FACOLTÁ DI SCIENZE MATEMATICHE, FISICHE E NATURALI
Dottorato di ricerca in matematica - Ciclo XXIX

Thurston's metric on Teichmüller space of semi-translation surfaces

Advisor:

Prof. Gabriele Mondello

Candidate:

Federico Wolenski

Introduction

The main objective of the present thesis is to define and study a pseudo-metric on Teichmüller spaces of semi-translation surfaces which is similar to the one William Thurston defined on Teichmüller spaces of Riemann surfaces in $[Th]$. The idea behind this work is that, since the Thurston's metric L is defined as the infimum of Lipschitz constants of diffeomorphisms between surfaces with hyperbolic metrics, it would be interesting to study what happens when one considers the singular flat metrics induced by quadratic differentials. W.A. Veech already did something similar in $[Ve2]$, claiming that he obtained a complete, complex-valued distance map on spaces of quadratic differentials: the proof of this fact should be contained in unpublished preprints $[Ve3]$.

We defined a pseudo-metric L_F which is slightly different from the real part of Veech's distance function and has some nice properties: for example it is symmetric, complete and greater than or equal to the Teichmüller metric between conformal structures underlying the holomorphic differentials.

One of the main properties of the Thurston's metric L is that it is equal to another metric K defined as the supremum of ratios of lengths of simple closed curves on hyperbolic surfaces (theorem 8.5 of $[Th]$). Motivated by this fact, we defined another symmetric pseudo-metric K_F as the supremum of ratios of lengths of saddle connections, but we were not able to adapt Thurston's method to prove the equality of the two metrics.

Instead, we found another approach which consists in the adaptation of a proof by F.A. Valentine of Kirszbraun's theorem in the real plane ($[Va]$). We used this method to prove that the equality of two asymmetric analogues L_F^a and K_F^a to L_F and K_F depends on two statements about 1-Lipschitz maps between planar polygons.

Finally, in the last chapter of this thesis, we proved that a known hermitian metric of signature $(2g, 0)$ on the moduli space of Abelian differentials with one zero on Riemann surfaces of genus $g \geq 2$ is not Kähler.

The first three chapters are devoted to the exposition of the main objects used in this thesis: Teichmüller spaces, the Thurston's metric and semi-translation surfaces.

In chapter 1 we defined the Teichmüller space $\mathcal{T}_g^n$ of an oriented surfaces S_g^n of genus $g \geq 2$ and n punctures as the space of hyperbolic metrics on S_g^n up to isotopy. Equivalently, it can be defined as the space of equivalence classes of marked Riemann surfaces $\mathcal{X} = [(X, \phi)]$, where X is a Riemann surface, $\phi : S_g^n \to X$ is a marking and $\mathcal{X} \sim \mathcal{X}'$ if and only if there is an isometry between X and X' homotopic to the change of marking.

The Teichmüller space $\mathcal{T}_g^n$ is homeomorphic to $\mathbb{R}^{6g-6+2n}$: theorem 1.1.4 for example shows the bijection provided by Fenchel-Nielsen coordinates. Beltrami differentials of maps between Riemann surfaces also endow $\mathcal{T}_g^n$ with the structure of a complex manifold: this is explained in section 1.2.1.
Given any two marked surfaces $\mathcal{X}, \mathcal{X}' \in \mathcal{T}_g^n$, their distance with respect to the Teichmüller metric $d_{\mathcal{T}}$ is defined as

$$d_{\mathcal{T}}(\mathcal{X}, \mathcal{X}') = \inf_{\varphi \approx \phi' \circ \phi^{-1}} \frac{1}{2} \log(K(\varphi)),$$

where $K(\varphi)$ is the complex dilatation of the diffeomorphisms φ.
The infimum of the definition of $d_{\mathcal{T}}(\mathcal{X}, \mathcal{X}')$ is always realized by the complex dilatation of a particular map $h : X \to X'$, called *Teichmüller map*: this result, along with the fact that $(\mathcal{T}_g^n, d_{\mathcal{T}})$ is a complete and geodesic space, is explained in section 1.3.
As we anticipated, given two hyperbolic surfaces $X, X' \in \mathcal{T}_g^n$, their distance with respect to the Thurston's metric is defined as

$$L(X, X') = \inf_{\varphi \in Diff_0^+(S_g^n)} \log(Lip(\varphi)_X^{X'}),$$

where $Diff_0^+(S_g^n)$ is the group of diffeomorphisms of S_g^n homotopic to the identity and $Lip(\varphi)_X^{X'}$ is the Lipschitz constant of φ computed with respect to the hyperbolic metrics of X and X'.
In $[Th]$ Thurston proved that for every $X, X' \in \mathcal{T}_g^n$ it results

$$L(X, X') = K(X, X'), \tag{1}$$

where K is another asymmetric metric on $\mathcal{T}_g^n$ defined as

$$K(X, X') = \sup_{\alpha \in \mathcal{S}} \log \left(\frac{\hat{l}_{X'}(\alpha)}{\hat{l}_X(\alpha)} \right)$$

with $\mathcal{S}$ being the set of homotopy classes of simple closed curves on S_g^n and $\hat{l}_X(\alpha)$ being the length of the geodesic representative for X of the homotopy class of α.
The equality (1) has been proved by Thurston using the properties of measured laminations on S_g^n: a brief explanation is given in section 2.2. Roughly, one could say that the idea of the proof is to triangulate the surface with hyperbolic triangles and then use the result of proposition 2.2.2: for any $K > 1$ there is a K-Lipschitz homeomorphism of a filled hyperbolic triangle to itself which maps each side to itself, multiplying arc length on the side by K.
The Teichmüller space endowed with the Thurston's metric is a geodesic space; A.Papadopoulos and G.Théret proved that it is also a complete asymmetric space ($[PT]$).

