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Algebras and Representation Theory

ISSN 1386-923X
Volume 15
Number 5

Algebr Represent Theor (2012)
15:977-1021
DOI 10.1007/s10468-011-9275-5

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Cluster Algebras of Type $A_2^{(1)}$

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Received: 19 June 2009 / Accepted: 14 January 2011 / Published online: 24 February 2011
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Abstract In this paper we study cluster algebras $\mathcal{A}$ of type $A_2^{(1)}$. We solve the recurrence relations among the cluster variables (which form a T-system of type $A_2^{(1)}$). We solve the recurrence relations among the coefficients of $\mathcal{A}$ (which form a Y-system of type $A_2^{(1)}$). In $\mathcal{A}$ there is a natural notion of positivity. We find linear bases $\mathbf{B}$ of $\mathcal{A}$ such that positive linear combinations of elements of $\mathbf{B}$ coincide with the cone of positive elements. We call these bases *atomic bases* of $\mathcal{A}$. These are the analogue of the "canonical bases" found by Sherman and Zelevinsky in type $A_1^{(1)}$. Every atomic basis consists of cluster monomials together with extra elements. We provide explicit expressions for the elements of such bases in every cluster. We prove that the elements of $\mathbf{B}$ are parameterized by $\mathbb{Z}^3$ via their $\mathbf{g}$-vectors in every cluster. We prove that the denominator vector map in every acyclic seed of $\mathcal{A}$ restricts to a bijection between $\mathbf{B}$ and $\mathbb{Z}^3$. We find explicit recurrence relations to express every element of $\mathcal{A}$ as linear combinations of elements of $\mathbf{B}$.

Keywords Cluster algebras · T and Y systems · Positivity

Mathematics Subject Classifications (2010) 13F60

1 Introduction

Cluster algebras were introduced in [16] by Fomin and Zelevinsky in order to define a combinatorial framework for studying canonical bases in quantum groups. In this perspective, it is an important problem, still widely open, to construct explicitly linear

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bases of cluster algebras. Among possible bases of a cluster algebra $\mathcal{A}$ (see [4, 5, 13, 14, 20]), there may exists a specific one of particular interest, that we call *atomic basis*. In previous versions of this paper and in the papers referring to it [7, 8, 13] the atomic bases were called canonically positive. We prefer to change the name after private conversations with Andrei Zelevinsky and other experts. It is not known in general if such a basis exists. For cluster algebras of type A_2 and $A_1^{(1)}$, Sherman and Zelevinsky were able to construct atomic bases in [30]. In a forthcoming paper the author will show that cluster monomials form an atomic of cluster algebras of finite type. By now these are the only known examples.

In this paper we prove the existence of atomic bases $\mathbf{B}$ for every cluster algebra of type $A_2^{(1)}$. We provide a completely explicit description of the elements of $\mathbf{B}$ which consists of cluster monomials together with extra elements. We find explicit straightening relations which provide a recursive way to compute the linear expansion in $\mathbf{B}$ of every element of $\mathcal{A}$. Our proof of the existence of an atomic basis $\mathbf{B}$ of $\mathcal{A}$ follows [30] but with an important technical difference: we prove that the powerful $\mathbf{g}$-vector parametrization of cluster monomials found in [17] extends to the elements of $\mathbf{B}$. In particular we work directly in the cluster algebra with coefficients in an arbitrary semifield. This allows us to provide explicit formulas for the elements of $\mathbf{B}$. Moreover we think that this approach can help in the study of the existence of atomic bases of cluster algebras of higher rank in view of results of [18] and [11] and of the geometric realization of $\mathbf{B}$ given in [8] and [13].

The paper is organized as follows: in Section 2 we collect the main results and the remaining sections are devoted to proofs.

2 Main Results

A *semifield* $\mathbb{P} = (\mathbb{P}, \cdot, \oplus)$ is an abelian multiplicative group $(\mathbb{P}, \cdot)$ endowed with an auxiliary addition $\oplus : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ which is associative, commutative and $a(b \oplus c) = ab \oplus ac$ for every $a, b, c \in \mathbb{P}$.

The main example of a semifield is a *tropical semifield*: by definition a tropical semifield $Trop(y_j : j \in J)$ is an abelian multiplicative group freely generated by the elements $\{y_j : j \in J\}$ (for some set of indexes J) endowed with the auxiliary addition $\oplus$ given by:

$$\prod_j y_j^{a_j} \oplus \prod_j y_j^{b_j} := \prod_j y_j^{\min(a_j, b_j)}.$$

It can be shown (see [16, Section 5]) that every semifield $\mathbb{P}$ is torsion-free as a multiplicative group and hence its group ring $\mathbb{Z}\mathbb{P}$ is a domain. Given a semifield $\mathbb{P}$, let $\mathbb{Q}\mathbb{P}$ be the field of fractions of $\mathbb{Z}\mathbb{P}$ and $\mathcal{F}_{\mathbb{P}} = \mathbb{Q}\mathbb{P}(x_1, x_2, x_3)$ be the field of rational functions in three independent variables with coefficients in $\mathbb{Q}\mathbb{P}$. We consider the $\mathbb{Z}\mathbb{P}$-subalgebra $\mathcal{A}_{\mathbb{P}}$ of $\mathcal{F}\mathbb{P}$ defined as follows: we choose arbitrarily three elements y_1, y_2 and y_3 of $\mathbb{P}$. We consider the family $\{y_{1;m} : m \in \mathbb{Z}\} \subset \mathbb{P}$ of coefficients defined by the initial conditions:

$$y_{1;0} = \frac{1}{y_3}, \quad y_{1;1} = y_1, \quad y_{1;2} = \frac{y_1 y_2}{y_1 \oplus 1}$$

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together with the recursive relations:

$$y_{1;m}y_{1;m+3} = \frac{y_{1;m+2}y_{1;m+1}}{(y_{1;m+1} \oplus 1)(y_{1;m+2} \oplus 1)} \tag{1}$$

We consider the following rational functions w, z of $\mathcal{F}_{\mathbb{P}}$:

$$w := \frac{y_2x_1 + x_3}{(y_2 \oplus 1)x_2}, \quad z := \frac{y_1y_3x_1x_2 + y_1 + x_2x_3}{(y_1y_3 \oplus y_1 \oplus 1)x_1x_3} \tag{2}$$

We consider the elements $x_m, m \in \mathbb{Z}$, of $\mathcal{F}_{\mathbb{P}}$ defined recursively by:

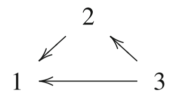
$$x_mx_{m+3} = \frac{x_{m+1}x_{m+2} + y_{1;m}}{y_{1;m} \oplus 1}. \tag{3}$$

By definition $\mathcal{A}_{\mathbb{P}}$ is the $\mathbb{Z}\mathbb{P}$-subalgebra of $\mathcal{F}_{\mathbb{P}}$ generated by the rational functions w, z and $x_m, m \in \mathbb{Z}$.

In Section 3.2 we prove that $\mathcal{A}_{\mathbb{P}}$ is the cluster algebra with initial seed (see Section 3.1 for some background on cluster algebras)

$$\Sigma := \left\{ H = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \mathbf{x} = \{x_1, x_2, x_3\}, \mathbf{y} = \{y_1, y_2, y_3\} \right\}. \tag{4}$$

The rational functions w, z and x_m, $m \in \mathbb{Z}$, are the *cluster variables* of $\mathcal{A}_{\mathbb{P}}$. To the matrix H is naturally associated the quiver

$Q :=$  (5)

whose underlying graph is an extended Dynkin graph of type $A_2^{(1)}$. The algebra $\mathcal{A}_{\mathbb{P}}$ is hence called a cluster algebra of type $A_2^{(1)}$ (see [4]). For $\mathbb{P} = \{1\}$ such cluster algebra appears in [16, Example 7.8].

The family of coefficients $\{y_{1;m}\}$ form a Y-system of type $A_2^{(1)}$ (see [25]). We solve the recursion (Eq. 1) in Eq. 45. The family of cluster variables form a T-system of type $A_2^{(1)}$. We solve the recurrence (Eq. 3) in Theorem 2.3.

The *clusters* of $\mathcal{A}_{\mathbb{P}}$ are the sets $\{x_m, x_{m+1}, x_{m+2}\}$, $\{x_{2m+1}, w, x_{2m+3}\}$ and the set $\{x_{2m}, z, x_{2m+2}\}$ for every $m \in \mathbb{Z}$. Every cluster variable s_1 can be completed to a cluster $\mathcal{C} = \{s_1, s_2, s_3\}$. Every cluster forms a free generating set of the field $\mathcal{F}_{\mathbb{P}}$, so that $\mathcal{F}_{\mathbb{P}} \simeq \mathbb{Q}\mathbb{P}(s_1, s_2, s_3)$. Figure 1 shows (a piece of) the "exchange graph" of $\mathcal{A}_{\mathbb{P}}$. By definition it has clusters as vertexes and an edge between two clusters if they share exactly two cluster variables. In this figure cluster variables are associated with regions: there are infinitely many bounded regions labeled by the x_m's, and there are two unbounded regions labeled respectively by w and z. Each cluster $\{s_1, s_2, s_3\}$ corresponds to the common vertex of the three regions s_1, s_2 and s_3.

We have already observed that $\mathcal{F}_{\mathbb{P}} \simeq \mathbb{Q}\mathbb{P}(s_1, s_2, s_3)$ for every cluster $\{s_1, s_2, s_3\}$ of $\mathcal{A}_{\mathbb{P}}$ and hence every element of $\mathcal{A}_{\mathbb{P}}$ can be expressed as a rational function in $\{s_1, s_2, s_3\}$. By the *Laurent phenomenon* proved in [16], such a rational function is actually a Laurent polynomial. Following [30] we say that an element of $\mathcal{A}_{\mathbb{P}}$ is *positive*

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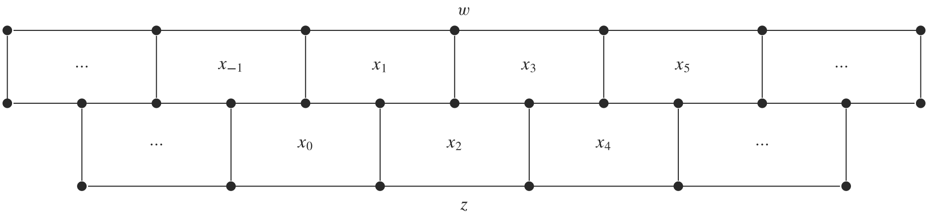

Fig. 1 The exchange graph of $\mathcal{A}_{\mathbb{P}}$

if its (irreducible) Laurent expansion in every cluster has coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$. Positive elements form a semiring, i.e. sums and products of positive elements are positive. We say that a $\mathbb{Z}\mathbb{P}$-basis $\mathbf{B}$ of $\mathcal{A}_{\mathbb{P}}$ is *atomic* if the semiring of positive elements coincides with the $\mathbb{Z}_{\geq 0}\mathbb{P}$-linear combinations of elements of it. If an atomic basis exists, it is composed by positive indecomposable elements, i.e. positive elements that cannot be written as a sum of positive elements. Moreover such a basis is unique up to rescaling by elements of $\mathbb{P}$.

Definition 2.1 Let $W := (y_2 \oplus 1)w$ and $Z := (y_1 y_3 \oplus y_1 \oplus 1)z$. We define elements $\{u_n | n \geq 0\}$ of $\mathcal{A}_{\mathbb{P}}$ by the initial conditions:

$$u_0 = 2,\ u_1 = ZW - y_1 y_3 - y_2 \tag{6}$$

together with the recurrence relation

$$u_{n+1} = u_1 u_n - y_1 y_2 y_3 u_{n-1},\ n \geq 1. \tag{7}$$

Theorem 2.1 *The set*

$$\mathbf{B} := \{\text{cluster monomials}\} \cup \{u_n w^k, u_n z^k | n \geqslant 1,\ k \geqslant 0\} \tag{8}$$

is an atomic basis of $\mathcal{A}_{\mathbb{P}}$. It is unique up to rescaling by elements of $\mathbb{P}$.

Theorem 2.1 shows that the atomic bases of $\mathcal{A}_{\mathbb{P}}$ consist of cluster monomials together with some extra elements. This is precisely the form of other linear bases constructed for cluster algebras of affine type: the generic basis introduced by G. Dupont (see [12, 14, 20]); the semicanonical basis introduced by Caldero and Zelevinsky [5] and the basis obtained by Dupont by using transverse quiver Grassmannians [13].

The proof of Theorem 2.1 is given in Section 6. We now collect some properties of the basis $\mathbf{B}$.

2.1 Parametrization of B by Denominator Vectors

By the already mentioned Laurent phenomenon [16, Theorem 3.1] every element b of $\mathcal{A}_{\mathbb{P}}$ is a Laurent polynomial in $\{x_1, x_2, x_3\}$ of the form $\frac{N_b(x_1,x_2,x_3)}{x_1^{d_1} x_2^{d_2} x_3^{d_3}}$ for some primitive, i.e. not divisible by any x_i, polynomial $N_b \in \mathbb{Z}\mathbb{P}[x_1, x_2, x_3]$ in x_1, x_2 and x_3, and some integers d_1, d_2, d_3. We consider the root lattice Q generated by an affine root system of type $A_2^{(1)}$. The choice of a simple system $\{\alpha_1, \alpha_2, \alpha_3\}$, with coordinates $\{e_1, e_2, e_3\}$, identifies Q with $\mathbb{Z}^3$. We usually write elements of Q as column vectors with integer coefficients and we denote them by a bold type letter.

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The map $b \mapsto \mathbf{d}(b) = (d_1, d_2, d_3)^t$ is hence a map between $\mathcal{A}_{\mathbb{P}}$ and Q; it is called the *denominator vector map in the cluster* $\{x_1, x_2, x_3\}$. The denominator vector of $x_i = 1/(x_i^{-1})$ equals $-\alpha_i$, $i = 1, 2, 3$. The following result provides a parametrization of $\mathbf{B}$ by Q via the denominator vector map. Recall that given $\delta := (1, 1, 1)^t$, the minimal positive imaginary root, and $\Pi^\circ = \{\alpha_1, \alpha_3\}$, a basis of simple roots for a root system Δ° of type A_2, the positive real roots of Q are of the form $\alpha + n\delta$ with $n \geq 0$ if α is a positive root of Δ° and $n \geq 1$ if α is a negative root of Δ° (see e.g. [24, Proposition 6.3]).

Theorem 2.2 *The denominator vector map* $\mathbf{d} : \mathcal{A}_{\mathbb{P}} \to Q : b \mapsto \mathbf{d}(b)$ *in the cluster* $\{x_1, x_2, x_3\}$ *restricts to a bijection between* $\mathbf{B}$ *and* Q. *Under this bijection positive real roots of the root system of type* $A_2^{(1)}$ *correspond to the set of cluster variables different from* x_1, x_2 *and* x_3 *together with the set* $\{u_n w, u_n z | n \geq 1\}$. *Moreover for every cluster* $\mathcal{C} = \{c_1, c_2, c_3\}$, *the set* $\{\mathbf{d}(c_1), \mathbf{d}(c_2), \mathbf{d}(c_3)\}$ *is a* $\mathbb{Z}$*-basis of* Q.

The denominator vector map in the cluster $\{x_1, w, x_3\}$ does not restrict to a bijection between $\mathbf{B}$ and the whole lattice $\mathbb{Z}^3$ (see Remark 3.2).

In Lemma 3.2 we show the denominator vectors of the elements of $\mathbf{B}$ in $\{x_1, x_2, x_3\}$. Figure 2 depicts the qualitative positions of denominator vectors of cluster variables (different from the initial ones) in Q: it shows the intersection between the plane $\mathcal{P} = \{e_1 + e_2 + e_3 = 1\}$ and the positive octant Q_+ of the real vector space $Q_{\mathbb{R}}$ generated by Q; a point labeled by $\mathbf{d}(s)$ denotes the intersection between $\mathcal{P}$ and the

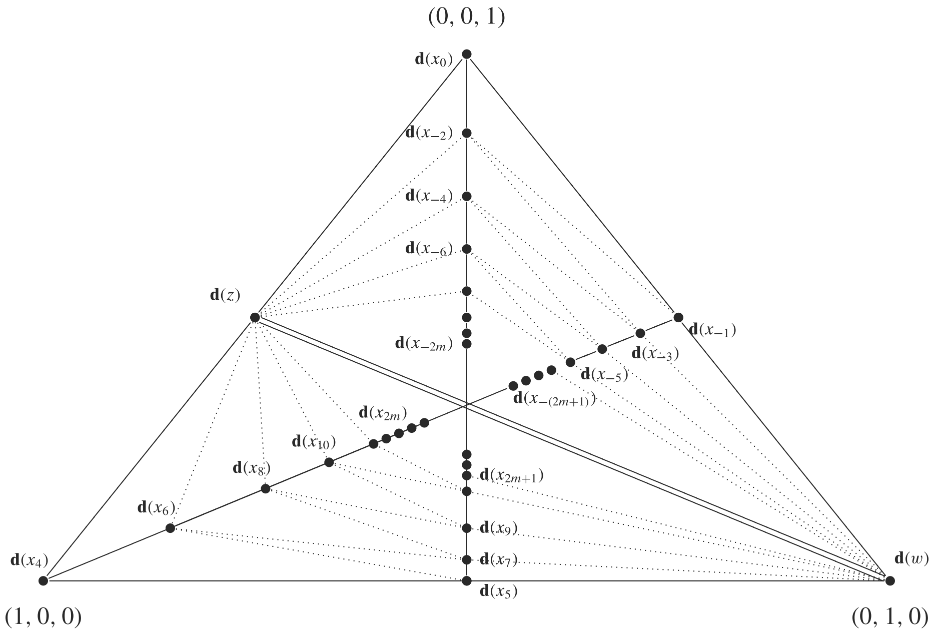

Fig. 2 "Cluster triangulation" of the intersection between the positive octant Q_+ and the plane $\mathcal{P} = \{e_1 + e_2 + e_3 = 1\}$. A *dotted line* joins two points corresponding to cluster variables belonging to the same cluster. The *double line* between w and z denotes the intersection with the "regular" cone which contains the denominator vector of all the elements $u_n w^k$ and $u_n z^k$

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line generated by the denominator vector of the cluster variable s. A dotted line joins two cluster variables that belong to the same cluster. In view of Theorem 2.2, these lines form mutually disjoint triangles. We notice that some of the points of $\mathcal{P} \cap Q_+$ do not lie in one of these triangles. All such points belong to the line between $\mathbf{d}(w)$ and $\mathbf{d}(z)$ that is denoted by a double line in the figure. This is the intersection between $\mathcal{P}$ and the "regular" cone generated by $\mathbf{d}(w)$ and $\mathbf{d}(z)$. The denominator vector of $u_n w^k$ and $u_n z^k$, for all $n, k \geq 0$, lie in the regular cone. Figure 2 appears also in [10] where the authors analyze the canonical decomposition of the elements of Q. Indeed denominator vectors of cluster variables are positive real Schur roots (see [2, 4]).

2.2 Explicit Expressions of the Elements of B

Our next result provides explicit formulas for the elements of $\mathbf{B}$ in every cluster of $\mathcal{A}_{\mathbb{P}}$. By the symmetry of the exchange relations it is sufficient to consider only the two clusters $\{x_1, x_2, x_3\}$ and $\{x_1, w, x_3\}$ and only cluster variables x_m with $m \geq 2$ (see Remark 3.1). We use the notation $\mathbf{x}^{\mathbf{e}} := x_1^{e_1} x_2^{e_2} x_3^{e_3}$.

