4.2. *Let $\gamma, \gamma' \in \mathcal{G}\Gamma$ such that $\gamma(\infty) \neq \gamma'(-\infty)$. There are sequences (ν_n) of geodesics in $\mathcal{G}\Gamma$, (g_n) of elements in G and (t_n) of numbers in $\mathbf{R}$ diverging to infinity with the following properties.*

(i) (ν_n) converges to a geodesic with the same endpoints at γ.

(ii) $(g_n\phi_{t_n}(\nu_n))$ converges to a geodesic with the same endpoints as γ'.

Proof. For simplicity we write ξ_+ and ξ_- for $\gamma(\infty)$ and $\gamma(-\infty)$. Similarly we define ξ'_+ and ξ'_-. Since $\xi_+ \neq \xi'_-$, there exists a sequence (g_n) of elements of G such that for some (hence any) $x \in \Gamma$ the sequence $(g_n x)$ and $(g_n^{-1}x)$ respectively converge to ξ'_- and ξ_+ – see for instance [18, Chapter III, Lemma 2.2.]. Nevertheless, the action of G on $\Gamma \cup \partial\Gamma$ is a convergence action. Up to replacing (g_n) by a subsequence, we can assume that for every for every $\xi \in \partial\Gamma \setminus \{\xi_+\}$, the sequence $(g_n\xi)$ converges to ξ'_- and for every $\xi \in \partial\Gamma \setminus \{\xi'_-\}$, the sequence $(g_n^{-1}\xi)$ converges to ξ_+ [37, Theorem 3A]. For every $n \in \mathbf{N}$, we denote by $\nu_n\colon \mathbf{R} \to \Gamma$ a bi-infinite geodesic joining ξ_- to $g_n^{-1}\xi'_+$. We observe that

- $\nu_n(-\infty)$ and $\nu_n(\infty)$ respectively converge to ξ_- and ξ_+, whereas
- $g_n\nu_n(-\infty)$ and $g_n\nu_n(\infty)$ respectively converge to ξ'_- and ξ'_+.

Geodesic triangles in $\Gamma \cup \partial\Gamma$ are 24δ-thin [13, Chapitre 2 Proposition 2.2]. Up to passing again to a subsequence we may assume that $d(\gamma(0), \nu_n) \leqslant 24\delta$ and $d(\gamma'(0), g_n\nu_n) \leqslant 24\delta$. By shifting if necessary the origin of ν_n we can assume that for every $n \in \mathbf{N}$, there exists $t_n \in \mathbf{R}$, such that $d(\gamma(0), \nu_n(0)) \leqslant 24\delta$ and $d(\gamma'(0), g_n\nu_n(t_n)) \leqslant 24\delta$. According to the Azelà-Ascoli theorem (ν_n) converges to a geodesic ν. It follows from our choice of (g_n) that ν has the same endpoints as γ. Similarly we obtain that $(g_n\phi_{t_n}(\nu_n))$ converges to a geodesic with the same endpoints as γ'. Note also that $d(\nu_n(t_n), g_n^{-1}\gamma'(0))$ and $d(\nu_n(0), \gamma(0))$ are uniformly bounded. As $(g_n^{-1}\gamma'(0))$ converges to ξ_+, the sequence (t_n) has to diverge to ∞. □

Proof of Proposition 4.1. It suffices to show that for every non-empty open subset $\widehat{U}$ and $\widehat{V}$ of $\widehat{\mathcal{G}}\Gamma$, there exists $s \in \mathbf{R}$, and $g \in G$ such that $g\psi_s(\widehat{U}) \cap \widehat{V} \neq \emptyset$. Let $\widehat{U}$ and $\widehat{V}$ be two non-empty open subsets of $\widehat{\mathcal{G}}\Gamma$. We denote by $\widehat{U}(\infty)$ the following subset of $\partial\Gamma$

$$\widehat{U}(\infty) = \left\{ \widehat{\gamma}(\infty) \;\middle|\; \widehat{\gamma} \in \widehat{U} \right\}.$$

The set $\widehat{V}(-\infty)$ is defined in a similar way. It follows from (F4) that $\widehat{U}(\infty)$ and $\widehat{V}(-\infty)$ are open non-empty subsets of $\partial\Gamma$. In particular they are not reduced to a point. Hence there exists $\widehat{\gamma} \in \widehat{U}$ and $\widehat{\gamma}' \in \widehat{V}$ such that $\widehat{\gamma}(\infty) \neq \widehat{\gamma}'(-\infty)$. Let $\gamma, \gamma' \in \mathcal{G}\Gamma$ be respective preimages of $\widehat{\gamma}$ and $\widehat{\gamma}'$. According to (F3), $\gamma(\infty) \neq \gamma'(-\infty)$. Applying Lemma 4.2, there are sequences (ν_n) of geodesics in $\mathcal{G}\Gamma$, (g_n) of elements in G and (t_n) of numbers in $\mathbf{R}$ diverging to infinity with the following properties: the geodesic ν_n converges to a geodesic ν with the same endpoints as γ; the geodesic $\nu'_n = g_n\phi_{t_n}(\nu_n)$ converges to a geodesic ν' with the same endpoints as γ'.

We now push these data in $\widehat{\mathcal{G}}\Gamma$ using the projection $p\colon \mathcal{G}\Gamma \twoheadrightarrow \widehat{\mathcal{G}}\Gamma$. For every $n \in \mathbf{N}$, we let $\widehat{\nu}_n = p(\nu_n)$ and $\widehat{\nu}'_n = p(\nu'_n)$. We define $\widehat{\nu}$ and $\widehat{\nu}'$ in the same way. Since p maps homeomorphically ϕ-orbits onto ψ-orbits, there exists a sequence (s_n) of numbers in $\mathbf{R}$, diverging to ∞ such that for every $n \in \mathbf{N}$, we have

$$p\left(\phi_{t_n}(\nu_n)\right) = \psi_{s_n}\left(\widehat{\nu}_n\right).$$

By construction the endpoints of $\widehat{\nu}$ are the same as those of $\widehat{\gamma}$. However there is a unique orbit of the flow ψ joining two distinct point of $\partial\Gamma$ – see (F4). Thus there exists $s \in \mathbf{R}$ such that $\widehat{\gamma} = \psi_s(\widehat{\nu})$. Recall that ν_n converges to ν. Since the projection p is continuous, $\widehat{\nu}_n$ converges to $\widehat{\nu}$, hence $\psi_s(\widehat{\nu}_n)$ converges to $\widehat{\gamma}$. Similarly we prove that there exists $s' \in \mathbf{R}$ such that $\psi_{s'}(\widehat{\nu}'_n) = g_n\psi_{s_n+s'}(\widehat{\nu}_n)$ converges to $\widehat{\gamma}'$. As $\widehat{U}$ and $\widehat{V}$ are open, there exists $n_0 \in \mathbf{N}$ such that for every $n \geqslant n_0$, the point $\psi_s(\widehat{\nu}_n)$ belongs to $\widehat{U}$ while $g_n\psi_{s_n+s'}(\widehat{\nu}_n)$ belongs to $\widehat{V}$. Hence

$$g_n\psi_{s_n+s'-s}(\widehat{U}) \cap \widehat{V} \neq \emptyset, \quad \forall n \geqslant n_0,$$

which completes the proof of the proposition. □

Corollary 4.3 (Compare Dal'bo [18, Chapter III, Theorem 4.2])**.** *There exists a point $\widehat{\gamma} \in \widehat{\mathcal{G}}\Gamma$ such that the image of $\{\psi_s(\widehat{\gamma}) \mid s \in \mathbf{R}_+\}$ in $\widehat{\mathcal{G}}\Gamma/G$ is dense.*

4.2 Dynamics on the space of horofunctions

Definition 4.4. A horofunction $h \in \mathfrak{H}(\Gamma)$ is *integral* if its restriction to the vertex set G takes integer values. We write $\mathfrak{H}_0(\Gamma)$ (or simply $\mathfrak{H}_0$) for the subset of $\mathfrak{H}(\Gamma)$ consisting of all integral horofunctions.

Note that $\mathfrak{H}_0$ is closed, G-invariant subset of $\mathfrak{H}(\Gamma)$. Moreover, the Busemann function of any geodesic ray starting at 1 is an integral horofunction. Consequently Proposition 3.4 leads to the following statement

Proposition 4.5. *The projection $\pi\colon \mathfrak{H}(\Gamma) \to \partial\Gamma$ maps $\mathfrak{H}_0$ onto $\partial\Gamma$.*

We now define a map $T\colon \mathfrak{H}_0 \to \mathfrak{H}_0$ which can be thought as an analogue of the first return map for a section of the geodesic flow. To that end we endow the generating set A with an arbitrary total order. The map $\theta\colon \mathfrak{H}_0 \to G$ is the one sending h to the smallest element $a \in A$ such that $h(a) = h(1) - 1$. The existence of such a generator $a \in A$ is ensured by [15, Lemma 5.1].

Definition 4.6. The map $T\colon \mathfrak{H}_0 \to \mathfrak{H}_0$ is the one sending h to $\theta(h)^{-1}h$.

This map is continuous [15, Proposition 5.6]. Before speaking of coding, let us recall a few properties of the dynamical system $(\mathfrak{H}_0, T)$. Recall that the edges of Γ are labelled by elements of A. Let $h \in \mathfrak{H}_0$ be a horofunction and x a vertex of Γ. The lexicographic order on $A^{\mathbf{N}}$ induces a total order on the set of h-gradient rays starting at x. This set admits a smallest element [15, Proposition 5.2] that we call the *minimal h-gradient ray starting at x*. Assume that $\gamma\colon \mathbf{R}_+ \to \Gamma$ is the minimal h-gradient ray starting at x. We have the following properties:

(i) for every $n \in \mathbf{N}$, the path $\gamma_n\colon \mathbf{R}_+ \to \Gamma$ defined by $\gamma_n(t) = \gamma(t+n)$ is the minimal h-gradient ray starting at $\gamma(n)$ [15, Proposition 5.3];

(ii) for every $g \in G$, $g\gamma$ is the minimal gh-gradient ray starting at gx [15, Proposition 5.4].

As usual we define for every $n \in \mathbf{N}$, the cocycle $\theta_n\colon \mathfrak{H}_0 \to G$ by

$$\theta_n(h) = \theta(h)\theta(Th)\cdots\theta(T^{n-1}h). \tag{8}$$

It follows from the previous two observations, that this cocycle has the following geometric interpretation: if $\gamma\colon \mathbf{R} \to \Gamma$ is the minimal h-gradient line starting at 1, then $\gamma(n) = \theta_n(h)$.

Connection with Gromov's geodesic flow. We relate here $(\mathfrak{H}_0, T)$ to Gromov's geodesic flow on Γ. Let $h \in \mathfrak{H}_0$ be an integral horofunction. A bi-infinite h-gradient line $\gamma\colon \mathbf{R} \to \Gamma$ is called *primitive* if the following holds

(i) $\gamma(0)$ is a vertex of Γ, and thus $\gamma(\mathbf{Z})$ is contained in G.

(ii) For every integer $m \in \mathbf{Z}$, the path $\gamma_m\colon \mathbf{R}_+ \to \Gamma$ sending t to $\gamma(m+t)$ is the minimal h-gradient ray starting at $\gamma(m)$.

Observe that in this case, for every $k \in \mathbf{Z}$, the path $\phi_k(\gamma)$ is also a primitive h-gradient line. Similarly for every $g \in G$, the path $g\gamma$ is a primitive gh-gradient line [16, Lemma 2.5]. Given $h \in \mathfrak{H}_0$ and $\xi \in \partial\Gamma$, there is always a primitive h-gradient line such that $\gamma(-\infty) = \xi$ [16, Proposition 5.2].

Let $\mathfrak{H}_{\mathbf{R}}$ be the subset of $\mathfrak{H}_0 \times \mathcal{G}\Gamma$ that consists of all pairs (h, γ) where $h \in \mathfrak{H}_0$ is an integral horofunction and $\gamma \in \mathcal{G}\Gamma$ a geodesic in the ϕ-orbit of a primitive h-gradient line [16, Definition 4.1]. We endow $\mathfrak{H}_{\mathbf{R}}$ with the topology induced by the product topology on $\mathfrak{H}_0 \times \mathcal{G}\Gamma$. The set $\mathfrak{H}_{\mathbf{R}}$ comes with a flow, that we still denote $\phi = (\phi_t)_{t\in\mathbf{R}}$, defined as follows: for every $(h, \gamma) \in \mathfrak{H}_{\mathbf{R}}$,

$$\phi_t(h, \gamma) = (h, \phi_t(\gamma)), \quad \forall t \in \mathbf{R}.$$

We call the dynamical system $(\mathfrak{H}_{\mathbf{R}}, \phi)$ the *horoflow* of Γ. This system shall not be confused with the horocyclic flow on a hyperbolic surface. The group G acts on $\mathfrak{H}_{\mathbf{R}}$ as follows: for every $(h, \gamma) \in \mathfrak{H}_{\mathbf{R}}$ we let

$$g(h, \gamma) = (gh, g\gamma), \quad \forall g \in G.$$

Note that the horoflow and the action of G commutes. In order to compare $(\mathfrak{H}_{\mathbf{R}}, \phi)$ with $(\widehat{\mathcal{G}}\Gamma, \psi)$, we define a G-equivariant continuous map

$$q \colon \mathfrak{H}_{\mathbf{R}} \to \widehat{\mathcal{G}}\Gamma,$$

by composing the map $\mathfrak{H}_{\mathbf{R}} \to \mathcal{G}\Gamma$ and the projection $p \colon \mathcal{G}\Gamma \to \widehat{\mathcal{G}}\Gamma$. By construction, given any $(h, \gamma) \in \mathfrak{H}_{\mathbf{R}}$, the map q maps homeomorphically the ϕ-orbit of (h, γ) onto the ψ-orbit of $q(h, \gamma) = p(\gamma)$.

Proposition 4.7. *The map $q \colon \mathfrak{H}_{\mathbf{R}} \to \widehat{\mathcal{G}}\Gamma$ is onto.*

Proof. Let η and ξ be two distinct points of $\partial\Gamma$. According to Proposition 3.4 there exists an integral horofunction $h \in \mathfrak{H}_0$ such that $\pi(h) = \xi$. On the other hand, there exists a primitive h-gradient line $\gamma \colon \mathbf{R} \to \Gamma$ such that $\gamma(-\infty) = \eta$ [16, Proposition 5.2]. Being an h-gradient line, γ is such that $\gamma(\infty) = \xi$. Hence q maps the ϕ-orbit of (h, γ) to the (unique) ψ-orbit in $\widehat{\mathcal{G}}\Gamma$ joining η to ξ. This works for any two distinct points $\eta, \xi \in \partial\Gamma$. Hence q is onto. □

Discretization of the flow. We denote by $\mathfrak{H}_{\mathbf{Z}}$ the closed subset of $\mathfrak{H}_{\mathbf{R}}$ containing all the pairs (h, γ) where γ is a primitive h-gradient line. Observe that $\mathfrak{H}_{\mathbf{Z}}$ is G-invariant. Moreover the time 1 flow ϕ_1 on $\mathfrak{H}_{\mathbf{R}}$ induces a homeomorphism of $\mathfrak{H}_{\mathbf{Z}}$ onto itself [16, Definition 2.6]. The system $(\mathfrak{H}_{\mathbf{Z}}, \phi_1)$ is called the *discrete horoflow* of Γ. Let $\bar{\mathfrak{H}}_{\mathbf{Z}}$ and $\bar{\mathfrak{H}}_{\mathbf{R}}$ the quotients $\mathfrak{H}_{\mathbf{Z}}/G$ and $\mathfrak{H}_{\mathbf{R}}/G$ respectively. As the flow ϕ and the action of G commute, ϕ induces a flow $\bar{\phi} = (\bar{\phi}_t)_{t \in \mathbf{R}}$ on the space $\bar{\mathfrak{H}}_{\mathbf{R}} = \mathfrak{H}_{\mathbf{R}}/G$. Moreover $\bar{\phi}_1$ induces a homeomorphism of $\bar{\mathfrak{H}}_{\mathbf{Z}}$ onto itself $\bar{\mathfrak{H}}_{\mathbf{Z}}$. One can check that $(\bar{\mathfrak{H}}_{\mathbf{R}}, \bar{\phi})$ is the suspension of the system $(\bar{\mathfrak{H}}_{\mathbf{Z}}, \bar{\phi}_1)$ [16, Proposition 4.8]. Let $r \colon \mathfrak{H}_{\mathbf{Z}} \to \mathfrak{H}_0$ be the map sending (h, γ) to $\gamma(0)^{-1}h$ (recall that γ is a primitive gradient line, hence $\gamma(0)$ is a vertex of Γ which corresponds to a unique element of G). We observe that r induces a map $\bar{r} \colon \bar{\mathfrak{H}}_{\mathbf{Z}} \to \mathfrak{H}_0$ such that $\bar{r} \circ \bar{\phi}_1 = T \circ \bar{r}$. Actually $(\bar{\mathfrak{H}}_{\mathbf{Z}}, \bar{\phi}_1)$ is conjugated to the canonical two sided shift induced by $(\mathfrak{H}_0, T)$ [16, Propositions 2.5 and 2.22].

4.3 Gromov's coding

In [25, Theorem 8.4.C] Gromov explains that $(\mathfrak{H}_0, T)$ is conjugated to a subshift of finite type. We recall here Gromov's coding as it is detailed by Coornaert and Papadopoulos in [15].

The alphabet. Fix a real number $R_0 \geqslant 100\delta + 1$ and an integer $L_0 \geqslant 2R_0 + 32\delta + 1$. Given a subset S of Γ and a number $r \in \mathbf{R}_+$, we denote by $\mathcal{N}_r(S)$ the r-neighborhood of S, i.e. the set

$$\mathcal{N}_r(S) = \{x \in \Gamma \mid d(x, S) \leqslant r\}.$$

Let $h \in \mathfrak{H}_0$ be a horofunction and $\gamma : \mathbf{R}_+ \to \Gamma$ the minimal h-gradient line starting at 1. The set $V(h)$ is the R_0-neighborhood of γ restricted to $[0\,, L_0]$. In addition we define the map

$$b(h) \colon V(h) \to \mathbf{R},$$

to be the restriction of h to $V(h)$ (recall that our horofunctions vanish at 1). The alphabet $\mathcal{B}$ is the set of functions $b(h) \colon V(h) \to \mathbf{R}$ where h runs over $\mathfrak{H}_0$. It is a finite set [15, Proposition 6.2].

The coding. Let $\sigma\colon \mathcal{B}^{\mathbf{N}} \to \mathcal{B}^{\mathbf{N}}$ the shift map, i.e. the map sending the sequence $(b_n)_{n\in\mathbf{N}}$ to $(b_{n+1})_{n\in\mathbf{N}}$. We define a map $\jmath\colon \mathfrak{H}_0 \to \mathcal{B}^{\mathbf{N}}$ by sending a horofunction $h \in \mathfrak{H}_0$ to the sequence (b_n) defined by

$$b_n = b(T^n h), \quad \forall n \in \mathbf{N}.$$

Let Σ be the image of $\jmath$. One observes that $\jmath \circ T = \sigma \circ \jmath$ [15, Lemma 6.3]. Moreover $\jmath\colon \mathfrak{H}_0 \to \mathcal{B}^{\mathbf{N}}$ induces a homeomorphism from $\mathfrak{H}_0$ onto Σ, which is a subshift of finite type of $\mathcal{B}^{\mathbf{N}}$ [15, Theorem 7.18].

Remark 4.8. From now on we implicitly identify $\mathfrak{H}_0$ with its image Σ. In particular, we say that $h_1, h_2 \in \mathfrak{H}_0$ belong to the same cylinder of length n, if $\jmath(h_1)$ an $\jmath(h_2)$ coincide on the first n letters. We endow $\mathfrak{H}_0$ with the canonical distance on $\mathcal{B}^{\mathbf{N}}$: for every $h_1, h_2 \in \mathfrak{H}_0$, we let $d(h_1, h_2) = e^{-n}$ where n is the largest integer such that h_1 and h_2 belong to the same cylinder of length n.

Let $h \in \mathfrak{H}_0$. By construction $b(h)$ completely determines the restriction of h to the ball of radius R_0 centered at 1. Hence $\theta(h)$ only depends on $b(h)$, i.e. the first letter of $\jmath(h)$. Consequently, if $h_1, h_2 \in \mathfrak{H}_0$ belong to the same cylinder of length n, then $\theta_n(h_1) = \theta_n(h_2)$, or said differently the minimal h_1- and h_2-gradient line starting at 1 coincide on $[0\,, n]$.

Choice of an irreducible component. Unlike the geodesic flow on a negatively curved compact surface the dynamical system $(\mathfrak{H}_0, T)$ is a priori not topologically mixing and even not topologically transitive. This can be a major issue to study its properties. The difficulty comes from the fact that two points in $\Gamma \cup \partial\Gamma$ may be joined by multiple geodesics. This pathology can be illustrated by the following simple example.