A. Belkhirat, A. Papadopoulos and M. Troyanov defined and studied in [*BPT*] two asymmetric pseudo-metrics λ and κ on the Teichmüller space $\mathcal{T}_1$ of the torus, which are similar to L and K. They showed the equality $\lambda = \kappa$: this result holds basically because, given two tori $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$, the minimal Lipschitz constant with respect to the metrics of the flat tori is realized by the affine map between marked lattices.
Chapter 3 is devoted to the explanation of the properties of *semi-translation surfaces*: they are closed surfaces S_g endowed with a semi-translation structure, that is:

(i) a finite set of points $\Sigma \subset S_g$ and an atlas of charts on $S_g \setminus \Sigma$ to $\mathbb{C}$ such that transition maps are of the form $z \mapsto \pm z + c$, with $c \in \mathbb{C}$,

(ii) a flat singular metric on S_g such that for each $p \in \Sigma$ there is a homeomorphism of a neighborhood of p with a neighborhood of a cone angle of $\pi(k+2)$ for some $k > 0$, which is an isometry away from p (we call such point a singular point of order k). Furthermore, charts of the atlas of (i) are isometries for the flat singular metric.

In particular, semi-translation surfaces are equivalent to the datum of a Riemann surfaces X and a holomorphic quadratic differential q: a singularity of order k of q corresponds to a cone angle of $\pi(k+2)$. When the quadratic differential is the square of an Abelian differential (and consequently the holonomy is trivial), the surface is denoted as a *translation surface*.
Given $g, m \in \mathbb{N}$, $g \geq 2$, $m \geq 1$ and any m-ple $\underline{k} \in \mathbb{N}^m$ such that $\sum_{i=1}^m k_i = 2g-2$, we fix a finite set of points $\Sigma = \{p_1, \ldots, p_m\} \subset S_g$ and denote by $\Omega_g(\underline{k}, \Sigma)$ the space of translation surfaces of genus g and with singularity on points of Σ prescribed by $\underline{k}$.
The Teichmüller and moduli space (denoted respectively $\mathcal{TH}_g(\underline{k})$ and $\mathcal{H}_g(\underline{k})$) of translation surfaces with singularities prescribed by $\underline{k}$ on points of Σ are defined as

$$\mathcal{TH}_g(\underline{k}) := \Omega_g(\underline{k}, \Sigma)/Diff_0^+(S_g, \Sigma), \quad \mathcal{H}_g(\underline{k}) := \Omega_g(\underline{k}, \Sigma)/Diff^+(S_g, \Sigma)$$

where $Diff_0^+(S_g, \Sigma)$ (resp. $Diff^+(S_g, \Sigma)$) is the subgroup of $Diff_0^+(S_g)$ (resp. $Diff^+(S_g)$) consisting of diffeomorphisms which fix points of Σ.
Theorem 3.0.1 ensues that $\mathcal{TH}_g(\underline{k})$ is a complex manifold of dimension $2g+m-1$, while $\mathcal{H}_g(\underline{k})$ is a complex orbifold of the same dimension.
We define in the same way Teichmüller and moduli spaces of semi-translation surfaces and denote them respectively as $\mathcal{TQ}_g(\underline{k}, \epsilon)$ and $\mathcal{Q}_g(\underline{k}, \epsilon)$. The only two differences are that $\underline{k}$ must be such that $\sum_{i=1}^m k_i = 4g-4$ and there is an additional constant $\epsilon \in \{\pm 1\}$ which indicates if the surfaces have trivial holonomy or not. Theorem 3.0.2 ensues that $\mathcal{TQ}_g(\underline{k}, 1)$ and $\mathcal{TQ}_g(\underline{k}, -1)$ respectively have the structure of complex manifold of dimension $2g+m-1$ and $2g+m-2$.

Chapters 4, 5 and 6 contain the original results of this thesis.
We defined the pseudo-metric L_F on $\mathcal{TQ}_g(\underline{k}, \epsilon)$ which is the symmetric analogue to the Thurston's metric.

$$L_F(q_1, q_2) := \inf_{\varphi \in Diff_0^+(S_g, \Sigma)} \mathcal{L}_{q_1}^{q_2}(\varphi),$$

$$\mathcal{L}_{q_1}^{q_2}(\varphi) := \sup_{p \in S_g \setminus \Sigma} \left(\sup_{v \in T_p S_g, ||v||_{q_1}=1} |\log(||d\varphi_p v||_{q_2})| \right).$$