Theorem 2.3 *Let $\mathbb{P}$ be any semifield. For a cluster variable with denominator vector $\mathbf{d} = (d_1, d_2, d_3)^t$ in the cluster $\{x_1, x_2, x_3\}$ we use the notation*

$$\varepsilon(\mathbf{e}) := (e_2 + e_3, d_1 - e_1 + e_3, d_1 + d_2 - e_1 - e_2).$$

For every $m, n \geq 1$ the following formulas hold:

$$x_{2m+1} = \frac{\sum_{\mathbf{e}} \binom{e_1-e_3}{e_2-e_3}\binom{m-1-e_3}{m-1-e_1}\binom{e_1-1}{e_3}\mathbf{y}^{\mathbf{e}}\mathbf{x}^{\varepsilon(\mathbf{e})} + x_2^{m-1}x_3^{2m-2}}{\left(\bigoplus_{\mathbf{e}} \binom{e_1-e_3}{e_2-e_3}\binom{m-1-e_3}{m-1-e_1}\binom{e_1-1}{e_3}\mathbf{y}^{\mathbf{e}} \oplus 1\right) \ x_1^{m-1}x_2^{m-1}x_3^{m-2}}$$

$$x_{2m+2} = \frac{\sum_{\mathbf{e}} \binom{e_1-1}{e_3}\binom{m-e_2}{e_1-e_2}\binom{m-1-e_3}{e_2-e_3}\mathbf{y}^{\mathbf{e}}\mathbf{x}^{\varepsilon(\mathbf{e})} + x_2^{m}x_3^{2m-1}}{\left(\bigoplus_{\mathbf{e}} \binom{e_1-1}{e_3}\binom{m-e_2}{e_1-e_2}\binom{m-1-e_3}{e_2-e_3}\mathbf{y}^{\mathbf{e}} \oplus 1\right) \ x_1^{m}x_2^{m-1}x_3^{m-1}}$$

$$u_n = \frac{y_1^n y_2^n y_3^n x_1^{2n} x_2^n + x_2^n x_3^{2n} + \sum_{\mathbf{e}} \binom{e_1-e_3}{e_1-e_2}\left[\binom{n-e_3}{n-e_1}\binom{e_1-1}{e_3} + \binom{n-e_3-1}{n-e_1}\binom{e_1-1}{e_3-1}\right]\mathbf{y}^{\mathbf{e}}\mathbf{x}^{\varepsilon(\mathbf{e})}}{x_1^n x_2^n x_3^n}$$

Let $\mathrm{p}_1 := y_1(y_2 \oplus 1)$, $\mathrm{p}_2 := \frac{1}{y_2}$ *and* $\mathrm{p}_3 := \frac{y_2 y_3}{y_2 \oplus 1}$. *For every $m, n \geq 1$:*

$$x_{2m+1} = \frac{\sum_{e_1,e_3} \binom{m-1-e_3}{m-1-e_1}\binom{e_1-1}{e_3}\mathrm{p}_1^{e_1}\mathrm{p}_3^{e_3} x_1^{2e_3} w^{e_1-e_3} x_3^{2m-2e_1-2} + x_3^{2m-2}}{\left(\bigoplus_{e_1,e_3} \binom{m-1-e_3}{m-1-e_1}\binom{e_1-1}{e_3}\mathrm{p}_1^{e_1}\mathrm{p}_3^{e_3} \oplus 1\right) x_1^{m-1}x_3^{m-2}}$$

$$x_{2m+2} = \frac{\sum_{\mathbf{e}} \binom{m-1-e_3+e_2}{m-1-e_1+e_2}\binom{e_1-1}{e_3}\binom{1}{e_2}\mathrm{p}^{\mathbf{e}} x_1^{2e_3+1-e_2} w^{e_1-e_3} x_3^{2m-2e_1-1+e_2} + x_3^{2m-1}(\mathrm{p}_2 x_3 + x_1)}{\left(\bigoplus_{\mathbf{e}} \binom{m-1-e_3+e_2}{m-1-e_1+e_2}\binom{e_1-1}{e_3}\binom{1}{e_2}\mathrm{p}^{\mathbf{e}} \oplus \mathrm{p}_2 \oplus 1\right) \ x_1^{m} w x_3^{m-1}}$$

$$u_n = \frac{\mathrm{p}_1^n \mathrm{p}_3^n x_1^{2n} + x_3^{2n} + \sum_{e_1,e_3}\left[\binom{n-e_3}{n-e_1}\binom{e_1-1}{e_3} + \binom{n-e_3-1}{n-e_1}\binom{e_1-1}{e_3-1}\right]\mathrm{p}_1^{e_1}\mathrm{p}_3^{e_3} x_1^{2e_3} w^{e_1-e_3} x_3^{2n-2e_1}}{x_1^n x_3^n}$$

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$$z = \frac{p_1p_2p_3x_1^2 + p_1p_2^2p_3x_1x_3 + p_1p_2w + p_2x_3^2 + x_1x_3}{(p_1p_2p_3 \oplus p_1p_2^2p_3 \oplus p_1p_2 \oplus p_2 \oplus 1)\ x_1wx_3} \tag{9}$$

2.3 Parametrization of **B** by **g**-Vectors

In [17, Section 7] it is shown that cluster monomials are parameterized by some integer vectors called **g**-vectors. In this section we extend such parametrization to all the elements of **B**. Given a polynomial $F \in \mathbb{Z}_{\geq 0}[z_1, \cdots, z_n]$ with positive coefficients in n variables and a semifield $\mathbb{P} = (\mathbb{P}, \cdot, \oplus)$, its *evaluation* $F|_{\mathbb{P}}(y_1, \cdots, y_n)$ at $(y_1, \cdots, y_n) \in \mathbb{P} \times \cdots \times \mathbb{P}$ is the element of $\mathbb{P}$ obtained by replacing the addition of $\mathbb{Z}[z_1, \cdots, z_n]$ with the auxiliary addition $\oplus$ of $\mathbb{P}$ in the expression $F(y_1, \cdots, y_n)$. This operation is well-defined since the resulting element of $\mathbb{P}$ does not depend on the particular coefficient-free expression for $F(y_1, \cdots, y_n)$ (see [17, Definition 2.1]). For example the evaluation of $F(z_1, z_2) := z_1 + 1$ at $(y_1, y_2) \in Trop(y_1, y_2) \times Trop(y_1, y_2)$ is 1. Let $\mathbb{P}$ be a semifield and let $\mathcal{A}_{\mathbb{P}}$ be a cluster algebra of type $A_2^{(1)}$ with coefficients in $\mathbb{P}$ and let $\Sigma = \{H, \mathcal{C}, \{y_1, y_2, y_3\}\}$ be a seed of $\mathcal{A}_{\mathbb{P}}$, so that H is an exchange matrix, $\mathcal{C}$ is a cluster and y_1, y_2 and y_3 are arbitrarily chosen elements of $\mathbb{P}$ (see Section 3.1). We use the notation $\mathbf{s}^{\mathbf{e}} := s_1^{e_1}s_2^{e_2}s_3^{e_3}$.

Proposition 2.1 *For every cluster monomial b of $\mathcal{A}_{\mathbb{P}}$ there exist a (unique) polynomial $F_b^{\mathcal{C}} \in \mathbb{Z}_{\geq 0}[z_1, z_2, z_3]$ in three variables, with non-negative coefficients and constant term 1 and a (unique) integer vector $\mathbf{g}_b^{\mathcal{C}} \in \mathbb{Z}^3$ such that the expansion of b in the cluster $\mathcal{C} = \{s_1, s_2, s_3\}$ is:*

$$b = \frac{F_b^{\mathcal{C}}\left(y_1\mathbf{s}^{\mathbf{h}_1}, y_2\mathbf{s}^{\mathbf{h}_2}, y_3\mathbf{s}^{\mathbf{h}_3}\right)}{F_b^{\mathcal{C}}|_{\mathbb{P}}(y_1, y_2, y_3)}\mathbf{s}^{\mathbf{g}_b^{\mathcal{C}}} \tag{10}$$

where $\mathbf{h}_i$ is the i-th column vector of the exchange matrix H. For every $n \geq 1$ there exist a (unique) polynomial $F_{u_n}^{\mathcal{C}} \in \mathbb{Z}_{\geq 0}[z_1, z_2, z_3]$ in three variables, with non-negative coefficients and constant term 1 and a (unique) integer vector $\mathbf{g}_{u_n}^{\mathcal{C}} \in \mathbb{Z}^3$ such that the expansion of u_n in the cluster $\mathcal{C}$ is:

$$u_n = F_{u_n}^{\mathcal{C}}(y_1\mathbf{s}^{\mathbf{h}_1}, y_2\mathbf{s}^{\mathbf{h}_2}, y_3\mathbf{s}^{\mathbf{h}_3})\mathbf{s}^{\mathbf{g}_b^{\mathcal{C}}}. \tag{11}$$

In view of Proposition 2.1, every element $b \in \mathbf{B}$ determines a polynomial $F_b^{\mathcal{C}}$ and an integer vector $\mathbf{g}_b^{\mathcal{C}}$, for every cluster $\mathcal{C}$ of $\mathcal{A}_{\mathbb{P}}$. The polynomial $F_b^{\mathcal{C}}$ and the vector $\mathbf{g}_b^{\mathcal{C}}$ are called respectively the F-polynomial and the **g**-vector of b in the cluster $\mathcal{C}$. The previous proposition is the key result for our proof of Theorem 2.3. In Section 5.1 we find the explicit expression of F-polynomials and **g**-vectors of every element of **B** in every cluster.

We notice that in view of Proposition 2.1 the F-polynomials of every cluster variable in every cluster has non-negative coefficients. This establishes the positivity conjecture formulated in [17] for every cluster algebra of type $A_2^{(1)}$. In particular this result follows from results of [27] since cluster algebras of type $A_2^{(1)}$ are associated to an unpunctured surface in the sense of [15].

We can define more intrinsically F-polynomials and **g**-vectors as follows: consider the tropical semifield $\mathbb{P} = Trop(y_1, y_2, y_3)$ generated by the coefficients of Σ and expand a cluster monomial $b \in \mathcal{A}_{\mathbb{P}}$ in the cluster $\mathcal{C}$. Since $F_b^{\mathcal{C}}$ has constant term 1,

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$F_b^{\mathcal{C}}|_{\mathbb{P}}(y_1, y_2, y_3) = 1$. It hence follows that if we replace s_1, s_2 and s_3 by 1 in Eq. 10 we get $F_b^{\mathcal{C}}(y_1, y_2, y_3)$.

The **g**-vector of b can be defined as follows: let $\mathbb{P} = Trop(y_1, y_2, y_3)$; following [17] we consider the *principal* $\mathbb{Z}^3$*-grading of* $\mathcal{A}_{\mathbb{P}}$ given by

$$deg(s_i) = \mathbf{e}_i,\ deg(y_i) = -\mathbf{h}_i,\ i = 1, 2, 3 \tag{12}$$

($\mathbf{e}_i$ is the i-th basis vector of $\mathbb{Z}^3$). The element $\hat{y}_i := y_i \mathbf{s}^{\mathbf{h}_i}$, $i = 1, 2, 3$, has degree zero with respect to such grading. Therefore, since $F_b^{\mathcal{C}}|_{\mathbb{P}}(y_1, y_2, y_3) = 1$, every element b of **B** is homogeneous with respect to such grading and the **g**-vector $\mathbf{g}_b^{\mathcal{C}}$ is its degree.

The relation between denominator vectors and **g**-vectors in the cluster $\{x_1, x_2, x_3\}$ is given by the following proposition.

Proposition 2.2 *Let b be an element of* **B** *not divisible by x_1, x_2 or x_3. The* **g***-vector* $\mathbf{g}_b$ *of b and its denominator vector* $\mathbf{d}(b)$ *both in the cluster* $\{x_1, x_2, x_3\}$ *are related by*

$$\mathbf{g}_b = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \mathbf{d}(b) \tag{13}$$

If b is a cluster monomial in the initial cluster $\{x_1, x_2, x_3\}$ *then* $\mathbf{g}_b = -\mathbf{d}(b)$.

Formula 13 between the **g**-vector and the denominator vector of a cluster variable in an "acyclic" seed can be deduced from results of [18] and [11]. We notice that the map (Eq. 13) is bijective, and hence by combining Proposition 2.2 and Theorem 2.2, we get that the map $b \mapsto \mathbf{g}_b$ restricts to a bijection between **B** and $\mathbb{Z}^3$. The denominator vector map in the cluster $\{x_1, w, x_3\}$ is not surjective; on the other hand, as expected for cluster monomials ([17, Conjecture 7.10], proved in [18, Theorem 6.3]), there is a bijective map (see Eq. 108) between the **g**-vectors of the elements of **B** in $\{x_1, x_2, x_3\}$ and in $\{x_1, w, x_3\}$. We have hence the following parametrization of **B** by $\mathbb{Z}^3$ via the **g**-vector map.

Proposition 2.3 *Given a cluster $\mathcal{C}$ of $\mathcal{A}$, the map* $b \mapsto \mathbf{g}_b^{\mathcal{C}}$ *which sends an element b of* **B** *to its* **g***-vector* $\mathbf{g}_b^{\mathcal{C}}$ *in the cluster $\mathcal{C}$, is a bijection between* **B** *and* $\mathbb{Z}^3$.

The last comment about Eq. 13 is the following: we note that the matrix in Eq. 13 equals $-E_Q$ where E_Q denotes the Euler matrix of the quiver Q given in Eq. 5. In [28] it is shown that the weight of the Schofield's semi-invariant c_V, where V is an indecomposable Q-representation of dimension vector $\mathbf{d} = \mathbf{d}(b)$, is given by $-E\mathbf{d}$ and in view of Eq. 13, is the **g**-vector of b. In particular one could deduce Proposition 2.3 from results of [21].

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3 Cluster Algebras of Type $A_2^{(1)}$

3.1 Background on Cluster Algebras

Let $\mathbb{P}$ be a semifield (see Section 2). Let $\mathcal{F}_{\mathbb{P}} = \mathbb{QP}(x_1, \cdots, x_n)$ be the field of rational functions in n independent variables $x_1, \cdots, x_n$. A *seed* in $\mathcal{F}_{\mathbb{P}}$ is a triple $\Sigma = \{H, \mathcal{C}, \mathbf{y}\}$ where H is an $n \times n$ integer matrix which is skew-symmetrizable, i.e. there exists a diagonal matrix $D = diag(d_1, \cdots, d_n)$ with $d_i > 0$ for all i such that DB is skew-symmetric; $\mathcal{C} = \{s_1, \cdots, s_n\}$ is an n-tuple of elements of $\mathcal{F}_{\mathbb{P}}$ which is a free generating set for $\mathcal{F}_{\mathbb{P}}$ so that $\mathcal{F}_{\mathbb{P}} \simeq \mathbb{QP}(s_1, \cdots, s_n)$; and finally $\mathbf{y} = \{y_1, \cdots, y_n\}$ is an n-tuple of elements of $\mathbb{P}$. The matrix H is called the *exchange matrix* of Σ, the set $\mathcal{C}$ is called the *cluster* of Σ, its elements are called *cluster variables* of Σ and the set $\mathbf{y}$ is called the *coefficients tuple* of the seed Σ.

We fix an integer $k \in [1, n]$. Given a seed Σ of $\mathcal{F}_{\mathbb{P}}$ we define another seed $\Sigma_k := \{H_k, \mathcal{C}_k, \mathbf{y}_k\}$ by the following *mutation rules* (see [17]):

(1) the exchange matrix $H_k = (h'_{ij})$ is obtained from $H = (h_{ij})$ by

$$h'_{ij} = \begin{cases} -h_{ij} & \text{if } i = k \text{ or } j = k \\ h_{ij} + sg(h_{ik})[h_{ik}h_{kj}]_+ & \text{otherwise} \end{cases} \tag{14}$$

where $[c]_+ := \max(c, 0)$ for every integer c and $sg(c)$ is the sign of c;

(2) the new coefficients tuple $\mathbf{y}_k = \{y'_1, \cdots, y'_n\}$ is given by:

$$y'_j := \begin{cases} \dfrac{1}{y_k} & \text{if } j = k \\ y_j y_k^{[h_{kj}]_+}(y_k \oplus 1)^{-h_{kj}} & \text{otherwise.} \end{cases} \tag{15}$$

(3) the new cluster $\mathcal{C}_k$ is given by $\mathcal{C}_k = \mathcal{C} \setminus \{s_k\} \cup \{s'_k\}$ where

$$s'_k := \frac{y_k \prod_i s_i^{[h_{ik}]_+} + \prod_i s_i^{[-h_{ik}]_+}}{(y_k \oplus 1)s_k}; \tag{16}$$

It is not hard to verify that Σ_k is again a seed of $\mathcal{F}_{\mathbb{P}}$. We say that the seed Σ_k is obtained from the seed Σ by a *mutation in direction* k. Every seed can be mutated in all the directions.

Given a seed Σ we consider the set $\chi(\Sigma)$ of all the cluster variables of all the seeds obtained by a sequence of mutations. The *rank n cluster algebra with initial seed* Σ *with coefficients in* $\mathbb{P}$ is by definition the $\mathbb{ZP}$-subalgebra of $\mathcal{F}_{\mathbb{P}}$ generated by $\chi(\Sigma)$; we denote it by $\mathcal{A}_{\mathbb{P}}(\Sigma)$.

The cluster algebra $\mathcal{A}_{\mathbb{P}}(\Sigma)$ is said to have *principal coefficients at* Σ and it is denoted by $\mathcal{A}_\bullet(\Sigma)$ if $\mathbb{P} = Trop(y_1, \cdots y_n)$.

If the semifield $\mathbb{P} = Trop(c_1, \cdots, c_r)$ is a tropical semifield, the elements of $\mathbb{P}$ are monomials in the c_j's. Therefore any coefficient y_j of a seed $\Sigma = \{H, \mathcal{C}, \{y_1, \cdots, y_n\}\}$ is a monomial of the form $y_j = \prod_{i=1}^r c_i^{h_{n+r,j}}$ for some integers $h_{n+1,j}, \cdots, h_{n+r,j}$. It is convenient to "complete" the exchange $n \times n$ matrix H to a rectangular $(n+r) \times n$ matrix $\tilde{H}$ whose (i, j)-th entry is h_{ij}. The seed Σ can hence be seen as a couple $\{\tilde{H}, \mathcal{C}\}$ and the mutation of the coefficients tuple (Eq. 15) translates into the mutation (Eq. 14) of the rectangular matrix $\tilde{H}$. We sometimes use this formalism.

We say that two seeds $\Sigma = \{H, \mathcal{C}, \mathbf{y}\}$ and $\Sigma' = \{H', \mathcal{C}', \mathbf{y}'\}$ of a cluster algebra of rank n are equivalent if there exists a permutation σ of the index set

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$[1,n] := \{1,\cdots,n\}$ such that $h_{\sigma(i)\sigma(j)} = h'_{ij}$, $s_{\sigma(i)} = s'_i$ and $y_{\sigma(i)} = y'_i$ for every $i, j \in [1,n]$. An equivalence class $[\Sigma]$ is called an *unlabeled* seed. In particular clusters and coefficient tuples of unlabeled seeds are sets, while in labeled seeds they are sequences. With abuse of language in both cases we denote them with braces $\{\ldots\}$. The mutation μ_i of an unlabeled seed $[\Sigma]$ in direction i can be defined as $[\mu_i(\Sigma)]$ and it hence depends on the particular choice of a representative Σ of $[\Sigma]$. It is not well-defined. We hence prefer to refer to a mutation of $[\Sigma]$ as one of the unlabeled seeds $\{[\mu_i\Sigma] : i = 1,\cdots,n\}$. We often consider unlabeled seeds in this paper. We use the cyclic representation of a permutation σ so that $\sigma = (i_1, i_2, \cdots, i_t)$ denotes the permutation σ such that $\sigma(i_k) = i_{k+1}$ if $k = 1,\cdots,t-1$ and $\sigma(i_t) = i_1$; all the other indices are fixed by σ.

Example 3.1 The seed

$$\Sigma := \left\{ H = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \mathbf{x} = \{x_1, x_2, x_3\}, \mathbf{y} = \{y_1, y_2, y_3\} \right\}$$

is equivalent to the seed

$$\Sigma' := \left\{ H' = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \mathbf{x}' = \{x_3, x_1, x_2\}, \mathbf{y}' = \{y_3, y_1, y_2\} \right\}$$

by the permutation $(1, 3, 2)$.

3.2 Algebraic Structure of $\mathcal{A}_{\mathbb{P}}$

Let $\mathbb{P} = (\mathbb{P}, \cdot, \oplus)$ be a semifield and $\mathcal{A}_{\mathbb{P}}$ be the cluster algebra with initial seed

$$\Sigma := \left\{ H = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \mathbf{x} = \{x_1, x_2, x_3\}, \mathbf{y} = \{y_1, y_2, y_3\} \right\}.$$

The following lemma gives the algebraic structure of $\mathcal{A}_{\mathbb{P}}$.

Lemma 3.1 *The unlabeled seeds of the cluster algebra $\mathcal{A}_{\mathbb{P}}$ with initial seed $\Sigma = \Sigma_1$ are the following:*

$$\Sigma_m := \{H_m, \{x_m, x_{m+1}, x_{m+2}\}, \{y_{1;m}, y_{2;m}, y_{3;m}\}\}, \tag{17}$$

$$\Sigma^{cyc}_{2m-1} := \left\{H^{cyc}_m, \{x_{2m-1}, w, x_{2m+1}\}, \left\{y^{cyc}_{1;2m-1}, y^{cyc}_{2;2m-1}, y^{cyc}_{3;2m-1}\right\}\right\}, \tag{18}$$

$$\Sigma^{cyc}_{2m} := \left\{H^{cyc}_m, \{x_{2m}, z, x_{2m+2}\}, \left\{y^{cyc}_{1;2m}, y^{cyc}_{2;2m}, y^{cyc}_{3;2m}\right\}\right\} \tag{19}$$

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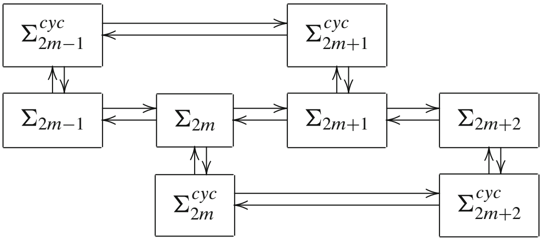
for every $m \in \mathbb{Z}$; they are mutually related by the following diagram of mutations:



(20)

where arrows from left to right (resp. from right to left) are mutations in direction 1 (resp. 3) and vertical arrows (in both directions) are mutations in direction 2. The seed Σ_m is not equivalent to the seed Σ_n if $m \neq n$, in particular the exchange graph of $\mathcal{A}_{\mathbb{P}}$ is given by Fig. 1 and every cluster $\mathcal{C}$ determines a unique seed $\Sigma^{\mathcal{C}}$. The exchange matrices H_m and H_m^{cyclic} are the following:

$$H_m = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \; H_m^{cyc} = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix} \tag{21}$$

for every $m \in \mathbb{Z}$. The exchange relations for the coefficient tuples are the following:

$$y_{1;1} = y_1; \quad y_{2;1} = y_2; \quad y_{3;1} = y_3 \tag{22}$$

$$y_{1;m+1} = \frac{y_{2;m}y_{1;m}}{y_{1;m} \oplus 1}; \quad y_{2;m+1} = \frac{y_{3;m}y_{1;m}}{y_{1;m} \oplus 1}; \quad y_{3;m+1} = \frac{1}{y_{1;m}} \tag{23}$$

$$y_{1;m-1} = \frac{1}{y_{3;m}}; \quad y_{2;m-1} = y_{1;m}(y_{3;m} \oplus 1); \quad y_{3;m-1} = y_{2;m}(y_{3;m} \oplus 1) \tag{24}$$

$$y^{cyc}_{1;m} = \frac{y_{1;m}}{y_{2;m} \oplus 1}; \quad y^{cyc}_{2;m} = \frac{1}{y_{2;m}}; \quad y^{cyc}_{3;m} = \frac{y_{3;m}y_{2;m}}{y_{2;m} \oplus 1} \tag{25}$$

$$y^{cyc}_{1;m+2} = y^{cyc}_{3;m}\left(y^{cyc}_{1;m} \oplus 1\right)^2; \quad y^{cyc}_{2;m+2} = \frac{y^{cyc}_{2;m}y^{cyc}_{1;m}}{y^{cyc}_{1;m} \oplus 1}; \quad y^{cyc}_{3;m+2} = \frac{1}{y^{cyc}_{1;m}} \tag{26}$$

$$y_{1;m} = \frac{y^{cyc}_{1;m}y^{cyc}_{2;m}}{y^{cyc}_{2;m} \oplus 1}; \quad y_{2;m} = \frac{1}{y^{cyc}_{2;m}}; \quad y_{3;m} = y^{cyc}_{3;m}\left(y^{cyc}_{2;m} \oplus 1\right) \tag{27}$$

$$y^{cyc}_{1;m-2} = \frac{1}{y^{cyc}_{3;m}}; \quad y^{cyc}_{2;m-2} = \frac{y^{cyc}_{2;m}y^{cyc}_{3;m}}{y^{cyc}_{3;m} \oplus 1}; \quad y^{cyc}_{3;m-2} = y^{cyc}_{1;m}\left(y^{cyc}_{3;m} \oplus 1\right)^2 \tag{28}$$

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The exchange relations for the cluster variables are the following:

$$x_m x_{m+3} = \frac{x_{m+1}x_{m+2} + y_{1;m}}{y_{1;m} \oplus 1} = \frac{y_{3;m+1}x_{m+1}x_{m+2} + 1}{y_{3;m+1} \oplus 1} \tag{29}$$

$$w x_{2m} = \frac{y_{2;2m-1}x_{2m-1} + x_{2m+1}}{y_{2;2m-1} \oplus 1} = \frac{x_{2m-1} + y^{cyc}_{2;2m-1}x_{2m+1}}{y^{cyc}_{2;2m-1} \oplus 1} \tag{30}$$

$$z x_{2m+1} = \frac{y_{2;2m}x_{2m} + x_{2m+2}}{y_{2;2m} \oplus 1} = \frac{x_{2m} + y^{cyc}_{2;2m}x_{2m+2}}{y^{cyc}_{2;2m} \oplus 1} \tag{31}$$

$$x_{2m-2}x_{2m+2} = \frac{x^2_{2m} + y^{cyc}_{1;2m-2}z}{y^{cyc}_{1;2m-2} \oplus 1} = \frac{y^{cyc}_{3;2m}x^2_{2m} + z}{y^{cyc}_{3;2m} \oplus 1} \tag{32}$$

$$x_{2m-1}x_{2m+3} = \frac{x^2_{2m+1} + y^{cyc}_{1;2m-1}w}{y^{cyc}_{1;2m-1} \oplus 1} = \frac{y^{cyc}_{3;2m+1}x^2_{2m+1} + w}{y^{cyc}_{3;2m+1} \oplus 1} \tag{33}$$

Proof of Lemma 3.1 We need to prove the diagram (Eq. 20) for every $m \in \mathbb{Z}$. Clearly Σ_1 coincides with the initial seed Σ. For $m \geq 2$ let Σ_m (resp. Σ_{-m+1}) be the seed of $\mathcal{A}_{\mathbb{P}}$ obtained from Σ_1 by applying $m-1$ times the following operation: first we mutate in direction 1 (resp. 3) and then we reorder the index set of the obtained seed by the permutation $(1, 3, 2)$ (resp. $1, 2, 3$) of the index set (as in Example 3.1). Assume that Σ_m (resp.Σ_{-m+1}) has the form (Eq. 17), i.e. its exchange matrix is H_m (resp. H_{-m+1}), its cluster is $\{x_m, x_{m+1}, x_{m+2}\}$ and its coefficient tuple is $\{y_{1;m}, y_{2;m}, y_{3;m}\}$. Then it is straightforward to check by induction on m that H_m is given by Eq. 21 and the exchange relations passing from Σ_m to Σ_{m+1} are given by Eq. 23 (resp. Eq. 24) for the coefficient tuple and by Eq. 29 for the cluster variables.

