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A FLAME PROPAGATION MODEL ON A NETWORK WITH APPLICATION TO A BLOCKING PROBLEM

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Abstract. We consider the Cauchy problem

$$\begin{cases} \partial_t u + H(x, Du) = 0 & (x,t) \in \Gamma \times (0,T) \\ u(x,0) = u_0(x) & x \in \Gamma \end{cases}$$

where Γ is a network and H is a positive homogeneous Hamiltonian which may change from edge to edge. In the first part of the paper, we prove that the Hopf-Lax type formula gives the (unique) viscosity solution of the problem. In the latter part of the paper we study a flame propagation model in a network and an optimal strategy to block a fire breaking up in some part of a pipeline; some numerical simulations are provided.

1. Introduction.

We study the Cauchy problem

$$\begin{cases} \partial_t u + H(x, Du) = 0 & (x,t) \in \Gamma \times (0,T) \\ u(x,0) = u_0(x) & x \in \Gamma \end{cases} \tag{1}$$

where Γ is a network and the operator $H : \Gamma \times \mathbb{R} \to \mathbb{R}$ may change from edge to edge and inside each edge it is continuous, nonnegative and positive homogeneous in the last variable (see assumption (8) below).

When the state variable varies in an Euclidean space $\mathbb{R}^n$, the problem (1) arises in flame propagation models and evolution of curves whose speed of propagation only depends on the normal direction. Existence, uniqueness and evolution of level sets of the solution of (1) have been extensively studied in the framework of viscosity

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solution theory (see [3, 4, 8, 20]). In this case, the unique viscosity solution of (1) is given by the Hopf-Lax formula

$$u(x,t) = \min\{u_0(y) :\ S(y,x) \le t\}, \qquad (x,t) \in \mathbb{R}^n \times (0,\infty), \tag{2}$$

where S is a distance function characterized by solving the associated stationary equation

$$H(x, Dw) = 1, \qquad x \in \mathbb{R}^n.$$

In the recent time, there is an increasing interest in the study of nonlinear differential equations on networks since they describe various phenomena as traffic flow, blood circulation, data transmission, electric networks, etc (see [10, 15]). Concerning Hamilton-Jacobi equations on networks, we mention the recent papers [1, 12, 13, 14, 17] where different notions of viscosity solution have been introduced; we refer to [5] for a comparison among some of them.

This paper is divided into two parts; in the former one, following the approach in [13, 14], we prove that the Hopf-Lax formula (2) can be extended to this framework. The main issue of the investigation is to tackle transition vertices (namely, points of the network where several edges meet each other). Actually, a suitable definition of viscosity solution at transition vertices (together with the standard one at points inside edges) will ensure the well posedness of the problem. Let us recall that this feature also happens for stationary first order equations (see [1, 12, 13, 17]) whereas, for second order equations, some transition conditions (the so-called Kirchhoff condition) need to be imposed (see [6, 15] and references therein).

In the second part of the paper we illustrate our results with a concrete application: the *blocking* problem. Suppose that a fire breaks up in some part of an oil pipeline. A central controller can stop the propagation of the fire by closing the junctions of the pipes, represented by the vertices of the network. The controller spends some time to reach the junctions which become unavailable when they are reached by the fire front. Therefore only a subset of the vertices can be closed on time to stop the fire. The aim is to find a strategy which maximizes the part of the network preserved by the fire. We give a characterization of the optimal strategy and we study the corresponding flame propagation in the network. Moreover we describe a numerical scheme for the solution of the problem and we present some numerical examples.

This paper is organized as follows: in the rest of the introduction we set our notations. Section 2 is devoted to the theoretical problem; Section 3 is devoted to the application to the blocking problem also providing some numerical simulations.

Notations: A network Γ is a connected subset of $\mathbb{R}^n$ formed by finite collections of points $V := \{x_i\}_{i\in I}$ and of edges $E := \{e_j\}_{j\in J}$. The vertices of V are connected by the continuous, not self-intersecting arcs of E. Each arc e_j is parametrized by a smooth function $\pi_j : [0, l_j] \to \mathbb{R}^n$, $l_j > 0$ and we set

$$e_j := \pi_j((0, l_j)) \qquad \text{and} \qquad \overline{e}_j := \pi_j([0, l_j]) \,.$$

For $i \in I$ we denote by $Inc_i := \{j \in J \mid\ e_j \text{ is incident to } x_i\}$ the set of arcs incident a same vertex x_i. We fix a set $I_B \subset I$ and we denote by $\partial\Gamma := \{x_i \in V \mid\ i \in I_B\}$, the set of boundary vertices of Γ.
A path $\xi : [0, t] \to \Gamma$ is said *admissible* if there are $t_0 = 0 < t_1 < \cdots < t_{M+1} = t$ such that, for any $m = 0, \dots, M$, $\xi([t_m, t_{m+1}]) \subset \overline{e}_{j_m}$ for some $j_m \in J$ and $\pi^{-1}_{j_m} \circ \xi \in C^1(t_m, t_{m+1})$. We denote by $B^t_{x,y}$ the set of the admissible path such that $\xi(0) = x$,

$\xi(t) = y$ and by $d : \Gamma \times \Gamma \to \mathbb{R}^+$ the path distance on Γ, i.e.

$$d(x, y) := \inf \left\{ \ell(\gamma) : \gamma \in B^t_{y,x} \right\}, \qquad x, y \in \Gamma \tag{3}$$

where $\ell(\gamma)$ is the length of γ. We assume that the network is connected, hence $d(x, y)$ is finite for any $x, y \in \Gamma$.
We shall always identify $x \in \overline{e}_j$ with $y = \pi_j^{-1}(x) \in [0, l_j]$. For any function $u : \Gamma \to \mathbb{R}$ and each $j \in J$ we denote by $u_j : [0, l_j] \to \mathbb{R}$ the restriction of u to $\overline{e}_j$, i.e. $u_j(y) = u(\pi_j(y))$ for $y \in [0, l_j]$. For a function $u : \Gamma \to \mathbb{R}$, we assume that $u_j(x_i) = u_k(x_i)$ for all $j, k \in Inc_i$, $i \in I$, i.e. the value of u at the vertex x_i is univocally defined.
The derivatives are always considered with respect to the parametrization of the arc, i.e. if $x \in e_j$, $y = \pi_j^{-1}(x)$ then $Du(x) := \frac{du_j}{dy}(y)$. At $x = x_i \in V$, we denote $D_j u(x_i)$ the internal derivative relative to the arc e_j, $j \in Inc_i$, i.e.

$$D_j u(x_i) = \begin{cases} \lim\limits_{h \to 0^+} \frac{u_j(h) - u_j(0)}{h}, & \text{if } x_i = \pi_j(0), \\ \lim\limits_{h \to 0^+} \frac{u_j(l_j - h) - u_j(l_j)}{h}, & \text{if } x_i = \pi_j(l_j). \end{cases}$$

The space $C(\Gamma \times (0, T))$ of continuous functions on $\Gamma \times (0, T)$ is the space of $u : \Gamma \times (0, T) \to \mathbb{R}$ such that $u_j \in C([0, l_j] \times (0, T))$ and $u_j(x_i, t) = u_k(x_i, t)$ for all $j, k \in Inc_i$, $t \in (0, T)$ and all $i \in I$.