A first, notable, inequality regarding L_F is given by proposition 4.1.2: it results

$$L_F(q_1, q_2) \geq d_{\mathcal{T}}(\mathcal{X}_1, \mathcal{X}_2),$$

where $\mathcal{X}_1, \mathcal{X}_2$ are the points in $\mathcal{T}_g$ corresponding to the conformal structures underlying the quadratic differentials.
The metric L_F endows $\mathcal{TQ}_g(\underline{k}, \epsilon)$ with the structure of proper and complete space (propositions 4.2.3 and 4.2.5) and the standard topology of $\mathcal{TQ}_g(\underline{k}, \epsilon)$ (which is the one induced by its structure of complex manifold) is finer than the topology induced by L_F (proposition 4.2.1). Furthermore, there is a metric $\mathbb{P}L_F$ on $\mathbb{P}\mathcal{TQ}_g(\underline{k}, \epsilon)$ induced by L_F, and the topology it induces is equal to the standard topology of the projectification of $\mathcal{TQ}_g(\underline{k}, \epsilon)$.
Motivated by Thurston's work, we defined another metric K_F on $\mathcal{TQ}_g(\underline{k}, \epsilon)$ through ratios of lengths of saddle connections:

$$K_F(q_1, q_2) := \max\{K_F^a(q_1, q_2), K_F^a(q_2, q_1)\},$$

$$K_F^a(q_1, q_2) := \sup_{\gamma \in SC(q_1)} \log\left(\frac{\hat{l}_{q_2}(\gamma)}{\hat{l}_{q_1}(\gamma)}\right),$$

where $SC(q_1)$ is the set of saddle connections of q_1 (geodesics for the flat metric meeting singular points only at their extremities), and $\hat{l}_{q_i}(\gamma)$ is the length of the geodesic representative for the metric $|q_i|$ of the homotopy class of γ with fixed endpoints.
While it is possible to prove $L_F(q_1, q_2) = K_F(q_1, q_2)$ if q_1 and q_2 are on the same orbit of the action of $GL(2, \mathbb{R})^+$ (proposition 4.3.2), in the general case we were not able to adapt Thurston's proof of $L = K$. This is mainly because, as it is explained in the end of chapter 4, we believe it is not possible to find a flat analogue to the large class of geodesics of L which Thurston uses in the proof of $L = K$.

In chapter 5 we introduced an asymmetric analogue L_F^a to L_F on $\mathcal{TQ}_g^{(1)}(\underline{k}, \epsilon)$ (which is the subset of $\mathcal{TQ}_g(\underline{k}, \epsilon)$ corresponding to surfaces of unitary area) defined as

$$L_F^a(q_1, q_2) := \inf_{\varphi \in \mathcal{D}} \log(Lip(\varphi)_{q_1}^{q_2}),$$

$$Lip(\varphi)_{q_1}^{q_2} = \sup_{p\in S_g\setminus\Sigma}\left(\sup_{v\in T_pS_g, ||v||_{q_1}=1} ||d\phi_p v||_{q_2}\right),$$

with $\mathcal{D}$ being the set of functions $\varphi : S_g \to S_g$ which are homotopic to the identity, differentiable almost everywhere and which fix the points of Σ.

We are able to reduce the proof of the equality of L_F^a and K_F^a on $\mathcal{TQ}_g^{(1)}(\underline{k}, \epsilon)$ to the proof of two statements (corresponding to following theorem 0.0.1 and conjecture 0.1) about 1-Lischitz maps between planar polygons. In order to give the reader an idea of the reasonings involved, we briefly state them in a slightly simplified version.

Consider two planar polygons Δ and Δ' such that there is an injective function

$$\iota : Vertices(\Delta) \to Vertices(\Delta')$$

which to every vertex v of Δ associates a unique vertex $\iota(v) = v'$. Suppose both Δ and Δ' have exactly three vertices with strictly convex internal angle, which we denote x_i and x_i', $i = 1, 2, 3$ respectively.
Suppose furthermore that for every $x, y \in Vertices(\Delta)$ it results

$$d_\Delta(x, y) \geq d_{\Delta'}(x', y'),$$

where d_Δ (resp. $d_{\Delta'}$) is the intrinsic Euclidean metric inside Δ (resp. Δ'): $d_\Delta(x, y)$ (resp. $d_{\Delta'}(x', y')$) is defined as the infimum of the lengths, computed with respect to the Euclidean metric, of all paths from x to y (resp. from x' to y') entirely contained in Δ (resp. in Δ').
We say that vertices of Δ and of $\iota(Vertices(\Delta))$ are *disposed in the same order* if it is possible to choose two parametrizations $\gamma : [0, 1] \to \partial\Delta$ and $\gamma_1 : [0, 1] \to \partial\Delta'$ such that $\gamma(0) = x_1$, $\gamma_1(0) = x_1'$ and γ, γ_1 meet respectively vertices of Δ and of Δ' in the same order.