Now it is straightforward to verify that, for every $m \in \mathbb{Z}$, Σ_m is obtained from Σ_{m+1} by mutating in direction 3 and then by reordering with the permutation $(1, 2, 3)$.

The central line of the diagram is hence proved.

Let Σ^{cyc}_m be the seed obtained from Σ_m by the mutation in direction 2. Suppose that Σ^{cyc}_m has the form $\left\{H^{cyc}_m, \{x_m, c_m, x_{m+2}\}, \{y^{cyc}_{1;m}, y^{cyc}_{2;m}, y^{cyc}_{3;m}\}\right\}$. Then it is straightforward to verify the following: that by Eq. 14 the exchange matrix H^{cyc}_m is given by Eq. 21; that by Eq. 15 the coefficient tuple satisfy Eq. 25 and by the exchange relation (Eq. 16) the cluster variable c_m is given by:

$$c_m = \frac{y_{2;m}x_m + x_{m+2}}{(y_{2;m} \oplus 1)x_{m+1}}. \tag{34}$$

By using Eqs. 23, 24 and 29, it is straightforward to verify that $c_m = c_{m+2}$ for every $m \in \mathbb{Z}$. We define $w := c_1$ and $z := c_2$ and we hence find that Σ^{cyc}_{2m-1} has the form (Eq. 18) and Σ^{cyc}_{2m} has the form (Eq. 19). Now since two cluster variables can belong to at most two clusters, we conclude that the mutation in direction 1 (resp. 3) of Σ^{cyc}_m is Σ^{cyc}_{m+2} (resp. Σ^{cyc}_{m-2}).

We have hence proved the diagram (Eq. 20) and the fact that every cluster determines a unique seed.

It remains to prove that the seeds $\{\Sigma_m\}$ (resp. $\{\Sigma^{cyc}_m\}$) are not equivalent for every $m \in \mathbb{Z}$. It is sufficient to prove that x_m is different from x_1, x_2 and x_3 for every $m \geq 4$.

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In view of the exchange relation (Eq. 29), the denominator vector of x_m, $m \geq 4$, in the cluster $\{x_1, x_2, x_3\}$ satisfies the initial conditions $\mathbf{d}(x_4) = \mathbf{e}_1$, $\mathbf{d}(x_5) = \mathbf{e}_1 + \mathbf{e}_2$ and

$$\mathbf{d}(x_{m+3}) + \mathbf{d}(x_m) = \mathbf{d}(x_{m+1}) + \mathbf{d}(x_{m+2}). \tag{35}$$

The solution of this recursion is given in Lemma 3.2 below and it is clearly not periodic. □

Remark 3.1 The expansion of a cluster variable x_{m+n} (resp. x_{2m+n}) in the cluster $\{x_m, c, x_{m+2}\}$ for $c = w$ or $c = x_{m+1}$, (resp. $\{x_{2m}, z, x_{2m+2}\}$) is obtained by the expansion of x_{1+n} (resp. x_{2m+1+n}) in the cluster $\{x_1, c, x_3\}$ (resp. $\{x_{2m+1}, w, x_{2m+3}\}$) by replacing x_1 with x_m, c with x_2 when $c \neq w$, x_3 with x_{m+2} and y_i with $y_{i;m}$ (resp. x_{2m+1} with x_{2m}, w with z, x_{2m+3} with x_{2m+2} and $y^{cyc}_{i;2m+1}$ with $y^{cyc}_{i;2m}$) for $i = 1, 2, 3$ and $n, m \in \mathbb{Z}$. Moreover the expansion of x_{-m+2} is obtained from the expansion of x_{m+2} by replacing x_1 with x_3, x_3 with x_1 and y_1 with y_3^{-1}, y_2 with y_2^{-1} and y_3 with y_1^{-1}.

In the sequel we abbreviate and we write:

$$\Sigma^{cyc} := \left\{ \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}, \{x_1, w, x_3\}, \{\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3\} \right\} \tag{36}$$

for the seed Σ_1^w obtained from the seed (Eq. 4) by a mutation in direction two. We sometimes call Σ^{cyc} the cyclic seed of $\mathcal{A}_{\mathbb{P}}$.

3.2.1 Denominator Vectors

In this subsection we compute denominator vectors of the elements of **B** in every cluster of $\mathcal{A}_{\mathbb{P}}$. By the symmetry in the exchange relations it is sufficient to consider only the two clusters $\{x_1, x_2, x_3\}$ and $\{x_1, w, x_3\}$.

Lemma 3.2 *Denominator vectors in the cluster* $\{x_1, x_2, x_3\}$ *of cluster variables different from* x_1, x_2 *and* x_3 *are the following: for* $m \geq 1$

$$\mathbf{d}(x_{2m+1}) = \begin{bmatrix} m-1 \\ m-1 \\ m-2 \end{bmatrix} \quad \mathbf{d}(x_{2m+2}) = \begin{bmatrix} m \\ m-1 \\ m-1 \end{bmatrix} \tag{37}$$

$$\mathbf{d}(x_{-2m+1}) = \begin{bmatrix} m-1 \\ m \\ m \end{bmatrix} \quad \mathbf{d}(x_{-2m+2}) = \begin{bmatrix} m-1 \\ m-1 \\ m \end{bmatrix} \tag{38}$$

$$\mathbf{d}(w) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{d}(z) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{39}$$

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For every $n \geq 1$ the denominator vector (in $\{x_1, x_2, x_3\}$) of u_n is given by

$$\mathbf{d}(u_n) = \begin{bmatrix} n \\ n \\ n \end{bmatrix}. \tag{40}$$

In particular the denominator vector map establishes a bijection between $\{$cluster variables$\} \cup \{u_n w,\ u_n z \,|\, n \geq 1\} \setminus \{x_1, x_2, x_3\}$ and the set of positive real roots of the root system of type $A_2^{(1)}$.

Proof In view of Remark 3.1 we have $\mathbf{d}(x_{-m+2}) = (1,3)\mathbf{d}(x_{m+2})$ for every $m \geq 2$. Then Eq. 38 follows from Eq. 37. We hence prove Eqs. 37, 39 and 40. By induction on $m \geq 4$, one verifies that Eq. 37 is the solution of the recurrence relation (Eq. 35) together with the initial condition $\mathbf{d}(x_4) = \mathbf{e}_1$, $\mathbf{d}(x_5) = \mathbf{e}_1 + \mathbf{e}_2$ ($\mathbf{e}_i$ being the i-th canonical basis vector of $\mathbb{Z}^3$).

The equalities in Eq. 39 follow from Eq. 2.

It remains to prove Eq. 40. We notice that the denominator vector map $s \mapsto \mathbf{d}(s)$ in a cluster $\mathcal{C}$ is additive, i.e. $\mathbf{d}(s_1 s_2) = \mathbf{d}(s_1) + \mathbf{d}(s_2)$. From its definition (Eq. 6), it hence follows that the denominator vector of u_1 is the sum of $\mathbf{d}(w)$ and of $\mathbf{d}(z)$, i.e. $\mathbf{d}(u_1) = \delta = (1,1,1)^t$. Then, by induction on $n \geq 1$, it follows from Eq. 7 that $\mathbf{d}(u_n) = n\mathbf{d}(u_1) = n\delta$.

The last sentence of the lemma follows from the structure of a root system of type $A_2^{(1)}$, recalled in Section 2. □

In view of a general well known result due to V. Kac [23], for every positive real root $\mathbf{d}$ of a root system of type $A_2^{(1)}$ there exists a unique (up to isomorphisms) indecomposable representation $M(\mathbf{d})$ of the quiver Q given in Eq. 5 whose dimension vector is $\mathbf{d}$. In particular, in view of Lemma 3.2, for every element b which is either a cluster variable or $u_n w$ or $u_n z$ ($n \geq 0$) there exists a unique Q-representation $M(b)$ whose dimension vector is $\mathbf{d}(b)$. If b is a cluster variable, this is a special case of [4, Theorem 3].

Corollary 3.1 *For every $m \geq 4$, $\mathbf{d}(x_m) + \mathbf{d}(u_1) = \mathbf{d}(x_{m+2})$. For every $m \leq 0$, $\mathbf{d}(x_m) + \mathbf{d}(u_1) = \mathbf{d}(x_{m-2})$.*

The following lemma shows the denominator vectors of the elements of $\mathbf{B}$ in the cyclic seed.

Lemma 3.3 *Let $\mathbf{d}^w(b)$ be the denominator vector of $b \in \mathbf{B}$ in the cluster $\{x_1, w, x_3\}$. The following formulas hold: for $m \geq 1$*

$$\mathbf{d}^w(x_{2m+1}) = \begin{bmatrix} m-1 \\ 0 \\ m-2 \end{bmatrix} \quad \mathbf{d}^w(x_{2m+2}) = \begin{bmatrix} m \\ 1 \\ m-1 \end{bmatrix} \tag{41}$$

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$$\mathbf{d}^w(x_{-2m+1}) = \begin{bmatrix} m-1 \\ 0 \\ m \end{bmatrix} \quad \mathbf{d}^w(x_{-2m+2}) = \begin{bmatrix} m-1 \\ 1 \\ m \end{bmatrix} \tag{42}$$

$$\mathbf{d}^w(x_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{d}^w(z) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{43}$$

For every $n \geq 1$ the denominator vector (in $\{x_1, w, x_3\}$) of u_n is given by

$$\mathbf{d}^w(u_n) = \begin{bmatrix} n \\ 0 \\ n \end{bmatrix}. \tag{44}$$

Proof In view of Remark 3.1, $\mathbf{d}^w(x_{-m+2}) = (1,3)\mathbf{d}^w(x_{m+2})$ for every $m \geq 2$. In particular Eq. 42 follows from Eq. 41. We hence prove Eqs. 41, 43 and 44. From the exchange relations it follows that $\mathbf{d}^w(x_4) = \mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{d}^w(x_5) = \mathbf{e}_1$. From the exchange relation (Eq. 29) it follows that the sequence of denominator vectors $\mathbf{d}^w(x_m)$ with $m \geq 4$ satisfies the relation:

$$\mathbf{d}^w(x_{m+3}) + \mathbf{d}^w(x_m) = \mathbf{d}^w(x_{m+1}) + \mathbf{d}^w(x_{m+2}).$$

By induction on $m \geq 4$ we verify that Eq. 41 is the unique solution of this recurrence.

The equalities in Eq. 43 follow by direct check.

The equality (Eq. 44) follows from the definition (Eq. 6). □

Many properties of denominator vectors of cluster variables can be found in [3] and [1]. Here we notice two of them.

Remark 3.2 The denominator vector map in the cluster $\{x_1, w, x_3\}$ does not restrict to a bijection between $\mathbf{B}$ and $\mathbb{Z}^3$ as noticed in [10]. For example the element $(1,2,3)^t$ cannot be expressed as a non-negative linear combination of denominator vectors of adjacent (in the exchange graph) cluster variables.

Definition 3.1 [17, Definition 6.12] A collection of vectors in $\mathbb{Z}^n$ (or in $\mathbb{R}^n$) are *sign-coherent* (to each other) if, for every $i \in \{1, \cdots, n\}$, the i-th coordinates of all of these vectors are either all non-negative or all non-positive.

Corollary 3.2 *Denominator vectors (in every cluster) of cluster variables belonging to the same cluster are sign-coherent.*

Proof By Lemmas 3.2 and 3.3 one verifies directly that for every cluster $\{s_1, s_2, s_3\}$ the sets $\{\mathbf{d}(s_1), \mathbf{d}(s_2), \mathbf{d}(s_3)\}$ and $\{\mathbf{d}^w(s_1), \mathbf{d}^w(s_2), \mathbf{d}^w(s_3)\}$ of corresponding denominator vectors are sign-coherent. □

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3.3 Explicit Expression of the Coefficient Tuples

In this section we solve the recurrence relations (Eqs. 22–28) for the coefficient tuples of the seeds of $\mathcal{A}_{\mathbb{P}}$, for every semifield $\mathbb{P}$. Such relations form a Y-system (see [25]). The solution of this recurrence is given in terms of denominator vectors and of F-polynomials in the cluster $\{x_1, x_2, x_3\}$ of the cluster variables of $\mathcal{A}_{\mathbb{P}}$. We assume Proposition 2.1, even if its proof will be given later in Section 5.1. Then F_s is a polynomial with positive coefficients. In particular we can consider its evaluation $F_s|_{\mathbb{P}}$ at $\mathbb{P}$.

Proposition 3.1

- *The family* $\{y_{1;m} : m \in \mathbb{Z}\}$ *is given by*

$$y_{1;m} = \frac{\mathbf{y}^{\mathbf{d}(x_{m+3})}}{F_{m+1}|_{\mathbb{P}}(\mathbf{y})F_{m+2}|_{\mathbb{P}}(\mathbf{y})}. \tag{45}$$

  where F_m *denotes the* F*-polynomial of the cluster variable* x_m *in the cluster* $\{x_1, x_2, x_3\}$ *and we use the notation* $F(\mathbf{y}) := F(y_1, y_2, y_3)$ *for every polynomial* F *in three variables.*
- *The family* $\{y_{2;m} : m \in \mathbb{Z}\}$ *is given by:*

$$y_{2;1} = y_2, \quad y_{2;0} = (y_3 \oplus 1)y_1, \quad y_{2;-1} = (y_2y_3 \oplus y_2 \oplus 1)\frac{1}{y_3}. \tag{46}$$

  For every $m \geq 1$

$$y_{2;2m} = \frac{F_{2m}|_{\mathbb{P}}(\mathbf{y})}{F_{2m+2}|_{\mathbb{P}}(\mathbf{y})}y_1y_3 \tag{47}$$

$$y_{2;-2m} = \frac{F_{-2m}|_{\mathbb{P}}(\mathbf{y})}{F_{-2m+2}|_{\mathbb{P}}(\mathbf{y})}\frac{1}{y_2} \tag{48}$$

$$y_{2;2m+1} = \frac{F_{2m+1}|_{\mathbb{P}}(\mathbf{y})}{F_{2m+3}|_{\mathbb{P}}(\mathbf{y})}y_2 \tag{49}$$

$$y_{2;-2m-1} = \frac{F_{-2m-1}|_{\mathbb{P}}(\mathbf{y})}{F_{-2m+1}|_{\mathbb{P}}(\mathbf{y})}\frac{1}{y_1y_3} \tag{50}$$

- *The family* $\{y_{3;m} | m \in \mathbb{Z}\}$ *is given by* $y_{3;m+1} = 1/y_{1;m}$.
- *The families* $\{y^{cyc}_{i;m} | m \in \mathbb{Z}\}$ *for* $i = 1, 2, 3$ *are given by*

$$y^{cyc}_{1;m} = \frac{y_{1;m}}{y_{2;m} \oplus 1}; \quad y^{cyc}_{2;m} = 1/y_{2;m}; \quad y^{cyc}_{3;m+2} = 1/y^{cyc}_{1;m} \tag{51}$$

Proof of Proposition 3.1 Formulas 46–51 follows directly from Eq. 45. We hence prove Eq. 45.

By Eq. 23 it follows that the family $\{y_{1;m} : m \in \mathbb{Z}\}$ is the sequence of elements of $\mathbb{P}$ uniquely determined by the initial data: $y_{1;m} = 1/y_{m+3}$ if $m = 0, -1, -2$, $y_{1;1} = y_1$, $y_{1;-3} = y_3$ together with the recurrence relations

$$y_{1;m}y_{1;m+3} = \frac{y_{1;m+2}y_{1;m+1}}{(y_{1;m+2} \oplus 1)(y_{1;m+1} \oplus 1)} \tag{52}$$

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Let us first suppose $\mathbb{P} = Trop(y_1, y_2, y_3)$. By induction on $m \geq 1$ and $m+3 \leq 1$ and by the relation (Eq. 35) one proves that the solution of Eq. 52 is the following: for every $m \in \mathbb{Z}$

$$y_{1;m} = \mathbf{y}^{\mathbf{d}(x_{m+3})} \tag{53}$$

where $\mathbf{d}(x_{m+3})$ is the denominator vector the cluster variable x_{m+3} in the cluster $\{x_1, x_2, x_3\}$ given in Lemma 3.2. In particular the exchange relations for $\mathcal{A}_{Trop(y_1,y_2,y_3)}$ are given in Corollary 3.3 below.

Let us assume now that $\mathbb{P}$ is any semifield. Recall that the F-polynomial F_m of the cluster variable x_m in the cluster $\{x_1, x_2, x_3\}$ is obtained from the Laurent expansion of the cluster variable $x_m \in \mathcal{A}_{Trop(y_1,y_2,y_3)}$ in $\{x_1, x_2, x_3\}$ by specializing $x_1 = x_2 = x_3 = 1$. In view of Corollary 3.3, the family $\{F_m : m \in \mathbb{Z}\}$ is hence recursively defined by the initial data: $F_1 = F_2 = F_3 = 1$

$$\begin{aligned} F_0(\mathbf{y}) = y_3 + 1 \qquad & F_{-1}(\mathbf{y}) = y_2y_3 + y_2 + 1 \\ F_{-2}(\mathbf{y}) = y_1y_2y_3^2 + 2&y_1y_2y_3 + y_1y_3 + y_1y_2 + y_1 + 1 \end{aligned} \tag{54}$$

together with the recurrence relations for $m \geq 1$ and $m \leq -3$:

$$F_m(\mathbf{y})F_{m+3}(\mathbf{y}) = F_{m+1}(\mathbf{y})F_{m+2}(\mathbf{y}) + \mathbf{y}^{\mathbf{d}(x_{m+3})} \tag{55}$$

Then an easy induction gives the desired Eq. 45. □

As a corollary of the previous proposition we get relations between principal cluster variables: we recall that given a cluster variable s the corresponding principal cluster variable in the cluster $\{x_1, x_2, x_3\}$ is the element $S := F_s|_{\mathbb{P}}(y_1, y_2, y_3)s$.

Corollary 3.3 *The principal cluster variables in the cluster $\{x_1, x_2, x_3\}$ of $\mathcal{A}_{\mathbb{P}}$ satisfy the following relations:*

$$X_0X_3 = y_3X_1X_2 + 1; \quad X_{-1}X_2 = y_2X_0X_1 + 1; \quad X_{-2}X_1 = y_1X_{-1}X_0 + 1. \tag{56}$$

$$X_mX_{m+3} = X_{m+1}X_{m+2} + \mathbf{y}^{\mathbf{d}(x_{m+3})} \ \textit{for } m \geq 1 \textit{ and } m \leq -3 \tag{57}$$

$$WX_{2m} = \begin{cases} y_2X_{2m-1} + X_{2m+1} & \textit{if } m \geq 1 \\ X_{-1} + y_3X_1 & \textit{if } m = 0 \\ X_{2m-1} + y_1y_3X_{2m+1} & \textit{if } m \leq -1 \end{cases} \tag{58}$$

$$ZX_{2m+1} = \begin{cases} y_1y_3X_{2m} + X_{2m+2} & \textit{if } m \geq 1 \\ y_1X_0 + X_2 & \textit{if } m = 0 \\ X_{2m} + y_2X_{2m+2} & \textit{if } m \leq -1 \end{cases} \tag{59}$$

$$X_{2m-2}X_{2m+2} = X_{2m}^2 + \mathbf{y}^{\mathbf{d}(x_{2m+1})}Z \tag{60}$$

$$X_{2m-1}X_{2m+3} = X_{2m+1}^2 + \mathbf{y}^{\mathbf{d}(x_{2m+2})}W \tag{61}$$

In particular, if $\mathbb{P} = Trop(y_1, y_2, y_3)$, these are precisely the exchange relations of the cluster algebra $\mathcal{A}_{\mathbb{P}} = \mathcal{A}_\bullet(\Sigma)$ with principal coefficients at the seed $\Sigma = \Sigma_1$.

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4 Proof of Theorem 2.2

In this section we prove that the denominator vector map $\mathbf{d}: \mathcal{A} \to Q$ in the initial cluster $\{x_1, x_2, x_3\}$ restricts to a bijection between $\mathbf{B}$ and the root lattice Q of type $A_2^{(1)}$.

Clearly the denominator vector of a cluster monomial $s_1^a s_2^b s_3^c$ is $a\mathbf{d}(s_1) + b\,\mathbf{d}(s_2) + c\mathbf{d}(s_3)$. We also know that $\mathbf{d}(u_1) = \mathbf{d}(w) + \mathbf{d}(z)$ and $\mathbf{d}(u_n) = n\mathbf{d}(u_1)$. Hence the cone $\mathcal{C}_{Reg} = \mathbb{Z}_{\geq 0}\mathbf{d}(w) + \mathbb{Z}_{\geq 0}\mathbf{d}(z)$ generated by $\mathbf{d}(w)$ and $\mathbf{d}(z)$ is in bijection with the set $\{\mathbf{d}(u_n w^k), \mathbf{d}(u_n z^k) | \, n, k \geq 0\}$. To complete the proof of Theorem 2.2 it is enough to show the following:

$$\text{For every cluster } \{s_1, s_2, s_3\}, \text{ the vectors } \mathbf{d}(s_1), \mathbf{d}(s_2) \text{ and } \mathbf{d}(s_3) \text{ form a } \mathbb{Z}\text{-basis of } Q. \tag{62}$$

$$\text{For every element } v \text{ of } Q \text{ which does not lie in the interior of } \mathcal{C}_{Reg} \text{ there exists a unique cluster } \{s_1, s_2, s_3\} \text{ such that } v \text{ belongs to the cone } \mathcal{C}_{\{s_1,s_2,s_3\}} := \mathbb{Z}_{\geq 0}\mathbf{d}(s_1) + \mathbb{Z}_{\geq 0}\mathbf{d}(s_2) + \mathbb{Z}_{\geq 0}\mathbf{d}(s_3). \tag{63}$$

This is done in the subsequent sections.