Example 4.9. Let $G = \mathbf{F}_2 \times B$ be the direct product of the free group generated by $\{a_1, a_2\}$ and a non-trivial finite group B. We choose for the generating set $A = \{a_1, a_1^{-1}, a_2, a_2^{-1}\} \cup B$ and write Γ for the corresponding Cayley graph. One can check easily that $\mathfrak{H}_0(\Gamma)$, contains one copy of $\partial\mathbf{F}_2$ (the usual Gromov boundary of $\mathbf{F}_2$) for each element $b \in B$. Said differently there is an embedding of $\partial\mathbf{F}_2 \times B$ into $\mathfrak{H}_0(\Gamma)$. This subset is invariant under the action of G. More precisely, for every $(h, b) \in \partial\mathbf{F}_2 \times B$, for every $g = (f, u)$ in G, we have

$$g \cdot (h, u) = (f \cdot h, ub).$$

Assume now that the order on A is such that the letters $a_1, a_1^{-1}, a_2, a_2^{-1}$ are smaller that the one of B. Then for every $b \in B$, the "layer" $\partial\mathbf{F}_2 \times \{b\}$ is invariant under T. On the contrary if every letter of B is smaller than $a_1, a_1^{-1}, a_2, a_2^{-1}$, then T maps $\partial\mathbf{F}_2 \times B$ onto $\partial\mathbf{F}_2 \times \{1\}$.

Nevertheless for our purpose, one does not need to work with the whole system $(\mathfrak{H}_0, T)$. It is sufficient to restrict our attention to an irreducible component of the system, as long as it visits almost all the group G. This is formalized by the visibility property (see Definition 2.1). The goal of this section is to prove that such an irreducible component exists (see Proposition 4.10). Our main tool is the space of the geodesic flow introduced by Gromov in [25, Section 8.3]. From now on, Γ is the Cayley graph of any hyperbolic group, as in the previous section.

We have seen that $(\mathfrak{H}_0, T)$ is conjugated to a subshift of finite type. We write $\mathfrak{I}_1, \dots, \mathfrak{I}_m$ for the irreducible components of $(\mathfrak{H}_0, T)$ (see Section 2.1). Recall that for every $h \in \mathfrak{H}_0$, there exists $i \in [\![1, m]\!]$ and $n_0 \in \mathbf{N}$ such that for every $n \geqslant n_0$, the horofunction $T^n h$ belongs to $\mathfrak{I}_i$ (which we call the asymptotic irreducible component of h). We can now state the main result of this section:

Proposition 4.10. *There exists an irreducible component $\mathfrak{I}_i$ of $\mathfrak{H}_0$ such that the extension of $(\mathfrak{I}_i, T)$ by θ has the visibility property.*

Proof. According to Corollary 4.3 there exists $\widehat{\gamma}$ in $\widehat{\mathcal{G}}\Gamma$ such that the positive orbit defined by

$$\{\psi_s(\widehat{\gamma}) \mid s \in \mathbf{R}_+\}$$

has a dense image in $\widehat{\mathcal{G}}\Gamma/G$. Recall that $(\mathfrak{H}_{\mathbf{R}}, \phi)$ is the horoflow of Γ introduced in Section 4.2. The map $q\colon \mathfrak{H}_{\mathbf{R}} \to \widehat{\mathcal{G}}\Gamma$ being onto (Proposition 4.7), we can fix a pre-image $(h, \gamma) \in \mathfrak{H}_{\mathbf{R}}$ of $\widehat{\gamma}$ by q. Without loss of generality we can assume that (h, γ) actually belongs to the space of the discrete horoflow $\mathfrak{H}_{\mathbf{Z}}$. We denote by h_0 the image of (h, γ) by the map $r\colon \mathfrak{H}_{\mathbf{Z}} \to \mathfrak{H}_0$, i.e. $h_0 = \gamma(0)^{-1}h$. We choose for $\mathfrak{I}_i$ the asymptotic irreducible component of h_0. In particular there exists $K \in \mathbf{N}$, such that for every integer $k \geqslant K$, we have $T^k(h_0) \in \mathfrak{I}_i$.

We now study the properties of the map $\theta\colon \mathfrak{H}_0 \to G$ restricted to $\mathfrak{I}_i$. It is an exercise of hyperbolic geometry to prove that there exists $R_0 \in \mathbf{R}_+$ with the following property: given any two points $y, y' \in \Gamma$, there exists a bi-infinite geodesic $\nu \in \mathcal{G}\Gamma$ such that $d(y, \nu) \leqslant R_0$ and $d(y', \nu) \leqslant R_0$. Recall that the map $\mathcal{G}\Gamma \to \Gamma$ sending ν to $\nu(0)$ as well as the projection $p\colon \mathcal{G}\Gamma \to \widehat{\mathcal{G}}\Gamma$ are quasi-isometries. Hence there exists $\kappa \geqslant 1$ and $\varepsilon \geqslant 0$ such that for every $\nu, \nu' \in \mathcal{G}\Gamma$ we have

$$d(\nu(0), \nu'(0)) \leqslant \kappa d(p(\nu), p(\nu')) + \varepsilon. \tag{9}$$

We define the finite set U by

$$U = \{u \in G \mid d(1, u) \leqslant \kappa + \varepsilon + R_0 + 1 + 50\delta\}.$$

We are going to prove that for every $g \in G$, there exists a horofunction $h \in \mathfrak{I}_i$, an integer $n \in \mathbf{N}$ and two elements $u_1, u_2 \in U$ such that $g = u_1\theta_n(h)u_2$.

Let $g \in G$. There exists a geodesic $\nu \in \mathcal{G}\Gamma$ such that $d(1, \nu) \leqslant R_0$ and $d(g, \nu) \leqslant R_0$. Up to changing the parametrization of ν, we can assume that $d(1, \nu(0)) \leqslant R_0$ and $d(g, \nu(m)) \leqslant R_0 + 1$ for some integer $m \in \mathbf{N}$. We denote by $\widehat{\nu}$ the image of ν in $\widehat{\mathcal{G}}\Gamma$. According to our choice of $\widehat{\gamma}$, there exist a sequence (g_n) of elements of G and a sequence (s_n) of times diverging to infinity, such that $(g_n\psi_{s_n}(\widehat{\gamma}))$ converges to $\widehat{\nu}$.

The projection $p\colon \mathcal{G}\Gamma \to \widehat{\mathcal{G}}\Gamma$ maps homeomorphically ϕ-orbits onto ψ-orbits. Hence there exists a sequence (t_n) of times diverging to infinity such that for every $n \in \mathbf{N}$, the map p sends $g_n\phi_{t_n}(\gamma)$ to $g_n\psi_{s_n}(\widehat{\gamma})$. Combining (9) with the fact that $(g_n\psi_{s_n}(\widehat{\gamma}))$ converges to $\widehat{\nu}$ we get that for sufficiently large integer n

$$d(g_n\gamma(t_n), \nu(0)) \leqslant \kappa + \varepsilon.$$

The convergence taking place in $\widehat{\mathcal{G}}\Gamma$ also tells us that $(g_n\gamma(\infty))$ and $(g_n\gamma(-\infty))$ converge to $\nu(\infty)$ and $\nu(-\infty)$ respectively. Recall that the metric on Γ is 8δ-quasi-convex [13, Chapitre 10, Corollaire 5.3] Combined with the previous inequality we get that for sufficiently large integer n

$$d(g_n\gamma(t_n + m), \nu(m)) \leqslant \kappa + \varepsilon + 50\delta.$$

For every $n \in \mathbf{N}$ we denote by k_n the integer the closest to t_n so that $|t_n - k_n| \leqslant 1$. Recall that ν has been chosen to pass close by 1 and g. Combining all these facts together we finally get that there exists $n_1 \in \mathbf{N}$ such that for all integer $n \geqslant n_1$ we have

$$d(1, g_n\gamma(k_n)) \leqslant \kappa + \varepsilon + R_0 + 1 \quad \text{and} \quad d(g, g_n\gamma(k_n + m)) \leqslant \kappa + \varepsilon + R_0 + 1 + 50\delta.$$

Let $n \geqslant n_1$. Recall that γ is a *primitive* h-gradient line, while k_n and m are integers. Hence $g_n\gamma(k_n)$ and $g_n\gamma(k_n+m)$ are elements of G. We let $u_n = g_n\gamma(k_n)$ and $u'_n = \gamma(k_n+m)^{-1}g_n^{-1}g$. The last inequalities tell us that u_n and u'_n belong to U. In addition we claim that

$$g = u_n\theta_m\left(T^{k_n}(h_0)\right)u'_n.$$

As we explained before, the map $r\colon \mathfrak{H}_{\mathbf{Z}} \to \mathfrak{H}_0$ induces a map $\bar{r}\colon \bar{\mathfrak{H}}_{\mathbf{Z}} \to \mathfrak{H}_0$ such that $\bar{r}\circ\bar{\phi}_1 = T\circ\bar{r}$. It follows that the ray $\rho_n\colon \mathbf{R}_+ \to \Gamma$ defined by $\rho_n(t) = \gamma(k_n)^{-1}\gamma(k_n+t)$ is the (unique) minimal $T^{k_n}(h_0)$-gradient ray starting at 1. Consequently

$$\theta_m\left(T^{k_n}(h_0)\right) = \rho_n(m) = \gamma(k_n)^{-1}\gamma(k_n+m) = u_n^{-1}g{u'_n}^{-1},$$

which completes the proof of our claim. This decomposition of g holds for any integer $n \geqslant n_1$. However k_n is diverging to infinity. Hence if n is sufficiently large $k_n \geqslant K$, thus $T^{k_n}(h_0)$ belongs to the irreducible component $\mathfrak{I}_i$. Thus g can be decomposed as announced in the proposition. □

The next lemmas formalize the fact that the irreducible component "visits almost" all the group G. The statements are quite technical. Nevertheless the idea one has to keep in mind is the following. Given a horofunction $h \in \mathfrak{I}$, one can follows its trajectory in G by looking at the corresponding minimal h-gradient line $\gamma\colon \mathbf{R}_+ \to \Gamma$. More precisely, given $n \in \mathbf{N}$, the point $\gamma(n)$ can be thought at the position in G at time n of the orbit of h. The next lemmas say that at time n the positions reached by the flow almost embed in (respectively almost cover) the sphere $S(n) \subset G$. These computations will be needed to estimate the spectral radii of various transfer operators (see Lemmas 5.6, 5.7 and 5.9). Before stating the lemmas we introduce a few notations. Let $n \in \mathbf{N}$. For every $h_0 \in \mathfrak{I}$, the set $S_{\mathfrak{I}}(h_0, n)$ is

$$S_{\mathfrak{I}}(h_0, n) = \{h \in \mathfrak{I} \mid T^n h = h_0\} = T^{-n}(h_0) \cap \mathfrak{I}.$$

In addition we define a map

$$\begin{array}{rccc} p_n\colon & \mathfrak{H}_0 \times G \times G & \to & G \\ & (h, u_1, u_2) & \to & u_1\theta_n(h)u_2 \end{array}$$

Lemma 4.11. *Let $h_0 \in \mathfrak{I}$. For every $n \in \mathbf{N}$, the map $\mathfrak{H}_0 \to G$ sending h to $\theta_n(h)$ induces an embedding from $S_{\mathfrak{I}}(h_0, n)$ into $S(n)$.*

Proof. Let $h \in \mathfrak{I}$ such that $T^n h = h_0$. Let $\gamma\colon \mathbf{R}_+ \to \Gamma$ be the minimal h-gradient line starting at 1. As we observed before $\theta_n(h) = \gamma(n)$. In particular $\theta_n(h)$ belongs to $S(n)$. On the other hand, it follows from the definition of the map T that $h = \theta_n(h)T^n h = \theta_n(h)h_0$. Hence h is completely determined by $\theta_n(h)$ which completes the proof of the lemma. □

Lemma 4.12. *There exist $R, N \in \mathbf{N}$ with the following property. For every $h_0 \in \mathfrak{I}$, for every $n \in \mathbf{N}$, the sphere $S(n) \subset G$ is contained in the image of the map*

$$\bigsqcup_{k=n}^{n+N} \Big(S_{\mathfrak{I}}(h_0, k) \times B(R) \times B(R)\Big) \to G$$

induced by the sequence (p_k).

Proof. We start by defining the constants R and N. Recall that the extension of $(\mathfrak{I}, T)$ by $\theta\colon \mathfrak{I} \to G$ has the visibility property, i.e. there is a finite set U such that for every $g \in G$, there exist a horofunction $h \in \mathfrak{I}$, an integer $n \in \mathbf{N}$, and two elements $u_1, u_2 \in U$ satisfying $g = u_1\theta_n(h)u_2$ (Proposition 4.10). We denote by L the maximal length (in Γ) of an element of U. As the system $(\mathfrak{I}, T)$ is an irreducible subshift of finite type, there exists K with the following property: for every $h_1, h_2 \in \mathfrak{I}$, for every $n \in \mathbf{N}$, there exists $h_1' \in \mathfrak{I}$ and $k \in [\![0, K]\!]$ such that $T^{n+k}h_1' = h_2$ and h_1 and h_1' belong to the same cylinder of length n. Finally we let

$$R = 5L + K \quad \text{and} \quad N = 2L + K.$$

We now fix a horofunction $h_0 \in \mathfrak{I}$ and an integer $n \in \mathbf{N}$. Let $g \in S(n)$. According to the visibility property there exist a horofunction $h \in \mathfrak{I}$, an integer $m \in \mathbf{N}$ and two elements $u_1, u_2 \in U$ such that $g = u_1\theta_m(h)u_2$. Recall that the length (in Γ) of $\theta_m(h)$ is m, hence $|n - m| \leqslant 2L$. As $(\mathfrak{I}, T)$ is irreducible, there exists $h' \in \mathfrak{I}$ and $k \in [\![0, K]\!]$ such that h and h' belongs to the same cylinder of length $n + 2L$ and

$$T^{n+2L+k}h' = h_0.$$

Since $m \leqslant n + 2L$, we have $\theta_m(h') = \theta_m(h)$. It follows that

$$\theta_{n+2L+k}(h') = \theta_m(h')\theta_{n+2L-m+k}(T^m h') = \theta_m(h)\theta_{n+2L-m+k}(T^m h')$$

Consequently

$$g = u_1\theta_{n+2L+k}(h')u_2',$$

where

$$u_2' = \left(\theta_{n+2L-m+k}(T^m h')\right)^{-1} u_2.$$

Observe that u_1 and u_2 belong to $B(L) \subset B(R)$. As we noticed $n \leqslant m+2L$, thus $n-m+2L+k \leqslant 4L + K$. Consequently u_2' belongs to $B(R)$. In other words p_{n+2L+k} maps the element (h', u_1, u_2') of $S_{\mathfrak{I}}(h_0, n + 2L + k) \times B(R) \times B(R)$ to g. □

5 Potential and transfer operator

The goal of this section is to prove the following statements.

Theorem 5.1. *Let G be a group acting properly co-compactly by isometries on a hyperbolic space X. We assume that one of the following holds. Either*

(i) X is the Cayley graph of G with respect to a finite generating set, or

(ii) X is a $\mathrm{CAT}(-1)$ space.

Let H be a subgroup of G. We denote by ω_G and ω_H the exponential growth rates of G and H acting on X. The subgroup H is co-amenable in G if and only if $\omega_H = \omega_G$. In particular if H is a normal subgroup of G, the quotient G/H is amenable if and only if $\omega_H = \omega_G$.

Theorem 5.2. *Let G be a group with Kazhdan's property (T) acting properly co-compactly by isometries on a hyperbolic space X. We assume that one of the following holds. Either*

(i) X is the Cayley graph of G with respect to a finite generating set, or

(ii) X is a CAT(-1) *space.*

There exists $\varepsilon > 0$ with the following property. Let H be a subgroup of G. We denote by ω_G and ω_H the exponential growth rates of G and H acting on X. If $\omega_H > \omega_G - \varepsilon$, then H is a finite index subgroup of G.

5.1 The data

Let G be a a group acting properly co-compactly by isometries on a hyperbolic space X. As in the statement of Theorems 5.1 and 5.2 we consider two cases.

Case 1. The space X is the Cayley graph of G with respect to a finite generating set A. In this situation we denote by Γ a copy of X.

Case 2. The space X is CAT(-1). In this situation we fix an arbitrary finite generating set A of G and denote by Γ the Cayley graph of G with respect to A.

In both cases we may assume without loss of generality that A is symmetric. As we work with two distinct metric spaces, namely the Cayley graph Γ and the space X, we use this section to emphasize which objects are related to one or the other space.

Data related to X. The space X is the one that will carry the geometric information. We denote by δ its hyperbolicity constant. We fix a base point $x_0 \in X$. This allows us to identify $C_*(X)$ with the set of continuous maps vanishing at x_0. We denote by $\pi_X \colon \mathfrak{H}(X) \twoheadrightarrow \partial X$ the projection studied in Proposition 3.4. We denote by ω_G the exponential growth rate of G acting on X.

Data related to Γ. The role of Γ is to provide a support for coding the geodesic flow. The space $\mathfrak{H}_0 \subset \mathfrak{H}(\Gamma)$ refers to the integral horofunctions on the Cayley graph Γ. We denote by $\pi_\Gamma \colon \mathfrak{H}_0 \twoheadrightarrow \partial\Gamma$ the projection coming from Proposition 4.5. The maps $\theta \colon \mathfrak{H}_0 \twoheadrightarrow G$ and $T \colon \mathfrak{H}_0 \to \mathfrak{H}_0$ are the ones defined at the beginning of Section 4.2. For simplicity we denote by $\mathfrak{I}$ the irreducible component of $(\mathfrak{H}_0, T)$ with the visibility property given by Proposition 4.10. Recall that for every $n \in \mathbf{N}$, the sets $S(n)$ and $B(n)$ are respectively the sphere and the ball of radius n, measured in Γ.

Comparing Γ and X. Since G acts properly co-compactly on X, the orbit maps $G \to X$ sending g to gx_0 leads a (κ, ℓ)-quasi-isometric embedding $f \colon \Gamma \to X$. This map induces a homeomorphism $\partial\Gamma \to \partial X$ between the respective Gromov boundary of Γ and X. For simplicity we implicitly identify $\partial\Gamma$ and ∂X.

5.2 Transfer operator for the irreducible component

Comparing horofunctions. The first task is to define a potential $F \colon \mathfrak{H}_0 \to \mathbf{R}_+^*$. This potential defined on the dynamical system $(\mathfrak{H}_0, T)$ should reflect to geometry of X. In the first case – when X is actually the Cayley graph Γ of X – the geometry coincides with the dynamics, and we can simply take for F the constant function equal to $e^{-\omega_G}$. In the second case – when X is an arbitrary CAT(-1) space – the situation is more subtle. Indeed $\mathfrak{H}(X)$ does not necessarily coincide with $\mathfrak{H}(\Gamma)$ nor $\mathfrak{H}_0$. As the space X is CAT(-1), the set $\mathfrak{H}(X)$ coincides with the usual visual boundary of X. More precisely the map $\pi_X \colon \mathfrak{H}(X) \to \partial X$ is a homeomorphism. On the other hand the projection $\pi_\Gamma \colon \mathfrak{H}_0 \to \partial\Gamma$ is not always injective. Nevertheless we are going to build a map comparing $\mathfrak{H}_0$ – the

horofunctions used for coding – to $\mathfrak{H}(X)$ – the horofunctions capturing the geometry of X. This is the purpose of the next proposition. Actually we develop a framework that covers both cases simultaneously.

Recall that we identify $(\mathfrak{H}_0, T)$ with an appropriate subshift of finite type of $(\mathcal{B}^{\mathbf{N}}, \sigma)$. This identification induces a distance on $\mathfrak{H}_0$ (Remark 4.8) as defined at the beginning of Section 2.1. Namely the distance between two horofunctions $h, h' \in \mathfrak{H}_0$ is $d(h, h') = e^{-n}$, where n is the largest integer such that the respective images of h and h' in $\mathcal{B}^{\mathbf{N}}$ have the same first n letters. We denote by $\mathcal{C}(\mathfrak{H}_0, \mathbf{C})$ for the space of continuous maps from $\mathfrak{H}_0$ to $\mathbf{C}$ while $H^\infty_\alpha(\mathfrak{H}_0, \mathbf{C})$ stands for the space of functions with bounded α-Hölder variations (see Section A.1).

Proposition 5.3. *There exists a G-equivariant* comparison map $\mathfrak{H}_0 \to \mathfrak{H}(X)$ *which we denote* $h \mapsto h_X$ *with the following properties.*

(i) The following diagram commutes

$$\begin{array}{ccc} \mathfrak{H}_0 & \longrightarrow & \mathfrak{H}(X) \\ \pi_\Gamma \downarrow & & \downarrow \pi_X \\ \partial\Gamma & \longrightarrow & \partial X \end{array}$$

(ii) There exists $\alpha \in \mathbf{R}_+^*$ such that the evaluation map *defined by*

$$\begin{array}{rccc} \varphi\colon & \mathfrak{H}_0 & \to & \mathbf{R} \\ & h & \to & h_X\Big(\theta(h)x_0\Big) \end{array}$$

belongs to $H^\infty_\alpha(\mathfrak{H}_0, \mathbf{R})$.