2. **Evolutive Hamilton-Jacobi equations on networks.** In this section we assume for simplicity that $\partial\Gamma = \emptyset$ (otherwise it is possible to introduce appropriate boundary condition on $\partial\Gamma$) and we consider the Hamilton-Jacobi equation

$$\partial_t u + H(x, Du) = 0 \qquad (x, t) \in \Gamma \times (0, T) \tag{4}$$

with the initial condition

$$u(x, 0) = u_0(x), \qquad x \in \Gamma. \tag{5}$$

The Hamiltonian H is given by a family $\{H_j\}_{j \in J}$, where $H_j : \overline{e}_j \times \mathbb{R} \to \mathbb{R}$, satisfies the following assumptions

$$H_j \in C(\overline{e}_j \times (0, T)); \tag{6}$$

$$|H_j(x, p) - H_j(y, p)| \le C d(x, y)(1 + |p|) \quad \text{for any } x,\, y \in \overline{e}_j,\, p \in \mathbb{R}; \tag{7}$$

$H_j(x, \cdot)$ is positive homogeneous in p for any $x \in \overline{e}_j$, namely

$$H_j(x, \lambda p) = \lambda H_j(x, p) \qquad \text{for any } \lambda \ge 0,\, (x, p) \in \overline{e}_j \times \mathbb{R}; \tag{8}$$

$$\inf\{H_j(x, p) : |p| = 1,\, x \in \overline{e}_j\} > 0. \tag{9}$$

Remark 1. A Hamiltonian satisfying the previous assumptions is given by

$$H_j(x, p) = \sup_{a \in A_j} \{-b_j(x, a)p\}$$

where A_j is a compact metric space, $b_j : \overline{e}_j \times A_j \to \mathbb{R}$ is a continuous function such that, for some $r > 0$, there holds $(-r, r) \subset \overline{\text{co}}\{b_j(x, a) : a \in A_j\}$. In particular, if $A_j = [-1, 1]$ and $b_j(x, a) = c_j(x)a$ with c bounded and strictly positive, then $H_j(x, p) = c_j(x)|p|$.

Remark 2. By (8) the equation is geometric and it is connected with front propagation (see [3, 4, 20]). Moreover, still (8) and the 1-dimensionality of the state imply that H_j must have the form: $H_j(x, p) = pH_j(x, 1)$ for $p \ge 0$ and $H_j(x, p) = -pH_j(x, -1)$ for $p < 0$.

Let us now recall the definition of viscosity solution introduced in [13, 14]. We consider the following class of test functions

$$C^1(\Gamma\times(0,T)) := \left\{\phi\in C(\Gamma\times(0,T)) \mid \phi_j\in C^1([0,l_j]\times(0,T)) \quad \forall j\in J\right\}.$$

Definition 2.1.

i) A function $u\in C(\Gamma\times(0,T))$ is said a (viscosity) subsolution to (4) if for every test function $\phi\in C^1(\Gamma\times(0,T))$ such that $u-\phi$ attains a local maximum at $(x,t)\in e_j\times(0,T)$, we have

$$\partial_t\phi_j(x,t)+H_j(x,D\phi_j(x,t))\le 0. \tag{10}$$

ii) A function $u\in C(\Gamma\times(0,T))$ is said a (viscosity) supersolution to (4) if for every test function $\phi\in C^1(\Gamma\times(0,T))$ such that $u-\phi$ attains a local minimum at $(x,t)\in\Gamma\times(0,T)$, we have

$$\begin{aligned} \partial_t\phi(x,t)+H(x,D\phi(x,t))\ge 0 \qquad & \text{if } x\notin V\\ \max_{j\in Inc_i}\{\partial_t\phi_j(x,t)+H_j(x,D_j\phi(x,t))\}\ge 0 \qquad & \text{if } x=x_i\in V. \end{aligned} \tag{11}$$

A function $u\in C(\Gamma\times(0,T))$ is said a (viscosity) solution of (4) if it is both a viscosity subsolution and a viscosity supersolution of (4).

Remark 3. We note that the continuity of ϕ at $x_i\in V$ implies: $\phi_j(x_i,t)=\phi_k(x_i,t)$ for every $t\in[0,T]$ and $j,k\in Inc_i$. Hence, the transition condition in (11) is equivalent to

$$\partial_t\phi(x_i,t)+\max_{j\in Inc_i}\{H_j(x_i,D_j\phi(x_i,t))\}\ge 0.$$

We first collect several properties of the viscosity solution of (4). The first result is a comparison principle for the equation (4) established in [13, 14].

Proposition 1. *Let $u,v\in C(\Gamma\times[0,T])$ be a subsolution and, respectively, a supersolution of (4) such that $u(x,0)\le v(x,0)$ for $x\in\Gamma$. Then $u\le v$ in $\Gamma\times[0,T]$.*

In the next Proposition 2 we state a regularity result for the solution of (4).

Proposition 2. *Let u be a solution to (4)-(5) where u_0 is Lipschitz continuous. Then, u is Lipschitz continuous in $\Gamma\times[0,T]$.*

The proof of the last proposition is a standard adaptation of the one in the Euclidean case (see [16, Prop.2.1]) and we skip it.

We now exploit the geometric character of the Hamiltonian (see (8)) to give a representation formula for the solution of (4). To this end, it is expedient to introduce some notations. Given the Hamiltonian $H=\{H_j\}_{j\in J}$, we define the support function of the sub-level set $\{p\in\mathbb{R}:\ H_j(x,p)\le 1\}$ as

$$s_j(x,q):=\sup\{p\,q:\ H_j(x,p)\le 1\}$$

and we set $s:=\{s_j\}_{j\in J}$. The function $s_j:\bar e_j\times\mathbb{R}\to\mathbb{R}$ is continuous, convex, positive homogeneous in q and nonnegative (see [19]). Note that s is *not* a function on Γ because it is not uniquely defined at internal vertices; it is a family of functions each one of them defined only on one edge. For example, if $H_j(x,p)=c_j(x)|p|$ then $s_j(x,q)=|q|/c_j(x)$.

We introduce a distance function related to the Hamiltonian H on the network. For any $x, y \in \Gamma$, we define

$$S(y,x) := \inf\left\{\int_0^t s(\xi(r),\dot{\xi}(r))dr : \, t>0,\, \xi \in B^t_{y,x}\right\}. \tag{12}$$

Let us note that this definition is well posed even if s is not uniquely defined at vertices. Actually, since an admissible path cannot stay on a vertex for a positive time with not null derivative, the intervals where ξ occupies a vertex give a null contribution to the integral in (12). Note that the distance defined by (12) coincides with the one defined by (3) for $H_j(x,p) = |p|$ for every $j \in J$; actually, in this case, we have $s_j(x,q) = |q|$ for every $j \in J$. Moreover, using the previous notation for an admissible path ξ, by the parametrization, we have

$$\int_0^t s(\xi(r),\dot{\xi}(r))dr = \sum_{m=0}^{M}\int_{t_m}^{t_{m+1}} s_{j_m}(\pi^{-1}\circ\xi(r),(\pi^{-1}\circ\xi)'(r))dr.$$

In the next lemma we collect some easy properties of S; for its proof we refer to [17, Prop.4.1] since their arguments easily adapt to our case.

Lemma 2.2. *The function S, defined in (12), verifies*

i) *symmetry: $S(x,y) = S(y,x)$ for any $x, y \in \Gamma$;*
ii) *subadditivity: $S(x,y) \le S(x,z) + S(z,y)$ for any $x, y, z \in \Gamma$;*
iii) *regularity: S is a Lipschitz continuous function on $\Gamma \times \Gamma$ and it is equivalent to the distance d, i.e. there exists $C > 0$ such that*

$$Cd(x,y) \le S(x,y) \le \frac{1}{C}\,d(x,y), \qquad \text{for any } x, y \in \Gamma. \tag{13}$$

For any closed set $K \subset \Gamma$, we define

$$S(K,x) := \inf_{y\in K} S(y,x). \tag{14}$$

The next proposition summarizes some properties of $S(K,\cdot)$ (for the definition of viscosity solution on a network in the stationary case, we refer the reader to the paper [13]).

Proposition 3. *For any closed $K \subset \Gamma$, the function $S(K,\cdot)$ defined in (14) is a subsolution in Γ and a supersolution in $\Gamma \setminus K$ of the Hamilton-Jacobi equation*

$$H(x,Du) = 1. \tag{15}$$

Proof. We first prove that, for a given $x_0 \in \Gamma$, $u(\cdot) = S(x_0,\cdot)$ is a viscosity subsolution in Γ and a viscosity supersolution in $\Gamma \setminus \{x_0\}$ of (15) in the sense of [13]. This amounts to prove that: (i) u is a viscosity subsolution in $\Gamma \setminus V$ and a viscosity supersolution in $(\Gamma \setminus \{x_0\}) \setminus V$ in the standard viscosity solution sense (see [2]), (ii) for any $x = x_i \in V \setminus \{x_0\}$ and for any test function $\phi \in C^1(\Gamma) := \left\{\phi \in C(\Gamma) \mid \phi_j \in C^1([0,l_j]) \quad \forall j \in J\right\}$ such that $u - \phi$ has a local minimum point at x, then

$$\max_{j\in Inc_i}\{H_j(x,D_j\phi(x))\} \ge 1. \tag{16}$$

For the proof that S is a viscosity solution of (15) inside the edges we refer to [19, Thm.2.1]. We show the the function u satisfies (16) at a vertex $x = x_i \in V \setminus \{x_0\}$. Without any loss of generality we assume that that $u - \phi$ has a *strict* local minimum at x.