Theorem 0.0.1. *If $Vertices(\Delta)$ and $\iota(Vertices(\Delta))$ are disposed in the same order, then there is a 1-Lipschitz map $f : \Delta \to \Delta'$ (with respect to the intrinsic Euclidean metrics of the polygons) which sends vertices to corresponding vertices.*

Conjecture 0.1. *If $Vertices(\Delta)$ and $\iota(Vertices(\Delta))$ are not disposed in the same order, then for every point $p \in \Delta$ there is a point $p' \in \Delta'$ such that*

$$d_\Delta(p, x_i) \geq d_{\Delta'}(p', x_i'), \quad i = 1, 2, 3.$$

We were able to prove theorem 0.0.1, which corresponds to theorem 5.3.3, but not conjecture 0.1, which corresponds to conjecture 5.1: we will still explain why we believe it must be true.
We proved the following theorem, which is the main result of this thesis.

Theorem 0.0.2. *If conjecture 0.1 is true, then for every* $q_1, q_2 \in \mathcal{TQ}_g^{(1)}(\underline{k}, \epsilon)$*, it results*

$$L_F^a(q_1, q_2) = K_F^a(q_1, q_2).$$

We proved theorem 0.0.2 adapting the idea of F.A.Valentine's proof ($[Va]$) of Kirszbraun's theorem for $\mathbb{R}^2$.

Theorem 0.0.3. *Let* $S \subset \mathbb{R}^2$ *be any subset and* $f : S \to \mathbb{R}^2$ *a 1-Lipschitz map. Given any set* T *which contains* S*, it is possible to extend* f *to a 1-Lipschitz map* $\hat{f} : T \to \mathbb{R}^2$ *such that* $\hat{f}(T)$ *is contained in the convex hull of* $f(S)$*.*

In the last chapter of the thesis we present a result which is not related to Thurston's metric, but even so has some significance on the theory of moduli spaces of translation surfaces.
There is a particular hermitian form h on $\mathcal{H}_g(2g-2)$, which restricts on every tangent space $T_\varphi\mathcal{H}_g(2g-2) \simeq H^1(X, \mathbb{C})$ to the hermitian product h_φ of signature $(2g, 0)$:

$$h_\varphi(\dot{\varphi}, \dot{\psi}) := \frac{1}{2}\int_{X_\varphi} \dot{\varphi} \wedge (*\overline{\dot{\psi}}) = \frac{i}{2}\left(\int_{X_\varphi} \dot{\varphi}^{1,0} \wedge \overline{\dot{\psi}^{1,0}} - \int_{X_\varphi} \dot{\varphi}^{0,1} \wedge \overline{\dot{\psi}^{0,1}}\right)$$

for every $\dot{\phi}, \dot{\psi} \in T_\varphi\mathcal{H}_g(2g-2)$. In the preceding expression of h_φ, we denoted by X_φ the complex structure underlying φ and by $*$ the Hodge operator of X_φ.
We proved that h is not Kähler, showing that the corresponding alternating form ω is not closed. In order to do so we used some formulas obtained by Royden ($[Ro]$) concerning the Hodge operator of Riemann surfaces obtained from the deformation given by a Beltrami differential.

Contents

Chapter 1

Teichmüller and moduli spaces

In this first chapter we define Teichmüller and moduli spaces of Riemann surfaces and review some known facts about them.

1.1 Teichmüller spaces

1.1.1 Uniformization theorem

The first fundamental result we state is the *Uniformization theorem*, originally due to Klein, Poincaré and Koebe (a proof can be found for example in [Ah]):

Theorem 1.1.1. *(Uniformization)*
The Riemann sphere $\hat{\mathbb{C}}$ *is the universal cover only of itself.*
The complex plane $\mathbb{C}$ *is the universal cover of itself, of the punctured plane and of all compact Riemann surfaces homeomorphic to a torus.*
The universal cover of all other Riemann surfaces is isomorphic to $\mathbb{H}$*.*

Riemann surfaces such that their universal cover is isomorphic to $\mathbb{H}$ are referred to as *hyperbolic*.
From this result one can deduce that every hyperbolic Riemann surface is biholomorphic to a quotient $\mathbb{H}/\Lambda$ where Λ is a group of holomorphic automorphisms of $\mathbb{H}$ acting freely and properly discontinuously. Such a group can be identified with a discrete subgroup of $PSL(2,\mathbb{R})$ and is referred to as a *Fuchsian group*.

Since the hyperbolic plane is endowed with the *Poincaré metric*

$$ds^2_{\mathbb{H}} = \frac{|dz|^2}{(Im(z))^2}$$

and $PSL(2,\mathbb{R})$ is the isometry group of $\mathbb{H}$, it follows that every hyperbolic Riemann surface is endowed with a unique complete Riemannian metric.
On the other hand, every Riemannian metric h on an oriented surface S provides a unique complex structure. This is a consequence of Gauss' theorem on the existence of *isothermal coordinates*: S admits a unique complex structure such that in local complex coordinates it results

$$h = f(z)|dz|^2,$$

where f is a smooth and positive local function.
For this reason, specifying a complex structure on the topological surface underlying a hyperbolic Riemann surface is equivalent to specifying a hyperbolic metric: we will use this fact in the definition of Teichmüller spaces.

1.1.2 Teichmüller space of the torus

The Teichmüller space of the torus is particularly simple and instructive: for these reasons we will treat it separately.
We indicate by $\mathcal{F}_1(T)$ the set of flat metrics on the torus T and by $Diff_0(T)$ the set of diffeomorphisms of the torus T which are isotopic to the identity: the group $Diff_0(T)$ acts on $\mathcal{F}_1(T)$ by pullback.
The group $\mathbb{R}_{>0}$ acts on $\mathcal{F}_1(T)$ rescaling the metrics: the quotient of $\mathcal{F}_1(T)$ by the action of $\mathbb{R}_{>0}$ is denoted by $\mathbb{P}(T)$.