Proof. We distinguish two cases depending whether X is a Cayley graph or a CAT(-1) space.

Case 1. Assume first that X is the Cayley graph of G with respect to the generating set A. In this situation we defined Γ to be exactly X. In particular $\mathfrak{H}_0$ can be see as a subset of $\mathfrak{H}(X)$. We simply define the comparison map $\mathfrak{H}_0 \to \mathfrak{H}(X)$ as the corresponding embedding. Point (i) becomes obvious. It follows from the very definition of θ that φ is constant equal to -1. Thus it belongs to $H^\infty_\alpha(\mathfrak{H}_0, \mathbf{R})$ for every $\alpha \in \mathbf{R}_+^*$.

Case 2. Assume now that X is a CAT(-1) space. In this situation Γ is the Cayley graph of G with respect to an arbitrary finite generating set. Given $h \in \mathfrak{H}_0$ we define h_X to be the Busemann function at $\xi = \pi_\Gamma(h)$ vanishing at x_0. By construction the diagram of Point (i) commutes.

Let $D = D(\kappa, \ell, \delta)$ be the parameter given by the Morse lemma (Proposition 2.3). Let $h, h' \in \mathfrak{H}_0$ such that $d(h, h') < 1$. We denote by $n \in \mathbf{N}^*$ the largest integer such that h and h' belong to the same cylinder of length n, so that $d(h, h') = e^{-n}$. We write $\gamma\colon \mathbf{R}_+ \to \Gamma$ for the minimal h-gradient ray starting at 1 and let $\xi = \gamma(\infty)$. Let $c\colon \mathbf{R}_+ \to X$ be the ray starting at x_0 such that $c(\infty) = \xi$. Recall that the map $f\colon \Gamma \to X$ induced by the orbit map is a (κ, ℓ)-quasi-isometry. In particular $f \circ \gamma$ is a (κ, ℓ)-quasi-geodesic between x_0 and ξ. It follows from the stability of quasi-geodesics (Proposition 2.3) that the Hausdorff distance between $f \circ \gamma$ and c is bounded above by D. In a similar way we associate to h' a gradient ray $\gamma'\colon \mathbf{R}_+ \to \Gamma$ as well as a geodesic ray $c'\colon \mathbf{R}_+ \to X$.

Since h and h' belong to the same cylinder of length n, the paths γ and γ' coincide on $[0\,,n]$ (see Remark 4.8). In particular, for every $k \in [\![0,n]\!]$, the point $y_k = f\circ\gamma(k) = f\circ\gamma'(k)$ lies in the D neighborhood of both c and c'. As f is a (κ,ℓ)-quasi-isometric embedding, we also observe that

$$\kappa^{-1}k - \ell \leqslant d(x_0,y_k) \leqslant \kappa k + \ell.$$

It follows that there exists $t \in \mathbf{R}_+$, such that $d(c(t),c'(t)) \leqslant 2D$ and $t \geqslant \kappa^{-1}n - \ell - D$. A standard exercise of CAT(-1) geometry shows that

$$d(c(t/2),c'(t/2)) \leqslant C_1 e^{-\frac{1}{2}t},$$

where C_1 is a parameter that only depends on D. Recall that $d(x_0,y_1) \leqslant \kappa+\ell$. Another exercise of CAT(-1) geometry shows that

$$\Big|h_X(y_1) - \Big[d(c(t/2),y_1) - t/2\Big]\Big| \leqslant C_2 e^{-\frac{1}{2}t} \quad \text{and} \quad \Big|h'_X(y_1) - \Big[d(c'(t/2),y_1) - t/2\Big]\Big| \leqslant C_2 e^{-\frac{1}{2}t},$$

where C_2 only depends on κ and ℓ. Consequently

$$|h_X(y_1) - h'_X(y_1)| \leqslant C_3 e^{-\frac{1}{2}t} \leqslant C_4 e^{-\frac{1}{2}\kappa^{-1}n} \leqslant C_4 d(h,h')^{\frac{1}{2}\kappa^{-1}}.$$

where C_3 and C_4 only depends on D, κ and ℓ. Nevertheless y_1 is the point $\theta(h)x_0 = \theta(h')x_0$. Hence the previous inequality exactly says that

$$|\varphi(h) - \varphi(h')| \leqslant C_4 d(h,h')^{\frac{1}{2}\kappa^{-1}}.$$

This inequality holds for every h,h' such that $d(h,h') < 1$. Consequently φ belongs to the space $H^\infty_\alpha(\mathfrak{H}_0,\mathbf{R})$ where $\alpha = \kappa^{-1}/2$, which proves Point (ii). □

The potential F. From now on, we fix the comparison map $\mathfrak{H}_0 \to \mathfrak{H}(X)$, $h \mapsto h_X$ given by Proposition 5.3. We keep the notations introduced in this statement. In particular the evaluation map $\varphi\colon \mathfrak{H}_0 \to \mathbf{R}$ sending h to $h_X(\theta(h)x_0)$ belongs to $H^\infty_\alpha(\mathfrak{H}_0,\mathbf{R})$. The potential $F\colon \mathfrak{H}_0 \to \mathbf{R}^*_+$ is the map defined by

$$F(h) = \exp\Big(\omega_G \varphi(h)\Big) = \exp\Big(\omega_G h_X\Big(\theta(h)x_0\Big)\Big), \quad \forall h \in \mathfrak{H}_0. \tag{10}$$

It directly follows from the previous proposition that $\ln F$ belongs to $H^\infty_\alpha(\mathfrak{H}_0,\mathbf{R})$

Remark. If X is simply the Cayley graph Γ of X, we previously observed that the evaluation map φ is constant equal to -1. Hence the potential becomes $F(h) = e^{-\omega_G}$.

Lemma 5.4. *For every $n \in \mathbf{N}$, for every $h \in \mathfrak{H}_0$, we have*

$$F_n(h) = \exp\Big(\omega_G h_X\Big(\theta_n(h)x_0\Big)\Big).$$

Proof. Let $h \in \mathfrak{H}_0$. It is sufficient to prove that for every $n \in \mathbf{N}$, we have

$$\sum_{k=0}^{n-1} \varphi\circ T^k(h) = h_X\Big(\theta_n(h)x_0\Big).$$

The proof is by induction on n. By convention $\theta_0(h) = 1$. Since h_X vanishes at x_0 the statement obviously holds for $n = 0$. Assume now that the claim holds for some $n \in \mathbf{N}$. It follows from the definition of T that $T^n(h) = \theta_n(h)^{-1}h$. As the comparison map $\mathfrak{H}_0 \to \mathfrak{H}(X)$ is G-equivariant we get

$$\begin{aligned}\varphi \circ T^n(h) = \left[\theta_n(h)^{-1}h_X\right]\left(\theta\left(T^n(h)\right)x_0\right) &= h_X\left(\theta_n(h)\theta\left(T^n(h)\right)x_0\right) - h_X\left(\theta_n(h)x_0\right)\\ &= h_X\left(\theta_{n+1}(h)x_0\right) - h_X\left(\theta_n(h)x_0\right).\end{aligned}$$

The statement for $n+1$ now follows from the induction hypotheses. □

Lemma 5.5. *There exists a constant $C \in \mathbf{R}_+^*$, such that for every $h \in \mathfrak{H}_0$, for every $n \in \mathbf{N}$, we have*

$$\frac{1}{C} \leqslant \frac{F_n(h)}{\exp\left(-\omega_G d(\theta_n(h)x_0, x_0)\right)} \leqslant C.$$

Proof. According to Lemma 5.4, it suffices to show that there exists $C' \in \mathbf{R}_+^*$ such that for every $h \in \mathfrak{H}_0$, for every $n \in \mathbf{N}$ we have

$$\left|h_X\left(\theta_n(h)x_0\right) + d\Big(\theta_n(h)x_0, x_0\Big)\right| \leqslant C'.$$

By the stability of quasi-geodesics, there exists $D \in \mathbf{R}_+$ such that the Hausdorff distance between two (κ, ℓ)-quasi-geodesics of X joining the same endpoints (possibly in ∂X) is at most D. Let $h \in \mathfrak{H}_0$. Let $c\colon \mathbf{R}_+ \to X$ be geodesic ray between x_0 and $\xi = \pi_X(h_X)$. We write $b\colon X \to \mathbf{R}$ for the corresponding Busemann function vanishing at x_0. Note that h_X and b are two horofunctions of X whose image by $\pi_X\colon \mathfrak{H}(X) \to \partial X$ is ξ. It follows that $\|h_X - b\|_\infty \leqslant 64\delta$ [15, Corollary 3.8]. Let $\gamma\colon \mathbf{R}_+ \to \Gamma$ be the minimal h-gradient line starting at 1. Observe that $f \circ \gamma$ is a (κ, ℓ) quasi-geodesic of X. Hence the Hausdorff distance between $f \circ \gamma$ and c is at most D.

Let $n \in \mathbf{N}$. By construction the element $\theta_n(h)$ lies on γ. Thus there exists $t \in \mathbf{R}_+$, such that $d(\theta_n(h)x_0, c(t)) \leqslant D$. It follows that

$$|d(\theta_n(h)x_0, x_0) + b(c(t))| \leqslant D. \tag{11}$$

Recall that the Busemann function b is a 1-Lipschitz. Combined with the fact that $\|h_X - b\|_\infty \leqslant 64\delta$ we get

$$|h_X\left(\theta_n(h)x_0\right) - b(c(t))| \leqslant D + 64\delta.$$

Hence (11) becomes

$$|h_X\left(\theta_n(h)x_0\right) + d(\theta_n(h)x_0, x_0)| \leqslant C',$$

where $C' = 2D + 64\delta$. Observes that C' neither depends on h of n, hence the proof is complete. □

The transfer operator. The transfer operator associated to the potential F is the operator $\mathcal{L}\colon \mathcal{C}(\mathfrak{I}, \mathbf{C}) \to \mathcal{C}(\mathfrak{I}, \mathbf{C})$ defined by

$$\mathcal{L}\Phi(h_0) = \sum_{Th = h_0} F(h)\Phi(h), \quad \forall \Phi \in \mathcal{C}(\mathfrak{I}, \mathbf{C}),\ \forall h_0 \in \mathfrak{I}. \tag{12}$$

Note that the restriction map $\mathcal{C}(\mathfrak{H}_0, \mathbf{C}) \to \mathcal{C}(\mathfrak{I}, \mathbf{C})$ induces a 1-Lipschitz map from $H^\infty_\alpha(\mathfrak{H}_0, \mathbf{C}) \to H^\infty_\alpha(\mathfrak{I}, \mathbf{C})$. Hence $\ln F$ restricted to $\mathfrak{I}$ belongs to $H^\infty_\alpha(\mathfrak{I}, \mathbf{C})$. As we observed in the appendix $\mathcal{L}$ induces a bounded operator of $H^\infty_\alpha(\mathfrak{I}, \mathbf{C})$. Since the system $(\mathfrak{I}, T)$ is irreducible, the spectral radii of $\mathcal{L}$ seen as an operator of $\mathcal{C}(\mathfrak{I}, \mathbf{C})$ or $H^\infty_\alpha(\mathfrak{I}, \mathbf{C})$ are the same (Theorem A.6). We denote it by ρ. According to (18) it can be computed as follows

$$\rho = \limsup_{n\to\infty} \sqrt[n]{\|\mathcal{L}^n \mathbb{1}\|_\infty}. \tag{13}$$

Computing ρ. The goal of this section is to prove that $\rho = 1$ (Proposition 5.8).

Lemma 5.6. *There exists $A_1 \in \mathbf{R}^*_+$ such that for every $n \in \mathbf{N}$, we have*

$$\|\mathcal{L}^n \mathbb{1}\|_\infty \leqslant A_1 \sum_{g\in S(n)} e^{-\omega_G d(gx_0, x_0)}.$$

Proof. We denote by C the constant given by Lemma 5.5. Let $n \in \mathbf{N}$. Let $h_0 \in \mathfrak{I}$. According to Lemma 5.4 we have

$$\mathcal{L}^n \mathbb{1}(h_0) = \sum_{T^n h = h_0} F_n(h) = \sum_{T^n h = h_0} \exp\Big(\omega_G h_X\left(\theta_n(h)x_0\right)\Big).$$

Applying Lemma 5.5 we get

$$\mathcal{L}^n \mathbb{1}(h_0) \leqslant C \sum_{T^n h = h_0} \exp\Big(-\omega_G d\Big(\theta_n(h)x_0, x_0\Big)\Big). \tag{14}$$

By Lemma 4.11 the map $\mathfrak{H}_0 \to G$ sending h to $\theta_n(h)$ induces an embedding of $\{h \in \mathfrak{I} \mid T^n h = h_0\}$ into $S(n)$. It follows that

$$\mathcal{L}^n \mathbb{1}(h_0) \leqslant C \sum_{g\in S(n)} e^{-\omega_G d(gx_0, x_0)}.$$

This inequality holds for every $h_0 \in \mathfrak{I}$, thus

$$\|\mathcal{L}^n \mathbb{1}\|_\infty \leqslant C \sum_{g\in S(n)} e^{-\omega_G d(gx_0, x_0)}. \qquad \square$$

Lemma 5.7. *There exists $A_2 \in \mathbf{R}^*_+$ such that for every $n \in \mathbf{N}$, we have*

$$\sum_{g\in S(n)} e^{-\omega_G d(gx_0, x_0)} \leqslant A_2 \|\mathcal{L}^n \mathbb{1}\|_\infty.$$

Proof. We write $C \in \mathbf{R}^*_+$, and $R, N \in \mathbf{N}$, for the constants given by Lemma 5.5 and Lemma 4.12 respectively. Recall that the map $f\colon \Gamma \to X$ induced by the orbit map is a (κ, ℓ)-quasi-isometric embedding. Let $h_0 \in \mathfrak{I}$. Let $n \in \mathbf{N}$. Let $k \in \mathbf{N}$. Applying Lemma 5.5 we observe that

$$\begin{aligned}\sum_{T^k h = h_0} \exp\Big(-\omega_G d\Big(\theta_k(h)x_0, x_0\Big)\Big) &\leqslant C \sum_{T^k h = h_0} \exp\Big(\omega_G h_X\left(\theta_k(h)x_0\right)\Big) \\ &\leqslant C\mathcal{L}^k \mathbb{1}(h_0).\end{aligned} \tag{15}$$

On the other hand if (u_1, u_2) belongs to $B(R) \times B(R)$, the triangle inequality tells us that

$$|d(u_1\theta_k(h)u_2x_0, x_0) - d(\theta_k(h)x_0, x_0)| \leqslant d(u_1x_0, x_0) + d(u_2x_0, x_0)\,.$$

However the map $f\colon \Gamma \to X$ being a (κ, ℓ)-quasi-isometric embedding, we get

$$|d(u_1\theta_k(h)u_2x_0, x_0) - d(\theta_k(h)x_0, x_0)| \leqslant 2(\kappa R + \ell).$$

Summing (15) when (u_1, u_2) runs over $B(R) \times B(R)$ and k over $[\![n, n+N]\!]$ gives

$$\begin{aligned}
&\sum_{k=n}^{n+N} \sum_{(u_1,u_2)\in B(R)\times B(R)} \sum_{T^k h = h_0} \exp\Big(- \omega_G d\Big(u_1\theta_k(h)u_2x_0, x_0\Big)\Big) \\
&\qquad \leqslant |B(R)|^2\, e^{2\omega_G(\kappa R+\ell)} \sum_{k=n}^{n+N} \sum_{T^k h = h_0} \exp\Big(- \omega_G d\Big(\theta_k(h)x_0, x_0\Big)\Big) \\
&\qquad \leqslant C\,|B(R)|^2\, e^{2\omega_G(\kappa R+\ell)} \sum_{k=n}^{n+N} \mathcal{L}^k\mathbb{1}(h_0).
\end{aligned}$$

Lemma 4.12 provides a lower bound of the triple sum in the left hand side of the inequality, leading to

$$\sum_{g\in S(n)} e^{-\omega_G d(gx_0,x_0)} \leqslant C\,|B(R)|^2\, e^{2\omega_G(\kappa R+\ell)} \sum_{k=n}^{n+N} \mathcal{L}^k\mathbb{1}(h_0).$$

As the potential F and the function $\mathbb{1}$ are positive we observe that for every $k \geqslant n$,

$$\mathcal{L}^k\mathbb{1}(h_0) \leqslant \big\|\mathcal{L}^k\mathbb{1}\big\|_\infty \leqslant \big\|\mathcal{L}^{k-n}\big\|_\infty \big\|\mathcal{L}^n\mathbb{1}\big\|_\infty\,.$$

Hence the previous inequality becomes

$$\sum_{g\in S(n)} e^{-\omega_G d(gx_0,x_0)} \leqslant C\,|B(R)|^2\, e^{2\omega_G(\kappa R+\ell)} \left(\sum_{k=0}^{N} \big\|\mathcal{L}^k\big\|_\infty\right) \big\|\mathcal{L}^n\mathbb{1}\big\|_\infty\,,$$

which is exactly the required inequality. □

Proposition 5.8. *The spectral radius of $\mathcal{L}$ is $\rho = 1$.*

Proof. We form the series

$$\Upsilon(s) = \sum_{n=0}^{\infty} e^{-sn} \big\|\mathcal{L}^n\mathbb{1}\big\|_\infty\,.$$

It follows from (13) that the critical exponent $\Upsilon(s)$ is $\ln\rho$. Hence it suffices to prove this critical exponent is 0. Let $s \in \mathbf{R}_+^*$. According to Lemmas 5.6 and 5.7 there exists A_1, A_2 such that for every $n \in \mathbf{N}$

$$A_1 \sum_{g\in S(n)} e^{-\omega_G d(gx_0,x_0)} \leqslant \big\|\mathcal{L}^n\mathbb{1}\big\|_\infty \leqslant A_2 \sum_{g\in S(n)} e^{-\omega_G d(gx_0,x_0)}.$$

Multiplying theses inequalities by e^{-sn} and summing over n gives

$$A_1 \sum_{n=0}^{\infty} \sum_{g\in S(n)} e^{-sn} e^{-\omega_G d(gx_0,x_0)} \leqslant \Upsilon(s) \leqslant A_2 \sum_{n=0}^{\infty} \sum_{g\in S(n)} e^{-sn} e^{-\omega_G d(gx_0,x_0)}.$$

The map $f\colon \Gamma \to X$ being a (κ,ℓ)-quasi-isometric embedding, for every $n \in \mathbf{N}$, for every $g \in S(n)$ we have

$$\kappa^{-1}\left[d(gx_0,x_0) - \ell\right] \leqslant n \leqslant \kappa\left[d(gx_0,x_0) + \ell\right].$$

Consequently

$$A_1 e^{-s\kappa\ell} \sum_{g\in G} e^{-(\omega_G+s\kappa)d(gx_0,x_0)} \leqslant \Upsilon(s) \leqslant A_2 e^{s\kappa^{-1}\ell} \sum_{g\in G} e^{-(\omega_G+s\kappa^{-1})d(gx_0,x_0)}.$$

This can be reformulated using the Poincaré series of G as

$$A_1 e^{-s\kappa\ell} \mathcal{P}_G(\omega_G + s\kappa) \leqslant \Upsilon(s) \leqslant A_2 e^{s\kappa^{-1}\ell} \mathcal{P}_G(\omega_G + s\kappa^{-1}).$$

Recall that $s \to \mathcal{P}_G(\omega_G + s)$ converges if $s > 0$ and diverges if $s < 0$. It follows that the critical exponent of $\Upsilon(s)$ equals 0. □

5.3 Twisted transfer operator associated to a subgroup

Data associated to a subgroup. Let H be a subgroup of G. We denote by Y the space of left cosets of H in G, i.e. $Y = H \setminus G$. We write y_0 for the image of 1 in Y. In other words y_0 is the coset H. We denote by $\mathcal{H} = \ell^2(Y)$ the space of square summable functions from Y to $\mathbf{C}$. The group G acts on Y by right translations. It induces a unitary representation $\lambda\colon G \to \mathcal{U}(\mathcal{H})$ defined

$$[\lambda(g)\phi]\, y = \phi(y \cdot g), \quad \forall g \in G,\ \forall \phi \in \mathcal{H}.$$

We call λ the *the regular representation of G relative to H*. We denote by ω_H the exponential growth rate of H *acting on* X.