We observe that the definition of S implies that ϵ-optimal paths must have finite length; in particular, they visit a finite number of vertices (independent of ϵ). Hence, possibly passing to a subsequence, we may assume that all the paths ξ_n (ξ_n is a $1/n$-optimal path) visit the same sequence of vertices and go through the same sequence of edges. We also observe that the integral in (12) is invariant by rescaling of parametrization of ξ. Therefore, without any loss of generality, we may assume: (1) all the ξ_n visit the same sequence of vertices and edges, (2) $t = 1$ in (12) for all $n \in \mathbb{N}$, (3) there exist $\eta > 0$ and $j \in Inc_i$ such that $\xi_n(r) \in \overline{e}_j$ for any $r \in [1-\eta, 1]$, $n \in \mathbb{N}$, (4) the point x is the endpoint of the parametrization of e_j namely $\pi_j(l_j) = x$. Fix $\delta = l_j$ if all the ξ_n's go through the whole e_j and $\delta = l_j - \pi_j^{-1}(x_0)$ if $x_0 \in e_j$ and the ξ_n's only go through the segment from x_0 to x. We define

$$u^n(z) := \int_0^{\tau_n(z)} s(\xi_n(r), \dot{\xi}_n(r))\, dr \qquad \forall z \in [l_j - \delta, l_j]$$

where $\tau_n(z)$ solves $\pi_j^{-1}(\xi_n(\tau)) = z$. We observe that $\tau_n(l_j) = 1$ (since $\xi_n(1) = x = \pi_j(l_j)$) and $\lim_{n\to+\infty} u^n(l_j) = u(x)$ (by the definition of ξ_n). Let x_n be a minimum point of $u^n - \phi_j$ on $[l_j - \delta, l_j]$. By the same arguments of Lemma [2, Lemma V.1.6], we get

$$x_n \to x \qquad \text{and} \qquad u^n(x_n) \to u(x) \qquad \text{as } n \to +\infty. \tag{17}$$

Since x_n is a minimum point for $u^n - \phi_j$, we get

$$\phi_j(x_n) - \phi_j(y) \geq u^n(x_n) - u^n(y) = \int_{\tau_n(y)}^{\tau_n(x_n)} s_j(\pi_j^{-1} \circ \xi_n(r), (\pi_j^{-1} \circ \xi_n)'(r))\, dr;$$

in particular, for $t_n := \tau_n(x_n) > \tau_n(y) =: t$ we infer

$$\frac{\phi_j(\pi_j^{-1} \circ \xi_n(t_n)) - \phi_j(\pi_j^{-1} \circ \xi_n(t))}{t_n - t} \geq \frac{1}{t_n - t} \int_t^{t_n} s_j(\pi_j^{-1} \circ \xi_n(r), (\pi_j^{-1} \circ \xi_n)'(r))\, dr.$$

Letting $t \to t_n^-$, for $q_n := (\pi_j^{-1} \circ \xi_n)'(t_n)$, we obtain $D\phi_j(x_n) q_n \geq s_j(x_n, q_n)$. Hence, the definition of s_j yields $H_j(x_n, D\phi_j(x_n)) \geq 1$. As $n \to +\infty$, by (17) we get the desired (16).

Having proved that $S(x_0, \cdot)$ is a viscosity subsolution in Γ and a viscosity supersolution in $\Gamma \setminus \{x_0\}$, then it is easy to prove that $S(K, \cdot)$ is a subsolution in Γ and a supersolution in $\Gamma \setminus K$ of (15) following the arguments of [17, Prop.6.2]. Actually, the fact that $S(K, \cdot)$ is a supersolution inside each edge is due to standard arguments for viscosity solutions. The fact that it is a supersolution at internal vertices follows from the same arguments of [17, Prop.3.4] and observing that (eventually passing to a subsequence) one may assume that all the $S(y_n, \cdot)$ approximating $S(K, \cdot)$ at a given vertex satisfy the transition condition on the same edge incident to the given vertex. Moreover, the fact that $S(K, \cdot)$ is a subsolution follows arguing as in [17, Prop.3.5] and noting that subsolutions to (15) are uniformly Lipschitz continuous. ☐

The following result gives a representation formula of Hopf-Lax type for the solution of (4)-(5).

Theorem 2.3. *Let $u_0 : \Gamma \to \mathbb{R}$ be a continuous function. Then the solution of (4)-(5) is given by*

$$u(x,t) = \min\{u_0(y) : S(y,x) \leq t\}. \tag{18}$$

In order to prove this result, let us first establish some preliminary lemmas.

Lemma 2.4. *If w is a subsolution (resp., supersolution) of (15) in Γ, then $u(x,t) = w(x) - t$ is a subsolution (resp., supersolution) of (4) in $\Gamma \times (0,T)$.*

The proof of the previous lemma is straightforward and we omit it.

Lemma 2.5.

(i) *A function $u \in C(\Gamma \times (0,T))$ is a subsolution of (4) if and only if for any $\alpha \in \mathbb{R}$ and for any admissible test function ϕ which has a local minimum on $\{u \geq \alpha\} \cap (\Gamma \times (0,T))$ at $(x,t) \in e_j \times (0,T)$, then (10) holds.*

(ii) *A function $v \in C(\Gamma \times (0,T))$ is a supersolution of (4) if and only if for any $\alpha \in \mathbb{R}$ and for any admissible test function ϕ which has a local maximum on $\{v \leq \alpha\} \cap (\Gamma \times (0,T))$ at $(x,t) \in \Gamma \times (0,T)$, then (11) holds.*

Proof. We use the arguments of [18, Lemma 3.1] adapted to the networks and we only consider the case $x = x_i \in V$.
We first assume that, for each $\alpha \in \mathbb{R}$ and test function ϕ which has a local maximum on $\{v \leq \alpha\} \cap (\Gamma \times (0,T))$ at (x,t), inequality (11) holds. Our aim is to prove that v satisfies the supersolution condition at (x,t). To this end, we assume by contradiction that there exists an admissible test function ϕ which verifies

$$0 = v(x,t) - \phi(x,t) < v(y,s) - \phi(y,s) \qquad \forall (y,s) \in B_r(x,t) \cap (\Gamma \times (0,T)) \tag{19}$$

$$\max_{j \in Inc_i} \{\partial_t \phi_j(x,t) + H_j(x, D_j\phi(x,t))\} < 0. \tag{20}$$

For $\alpha = \phi(x,t)$, we set $\Delta := \{v \leq \alpha\} \cap (\Gamma \times (0,T))$. Inequality (19) and the definition of Δ yield

$$\phi(y,s) \leq v(y,s) \leq \alpha = \phi(x,t) \qquad \forall (y,s) \in \Delta \cap B_r(x,t);$$

namely, the function ϕ attains a local maximum in (x,t) with respect to Δ. Invoking our assumption, we infer inequality (11) which amounts to the desired contradiction. Hence, the first implication of the statement is achieved.

We now prove the reverse implication and we assume that v is a supersolution to (4). Fix $\alpha \in \mathbb{R}$, $x = x_i \in V$ and let ϕ be an admissible test function which attains a local maximum on $\Delta := \{v \leq \alpha\} \cap (\Gamma \times (0,T))$ at (x,t). We recall that by the geometric character of the equation (4) it follows that if $u \in C(\Gamma \times (0,T))$ is a subsolution (resp., supersolution) to (4), then for any function $\theta : \mathbb{R} \to \mathbb{R}$ continuous and nondecreasing, the function $\theta \circ u$ is still a subsolution (resp., supersolution) to (4). Hence, by the previous observation, it suffices to prove that there exists a continuous nondecreasing function θ such that $\theta \circ v - \phi$ attains a local minimum at (x,t) relatively to $\Gamma \times (0,T)$.