Definition 1.1. *The Teichmüller space $\mathcal{T}_1$ of the torus T is defined as the quotient*

$$\mathcal{T}_1 := \mathbb{P}(T)/Diff_0(T).$$

There is another equivalent definition of $\mathcal{T}_1$ which involves the use of markings: a *marking* of the torus T is a choice of a group isomorphism $\phi : \mathbb{Z}^2 \to \pi_1(T)$.

Definition 1.2. *The Teichmüller space $\mathcal{T}_1$ of the torus T is defined as the space of equivalence classes $[(\sigma, \phi)]$, where σ is a flat metric on T and ϕ is a marking. The couples (σ, ϕ) and (σ', ϕ') are equivalent if there is a diffeomorphism f of the torus and a constant $c > 0$ such that $f^*\sigma' = c\sigma$ and $f_*\phi = \phi'$.*

If the equivalence between the two definitions is not clear we refer the reader to the explanation of the equivalence of the two definitions of the Teichmüller space in the following section.

The Teichmüller space $\mathcal{T}_1$ can be explicitly described using *marked lattices.*

Definition 1.3. *A marked lattice of* $\mathbb{C}$ *is a lattice* Λ *of* $\mathbb{C}$ *(i.e. discrete subgroup* $\Lambda < \mathbb{C}$ *such that* $\mathbb{C}/\Lambda$ *is compact) together with an ordered set of generators.*

We denote by $\mathcal{L}$ the set of marked lattices of $\mathbb{C}$ up to the equivalence relation where two marked lattices are equivalent if they differ by an Euclidean isometry or a homothety. An element of $\mathcal{L}$ can thus be represented by an ordered pair $[(u_1, u_2)]$ of generators, where u_1 and u_2 are linearly independent and form a matrix with determinant equal to one.

Using the second definition of $\mathcal{T}_1$ one can show that there is a bijective correspondence

$$\Psi : \mathcal{L} \to \mathcal{T}_1$$

defined as follows.
Given $[(u_1, u_2)] \in \mathcal{L}$, we denote by Λ the lattice they generate and set

$$\Psi([u_1, u_2]) = [(\sigma, \phi)],$$

where σ is the canonical flat metric of $\mathbb{C}/\Lambda$ and ϕ is the isomorphism which sends $(1, 0)$ and $(0, 1)$ respectively to u_1 and u_2.
Conversely, to any element $[(\sigma, \phi)] \in \mathcal{T}_1$ one associates the lattice corresponding to the group of deck transformations and the marking given by $(\phi(1, 0), \phi(0, 1))$.

It is a standard fact that the space $\mathcal{L}$ in fact corresponds to the hyperbolic plane $\mathbb{H}$: combining these two results one obtains the following proposition.

Proposition 1.1.2. *There is a bijective correspondence*

$$\Phi_{\mathbb{H}} : \mathbb{H} \to \mathcal{T}_1, \quad \Phi_{\mathbb{H}}(\xi) = [(\sigma_\xi, \phi_\xi)],$$

where σ_ξ *is the canonical flat metric of* $\mathbb{C}/(\mathbb{Z} + \xi\mathbb{Z})$ *and* ϕ_ξ *sends* $(1, 0)$ *to itself and* $(0, 1)$ *to* ξ.

In section 1.4 we will see that the bijective correspondence $\Phi_{\mathbb{H}}$ is in fact an isometry between $\mathbb{H}$ and $\mathcal{T}_1$ endowed with a metric called *Teichmüller metric.*

Finally, notice that $\mathcal{T}_1$ can also be identified with the space $SL(2, \mathbb{R})/SO(2)$, since if u_1, u_2 generate a lattice Λ such that $Area(\mathbb{C}/\Lambda)$=1 (and this is always the case, up to rescaling), then they also define an element of $SL(2, \mathbb{R})$.

1.1.3 Teichmüller space $\mathcal{T}_g^n$

Denote by S_g a closed and oriented surface of genus $g \geq 2$ and by S_g^n a surface obtained from S_g removing n points (referred to as punctures).

We say that a Riemann surface X of genus $g \geq 2$ is of *finite type* if it corresponds to a conformal structure on S_g^n, that is, if every puncture has an open neighborhood $U_i \subset X$ such that $X \setminus (U_1 \cup \cdots \cup U_n)$ is compact and for every $i = 1, \ldots, n$ there is a homeomorphism z_i to the punctured disk

$$z_i : U_i \to \mathbb{D} := \{z \in \mathbb{C} \mid 0 < |z| < 1\}$$

which is holomorphically compatible with the complex structure of X.
Riemann surfaces of finite type are often referred to as *punctured Riemann surfaces*: these are the only types of Riemann surfaces we will consider in the present thesis (compact Riemann surfaces are just punctured Riemann surfaces with zero punctures).
Denote by $Diff_0^+(S_g^n)$ the group of diffeomorphisms of S_g^n which are homotopic to the identity and which fix the punctures. The set $Hyp(S_g^n)$ is composed of complete, finite-area hyperbolic metrics on S_g^n with cusps in the punctures.