Twisted transfer operator. We denote by $\mathcal{C}(\mathfrak{I},\mathcal{H})$ the set of continuous function from $\mathfrak{I}$ to $\mathcal{H}$. Similarly $H^\infty_\alpha(\mathfrak{I},\mathcal{H})$ stands for the space of functions with bounded α-Hölder variations (see Section A.1). As explained in this appendix, the representation λ leads to a twisted transfer operator $\mathcal{L}_\lambda\colon C(\mathfrak{I},\mathcal{H}) \to C(\mathfrak{I},\mathcal{H})$ defined by

$$\mathcal{L}_\lambda \Phi(h_0) = \sum_{Th=h_0} F(h)\lambda(\theta(h))^{-1}\Phi(h), \quad \forall \Phi \in C(\mathfrak{I},\mathcal{H}),\ \forall h_0 \in \mathfrak{I}.$$

This operator induces a bounded operator of $H^\infty_\alpha(\mathfrak{I},\mathcal{H})$ (Proposition A.8). We write ρ_λ for the spectral radius of $\mathcal{L}_\lambda$ seen as an operator of $H^\infty_\alpha(\mathfrak{I},\mathcal{H})$.

Computing ρ_λ. Our goal is to provide an estimate of ρ_λ in terms of ω_G and ω_H. Let us first remark that $\rho_\lambda \leqslant \rho$ (Corollary A.9) that is in our setting $\rho_\lambda \leqslant 1$ (Proposition 5.8). We now provide a lower bound for ρ_λ. The proof follows the same strategy as the one of Proposition 5.8.

Lemma 5.9. *There exist $B_2 \in \mathbf{R}_+^*$ and a function $\Psi \in H_\alpha^\infty(\mathfrak{I}, \mathcal{H})$ such that for every $n \in \mathbf{N}$, we have*

$$\sum_{g \in S(n) \cap H} e^{-\omega_G d(gx_0, x_0)} \leqslant B_2 \left\| \mathcal{L}_\lambda^n \Psi \right\|_\infty .$$

Proof. We write $C \in \mathbf{R}_+^*$, and $R, N \in \mathbf{N}$, for the constants given by Lemma 5.5 and Lemma 4.12 respectively. Recall that the map $f \colon \Gamma \to X$ induced by the orbit map is a (κ, ℓ)-quasi-isometric embedding. We denote by Z the following finite subset of Y.

$$Z = \{ y_0 \cdot u \mid u \in B(R) \} .$$

The map $\Psi \colon \mathfrak{I} \to \mathcal{H}$ is the constant function equal to the characteristic function $\mathbb{1}_Z$ of Z. One observes easily that Ψ belongs to $H_\alpha^\infty(\mathfrak{I}, \mathcal{H})$.

We now fix $h_0 \in \mathfrak{I}$ and $n \in \mathbf{N}$. Let $k \in \mathbf{N}$ and $u_2 \in B(R)$. Let us compute $\mathcal{L}_\lambda^k \Psi(h_0)$ at the point $y_0 \cdot u_2^{-1}$. By definition we have

$$\left[\mathcal{L}_\lambda^k \Psi(h_0) \right] \left(y_0 \cdot u_2^{-1} \right) = \sum_{T^k h = h_0} F_k(h) \left[\lambda \left(\theta_k(h)^{-1} \right) \Psi \right] \left(y_0 \cdot u_2^{-1} \right) = \sum_{T^k h = h_0} F_k(h) \mathbb{1}_Z \left(y_0 \cdot u_2^{-1} \theta_k(h)^{-1} \right) .$$

Recall that $Z = y_0 \cdot B(R)$. Hence the term

$$\mathbb{1}_Z \left(y_0 \cdot u_2^{-1} \theta_k(h)^{-1} \right)$$

equals 1 if there exists $u_1 \in B(R)$ such that $u_1 \theta_k(h) u_2$ belongs to H and zero otherwise. Hence

$$\sum_{u_1 \in B(R)} \sum_{\substack{T^k h = h_0, \\ u_1 \theta_k(h) u_2 \in H}} F_k(h) \leqslant |B(R)| \left[\mathcal{L}_\lambda^k \Psi(h_0) \right] \left(y_0 \cdot u_2^{-1} \right) .$$

Recall that the potential F and the vector $\mathbb{1}_Z$ are non-negative. It follows that

$$\sum_{u_1 \in B(R)} \sum_{\substack{T^k h = h_0, \\ u_1 \theta_k(h) u_2 \in H}} F_k(h) \leqslant |B(R)| \left\| \mathcal{L}_\lambda^k \Psi(h_0) \right\|_{\ell^2(Y)} \leqslant |B(R)| \left\| \mathcal{L}_\lambda^k \Psi \right\|_\infty .$$

Applying Lemma 5.5 to the previous inequality, we observe (as in the proof of Lemma 5.7) that

$$\sum_{u_1 \in B(R)} \sum_{\substack{T^k h = h_0, \\ u_1 \theta_k(h) u_2 \in H}} \exp \left(- \omega_G d\Big(\theta_k(h) x_0, x_0 \Big) \right) \leqslant C \, |B(R)| \left\| \mathcal{L}_\lambda^k \Psi \right\|_\infty . \tag{16}$$

On the other hand given $(u_1, u_2) \in B(R) \times B(R)$, the triangle inequality tells us that

$$|d(u_1 \theta_k(h) u_2 x_0, x_0) - d(\theta_k(h) x_0, x_0)| \leqslant d(u_1 x_0, x_0) + d(u_2 x_0, x_0) .$$

The map $f \colon \Gamma \to X$ being a (κ, ℓ)-quasi-isometric embedding, we get

$$|d(u_1 \theta_k(h) u_2 x_0, x_0) - d(\theta_k(h) x_0, x_0)| \leqslant 2(\kappa R + \ell).$$

Summing (16) when u_2 runs over $B(R)$ and k over $[\![n, n+N]\!]$ yields

$$\sum_{k=n}^{n+N} \sum_{(u_1,u_2)\in B(R)\times B(R)} \sum_{\substack{T^k h=h_0,\\ u_1\theta_k(h)u_2\in H}} \exp\Big(-\omega_G d\Big(u_1\theta_k(h)u_2x_0, x_0\Big)\Big)$$
$$\leqslant C\,|B(R)|^2\, e^{2\omega_G(\kappa R+\ell)} \sum_{k=n}^{n+N} \left\|\mathcal{L}_\lambda^k\Psi\right\|_\infty .$$

Lemma 4.12 provides a lower bound of the triple sum in the left hand side of the inequality, leading to

$$\sum_{g\in S(n)\cap H} e^{-\omega_G d(gx_0,x_0)} \leqslant C\,|B(R)|^2\, e^{2\omega_G(\kappa R+\ell)} \sum_{k=n}^{n+N} \left\|\mathcal{L}_\lambda^k\Psi\right\|_\infty .$$

Observe that for every $k \geqslant n$ we have,

$$\left\|\mathcal{L}\lambda^k\Psi\right\|_\infty \leqslant \left\|\mathcal{L}_\lambda^{k-n}\right\|_\infty \left\|\mathcal{L}_\lambda^n\Psi\right\|_\infty .$$

Hence the previous inequality becomes

$$\sum_{g\in S(n)\cap H} e^{-\omega_G d(gx_0,x_0)} \leqslant C\,|B(R)|^2\, e^{2\omega_G(\kappa R+\ell)} \left(\sum_{k=0}^{N} \left\|\mathcal{L}_\lambda^k\right\|_\infty\right) \left\|\mathcal{L}_\lambda^n\Psi\right\|_\infty ,$$

which is exactly the required inequality. □

Proposition 5.10. *The spectral radius ρ_λ of $\mathcal{L}_\lambda$ satisfies the following inequality*

$$\rho_\lambda \geqslant \exp\left(\frac{\omega_H - \omega_G}{\kappa}\right).$$

Proof. According to Lemma 5.9 there exist $B_2 \in \mathbf{R}_+^*$ and a function $\Psi \in H_\alpha^\infty(\mathfrak{I}, \mathcal{H})$ such that for every $n \in \mathbf{N}$ we have

$$\sum_{g\in S(n)\cap H} e^{-\omega_G d(gx_0,x_0)} \leqslant B_2 \left\|\mathcal{L}_\lambda^n\Psi\right\|_\infty . \tag{17}$$

Recall that the canonical map $H_\alpha^\infty(\mathfrak{I}, \mathcal{H}) \to C(\mathfrak{I}, \mathcal{H})$ is 1-Lipschitz (see Section A.1). Hence for every $n \in \mathbf{N}$, we have

$$\left\|\mathcal{L}_\lambda^n\Psi\right\|_\infty \leqslant \left\|\mathcal{L}_\lambda^n\Psi\right\|_{\infty,\alpha} \leqslant \left\|\mathcal{L}_\lambda^n\right\|_{\infty,\alpha} \left\|\Psi\right\|_{\infty,\alpha} .$$

Hence

$$\limsup_{n\to\infty} \frac{1}{n} \ln \left\|\mathcal{L}_\lambda^n\Psi\right\|_\infty \leqslant \limsup_{n\to\infty} \frac{1}{n} \ln \left\|\mathcal{L}_\lambda^n\right\|_{\infty,\alpha} \leqslant \ln \rho_\lambda .$$

The left hand side of the inequality can be interpreted as the critical exponent of the series

$$\Upsilon_H(s) = \sum_{n=0}^{\infty} e^{-sn} \left\|\mathcal{L}_\lambda^n\Psi\right\|_\infty .$$

We use (17) exactly as we did in Proposition 5.8 to prove that for every $s \in \mathbf{R}$,

$$\mathcal{P}_H(\omega_G + s\kappa) \leqslant B_2 e^{s\kappa\ell} \Upsilon_H(s),$$

where $\mathcal{P}_H$ stands for the Poincaré series of H. Recall that $s \to \mathcal{P}_H(s)$ diverges whenever $s < \omega_H$. Consequently the critical exponent of $\Upsilon_H(s)$ is bounded below by $(\omega_H - \omega_G)/\kappa$, hence the result. □

Corollary 5.11. *If $\omega_H = \omega_G$, then $\rho_\lambda = 1$.*

Proof. It directly follows from the observation that $\rho_\lambda \leqslant 1$. □

Remark. The converse statement actually holds. It is a consequence of Theorems A.25 and B.1. Indeed if $\rho_\lambda = 1$, then the group H is co-amenable in G (Theorem A.25), hence $\omega_H = \omega_G$ (Theorem B.1). Nevertheless we are not aware of an upper bound of the spectral radius ρ_λ in the spirit of Proposition 5.10 which would directly leads to the converse direction (and an alternative proof of Roblin's theorem).

5.4 Proofs of the theorems

We are now in position to prove Theorems 5.1 and 5.2.

Proof of Theorem 5.1. The Cayley graph Γ of G is defined as in Section 5.1. This provides a subshift of finite type $(\mathfrak{H}_0, T)$ as detailed in Section 4.3 together with the labelling map $\theta\colon \mathfrak{H}_0 \to G$ defined in Section 4.2. We extract from this dynamical system an irreducible component $\mathfrak{I}$, such that the extension of $(\mathfrak{I}, T)$ by θ has the visibility property (Proposition 4.10). Using the strategy developed in Section 5.2 we define a potential $F\colon \mathfrak{I} \to \mathbf{R}_+^*$ which belongs to $H_\alpha^\infty(\mathfrak{I}, \mathbf{R})$ for some $\alpha \in \mathbf{R}_+^*$. We denote by $\mathcal{L}\colon H_\alpha^\infty(\mathfrak{I}, \mathbf{C}) \to H_\alpha^\infty(\mathfrak{I}, \mathbf{C})$ the corresponding transfer operator. Its spectral radius is $\rho = 1$ (Proposition 5.8).

Let H be a subgroup. We consider the set Y of left H-cosets of G. The group G acts on Y by right translations. If the action is amenable, then it follows from Roblin's Theorem (Theorem B.1) that $\omega_H = \omega_G$. Let us assume now that $\omega_H = \omega_G$. The action of G on Y induces a unitary representation $\lambda\colon G \to \mathcal{U}(\mathcal{H})$ where $\mathcal{H}$ stands for $\ell^2(Y)$. This leads to a twisted transfer operator $\mathcal{L}_\lambda\colon H_\alpha^\infty(\mathfrak{I}, \mathcal{H}) \to H_\alpha^\infty(\mathfrak{I}, \mathcal{H})$. Since $\omega_H = \omega_G$, Corollary 5.11 tells us that the spectral radius of $\mathcal{L}_\lambda$ is $\rho_\lambda = 1$. In particular $\rho_\lambda = \rho$. It follows from the amenability criterion (Theorem A.25) that the action of G on Y is amenable. □

Proof of Theorem 5.2. The dynamical system $(\mathfrak{I}, T)$, the extension map $\theta\colon \mathfrak{I} \to G$, the potential $F \in H_\alpha^\infty(\mathfrak{I}, \mathbf{R})$ as well as the transfer operator $\mathcal{L}\colon H_\alpha^\infty(\mathfrak{I}, \mathbf{C}) \to H_\alpha^\infty(\mathfrak{I}, \mathbf{C})$ are build as in the previous proof. In particular the spectral radius of $\mathcal{L}$ is $\rho = 1$. Let $\eta \in \mathbf{R}_+^*$, be the constant given by Theorem A.28.

Let H be an infinite index subgroup of G. We consider the set Y of left H-cosets of G. The group G acts transitively on Y by right translations. It induces a unitary representation $\lambda\colon G \to \mathcal{U}(\mathcal{H})$ where $\mathcal{H}$ stands for $\ell^2(Y)$. This leads to a twisted transfer operator $\mathcal{L}_\lambda\colon H_\alpha^\infty(\mathfrak{I}, \mathcal{H}) \to H_\alpha^\infty(\mathfrak{I}, \mathcal{H})$. Since H has infinite index in G, Theorem A.28 tells us that the spectral radius ρ_λ of $\mathcal{L}_\lambda$ is bounded above by $1 - \eta$. However Proposition 5.10 provides a lower bound for ρ_λ in termes of ω_G and ω_H. It yields $\omega_H \leqslant \omega_G - \varepsilon$ where

$$\varepsilon = \kappa \left|\ln(1 - \eta)\right| .$$

Note that ε does not depends on H, which completes the proof of the theorem. □

A An extension of Kesten's criterion

As explained in the introduction, this appendix is deeply inspired by the work of Stadlbauer. We prove a variation of his amenability criterion. Our approach makes an explicit use of representation theory and operator algebra, which were somehow hidden in [36]. In addition it provides precise estimates that can be use to analyse groups with Kazhdan's property (T) (see Theorem A.28). Similar results were also obtained by Dougall in [21].

In this section (Σ, σ) is a subshift of finite type of the alphabet $\mathcal{A}$. We use the same notations as in Section 2.1.

A.1 Function spaces

In order to study the dynamical system (Σ, σ) and its extensions, we define a family of function spaces. To that end, we fix a Banach space $(E, \|\,.\,\|)$. We endow the space $\mathcal{C}(\Sigma, E)$ of continuous function $\Phi\colon \Sigma \to E$ with the norm $\|\,.\,\|_\infty$ defined by

$$\|\Phi\|_\infty = \sup_{x\in\Sigma} \|\Phi(x)\|\,.$$

For our purpose, it will be rather convenient to work with smooth functions. Given $\alpha > 0$ we measure the *α-Hölder variations* of Φ by the quantity

$$\Delta_\alpha(\Phi) = \sup_{x\neq y} \frac{\|\Phi(x) - \Phi(y)\|}{d(x,y)^\alpha}.$$

We define the norm $\|\,.\,\|_{\infty,\alpha}$ of Φ by

$$\|\Phi\|_{\infty,\alpha} = \|\Phi\|_\infty + \Delta_\alpha(\Phi).$$

Definition A.1 (Functions with bounded variations)**.** Let $\alpha > 0$. The space $H^\infty_\alpha(\Sigma, E)$ is the set of all maps $\Phi\colon \Sigma \to E$ satisfying $\|\Phi\|_{\infty,\alpha} < \infty$.

It is a standard exercise to prove that $H^\infty_\alpha(\Sigma, E)$ is a Banach space. Moreover the canonical map $H^\infty_\alpha(\Sigma, E) \to \mathcal{C}(\Sigma, E)$ is a 1-Lischitz embedding. Sometimes it is more convenient to focus on the *local* α-Hölder variations of a function. Given $r \in \mathbf{R}^*_+$ and $\Phi \in \mathcal{C}(\Sigma, E)$ we let

$$\Delta_{\alpha,r}(\Phi) = \sup_{\substack{d(x,y)<r,\\ x\neq y}} \frac{\|\Phi(x) - \Phi(y)\|}{d(x,y)^\alpha}.$$

The next lemma explains how $\Delta_{\alpha,r}(\Phi)$ depends on r.

Lemma A.2. *Let $\alpha > 0$ and $r \in \mathbf{R}^*_+$. For every $\Phi \in \mathcal{C}(\Sigma, E)$ we have*

$$\Delta_\alpha(\Phi) - 2r^{-\alpha}\,\|\Phi\|_\infty \leqslant \Delta_{\alpha,r}(\Phi) \leqslant \Delta_\alpha(\Phi).$$

Remark A.3. In particular $\Delta_\alpha(\Phi)$ is finite if and only if so is $\Delta_{\alpha,r}(\Phi)$ for some (hence any) $r \in \mathbf{R}^*_+$.

Proof. Let $\Phi \in \mathcal{C}(\Sigma, E)$. The second inequality is obvious. Let us focus on the first one. Let $x, y \in \Sigma$. If $d(x,y) < r$, then by definition we have

$$\|\Phi(x) - \Phi(y)\| \leqslant \Delta_{\alpha,r}(\Phi) d(x,y)^{\,\alpha}.$$

On the other hand if $d(x,y) \geqslant r$, the triangle inequality yields

$$\|\Phi(x) - \Phi(y)\| \leqslant 2\,\|\Phi\|_\infty \leqslant 2r^{-\alpha}\,\|\Phi\|_\infty\, d(x,y)^{\,\alpha}.$$

Consequently for every distinct $x, y \in \Sigma$ we have

$$\frac{\|\Phi(x) - \Phi(y)\|}{d(x,y)^\alpha} \leqslant \Delta_{\alpha,r}(\Phi) + 2r^{-\alpha}\,\|\Phi\|_\infty\,. \qquad \square$$

The next lemmas are straightforward. Their proof is left to the reader.

Lemma A.4. *Let $\Phi \in \mathcal{C}(\Sigma, \mathbf{R})$ such that $\Phi(x) > 0$, for every $x \in \Sigma$. The map Φ belongs to $H^\infty_\alpha(\Sigma, \mathbf{C})$ if and only if so does $\ln \Phi$.*

Lemma A.5. *Let $f \in H^\infty(\Sigma, \mathbf{R})$ and $\Phi \in H^\infty_\alpha(\Sigma, E)$. The pointwise product function $f\Phi$ belongs to $H^\infty_\alpha(\Sigma, E)$. Moreover*

$$\|f\Phi\|_{\infty,\alpha} \leqslant \|f\|_{\infty,\alpha}\,\|\Phi\|_{\infty,\alpha}\,.$$

A.2 Ruelle's Perron Frobenius Theorem

Transfer operator. We fix a potential $F\colon \Sigma \to \mathbf{R}_+^*$ and assume that there exists $\alpha > 0$ such that $\ln F \in H^\infty_\alpha(\Sigma, \mathbf{R})$. For every $n \in \mathbf{N}$, for every $x \in \Sigma$, we let

$$F_n(x) = F(x)F(\sigma x)\cdots F\left(\sigma^{n-1}x\right).$$

By convention $F_0 = \mathbb{1}$. To such a potential we associate a transfer operator $\mathcal{L}\colon \mathcal{C}(\Sigma, \mathbf{C}) \to \mathcal{C}(\Sigma, \mathbf{C})$ defined by

$$\mathcal{L}\Phi(x) = \sum_{\sigma y = x} F(y)\Phi(y) = \sum_{a \in \mathcal{A}} \mathbb{1}_{\sigma[a]}(x) F(ax)\Phi(ax).$$

One checks easily that the powers of $\mathcal{L}$ are given by the following formula

$$\mathcal{L}^n\Phi(x) = \sum_{\sigma^n y = x} F_n(y)\Phi(y) = \sum_{w \in \mathcal{W}^n} \mathbb{1}_{\sigma^n[w]}(x) F_n(wx)\Phi(wx).$$

It is a standard fact that $\mathcal{L}$ defines a bounded operator of both $\mathcal{C}(\Sigma, \mathbf{C})$ and $H^\infty_\alpha(\Sigma, \mathbf{C})$ [26, Section XII.2]. We write ρ_∞ and ρ for the spectral radius of $\mathcal{L}$ seen as an operator of $\mathcal{C}(\Sigma, \mathbf{C})$ and $H^\infty_\alpha(\Sigma, \mathbf{C})$ respectively. One observes easily (see for instance [26, Section XII.2]) that

$$\rho_\infty = \lim_{n\to\infty} \sqrt[n]{\|\mathcal{L}^n\mathbb{1}\|_\infty}. \tag{18}$$

Recall that by the Riesz representation theorem, the dual space of $\mathcal{C}(\Sigma, \mathbf{R})$ can be identified with the set of measures on Σ. We write $\mathcal{L}^*$ for the dual operator of $\mathcal{L}$. From now on, unless mentioned otherwise, we see $\mathcal{L}$ as an operator on $H^\infty_\alpha(\Sigma, \mathbf{C})$ rather than $\mathcal{C}(\Sigma, \mathbf{C})$. For a proof of the following version of the Ruelle Perron-Frobenius Theorem we refer the reader to [26, Theorem XII.6] or [1, Theorem 1.5].