Let $W \subset \Gamma \times (0,T)$ be a compact neighborhood of (x,t) such that $\phi \leq \phi(x,t)$ in $W \cap \Delta$. We set

$$E_0 := \{(y,s) \in W \mid \phi(y,s) > \phi(x,t)\}$$

(observe that the definition of W ensures: $E_0 \subset \Gamma \times (0,T)$ and $E_0 \cap \Delta = \emptyset$).

We now define the function θ according to the following cases: (a), $E_0 = \emptyset$; (b), $E_0 \neq \emptyset$ and $\beta := \inf_{E_0} v > \alpha$; (c), $E_0 \neq \emptyset$ and $\beta = \alpha$.

Case-(a). For $\theta(s) := \phi(x,t)$ (a constant function), one can easily check that $\theta \circ v - \phi$ attains a local minimum at (x,t) relatively to $\Gamma \times (0,T)$.

Case-(b). We define

$$\theta(s) := \begin{cases} \phi(x,t) & \text{for } s \in (-\infty, \alpha) \\ \sup_W \phi & \text{for } s \in [\beta, +\infty) \\ (\beta - \alpha)^{-1} \left[(s-\alpha)\sup_W \phi + (\beta - s)\phi(x,t)\right] & \text{for } s \in (\alpha, \beta). \end{cases}$$

Now, we want to prove

$$\theta(v(y,s)) - \phi(y,s) \geq \theta(v(x,t)) - \phi(x,t) = 0 \tag{21}$$

for every $(y,s) \in \Gamma \times (0,T)$ in some neighborhood of (x,t). To this end, we shall consider separately the cases when (y,s) belongs to $W \setminus (E_0 \cup \Delta)$, E_0, $W \cap \Delta$. For $(y,s) \in W \setminus (E_0 \cup \Delta)$, by the monotonicity of θ and the definition of Δ and E_0, we have

$$\theta(v(y,s)) \geq \theta(\alpha) = \phi(x,t) \geq \phi(y,s)$$

which amounts to inequality (21). For $(y,s) \in E_0$, taking into account the definition of β, we have

$$\theta(v(y,s)) \geq \theta(\beta) = \sup_W \phi \geq \phi(x,t).$$

For $(y,s) \in W \cap \Delta$, by the definition of Δ (recall that ϕ attains a local maximum on Δ at (x,t)), there holds

$$\theta(v(y,s)) = \phi(x,t) \geq \phi(y,s);$$

hence our claim (21) is completely proved.
Case-(c). For any $n \in \mathbb{N}$, we introduce

$$E_n := \{(y,s) \mid \phi(y,s) \geq \phi(x,t) + 1/n\}, \qquad \beta_n := \begin{cases} \inf_{E_n} v & \text{if } E_n \neq \emptyset \\ +\infty & \text{if } E_n = \emptyset; \end{cases}$$

we observe that the sets E_n are not empty for n sufficiently large with $E_n \subset E_{n+1}$ and $E_0 = \cup_n E_n$. Moreover, the sequence $\{\beta_n\}$ is decreasing and, by the compactness of E_n, it fulfills: $\beta_n > \alpha$ and $\beta_n \to \alpha$ as $n \to +\infty$. Hence, there exists an increasing sequence $\{n_m\}$ such that β_{n_m} is decreasing and $E_{n_{m+1}} \setminus E_{n_m} \neq \emptyset$. We set

$$\theta(s) := \begin{cases} \phi(x,t) & \text{for } s \in (-\infty, \alpha] \\ \phi(x,t) + 1/n_{m-1} & \text{for } s = \beta_{n_m} \\ \sup_W \phi & \text{for } s \in [\beta_{n_1}, +\infty) \\ \text{linear function} & \text{for } s \in (\beta_{n_m}, \beta_{n_m-1}). \end{cases}$$

We want to prove (21) studying separately the cases when (y,s) belongs to $W \setminus (E_0 \cup \Delta)$, E_{n_1}, $E_{n_{m+1}} \setminus E_{n_m}$ ($m \in \mathbb{N}$) and $W \cap \Delta$. In the first and last cases, inequality (21) follows by the same arguments of point (b). For $(y,s) \in E_{n_1}$ there holds $u(y,s) \geq \beta_{n_1}$ and, by the monotonicity of θ,

$$\theta(v(y,s)) \geq \theta(\beta_{n_1}) = \sup_W \phi \geq \phi(y,s).$$

For $(y,s) \in E_{n_{m+1}} \setminus E_{n_m}$, there holds $v(y,s) \geq \beta_{n_{m+1}}$ and $\phi(y,s) < \phi(x,t) + 1/n_m$; therefore, we infer

$$\theta(v(y,s)) \geq \theta(\beta_{n_{m+1}}) = \phi(x,t) + 1/n_m > \phi(y,s).$$

Whence, inequality (21) is established and statement is completely proved. □

Proof of Theorem 2.3. We first prove that u is continuous. Given $(x_0,t_0) \in \Gamma \times [0,T]$, let $(x_n,t_n) \in \Gamma \times [0,T]$ be such that $\lim_{n\to\infty}(x_n,t_n) = (x_0,t_0)$ and set $\delta_n = |t_n - t_0| + C^{-1}d(x_n,x_0)$ where C is as in (13). We claim that

$$\{y \in \Gamma : S(y,x_0) \leq t_0\} \subset \{y \in \Gamma : S(y,x_n) \leq t_n + \delta_n\} \tag{22}$$

Indeed, if $S(y,x_0) \leq t_0$, then, by (13) and by subadditivity of S (see Lemma 2.2-(ii)), we get

$$S(y,x_n) \leq S(y,x_0) + S(x_0,x_n) \leq t_0 + C^{-1}d(x_0,x_n) \leq t_n + \delta_n$$

and therefore (22). Moreover, we claim

$$\{y\in\Gamma:\, S(y,x_n)\le t_n+\delta_n\}\subset\big\{y\in\Gamma:\, d\,(y,\{z:\, S(z,x_n)\le t_n\})\le\delta_n/C\big\}. \quad (23)$$

Let us prove this relation; clearly, for y with $S(y,x_n)\le t_n$ there is nothing to prove. Consider $y\in\Gamma$ with $S(y,x_n)\in(t_n,t_n+\delta_n]$. By definition of S, there exists a sequence of curves $\xi_m\in B^1_{y,x_n}$ such that $\int_0^1 s(\xi_m(r),\dot\xi_m(r))dr\le t_n+\delta_n+1/m$ (wlog we may assume $t=1$ because the integral is invariant by a rescaling of the parametrization). We denote τ_m the lowest value of $(0,1)$ such that $\int_{\tau_m}^1 s(\xi_m(r),\dot\xi_m(r))dr=t_n$. Since ξ_m restricted to $[\tau_m,1]$ is an admissible path joining $\xi_m(\tau_m)$ to x_n, we get $S(\xi_m(\tau_m),x_n)\le t_n$. Hence, $\xi_m(\tau_m)\in\{z:\, S(z,x_n)\le t_n\}$. Similarly, ξ_m restricted to $[0,\tau_m]$ is an admissible path joining y to $\xi_m(\tau_m)$; hence, $S(y,\xi_m(\tau_m))\le\delta_n+1/m$ and, by (13), $d(y,\xi_m(\tau_m))\le\frac{\delta_n+1/m}{C}$. Letting $m\to+\infty$, we get our claim (23). Therefore, by (22) and (23), we deduce

$$\begin{aligned}u(x_0,t_0)&\ge\min\{u_0(y):\, S(y,x_n)\le t_n+\delta_n\}\\&\ge\min\{u_0(y):\, d\,(y,\{z:\, S(z,x_n)\le t_n\})\le\delta_n/C\}\\&\ge u(x_n,t_n)-\omega(\delta_n/C)\end{aligned}$$

where ω is the modulus of continuity for u_0 in a neighborhood of x_0. This gives

$$u(x_0,t_0)\ge\limsup_{(x_n,t_n)\to(x_0,t_0)}u(x_n,t_n).$$

By $\{y\in\Gamma:\, S(y,x_n)\le t_n\}\subset\{y\in\Gamma:\, S(y,x_0)\le t_0+\delta_n\}$ we get in a similar way

$$u(x_0,t_0)\le\liminf_{(x_n,t_n)\to(x_0,t_0)}u(x_n,t_n).$$

We now prove that u is a solution of (4). We only prove that u is a supersolution at (x_0,t_0) with $x_0=x_i\in V$, since the other cases can be proved as in the Euclidean case. Assume by contradiction that there is an admissible test function ϕ such that $u-\phi$ has a local minimum at (x_0,t_0) with $\phi(x_0,t_0)=u(x_0,t_0)=\alpha$ and such that

$$\max_{j\in Inc_i}\{\partial_t\phi_j(x_0,t_0)+H_j(x_0,D_j\phi(x_0,t_0))\}\le-\delta<0. \quad (24)$$