Proof of Eq. 62 By the explicit formulas of denominator vectors of cluster variables given in Lemma 3.2 one checks directly that $|\det(\mathbf{d}(s_1), \mathbf{d}(s_2), \mathbf{d}(s_3))| = 1$ for every cluster $\{s_1, s_2, s_3\}$. □

Proof of Eq. 63 We consider the basis of simple roots $\alpha_1, \alpha_2, \alpha_3$ of Q and the corresponding coordinate system (e_1, e_2, e_3). In Corollary 3.2 it is shown that given a cluster $\mathcal{C} = \{s_1, s_2, s_3\}$ the set of corresponding denominator vectors $\{\mathbf{d}(s_1), \mathbf{d}(s_2), \mathbf{d}(s_3)\}$ is sign-coherent. In particular if the initial cluster variable x_i lies in the cluster $\mathcal{C}$, then the i-th coordinate of the denominator vector of the other two elements of $\mathcal{C}$ is zero. By Lemma 3.2 also the opposite holds: if the i-th coordinate of the denominator vector of a cluster variable is zero, then it lies in a same cluster as x_i. There are hence precisely nine cluster variables with this property and the corresponding denominator vectors are shown in Fig. 3. Notice that the denominator vectors $\{\mathbf{d}(x_4), \mathbf{d}(w), \mathbf{d}(x_0)\}$ form the canonical basis of $\mathbb{Z}^3$ and the positive octant is given by $Q_+ = \mathbb{Z}_{\geq 0}\mathbf{d}(x_4) + \mathbb{Z}_{\geq 0}\mathbf{d}(w) + \mathbb{Z}_{\geq 0}\mathbf{d}(x_0)$. The corresponding cones $\mathcal{C}_{\{s_1,s_2,s_3\}}$ satisfy property (Eq. 63), i.e. they do not overlap. Moreover their union is the whole lattice except the interior of the positive octant Q_+. □

We hence consider the denominator vectors of cluster variables contained in Q_+. We suggest to use Fig. 2 to visualize the situation. By using Lemma 3.2, we notice that there are four affine lines in $Q_{\mathbb{R}}$ which contain the denominator vectors of all the cluster variables different from x_2, w and z. They contain respectively "negative odd", "positive odd", "negative even" and "positive even" cluster variables: they are

$$\ell^-_{\text{odd}} := \begin{cases} e_2 = e_3 \\ e_1 = e_2 - 1 \end{cases}; \quad \ell^+_{\text{odd}} := \begin{cases} e_1 = e_2 \\ e_3 = e_2 - 1 \end{cases};$$
$$\ell^-_{\text{even}} := \begin{cases} e_1 = e_2 \\ e_3 = e_2 + 1 \end{cases}; \quad \ell^+_{\text{even}} := \begin{cases} e_2 = e_3 \\ e_1 = e_2 + 1 \end{cases}.$$

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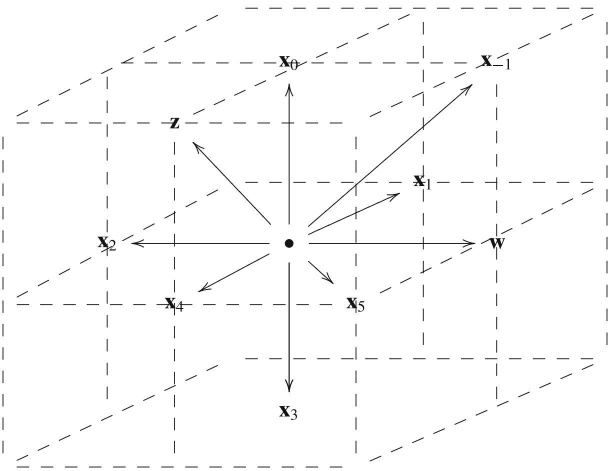

Fig. 3 Denominator vectors of cluster variables having at least one coordinate equal to zero. We wrote x_m for $\mathbf{d}(x_m)$. The clusters involved here form a fan whose union is $Q \setminus Q_+$

We define the two-dimensional subspaces P and T of $Q_{\mathbb{R}}$ containing respectively both ℓ^+_{odd} and ℓ^-_{even} and both ℓ^-_{odd} and ℓ^+_{even} of equation: $P := \{e_1 = e_2\}$ and $T := \{e_2 = e_3\}$.

Let $\mathcal{C}_P$ be the (open) cone inside $P \cap Q_+$ defined by $\mathcal{C}_P := \{0 < e_3 < e_1\} \cup \{0\}$. By Eq. 37, $\mathbf{d}(x_{2n+1}) \in \mathcal{C}_P$ for every $n \geq 2$. The vectors $v_1 := (1, 1, 0)^t = \mathbf{d}(x_5)$ and $v_2 = (0, 0, 1)^t = \mathbf{d}(x_0)$ form a $\mathbb{Z}$-basis of P such that $\mathcal{C}_P$ is contained in $\mathbb{Z}_{\geq 0} v_1 + \mathbb{Z}_{\geq 0}(v_1 + v_2)$. In this basis $\mathbf{d}(x_{2n+1}) = a_{n1} v_1 + a_{n2} v_2$ where $a_{n1} = n - 1$ and $a_{n2} = n - 2$. The sequence a_{n2}/a_{n1} is strictly increasing. It has limit $\lim_{n \to \infty} \frac{a_{n2}}{a_{n1}} = 1$. We conclude that

$$\mathcal{C}_P = \bigcup_{n \geq 2} \mathcal{C}_{\{x_{2n+1}, x_{2n+3}\}} \tag{64}$$

(here and in the sequel we set $\mathcal{C}_{\{s_1, \cdots, s_k\}} := \mathbb{Z}_{\geq 0}\mathbf{d}(s_1) + \cdots + \mathbb{Z}_{\geq 0}\mathbf{d}(s_k)$). Moreover the interior of two different cones in the right hand side are disjoint. In particular we have

$$\bigcup_{n \geq 2} \mathcal{C}_{\{x_{2n+1}, w, x_{2n+3}\}} = \mathbb{Z}_{\geq 0}\mathbf{d}(w) + \mathcal{C}_P$$

and the cones in the left hand side have no common interior points.

Similarly let $\mathcal{C}_T$ be the (open) cone inside $T \cap Q_+$ defined by $\mathcal{C}_T = \{0 < e_2 < e_1\} \cup \{0\}$. By Eq. 37, $\mathbf{d}(x_{2n}) \in \mathcal{C}_T$ for every $n \geq 2$. The vectors $w_1 = (1, 0, 0)^t = \mathbf{d}(x_4)$ and $w_2 = (0, 1, 1)^t = \mathbf{d}(x_{-1})$ form a $\mathbb{Z}$-basis of T such that $\mathcal{C}_T$ is contained in $\mathbb{Z}_{\geq 0} w_1 + \mathbb{Z}_{\geq 0}(w_1 + w_2)$. In this basis $\mathbf{d}(x_{2n}) = b_{n1} w_1 + b_{n2} w_2$ with $b_{n1} = n - 1$ and $b_{n2} = n - 2$. The strictly increasing sequence $\{b_{n2}/b_{n1}\}$ has limit 1 for $n \to \infty$. We conclude that

$$\mathcal{C}_T = \bigcup_{n \geq 2} \mathcal{C}_{\{x_{2n}, x_{2n+2}\}} \tag{65}$$

and the interiors of the cones in the right hand side are mutually disjoint. In particular we have

$$\bigcup_{n \geq 2} \mathcal{C}_{\{x_{2n}, z, x_{2n+2}\}} = \mathbb{Z}_{\geq 0}\mathbf{d}(z) + \mathcal{C}_T.$$

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and the cones in the left hand side have no common interior points.

We now prove that

$$\mathcal{C}_P + \mathcal{C}_T = \bigcup_{m\geq 4} \mathcal{C}_{\{x_m, x_{m+1}, x_{m+2}\}}$$

and that the interiors of two different cones in the right hand side are disjoint. By definition $\mathcal{C}_P + \mathcal{C}_T$ is contained in $\mathbb{Z}\mathbf{d}(u_1) + \mathbb{Z}\mathbf{d}(x_4) + \mathbb{Z}\mathbf{d}(x_5)$ and consists of all integer vectors $v = (l, m, n)^t$ such that $l \geq m \geq n \geq 0$ not all equals (see Fig. 2). The expansion of such vector in the basis $\{\mathbf{d}(u_1), \mathbf{d}(x_4), \mathbf{d}(x_5)\}$ is:

$$v = n\mathbf{d}(u_1) + (l - m)\mathbf{d}(x_4) + (m - n)\mathbf{d}(x_5). \tag{66}$$

We are going to provide an algorithm which gives the precise expression of v as an element of precisely one cone $\mathcal{C}_{\{x_m, x_{m+1}, x_{m+2}\}}$ for some $m \geq 4$. This is done in two steps. The first step provides the explicit expression of an element of $\mathcal{C}_{\{x_4, x_5, x_6, x_7\}}$ (this is the quadrilateral at the bottom of Fig. 2) in one of the two cones $\mathcal{C}_{\{x_4, x_5, x_6\}}$ or $\mathcal{C}_{\{x_5, x_6, x_7\}}$. The second step reduces to the case in which v belongs to the cone $\mathcal{C}_{\{x_4, x_5, x_6, x_7\}}$.

First Step The following lemma gives arithmetic conditions on $v = (l, m, n)^t$ to belong to $\mathcal{C}_{\{x_4, x_5, x_6, x_7\}}$.

Lemma 4.1 *An element $v = (l, m, n)^t \in \mathcal{C}_P + \mathcal{C}_T$ belongs to $\mathcal{C}_{\{x_4, x_5, x_6, x_7\}}$ if and only if $l \geq 2n$. In this case if $n \leq l - m$ then*

$$v = n\mathbf{d}(x_6) + (l - m - n)\mathbf{d}(x_4) + (m - n)\mathbf{d}(x_5) \tag{67}$$

and $v \in \mathcal{C}_{\{x_4, x_5, x_6\}}$. If $n \geq l - m$, then

$$v = (l - 2n)\mathbf{d}(x_5) + (l - m)\mathbf{d}(x_6) + (n - l + m)\mathbf{d}(x_7) \tag{68}$$

and $v \in \mathcal{C}_{\{x_5, x_6, x_7\}}$.

Proof If $l < 2n$ then $l - n = (l - m) + (m - n) < n$ and there exists $k > 0$ such that $n = (l - m) + (m - n) + k$. By Eq. 66 we have

$$v = (k - 1)\mathbf{d}(u_1) + (l - m)\mathbf{d}(x_8) + (m - n)\mathbf{d}(x_7).$$

which, in view of Corollary 3.1, does not belong to $\mathcal{C}_{\{x_4, x_5, x_6, x_7\}}$. On the other hand let us assume $l \geq 2n$. Then $(l - m) + (m - n) > n$ and there are two cases: either $n \leq l - m$ or $n > l - m$. In the first case v has the expansion (Eq. 67) and hence $v \in \mathcal{C}_{\{x_4, x_5, x_6\}}$; in the second case v has the expansion (Eq. 68) and hence $v \in \mathcal{C}_{\{x_5, x_6, x_7\}}$. □

Corollary 4.1 *The interiors of the two cones $\mathcal{C}_{\{x_4, x_5, x_6\}}$ and $\mathcal{C}_{\{x_5, x_6, x_7\}}$ are disjoint*

Proof Let $v = (l, m, n)^t$ be an element of both $\mathcal{C}_{\{x_4, x_5, x_6\}}$ and $\mathcal{C}_{\{x_5, x_6, x_7\}}$. Then, by Lemma 4.1, $l \geq 2n$, $n = l - m$ and $v = n\mathbf{d}(x_6) + (m - n)\mathbf{d}(x_5)$. It hence follows that v does not belong to the interior of the two cones. □

By Lemma 4.1 we notice that an element v of $\mathcal{C}_{\{x_4, x_5, x_6, x_7\}}$ has the form:

$$v = \alpha_1\mathbf{d}(x_4) + \alpha_2\mathbf{d}(x_5) + \alpha_3\mathbf{d}(x_6) + \alpha_4\mathbf{d}(x_7) \tag{69}$$

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for some non negative integers α_1, α_2, α_3 and α_4 such that $\alpha_1\alpha_4 = 0$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = l - n$.

Second Step Let $v = (l, m, n)^t$ not be an element of $\mathcal{C}_{\{x_4,x_5,x_6,x_7\}}$. By Lemma 4.1 this implies that $l - n < n$. We divide n by $l - n$ and we find k and r such that $0 \le r < (l - n)$ and $n = k(l - n) + r$. We consider the vector

$$v' := v - k(l-n)\mathbf{d}(u_1) = (l - k(l-n), m - k(l-n), n - k(l-n))^t.$$

By using Lemma 4.1, one checks easily that v' belongs to $\mathcal{C}_{\{x_4,x_5,x_6,x_7\}}$. Then by the first step v' has the form (Eq. 69). Then we get

$$\begin{aligned} v &= v' + k(l-n)\mathbf{d}(u_1) \\ &= \alpha_1(\mathbf{d}(x_4) + k\mathbf{d}(u_1)) + \alpha_2(\mathbf{d}(x_5) + k\mathbf{d}(u_1)) \\ &\quad + \alpha_3(\mathbf{d}(x_6) + k\mathbf{d}(u_1)) + \alpha_4(\mathbf{d}(x_7) + k\mathbf{d}(u_1)) \\ &= \alpha_1\mathbf{d}(x_{4+2k}) + \alpha_2\mathbf{d}(x_{5+2k}) + \alpha_3\mathbf{d}(x_{6+2k}) + \alpha_4\mathbf{d}(x_{7+2k}) \end{aligned}$$

which is the desired expansion. In particular we have that $v = (l, m, n)^t$ lies in $\mathcal{C}_{\{x_{4+k},x_{5+k},x_{6+k},x_{7+k}\}}$ if and only if $n = k(l - n) + r$ for some $0 \le r < (l - n)$; in this case v belongs to exactly one cone, either $\mathcal{C}_{\{x_{4+k},x_{5+k},x_{6+k}\}}$ or $\mathcal{C}_{\{x_{5+k},x_{6+k},x_{7+k}\}}$.

By now in Fig. 2 we have obtained all the elements of the cone $\mathbb{Z}_{\ge 0}\mathbf{d}(z) + \mathbb{Z}_{\ge 0}\mathbf{d}(w) + \mathbb{Z}_{\ge 0}\mathbf{d}(x_4) = \mathcal{C}_{Reg} + \mathbb{Z}_{\ge 0}\mathbf{d}(x_4)$. We consider the orthogonal reflection r_{Reg} with respect to the regular cone $\mathcal{C}_{Reg}$: this is the $\mathbb{Z}$-linear isomorphism of Q which exchanges the first coordinate with the third one. In particular it fixes $\mathcal{C}_{Reg}$ pointwise. By Remark 3.1, r_{Reg} sends $\mathbf{d}(x_m)$ to $\mathbf{d}(x_{-m+4})$ for $m \ge 4$ and hence induces a bijection between $\mathcal{C}_{Reg} + \mathbb{Z}_{\ge 0}\mathbf{d}(x_4)$ and $\mathcal{C}_{Reg} + \mathbb{Z}_{\ge 0}\mathbf{d}(x_0)$. This concludes the proof of Theorem 2.2.

Remark 4.1 The proof of Theorem 2.2 contains an algorithm to compute the "virtual" canonical decomposition of every element of Q. There are several more effective algorithms than this in much more generality (see [9, 10, 22, 23, 29]).

5 Proof of Theorem 2.3

The proof of Theorem 2.3 is based on Proposition 2.1: we find the explicit expression of the F-polynomial and of the $\mathbf{g}$-vector of every element of $\mathbf{B}$ in every cluster of $\mathcal{A}_{\mathbb{P}}$. Once again it is sufficient to consider only the two clusters $\{x_1, x_2, x_3\}$ and $\{x_1, w, x_3\}$ (see Remark 3.1). The F-polynomials (resp. the $\mathbf{g}$-vectors) in the cluster $\{x_1, x_2, x_3\}$ and $\{x_1, w, x_3\}$ are given respectively in Proposition 5.1 (resp. Proposition 5.2) and Proposition 5.3 (resp. Proposition 5.4).

Remark 5.1 Our first proof of Theorem 2.3 uses the theory of cluster categories by computing explicitly cluster characters [6]. We do not use this approach here. Instead the strategy of our proof uses a parametrization of the elements of $\mathbf{B}$ shown in the next section.

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5.1 Proof of Proposition 2.1

By [17, Corollary 6.3] the expansion of all cluster variables and hence of all cluster monomials in every cluster has the form (Eq. 10). In Section 5.2 it is shown that the F-polynomial of every cluster variable in every cluster has positive integer coefficients. It remains to deal with the u_n's. We prove that for every $n \geq 1$, u_n has the form (Eq. 11) in both the clusters $\{x_1, x_2, x_3\}$ and $\{x_1, w, x_3\}$; by the symmetry of the exchange relations this implies that u_n has the form (Eq. 11) in every cluster of $\mathcal{A}$.

Let $\mathbb{P}$ be a semifield. Let $\mathcal{M}$ be the set of all the elements b of $\mathcal{F}_{\mathbb{P}}$ that can be written in the form

$$b = F_b(\hat{y}_1, \hat{y}_2, \hat{y}_3)\mathbf{x}^{\mathbf{g}_b} \tag{70}$$

where F_b is a polynomial with integer coefficients and

$$\hat{y}_1 := \frac{y_1}{x_2 x_3} = y_1 \mathbf{x}^{\mathbf{h}_1} \ \hat{y}_2 := \frac{y_2 x_1}{x_3} = y_2 \mathbf{x}^{\mathbf{h}_2} \ \hat{y}_3 := y_3 x_1 x_2 = y_3 \mathbf{x}^{\mathbf{h}_3} \tag{71}$$

Both the principal cluster variables W and Z belong to $\mathcal{M}$ and by Eq. 2 their F-polynomials and $\mathbf{g}$-vectors are respectively

$$F_w(\mathbf{y}) = y_2 + 1, \ \mathbf{g}_w = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$F_z(\mathbf{y}) = y_1 y_3 + y_1 + 1 \ \ \mathbf{g}_z = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \tag{72}$$

By Definition 2.1 $u_1 = ZW - y_1 y_3 - y_2$. By inverting the equalities in Eq. 71 we get:

$$u_1 = \left[F_z(\hat{y}_1, \hat{y}_2, \hat{y}_3) F_w(\hat{y}_1, \hat{y}_2, \hat{y}_3) - \hat{y}_1 \hat{y}_3 - \hat{y}_2\right] \frac{x_3}{x_1} \tag{73}$$

and hence u_1 belongs to $\mathcal{M}$. We define:

$$F_{u_1}(y_1, y_2, y_3) := y_1 y_2 y_3 + y_1 y_2 + y_1 + 1 \ \mathbf{g}_{u_1} := \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \tag{74}$$

Since the polynomial F_{u_1} has constant term 1, it is the unique polynomial F such that

$$u_1 = F(\hat{y}_1, \hat{y}_2, \hat{y}_3)\mathbf{x}^{\mathbf{g}_{u_1}}$$

and hence F_{u_1} is the F-polynomial of u_1 (in the seed Σ) mentioned in Proposition 2.1 and $\mathbf{g}_{u_1}$ is the $\mathbf{g}$-vector of u_1 (in the seed Σ). Similarly, by Definition 2.1 and induction on n, we get:

$$u_2 = \left[F_{u_1}(\hat{y}_1, \hat{y}_2, \hat{y}_3)^2 - 2\hat{y}_1 \hat{y}_2 \hat{y}_3\right] \left(\frac{x_3}{x_1}\right)^2$$

$$u_{n+1} = \left[F_{u_1} F_{u_n}(\hat{y}_1, \hat{y}_2, \hat{y}_3) - \hat{y}_1 \hat{y}_2 \hat{y}_3 F_{u_{n-1}}(\hat{y}_1, \hat{y}_2, \hat{y}_3)\right] \left(\frac{x_3}{x_1}\right)^{n+1} \tag{75}$$

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We hence have that u_n belongs to $\mathcal{M}$. We define $\mathbf{g}_{u_n} := (-n, 0, n)^t$. We recursively define the polynomial F_{u_n} by the initial condition (Eq. 74) together with the recurrence relations for $n \geq 2$:

$$F_{u_2}(y_1, y_2, y_3) = F^2_{u_1}(y_1, y_2, y_3) - 2y_1y_2y_3 \tag{76}$$

$$F_{u_{n+1}}(y_1, y_2, y_3) = F_{u_1}F_{u_n}(y_1, y_2, y_3) - y_1y_2y_3F_{u_{n-1}}(y_1, y_2, y_3) \tag{77}$$

By induction on $n \geq 1$, F_{u_n} has constant term 1 and hence it is the F-polynomial of u_n (in the seed Σ) and $\mathbf{g}_{u_n}$ is its $\mathbf{g}$-vector (in the seed Σ). In Proposition 5.1 below we show that the polynomial F_{u_n}, for every $n \geq 1$, has non-negative integer coefficients.