Theorem A.6 (Ruelle's Perron-Frobenius Theorem)**.** *If (Σ, σ) is topologically transitive, then the following holds*

(i) $\rho = \rho_\infty$ *is positive.*

(ii) There exists a probability measure μ on Σ whose support is Σ such that $\mathcal{L}^*\mu = \rho\mu$.

(iii) ρ is an eigenvalue of $\mathcal{L}$; the corresponding eigenspace has dimension 1; it is spanned by a function $h \in H^\infty_\alpha(\Sigma, \mathbf{C})$ such that $h(x) > 0$, for every $x \in \Sigma$ and

$$\int_\Sigma h d\mu = 1.$$

(iv) $\mathcal{L}$ has only finitely many eigenvalue of modulus ρ; the corresponding eigenspaces are finite dimensional; the rest of the spectrum of $\mathcal{L}$ is included in a disc of radius strictly less than ρ.

A.3 Twisted transfer operator

Let G be a finitely generated group. We fix a locally constant map $\theta\colon \Sigma \to G$. For every $n \in \mathbf{N}$, for every $x \in \Sigma$ we write

$$\theta_n(x) = \theta(x)\theta(\sigma x)\cdots\theta\left(\sigma^{n-1}x\right).$$

By convention θ_0 is the constant map sending x to the identity $1 \in G$. We use this data to produce an extension $(\Sigma_\theta, \sigma_\theta)$ of (Σ, σ) as follows. We let $\Sigma_\theta = \Sigma \times G$ and define $\sigma_\theta\colon \Sigma_\theta \to \Sigma_\theta$ by

$$\sigma_\theta(x, g) = (\sigma x, g\theta(x)).$$

Recall that the extension $(\Sigma_\theta, \sigma_\theta)$ has the *visibility property* if there exists a finite subset U of G such that for every $g \in G$, there exists two elements $u_1, u_2 \in U$, a point $x \in \Sigma$, and an integer $n \in \mathbf{N}$, satisfying $g = u_1\theta_n(x)u_2$.

We now fix a Banach space $(E, \|\,.\,\|)$. We denote by $\mathcal{B}(E)$ the space of bounded operators on E endowed with the operator norm, while $\mathrm{Isom}(E)$ stands for the set of *linear* isometries of E. Let $\lambda\colon G \to \mathrm{Isom}(E)$ be a homomorphism. As the $\theta\colon \Sigma \to G$ is locally constant the composition $\lambda \circ \theta\colon \Sigma \to \mathcal{B}(E)$ has α-Hölder bounded variations. For simplicity we make the following abuse of notations: given $x \in \Sigma$ and $n \in \mathbf{N}$, we write $\lambda(x)$ for $\lambda \circ \theta(x)$ and $\lambda_n(x)$ for $\lambda \circ \theta_n(x)$. In particular we have

$$\lambda_n(x) = \lambda(x)\lambda(\sigma x)\cdots\lambda\left(\sigma^{n-1}x\right).$$

The representation λ allows us to define a *twisted transfer operator* $\mathcal{L}_\lambda\colon \mathcal{C}(\Sigma, E) \to \mathcal{C}(\Sigma, E)$ as follows

$$\mathcal{L}_\lambda\Phi(x) = \sum_{\sigma y = x} F(y)\lambda(y)^{-1}\Phi(y) = \sum_{a\in\mathcal{A}} \mathbb{1}_{\sigma[a]}(x)F(ax)\lambda(ax)^{-1}\Phi(ax).$$

A standard computation shows that the n-th power of $\mathcal{L}_\lambda$ is given by

$$\mathcal{L}^n_\lambda\Phi(x) = \sum_{\sigma^n y = x} F_n(y)\lambda_n(y)^{-1}\Phi(y) = \sum_{w\in\mathcal{W}^n} \mathbb{1}_{\sigma^n[w]}(x)F_n(wx)\lambda_n(wx)^{-1}\Phi(wx).$$

If λ is the trivial representation of G in $\mathbf{C}$, we recover the usual Ruelle Perron-Frobenius operator defined in the previous section.

Lemma A.7. *The operator $\mathcal{L}_\lambda \colon \mathcal{C}(\Sigma, E) \to \mathcal{C}(\Sigma, E)$ is bounded. More precisely for every $n \in \mathbf{N}$, we have*

$$\|\mathcal{L}_\lambda^n\|_\infty \leqslant \|\mathcal{L}^n \mathbb{1}\|_\infty .$$

Proof. Let $n \in \mathbf{N}$. Let $\Phi \in \mathcal{C}(\Sigma, E)$. It follows from the triangle inequality that for every $x \in \Sigma$,

$$\|\mathcal{L}_\lambda^n \Phi(x)\| \leqslant \sum_{w \in \mathcal{W}^n} \mathbb{1}_{\sigma^n[w]}(x) F(wx) \left\|\lambda_n(wx)^{-1} \Phi(wx)\right\| \leqslant \mathcal{L}^n \mathbb{1}(x) \|\Phi\|_\infty .$$

Hence $\mathcal{L}_\lambda^n$ is a bounded operator and its operator norm is at most $\|\mathcal{L}^n \mathbb{1}\|_\infty$. □

Proposition A.8. *For every $n \in \mathbf{N}$, there exists $C_n \in \mathbf{R}_+^*$ such that for every linear representation $\lambda \colon G \to \mathrm{Isom}(E)$ into a Banach space $(E, \|\,.\,\|)$, for every $\Phi \in H_\alpha^\infty(\Sigma, E)$ we have*

$$\Delta_\alpha(\mathcal{L}_\lambda^n \Phi) \leqslant e^{-n\alpha} \|\mathcal{L}^n \mathbb{1}\|_\infty \Delta_\alpha(\Phi) + C_n \|\Phi\|_\infty .$$

Proof. As (Σ, σ) is a subshift of finite type, there exists $r > 0$ with the following property: for every $x, y \in \Sigma$, if $d(x, y) < r$, then for every $n \in \mathbf{N}$, for every $w \in \mathcal{W}^n$, $\mathbb{1}_{\sigma^n[w]}(x) = \mathbb{1}_{\sigma^n[w]}(y)$. Said differently the words that can be added in front of x or y are the same. We will take advantage of this fact to estimate the *local* Hölder variations of the twisted transfer operator. Recall that $\theta \colon \Sigma \to G$ is locally constant. Hence there exists $m \in \mathbf{N}$ such that θ is constant on every cylinder of length m.

Let $\lambda \colon G \to \mathrm{Isom}(E)$ be a linear representation. Let $n \in \mathbf{N}$. Let $\Phi \in H_\alpha^\infty(\Sigma, E)$. Let $x, y \in \Sigma$ be two distinct points such that $d(x, y) < r$. We fix $w \in \mathcal{W}^n$ such that $\mathbb{1}_{\sigma^n[w]}(x) = \mathbb{1}_{\sigma^n[w]}(y)$ equals 1. A standard computation tells us that

$$\begin{aligned} F_n(wx)\lambda_n(wx)^{-1}\Phi(wx) - F_n(wy)\lambda_n(wy)^{-1}\Phi(wy) &= F_n(wx)\lambda_n(wx)^{-1}\left(\Phi(wx) - \Phi(wy)\right) \\ &\quad + F_n(wx)\left(\lambda_n(wx)^{-1} - \lambda_n(wy)^{-1}\right)\Phi(wy) \\ &\quad + \left(F_n(wx) - F_n(wy)\right)\lambda_n(wy)^{-1}\Phi(wy). \end{aligned}$$

Recall that the image of λ is contained in the isometry group of E. On the other hand we observe that $d(wx, wy) = e^{-n} d(x, y)$. Hence the triangle inequality yields

$$\begin{aligned} &\left\|F_n(wx)\lambda_n(wx)^{-1}\Phi(wx) - F_n(wy)\lambda_n(wy)^{-1}\Phi(wy)\right\| \\ &\quad \leqslant \left(F_n(wx)\Delta_\alpha(\Phi) + F_n(wx)\Delta_\alpha(\lambda_n)\|\Phi\|_\infty + \Delta_\alpha(F_n)\|\Phi\|_\infty\right) e^{-n\alpha} d(x, y)^{\,\alpha}. \end{aligned}$$

This inequality holds for every $w \in \mathcal{W}^n$ such that $\mathbb{1}_{\sigma^n[w]}(x)$ – which equals $\mathbb{1}_{\sigma^n[w]}(y)$ – does not vanish. We sum these inequalities to get

$$\frac{\|\mathcal{L}_\lambda^n \Phi(x) - \mathcal{L}_\lambda^n \Phi(y)\|}{d(x, y)^\alpha} \leqslant e^{-n\alpha} \|\mathcal{L}^n \mathbb{1}\|_\infty \Delta_\alpha(\Phi) + e^{-n\alpha}\left(|\mathcal{W}^n| \Delta_\alpha(F_n) + \|\mathcal{L}^n \mathbb{1}\|_\infty \Delta_\alpha(\lambda_n)\right)\|\Phi\|_\infty .$$

Recall that θ is constant on any cylinder of length m. Hence θ_n is constant on any cylinder of length $n + m$. It follows from Lemma A.2 that $\Delta_\alpha(\lambda_n) \leqslant 2e^{(n+m)\alpha}$. Hence

$$\frac{\|\mathcal{L}_\lambda^n \Phi(x) - \mathcal{L}_\lambda^n \Phi(y)\|}{d(x, y)^\alpha} \leqslant e^{-n\alpha} \|\mathcal{L}^n \mathbb{1}\|_\infty \Delta_\alpha(\Phi) + \left(e^{-n\alpha} |\mathcal{W}^n| \Delta_\alpha(F_n) + 2e^{m\alpha} \|\mathcal{L}^n \mathbb{1}\|_\infty\right)\|\Phi\|_\infty .$$

This estimation holds for every distinct $x, y \in \Sigma$ satisfying $d(x,y) < r$, hence the right hand side of the last inequality is an upper bound of $\Delta_{\alpha,r}(\mathcal{L}_\lambda^n \Phi)$. Combined with Lemma A.2 we get that

$$\Delta_\alpha(\mathcal{L}_\lambda \Phi) \leqslant e^{-n\alpha} \left\|\mathcal{L}^n \mathbb{1}\right\|_\infty \Delta_\alpha(\Phi) + C_n \left\|\Phi\right\|_\infty ,$$

where

$$C_n = e^{-n\alpha} \left|\mathcal{W}^n\right| \Delta_\alpha(F_n) + 2e^{m\alpha} \left\|\mathcal{L}^n \mathbb{1}\right\|_\infty + 2r^{-\alpha}.$$

Observe that the parameter C_n neither depends on λ nor on Φ, thus the proof of the proposition is complete. □

Combined with Lemma A.7, the previous proposition tells us that for every $n \in \mathbf{N}$, for every $\Phi \in H_\alpha^\infty(\Sigma, E)$ we have

$$\left\|\mathcal{L}_\lambda^n \Phi\right\|_{\infty,\alpha} \leqslant e^{-n\alpha} \left\|\mathcal{L}^n \mathbb{1}\right\|_\infty \left\|\Phi\right\|_{\infty,\alpha} + (C_n + \left\|\mathcal{L}^n \mathbb{1}\right\|_\infty) \left\|\Phi\right\|_\infty . \tag{19}$$

Hence we can view $\mathcal{L}_\lambda$ as a bounded operator on $H_\alpha^\infty(\Sigma, E)$. This is the point of view that we will adopt in the remainder of this appendix. In particular we denote by $\|\mathcal{L}_\lambda\|_{\infty,\alpha}$ its corresponding operator norm and ρ_λ its spectral radius. Recall that ρ_∞ stands for the spectral radius of $\mathcal{L}$ seen as an operator of $\mathcal{C}(\Sigma, \mathbf{C})$.

Corollary A.9. *The spectral radius ρ_λ of $\mathcal{L}_\lambda$ is bounded above by ρ_∞.*

Proof. Let $\beta > \ln \rho_\infty$. It follows from (18) that there exists $n_0 \in \mathbf{N}$, such that for every $n \geqslant n_0$.

$$\left\|\mathcal{L}^n \mathbb{1}\right\|_\infty \leqslant e^{n\beta}.$$

We now fix $n \geqslant n_0$. According to Proposition A.8 there exists $C \in \mathbf{R}_+$ such that for every $\Phi \in H_\alpha^\infty(\Sigma, E)$ we have

$$\left\|\mathcal{L}_\lambda^n \Phi\right\|_{\infty,\alpha} \leqslant e^{n(\beta-\alpha)} \left\|\Phi\right\|_{\infty,\alpha} + C \left\|\Phi\right\|_\infty .$$

Let $m \in \mathbf{N}$. We are going to estimate $\|\mathcal{L}_\lambda^{nm}\|_{\infty,\alpha}$. To that end we choose $\Phi \in H_\alpha^\infty(\Sigma, E)$ and $k \in [\![0, m-1]\!]$. Applying the previous inequality to $\mathcal{L}_\lambda^{kn}\Phi$ we get

$$\left\|\mathcal{L}_\lambda^{(k+1)n} \Phi\right\|_{\infty,\alpha} \leqslant e^{n(\beta-\alpha)} \left\|\mathcal{L}_\lambda^{kn} \Phi\right\|_{\infty,\alpha} + C \left\|\mathcal{L}_\lambda^{kn} \Phi\right\|_\infty .$$

which combined with Lemma A.7 becomes

$$\left\|\mathcal{L}_\lambda^{(k+1)n} \Phi\right\|_{\infty,\alpha} \leqslant e^{n(\beta-\alpha)} \left\|\mathcal{L}_\lambda^{kn} \Phi\right\|_{\infty,\alpha} + Ce^{kn\beta} \left\|\Phi\right\|_\infty .$$

We multiply these inequalities by $e^{-(k+1)n(\beta-\alpha)}$ and sum them when k runs over $[\![0, m-1]\!]$. It gives

$$\left\|\mathcal{L}_\lambda^{nm} \Phi\right\|_{\infty,\alpha} \leqslant e^{nm(\beta-\alpha)} \left\|\Phi\right\|_{\infty,\alpha} + e^{nm\beta} \frac{Ce^{-n(\beta-\alpha)}}{e^{n\alpha} - 1} \left\|\Phi\right\|_\infty .$$

Recall that $\|\Phi\|_\infty \leqslant \|\Phi\|_{\infty,\alpha}$. So we have proved that there exists $D_n \in \mathbf{R}_+^*$ such that for every $m \in \mathbf{N}$, for every $\Phi \in H^\infty(\Sigma, E)$

$$\left\|\mathcal{L}_\lambda^{nm} \Phi\right\|_{\infty,\alpha} \leqslant e^{nm\beta} D_n \left\|\Phi\right\|_{\infty,\alpha} .$$

Hence for every $m \in \mathbf{N}$, we get

$$\left\|\mathcal{L}_\lambda^{nm}\right\|_{\infty,\alpha} \leqslant e^{nm\beta} D_n.$$

Passing to the limit when m tends to infinity we obtain

$$\ln \rho_\lambda = \lim_{m\to\infty} \frac{1}{nm} \ln \left(\left\|\mathcal{L}_\lambda^{nm}\right\|_{\infty,\alpha}\right) \leqslant \beta.$$

This inequality holds for every $\beta \in \mathbf{R}$ such that $\beta > \ln \rho_\infty$, thus $\rho_\lambda \leqslant \rho_\infty$. □

A.4 Renormalization

In this section we study how the transfer operators are affected when replacing the potential F by a rescaled and/or an homologous potential. This will allow us later to assume that $\mathcal{L}$ has spectral radius 1 and fixes the constant map $\mathbb{1}$, which will considerably lighten the notations.

Rescaling a homologous potential. Assume that the system (Σ, σ) is topologically transitive. Let $h \in H^\infty_\alpha(\Sigma, \mathbf{C})$ be the positive eigenvector given by Theorem A.6. We define a new potential $F' \colon \Sigma \to \mathbf{R}_+^*$ by

$$F'(x) = \frac{1}{\rho} \cdot \frac{h(x)}{h \circ \sigma(x)} F(x), \tag{20}$$

so that

$$\ln F'(x) = -\ln \rho + \ln\left(h(x)\right) - \ln\left(h \circ \sigma(x)\right) + \ln F(x).$$

We know that $\ln F$ belongs to $H^\infty_\alpha(\Sigma, \mathbf{C})$. The same holds for h. It follows that $\ln F'$ belongs to $H^\infty_\alpha(\Sigma, \mathbf{C})$ (Lemma A.4). In particular, F' satisfies the same assumptions as F. We write $\mathcal{L}' \colon H^\infty_\alpha(\Sigma, \mathbf{C}) \to H^\infty_\alpha(\Sigma, \mathbf{C})$ and $\mathcal{L}'_\lambda \colon H^\infty_\alpha(\Sigma, E) \to H^\infty_\alpha(\Sigma, E)$ for the corresponding usual and twisted transfer operators.

Comparing the operators. Let $\Phi \in H^\infty_\alpha(\Sigma, E)$. We observe that for every $x \in \Sigma$

$$\mathcal{L}'_\lambda \Phi(x) = \sum_{\sigma y = x} \frac{h(y)}{\rho h(x)} F(y) \lambda(y)^{-1} \Phi(y) = \frac{1}{\rho h(x)} \mathcal{L}_\lambda(h\Phi)(x).$$

In particular, since h is a positive eigenvector of $\mathcal{L}$ for the eigenvalue ρ, we get that $\mathcal{L}'\mathbb{1} = \mathbb{1}$. It follows from (18) combined with Theorem A.6 that the spectral radius of $\mathcal{L}'$ is 1. Let $\Phi \in H^\infty_\alpha(\Sigma, E)$. A proof by induction shows that for every $n \in \mathbf{N}$,

$$\rho^n h {\mathcal{L}'_\lambda}^n(\Phi) = \mathcal{L}^n_\lambda(h\Phi). \tag{21}$$

Since h is a positive continuous map on a compact set, there exists $m, M \in \mathbf{R}_+^*$ such that for every $x \in \Sigma$, we have $m \leqslant h(x) \leqslant M$. Hence $1/h$ also belongs to $H^\infty_\alpha(\Sigma, \mathbf{C})$. Combining (21) with Lemma A.5 we observe that there exists $A_1, A_2 \in \mathbf{R}_+^*$ such that for every $n \in \mathbf{N}$,

$$A_1 \left\|\mathcal{L}^n_\lambda\right\|_{\infty,\alpha} \leqslant \rho^n \left\|{\mathcal{L}'_\lambda}^n\right\|_{\infty,\alpha} \leqslant A_2 \left\|\mathcal{L}^n_\lambda\right\|_{\infty,\alpha}.$$

Hence the spectral radius of $\mathcal{L}'_\lambda$ is ρ_λ/ρ.

A.5 Invariant and almost invariant vectors

In this section we suppose that the system (Σ, σ) is topologically transitive. In addition we assume that $\mathcal{L}$ has spectral radius 1 and fixes $\mathbb{1}$. Under these assumptions, the operator norm of $\mathcal{L}_\lambda \colon \mathcal{C}(\Sigma, E) \to \mathcal{C}(\Sigma, E)$ is at most 1 (Lemma A.7). Similarly Proposition A.8 yields

Proposition A.10. *There exists $C_1 \in \mathbf{R}_+^*$ such that for every linear representation $\lambda \colon G \to \mathrm{Isom}(E)$ into a Banach space $(E, \|\,.\,\|)$, for every $\Phi \in H^\infty_\alpha(\Sigma, E)$ we have*

$$\Delta_\alpha(\mathcal{L}_\lambda \Phi) \leqslant e^{-\alpha} \Delta_\alpha(\Phi) + C_1 \|\Phi\|_\infty, \tag{22}$$

We would like to understand the behavior $\mathcal{L}_\lambda$ when 1 is a spectral value. To that end we study invariant and almost invariant vectors of the twisted transfer operator. Given $\varepsilon \in \mathbf{R}_+^*$ we say that $\Phi \in H_\alpha^\infty(\Sigma, E)$ is an *ε-invariant vector* if

$$\|\mathcal{L}_\lambda \Phi - \Phi\|_{\infty,\alpha} < \varepsilon \|\Phi\|_{\infty,\alpha}.$$

Almost invariant vectors. We start with some preliminary properties of almost invariants vectors for the twisted transfer operator.