Definition 1.4. *The Teichmüller space $\mathcal{T}_g^n$ of S_g^n is defined as the set of isotopy classes of hyperbolic metrics on S_g^n:*

$$\mathcal{T}_g^n := Hyp(S_g^n)/Diff_0^+(S_g^n).$$

In case the reference surface has no punctures, we will write $\mathcal{T}_g$ instead of $\mathcal{T}_g^0$.

Notice that, due to the correspondence of section 1.1.1, this definition could be restated in terms of conformal structures.
Since there is no possible ambiguity, we will denote by X both the Riemann surface and the correspondent *hyperbolic surface*, which is the datum of a hyperbolic metric on S_g^n.

There is also an alternative definition of $\mathcal{T}_g^n$ which involves the use of marked surfaces: a *marking* of X is a diffeomorphism $\phi : S_g^n \to X$.

Definition 1.5. *The Teichmüller space $\mathcal{T}_g^n$ can be defined as the set of equivalence classes $\mathcal{X} = [(X, \phi)]$ of marked surfaces. Two marked surfaces (X, ϕ) and (X', ϕ') are said to be equivalent if there is an isometry $f : X \to X'$ which is homotopic to $\phi' \circ \phi^{-1}$.*

The equivalence of the two definitions can be easily understood. Indeed, suppose that (X, ϕ) and (X', ϕ') are such that $\phi = \phi' = Id$, then it is trivial that $\phi' \circ \phi^{-1}$ is homotopic to an isometry if and only if there is an isometry $f : X \to X'$ homotopic to the identity. On the other hand, note that every marking $\phi : S_g^n \to X$ is equivalent to another one $Id : S_g^n \to \phi^* X$ with the identity function between the topological surfaces.

Given any simple closed curve α in S_g^n and $\mathcal{X} \in \mathcal{T}_g^n$, we denote by $\hat{l}_{\mathcal{X}}(\alpha)$ the length of the geodesic representative in the homotopy class of $\phi(\alpha)$ for the hyperbolic metric of X.

1.1.4 Topology of $\mathcal{T}_g^n$

There are many ways to endow $\mathcal{T}_g^n$ with a topology: one of them consists in identifying $\mathcal{T}_g^n$ with a quotient of a space of representations.
Denote by $Rep(\pi_1(S_g^n), PSL(2,\mathbb{R}))$ the space of discrete and faithful representations $\rho : \pi_1(S_g^n) \to PSL(2,\mathbb{R})$ and notice that the group $PGL(2,\mathbb{R})$ acts on the space $Rep(\pi_1(S_g^n), PSL(2,\mathbb{R}))$ by conjugation:

$$(g \cdot \rho)(\gamma) := g\rho(\gamma)g^{-1}$$

for every $\rho \in Rep(\pi_1(S_g^n), PSL(2,\mathbb{R})), g \in PGL(2,\mathbb{R})$ and $\gamma \in \pi_1(S_g)$.

It is thus possible to define a function

$$\Theta : \mathcal{T}_g^n \to Rep(\pi_1(S_g^n), PSL(2,\mathbb{R}))/PGL(2,\mathbb{R})$$

which associates to $[(X,\phi)] \in \mathcal{T}_g^n$ the representation $\Theta([(X,\phi)])$ obtained fixing a metric universal cover $\widetilde{X}$ of X, the monodromy representation

$$\hat{\rho}_X : \pi_1(X) \to Isom^+(\widetilde{X})$$

and considering the isomorphisms

$$\phi_* : \pi_1(S_g^n) \simeq \pi_1(X), \quad \tau : \mathbb{H} \simeq \widetilde{X}, \quad Isom^+(\mathbb{H}) \simeq PSL(2,\mathbb{R}).$$

In other words, for every $\gamma \in \pi_1(S_g^n)$, we set

$$\Theta([(X,\phi)])(\gamma) := \tau^*(\rho_X(\phi_*(\gamma))).$$

In this definition of Θ we made several choices: the choice of (X,ϕ) in its equivalence class, the choice of τ and the choice of $\hat{\rho}_X$. The function Θ il well-defined since a change in any of these choices produces a conjugation of $\Theta([(X,\phi)])$ by an element of $PGL(2,\mathbb{R})$.

Since the function Θ is a bijection onto its image (see for example $[FM]$), it follows that the Teichmüller space inherits the compact-open topology of the space $Rep(\pi_1(S_g^n), PSL(2,\mathbb{R}))$.

There is another notable characterization of $\mathcal{T}_g^n$, which in fact produces a bijection $\mathcal{T}_g^n \simeq \mathbb{R}^{6g-6+2n}$: it is given by Fenchel-Nielsen coordinates. We will briefly explain

how these coordinates are obtained, for proofs and a deeper explanation one could refer to $[FM]$ section 10.6.2 or $[Hu]$ section 7.6.
First one needs to consider the Teichmüller space $\mathcal{T}(P)$ of a pair of pants P, which is defined as the set of hyperbolic metrics with geodesic border on P up to isotopies. Denote by $\gamma_1, \gamma_2, \gamma_3$ the curves which make up the boundary components of P, then one gets the following result.