We now prove that, for every $n \geq 1$, u_n has the form (Eq. 11) in the seed $\Sigma^{cyc} = \{H^{cyc}, \{x_1, w, x_3\}, \{\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3\}\}$ defined in Eq. 36. We introduce the elements $\hat{\mathrm{p}}_1$, $\hat{\mathrm{p}}_2$ and $\hat{\mathrm{p}}_3$ of $\mathcal{F}_{\mathbb{P}}$ in analogy with Eq. 71 as follows:

$$\hat{\mathrm{p}}_1 := \frac{\mathrm{p}_1 w}{x_3^2} \quad \hat{\mathrm{p}}_2 := \frac{\mathrm{p}_2 x_3}{x_1} \quad \hat{\mathrm{p}}_3 := \frac{\mathrm{p}_3 x_1^2}{w} \tag{78}$$

We hence prove that for every $n \geq 1$ there exist both a polynomial $F^w_{u_n}(y_1, y_2, y_3)$ with non-negative integer coefficients and constant term 1 and an integer vector $\mathbf{g}^w_{u_n} = (g_1, g_2, g_3)^t \in \mathbb{Z}^3$ such that the expansion of u_n in Σ^{cyc} is given by:

$$u_n = F^w_{u_n}(\hat{\mathrm{p}}_1, \hat{\mathrm{p}}_2, \hat{\mathrm{p}}_3)x_1^{g_1}w^{g_2}x_3^{g_3}.$$

By definition of the coefficient mutation (Eq. 15) in direction 2, the coefficients y_1, y_2 and y_3 of the seed $\Sigma = \mu_2(\Sigma^{cyc})$ in the semifield $\mathbb{P}$, are given by:

$$y_1 = \frac{\mathrm{p}_1\mathrm{p}_2}{\mathrm{p}_2 \oplus 1} \quad y_2 = \frac{1}{\mathrm{p}_2} \quad y_3 = \mathrm{p}_3(\mathrm{p}_2 \oplus 1) \tag{79}$$

The following lemma shows that the elements $\{\hat{\mathrm{p}}_i\}$ are obtained from $\{\hat{y}_i\}$ by the mutation (Eq. 15) in direction 2 (in the terminology of [17] this means that the families $\{\hat{y}_{i;C}\}$ form a Y-pattern):

Lemma 5.1

$$\hat{y}_1 = \frac{\hat{\mathrm{p}}_1\hat{\mathrm{p}}_2}{\hat{\mathrm{p}}_2 + 1} \quad \hat{y}_2 = \frac{1}{\hat{\mathrm{p}}_2} \quad \hat{y}_3 = \hat{\mathrm{p}}_3(\hat{\mathrm{p}}_2 + 1)$$

Proof By definition we have

$$\hat{y}_1 := \frac{y_1}{x_2x_3} \quad \hat{y}_2 = \frac{y_2x_1}{x_3} \quad \hat{y}_3 := y_3x_1x_2 \quad x_2 = \frac{x_1 + \mathrm{p}_2x_3}{(\mathrm{p}_2 \oplus 1)w}$$

and hence the proof follows by direct check. □

In view of Lemma 5.1 the expansion of u_n in Σ^{cyc} is given by:

$$\begin{aligned} u_n &= F_{u_n}(\hat{y}_1, \hat{y}_2, \hat{y}_3)\left(\frac{x_3}{x_1}\right)^n \\ &= F_{u_n}\left(\frac{\hat{\mathrm{p}}_1\hat{\mathrm{p}}_2}{\hat{\mathrm{p}}_2 + 1}, \frac{1}{\hat{\mathrm{p}}_2}, \hat{\mathrm{p}}_3(\hat{\mathrm{p}}_2 + 1)\right)\left(\frac{x_3}{x_1}\right)^n \end{aligned} \tag{80}$$

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Lemma 5.2 *For every* $n \geq 1$, $F_{u_n}\left(\frac{y_1 y_2}{y_2+1}, \frac{1}{y_2}, y_3(y_2+1)\right) \in \mathbb{Z}[y_1, y_2, y_3]$

Proof By Eq. 74 the statement holds for $n = 1$. By Eqs. 76 and 77 an easy induction shows the result. □

Definition 5.1 For every $n \geq 1$ we define

$$F^w_{u_n}(\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3) := F_{u_n}\left(\frac{\mathrm{p}_1\mathrm{p}_2}{\mathrm{p}_2+1}, \frac{1}{\mathrm{p}_2}, \mathrm{p}_3(\mathrm{p}_2+1)\right). \tag{81}$$

We define the vector $\mathbf{g}^w_{u_n} := (-n, 0, n)^t \in \mathbb{Z}^3$

In view of Lemma 5.2 and Definition 5.1, Eq. 80 is the desired expansion. Proposition 5.3 provides the explicit formulas of $F^w_{u_n}$ which is hence a polynomial with non-negative integer coefficients and constant term 1. Up to Propositions 5.1 and 5.3, this concludes the proof of Proposition 2.1.

5.2 F-Polynomials and g-Vectors of the Elements of B

In this section we provide explicit formulas for the F-polynomial and the **g**-vector of every element of **B** in every cluster of $\mathcal{A}$.

Proposition 5.1 *The F-polynomial F_m of a cluster variable x_m ($m \geq 1$) in $\{x_1, x_2, x_3\}$ is: for every $m \geq 0$*

$$F_{2m+1}(\mathbf{y}) = \sum_{\mathbf{e}} \binom{e_1-1}{e_3}\binom{m-1-e_2}{e_1-e_2}\binom{m-1-e_3}{e_2-e_3}\mathbf{y}^{\mathbf{e}} + 1. \tag{82}$$

$$F_{2m+2}(\mathbf{y}) = \sum_{\mathbf{e}} \binom{e_1-1}{e_3}\binom{m-e_2}{e_1-e_2}\binom{m-1-e_3}{e_2-e_3}\mathbf{y}^{\mathbf{e}} + 1. \tag{83}$$

$$F_{-(2m+1)}(\mathbf{y}) = \sum_{\mathbf{e}} \binom{m-e_3}{e_1-e_3}\binom{e_2}{e_3}\binom{e_1+1}{e_2}\mathbf{y}^{\mathbf{e}} + y_1^m y_2^{m+1} y_3^{m+1}. \tag{84}$$

$$F_{-2m}(\mathbf{y}) = \sum_{\mathbf{e}} \binom{m-e_3}{e_1-e_3}\binom{e_2+1}{e_3}\binom{e_1}{e_2}\mathbf{y}^{\mathbf{e}} + y_1^m y_2^m y_3^{m+1}. \tag{85}$$

For every $n \geq 1$ the F-polynomial of u_n is the following:

$$F_{u_n}(\mathbf{y}) = \mathbf{y}^{n\delta} + \sum_{\mathbf{e}} \binom{e_1-e_3}{e_2-e_3}\left[\binom{e_1-1}{e_3}\binom{n-e_3}{n-e_1} + \binom{e_1-1}{e_3-1}\binom{n-e_3-1}{n-e_1}\right]\mathbf{y}^{\mathbf{e}} + 1. \tag{86}$$

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Proof By Eq. 2 the F-polynomial of w and z is respectively $F_w(\mathbf{y}) = y_2 + 1$, and $F_z(\mathbf{y}) = y_1y_3 + y_1 + 1$. By Eqs. 58 and 59, the F-polynomials satisfy the following recurrence relations: for $m \geq 1$

$$F_{2m+2} = F_z F_{2m+1} - y_1 y_3 F_{2m} \tag{87}$$

$$F_{2m+1} = F_w F_{2m} - y_2 F_{2m-1} \tag{88}$$

for which Eqs. 82 and 83 follow by induction on $m \geq 1$. By Eqs. 58 and 59 we have that for every $m \geq 1$

$$F_{-2m} = F_z F_{-(2m-1)} - y_2 F_{-(2m-2)} \tag{89}$$

$$F_{-(2m+1)} = F_w F_{-2m} - y_1 y_3 F_{-(2m-1)} \tag{90}$$

from which Eqs. 84 and 85 follow by induction on $m \geq 1$.

In order to get Eq. 86 we proceed by induction on $n \geq 1$. By direct check one verifies that the right-hand side of Eq. 86 satisfies the initial condition (Eq. 74) together with the recurrence relations (Eqs. 76 and 77). □

Proposition 5.2 *The* $\mathbf{g}$*-vector* $\mathbf{g}_m$ *in* $\{x_1, x_2, x_3\}$ *of a cluster variable* x_m *is given by: for every* $m \geq 0$*:*

$$\mathbf{g}_{2m+1} = \begin{bmatrix} 1-m \\ 0 \\ m \end{bmatrix} \quad \mathbf{g}_{2m+2} = \begin{bmatrix} -m \\ 1 \\ m \end{bmatrix} \tag{91}$$

$$\mathbf{g}_{-(2m+1)} = \begin{bmatrix} -m \\ -1 \\ m \end{bmatrix} \quad \mathbf{g}_{-2m} = \begin{bmatrix} -m \\ 0 \\ m-1 \end{bmatrix} \tag{92}$$

For every $n \geq 1$ *the* $\mathbf{g}$*-vector of* u_n *is the following*

$$\mathbf{g}_{u_n} = \begin{bmatrix} -n \\ 0 \\ n \end{bmatrix} \tag{93}$$

Proof By the exchange relation (Eq. 57) the family $\{\mathbf{g}_m : m \geq 1\} \subset \mathbb{Z}^3$ satisfies the initial conditions $\mathbf{g}_i = \mathbf{e}_i$, for $i = 1, 2, 3$, together with the recurrence relations: for $m \geq 1$

$$\mathbf{g}_{m+3} + \mathbf{g}_m = \mathbf{g}_{m+1} + \mathbf{g}_{m+2} \tag{94}$$

The proof is hence by induction on $m \geq 1$ and $m \leq 1$. The equality (Eq. 93) follows from Eq. 75. □

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Proposition 5.3 *For every $m \in \mathbb{Z}$ the F-polynomial F_m^w in $\{x_1, w, x_3\}$ of a cluster variable x_m is given by: for $m \geq 0$*

$$F_{2m+1}^w(\mathbf{y}) = \sum_{e_1,e_3} \binom{e_1-1}{e_3}\binom{m-1-e_3}{e_1-e_3} y_1^{e_1} y_3^{e_3} + 1; \tag{95}$$

$$F_{2m+2}^w(\mathbf{y}) = \sum_{\mathbf{e}} \binom{e_1-1}{e_3}\binom{m-1-e_3+e_2}{e_1-e_3}\binom{1}{e_2} \mathbf{y}^{\mathbf{e}} + y_2 + 1. \tag{96}$$

$$F_{-(2m+1)}^w(\mathbf{y}) = \sum_{e_1,e_3} \binom{m-e_3}{e_1-e_3}\binom{e_1+1}{e_3} y_1^{e_1} y_3^{e_3} + y_1^m y_3^{m+1}; \tag{97}$$

$$F_{-2m}^w(\mathbf{y}) = \sum_{\mathbf{e}} \binom{m-e_3}{e_1-e_3}\binom{e_1+1-e_2}{e_3-e_2}\binom{1}{e_2} \mathbf{y}^{\mathbf{e}} + y_1^m y_3^{m+1}(y_2+1). \tag{98}$$

$$F_z^w(\mathbf{y}) = y_1 y_2^2 y_3 + y_1 y_2 y_3 + y_1 y_2 + y_2 + 1. \tag{99}$$

For every $n \geq 1$:

$$F_{u_n}^w(\mathbf{y}) = y_1^n y_3^n + \sum_{e_1,e_3} \left[\binom{n-e_3}{n-e_1}\binom{e_1-1}{e_3} + \binom{n-e_3-1}{n-e_1}\binom{e_1-1}{e_3-1}\right] y_1^{e_1} y_3^{e_3} + 1. \tag{100}$$

Proof Let $\mathbb{P} = \mathrm{Trop}(\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3)$ and let $\mathcal{A}_\bullet(\Sigma^{cyc})$ be the cluster algebra with principal coefficients at the seed $\Sigma^{cyc} = \{H^{cyc}, \{x_1, w, x_3\}, \{\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3\}\}$ defined in Eq. 36. In view of Proposition 2.1 and Lemma 5.1 the expansion of a cluster variable x in this seed is given by:

$$\begin{aligned} x &= \frac{F_x(\hat{y}_1, \hat{y}_2, \hat{y}_3)}{F_x|_{\mathbb{P}}(y_1, y_2, y_3)} x_1^{g_1} x_2^{g_2} x_3^{g_3} \\ &= \frac{F_x\left(\frac{\hat{\mathrm{p}}_1\hat{\mathrm{p}}_2}{\hat{\mathrm{p}}_2+1}, \frac{1}{\hat{\mathrm{p}}_2}, \hat{\mathrm{p}}_3(\hat{\mathrm{p}}_2+1)\right)}{F_x|_{\mathbb{P}}\left(\mathrm{p}_1\mathrm{p}_2, \frac{1}{\mathrm{p}_2}, \mathrm{p}_3\right)} x_1^{g_1} \left(\frac{x_1+\mathrm{p}_2 x_3}{w}\right)^{g_2} x_3^{g_3} \end{aligned} \tag{101}$$

where $\mathbf{g}_x := (g_1, g_2, g_3)^t$ is the $\mathbf{g}$-vector of x in the cluster $\{x_1, x_2, x_3\}$. In this expression we replace $x_1 = w = x_3 = 1$ and we get the relation:

$$F_x^w(\mathrm{p}) = \frac{F_x\left(\frac{\mathrm{p}_1\mathrm{p}_2}{1+\mathrm{p}_2}, \frac{1}{\mathrm{p}_2}, \mathrm{p}_3(1+\mathrm{p}_2)\right)}{F_x|_{\mathbb{P}}(\mathrm{p}_1\mathrm{p}_2, \frac{1}{\mathrm{p}_2}, \mathrm{p}_3)} \cdot (1+\mathrm{p}_2)^{g_2} \tag{102}$$

By direct check, using Proposition 5.1, we get that

$$F_x|_{\mathbb{P}}\left(\mathrm{p}_1\mathrm{p}_2, \frac{1}{\mathrm{p}_2}, \mathrm{p}_3\right) = \begin{cases} \dfrac{1}{\mathrm{p}_2} & \text{if } x = x_{-(2m+1)}\ m \geq 0, \text{ or } x = w \\ 1 & \text{otherwise.} \end{cases} \tag{103}$$

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By the explicit formulas for the **g**-vectors given in Proposition 5.2 and in Eq. 72 we hence have that for every cluster variable x the F-polynomials F_x and F_x^w are related by the following formula:

$$F_x^w(\mathrm{p}) = \begin{cases} F_x\left(\dfrac{\mathrm{p}_1\mathrm{p}_2}{1+\mathrm{p}_2}, \dfrac{1}{\mathrm{p}_2}, \mathrm{p}_3(1+\mathrm{p}_2)\right) \cdot \dfrac{\mathrm{p}_2}{1+\mathrm{p}_2} & \text{if } x = x_{-(2m+1)} \\ & \text{or } x = w, \\ F_x\left(\dfrac{\mathrm{p}_1\mathrm{p}_2}{1+\mathrm{p}_2}, \dfrac{1}{\mathrm{p}_2}, \mathrm{p}_3(1+\mathrm{p}_2)\right) \cdot (1+\mathrm{p}_2)^{\mathbf{g}_2} & \text{otherwise.} \end{cases} \tag{104}$$

The proof of Eqs. 95–99 now follows from Proposition 5.1 by direct check.

Equation 100 follows from Eq. 86 by using Eq. 81. □

Corollary 5.1 *For every element b of* **B**, *F_b^w has constant term* 1.

Remark 5.2 We notice that

$$F_z^w|_{\mathrm{Trop}(\mathrm{p}_1,\mathrm{p}_2,\mathrm{p}_3)}\left(\frac{1}{\mathrm{p}_1}, \frac{1}{\mathrm{p}_2}, \frac{1}{\mathrm{p}_3}\right) = \frac{1}{\mathrm{p}_1\mathrm{p}_2^2\mathrm{p}_3}.$$

In [17, Conjecture 7.17] it was expected the right-hand side to be $\mathrm{p}^{-\mathbf{d}^w(z)} = \frac{1}{\mathrm{p}_1\mathrm{p}_2\mathrm{p}_3}$. This counterexample appears also in [3] and in [18].

Proposition 5.4 *For every $m \in \mathbb{Z}$ the* **g**-*vector $\mathbf{g}_m^w$ of a cluster variable x_m in the cluster $\{x_1, w, x_3\}$ is the following: for every $m \geq 0$*

$$\mathbf{g}_{2m+1}^w = \begin{bmatrix} 1-m \\ 0 \\ m \end{bmatrix} \quad \mathbf{g}_{2m+2}^w = \begin{bmatrix} 1-m \\ -1 \\ m \end{bmatrix} \tag{105}$$

$$\mathbf{g}_{-(2m+1)}^w = \begin{bmatrix} -m \\ 1 \\ m-1 \end{bmatrix} \quad \mathbf{g}_{-2m}^w = \begin{bmatrix} -m \\ 0 \\ m-1 \end{bmatrix} \tag{106}$$

For every $n \geq 1$

$$\mathbf{g}_w^w = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{g}_z^w = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{g}_{u_n}^w = \begin{bmatrix} -n \\ 0 \\ n \end{bmatrix} \tag{107}$$

Proof Let x be a cluster variable. The following lemma shows a formula which relates the **g**-vector of x in $\{x_1, x_2, x_3\}$ and in $\{x_1, w, x_3\}$.

Lemma 5.3 *For every $b \in$* **B** *the* **g**-*vector $\mathbf{g}_b = (g_1, g_2, g_3)^t$ of b in $\{x_1, x_2, x_3\}$ and the* **g**-*vector $\mathbf{g}_b^w = (g_1^w, g_2^w, g_3^w)^t$ of b in $\{x_1, w, x_3\}$ are related by the following formula*

$$g_1^w = g_1 + g_2 - \min(g_2, 0), \quad g_2^w = -g_2, \quad g_3^w = g_3 + \min(g_2, 0) \tag{108}$$

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Proof of Lemma 5.3 Let $\mathbb{P} = \mathrm{Trop}(\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3)$ and let $\mathcal{A}_\bullet(\Sigma^{cyc})$ be the cluster algebra with principal coefficients at the seed Σ^{cyc} defined in Eq. 36. Let x be a cluster variable of $\mathcal{A}_\bullet(\Sigma^{cyc})$. We expand x in $\{x_1, w, x_3\}$ as in both Eqs. 101 and 11 and we find the equality

$$\frac{F_x\left(\frac{\hat{\mathrm{p}}_1\hat{\mathrm{p}}_2}{\hat{\mathrm{p}}_2+1}, \frac{1}{\hat{\mathrm{p}}_2}, \hat{\mathrm{p}}_3(\hat{\mathrm{p}}_2+1)\right)}{F_x|_{\mathbb{P}}\left(\mathrm{p}_1\mathrm{p}_2, \frac{1}{\mathrm{p}_2}, \mathrm{p}_3\right)} x_1^{g_1}\left(\frac{x_1+\mathrm{p}_2x_3}{w}\right)^{g_2} x_3^{g_3} = F_x^w(\hat{\mathrm{p}}_1, \hat{\mathrm{p}}_2, \hat{\mathrm{p}}_2) x_1^{g_1^w} w^{g_2^w} x_3^{g_3^w}.$$

By Eqs. 103 and 104 we get:

$$\begin{cases} \mathrm{p}_2 x_1^{g_1}\left(\frac{x_1+\mathrm{p}_2x_3}{w}\right)^{g_2} x_3^{g_3} = \frac{\hat{\mathrm{p}}_2}{1+\hat{\mathrm{p}}_2} x_1^{g_1^w} w^{g_2^w} x_3^{g_3^w} & \text{if } x = x_{-(2m+1)} \text{ or } w \\ x_1^{g_1}\left(\frac{x_1+\mathrm{p}_2x_3}{w}\right)^{g_2} x_3^{g_3} = (1+\hat{\mathrm{p}}_2)^{g_2} x_1^{g_1^w} w^{g_2^w} x_3^{g_3^w} & \text{otherwise} \end{cases}$$

from which Eq. 108 follows by using Proposition 5.4. Let now $b = s_1^a s_2^b s_3^c$ be a cluster monomial. In Proposition 5.4 we notice that the second entry of the **g**-vector of both s_1, s_2 and s_3 have the same sign and hence the transformation (Eq. 108) is linear. Then $\mathbf{g}_b^w$ is given by Eq. 108. The same argument works if $b = u_n w^k$ or $b = u_n z^k$. □

The proof of Proposition 5.4 follows from Proposition 5.2 by Lemma 5.3. □

Formula 108 between **g**-vectors of cluster monomials in two adjacent clusters was conjectured in [17] and proved in [18] in much more generality.

5.3 Proof of Proposition 2.2

We denote by

$$f := \begin{pmatrix} -1 & 0 & 0 \\ [?] & -1 & 0 \\ [?] & [?] & -1 \end{pmatrix}$$

the map $f : Q \to Q$ which acts on Q as follows

$$f \cdot (a, b, c)^t := (-a, -b + [a]_+, -c + [a]_+ + [b]_+)$$

where $[b]_+ := \max(b, 0)$. Proposition 2.2 says that for every element b of $\mathbf{B}$ the corresponding **g**-vector $\mathbf{g}_b$ and denominator vector $\mathbf{d}(b)$ are related by

$$\mathbf{g}_b = f \cdot \mathbf{d}(b) \tag{109}$$

We hence prove Eq. 109. By the explicit formulas for the **g**-vectors given in Proposition 5.2 and from the explicit formulas for the denominator vectors given in Eq. 37, formula 109 holds for cluster variables and for the u_n's.

By Corollary 3.2 denominator vectors of cluster variables belonging to the same cluster are sign-coherent. It is clear that if v_1 and v_2 are sign-coherent then $f \cdot (v_1 + v_2) = f \cdot v_1 + f \cdot v_2$. Moreover f is injective and hence Eq. 109 holds for cluster

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monomials. By Lemma 3.3 denominator vectors of the u_n 's, w and z lie in the positive octant Q_+ in which f is linear. The claim is hence true for $u_n w^k$ and $u_n z^k, n, k \geq 1$. □

5.4 Proof of Proposition 2.3

By Proposition 2.2 the map $\mathbf{d}(b) \mapsto \mathbf{g}_b$ is bijective. By Theorem 2.2 the map $b \mapsto \mathbf{d}(b)$ is a bijection between $\mathbf{B}$ and Q. Then the composition $b \mapsto \mathbf{d}(b) \mapsto \mathbf{g}_b$ is a bijection between $\mathbf{B}$ and Q.

The map $\mathbf{g}_b \mapsto \mathbf{g}_b^w$ given by Eq. 108 is bijective and hence the map $b \mapsto \mathbf{g}_b^w$ is bijective.

6 Proof of Theorem 2.1

Let $\mathbb{P}$ be a tropical semifield. In this section we prove that the set $\mathbf{B}$ of cluster monomials and of the elements $\{u_n w^k, u_n z^k | n \geq 1, k \geq 0\}$ of the cluster algebra $\mathcal{A}_{\mathbb{P}}$ has the following properties:

- $\mathbf{B}$ is a linearly independent set over $\mathbb{ZP}$ (Section 6.1);
- the elements of $\mathbf{B}$ are positive (Section 6.2);
- $\mathbf{B}$ spans $\mathcal{A}_{\mathbb{P}}$ over $\mathbb{ZP}$ (Section 6.3);
- the elements of $\mathbf{B}$ are positive indecomposable (Section 6.4).

and hence $\mathbf{B}$ is an atomic basis of $\mathcal{A}_{\mathbb{P}}$.