Observe that

$$\{(x,t):\, u(x,t)\le\alpha\}=\{(x,t):S(\{u_0\le\alpha\},x)\le t\}. \quad (25)$$

Assume first that $S(\{u_0\le\alpha\},x_0)>0$ and define $w(x,t)=S(\{u_0\le\alpha\},x)-t$. We claim that ϕ has a local maximum on the set $\{w\le 0\}$ at (x_0,t_0). In fact if $(x,t)\in\{w\le 0\}$ then by (25), $u(x,t)\le\alpha$ and since

$$0=u(x_0,t_0)-\phi(x_0,t_0)\le u(x,t)-\phi(x,t) \quad (26)$$

we get $\phi(x,t)\le u(x,t)\le\alpha\le\phi(x_0,t_0)$ and the claim is proved. By Proposition 3 and Lemma 2.4 w is supersolution to (4) at x_0 and therefore Lemma 2.5 gives a contradiction to (24).

If $S(\{u_0\le\alpha\},x_0)=0$, we claim that (x_0,t_0) is a local maximum point for u. In fact, $S(\{u_0\le\alpha\},x_0)=0\le t_0-\eta$ for some $\eta>0$. If (x,t) is such that $\max\{S(x,x_0),|t-t_0|\}\le\delta/2$ with $\delta<\eta$, then

$$S(\{u_0\le\alpha\},x)\le S(\{u_0\le\alpha\},x_0)+S(x_0,x)\le t_0-\eta+\delta/2\le t$$

hence $u(x,t)\le\alpha=u(x_0,t_0)$ and the claim is proved. By (26) (x_0,t_0) is also a local maximum point for ϕ. By (9) and (24), we get $\phi_t(x_0,t_0)<0$ and therefore a contradiction to (x_0,t_0) being a local maximum point for ϕ. ∎

3. **An application: the blocking problem.** In this section we provide a concrete application of our results: now, the network Γ represents an oil pipeline (a network of computers, the circulatory system, etc.) and at initial time a fire breaks up in the region $R_0 \subset \Gamma$ (a virus is detected in a subnet, an embolus occurs in some vessel). The speed of propagation of the fire is known but it may depend on the state variable (and, in particular, on the edge of the network). Our aim is to determine an optimal strategy to stop the fire and to minimize the burnt region.

As in the flame propagation model described in [3], let R_0 be the initial burnt region and R_t the region burnt at time t. Assume that the front ∂R_t propagates in the outward normal direction to the front itself. Then R_t is given by the 0-sublevel set of a viscosity solution of (4)-(5) where the initial datum u_0 satisfies $R_0 = \{x \in \Gamma : u_0(x) \le 0\}$.

Recalling the representation formula (18) we observe that the 0-sublevel set of the solution of (4)-(5) is given by

$$R_t = \{x \in \Gamma : \, S(R_0, x) \le t\}$$

where S is defined as (12). Note that, since Γ is composed by a finite number of bounded edges and therefore its total length is finite, then the burnt region

$$R = \{x \in \Gamma : \, S(R_0, x) < \infty\} = \cup_{t \ge 0} R_t$$

coincides with Γ. In other words, without any external intervention, the pipeline will be completely burnt in a finite time.

We assume that an operator, located at $x_0 \in V$ (the "operation center"), can block the fire by closing the junctions of the pipeline (i.e., vertices of the network) and that this operation is effective only after a delay which depends on the distance of the junction from x_0.

Our problem is reminiscent of other models described in literature (for instance, see [11, 21] and references therein) which concern the control of some diffusion in a network (e.g. minimizing the spread of a virus or maximizing the spread of an information). In this framework, let us stress the main novelties of our setting: in our model, the diffusion has positive finite speed and it affects both vertices and edges, the spread is not reversible (namely, "infected" points cannot become again "healthy") and the effect of the operator's action has finite speed (in other words, it is effective after a delay depending on the distance from the operation center).

Definition 3.1. An admissible strategy σ is a subset of V such that

$$S(R_0, x_i) \ge \delta d(x_0, x_i) \qquad \forall x_i \in \sigma \tag{27}$$

where δ is a given nonnegative constant. We denote by V_{ad} the set of the vertices which satisfy the admissibility condition (27) and by Σ_{ad} the set of the admissible strategies.

Remark 4. Condition (27) means that the time to reach the vertex $x_i \in \sigma$ from x_0 at the velocity $1/\delta$ is less than or equal to the time the fire front reaches x_i and therefore the junction x_i can be blocked before the front goes through it. Hence an admissible strategy is a subset of the set V_{ad} of the vertices that the operator can reach before the fire.

Given a strategy $\sigma \in \Sigma_{ad}$, we denote $S^\sigma : \Gamma \times \Gamma \to [0, \infty]$ the distance restricted to the trajectories not going through a vertex in σ, i.e.

$$S^\sigma(y, x) := \inf \left\{ \int_0^t s(\xi(r), \dot{\xi}(r)) dr \, : \, t > 0, \, \xi \in B^t_{y,x} \text{ s.t. } \xi(r) \notin \sigma \ \forall r \in [0, t] \right\}$$

with $S^\sigma(y,x)=\infty$ if there is no admissible curve joining y to x. We also set

$$R_t^\sigma := \{x\in\Gamma:\ S^\sigma(R_0,x)\le t\}$$
$$R^\sigma := \cup_{t\ge 0}R_t^\sigma = \{x\in\Gamma:\ S^\sigma(R_0,x)<\infty\}$$

which are respectively the region burnt at time t and the total burnt region using the strategy σ. Observe that

- if δ is very small, then the optimal strategy is given by the endpoints of the edges containing R_0;
- if δ is very large and $x_0\in R_0$, then every strategy is useless since the whole pipeline will burn whatever the operator does.

Aside the previous simple cases an optimal strategy for the blocking problem may be not obvious and we aim to find an efficient way to compute it. To find a strategy which minimizes the burnt region, we first give a characterization of R^σ in terms of a problem satisfied by the distance $S^\sigma(R_0,\cdot)$.

Proposition 4. *Given $\sigma\in\Sigma_{ad}$, set $u(x)=S^\sigma(R_0,x)$ and $\mathcal{R}=R^\sigma$. Then*

i) *$u\in C^0(\mathcal{R})$ and $u=+\infty$ in $\Gamma\setminus\mathcal{R}$. Moreover if $x_i\in\sigma$ and $j\in Inc_i$ is such that $e_j\subset\mathcal{R}$, then $\lim_{x\to x_i,\,x\in e_j}u(x)=u_j(x_i)<\infty$;*

ii) *u is a viscosity solution of the problem*

$$\begin{cases} H(x,Du)=1, & x\in\mathcal{R}\setminus(\ R_0\cup\sigma)\\ u=0, & x\in R_0;\end{cases}\tag{28}$$

iii) *let $\tilde{\mathcal{R}}$ be an open set containing R_0 and $w\in C(\tilde{\mathcal{R}})$ such that*

$$\begin{cases} H(x,Dw)\le 1, & x\in\tilde{\mathcal{R}}\setminus R_0\\ w=0, & x\in R_0,\end{cases}$$

then $\mathcal{R}\subset\tilde{\mathcal{R}}$ and $w\le u$ in Γ.

Proof. Note that, for $j\in J$, either $e_j\subset\mathcal{R}$ or $e_j\cap\mathcal{R}=\emptyset$, i.e. an edge is either completely burnt or it cannot be reached by the fire. The function u can be discontinuous at $x_i\in\sigma$ and

- if $x_i\in V\setminus\sigma$, then either $u_j(x_i)=\infty$ for all $j\in Inc_i$ if $x_i\in\Gamma\setminus\mathcal{R}$ or $u_j(x_i)<\infty$ for all $j\in Inc_i$ if $x_i\in\mathcal{R}$;
- if $x_i\in\sigma$, then either $u_j(x_i)=\infty$ for all $j\in Inc_i$ if $x_i\in\Gamma\setminus\mathcal{R}$ or there exists $j\in Inc_i$ such that $u_j(x_i)<\infty$ if $x_i\in\mathcal{R}$ and in this case $u_j(x_i)=\sup_{e_j}u_j$.