Proposition A.11. *There exists $D_2, \eta \in \mathbf{R}_+^*$ with the following property. For every linear representation $\lambda\colon G \to \mathrm{Isom}(E)$ into a Banach space $(E, \|\,.\,\|)$, if $\Phi \in H_\alpha^\infty(\Sigma, E)$ is an η-invariant vector of $\mathcal{L}_\lambda$, then $\Delta_\alpha(\Phi) \leqslant D_2 \|\Phi\|_\infty$.*

Proof. Let C_1 be the parameter given by Proposition A.10. We fix $\eta \in \mathbf{R}_+^*$ such that $\eta < 1 - e^{-\alpha}$. Let $\lambda\colon G \to \mathrm{Isom}(E)$ be a linear representation into a Banach space $(E, \|\,.\,\|)$. Let $\Phi \in H_\alpha^\infty(\Sigma, E)$ be an η-invariant vector of $\mathcal{L}_\lambda$. The triangle inequality combined with (22) yields

$$\Delta_\alpha(\Phi) \leqslant \Delta_\alpha(\mathcal{L}_\lambda \Phi) + \Delta_\alpha(\mathcal{L}_\lambda \Phi - \Phi) \leqslant e^{-\alpha} \Delta_\alpha(\Phi) + C_1 \|\Phi\|_\infty + \eta \|\Phi\|_{\infty,\alpha}.$$

Hence

$$\Delta_\alpha(\Phi) \leqslant \frac{C_1 + \eta}{1 - (e^{-\alpha} + \eta)} \|\Phi\|_\infty. \qquad \square$$

Invariants vectors. We now detail the behavior of invariant vectors for the twisted transfer operator.

Lemma A.12. *Let $\Phi \in H_\alpha^\infty(\Sigma, E)$. If $\mathcal{L}_\lambda \Phi = \Phi$, then the map $\Sigma \to \mathbf{R}$ sending x to $\|\Phi(x)\|$ is constant.*

Proof. We denote by $\Psi\colon \Sigma \to \mathbf{R}$ the map defined by $\Psi(x) = \|\Phi(x)\|$. It is an element of $H_\alpha^\infty(\Sigma, \mathbf{R})$. Let $x_0 \in \Sigma$ such that

$$\Psi(x_0) = \sup_{x \in \Sigma} \Psi(x).$$

Such a point exists as Ψ is a continuous function on a compact set. Let $x \in \Sigma$. Let $\varepsilon \in \mathbf{R}_+^*$. Since the system (Σ, σ) is topologically transitive, there exists $n \in \mathbf{N}$ and $w_0 \in \mathcal{W}^n$ such that $w_0 x_0$ belongs to Σ and $d(x, w_0 x_0) \leqslant \varepsilon$. According to the triangle inequality we have

$$\Psi(x_0) \leqslant \mathcal{L}^n \Psi(x_0) \leqslant \sum_{w \in \mathcal{W}^n} \mathbb{1}_{\sigma^n[w]}(x_0) F_n(w x_0) \Psi(w x_0).$$

Recall that $\mathcal{L}^n \mathbb{1} = \mathbb{1}$. Hence the right hand side is a convex combination of terms of the form $\Psi(w x_0)$, all of them being bounded above by $\Psi(x_0)$. Consequently $\Psi(w_0 x_0) = \Psi(x_0)$. Since Ψ has bounded variations we get

$$\|\Psi(x) - \Psi(x_0)\| = \|\Psi(x) - \Psi(w_0 x_0)\| \leqslant \Delta_\alpha(\Psi) d(x, w_0 x_0)^\alpha \leqslant \Delta_\alpha(\Psi) \varepsilon^\alpha.$$

This inequality holds for every sufficiently small positive ε hence $\Psi(x) = \Psi(x_0)$. This proves that Ψ is a constant function equal to $\Psi(x_0)$. $\square$

Lemma A.13. *Assume that E is strictly convex. Let $\Phi \in H^\infty_\alpha(\Sigma, E)$. If $\mathcal{L}_\lambda \Phi = \Phi$, then for every $x \in \Sigma$, for every $n \in \mathbf{N}$, we have*

$$\lambda_n(x)\Phi(\sigma^n x) = \Phi(x).$$

Proof. It follows from Lemma A.12 that for every $x \in \Sigma$, we have $\|\Phi(x)\| = \|\Phi\|_\infty$. Let $x \in \Sigma$ and $n \in \mathbf{N}$. We let $z = \sigma^n x$. Observe that

$$\Phi(z) = \mathcal{L}^n_\lambda \Phi(z) = \sum_{\sigma^n y = z} F_n(y)\lambda_n(y)^{-1}\Phi(y).$$

Recall that $\mathcal{L}^n \mathbb{1} = \mathbb{1}$. Hence the right hand side is a convex combination of vectors of the form $\lambda_n(y)^{-1}\Phi(y)$. Since the image of λ is contained in the isometry group of E, their norm is the same as the one of $\Phi(z)$, namely $\|\Phi\|_\infty$. The space E being strictly convex, we get that $\Phi(z) = \lambda_n(y)^{-1}\Phi(y)$, for every $y \in \sigma^{-n}(\{z\})$. This holds in particular for $y = x$, hence the result. □

From the transfer operator to the representation. The goal is now to prove that if $\mathcal{L}$ and $\mathcal{L}_\lambda$ have the same spectral radius, then λ admits almost invariant vectors. We first cover the case when 1 is an *eigenvalue* of $\mathcal{L}_\lambda$ (Proposition A.14). In this situation we combine a convexity argument taking place in E with the visibility property to prove the existence of a non-zero vector $\phi_0 \in E$ that is fixed by a finite index subgroup G_0 of G. The second step deals with the general situation, i.e. when $\mathcal{L}_\lambda$ admits almost invariant vectors (Proposition A.18). The proof of this proposition is by contradiction. Negating the statement provides a family of counterexamples. Then, using an ultra-limit argument, we are able two produce a new twisted transfer operator for which 1 is an eigenvalue, therefore reducing the general case to the previous one.

Proposition A.14. *We assume that the extension of (Σ, σ) by θ has the visibility property. There exists a finite index subgroup G_0 of G such that for every representation $\lambda \colon G \to \mathrm{Isom}(E)$ into a uniformly convex Banach space $(E, \|\,.\,\|)$ the following holds. If 1 is an eigenvalue of the twisted transfer operator $\mathcal{L}_\lambda$, then the representation λ restricted to G_0 has a non-zero invariant vector $\phi_0 \in E$.*

Proof. We start the proof by introducing a few auxiliary objects that will lead to the definition of G_0. The most important point is that these objects do not involve the representation of G. Let $D_2 \in \mathbf{R}$ be the constant given by Proposition A.11. We fix an integer $m \in \mathbf{N}$ such that $e^{-m\alpha} D_2 < 1/2$. Up to increasing the value of m we can assume that $\theta \colon \Sigma \to G$ is constant when restricted to any cylinder of length m. Since (Σ, σ) is an irreducible subshift of finite type, there exists a finite subset $\mathcal{W}_0 \subset \mathcal{W}$ with the following property: for every $w, w' \in \mathcal{W}$, there exists $w_0 \in \mathcal{W}_0$ such that ww_0w' is admissible. We denote by N the length of the longest word in $\mathcal{W}_0$. As θ is locally constant, the set

$$U = \{\theta_k(x) \mid x \in \Sigma,\ k \in [\![0, m+N]\!]\}$$

is finite. It follows from the visibility property, there exists a finite subset U' of G with the following property: for every $g \in G$, there exists two elements $u'_1, u'_2 \in U'$, a point $x \in \Sigma$, and an integer $n \in \mathbf{N}$, satisfying $g = u'_1\theta_n(x)u'_2$. Finally, we let

$$K = |U|^2 \, |U'|^2 .$$

Since G is finitely generated, it has only finitely many finite index subgroups whose index does not exceed K. We define G_0 as the intersection of these subgroups. It is a finite index subgroup of G.

Let us now fix a representation $\lambda\colon G \to \mathrm{Isom}(E)$ into a uniformly convex Banach space $(E, \|\,.\,\|)$ such that 1 is an eigenvalue of the twisted transfer operator $\mathcal{L}_\lambda$. We choose a non-zero eigenvector $\Phi \in H^\infty_\alpha(\Sigma, E)$ of $\mathcal{L}_\lambda$, i.e. $\mathcal{L}_\lambda \Phi = \Phi$. It follows that the map $x \to \|\Phi(x)\|$ is constant (Lemma A.12) and $\Delta_\alpha(\Phi) \leqslant D_2 \|\Phi\|_\infty$ (Proposition A.11).

We write $v_1, \dots, v_\ell$ for the collection of admissible words of length m. Let $i \in [\![1, \ell]\!]$. We denote by C_i the closure of the convex hull of $\Phi([v_i])$. Since E is uniformly convex, the zero vector $0 \in E$ admits a unique projection on C_i that we denote by ϕ_i. It follows from our choice of m, that for every $x, x' \in [v_i]$, we have

$$\|\Phi(x) - \Phi(x')\| \leqslant \Delta_\alpha(\Phi) e^{-m\alpha} < \frac{1}{2} \|\Phi(x)\|\,.$$

Consequently C_i does not contains the zero vector and therefore ϕ_i is non-zero. We define H_i as the pre-image by $\lambda\colon G \to \mathrm{Isom}(E)$ of the stabilizer of ϕ_i. We are going to show that H_i is a finite index subgroup of G. To that end we start by proving the following lemma.

Lemma A.15. *For every $x \in \Sigma$, for every $n \in \mathbf{N}$, there exists two elements $u_1, u_2 \in U$ such that $u_1 \theta_n(x) u_2 \in H_i$.*

Proof. Let $x \in \Sigma$ and $n \in \mathbf{N}$. For simplicity we let $g = \theta_n(x)$. Let w be the prefix of length $n + m$ of x. Since $\theta\colon \Sigma \to G$ is constant when restricted on any cylinder of length m, we note that $\theta_n(x') = \theta_n(x)$ for every $x' \in [w]$. By the very definition of $\mathcal{W}_0$, there exists $w_1, w_2 \in \mathcal{W}_0$ such that $v_i w_1 w w_2 v_i$ is admissible. We denote by p and q the length of $v_i w_1$ and $w w_2$ respectively and observe that $p \leqslant m + N$ and $q \leqslant n + m + N$.

We now claim that there exist $u_1, u_2 \in U$ such that for every $y_1 \in [v_i]$, there exists $y_2 \in [v_i]$ satisfying

$$\lambda(u_1 g u_2) \Phi(y_1) = \Phi(y_2).$$

Choose $y_1 \in [v_i]$. It follows from our choice of w_1 and w_2 that $y_2 = v_i w_1 w w_2 y_1$ and $z = w w_2 y_1$ are two well defined points of Σ. Moreover $z = \sigma^p y_2$ and $y_1 = \sigma^q z$. According to Lemma A.13 we have

$$\lambda_{p+q}(y_2) \Phi(y_1) = \lambda_{p+q}(y_2) \Phi(\sigma^{p+q} y_2) = \Phi(y_2).$$

Note that y_2 belongs to $[v_i]$. Hence it suffices to show that $\theta_{p+q}(y_2)$ can be written $u_1 g u_2$ where u_1 and u_2 do not depend on y_1. The cocycle relation that θ_{p+q} satisfies yields

$$\theta_{p+q}(y_2) = \theta_p(y_2) \theta_n(\sigma^p y_2) \theta_{q-n}(\sigma^{p+n} y_2) = \theta_p(y_2) \theta_n(z) \theta_{q-n}(\sigma^n z).$$

By construction z belongs to $[w]$, hence $\theta_n(z) = \theta_n(x) = g$. Observe that p and $q - n$ are bounded above by $m + N$. Hence $u_1 = \theta_p(y_2)$ and $u_2 = \theta_{q-n}(\sigma^n z)$ belong to U. The proof of the claim will be complete if we can prove that u_1 and u_2 do not depend on y_1. Consider y_1' another element of $[v_i]$ and let as previously $y_2' = v_i w_1 w w_2 y_1'$ and $z = w w_2 y'$. We observe that z and z' belong to the same cylinder, namely $[w w_2 v_i]$ whose length is bounded below by $q + m$. Hence $\sigma^n z$ and $\sigma^n z'$ coincide on the first $q - n + m$ letters. Since $\theta\colon \Sigma \to G$ is constant on any cylinder of length m, we get $\theta_{q-n}(\sigma^n z) = \theta_{q-n}(\sigma^n z')$. The same argument shows that $\theta_p(y_2) = \theta_p(y_2')$, which completes the proof of our claim.

It follows from the claim that $\lambda(u_1 g u_2)$ maps C_i into itself as well. In particular, $\lambda(u_1 g u_2)\phi_i$ belongs to C_i. However $\lambda(u_1 g u_2)$ being a linear isometry we have $\|\lambda(u_1 g u_2)\phi_i\| = \|\phi_i\|$. Recall

that we defined ϕ_i as the projection of 0 onto C_i. By unicity of the projection (E is uniformly convex) we get $\lambda(u_1 g u_2)\phi_i = \phi_i$. In other words $u_1 g u_2 = u_1 \theta_n(x) u_2$ belongs to H_i which completes the proof of the lemma. □

As the extension of (Σ, σ) by θ satisfies the visibility property, the previous lemma has the following consequence:

$$G = \bigcup_{u_1,u_2 \in U, u_1', u_2' \in U'} u_1' u_1^{-1} H_i u_2^{-1} u_2' = \bigcup_{u_1,u_2 \in U, u_1', u_2' \in U'} \left[\left(u_1' u_1^{-1}\right) H_i \left(u_1' u_1^{-1}\right)^{-1} \right] \left(u_1' u_1^{-1} u_2^{-1} u_2'\right).$$

In other words G can be covered by finitely many cosets of conjugates of H_i. Moreover the number of these cosets is bounded above by

$$K = |U|^2 \, |U'|^2 .$$

According to [31, Lemma 4.1], H_i is a finite index subgroup of G and $[G : H_i] \leqslant K$. It follows from its definition that G_0 is a subgroup of H_i, thus the representation λ restricted to G_0 has a non-zero invariant vector, namely ϕ_i. □

Definition A.16. We say that a collection $\mathcal{E}$ of Banach spaces is *uniformly convex* if for every $\varepsilon > 0$ there exits $\eta > 0$ such that for every space $(E, \|\,.\,\|)$ of $\mathcal{E}$, for every unit vectors $\phi, \phi' \in E$, if $\|\phi - \phi'\| \geqslant \varepsilon$, then $\|\phi + \phi'\| \leqslant 2(1 - \eta)$.

Remark. This definition not only asks that each space $E \in \mathcal{E}$ is uniformly convex, but also that the parameters quantifying their rotundity work for all spaces simultaneously.

Definition A.17. Let $\lambda \colon G \to \mathrm{Isom}(E)$ be a representation of G into a Banach space. Let S be a finite subset of G and $\varepsilon \in \mathbf{R}_+^*$. A vector $\phi \in E$ is (S, ε)-*invariant* (with respect to λ) if

$$\sup_{s \in S} \|\lambda(s)\phi - \phi\| < \varepsilon \, \|\phi\| .$$

The representation $\lambda \colon G \to \mathrm{Isom}(E)$ *almost has invariant vectors* if for every finite subset S of G, for every $\varepsilon \in \mathbf{R}_+^*$ there exists an (S, ε)-invariant vector.

Proposition A.18. *We assume that the extension of (Σ, σ) by θ has the visibility property. There exists a finite index subgroup G_0 of G with the following property. Let $\mathcal{E}$ be a uniformly convex collection of Banach spaces. For every finite subset S_0 of G_0, for every $\varepsilon \in \mathbf{R}_+^*$, there exists $\eta \in \mathbf{R}_+^*$, such that the following holds. Let $\lambda \colon G \to \mathrm{Isom}(E)$ be a representation of G into a Banach space $(E, \|\,.\,\|)$ of $\mathcal{E}$. If the twisted transfer operator $\mathcal{L}_\lambda$ has an η-invariant vector, then the representation λ admits an (S_0, ε)-invariant vector.*

Before giving the proof, we recall some useful material regarding ultra-limit of Banach spaces.

Ultra-limit of Banach spaces. Let $\omega \colon \mathcal{P}(\mathbf{N}) \to \{0, 1\}$ be a *non-principal ultra-filter*, i.e. a finitely additive map which vanishes on any finite subset of $\mathbf{N}$ and such that $\omega(\mathbf{N}) = 1$. We say that a property $\mathcal{P}_n$ is true ω-*almost surely* (ω-as) if $\omega(\{n \in \mathbf{N} \mid \mathcal{P}_n \text{ holds}\}) = 1$. A real sequence (u_n) is ω-*essentially bounded* (ω-eb) if there exists $M \in \mathbf{R}$ such that $|u_n| \leqslant M$ ω-as. We say that the ω-limit of (u_n) is $\ell \in \mathbf{R}$ and write $\lim_\omega u_n = \ell$, if for every $\varepsilon > 0$, we have $d(u_n, \ell) < \varepsilon$ ω-as. Any real sequence which is ω-eb admits an ω-limit [4].

Let $(E_n, \|\,.\,\|)$ be a sequence of Banach spaces. We define the ultra-product $\Pi_\omega E_n$ as the set

$$\Pi_\omega E_n = \{(\phi_n) \in \Pi_{\mathbf{N}} E_n \mid \|\phi_n\| \,\omega\text{-eb}\}\,.$$

We endow $\Pi_\omega E_n$ with the following equivalence relation: $(\phi_n) \equiv (\phi'_n)$ if $\lim_\omega \|\phi_n - \phi'_n\| = 0$.

Definition A.19. The ω-limit of the sequence $(E_n, \|\,.\,\|)$ that we denote $\lim_\omega E_n$ is the quotient of $\Pi_\omega E_n$ by the equivalence relation $\equiv$.

If (ϕ_n) is an element of $\Pi_\omega E_n$ we write $\lim_\omega \phi_n$ for its image in $\lim_\omega E_n$. The set $\lim_\omega E_n$ naturally comes with a structure of vector space characterized as follows.

(i) $(\lim_\omega \phi_n) + (\lim_\omega \phi'_n) = \lim_\omega(\phi_n + \phi'_n)$, for every $(\phi_n), (\phi'_n) \in \Pi_\omega E_n$;

(ii) $c(\lim_\omega \phi_n) = \lim_\omega(c\phi_n)$, for every $(\phi_n) \in \Pi_\omega E_n$ and $c \in \mathbf{C}$.

The space $\lim_\omega E_n$ carries also a norm characterized by

$$\left\|\lim_\omega \phi_n\right\| = \lim_\omega \|\phi_n\|\,, \quad \forall (\phi_n) \in \Pi_\omega E_n.$$

One can check that $(\lim_\omega E_n, \|\,.\,\|)$ is a Banach space. The next lemma is a straightforward exercise.

Lemma A.20. *If $(E_n, \|\,.\,\|)$ is a uniformly convex collection of Banach spaces, then $(\lim_\omega E_n, \|\,.\,\|)$ is uniformly convex.*

Proof of Proposition A.18. We denote by G_0 the finite index subgroup of G given by Proposition A.14. We denote by D_2 and η the parameters given by Proposition A.11. Let $\mathcal{E}$ be a uniformly convex collection of Banach spaces.

A family of counterexamples. Let S_0 be a finite subset of G_0 and $\varepsilon \in \mathbf{R}_+^*$. Let (η_n) be a sequence of positive real numbers converging to zero. Assume that the proposition is false. It means that for every $n \in \mathbf{N}$, there exists a representation $\lambda_n \colon G \to \mathrm{Isom}(E_n)$ where $(E_n, \|\,.\,\|)$ belongs to $\mathcal{E}$, with the following properties:

(i) the twisted transfer operator $\mathcal{L}_{\lambda_n}$ has an η_n-invariant vector $\Phi_n \in H^\infty_\alpha(\Sigma, E_n)$;

(ii) the representation λ_n does not admit any (S_0, ε)-invariant vector.

In the remainder of the proof, we write for simplicity $\mathcal{L}_n$ instead of $\mathcal{L}_{\lambda_n}$.

Almost invariant vectors for the twisted operator. Without loss of generality we can assume that $\|\Phi_n\|_\infty = 1$, for every $n \in \mathbf{N}$. Recall that $D_2, \eta \in \mathbf{R}_+^*$ are the parameter given by Proposition A.11. Since (η_n) converges to 0, up to throwing away the first terms of the sequence, we can assume that $\eta_n \leqslant \eta$ for every $n \in \mathbf{N}$. It follows now from Proposition A.11 that for every $n \in \mathbf{N}$, we have

$$\Delta_\alpha(\Phi_n) \leqslant D_2 \|\Phi_n\|_\infty \leqslant D_2. \tag{23}$$

We now fix $k \in \mathbf{N}$ such that $e^{-k\alpha} D_2 < 1/2$. We write $v_1, \dots, v_\ell$ for the collection of all admissible words of length k. Up to passing to a subsequence we may assume that there exists $i \in [\![1, \ell]\!]$ such that the map $x \to \|\Phi_n(x)\|$ achieved its maximum in $[v_i]$. By reordering the elements $v_1, \dots, v_\ell$ we can actually assume that $i = 1$. In particular it follows from (23) and our choice of k, that for every $n \in \mathbf{N}$, for every $x \in [v_1]$, we have $\|\Phi_n(x)\| \geqslant 1/2$.