6.1 Linear Independence of B

Let $\mathbb{P}$ be an arbitrary semifield and let $\mathcal{A}_{\mathbb{P}}$ be the cluster algebra with initial seed Σ given by Eq. 4. In view of Proposition 2.1 the expansion of an element b of $\mathbf{B}$ in the seed Σ has the form:

$$b = \frac{F_b(y_1\mathbf{x}^{\mathbf{h}_1}, y_2\mathbf{x}^{\mathbf{h}_2}, y_3\mathbf{x}^{\mathbf{h}_3})\mathbf{x}^{\mathbf{g}_b}}{F_b|_{\mathbb{P}}(y_1, y_2, y_3)} \tag{110}$$

where F_b and $\mathbf{g}_b$ are respectively the polynomial and the vector given in Section 5.1, $\mathbf{h}_i$ is the i-th column vector of the exchange matrix H of the seed Σ. Moreover the polynomial F_b has the form: $F_b(y_1, y_2, y_3) = 1 + \sum_{\mathbf{e}} \chi_{\mathbf{e}}(b)\mathbf{y}^{\mathbf{e}}$ where the sum is over a finite set of non-negative integer vectors and the coefficients $\{\chi_{\mathbf{e}}(b)\}$ are positive integer numbers. We denote by $E(b) = \left\{\mathbf{e} \in \mathbb{Z}^3_{\geq 0} \setminus \{0\} | \chi_{\mathbf{e}}(b) \neq 0\right\}$ the support of F_b. The expansion of b has hence the form:

$$b = \frac{\mathbf{x}^{\mathbf{g}_b} + \sum_{\mathbf{e}=(e_1,e_2,e_3)} \chi_{\mathbf{e}}(b)\mathbf{y}^{\mathbf{e}}\mathbf{x}^{\mathbf{g}_b + \sum_{i=1}^3 e_i\mathbf{h}_i}}{1 \oplus \bigoplus_{\mathbf{e}} \chi_{\mathbf{e}}(b)\mathbf{y}^{\mathbf{e}}} \tag{111}$$

We say that a monomial $\mathbf{x}^{\mathbf{c}}$ is a *summand* of an element b of $\mathbf{B}$ in the cluster $\{x_1, x_2, x_3\}$ if it appears with non zero coefficients in the expansion (Eq. 111) of b in the cluster $\{x_1, x_2, x_3\}$.

We introduce in $\mathbb{Z}^3$ the following binary relation: given $\mathbf{s}, \mathbf{t} \in \mathbb{Z}^3$ we say that $\mathbf{s} \leq_H \mathbf{t}$ if and only if there exist non-negative integers e_1, e_2, e_3 such that $\mathbf{t} = \mathbf{s} + \sum_{i=1}^3 e_i\mathbf{h}_i$. The vectors $\mathbf{h}_1, \mathbf{h}_2$ and $\mathbf{h}_3$ form a pointed cone, i.e. a non-negative linear combination of them is zero if and only if all the coefficients are zero. The relation $\leq_H$ is hence

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a partial order on $\mathbb{Z}^3$. The map $b \mapsto \mathbf{g}_b$ between $\mathbf{B}$ and $\mathbb{Z}^3$ is injective (actually bijective) by Proposition 2.3 and hence the partial order $\leq_H$ induces a partial order on $\mathbf{B}$ given by:

$$b \leq b' \Longleftrightarrow \mathbf{g}_b \leq_H \mathbf{g}_{b'}. \tag{112}$$

In particular every finite subset $\mathbf{B}'$ of $\mathbf{B}$ has a minimal object b_0. Then the monomial $\mathbf{x}^{\mathbf{g}_{b_0}}$ is not a summand of any other element of $\mathbf{B}'$. We conclude that $\mathbf{B}$ is a linearly independent set over $\mathbb{ZP}$.

Remark 6.1 Linear independence of cluster monomials for cluster algebras with an "acyclic" seed was proved in [4] and [19] and recently in [26].

6.2 Positivity of the Elements of B

In this section we show that the elements of the set $\mathbf{B}$ defined in Theorem 2.1 are positive, i.e. their Laurent expansion in every cluster of $\mathcal{A}_{\mathbb{P}}$ has coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$. In view of Remark 3.1 it is sufficient to show that they have such property only in the two clusters $\{x_1, x_2, x_3\}$ and $\{x_1, w, x_3\}$. By Theorem 2.3 and by the symmetries of the exchange relations, the elements w, z , u_n and x_m with $m \in \mathbb{Z}$ and $n \geq 1$ have such property. Since a product of positive elements is positive, we conclude that the cluster monomials and the elements $\{u_n w^k, u_n z^k : n, k \geq 1\}$ are positive.

6.3 The Set B spans $\mathcal{A}_{\mathbb{P}}$ Over $\mathbb{ZP}$

In this section we show the set $\mathbf{B}$ defined in Theorem 2.1 spans $\mathcal{A}_{\mathbb{P}}$ over $\mathbb{ZP}$. The strategy of the proof is the following: since $\mathbf{B}$ contains cluster variables, the monomials in its elements span $\mathcal{A}_{\mathbb{P}}$ over $\mathbb{ZP}$. It is then sufficient to express every such monomial as a $\mathbb{ZP}$-linear combination of elements of $\mathbf{B}$. Following [30] we write a monomial M as $M = u_{n_1}^{a_1} \cdots u_{n_s}^{a_s} x_{m_1}^{b_1} \cdots x_{m_t}^{b_t} w^c z^d$ where $0 < n_1 < \cdots < n_s, m_1 < \cdots < m_t$ and the exponents are positive integers, and we define the multi-degree $\mu(M) = (\mu_1(M), \mu_2(M), \mu_3(M)) \in \mathbb{Z}^3_{\geq 0}$ by setting

$$\begin{cases} \mu_1(M) := \sum_{i=1}^s a_i + \sum_{j=1}^t b_j + c + d \\ \mu_2(M) := m_t - m_1 \\ \mu_3(M) := b_1 + b_t \end{cases} \tag{113}$$

The lexicographic order of $\mathbb{Z}^3$ makes it into a well ordered set (i.e., every non-empty subset of $\mathbb{Z}^3_{\geq 0}$ has a smallest element). In Section 6.3.2 we show that M can be expressed as a linear combination of monomials of (lexicographically) smaller multi-degree. In Section 6.3.1 we find the minimal monomials which do not belong to $\mathbf{B}$ and express them as linear combinations of elements of $\mathbf{B}$. We refer to such expressions as straightening relations.

6.3.1 Straightening Relations

In this section we express the monomials in the elements of $\mathbf{B}$ which do not belong to $\mathbf{B}$ and are minimal with respect to the multi-degree Eq. 113 as $\mathbb{ZP}$-linear combinations of elements of $\mathbf{B}$. We notice that the multi-degree Eq. 113 does not depend on coefficients, and hence a cluster variable s and the relative principal cluster

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variable $S := F_s|_{\mathbb{P}}(y_1, y_2, y_3)s$ in the initial seed Σ have the same multi-degree. It is convenient to consider principal cluster variables in the initial seed Σ since the exchange relations are simpler and are given by Eqs. 56, 57, 58, 59, 60 and 61.

Such minimal monomials are the following:

$$u_n u_p; \quad u_n X_m; \quad X_m X_{m+2+n}; \quad X_{2m} W; \quad X_{2m+1} Z; \quad ZW \tag{114}$$

for every $n, p \geq 1$ and $m \in \mathbb{Z}$. Indeed every monomial M in Eq. 114 satisfies $\mu_1(M) = 2$ and hence they are minimal (it follows from the definition that $\mu_1(M) = 1$ if and only if M is either a cluster variable or one of the u_n's). Moreover they are the only monomials not belonging to **B** with this property.

The straightening relations for the monomials $X_{2m}W$, $X_{2m+1}Z$, ZW are given respectively by Eqs. 58, 59 and 6. Propositions 6.1 and 6.2 give the remaining ones.

Proposition 6.1 *For every $n, p \geq 1$:*

$$u_n u_p = \begin{cases} u_{n+p} + \mathbf{y}^{p\delta} u_{n-p} & \text{if } n > p \\ u_{2n} + 2\mathbf{y}^{n\delta} & \text{if } n = p \end{cases} \tag{115}$$

where $\delta := (1, 1, 1)^t$.

Proof We use the definition of the u_n's given in Eq. 7. For simplicity we assume now that $u_0 := 2$, so that the relation $u_1 u_n = u_{n+1} + \mathbf{y}^{\delta} u_{n-1}$ holds for every $n \geq 1$ (instead of holding only for $n \geq 2$). Moreover, with this convention, we have to prove that for every $p : 1 \leq p \leq n$ we have:

$$u_n u_p = u_{n+p} + \mathbf{y}^{p\delta} u_{n-p} \tag{116}$$

If $n = p = 1$ then Eq. 116 is the definition (Eq. 6) of u_2; we assume $n \geq 2$ and we proceed by induction on $p \geq 1$: if $p = 1$, then Eq. 116 is just the definition (Eq. 7) of u_{n+1}. We then assume $2 \leq p + 1 \leq n$ and we get:

$$\begin{aligned} u_n u_{p+1} &= u_n[u_1 u_p - \mathbf{y}^{\delta} u_{p-1}] \\ &= u_1[u_{n+p} + \mathbf{y}^{p\delta} u_{n-p}] - \mathbf{y}^{\delta}[u_{n+p-1} + \mathbf{y}^{(p-1)\delta} u_{n-p+1}] \\ &= u_{n+1+p} + \mathbf{y}^{\delta} u_{n+p-1} + \mathbf{y}^{p\delta}[u_{n+1-p} + \mathbf{y}^{\delta} u_{n-p-1}] - \mathbf{y}^{\delta}[u_{n+p-1} + \mathbf{y}^{(p-1)\delta} u_{n-p+1}] \\ &= u_{n+p+1} + \mathbf{y}^{(p+1)\delta} u_{n-(p+1)} \end{aligned}$$

□

In order to give the remaining straightening relations we need to introduce the following coefficients. We use the notation:

$$\mathbf{y}^{\mathbf{e}} \overset{\min}{\oplus} \mathbf{y}^{\mathbf{d}} := y_1^{\min(e_1,d_1)} y_2^{\min(e_2,d_2)} y_3^{\min(e_3,d_3)}$$

Definition 6.1 For every $m \in \mathbb{Z}$ we define

$$\xi_m := \begin{cases} \mathbf{y}^{\mathbf{d}(x_{m+3})} = y_{1;m} & \text{if } m \geq 1 \\ \mathbf{y}^{\mathbf{d}(x_m)} = y_{1;m-3} & \text{if } m \leq 0 \end{cases} \tag{117}$$

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and also

$$\zeta_n^-(m) = \begin{cases} \xi_m \overset{\min}{\oplus} \mathbf{y}^{n\delta} & \text{if } m \geq 1 \\ 1 & \text{if } m \leq 0 \end{cases} \qquad \zeta_n^+(m) = \begin{cases} 1 & \text{if } m \geq 1 \\ \xi_m \overset{\min}{\oplus} \mathbf{y}^{n\delta} & \text{if } m \leq 0. \end{cases} \tag{118}$$

For every integer $k \geq 0$ we define

$$\gamma_1(k) = y_1^{\lceil \frac{k}{2} \rceil} y_2^{\lfloor \frac{k}{2} \rfloor} y_3^{\lceil \frac{k}{2} \rceil}; \quad \gamma_2(k) = y_1^{\lfloor \frac{k}{2} \rfloor} y_2^{\lceil \frac{k}{2} \rceil} y_3^{\lfloor \frac{k}{2} \rfloor}; \quad \gamma_3(k) = \begin{cases} \mathbf{y}^{\frac{k}{2}\delta} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases} \tag{119}$$

where $\lceil t \rceil$ (resp. $\lfloor t \rfloor$) denotes the smallest (resp. largest) integer larger (resp. smaller) than t. For $i = 1, 2, 3$ we define the following elements of $\mathcal{A}_{\mathbb{P}}$:

$$\Gamma_i(n) = \sum_{k \geq 0} \left(\left\lfloor \frac{k}{2} \right\rfloor + 1 \right) \cdot \gamma_i(k) \cdot u_{n-k}.$$

We also define for every $m \in \mathbb{Z}$ and $m_1 \geq 0$:

$$\eta^-_{m;m_1} := \begin{cases} \xi_m \overset{\min}{\oplus} \xi_{m+m_1} & \text{if } m \leq 0 < m + m_1 \\ 1 & \text{otherwise} \end{cases}$$

and

$$\eta^+_{m;m_1} := \begin{cases} 1 & \text{if } m \leq 0 < m + m_1 \\ \xi_m \overset{\min}{\oplus} \xi_{m+m_1} & \text{otherwise} \end{cases}$$

Proposition 6.2 *With the notations of Definition* 6.1 *the following equalities hold.*

(i) *: For every $m \in \mathbb{Z}$ and $n \geq 1$:*

$$u_n X_m = \zeta_n^-(m) X_{m-2n} + \zeta_n^+(m) X_{m+2n} \tag{120}$$

(ii) *: For every $m \in \mathbb{Z}$ even and $n \geq 0$:*

$$X_m X_{m+2n+3} = \eta^-_{m;2n+3} X_{m+n+1} X_{m+n+2} + \eta^+_{m;2n+3} \Gamma_1(n) \tag{121}$$

(iii) *: For every $m \in \mathbb{Z}$ odd and $n \geq 0$:*

$$X_m X_{m+2n+3} = \eta^-_{m;2n+3} X_{m+n+1} X_{m+n+2} + \eta^+_{m;2n+3} \Gamma_2(n) \tag{122}$$

(iv) *: For every $m \in \mathbb{Z}$ even and $n \geq 2$:*

$$X_m X_{m+2n} = \eta^-_{m;2n} X_{m+2\lfloor \frac{n}{2} \rfloor} X_{m+2\lceil \frac{n}{2} \rceil} + \eta^+_{m;2n} \Gamma_3(n-2) Z \tag{123}$$

(v) *: For every $m \in \mathbb{Z}$ odd and $n \geq 2$:*

$$X_m X_{m+2n} = \eta^-_{m;2n} X_{m+2\lfloor \frac{n}{2} \rfloor} X_{m+2\lceil \frac{n}{2} \rceil} + \eta^+_{m;2n} \Gamma_3(n-2) W \tag{124}$$

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Proof We prove part (i) by induction on $n \geq 1$. By using relations (Eqs. 58 and 59) it is easy to verify that for every $m \in \mathbb{Z}$ we have:

$$u_1 X_m = \begin{cases} \mathbf{y}^\delta X_{m-2} + X_{m+2} & \text{if } m \geq 2 \\ y_1 X_{-1} + X_3 & \text{if } m = 1 \\ X_{-2} + y_3 X_2 & \text{if } m = 0 \\ X_{-3} + y_2 y_3 X_1 & \text{if } m = -1 \\ X_{m-2} + \mathbf{y}^\delta X_{m+2} & \text{if } m \leq -2 \end{cases}$$

and Eq. 120 holds for $n = 1$. We now proceed by induction on $n \geq 1$. We use the convention that $u_0 = 2$ so that the relation $u_{n+1} = u_1 u_n - \mathbf{y}^\delta u_{n-1}$ (given in Definition 2.1) holds for every $n \geq 1$. Moreover, with this convention, since $\zeta_0^{\pm}(m) = 1$, Eq. 120 still holds for n = 0. We have

$$\begin{aligned} u_{n+1} X_m &= u_1 u_n X_m - \mathbf{y}^\delta u_{n-1} X_m \\ &= u_n\left[\zeta_1^-(m) X_{m-2} + \zeta_1^+(m) X_{m+2}\right] \\ &\quad + -\mathbf{y}^\delta\left[\zeta_{n-1}^-(m) X_{m-2n+2} + \zeta_{n-1}^+(m) X_{m+2n-2}\right] \\ &= \zeta_1^-(m)\left[\zeta_n^-(m-2) X_{m-2-2n} + \zeta_n^+(m-2) X_{m-2+2n}\right] \\ &\quad + \zeta_1^+(m)\left[\zeta_n^-(m+2) X_{m+2-2n} + \zeta_n^+(m+2) X_{m+2+2n}\right] \\ &\quad + -\mathbf{y}^\delta \zeta_{n-1}^-(m) X_{m+2-2n} - \mathbf{y}^\delta \zeta_{n-1}^+(m) X_{m-2+2n} \\ &= X_{m-2-2n}\left[\zeta_1^-(m)\zeta_n^-(m-2)\right] + X_{m-2+2n}\left[\zeta_1^-(m)\zeta_n^+(m-2) - \mathbf{y}^\delta \zeta_{n-1}^+(m)\right] \\ &\quad + X_{m+2-2n}\left[\zeta_1^+(m)\zeta_n^-(m+2) - \mathbf{y}^\delta \zeta_{n-1}^-(m)\right] + X_{m+2+2n}\left[\zeta_1^+(m)\zeta_n^+(m+2)\right] \end{aligned}$$

The claim follows by Lemma 6.1 below.

Lemma 6.1 *For every $m \in \mathbb{Z}$ and $n \geq 1$ the following equalities hold:*

(1) $\zeta_1^-(m)\zeta_n^-(m-2) = \zeta_{n+1}^-(m)$;
(2) $\zeta_1^-(m)\zeta_n^+(m-2) - \mathbf{y}^\delta \zeta_{n-1}^+(m) = 0$;
(3) $\zeta_1^+(m)\zeta_n^-(m+2) - \mathbf{y}^\delta \zeta_{n-1}^-(m) = 0$;
(4) $\zeta_1^+(m)\zeta_n^+(m+2) = \zeta_{n+1}^+(m)$.

Proof The proof of Lemma 6.1 is by direct check. □

We prove parts (ii) and (iii) together. It is convenient to prove that the following relation holds for every $m \in \mathbb{Z}$, $n \geq 0$ and $i = i_m = 1$ if m is even and 2 if m is odd:

$$\eta^+_{m;2n+3}\Gamma_i(n) = X_m X_{m+2n+3} - \eta^-_{m;2n+3} X_{m+n+1} X_{m+n+2} \qquad (125)$$

We proceed by induction on $n \geq 0$. We first prove Eq. 125 for $n = 0$. In this case $\Gamma_1(0) = \Gamma_2(0) = 1$. By the exchange relations (Eqs. 56 and 57) we know that for every $m \in \mathbb{Z}$ the following relation holds:

$$X_m X_{m+3} = \begin{cases} y_{m+3} X_{m+1} X_{m+2} + 1 & \text{if } m = 0, -1, -2; \\ X_{m+1} X_{m+2} + y_{1;m} & \text{otherwise.} \end{cases} \qquad (126)$$

We need the following lemma.

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Lemma 6.2 *With notations of Definition* 6.1 *we have the following:*

(1) *For every* $m \in \mathbb{Z}$ *and* $k \geq 0$*:*

$$\xi_m \overset{\min}{\oplus} \xi_{m+k} = \begin{cases} \xi_m & \text{if } m \geq 1, \\ \xi_{m+k} & \text{if } m+k \leq 0. \end{cases} \tag{127}$$

$$\xi_m \overset{\min}{\oplus} \mathbf{y}^\delta = \begin{cases} \xi_m & \text{if } -1 \leq m \leq 2, \\ \mathbf{y}^\delta & \text{otherwise.} \end{cases} \tag{128}$$

If $m \geq 0$ *and* $n \geq 1$ *we get*

$$\xi_{-m} \overset{\min}{\oplus} \xi_n = \begin{cases} \xi_{-m} & \text{if } m < n-1, \\ \mathbf{y}^{k\delta} & \text{if } m = n-1 = 2k, \\ y_2\mathbf{y}^{k\delta} & \text{if } m = n-1 = 2k+1, \\ \xi_n & \text{if } m > n-1. \end{cases} \tag{129}$$

(2) *For every* $m \in \mathbb{Z}$ *and* $n \geq 1$ *the following relation holds:*

$$\zeta_n^+(m) = (1,3)\zeta_n^-(1-m); \ \zeta_n^-(m) = (1,3)\zeta_n^+(1-m) \tag{130}$$

where $(1,3)$ *is the automorphism of* $\mathbb{P}$ *that exchanges* y_1 *with* y_3.

(3) *For every* $n \geq 1$ *and* $i = 1, 2, 3$ *we have:*

$$u_1\Gamma_i(n) = \Gamma_i(n+1) + \mathbf{y}^\delta\Gamma_i(n-1) - \gamma_i(n+1) \tag{131}$$

Proof of Lemma 6.2 Equations 127 and 128 follow directly by the definition of ξ_m and ξ_{m+k} by using Lemma 3.2: indeed one can see that $\mathbf{d}(x_m) \leq \mathbf{d}(x_{m+k})$ (resp. $\mathbf{d}(x_m) \geq \mathbf{d}(x_{m+k})$) if $m \geq 1$ (resp. $m+k \leq 0$). (Here $\leq$ is understood term by term). We now compute $\xi_{-m} \overset{\min}{\oplus} \xi_n := \mathbf{y}^{\mathbf{d}(x_{-m})} \overset{\min}{\oplus} \mathbf{y}^{\mathbf{d}(x_{n+3})}$. By Remark 3.1, $\mathbf{d}(x_{-m}) = (1,3)\mathbf{d}(x_{m+4})$, where $(1,3)$ is the linear operator on $\mathbb{Z}^3$ that exchanges the first entry with the third one. We now consider all the possible cases:

If $m+4 < n+3$ then $\mathbf{d}(x_{m+4}) \leq \mathbf{d}(x_{n+3})$; since $m+4$ and $n+3$ are positive integers, $\mathbf{d}(x_{m+4})$ and $\mathbf{d}(x_{-m})$ have respectively the form (d_3+1, d_2, d_3) and $(d_3'+1, d_2', d_3')$ for some $d_2, d_3, d_2', d_3' \geq 0$; in particular $\mathbf{d}(x_{-m}) = (d_3, d_2, d_3+1)$. Since by hypothesis $d_3 < d_3'$ and $d_2 < d_2'$, we conclude $\mathbf{d}(x_{-m}) \leq \mathbf{d}(x_{n+3})$ so that $\mathbf{y}^{\mathbf{d}(x_{-m})} \overset{\min}{\oplus} \mathbf{y}^{\mathbf{d}(x_{n+3})} = \mathbf{y}^{\mathbf{d}(x_{-m})}$.

If $m+4 = n+3 = 2k+4$ for some $k \geq 0$, then by Eq. 37, $\mathbf{d}(x_{m+4}) = (k+1, k, k)^t$ so that $(1,3)\mathbf{d}(x_{m+4}) = (k, k, k+1)^t$; then $\mathbf{y}^{\mathbf{d}(x_{-m})} \overset{\min}{\oplus} \mathbf{y}^{\mathbf{d}(x_{n+3})} = \mathbf{y}^{k\delta}$.

If $m+4 = n+3 = 2k+5$ for some $k \geq 0$, then by Eq. 37, $\mathbf{d}(x_{m+4}) = (k+1, k+1, k)^t$ so that $(1,3)\mathbf{d}(x_{m+4}) = (k, k+1, k+1)^t$; then $\mathbf{y}^{\mathbf{d}(x_{-m})} \overset{\min}{\oplus} \mathbf{y}^{\mathbf{d}(x_{n+3})} = y_2\mathbf{y}^{k\delta}$.