Actually, if $x_i\in\sigma$ and $u_j(x_i)<\infty$, an admissible trajectory for S^σ connecting x_i to R_0 and containing the edge e_j, $j\in Inc_i$, necessarily enters from x_i into e_j. Hence $u(x)$ is increasing for $x\in e_j$, $x\to x_i$ and $\lim_{x\in e_j,\,x\to x_i}u_j(x)=u_j(x_i)$.

In $\mathcal{R}\setminus\sigma$, S^σ locally behaves as the distance S defined in (12). Therefore the continuity of u in $\mathcal{R}\setminus\sigma$ and the sub- and supersolution properties in the open set $\mathcal{R}\setminus(\ R_0\cup\sigma)$ are obtained by the same arguments of [17, Prop.4.1].

To prove *iii)*, assume by contradiction that there exists $x_0\in\mathcal{R}$ such that $u(x_0)<w(x_0)$. For any $x,y\in\mathcal{R}$ such that $S^\sigma(x,y)<\infty$, a minimizing trajectory always exists since (up to reparametrization) there is only a finite number of trajectories connecting the two points.

Hence let ξ be an admissible curve for S^σ such that $\xi(0)=y_0\in R_0$, $\xi(T)=x_0$ and $u(x_0)=\int_0^T s(\xi(r),\dot\xi(r))ds$. Let $t_0=0<t_1<\dots<t_{M+1}=T$ such that, for any $m=0,\dots,M$, $\xi([t_m,t_{m+1}])\subset e_{j_m}$ for some $j_m\in J$, $\xi(t_{i_m})=x_{i_m}\in V$ and $\pi_{j_m}^{-1}\circ\xi\in C^1(t_m,t_{m+1})$. Clearly $u(\xi(t))=\int_0^t s(\xi(r),\dot\xi(r))dr$ for $t\in[0,T]$.

If w is a subsolution to (28), then by the coercivity of H, w is Lipschitz continuous in $\mathcal{R}_w \setminus R_0$ and therefore $H(x, Dw) \leq 1$ a.e. on $\mathcal{R}_w \setminus R_0$. Moreover, by the definition of the support function s, we have $H(x, p) \leq 1$ if and only if $\sup_{q\in\mathbb{R}}\{pq - s(x, q)\} \leq 0$. Hence

$$\dot{\xi}(r)Dw(\xi(r)) \leq s(\xi(r), \dot{\xi}(r)) \qquad \forall r \in [t_m, t_{m+1}],\ m = 0, \dots, M.$$

Integrating the previous relation in $[0, t_1]$ and recalling that $u(y_0) = w(y_0) = 0$, we get

$$w(\xi(t_1)) \leq \int_0^{t_1} s(\xi(r), \dot{\xi}(r))dr = u(\xi(t_1)).$$

Iterating the same argument in $[t_m, t_{m+1}]$ we finally get $w(\xi(T)) \leq u(\xi(T))$ and therefore a contradiction since $\xi(T) = x_0$. We conclude that $x_0 \in \mathcal{R}_w$ and $w(x_0) \leq u(x_0)$. □

We now show that the strategy composed by all the admissible nodes which are adjacent to a non admissible node is optimal, in the sense that it maximizes the preserved region.

Proposition 5. *The admissible strategy*

$$\sigma_{opt} = \{x_i \in V_{ad} : \exists x_j \in V \setminus V_{ad},\ e_k \in E\, s.t.\, x_i,\ x_j \in \overline{e}_k\}$$

satisfies: for any $\sigma \in \Sigma_{ad}$, $R^{\sigma_{opt}} \subset R^\sigma$.

Proof. Assume by contradiction that there exist $\sigma \in \Sigma_{ad}$ and $x_0 \in R^{\sigma_{opt}} \setminus R^\sigma$. Hence there exists an admissible trajectory ξ for σ_{opt} connecting x_0 to R_0, i.e. there exists $y_0 \in R_0$, $t > 0$ and $\xi \in B^t_{y_0,x_0}$ such that $\xi(r) \notin \sigma_{opt}$ for $r \in [0, t]$. Note that σ_{opt} disconnects the subgraph containing the admissible vertices V_{ad} by the one containing the non admissible vertices $V \setminus V_{ad}$ and therefore $\xi([0, t])$ is contained in the subgraph with vertices $(V \setminus V_{ad}) \cup \sigma_{opt}$. Since $\sigma \subset V_{ad}$, then ξ is also admissible for $S^\sigma(y_0, x_0)$ and therefore a contradiction to $x_0 \notin R^\sigma$. □

Remark 5. It is possible to consider a cost functional on the set of the admissible strategies Γ_{ad} which takes into account not only the part of the network destroyed by the fire but also other terms, such as the cost of blocking a given junction. Consider the cost functional $\mathcal{I} : \Gamma_{ad} \to \mathbb{R}$ given by

$$\mathcal{I}(\sigma) = \sum_{x_i\in\sigma} \alpha_i + \sum_{e_j\subset R^\sigma} \beta_j$$

(recall that either $e_j \subset R^\sigma$ or $e_j \cap R^\sigma = \emptyset$). The first term represent the cost of blocking the node x_i and can depend on various parameters (distance of the node from x_0, accessibility of x_i, cost of blocking x_i, etc.) while the second term is the cost of the burnt region with a given cost β_j for each arc. Clearly, the minimum of $\mathcal{I}(\cdot)$ exists since Σ_{ad} is finite, but it seems more difficult to characterize the optimal strategy.

3.1. **Numerical simulations.** In this section we propose a numerical method to compute the optimal strategy for the blocking problem. The scheme is based on a finite difference approximation of the stationary problem (28); for simplicity, we only consider the case of the eikonal Hamiltonian $H(x, p) = |p|/c(x)$.

On each interval $[0, l_j]$ parametrizing the arc e_j, we consider an uniform partition $y_{j,m} = mh_j$ with $M_j = l_j/h_j \in \mathbb{N}$ and $m = 0, \dots, M_j$. In this way we obtain a grid $\mathcal{G}^h = \{x_{j,m} = \pi_j(y_{j,m}),\ j \in J,\ m = 0, \dots, M_j\}$ on the network Γ. We define

$\mathcal{R}_0^h = \mathcal{G}^h \cap R_0$, the set of the nodes in the initial front. For $x_1, x_2 \in \mathcal{G}^h$, we say that x_1 and x_2 are adjacent and we write $x_1 \sim x_2$ if and only if they are the image of two adjacent grid points, i.e. $x_k = \pi_j(y_k)$, for $y_k \in [0, l_j]$, $k = 1, 2$, $j \in J$ and $|y_1 - y_2| = h_j$. Note that if $x_i \in V$ is a vertex of Γ, then the nodes of the grid $\mathcal{G}^h$ adjacent to x_i may belong to different arcs.
We compute the optimal strategy by means of the following Algorithm, based on the results of Prop. 5.