The Banach space E_∞. We now fix a non-principal ultra-filter ω. The limit space $E_\infty = \lim_\omega E_n$ is a uniformly convex Banach space (Lemma A.20). The next step is to define a representation $\lambda_\infty \colon G \to \mathrm{Isom}(E_\infty)$. Given $g \in G$ and a vector $\phi = \lim_\omega \phi_n$ of E_∞, we let

$$\lambda_\infty(g)\phi = \lim_\omega \left[\lambda_n(g)\phi_n\right].$$

One easily checks that $\lambda_\infty(g)$ is a well-defined linear isometry of E_∞. Moreover that map $\lambda_\infty \colon G \to \mathrm{Isom}(E_\infty)$ obtained in this way is a homomorphism. In particular, one can consider the twisted transfer operator

$$\mathcal{L}_{\lambda_\infty} \colon H^\infty_\alpha(\Sigma, E_\infty) \to H^\infty_\alpha(\Sigma, E_\infty).$$

that for simplicity we denote $\mathcal{L}_\infty$.

The eigenvector Φ_∞. We now use the sequence (Φ_n) to produce an eigenvector of $\mathcal{L}_\infty$. Recall that for every $n \in \mathbf{N}$, for every $x \in \Sigma$, we have $\|\Phi_n(x)\| \leqslant 1$. Hence we can define a map $\Phi_\infty \colon \Sigma \to E_\infty$ as follows

$$\Phi_\infty(x) = \lim_\omega \Phi_n(x), \quad \forall x \in \Sigma.$$

Note that Φ_∞ is bounded. More precisely, $\|\Phi_\infty\|_\infty \leqslant 1$. Recall that for every $\Delta_\alpha(\Phi_n) \leqslant D_2$ for every $n \in \mathbf{N}$. It directly follows that $\Delta_\alpha(\Phi_\infty) \leqslant D_2$. Hence Φ_∞ belongs to $H^\infty_\alpha(\Sigma, E_\infty)$. We also observe that Φ_∞ is non-trivial. Indeed by construction $\|\Phi_n(x)\| \geqslant 1/2$, for every $n \in \mathbf{N}$, for every $x \in [v_1]$. It follows that Φ_∞ restricted to $[v_1]$ does not vanish. Finally, we claim that $\mathcal{L}_{\lambda_\infty}\Phi_\infty = \Phi_\infty$. Let $x \in \Sigma$. Since $\mathcal{L}\mathbb{1} = \mathbb{1}$ we can write

$$\mathcal{L}_n\Phi_n(x) - \Phi_n(x) = \sum_{\sigma y = x} F(y)\left[\,\lambda_n(y)^{-1}\Phi_n(y) - \Phi_n(x)\right], \quad \forall n \ \in \mathbf{N}$$

and

$$\mathcal{L}_\infty\Phi_\infty(x) - \Phi_\infty(x) = \sum_{\sigma y = x} F(y)\left[\,\lambda_\infty(y)^{-1}\Phi_\infty(y) - \Phi_\infty(x)\right].$$

It follows from the definition of λ_∞ and Φ_∞ that

$$\mathcal{L}_\infty\Phi_\infty(x) - \Phi_\infty(x) = \lim_\omega \left[\,\mathcal{L}_n\Phi_n(x) - \Phi_n(x)\right] = 0,$$

which completes the proof of our claim.

Almost invariant vector for λ_n. The previous discussion shows that 1 is an eigenvalue of $\mathcal{L}_\infty \colon H^\infty_\alpha(\Sigma, E_\infty) \to H^\infty_\alpha(\Sigma, E_\infty)$. It follows from Proposition A.14 that the limit representation $\lambda_\infty \colon G \to \mathrm{Isom}(E_\infty)$ restricted to the finite index subgroup G_0 admits a non-zero invariant ϕ_∞. Such a vector can be written $\phi_\infty = \lim_\omega \phi_n$, where $(\phi_n) \in \Pi_\omega E_n$. Since ϕ_∞ is non zero, we can assume without loss of generality that $\|\phi_n\| = 1$, for every $n \in \mathbf{N}$. Since S_0 is contained in G_0, for every $g_0 \in S_0$, we have

$$\lim_\omega \left[\lambda_n(g_0)\phi_n\right] = \lambda_\infty(g_0)\phi_\infty = \phi_\infty = \lim_\omega \phi_n.$$

The set S_0 being finite, the vector ϕ_n is an (S_0, ε)-invariant vector (with respect to λ_n) ω-as. This contradicts our initial assumption and completes the proof of the proposition. □

Corollary A.21. *We assume that the extension of (Σ, σ) by θ has the visibility property. There exists a finite index subgroup G_0 of G with the following property. Let $\mathcal{E}$ be a uniformly convex collection of Banach spaces. For every finite subset S_0 of G_0, for every $\varepsilon \in \mathbf{R}_+^*$ there exists $\eta \in \mathbf{R}_+^*$ such that the following holds. Let $\lambda\colon G \to \mathrm{Isom}(E)$ be a representation of G into a Banach space $(E, \|\,.\,\|)$ of $\mathcal{E}$. If the spectral radius ρ_λ of the twisted transfer operator $\mathcal{L}_\lambda$ satisfies $\rho_\lambda > 1-\eta$, then there exists $\beta \in \mathbf{R}$ such that the representation $e^{i\beta}\lambda$ admits an (S_0, ε)-invariant vector.*

Proof. Let $\lambda\colon G \to \mathrm{Isom}(E)$ be a representation of G into a Banach space $(E, \|\,.\,\|)$. Let $\eta > 0$. We claim that if $\rho_\lambda > 1-\eta$, then there exists $\beta \in \mathbf{R}$, such that the operator $e^{i\beta}\mathcal{L}_\lambda$ admits an η-invariant vector. Since ρ_λ is the spectral radius of $\mathcal{L}_\lambda$, there exists $\beta \in \mathbf{R}$, such that $\rho_\lambda e^{-i\beta}$ is a point in the boundary of $\mathrm{Spec}(\mathcal{L}_\lambda)$. According to [11, Proposition 6.7], there exists $\Phi \in H^\infty_\alpha(\Sigma, E)$ such that

$$\left\|\mathcal{L}_\lambda \Phi - \rho_\lambda e^{-i\beta}\Phi\right\|_{\infty,\alpha} < (\rho_\lambda - 1 + \eta)\left\|\Phi\right\|_{\infty,\alpha}. \tag{24}$$

Combined with the triangle inequality, it yields

$$\begin{aligned}\left\|e^{i\beta}\mathcal{L}_\lambda \Phi - \Phi\right\|_{\infty,\alpha} &\leqslant \left\|\mathcal{L}_\lambda \Phi_n - \rho_\lambda e^{-i\beta}\Phi\right\|_{\infty,\alpha} + \left\|\rho_\lambda \Phi - \Phi\right\|_{\infty,\alpha} \\ &< (\rho_\lambda - 1 + \eta)\left\|\Phi\right\|_{\infty,\alpha} + (1-\rho_\lambda)\left\|\Phi\right\|_{\infty,\alpha}.\end{aligned}$$

Hence $\|e^{i\beta}\mathcal{L}_\lambda\Phi - \Phi\|_{\infty,\alpha} < \eta\|\Phi\|_{\infty,\alpha}$, which completes the proof of our claim. Observe that the operator $e^{i\beta}\mathcal{L}_\lambda$ can be seen as the twisted transfer operator $\mathcal{L}_{\lambda'}$ associated to the representation $\lambda'\colon G \to \mathrm{Isom}(E)$ defined by $\lambda'(g) = e^{i\beta}\lambda(g)$. The corollary is now a direct consequence of Proposition A.18 □

A.6 Amenability and Kazhdan property (T)

In this section we focus on representations induced by a group actions. Let Y be a set. Let $\mathcal{H} = \ell^2(Y)$ be the set of functions $\phi\colon Y \to \mathbf{C}$ which are square summable. It carries a natural structure of Hilbert space. A vector $\phi \in \mathcal{H}$ is *non-negative*, if $\phi(y) \in \mathbf{R}_+$ for every $y \in Y$. Given any vector $\phi \in \mathcal{H}$, we defined its *modulus* to be the vector $|\phi| \in \mathcal{H}$ defined by $|\phi|(y) = |\phi(y)|$ for every $y \in Y$. Observe that $\||\phi|\| = \|\phi\|$.

Let G be a group acting on Y. The action of G induces a unitary representation $\lambda\colon G \to \mathcal{U}(\mathcal{H})$ defined as follows: for every $g \in G$, for every $\phi \in \mathcal{H}$,

$$[\lambda(g)\phi](y) = \phi(g^{-1}y), \quad \forall y \in Y.$$

Observe that for every $\phi \in \mathcal{H}$, for every $g \in G$ we have $|\lambda(g)\phi| = \lambda(g)|\phi|$.

Lemma A.22. *Let Y be a metric space endowed with an action of G. Let $\mathcal{H} = \ell^2(Y)$ and $\lambda\colon G \to \mathcal{U}(\mathcal{H})$ be the unitary representation induced by the action of G. Let $\beta \in [0, 2\pi)$. Let S be a finite subset of G and $\varepsilon \in \mathbf{R}_+^*$. If $\phi \in \mathcal{H}$ is (S, ε)-invariant with respect to $e^{i\beta}\lambda$, then $|\phi|$ is* non-negative *and (S, ε)-invariant with respect to λ.*

Proof. Given any two vectors $\phi_1, \phi_2 \in \mathcal{H}$, one checks easily that their modulus satisfies

$$\||\phi_1| - |\phi_2|\| \leqslant \|\phi_1 - \phi_2\|.$$

Let $\phi \in \mathcal{H}$, be an (S,ε)-invariant with respect to $e^{i\beta}\lambda$ Combining our various observations on modulus vector, we get that for every $g \in S$,

$$\|\lambda(g)\,|\phi| - |\phi|\| = \big\|\big|e^{i\beta}\lambda(g)\phi\big| - |\phi|\big\| \leqslant \big\|e^{i\beta}\lambda(g)\phi - \phi\big\| \leqslant \varepsilon\,\|\phi\| = \varepsilon\,\||\phi|\|\,.$$

Hence $|\phi|$ is a non-negative (S,ε)-invariant with respect to λ □

The main result of this section is the following statement.

Theorem A.23. *Let (Σ,σ) be an irreducible subshift of finite type. Let $F\colon \Sigma \to \mathbf{R}_+^*$ be a potential with α-bounded Hölder variations for some $\alpha \in \mathbf{R}_+^*$. Let $\mathcal{L}$ be the corresponding transfer operator and ρ its spectral radius. Let G be a finitely generated group and $\theta\colon \Sigma \to G$ be a locally constant map. We assume that the corresponding extension $(\Sigma_\theta,\sigma_\theta)$ has the visibility property. For every finite subset S of G and every $\varepsilon \in \mathbf{R}_+^*$ there exists $\eta \in \mathbf{R}_+^*$ with the following property.*

Let Y be a set endowed with an action of G and $\lambda\colon G \to \mathcal{U}(\mathcal{H})$ be the induced unitary representation, where $\mathcal{H} = \ell^2(Y)$. Let ρ_λ be the spectral radius of the twisted transfer operator $\mathcal{L}_\lambda\colon H_\alpha^\infty(\Sigma,\mathcal{H}) \to H_\alpha^\infty(\Sigma,\mathcal{H})$. If $\rho_\lambda > (1-\eta)\rho$, then the representation λ admits an (S,ε) invariant vector.

Proof of Theorem A.23. The strategy of the proof is the following. First we renormalize the potential so that we can assume that the spectral radius of $\mathcal{L}$ is $\rho = 1$ and $\mathbb{1}$ is an invariant vector of $\mathcal{L}$. Applying Corollary A.21 we get a finite index subgroup G_0 of G such that the representation λ when restricted to G_0 admits a certain almost invariant vector ϕ. We finally take advantage of the structure of $\ell^2(Y)$ to average the orbit of ϕ, and thus get an almost invariant vector with respect to λ.

Renormalization of the potential. We start with a reduction argument: we claim that without loss of generality we can assume that $\mathcal{L}$ has spectral radius 1 and fixes $\mathbb{1}$. Assume indeed that the result has be proved in this context and let us explain how to deduce the general case. Let $h \in H_\alpha^\infty(\Sigma,\mathbf{C})$ be the positive eigenvector of $\mathcal{L}$ given by the Ruelle Perron-Frobenius Theorem (Theorem A.6). Following the strategy of Section A.4 we define a new potential $F'\colon \Sigma \to \mathbf{R}_+^*$ by

$$F'(x) = \frac{1}{\rho}\cdot\frac{h(x)}{h\circ\sigma(x)}F(x).$$

We write $\mathcal{L}'$ for the corresponding transfer operator. As we observed $\mathcal{L}'$ has spectral radius 1 and fixes $\mathbb{1}$. It follows from our assumption that we can apply Theorem A.23 to this operator.

Let S be a finite subset of G and $\varepsilon \in \mathbf{R}_+^*$. Let $\eta \in \mathbf{R}_+^*$ be the parameter given by Theorem A.23 (with the additional assumption that the transfer operator has spectral radius 1 and $\mathbb{1}$ as an eigenvector) applied to the potential F'. Suppose now that Y is a space endowed with an action of G and denote by $\lambda\colon G \to \mathcal{U}(\mathcal{H})$ the induced unitary representation, where $\mathcal{H} = \ell^2(Y)$. Assume that the spectral radius of $\mathcal{L}_\lambda$ satisfies $\rho_\lambda > (1-\eta)\rho$. It follows from the discussion of Section A.4 that the spectral radius ρ'_λ of $\mathcal{L}'_\lambda$ satisfies $\rho'_\lambda = \rho_\lambda/\rho$. In particular $\rho'_\lambda > 1-\eta$. Theorem A.23 applied to the potential F' tells us that λ admits an (S,ε) invariant vector, which completes the proof of our claim.

Finite index subgroup with almost invariant vectors. From now on we assume that $\mathcal{L}$ has spectral radius 1 and fixes $\mathbb{1}$. Let S be a finite subset of G and $\varepsilon \in \mathbf{R}_+^*$. We denote by G_0 the finite index subgroup of G given by Corollary A.21. We denote by $u_1, \dots, u_m$ a set of representatives of G/G_0. For every $g \in S$, there exists a permutation $\sigma_g \colon [\![1, m]\!] \to [\![1, m]\!]$ such that for all $i \in [\![1, m]\!]$, we have

$$u_{\sigma_g(i)}^{-1} g u_i \in G_0.$$

We now define a finite subset S_0 of G_0 as

$$S_0 = \left\{ u_{\sigma_g(i)}^{-1} g u_i \;\middle|\; g \in S,\ i \in [\![1, m]\!] \right\}.$$

Note that the set of all Hilbert spaces is a uniformly convex collection of Banach spaces. According to Corollary A.21, there exists $\eta \in \mathbf{R}_+^*$ with the following property. Let $\lambda \colon G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of G. If the spectral radius of the twisted transfer operator $\mathcal{L}_\lambda$ is larger than $1 - \eta$, then there exists $\beta \in \mathbf{R}$ such that the representation $e^{i\beta}\lambda$ admits an $(S_0, \varepsilon/\sqrt{m})$-invariant vector.

Representation induced by an action. Let Y be a set endowed with an action of G and $\lambda \colon G \to \mathcal{U}(\mathcal{H})$ be the induced unitary representation, where $\mathcal{H} = \ell^2(Y)$. Assume that the spectral radius ρ_λ of the twisted transfer operator $\mathcal{L}_\lambda$ is at least $1 - \eta$. According to the very definition of η, there exists $\beta \in \mathbf{R}$, such that the representation $e^{i\beta}\lambda$ admits an $(S_0, \varepsilon/\sqrt{m})$-invariant vector ϕ. It follows from Lemma A.22 that $|\phi|$ is an $(S_0, \varepsilon/\sqrt{m})$-invariant vector with respect to the representation λ. We now let

$$\bar{\phi} = \frac{1}{m} \sum_{i=1}^{m} \lambda_n(u_i) \, |\phi| \,.$$

Let $g \in S$. The computation yields

$$m\lambda(g)\bar{\phi} = \sum_{i=1}^{m} \lambda\left(g u_i\right) |\phi| = \sum_{i=1}^{m} \lambda\left(u_{\sigma_g(i)}\right) \lambda\left(u_{\sigma_g(i)}^{-1} g u_i\right) |\phi| \,.$$

On the other hand, reindexing the sum defining $\bar{\phi}$ gives

$$m\bar{\phi} = \sum_{i=1}^{m} \lambda\left(u_{\sigma_g(i)}\right) |\phi|$$

Recall that for every $i \in [\![1, m]\!]$, the element $u_{\sigma_g(i)}^{-1} g u_i$ belongs to S_0. Thus the triangle inequality yields

$$\left\| \lambda(g)\bar{\phi} - \bar{\phi} \right\| \leqslant \frac{1}{m} \sum_{i=1}^{m} \left\| \lambda\left(u_{\sigma_g(i)}^{-1} g u_i\right) |\phi| - |\phi| \right\| < \frac{\varepsilon}{\sqrt{m}}.$$

This inequality holds for every $g \in G$. Observe that $\bar{\phi}$ is obtained by averaging non-negative vectors of $\mathcal{H}$ all of them having norm 1. It follows that the norm of $\bar{\phi}$ is bounded below by $1/\sqrt{m}$. Hence $\bar{\phi}$ is an (S, ε)-invariant vector with respect to λ. □

Amenability. There are numerous equivalent definition of amenability. The one that is the most adapted four our purpose can be formulated in terms of regular representation.

Definition A.24. The action of G on Y is amenable if and only if the representation $\lambda\colon G \to \mathcal{U}(\mathcal{H})$ admits almost invariant vectors. The group G is *amenable* if its action on itself is amenable.

Theorem A.25 (Amenability criterion)**.** *Let (Σ, σ) be an irreducible subshift of finite type. Let $F\colon \Sigma \to \mathbf{R}_+^*$ be a potential with α-bounded Hölder variations for some $\alpha \in \mathbf{R}_+^*$. Let $\mathcal{L}$ be the corresponding transfer operator and ρ its spectral radius. Let G be a finitely generated group and $\theta\colon \Sigma \to G$ be a locally constant map. We assume that the corresponding extension $(\Sigma_\theta, \sigma_\theta)$ has the visibility property. Let Y be a set endowed with an action of G and $\lambda\colon G \to \mathcal{U}(\mathcal{H})$ be the induced unitary representation, where $\mathcal{H} = \ell^2(Y)$. Let ρ_λ be the spectral radius of the twisted transfer operator $\mathcal{L}_\lambda\colon H_\alpha^\infty(\Sigma, \mathcal{H}) \to H_\alpha^\infty(\Sigma, \mathcal{H})$ defined by*

$$\mathcal{L}_\lambda \Phi(x) = \sum_{\sigma y = x} F(y)\lambda(y)^{-1}\Phi(y).$$

The following statements are equivalent.

(i) The action of G on Y is amenable.

(ii) ρ belongs to $\mathrm{Spec}(\mathcal{L}_\lambda)$.

(iii) $\rho_\lambda = \rho$.

Proof. Reasoning as in the beginning of the proof of Theorem A.23 we observe that without loss of generality we can assume that $\mathcal{L}$ has spectral radius 1 and fixes $\mathbb{1}$. We start with (ii)⇒(iii) Recall that $\rho_\lambda \leqslant 1$ (Corollary A.9). Hence if 1 belongs to $\mathrm{Spec}(\mathcal{L}_\lambda)$, then $\rho_\lambda = 1$. We now focus on (iii)⇒(i). Assume that $\rho_\lambda = 1$. It follows from Theorem A.23 that λ almost admits invariants vectors. Hence the action of G on Y is amenable. We are left to prove (i)⇒(ii). Assume that the action of G on Y is amenable. According to Corollary A.9 it is sufficient to prove that $\rho_\lambda \geqslant 1$. Let $n \in \mathbf{N}$. Let $\varepsilon > 0$. Since $\theta\colon \Sigma \to G$ is locally constant, the set

$$S = \{\theta_n(x) \mid x \in \Sigma\}$$

is finite. Since the action of G is amenable, there exists an (S, ε)-invariant vector $\phi \in \mathcal{H} \setminus \{0\}$. Without loss of generality we can assume that $\|\phi\| = 1$. We define a map $\Phi\colon \Sigma \to \mathcal{H}$ by letting $\Phi(x) = \phi$, for every $x \in \Sigma$. Obviously $\|\Phi\|_\infty = 1$ and $\Delta_\alpha(\Phi) = 0$, hence Φ belongs to $H_\alpha^\infty(\Sigma, \mathcal{H})$. Using the fact that $\mathcal{L}\mathbb{1} = \mathbb{1}$, we can write for every $x \in \Sigma$,

$$\|\mathcal{L}_\lambda^n \Phi(x) - \Phi(x)\| \leqslant \sum_{\sigma^n y = x} F_n(y) \left\|\lambda_n(y)^{-1}\Phi(y) - \Phi(x)\right\| = \sum_{\sigma^n y = x} F_n(y) \left\|\lambda_n(y)\phi - \phi\right\| < \varepsilon.$$

This proves that $\|\mathcal{L}_\lambda^n \Phi - \Phi\|_\infty < \varepsilon$. In particular, we get

$$\|\mathcal{L}_\lambda^n \Phi\|_{\infty,\alpha} \geqslant \|\mathcal{L}_\lambda^n \Phi\|_\infty > \|\Phi\|_\infty - \varepsilon \geqslant 1 - \varepsilon.$$

Recall that $\|\Phi\|_{\infty,\alpha} = 1$. Thus we have proved that the norm of $\mathcal{L}_\lambda^n$ – see as an operator of $H_\alpha^\infty(\Sigma, \mathcal{H})$ – is larger than $1 - \varepsilon$. This holds for every $\varepsilon > 0$. Hence for every $n \in \mathbf{N}$, we have

$$\|\mathcal{L}_\lambda^n\|_{\infty,\alpha} \geqslant 1.$$

Consequently

$$\rho_\lambda = \lim_{n\to\infty} \sqrt[n]{\|\mathcal{L}_\lambda^n\|_{\infty,\alpha}} \geqslant 1. \qquad \square$$

Kazhdan property (T). Let us recall first the definition of property (T).