If $m+4 > n+3$ then $\mathbf{d}(x_{m+4}) \geq \mathbf{d}(x_{n+3})$, then also $(1,3)\mathbf{d}(x_{m+4}) \geq \mathbf{d}(x_{n+3})$.

and Eq. 129 is proved. Formula 130 follows from the definition and from Remark 3.1. Formula 131 follows by Eq. 115. □

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By part 1 of Lemma 6.2, it is immediate to verify that

$$\eta^-_{m;3} = \begin{cases} y_{m+3} & \text{if } m = 0, -1, -2 \\ 1 & \text{otherwise} \end{cases} ; \quad \eta^+_{m;3} = \begin{cases} 1 & \text{if } m = 0, -1, -2 \\ y_{1;m} & \text{otherwise} \end{cases}$$

so that Eq. 125 specializes to Eq. 126 when $n = 0$, i.e. for every $m \in \mathbb{Z}$ the following relation holds

$$X_m X_{m+3} = \eta^-_{m;3} X_{m+1} X_{m+2} + \eta^+_{m;3}. \tag{132}$$

We now assume $n \geq 1$. In this case, by the inductive hypothesis we have:

$$\begin{aligned}
\Gamma_i(n+1) &= u_1 \Gamma_i(n) - \mathbf{y}^\delta \Gamma_i(n-1) + \gamma_i(n+1) \\
&= \frac{u_1}{\eta^+_{m;2n+3}} \cdot \left[X_m X_{m+2n+3} - \eta^-_{m;2n+3} X_{m+n+1} X_{m+n+2}\right] + \\
&\quad - \frac{\mathbf{y}^\delta}{\eta^+_{m;2n+1}} \cdot \left[X_m X_{m+2n+1} - \eta^-_{m;2n+1} X_{m+n} X_{m+n+1}\right] + \gamma_i(n+1) \\
&= \frac{X_m}{\eta^+_{m;2n+3}} \cdot \left[\zeta_1^-(m+2n+3) X_{m+2n+1} + \zeta_1^+(m+2n+3) X_{m+2n+5}\right] + \\
&\quad - \frac{\eta^-_{m;2n+3}}{\eta^+_{m;2n+3}} \cdot \left[\zeta_1^-(m+n+2) X_{m+n} X_{m+n+1} + \zeta_1^+(m+n+2) X_{m+n+1} X_{m+n+4}\right] + \\
&\quad - \frac{\mathbf{y}^\delta}{\eta^+_{m;2n+1}} \cdot \left[X_m X_{m+2n+1} - \eta^-_{m;2n+1} X_{m+n} X_{m+n+1}\right] + \gamma_i(n+1) \\
&= X_m X_{m+2n+1} \left[\frac{\zeta_1^-(m+2n+3)\eta^+_{m;2n+1} - \mathbf{y}^\delta \eta^+_{m;2n+3}}{\eta^+_{m;2n+1}\eta^+_{m;2n+3}}\right] \\
&\quad + X_m X_{m+2n+5} \left[\frac{\zeta_1^+(m+2n+3)}{\eta^+_{m;2n+3}}\right] \\
&\quad + X_{m+n} X_{m+n+1} \left[\frac{\mathbf{y}^\delta \eta^-_{m;2n+1}\eta^+_{m;2n+3} - \eta^-_{m;2n+3}\zeta_1^-(m+n+2)\eta^+_{m;2n+1}}{\eta^+_{m;2n+1}\eta^+_{m;2n+3}}\right] + \\
&\quad - \frac{\zeta_1^+(m+n+2)\eta^-_{m;2n+3}}{\eta^+_{m;2n+3}} X_{m+n+1} X_{m+n+4} + \gamma_i(n+1) \\
&= X_m X_{m+2n+1} \left[\frac{\zeta_1^-(m+2n+3)\eta^+_{m;2n+1} - \mathbf{y}^\delta \eta^+_{m;2n+3}}{\eta^+_{m;2n+1}\eta^+_{m;2n+3}}\right] \\
&\quad + X_m X_{m+2n+5} \left[\frac{\zeta_1^+(m+2n+3)}{\eta^+_{m;2n+3}}\right] \\
&\quad + X_{m+n} X_{m+n+1} \left[\frac{\mathbf{y}^\delta \eta^-_{m;2n+1}\eta^+_{m;2n+3} - \eta^-_{m;2n+3}\zeta_1^-(m+n+2)\eta^+_{m;2n+1}}{\eta^+_{m;2n+1}\eta^+_{m;2n+3}}\right] \\
&\quad + \frac{\zeta_1^+(m+n+2)\eta^-_{m;2n+3}}{\eta^+_{m;2n+3}} \left[\eta^+_{m+n+1;3} X_{m+n+2} X_{m+n+3} + \eta^+_{m+n+1;3}\right] + \gamma_i(n+1)
\end{aligned}$$

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Lemma 6.3 below shows that this polynomial is equal to

$$\frac{1}{\eta^+_{m;2n+5}}[X_m X_{m+2n+5} - \eta^-_{m;2n+5} X_{m+n+2} X_{m+n+3}]$$

and we are done.

We prove (iii) and (iv) together. In order to do that we introduce the variable $c = c(m)$ depending on $m \in \mathbb{Z}$ in the following way: c is w if m is odd and c is z if m is even. With this convention, both Eqs. 123 and 124 are equivalent to the following:

$$c(m)\Gamma_3(n-2) = \frac{1}{\eta^+_{m;2n}}[X_m X_{m+2n} - \eta^-_{m;2n} X_{m+2\lfloor\frac{n}{2}\rfloor} X_{m+2\lceil\frac{n}{2}\rceil}]. \tag{133}$$

In order to prove Eq. 133 we proceed by induction on $n \geq 2$. We verify directly the formula for $n = 2$ and $n = 3$. We then assume $n \geq 4$. By using Eq. 131 and the inductive hypothesis we get the following equality:

$$\begin{aligned} c(m)\Gamma_3(n-2) = {} & X_m X_{m+2n+4}\left[\frac{\zeta_1^-(m+2n-2)\eta^+_{m;2n-4} - \mathbf{y}^\delta \eta^+_{m;2n-2}}{\eta^+_{m;2n-2}\eta^+_{m;2n-4}}\right] \\ & + X_m X_{m+2n}\left[\frac{\zeta_1^+(m+2n-2)}{\eta^+_{m;2n-2}}\right] + X_{m+2\lfloor\frac{n-2}{2}\rfloor} X_{m+2\lceil\frac{n-2}{2}\rceil} \\ & \times \left[\frac{\mathbf{y}^\delta \eta^-_{m;2n-4}\eta^+_{m;2n-2} - \eta^-_{m;2n-2}\eta^+_{m;2n-4}\zeta_1^-(m+2\lceil\frac{n-1}{2}\rceil)}{\eta^+_{m;2n-2}\eta^+_{m;2n-4}}\right] \\ & - \frac{\eta^-_{m;2n-2}\zeta_1^+(m+2\lceil\frac{n-1}{2}\rceil)}{\eta^+_{m;2n-2}} X_{m+2\lfloor\frac{n-1}{2}\rfloor} X_{m+2\lceil\frac{n-1}{2}\rceil} + c(m)\gamma_3(n-2) \end{aligned}$$

the following Lemma 6.3 concludes the proof.

Lemma 6.3 *For every $n \geq 1$, $m_1 \geq 3$ and $m \in \mathbb{Z}$ the following equalities hold in $\mathbb{ZP}$:*

(1) $\zeta_1^-(m+m_1+2)\eta^+_{m;m_1} - \mathbf{y}^\delta \eta_{m;m_1+2} = 0$;
(2) $\zeta_1^+(m+m_1) = \eta^+_{m;m_1}/\eta^+_{m;m_1+2}$;
(3) $\mathbf{y}^\delta \eta^-_{m;m_1}\eta^+_{m;m_1+2} - \eta^-_{m;m_1+2}\eta^+_{m;m_1}\zeta_1^-(m+\lceil\frac{m_1+2}{2}\rceil) = 0$;
(4) $\zeta_1^+(m+n+2)\eta^-_{m;2n+3}\eta^-_{m+n+1;3} = \eta^+_{m;2n+3}\eta^-_{m;2n+5}/\eta^+_{m;2n+5}$;
(5) *For $i = i(m) := 1$ if m is even and $i = i(m) := 2$ if m is odd we have for every $n \geq 1$:*

$$\gamma_i(n+1)\eta^+_{m;2n+3} - \zeta_1^+(m+n+2)\eta^-_{m;2n+3}\eta^+_{m+n+1;3} = 0;$$

(6) $\frac{\eta^-_{m;2n-2}\zeta_1^+(m+2\lceil\frac{n-1}{2}\rceil)}{\eta^+_{m;2n-2}} X_{m+2\lfloor\frac{n-1}{2}\rfloor} X_{m+2\lceil\frac{n-1}{2}\rceil} - c\gamma_3(n-2) = \frac{\eta^-_{m;2n}}{\eta^+_{m;2n}} X_{m+2\lfloor\frac{n}{2}\rfloor} X_{m+2\lceil\frac{n}{2}\rceil}$

Proof The proof of Lemma 6.3 follows by direct check. □

6.3.2 Span Property

In this section we prove that a monomial of the form $M = u^{a_1}_{n_1}\cdots u^{a_s}_{n_s} x^{b_1}_{m_1}\cdots x^{b_t}_{m_t} w^c z^d$ in the elements of $\mathbf{B}$ is a $\mathbb{ZP}$-linear combination of elements of $\mathbf{B}$. We proceed by

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induction on the multi-degree $\mu(M)$ (defined in Eq. 113). If $\mu_1(M) = 1$ then M is either a cluster variable or one of the u_n's. If $\sum_{i=1}^{s} a_i \geq 2$ (resp. 1) then one can apply Eq. 115 (resp. Eq. 120), expressing M as a linear combination of monomials with smaller value of μ_1. So we can assume that $M = x_{m_1}^{b_1} \cdots x_{m_t}^{b_t} w^c z^d$. If both c and d are positive, by using the fact that $ZW = u_1 + y_1 y_3 + y_2$, one obtains again a sum of two monomials with smaller value of μ_1. So we assume that $d = 0$ (resp. $c = 0$) and that we can apply the relation (Eq. 58) (resp. Eq. 59), i.e. some m_i is odd (resp. even). We again obtain a sum of two monomials having smaller value of μ_1 than the initial one. So we can assume that M has one of the following forms: $M_1 := \left(\prod_{m_i \text{ even}} x_{m_i}^{b_i}\right) w^c$ or $M_2 := \left(\prod_{m_i \text{ odd}} x_{m_i}^{b_i}\right) z^d$ or $M_3 := x_{m_1}^{b_1} \cdots x_{m_t}^{b_t}$ with $m_t - m_1 \geq 3$. We apply either Eqs. 123 or 124 or 132 to the product $x_{m_1} x_{m_t}$. By inspection, in the resulting expression for both M_1 and M_2, all the monomials except at most one that has smaller value of μ_1 have the same value of μ_1. By further inspection, for every such monomial M', if $\min(b_1, b_t) = 1$ (resp. $\min(b_1, b_t) \geq 2$) then $\mu_2(M') < \mu_2(M)$ (resp. $\mu_2(M') = \mu_2(M)$ and $\mu_3(M') = \mu_3(M) - 2$). Analogously in the resulting expression for M_3 , there is precisely one monomial M with $\mu_1(M') = \mu_1(M)$, while the rest of the terms have smaller value of μ_1. Moreover if $\min(b_1, b_t) = 1$ (resp. $\min(b_1, b_t) \geq 2$) then $\mu_2(M') < \mu_2(M)$ (resp. $\mu_2(M') = \mu_2(M)$ and $\mu_3(M') = \mu_3(M) - 2$). We have hence expressed M as a linear combinations of monomials of smaller multidegree. By inductive hypothesis M is a linear combinations of elements of $\mathbf{B}$.

Example 6.1 As an exercise one can apply the argument above to expand the monomial $M := u_1 u_2 x_3 x_5 z w$ in the atomic basis of the coefficient free cluster algebra $\mathcal{A}_{\{1\}}$. The expansion is the following

$$M = x_{-1}x_1 + x_7x_9 + 2x_3x_5 + 2x_1x_3 + 2x_5x_7 + 2x_1^2 + 2x_7^2 + 2x_3^2 + 2x_5^2 + \\ + u_3 w + 2u_2 w + 5u_1 w + 8w$$

6.4 The Elements of B are Positive Indecomposable

In the previous sections we proved that the set $\mathbf{B} = \{\text{cluster monomials}\} \cup \{u_n w^k,\ u_n z^k : n \geq 1,\ k \geq 0\}$ is a $\mathbb{ZP}$-basis of $\mathcal{A}_{\mathbb{P}}$ (for every semifield $\mathbb{P}$) and its elements are positive, i.e. their Laurent expansion in every cluster has coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$. In this section we prove that given a positive element of $\mathcal{A}_{\mathbb{P}}$ its expansion in $\mathbf{B}$ has coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$. Let hence p be a positive element of $\mathcal{A}_{\mathbb{P}}$. We express $p = \sum_{b \in \mathbf{B}'} a_b b$ as a $\mathbb{ZP}$-linear combination of elements of a (finite) subset $\mathbf{B}' \subset \mathbf{B}$.

Definition 6.2 Let $\mathcal{C} = \{s_1, s_2, s_3\}$ be a cluster of $\mathcal{A}_{\mathbb{P}}$. A Laurent monomial $s_1^a s_2^b s_3^c$ is called *proper* if either $a < 0$ or $b < 0$ or $c < 0$.

Lemma 6.4 *Let $\mathcal{C}$ be a cluster of $\mathcal{A}_{\mathbb{P}}$ and let $b \in \mathbf{B}$ not be a cluster monomial in $\mathcal{C}$. The Laurent expansion of b in $\mathcal{C}$ is a $\mathbb{ZP}$-linear combination of proper Laurent monomials.*

Proof The proof of Lemma 6.4 will be given in Section 6.4.2. □

Now suppose that a cluster monomial b in some cluster $\mathcal{C}$ appears in the expansion of p in $\mathbf{B}$ with coefficient a_b. We expand p in the cluster $\mathcal{C}$ and in view of Lemma 6.4 the monomial b appears with coefficient a_b in this expansion. Since p is positive

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we conclude that $a_b \in \mathbb{Z}_{\geq 0}\mathbb{P}$. It remains to deal with elements u_n, $u_n w^k$ and $u_n z^k$. Without loss of generality we can assume that the cluster monomials in $\mathbf{B}'$ are of the form $x_m^a s^b x_{m+2}^c$ with $m \geq 1$ and $s = x_{m+1}$ or w or z. We have the following lemma.

Lemma 6.5 *Let b be an element of $\mathbf{B}$ which is not divisible by cluster variables x_m with $m \leq 0$.*

(1) *The (proper) Laurent monomial $\frac{x_1^n}{x_3^n}$ is a summand of u_n in the cluster $\{x_1, x_2, x_3\}$ but it is not a summand of b in $\{x_1, x_2, x_3\}$. Moreover the coefficient of $\frac{x_1^n}{x_3^n}$ in this expansion is an element t of $\mathbb{P}$.*
(2) *The (proper) Laurent monomial $\frac{x_1^n w^k}{x_3^n}$ is a summand of $u_n w^k$ in the cluster $\{x_1, w, x_3\}$ but it is not a summand of b in $\{x_1, w, x_3\}$. Moreover the coefficient of $\frac{x_1^n w^k}{x_3^n}$ in this expansion is an element t of $\mathbb{P}$.*
(3) *The (proper) Laurent monomial $\frac{x_0^n z^k}{x_2^n}$ is a summand of $u_n z^k$ in the cluster $\{x_0, z, x_2\}$ but it is not a summand of b in $\{x_0, z, x_2\}$. Moreover the coefficient of $\frac{x_0^n z^k}{x_2^n}$ in this expansion is an element t of $\mathbb{P}$.*

Proof The proof of Lemma 6.5 will be given in Section 6.4.3. □

Now assume that u_n (resp. $u_n w^k$, $u_n z^k$) appears with coefficient a in the expansion of p in $\mathbf{B}$. We expand p in the cluster $\{x_1, x_2, x_3\}$ (resp. $\{x_1, w, x_3\}$, $\{x_0, z, x_2\}$) and in view of Lemma 6.5 (1) (resp. (2), (3)) we find that the Laurent monomial $\frac{x_1^n}{x_3^n}$ $\left(\text{resp.} \frac{x_1^n w^k}{x_3^n}, \frac{x_0^n z^k}{x_2^n}\right)$ has coefficient at in this expansion. Since p is positive we conclude that $at \in \mathbb{Z}_{\geq 0}\mathbb{P}$. Since $t \in \mathbb{P}$ we conclude that $a \in \mathbb{Z}_{\geq 0}\mathbb{P}$.

In order to prove Lemmas 6.4 and 6.5 we use "Newton polytopes" of the elements of $\mathbf{B}$ in every cluster of $\mathcal{A}_{\mathbb{P}}$. This is the subject of the next section.

6.4.1 Newton Polytopes of the Elements of B

The Newton polytope of a Laurent polynomial $x \in \mathbb{Z}[s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1}]$ with respect to the ordered set $\mathcal{C} = \{s_1, s_2, s_3\}$ is the convex hull in $Q_{\mathbb{R}} = \mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2 \oplus \mathbb{R}\mathbf{e}_3$ of all lattice points $\mathbf{g} = (g_1, g_2, g_3)^t$ such that the monomial $\mathbf{s}^{\mathbf{g}} := s_1^{g_1} s_2^{g_2} s_3^{g_3}$ appears with a non-zero coefficient in x. We denote it by $\mathrm{Newt}_{\mathcal{C}}(x)$. In this section we find the Newton polytopes of the elements of $\mathbf{B}$ in every cluster of $\mathcal{A}_{\mathbb{P}}$. By the symmetry of the exchange relations it is sufficient to consider only the two clusters $\{x_1, x_2, x_3\}$ and $\{x_1, w, x_3\}$. Moreover in such clusters the Newton polytope of the cluster variable x_{-m} is obtained from the Newton polytope of the cluster variable x_{m+4} by the automorphism $(1, 3)$ of $Q_{\mathbb{R}}$ that exchanges the first coordinate with the third one (see Remark 3.1). It is hence sufficient to consider only cluster variables x_m with $m \geq 2$. Before doing that we notice the following interesting fact.

Lemma 6.6 *The algebra $\mathcal{A}_{\mathbb{P}}$ is $\mathbb{Z}$-graded by the following grading:* $\deg(w) = 2$, $\deg(x_{2m+1}) = 1$, $\deg(u_n) = 0$, $\deg(x_{2m}) = -1$, $\deg(z) = -2$ *for* $m \in \mathbb{Z}$, $\deg(y) = 0$ *for every* $y \in \mathbb{P}$

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Proof The exchange relations (Eqs. 29–33) are homogeneous with respect to such grading. The fact that $\deg(u_n) = 0$ follows from Definition 2.1. □

The elements of **B** are homogeneous with respect to the grading given in Lemma 6.6. This implies that the Newton polytopes of the elements of **B** are actually polygons. Indeed let $\mathcal{C} = \{s_1, s_2, s_3\}$ be a cluster of $\mathcal{A}_{\mathbb{P}}$, $b \in \mathbf{B}$ and let $P_b^{\mathcal{C}} := \{(e_1, e_2, e_3) \in Q_{\mathbb{R}} | \deg(s_1)e_1 + \deg(s_2)e_2 + \deg(s_3)e_3 = \deg(b)\}$. Then $\mathrm{Newt}_{\mathcal{C}}(b) \subset P_b^{\mathcal{C}}$.

The following proposition gives the Newton polygons of the elements of **B** in the cluster $\{x_1, x_2, x_3\}$.

Proposition 6.3 *For every $m \geq 2$ and $n \geq 1$ we have:*

$$Newt_{\{x_1,x_2,x_3\}}(x_{2m+1}) = \mathrm{Conv}\left\{\begin{bmatrix}1-m\\0\\m\end{bmatrix}, \begin{bmatrix}1-m\\1-m\\1\end{bmatrix}, \begin{bmatrix}0\\1-m\\2-m\end{bmatrix}, \begin{bmatrix}m-2\\-1\\2-m\end{bmatrix}\right\}$$

$$Newt_{\{x_1,x_2,x_3\}}(x_{2m}) = \mathrm{Conv}\left\{\begin{bmatrix}1-m\\1\\m-1\end{bmatrix}, \begin{bmatrix}1-m\\2-m\\0\end{bmatrix}, \begin{bmatrix}-1\\2-m\\2-m\end{bmatrix}, \begin{bmatrix}m-3\\0\\2-m\end{bmatrix}\right\} \tag{134}$$

$$Newt_{\{x_1,x_2,x_3\}}(u_n) = \mathrm{Conv}\left\{\begin{bmatrix}-n\\0\\n\end{bmatrix}, \begin{bmatrix}-n\\-n\\0\end{bmatrix}, \begin{bmatrix}0\\-n\\-n\end{bmatrix}, \begin{bmatrix}n\\0\\-n\end{bmatrix}\right\} \tag{135}$$

$$Newt_{\{x_1,x_2,x_3\}}(w) = \mathrm{Conv}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}\right\} \tag{136}$$

$$Newt_{\{x_1,x_2,x_3\}}(z) = \mathrm{Conv}\left\{\begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\0\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\} \tag{137}$$

where Conv *means convex hull in $Q_{\mathbb{R}}$.*

Proof Formulas 136 and 137 follow from Eq. 2. By Proposition 2.1, up to a factor in $\mathbb{P}$ which does not modify the Newton polygon, every element b of **B** has the form

$$b = \sum_{\mathbf{e} \in E(b)} \chi_{\mathbf{e}}(b) \mathbf{y}^{\mathbf{e}} \mathbf{x}^{\mathbf{g}_b + H\mathbf{e}}$$

where $\chi_{\mathbf{e}}(b)$ is the coefficient of $\mathbf{y}^{\mathbf{e}}$ in the F-polynomial of b given in Proposition 5.1, $E(b) := \mathrm{Conv}\{e \in \mathbb{Z}^3 | \chi_{\mathbf{e}}(b) \neq 0\}$ is the convex hull in $Q_{\mathbb{R}}$ of the support of $\mathbf{e} \mapsto \chi_{\mathbf{e}}(b)$, $\mathbf{g}_b$ is the $\mathbf{g}$-vector of b given in Proposition 5.2 and H is the exchange matrix given in Eq. 4. The affine map $N_b : \mathbf{e} \mapsto \mathbf{g}_b + H\mathbf{e}$ sends convex sets to convex sets. In particular if $E(b) = \mathrm{Conv}\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$, then $\mathrm{Newt}(b) = \mathrm{Conv}\{N_b(\mathbf{e}_1), \cdots, N_b(\mathbf{e}_n)\}$. The proof is based on the following lemma.