Blocking strategy [B]

1. In the first step we solve the front propagation problem on the network computing the approximated time $u^h(x)$ at which a node $x_{j,m} \in \mathcal{G}^h$ gets burnt

$$\begin{cases} \max_{x\in\mathcal{G}^h,\, x\sim x_{j,m}} \left\{-\frac{1}{h_j}(u^h(x) - u^h(x_{j,m}))\right\} - c(x_{j,m}) = 0 & x_{j,m} \in \mathcal{G}^h \\ u^h(x_{j,m}) = 0 & x_{j,m} \in \mathcal{R}_0^h. \end{cases} \tag{29}$$

   Note that if $x_{j,m}$ coincides with a vertex $x_i \in V$, the approximating equation reads as

$$\max_{j\in Inc_i} \max_{x\in\mathcal{G}^h\cap e_j,\, x\sim x_{j,m}} \left\{-\frac{1}{h_j}(u^h(x) - u^h(x_{j,m}))\right\} - c(x_{j,m}) = 0.$$

   The discrete function $u^h : \mathcal{G}^h \to \mathbb{R}$ is such that $u^h(x_{j,m}) \simeq u(x_{j,m})$, where $u(x) = S(R_0, x)$.
2. In the second step we determine the vertices which satisfy the admissibility condition (27). We define $V_{ad}^h = \{x_i \in V : w^h(x_i) < u^h(x_i)\}$, where $w^h : \mathcal{G}^h \to \mathbb{R}$ represents the approximated time to reach a node $x \in \mathcal{G}^h$, starting from the operation center x_0 and moving with a constant speed $1/\delta$. The function w^h is computed by means of the finite difference scheme

$$\begin{cases} \max_{x\in\mathcal{G}^h,\, x\sim x_{j,m}} \left\{-\frac{1}{h_j}(w^h(x) - w^h(x_{j,m}))\right\} - \frac{1}{\delta} = 0 & x_{j,m} \in \mathcal{G}^h \setminus \{x_0\} \\ w^h(x_0) = 0. \end{cases} \tag{30}$$

3. We define the approximated optimal strategy by setting

$$\sigma_{opt}^h = \{x_i \in V_{ad}^h : \exists x_j \in V \setminus V_{ad}^h,\, e_k \in E \,\text{s.t.}\, x_i,\, x_j \in e_k\}$$

   and, for $\sigma = \sigma_{opt}^h$, we compute the corresponding approximate distance by solving the following finite difference scheme

$$\begin{cases} \max\limits_{x\in\mathcal{G}^h,\, x\sim x_{j,m}} \left\{-\frac{1}{h_j}(u_\sigma^h(x) - u_\sigma^h(x_{j,m}))\right\} - c(x_{j,m}) = 0 & x_{j,m} \in \mathcal{G}^h \setminus (\mathcal{R}_0^h \cup \sigma) \\ u_\sigma^h(x_{j,m}) = 0 & x_{j,m} \in \mathcal{R}_0^h \\ -\frac{1}{h_j}(u_{j,\sigma}^h(x) - u_{j,\sigma}^h(x_{j,m})) - c(x_{j,m}) = 0 & x_{j,m} \in \sigma,\, x \in \mathcal{G}^h,\, x \sim x_{j,m}. \end{cases}$$

   The discrete function $u^h : \mathcal{G}^h \to \mathbb{R}$ is such that $u_\sigma^h(x_{j,m}) \simeq u(x_{j,m})$, where $u(x) = S^\sigma(R_0, x)$ solves (28). Note that as in the continuous case, the value of u^h at $x_i \in \sigma$ can depend on the edge e_j and in general the function is discontinuous at these points.

Remark 6. In this paper we do not analyze the properties of the previous finite difference schemes. In any case, at least for (29) and (30), the well-posedness and the convergence of the schemes can be studied by adapting the techniques in [7].

3.2. **Example 1: a simple network.** We consider a network with a simple structure where the fire starts in one vertex, $R_0 = \{(0,0)\}$ and propagates with speed $c = 1$.

We first perform step $i)$ of Algorithm [**B**] and we compute the approximated time $u^h(x)$ at which a node $x \in \mathcal{G}^h$ get burnt. The results are shown in Fig.1 together with the graph structure.

Next, we perform step $ii)$ of Algorithm [**B**] . We suppose the operation center x_0

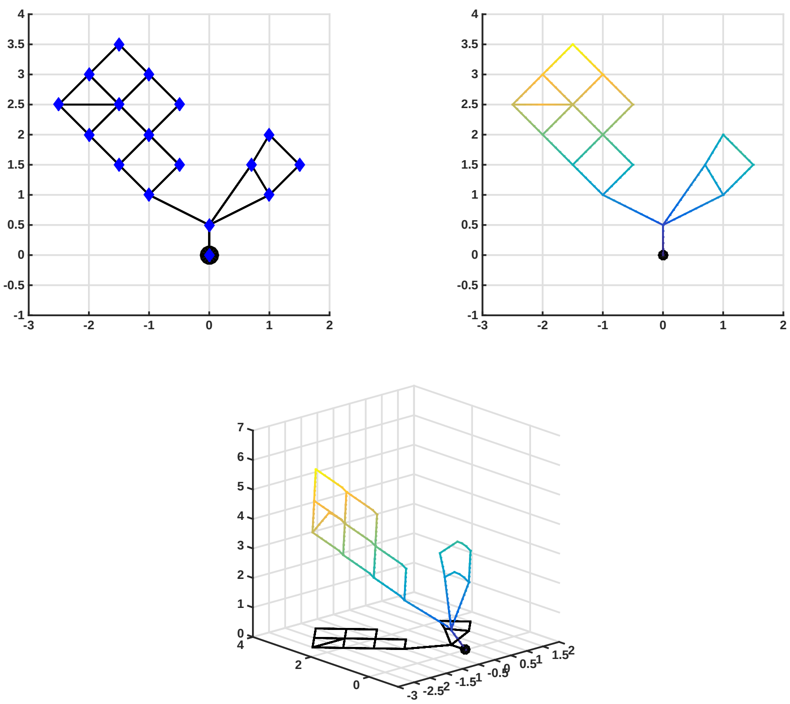

FIGURE 1. Test1. Graph structure where R_0 is represented by the circle marker and the vertices by the rhombus markers (Top Left). Color map of the time $u_h(x)$ at which a node x get burnt, computed by (29), (Top Right) and its 3D view (Bottom).

is located on the vertex $(-1.5, 2.5)$ and the velocity to reach a node x_i from x_0 is $\frac{1}{\delta} = 1$. Using (30), we compute the set of nodes V^h_{ad}. The result is shown in Fig.2, the set of nodes in V^h_{ad} are represented by the square markers.

Once computed the set of admissible nodes V^h_{ad}, we can compute the optimal strategy, following step $iii)$. The result is shown in Fig.3. It is clear, from the simple structure of the network, that any other choice of σ^h_{opt} would lead to a greater burnt region and consequently to a smaller preserved network region.

3.3. **Example 2: a more complex network.** We consider a more complex network, with 20 vertices and 32 arcs. We suppose the fire starts in two vertices and propagates with a not constant normal speed $c(x) = |x|$.

We proceed as in the first example and we compute the approximated time $u^h(x)$

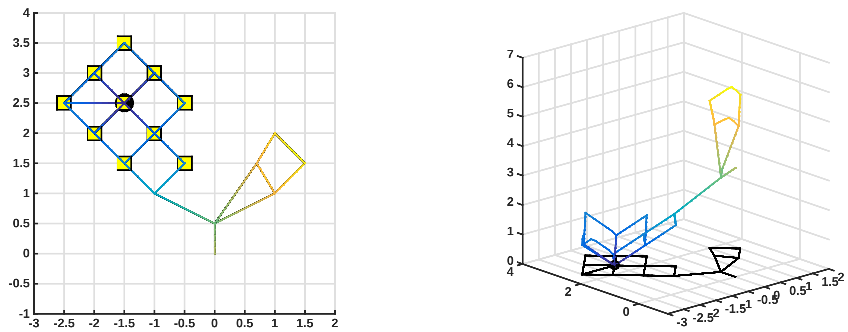

FIGURE 2. Test1. Time to reach a point x from the operation center x_0 (circle marker) and set of the admissible nodes V_{ad}^h (square marker). 2D view (Left) and 3D view (Right).

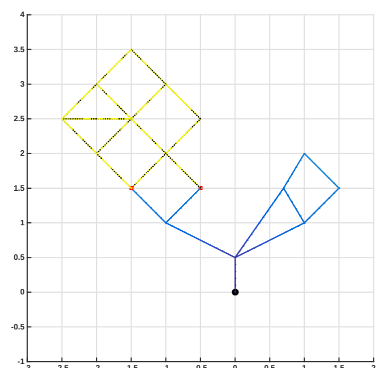

FIGURE 3. Test1. Optimal blocking strategy σ_{opt}^h (square marker), preserved network region (cross marker) and minimum burnt network region (continuum line) starting from R_0 (circle marker).

at which a node $x \in \mathcal{G}^h$ get burnt. The results are shown in Fig.4 together with the graph structure.