Lemma 1.1.3. *The map which associates to each $\mathcal{X} \in \mathcal{T}(P)$ the triple $(\hat{l}_{\mathcal{X}}(\gamma_1), \hat{l}_{\mathcal{X}}(\gamma_2), \hat{l}_{\mathcal{X}}(\gamma_3)) \in \mathbb{R}^3_+$ is a bijection.*

The idea behind Fenchel-Nielsen coordinates is to decompose the surface S_g^n into pairs of pants by cutting it along a set of $3g-3+n$ simple closed curves γ_i (the lengths of which determine the hyperbolic structure on each pair of pants, as the previous lemma states) and then use $3g-3+n$ twisting parameters to determine how the pairs of pants are glued together.
The twisting parameters depend on the choice of another set of curves σ_j such that the intersection of their union with each pair of pants P of the decomposition gives three arcs connecting each pair of boundary components of P. Since Fenchel-Nielsen coordinates depend on the choice of the two sets of curves γ_i and σ_j, the union of such sets is called a *coordinate system of curves on S_g^n*.
The twisting parameter of the curve γ_i and $\mathcal{X} = [(X, \phi)]$, denoted by $\theta_i(\mathcal{X})$, is defined as

$$\theta_i(\mathcal{X}) := 2\pi \frac{d_1 - d_2}{\hat{l}_{\mathcal{X}}(\gamma_i)},$$

where d_1, d_2 are two displacement parameters computed as we now explain.
Choose one curve $\phi(\sigma_j)$ which crosses $\phi(\gamma_i)$ and consider the two pairs of pants P_1, P_2 which have $\phi(\gamma_i)$ as a boundary component: the intersection of $\phi(\sigma_j)$ with P_1 (resp. P_2) is an arc, denoted $\phi(\sigma_j)^1$ (resp. $\phi(\sigma_j)^2$), which connects $\phi(\gamma_i)$ with $\phi(\gamma_l)$ (resp. $\phi(\gamma_m)$). Let V_i, V_l, V_m be regular metric neighborhoods respectively of $\phi(\gamma_i), \phi(\gamma_l), \phi(\gamma_m)$ and α_1 (resp. α_2) be the unique shortest arc connecting $\phi(\gamma_i)$ and $\phi(\gamma_l)$ (resp.$\phi(\gamma_m)$). Denote by $\hat{\phi}(\tau_j)^1$ (resp. $\hat{\phi}(\tau_j)^2$) the arc obtained modifying $\phi(\tau_j)^1$ (resp. $\phi(\tau_j)^2$) by an isotopy which leaves the endpoints fixed in such a way that it coincides with α_1 (resp. α_2) outside $V_i \cup V_l$ (resp. $V_i \cup V_m$). Then d_1 (resp. d_2) is defined as the signed horizontal displacement of the endpoints of $\hat{\phi}(\tau_j)^1 \cap \partial V_i$ (resp. $\hat{\phi}(\tau_j)^2 \cap \partial V_i$).
One then can prove that considering the other curve $\phi(\sigma_t)$ of the collection which crosses $\phi(\gamma_i)$ it is possible to obtain the same twisting parameter.

Theorem 1.1.4. *Fenchel-Nielsen coordinates relative to any coordinate system of curves on S_g^n give a bijection*

$$FN : \mathcal{T}_g^n \to \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}$$

defined by

$$FN(\mathcal{X}) = (\hat{l}_{\mathcal{X}}(\gamma_1), \ldots, \hat{l}_{\mathcal{X}}(\gamma_{3g-3+n}), \theta_1(\mathcal{X}), \ldots, \theta_{3g-3+n}(\mathcal{X})).$$

The two identifications of $\mathcal{T}_g^n$ we just presented induce the same topology.
In fact, more is true: this topology is also equivalent to the one induced by the Teichmüller metric which we will introduce in section 1.4 (this fact is proved for example in [*Abi*]).

Lastly, we mention the characterization of the Teichmüller space $\mathcal{T}_g^n$ as a subset of $\mathbb{R}^{\mathcal{S}}$, the set of real valued functions on the set $\mathcal{S}$ of isotopy classes of simple closed curves on S_g^n.
If $\mathbb{R}^{\mathcal{S}}$ is endowed with the pointwise convergence, it is possible to show (see [*Lei*]) that the map

$$l_* : \mathcal{T}_g^n \to \mathbb{R}^{\mathcal{S}}, \quad l_*(\mathcal{X})(\alpha) := \hat{l}_{\mathcal{X}}(\alpha)$$

is a proper embedding.

Furthermore, if $n = 0$ then there are $9g - 9$ simple closed curves such that $l_* : \mathcal{T}_g \to \mathbb{R}^{9g-9}$ is a proper embedding.

1.2 Complex structure of $\mathcal{T}_g^n$

The Teichmüller space can be endowed with the structure of complex manifold using Beltrami differentials: we thus introduce them and state their properties.