Definition A.26. A discrete group G has *Kazhdan property* (T), if there exists a finite subset S of G and $\varepsilon \in \mathbf{R}_+^*$ with the following property. Any unitary representation $\pi\colon G \to \mathcal{U}(\mathcal{H})$ into a Hilbert space which admits (S,ε)-invariant vectors has a non-zero invariant vector. Such a pair (S,ε) is called a Kazhdan pair.

Let us also recall the following useful statement.

Lemma A.27. *Assume that the action of G on Y is transitive. The set Y is finite if and only if the representation $\lambda\colon G \to \mathcal{U}(\mathcal{H})$ admits a non-zero invariant vector.*

Theorem A.28. *Let (Σ,σ) be an irreducible subshift of finite type. Let $F\colon \Sigma \to \mathbf{R}_+^*$ be a potential with α-bounded Hölder variations for some $\alpha \in \mathbf{R}_+^*$. Let $\mathcal{L}$ be the corresponding transfer operator and ρ its spectral radius. Let G be a finitely generated group with Kazhdan property (T) and $\theta\colon \Sigma \to G$ be a locally constant map. We assume that the corresponding extension $(\Sigma_\theta,\sigma_\theta)$ has the visibility property. There exists $\eta > 0$ with the following property.*

Let Y be an infinite set endowed with an transitive action of G and $\lambda\colon G \to \mathcal{U}(\mathcal{H})$ be the induced unitary representation, where $\mathcal{H} = \ell^2(Y)$. Let ρ_λ be the spectral radius of the twisted transfer operator $\mathcal{L}_\lambda\colon H_\alpha^\infty(\Sigma,\mathcal{H}) \to H_\alpha^\infty(\Sigma,\mathcal{H})$ defined by

$$\mathcal{L}_\lambda \Phi(x) = \sum_{\sigma y = x} F(y)\lambda(y)^{-1}\Phi(y).$$

Then $\rho_\lambda \leqslant (1-\eta)\rho$.

Proof. Let (S,ε) be a Kazhdan pair of G. Let $\eta > 0$ be the constant given by Theorem A.23. Let Y be a set endowed with an transitive action of G and $\lambda\colon G \to \mathcal{U}(\mathcal{H})$ be the induced unitary representation, where $\mathcal{H} = \ell^2(Y)$. Let $\mathcal{L}_\lambda$ be the corresponding twisted transfer operator. Assume that $\rho_\lambda > (1-\eta)\rho$. It follows from Theorem A.23 that λ has an (S,ε)-invariant vector. Since (S,ε) is a Kazhdan pair, it follows that λ has a non-zero invariant vector. However the action of G on Y is transitive. Hence Y is finite (Lemma A.27). □

B Roblin's theorem

In this appendix, we provide a proof of Roblin's Theorem. Note that the statement below does not require H to be a normal subgroup. The proof is probably well-known from the specialists in the field, however we did not find it in the literature. It relies on a rather simple counting argument in a hyperbolic space.

Theorem B.1 (compare with Roblin [34, Théorème 2.2.2])**.** *Let G be a group acting properly co-compactly on a hyperbolic space X. Let H be a subgroup of G. We denote by ω_G and ω_H the exponential growth rates of G and H acting on X. If H is co-amenable in G, then $\omega_H = \omega_G$.*

Let G be a group acting properly co-compactly on a hyperbolic space. Let ω_G be the exponential growth rate of G acting on X. We fix a base point $o \in X$. Given $r \in \mathbf{R}_+$ we define the *ball* of radius r to be

$$B(r) = \{g \in G \mid d(go,o) \leqslant r\}.$$

Coornaert [12] proved that there exists $C_1 \in \mathbf{R}_+^*$ such that for every $r \in \mathbf{R}_+$,

$$e^{\omega_G r} \leqslant |B(r)| \leqslant C_1 e^{\omega_G r}. \tag{25}$$

Let $\delta \in \mathbf{R}_+$ be the hyperbolicity constant of X. Up to increasing the value of δ we can always assume that the following holds:

(i) The diameter of X/G is at most δ. In particular, for every $x, y \in X$, there exists $g \in G$ such that $d(x, gy) \leqslant \delta$.

(ii) $1 - C_1 e^{-\omega_G \delta} > 0$.

For every $r \in \mathbf{R}_+$, we denote by $S(r) = B(r) \setminus B(r-\delta)$ the *sphere* of radius r.

Lemma B.2. *There exists $C_2 \in \mathbf{R}_+^*$ with the following property. Given $\ell, r \in \mathbf{R}_+$ and $x \in X$, we denote by U the set of elements $g \in B(\ell)$ such that $\langle go, x\rangle_o \geqslant r$. The cardinality of U is bounded above by*

$$|U| \leqslant C_2 e^{\omega_G(\ell - r)}.$$

Proof. We fix a geodesic $[o, x]$ from o to x and write y for the point of $[o, x]$ at distance r from o. According to our choice of δ that there exists $h \in G$ such that $d(y, ho) \leqslant \delta$. Note that $d(ho, o) \geqslant r - \delta$. Let $g \in U$. It follows from the four point inequality (4) that

$$\langle go, o\rangle_{ho} \leqslant \langle go, o\rangle_y + \delta \leqslant 2\delta$$

Consequently

$$d\big(h^{-1}go, o\big) = d(go, ho) = d(go, o) - d(ho, o) + 2\langle go, o\rangle_{ho} \leqslant \ell - r + 5\delta.$$

Thus $h^{-1}U$ is contained in $B(\ell - r + 5\delta)$ and the result follows from (25). □

Let $\ell \in \mathbf{R}_+$. We denote by μ_ℓ the probability measure on G which is uniformly distributed on $S(\ell)$. It follows from (25) that for every $g \in S(\ell)$ we have

$$\frac{1}{C_1} e^{-\omega_G \ell} \leqslant \mu_\ell(g) \leqslant \frac{1}{C_3} e^{-\omega_G \ell}, \tag{26}$$

where $C_3 = 1 - C_1 e^{-\omega_G \delta}$. Our first task is to provide an estimate for the n-th convolution product of μ_ℓ. Later we will let ℓ tends to infinity. Thus we will be particularly careful to control these estimates in terms of ℓ. More precisely we are going to prove the following statement.

Proposition B.3. *There exists $D \in \mathbf{R}_+^*$ such that for every $\ell \in \mathbf{R}_+$, for every $n \in \mathbf{N}$, for every $g \in G$, we have*

$$\mu_\ell^{*n}(g) \leqslant D^n \left(\frac{\ell}{\delta} + 1\right)^n \exp\left(-\omega_G \frac{n\ell + d(go, o)}{2}\right).$$

In order to prove this proposition, we introduce the following sets that will allow us to track the orbits of the random walk. For every $\ell \in \mathbf{R}_+$, for every $n \in \mathbf{N}$, for every $g \in G$, we let

$$\mathcal{O}_\ell(g, n) = \{(u_1, \dots, u_n) \in S(\ell)^n \mid u_1 \cdots u_n = g\}.$$

Note that if $d(go, o) > n\ell$, then $\mathcal{O}_\ell(g, n)$ is empty. We adopt the convention that a product of elements of G indexed by the empty set is trivial. It follows that $\mathcal{O}_\ell(g, 0)$ is empty if g is non trivial and reduced to a single element (the empty tuple) if $g = 1$.

Lemma B.4. *There exists $D_0 \in \mathbf{R}_+^*$ such that for every $\ell \in \mathbf{R}_+$, for every $n \in \mathbf{N}$, for every $g \in G$, we have*

$$|\mathcal{O}_\ell(g,n)| \leqslant D_0^n \left(\frac{\ell}{\delta}+1\right)^n \exp\left(\omega_G \frac{n\ell - d(go,o)}{2}\right).$$

Proof. Let C_2 be the constant given by Lemma B.2. We let

$$D_0 = C_2 e^{2\omega_G \delta}.$$

Let $\ell \in \mathbf{R}_+$. We are going to prove the result by induction on n. If $n = 0$, it follows from our convention that for every $g \in G$, the set $\mathcal{O}_\ell(g,0)$ contains at most 1 element, hence the result. Assume now that the statement holds for some $n \in \mathbf{N}$. Let $g \in G$. For every element $u = (u_1, \dots, u_{n+1})$ of $\mathcal{O}_\ell(g, n+1)$ we let $g_u = u_1 \cdots u_n = g u_{n+1}^{-1}$ (according to our convention g_u is trivial is $n = 0$). For every $k \in \mathbf{N}$ such that $k\delta \leqslant \ell$, we denote by P_k the set of elements $u \in \mathcal{O}_\ell(g, n+1)$ such that

$$k\delta \leqslant \langle g_u o, o\rangle_{go} < (k+1)\delta.$$

Note that if $u = (u_1, \dots, u_{n+1})$ is an element of $\mathcal{O}_\ell(g, n+1)$, then $\langle g_u o, o\rangle_{go} \leqslant d(u_{n+1}o, o) \leqslant \ell$. Hence the collection (P_k) forms a partition of $\mathcal{O}_\ell(g, n+1)$. We are now going to estimate the cardinality of each of these sets.

Let $k \in \mathbf{N}$ such that $k\delta \leqslant \ell$. We write U_k for the image of P_k by the projection $P_k \to S(\ell)$ sending $(u_1, \dots, u_{n+1})$ to u_{n+1}. It follows from Lemma B.2 that

$$|U_k| \leqslant C_2 e^{\omega_G(\ell - k\delta)} \leqslant D_0 e^{-\omega_G(k+2)\delta} e^{\omega_G \ell}.$$

Let u_{n+1} be an element of U_k and $u = (u_1, \dots, u_{n+1})$ a pre-image of u_{n+1} in P_k. By definition $(u_1, \dots, u_n)$ is an element of $\mathcal{O}_\ell(g u_{n+1}^{-1}, n)$, whose cardinality can be bounded from above using the induction hypotheses. It follows that

$$\begin{aligned} |P_k| &\leqslant \sum_{u_{n+1} \in U_k} \left|\mathcal{O}_\ell(g u_{n+1}^{-1}, n)\right| \\ &\leqslant \sum_{u_{n+1} \in U_k} D_0^n \left(\frac{\ell}{\delta}+1\right)^n \exp\left(\omega_G \frac{n\ell - d(g u_{n+1}^{-1} o, o)}{2}\right). \end{aligned} \tag{27}$$

Observe that for any $u_{n+1} \in U_k$ we have $\langle g u_{n+1}^{-1} o, o\rangle_{go} < (k+1)\delta$ Hence

$$d\big(g u_{n+1}^{-1} o, o\big) \geqslant d(go, o) + d(u_{n+1}o, o) - 2\left\langle g u_{n+1}^{-1} o, o\right\rangle_{go} \geqslant d(go,o) + \ell - 2(k+2)\delta.$$

Consequently (27) becomes

$$\begin{aligned} |P_k| &\leqslant \sum_{u_{n+1} \in U_k} e^{\omega_G(k+2)\delta} D_0^n \left(\frac{\ell}{\delta}+1\right)^n \exp\left(\omega_G \frac{(n-1)\ell - d(go,o)}{2}\right) \\ &\leqslant |U_k|\, e^{\omega_G(k+2)\delta} D_0^n \left(\frac{\ell}{\delta}+1\right)^n \exp\left(\omega_G \frac{(n-1)\ell - d(go,o)}{2}\right). \end{aligned}$$

We now use the above estimate of $|U_k|$ to get

$$|P_k| \leqslant D_0^{n+1} \left(\frac{\ell}{\delta}+1\right)^n \exp\left(\omega_G \frac{(n+1)\ell - d(go,o)}{2}\right).$$

Note that this estimate does not depends on k. Moreover there are at most $\ell/\delta + 1$ integer $k \in \mathbf{N}$ such that $k\delta \leqslant \ell$. Since (P_k) forms a partition of $\mathcal{O}_\ell(g, n+1)$ we obtain

$$|\mathcal{O}_\ell(g, n+1)| \leqslant \sum_{k\delta \leqslant \ell} |P_k| \leqslant D_0^{n+1} \left(\frac{\ell}{\delta} + 1\right)^{n+1} \exp\left(\omega_G \frac{(n+1)\ell - d(go, o)}{2}\right).$$

Hence the statement holds for $n+1$, which completes the proof of the proposition. □

Proof of Proposition B.3. We denote by C_3 and D_0 the constants given by (26) and Lemma B.4 respectively and let $D = D_0/C_3$. Let $\ell \in \mathbf{R}_+$. Let $n \in \mathbf{N}$ and $g \in G$. It follows from the definition of the convolution that

$$\mu_\ell^{*n}(g) = \sum_{(u_1,\dots,u_n) \in O_\ell(g,n)} \mu_\ell(u_1) \cdots \mu_\ell(u_n).$$

Combining (26) and Lemma B.4, the previous equality becomes

$$\begin{aligned} \mu_\ell^{*n}(g) &\leqslant \left(\frac{D_0}{C_3}\right)^n \left(\frac{\ell}{\delta} + 1\right)^n \exp\left(-\omega_G \frac{n\ell + d(go, o)}{2}\right) \\ &\leqslant D^n \left(\frac{\ell}{\delta} + 1\right)^n \exp\left(-\omega_G \frac{n\ell + d(go, o)}{2}\right). \end{aligned}$$

□

We now fix a subgroup H of G and write ω_H for the exponential growth rate of H acting on X. We denote by Y the set of left H-cosets in G. The group G acts on Y by right translations. We write $\mathcal{H} = \ell^2(Y)$ for the set of square summable functions from Y to $\mathbf{C}$ and $\lambda\colon G \to \mathcal{U}(\mathcal{H})$ for the regular representation of G relative to H. Given $\ell \in \mathbf{R}_+$, we consider the random walk on Y associated to the probability measure μ_ℓ. Said differently for every $y \in Y$ and $g \in G$ the probability of going from y to $y \cdot g$ is $\mu_\ell(g)$. Let y_0 be the point of Y corresponding to H. Note that its stabilizer is exactly H. Hence the probability $p_\ell(n)$ that after n-step, the random walk starting to y_0 goes back to y_0 is exactly

$$p_\ell(n) = \mu_\ell^{*n}(H).$$

We associate to this random walk a Markov operator M_ℓ on $\mathcal{H}$.

$$M_\ell \phi = \sum_{g \in G} \mu_\ell(g) \lambda(g) \phi, \quad \forall \phi \in \mathcal{H}.$$

Since μ_ℓ is symmetric, M_ℓ is a self-adjoint operator. It follows that its spectral radius ρ_ℓ can be computed as follows – see for instance [39, Lemma 10.1].

$$\rho_\ell = \limsup_{n\to\infty} \sqrt[n]{p_\ell(n)} = \limsup_{n\to\infty} \sqrt[n]{\mu_\ell^{*n}(H)}.$$

The next proposition relates the spectral radius ρ_ℓ to the critical exponents ω_H and ω_G.

Proposition B.5. *The growth rates of H and G acting on X satisfy the following inequality*

$$\ln \rho_\infty \leqslant \max\left\{-\frac{1}{2}\omega_G, \omega_H - \omega_G\right\},$$

where $\rho_\infty = \limsup_{\ell\to\infty} \sqrt[\ell]{\rho_\ell}$.

Proof. We fix $\varepsilon \in \mathbf{R}_+^*$ such that if $\omega_H < \omega_G/2$, then $\omega_H + \varepsilon < \omega_G/2$. It follows from the definition of exponential growth rate that there exists $A \in \mathbf{R}_+^*$ such that for every $r \in \mathbf{R}_+$,

$$|H \cap S(r)| \leqslant |H \cap B(r)| \leqslant Ae^{(\omega_H+\varepsilon)r}. \tag{28}$$

We write D for the constant given by Proposition B.3. Let $\ell \in \mathbf{R}_+$ and $n \in \mathbf{N}$. Our first task is to bound $\mu_\ell^{*n}(H)$ from above. To that end we partition H according to the length of its elements.

$$\mu_\ell^{*n}(H) = \sum_{k\in\mathbf{N}} \sum_{h\in H\cap S(k\delta)} \mu_\ell^{*n}(h).$$

Note that if $n\ell > k\delta$, then the probabilities $\mu_\ell^{*n}(h)$ vanish. Using Proposition B.3, we get

$$\mu_\ell^{*n}(H) \leqslant \sum_{k\delta\leqslant n\ell} |H \cap S(k\delta)|\, D^n \left(\frac{\ell}{\delta}+1\right)^n \exp\left(-\omega_G \frac{n\ell + (k-1)\delta}{2}\right).$$

Combined with (28) it yields

$$\mu_\ell^{*n}(H) \leqslant Ae^{\frac{1}{2}\omega_G\delta} D^n \left(\frac{\ell}{\delta}+1\right)^n e^{-\frac{1}{2}\omega_G n\ell} \sum_{k\delta\leqslant n\ell} e^{\left(\omega_H+\varepsilon-\frac{1}{2}\omega_G\right)k\delta}. \tag{29}$$

We now distinguish two cases. Assume first that $\omega_H < \omega_G$. It follows from our choice of ε that $\omega_H + \varepsilon - \omega_G/2 < 0$. Hence a (rather brutal !) majoration in (29) gives

$$\mu_\ell^{*n}(H) \leqslant Ae^{\frac{1}{2}\omega_G\delta} D^n \left(\frac{\ell}{\delta}+1\right)^n \left(\frac{n\ell}{\delta}+1\right) e^{-\frac{1}{2}\omega_G n\ell}.$$

Consequently

$$\ln \rho_\ell \leqslant \ln D + \ln\left(\frac{\ell}{\delta}+1\right) - \frac{1}{2}\omega_G\ell.$$

This inequality holds for every $\ell \in \mathbf{R}_+$. Consequently

$$\ln \rho_\infty = \limsup_{\ell\to\infty} \frac{1}{\ell} \ln \rho_\ell \leqslant -\frac{1}{2}\omega_G,$$

which completes the first case. Assume now that $\omega_H \geqslant \omega_G/2$. Computing the sum in (29) we get

$$\mu_\ell^{*n}(H) \leqslant Ae^{\frac{1}{2}\omega_G\delta} D^n \left(\frac{\ell}{\delta}+1\right)^n e^{-\frac{1}{2}\omega_G n\ell} \frac{e^{\left(\omega_H+\varepsilon-\frac{1}{2}\omega_G\right)(n\ell+\delta)} - 1}{e^{\left(\omega_H+\varepsilon-\frac{1}{2}\omega_G\right)\delta} - 1}.$$

Consequently

$$\ln \rho_\ell \leqslant \ln D + \ln\left(\frac{\ell}{\delta}+1\right) + (\omega_H + \varepsilon - \omega_G)\ell.$$

Since this inequality holds for every $\ell \in \mathbf{R}_+$, we get $\ln \rho_\infty \leqslant \omega_H + \varepsilon - \omega_G$. This last inequality holds for every $\varepsilon > 0$, hence the result. □

Proof of Theorem B.1. Assume now that H is co-amenable in G. According to Kesten's criterion, the spectral radius of any of the Markov operator M_ℓ is 1. It follows from Proposition B.5 that $\omega_H \geqslant \omega_G$. The other inequality is obvious. □

Remark. The exact same strategy can be used to provide a lower bound for μ_ℓ^{*n} of the same kind than the one given in Proposition B.3. This leads to the more general version of Proposition B.5

Proposition B.6. *The limit* $\rho_\infty = \lim_{\ell\to\infty} \sqrt[\ell]{\rho_\ell}$ *exists. Moreover*

$$\ln \rho_\infty = \max\left\{-\frac{1}{2}\omega_G, \omega_H - \omega_G\right\}.$$

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