We suppose the operation center x_0 is located on the vertex $x_0 = (3.8, 6.5)$ and the velocity to reach a node x_i from x_0 is $\frac{1}{\delta} = 1/5$. Using (30), we compute the set of nodes V_{ad}^h. The result is shown in Fig. 5, the set of nodes in V_{ad}^h are represented by the square markers. Once computed the set of admissible nodes V_{ad}^h, we can compute the optimal strategy, following step iii). The result is shown in Fig.6. In this case we get the optimal solution blocking only three vertices. By changing the set R_0 as in Fig.7, the region of the admissible node V_{ad}^h, shown in Fig. 8, turns out to be much smaller. In this case the optimal strategy is formed by all the vertices in V_{ad}^h, and the preserved region becomes smaller than the previous case, see Fig. 9.

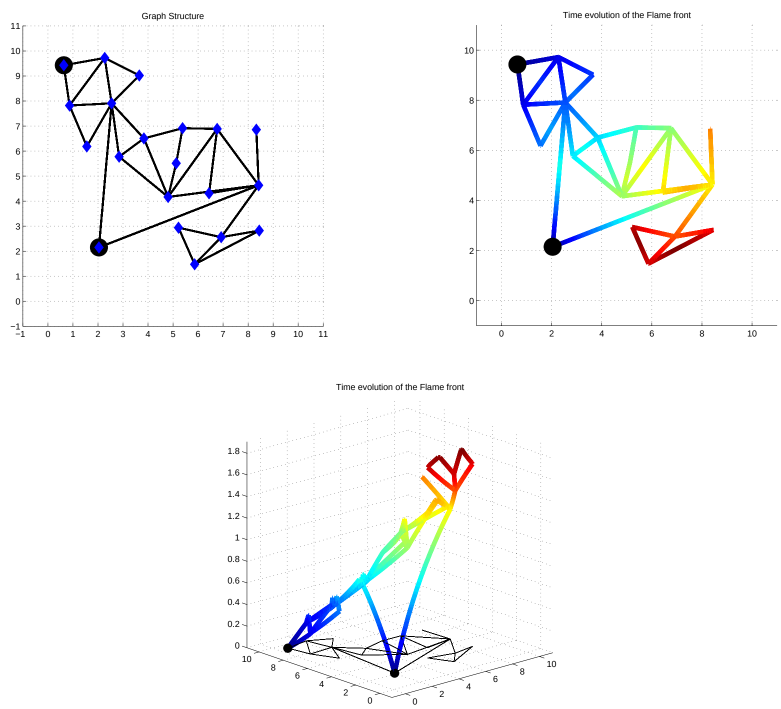

FIGURE 4. Test2. Graph structure where R_0 is represented by the circle markers and the vertices by the rhombus markers (Top Left). Color map of the time $u_h(x)$ at which a node x get burnt, computed by (29) (Top Right), and its 3D view (Bottom).

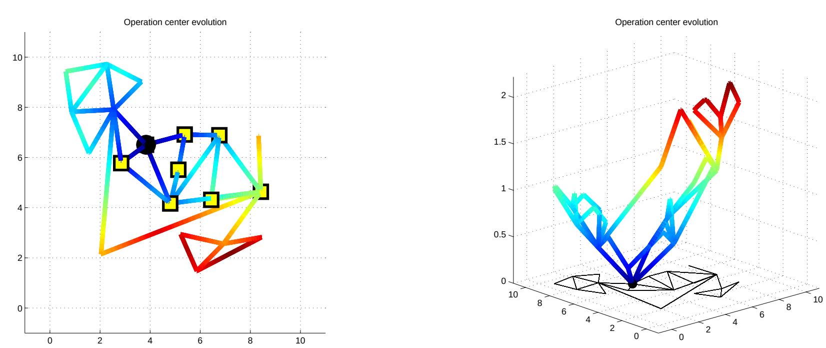

FIGURE 5. Test2. Time to reach a point x from the operation center x_0 (circle marker) and set of the admissible nodes V_{ad}^h (square markers). 2D view(Left) and 3D view (Right).

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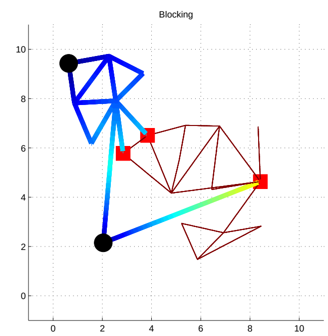

FIGURE 6. Test2. Optimal blocking strategy σ^h_{opt} (square markers), preserved network region (thin line) and minimum burnt network region (thick line) starting from R_0 (circle markers).

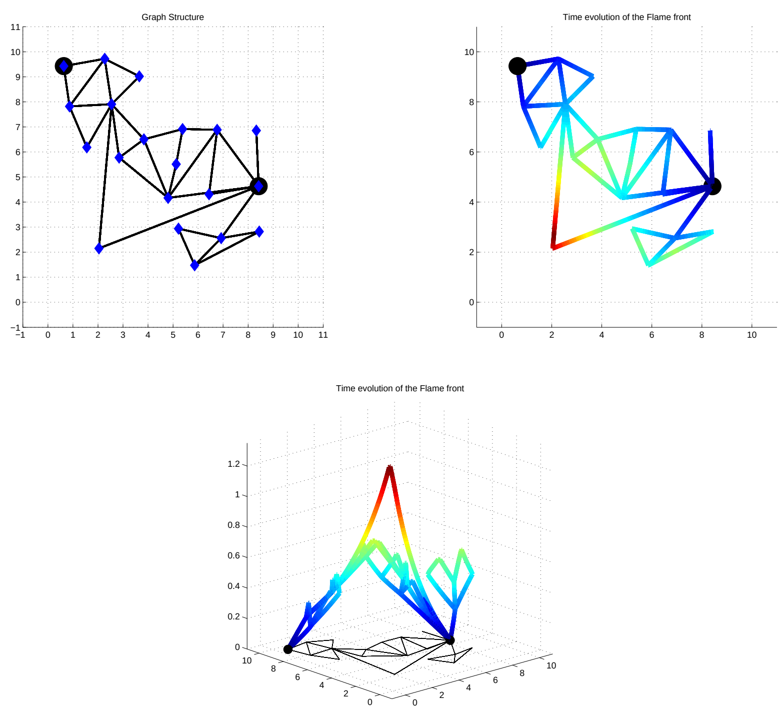

FIGURE 7. Test3. Graph structure where R_0 is represented by the circle markers and the vertices by the rhombus markers (Top Left). Color map of the time $u_h(x)$ at which a node x get burnt, computed by (29), (Top Right) and its 3D view (Bottom).

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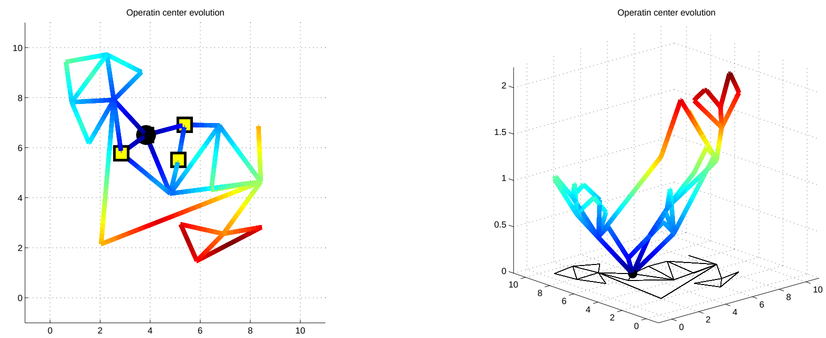

FIGURE 8. Test3. Time to reach a point x from the operation center x_0 (circle marker) and set of the admissible nodes V^h_{ad} (square markers). 2D view(Left) and 3D view (Right).

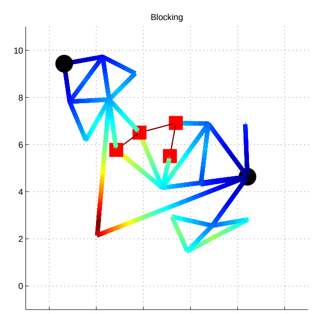

FIGURE 9. Test3. Optimal blocking strategy σ^h_{opt} (square marker), preserved network region (thin line) and minimum burnt network region (thick line) starting from R_0 (circle marker).

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