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Lemma 6.7 *For every $m \geq 2$ and $n \geq 1$ we have:*

$$E(x_{2m+1}) = \mathrm{Conv}\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}m-1\\0\\0\end{bmatrix}, \begin{bmatrix}m-1\\m-1\\0\end{bmatrix}, \begin{bmatrix}m-1\\m-1\\m-2\end{bmatrix}\right\};$$

$$E(x_{2m}) = \mathrm{Conv}\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}m-1\\0\\0\end{bmatrix}, \begin{bmatrix}m-2\\m-2\\0\end{bmatrix}, \begin{bmatrix}m-1\\m-2\\0\end{bmatrix}, \begin{bmatrix}m-1\\m-2\\m-2\end{bmatrix}, \begin{bmatrix}m-2\\m-2\\m-3\end{bmatrix}\right\};$$

$$E(u_n) = \mathrm{Conv}\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}n\\0\\0\end{bmatrix}, \begin{bmatrix}n\\n\\0\end{bmatrix}, \begin{bmatrix}n\\n\\n\end{bmatrix}\right\}.$$

Proof The proof follows from Proposition 5.1 by direct check. □

In order to finish the proof of Proposition 6.3 we apply the affine transformation N_b to every generator of $E(b)$ given in Lemma 6.7. If $b = x_{2m+1}$ or $b = u_n$ we find the desired expression. If $b = x_{2m}$ we apply $N_{x_{2m}}$ to $E(x_{2m})$ and we get

$$\begin{aligned}&\mathrm{Newt}(x_{2m})\\ &= \mathrm{Conv}\left\{\begin{bmatrix}1-m\\1\\m-1\end{bmatrix}, \begin{bmatrix}1-m\\2-m\\0\end{bmatrix}, \begin{bmatrix}-1\\3-m\\3-m\end{bmatrix}, \begin{bmatrix}-1\\2-m\\2-m\end{bmatrix}, \begin{bmatrix}m-3\\0\\2-m\end{bmatrix}, \begin{bmatrix}m-4\\0\\3-m\end{bmatrix}\right\}.\end{aligned}$$

To conclude Eq. 134 we show that the third and the last generators are convex combinations of the others: indeed let $v_1, \cdots, v_6$ be the generators enumerated from left to right. Then $v_3 = \frac{1}{2m-3}v_1 + \frac{2m-5}{2m-3}v_4 + \frac{1}{2m-3}v_5$ and $v_6 = \frac{1}{(m-1)(2m-4)}[(m-2)v_1 + v_2 + (m-1)(2m-5)v_5]$. □

The following proposition gives the Newton polygons of the elements of **B** in the cluster $\{x_1, w, x_3\}$.

Proposition 6.4 *For every $m \geq 2$ we have:*

$$\mathrm{Newt}_{\{x_1,w,x_3\}}(x_{2m+1}) = \mathrm{Conv}\left\{\begin{bmatrix}1-m\\0\\m\end{bmatrix}, \begin{bmatrix}m-3\\1\\2-m\end{bmatrix}, \begin{bmatrix}1-m\\m-1\\2-m\end{bmatrix}\right\}; \quad (138)$$

For every $m \geq 1$ we have:

$$\mathrm{Newt}_{\{x_1,w,x_3\}}(x_{2m+2}) = \mathrm{Conv}\left\{\begin{bmatrix}1-m\\-1\\m\end{bmatrix}, \begin{bmatrix}m-2\\0\\1-m\end{bmatrix}, \begin{bmatrix}-m\\-1\\m+1\end{bmatrix}, \begin{bmatrix}-m\\m-1\\1-m\end{bmatrix}\right\};$$

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For every $n \geq 1$ we have:

$$\mathrm{Newt}_{\{x_1,w,x_3\}}(u_n) = \mathrm{Conv}\left\{\begin{bmatrix} n \\ 0 \\ -n \end{bmatrix}, \begin{bmatrix} -n \\ 0 \\ n \end{bmatrix}, \begin{bmatrix} -n \\ n \\ -n \end{bmatrix}\right\}; \tag{139}$$

$$\mathrm{Newt}_{\{x_1,w,x_3\}}(z) = \mathrm{Conv}\left\{\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}; \tag{140}$$

$$\mathrm{Newt}_{\{x_1,w,x_3\}}(x_2) = \mathrm{Conv}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}; \tag{141}$$

where Conv *means convex hull in $Q_{\mathbb{R}}$.*

Proof Formula 140 follows from Eq. 9. We check (Eq. 139) by using the explicit formula:

$$x_4 = \frac{x_1x_3 + x_3^2 + w}{x_1 w} \in \mathcal{A}_{\{1\}}. \tag{142}$$

We hence prove (Eq. 138) and (Eq. 139) for $m \geq 2$. Formula 141 follows from Eq. 30. By Proposition 2.1, up to a factor in $\mathbb{P}$, every element b of $\mathbf{B}$ has the form

$$b = \sum_{\mathbf{e}\in E^w(b)} \chi_{\mathbf{e}}^w(b)\mathrm{p}^{\mathbf{e}}\mathbf{x}^{\mathbf{g}_b^w + H^{cyc}\mathbf{e}}$$

where $\chi_{\mathbf{e}}^w(b)$ is the coefficient of $\mathrm{p}^{\mathbf{e}}$ in the F-polynomial F_b^w of b given in Proposition 5.3, $E^w(b) = \mathrm{Conv}\{\mathbf{e} \in \mathbb{Z}^3 | \chi_{\mathbf{e}}^w(b) \neq 0\}$ is the convex hull in $Q_{\mathbb{R}}$ of the support of $\mathbf{e} \mapsto \chi_{\mathbf{e}}^w(b)$, $\mathbf{g}_b^w$ is the $\mathbf{g}$-vector in the cluster $\{x_1, w, x_3\}$ of b given in Proposition 5.4 and H^{cyc} is the exchange matrix given in Eq. 36.

The affine map $N_b^w : e \mapsto \mathbf{g}_b^w + H^{cyc}\mathbf{e}$ sends convex sets to convex sets. In particular if $E^w(b) = \mathrm{Conv}\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$, then $\mathrm{Newt}_{\{x_1,w,x_3\}}(b) = \mathrm{Conv}\{N_b^w(\mathbf{e}_1), \cdots, N_b^w(\mathbf{e}_n)\}$. The proof is hence based on the following lemma.

Lemma 6.8 *For every $m \geq 2$ and $n \geq 1$ we have:*

$$E^w(x_{2m+1}) = \mathrm{Conv}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} m-1 \\ 0 \\ m-2 \end{bmatrix}, \begin{bmatrix} m-1 \\ 0 \\ 0 \end{bmatrix}\right\};$$

$$E^w(x_{2m+2}) = \mathrm{Conv}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} m-1 \\ 0 \\ m-2 \end{bmatrix}, \begin{bmatrix} m-1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} m \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} m \\ 1 \\ m-1 \end{bmatrix}\right\};$$

$$E^w(u_n) = \mathrm{Conv}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} n \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} n \\ 0 \\ n \end{bmatrix}\right\}.$$

Proof It follows from Proposition 5.3 by a case by case inspection. □

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In order to finish the proof of Proposition 6.4 we apply the affine transformation N_b^w to every generator of $E^w(b)$ given in Lemma 6.8. If $b = x_{2m+1}$ or $b = u_n$ we find the desired expression. If $b = x_{2m+2}$ we apply N_b^w to $E^w(b)$ and we get

$$\mathrm{Conv}\left\{\begin{bmatrix}1-m\\-1\\m\end{bmatrix},\begin{bmatrix}m-3\\0\\2-m\end{bmatrix},\begin{bmatrix}1-m\\m-2\\2-m\end{bmatrix},\begin{bmatrix}-m\\-1\\m+1\end{bmatrix},\begin{bmatrix}-m\\m-1\\1-m\end{bmatrix},\begin{bmatrix}m-2\\0\\1-m\end{bmatrix}\right\}.$$

To conclude Eq. 138 we show that the second and the third generators are convex combinations of the others: indeed let $v_1, \cdots, v_6$ be the generators enumerated from left to right. Then $v_2 = \frac{1}{2m}v_4 + \frac{1}{2m(m-1)}v_5 + \frac{2m-3}{2m-2}v_6$ and $v_3 = \frac{1}{2m-1}v_1 + \frac{2m-3}{2m-1}v_5 + \frac{1}{2m-1}v_6$. □

6.4.2 Proof of Lemma 6.4

By the symmetry of the exchange relations we prove the lemma only for the two clusters $\{x_1, x_2, x_3\}$ and $\{x_1, w, x_3\}$. Let hence $\mathcal{C}$ be either the cluster $\{x_1, x_2, x_3\}$ or $\{x_1, w, x_3\}$. Let $b \in \mathbf{B}$ not be a cluster monomial in $\mathcal{C}$. We prove that the Newton polygon of b in $\mathcal{C}$ does not intersect the positive octant Q_+. Since $\mathrm{Newt}(s_1^p s_2^q s_3^r) = p\mathrm{Newt}(s_1) + q\mathrm{Newt}(s_2) + r\mathrm{Newt}(s_3)$, it is sufficient to find a non-zero linear form $\varphi_b^{\mathcal{C}} : Q \to \mathbf{R}$, $\varphi_b^{\mathcal{C}}(e_1, e_2, e_3) = \alpha e_1 + \beta e_2 + \gamma c_3$, such that $\alpha, \beta, \gamma \geq 0$ and that takes negative values on the vertexes of both $\mathrm{Newt}(s_1)$, $\mathrm{Newt}(s_2)$ and $\mathrm{Newt}(s_3)$. If b is a cluster monomial not divisible by cluster variables x_m with $m \leq 0$, Tables 1 and 2 show a linear form with the desired property (this can be checked directly by using Lemmas 6.3 and 6.4). For $m \geq 0$ $Newt_{\mathcal{C}}(x_{-m})$ is obtained from $Newt_{\mathcal{C}}(x_{m+4})$ by

Table 1 Every summand $s = x_1^{e_1}x_2^{e_2}x_3^{e_3}$ of b in $\{x_1, x_2, x_3\}$ satisfies $\varphi_b(e_1, e_2, e_3) < 0$

b	φ_b	
$x_{2m+1}^p x_{2m+2}^q x_{2m+3}^r$	$\left[m(m-1), m(m-1), \left(m^2-2m+\frac{1}{2}\right)\right]$	$m \geq 2$
$x_3^p x_4^q x_5^r$	$[1, 1, 0]$	
$x_{2m}^p x_{2m+1}^q x_{2m+2}^r$	$\left[m(m-1), \left(m-\frac{1}{2}\right)\left(m-\frac{3}{2}\right), \left(m^2-2m+\frac{1}{2}\right)\right]$	$m \geq 4$
$x_2^p x_3^q x_4^r$	$[1, 0, 0]$	$r > 0$
$x_4^p x_5^q x_6^r$	$[6, 3, 2]$	
$x_6^p x_7^q x_8^r$	$[9, 5, 5]$	
$x_{2m+1}^p w^q x_{2m+3}^r$	$[m, 2m, (m-2)]$	$m \geq 2$
$x_1^p w^q x_3^r$	$[0, 1, 0]$	$q > 0$
$x_3^p w^q x_5^r$	$[0, 1, 0]$	$q > 0$
$x_{2m}^p z^q x_{2m+2}^r$	$\left[m(m-1), \frac{m}{4}(m-1), m\left(m-\frac{3}{2}\right)\right]$	$m \geq 2$
$x_2^p z^q x_4^r$	$[1, 0, 0]$	
$u_n w^k$	$[1, 2, 1]$	$n, k > 0$
$u_n z^k$	$[2, 1, 2]$	$n, k > 0$

In the second column we write the (row) vector that defines φ_b. In the first column b is assumed to be not divisible by x_m, $m \leq 0$; it hence satisfies the conditions given in the third column on the right. We abbreviate $\varphi_b := \varphi_b^{\{x_1,x_2,x_3\}}$

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Table 2 Every summand $s = x_1^{e_1} w^{e_2} x_3^{e_3}$ of b in $\{x_1, w, x_3\}$ satisfies $\varphi_b(e_1, e_2, e_3) < 0$

b	φ_b^w	
$x_{2m+1}^p x_{2m+2}^q x_{2m+3}^r$	$\left[m(m-1), \frac{1}{4}(m-1), \left(m^2 - 2m + \frac{1}{2}\right)\right]$	$m \geq 2$
$x_1^p x_2^q x_3^r$	$[0, 1, 0]$	$q > 0$
$x_3^p x_4^q$	$[1, 1, 0]$	$q > 0$
$x_3^p x_4^q x_5^r$	$[1, 0, 0]$	$r > 0$
$x_{2m}^p x_{2m+1}^q x_{2m+2}^r$	$\left[m(m-1), (m-1), \left(m^2 - 2m + \frac{1}{2}\right)\right]$	$m \geq 2$
$x_2^p x_3^q x_4^r$	$[0, 1, 0]$	
$x_{2m+1}^p w^q x_{2m+3}^r$	$\left[m(m-1), 0, \left(m^2 - 2m + \frac{1}{2}\right)\right]$	$m \geq 2$
$x_3^p w^q x_5^r$	$[1, 0, 0]$	
$x_{2m}^p z^q x_{2m+2}^r$	$[1, 2, 1]$	$m \geq 1$

In the second column we write the (row) vector that defines φ_b. In the first column b is assumed to be not divisible by x_m, $m \leq 0$; it hence satisfies the conditions given in the third column on the right. We abbreviate $\varphi_b^w := \varphi_b^{\{x_1,w,x_3\}}$

exchanging the first coordinate with the third one. In particular if $Newt_{\mathcal{C}}(x_{m+4})$ does not intersect the positive octant the same holds for $Newt_{\mathcal{C}}(x_{-m})$.

Let $\mathcal{C} = \{x_1, x_2, x_3\}$. If $b = u_n w^k$ or $b = u_n z^k$, $n, k > 0$, Table 1 shows a linear form φ_b which takes negative values on the vertexes of $\mathrm{Newt}_{\mathcal{C}}(b)$. We notice that both such forms satisfy $\varphi_b(\gamma) \leq 0$ for every vertex γ of $\mathrm{Newt}_{\mathcal{C}}(u_n)$. It hence remains to check that 1 is not a summand of u_n in $\{x_1, x_2, x_3\}$. This is done by direct check using Theorem 2.3.

Let $\mathcal{C} = \{x_1, w, x_3\}$. We notice that the grading of a monomial in the elements of $\mathcal{C}$ is non-negative (see Lemma 6.6). Now $\deg(u_n z^k) = -2k$ and hence, for $k > 0$, b is a sum of proper Laurent monomials in $\mathcal{C}$. If $b = u_n$ then $deg(u_n) = 0$ and hence the only monomial in $\mathcal{C}$ that could appear in the expansion of u_n in $\mathcal{C}$ is 1. By using Theorem 2.3 we check that this is not the case. Let hence $b = u_n w^k$ with $k > 0$. In view of Eq. 139 the linear form $\varphi_b^w(e_1, e_2, e_3) := e_1 + e_3$ satisfies $\varphi_b^w(e_1, e_2, e_3)(\gamma) \leq 0$ for every vertex γ of $\mathrm{Newt}_{\{x_1,w,x_3\}}(u_n)$ and $\varphi_b^w(0, 1, 0) = 0$. It follows that the only possible monomial in $\mathcal{C}$ in the expansion of b in $\mathcal{C}$ is 1. But 1 is a summand of $u_n w^k$ in $\mathcal{C}$ if and only if $(0, -k, 0)^t$ is an element of $\mathrm{Newt}_{\{x_1,w,x_3\}}(u_n)$ which is not the case in view of Eq. 139.

6.4.3 Proof of Lemma 6.5

Part (3) follows by part (2) by the symmetry of the exchange relations. By the explicit formula of u_n in both the clusters $\{x_1, x_2, x_3\}$ and $\{x_1, w, x_3\}$ given in Theorem 2.3 we find that the Laurent monomial x_1^n/x_3^n has coefficient in $\mathbb{P}$ in these expansions and it is not a summand of u_p for $p \neq n$. In particular the monomial $x_1^n w^k/x_3^n$ appears in the expansion of $u_n w^k$ in $\{x_1, w, x_3\}$ with coefficient in $\mathbb{P}$ and it is not a summand of $u_p w^k$ for $p \neq n$. The monomial x_1^n/x_3^n is not a summand of $u_p w^q$ and $u_p z^q$ if $q > 0$, because $\deg(x_1^n/x_3^n) = 0$ whereas $\deg(u_p w^q) = 2q$ and $\deg(u_p z^q) = -2q$ (see Lemma 6.6). Similarly the monomial $x_1^n w^k/x_3^n$ is not a summand of u_p, $u_p w^r$ and $u_p z^q$ if $p, r, q > 0$ and $r \neq k$, because $\deg(x_1^n w^k/x_3^n) = 2k$ whereas $\deg(u_p) = 0$, $\deg(u_p w^r) = 2r$ and $\deg(u_p z^q) = -2q$. Finally the monomial x_1^n/x_3^n (resp. $x_1^n w^k/x_3^n$) is not a summand

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of a cluster monomial b not divisible by x_m, $m \leq 0$, in the cluster $\{x_1, x_2, x_3\}$ (resp. $\{x_1, w, x_3\}$) because after a glance at Table 1 (resp. Table 2) $\varphi_b(n, 0, -n) \geq 0$ (resp. $\varphi_b^w(n, k, -n) > 0$). This concludes the proof. □

Acknowledgements This paper is part of my Ph.D thesis [6] developed at the "Università degli studi di Padova" under the supervision of Professor Andrei Zelevinsky. I thank the director of the doctoral school, Professor Bruno Chiarellotto, for his support. I thank Professor Andrei Zelevinsky for the patience he had in introducing me to this subject and for many advices. I thank Alberto Tonolo and Silvana Bazzoni, for many conversations about this topic. I thank the Department of mathematics of Northeastern University of Boston for its kind hospitality during the three semesters I spent there; in particular Sachin, Shih-Wei and Daniel for several conversations about cluster algebras. I thank the referee of the previous version of this paper for his deep suggestions and comments.

References

1. Buan, A., Marsh, R.J., Reiten, I.: Denominators of cluster variables. J. Lond. Math. Soc. (2) **79**(3), 589–611 (2009)
2. Buan, A., Marsh, R.J., Reiten, I., Todorov, G.: Clusters and seeds in acyclic cluster algebras. Proc. Am. Math. Soc. **135**(10), 3049–3060 (electronic) (2007). With an appendix coauthored in addition by P. Caldero and B. Keller
3. Buan, A., Marsh, R.J.: Denominators in cluster algebras of affine type. J. Algebra **323**(8), 2083–2102 (2010)
4. Caldero, P., Keller, B.: From triangulated categories to cluster algebras. II. Ann. Sci. École Norm. Sup. (4) **39**(6), 983–1009 (2006)
5. Caldero, P., Zelevinsky, A.: Laurent expansions in cluster algebras via quiver representations. Mosc. Math. J. **6**(3), 411–429 (2006)
6. Cerulli Irelli, G.: Structural Theory of Rank Three Cluster Algebras of Affine Type. PhD thesis, Università degli studi di Padova (2008)
7. Cerulli Irelli, G., Dupont, G., Esposito, F.: A Homological Interpretation of the Transverse Quiver Grassmannians. ArXiv:math/1005.1405 (2010)
8. Cerulli Irelli, G., Esposito, F.: Geometry of quiver Grassmannians of Kronecker type and applications to cluster algebras. Algebra & Number Theory. arXiv:1003.3037v2 (2010)
9. Derksen, H., Schofield, A., Weyman, J.: On the number of subrepresentations of a general quiver representation. J. Lond. Math. Soc. (2) **76**(1), 135–147 (2007)
10. Derksen, H., Weyman, J.: On the canonical decomposition of quiver representations. Compos. Math. **133**(3), 245–265 (2002)
11. Derksen, H., Weyman, J., Zelevinsky, A.: Quivers with potentials and their representations II: applications to cluster algebras. J. Am. Math. Soc. **23**(3), 749–790 (2010)
12. Ding, M., Xiao, J., Xu, F.: Integral Bases of Cluster Algebras and Representations of Tame Quivers. ArXiv:math/0901.1937 (2009)
13. Dupont, G.: Transverse quiver Grassmannians and bases in affine cluster algebras. Algebra and Number Theory **4**(5), 599–624 (2010)
14. Dupont, G.: Generic variables in acyclic cluster algebras. J. Pure Appl. Algebra **215**(4), 628–641 (2011)
15. Fomin, S., Shapiro, M., Thurston, D.: Cluster algebras and triangulated surfaces. I. Cluster complexes. Acta Math. **201**(1), 83–146 (2008)
16. Fomin, S., Zelevinsky, A.: Cluster algebras. I. Foundations. J. Am. Math. Soc. **15**(2), 497–529 (electronic) (2002)
17. Fomin, S., Zelevinsky, A.: Cluster algebras. IV. Coefficients. Compos. Math. **143**(1), 112–164 (2007)
18. Fu, C., Keller, B.: On cluster algebras with coefficients and 2-Calabi-Yau categories. Trans. Am. Math. Soc. **362**(2), 859–895 (2010)
19. Geiss, C., Leclerc, B., Schröer, J.: Cluster Algebra Structures and Semicanoncial Bases for Unipotent Groups. ArXiv:math/0703039 (2007)
20. Geiß, C., Leclerc, B., Schröer, J.: Generic Bases for Cluster Algebras and the Chamber Ansatz. ArXiv:math/1004.2781 (2010)

Author's personal copy

21. Igusa, K., Orr, K., Todorov, G., Weyman, J.: Cluster complexes via semi-invariants. Compos. Math. **145**(4), 1001–1034 (2009)
22. Kac, V.: Infinite root systems, representations of graphs and invariant theory. Inv. Math. **56**, 57–92 (1980)
23. Kac, V.G.: Infinite root systems, representations of graphs and invariant theory. II. J. Algebra **78**(1), 141–162 (1982)
24. Kac, V.G.: Infinite–Dimensional Lie Algebras, 3rd edn. Cambridge University Press, Cambridge (1990)
25. Kuniba, A., Nakanishi, T., Suzuki, J.: T-Systems and Y-Systems in Integrable Systems. ArXiv:math/1010.1344 (2010)
26. Plamondon, P.-G.: Cluster Algebras via Cluster Categories with Infinite–Dimensional Morphism Spaces. ArXiv:math/1004.0830 (2010)
27. Schiffler, R.: On cluster algebras arising from unpunctured surfaces. II. Adv. Math. **223**(6), 1885–1923 (2010)
28. Schofield, A.: Semi-invariants of quivers. J. Lond. Math. Soc. (2) **43**(3), 385–395 (1991)
29. Schofield, A.: General representations of quivers. Proc. Lond. Math. Soc. (3) **65**(1), 46–64 (1992)
30. Sherman, P., Zelevinsky, A.: Positivity and canonical bases in rank 2 cluster algebras of finite and affine types. Mosc. Math. J. **4**(4), 947–974, 982 (2004)