1.2.1 Beltrami differentials

Definition 1.6. *Let U, V be two open subsets of $\mathbb{C}$ and $f : U \to V$ an orientation-preserving homeomorphism which is smooth outside of a finite number of points. Let $p \in U$ be a point at which f is differentiable.*
The complex dilatation $\mu_f(p)$ of f at p is defined as

$$\mu_f(p) := \left(\frac{\partial f}{\partial \overline{z}}(p)\right) \Big/ \left(\frac{\partial f}{\partial z}(p)\right).$$

The dilation $K_f(p)$ of f at p is defined as

$$K_f(p) := \frac{|f_z(p)| + |f_{\overline{z}}(p)|}{|f_z(p)| - |f_{\overline{z}}(p)|} = \frac{1 + |\mu_f(p)|}{1 - |\mu_f(p)|}.$$

The complex dilatation gives an important information about f, since the direction of the maximal stretching of df_p is given by $\frac{1}{2}\arg(\mu_f(p))$. Furthermore it results

$||\mu_f||_\infty < 1$ if and only if f is orientation preserving.
There is also a very clear geometric interpretation of $K_f(p)$. The differential df_p maps the unit circle of T_pU to an ellipse $E \subset T_pV$ with major and minor axis of length respectively M and m. Then it results

$$K_f(p) = \frac{M}{m}.$$

The proof of this fact is straightforward: first observe that the ellipse E can be parametrized as $E(t) = f_z(p)e^{it} + f_{\overline{z}}(p)e^{-it}$ for $t \in [0, 2\pi]$. Then note that

$$|E(t)| = |f_z(p)||1 + \mu_f(p)e^{-2it}|$$

and

$$1 - |\mu_f(p)| \leq |1 + \mu_f(p)e^{-2it}| \leq 1 + |\mu_f(p)|.$$

We can define the *global dilatation* K_f of the function f to be

$$K_f := \sup_{p \in \hat{U}} K_f(p),$$

where $\hat{U} \subset U$ is the subset where f is differentiable. Clearly $1 \leq K_f \leq \infty$.

An orientation-preserving homeomorphism $f : U \to V$ which is smooth outside of a finite number of points and such that $K_f < \infty$ is said to be *K_f-quasiconformal* or simply *quasiconformal*.

Quasiconformal homeomorphisms have some nice properties which we now list.

Lemma 1.2.1. *Let $f, g : U \to V$ be two quasiconformal homeomorphisms, then the following facts are true.*

(i) The function f is 1-quasiconformal if and only if it is holomorphic.

(ii) The composition $f \circ g$ is quasiconformal and

$$K_{f\circ g} \leq K_f K_g.$$

(iii) The inverse f^{-1} is quasiconformal and

$$K_{f^{-1}} = K_f.$$

(iv) If g is holomorphic then

$$K_{f\circ g} = K_{g\circ f} = K_f.$$

In particular property (*iv*) allows us to consider quasiconformal maps between Riemann surfaces.
It is useful to consider orientation preserving homeomorphisms f which are not necessarily differentiable, but which satisfy the Beltrami equation $f_{\overline{z}} = \mu f_z$ in some sense. For this reason we extend the definition of quasi-conformal maps to oriented homeomorphisms $f : U \to V$ such that:

- the distributional partial derivatives of f with respect to z and $\overline{z}$ can be represented by locally integrable functions,
- the function $\mu_f := f_{\overline{z}}/f_z$ is such that $||\mu||_\infty \in [0, 1]$.

Properties of lemma 1.2.1 remain valid and in particular, due to the following lemma by Weil, it is still true $\mu_f = 0$ if and only if f is holomorphic.

Lemma 1.2.2. *(Weil)*
Let U be an open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$, $f \in L^1_{loc}(U)$. If $f_{\overline{z}} \equiv 0$, then f is holomorphic.

The complex dilatation of a map $f : X \to X'$ between Riemann surfaces can be actually regarded as a form in $\mathcal{A}^{0,1}_{\mathbb{C}}(X, T^{1,0}_X)$. Indeed, denote by μ^z_f the complex dilatation of f computed in the local coordinate z of X. Notice that if z and w are two overlapping local coordinates of X then

$$\mu^w_f = \mu^z_f \varphi_{zw}, \quad \varphi_{zw} = \overline{\left(\frac{dz}{dw}\right)} \Big/ \left(\frac{dz}{dw}\right)$$

that is, μ_f changes under variation of coordinates as forms in $\mathcal{A}^{0,1}_{\mathbb{C}}(X, T^{1,0}_X)$ do. What is more, $|\varphi_{zw}| = 1$.
The form μ_f associated to f is called a *Beltrami differential*.
We denote by $\mathcal{B}(X)$ the space of Beltrami differentials on X: it is a Banach space with the L^∞-norm.

One could wonder wether it is also possible to establish the inverse association. Specifically, given any $\mu \in L^\infty(\mathbb{C})$ is there always a quasi-conformal homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that it satisfies the Beltrami equation

$$\mu \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \overline{z}}$$

almost everywhere?
The following theorem (proved by Ahlfors-Bers in [*AB*]) gives a positive answer to this question and establishes also an analytic dependence of the solution to μ.

Theorem 1.2.3. *(Measurable Riemann mapping theorem) For every $\mu \in L^\infty(\mathbb{C})$ such that $||\mu||_\infty < 1$ there is a unique quasi-conformal homeomorphism $f^\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which fixes $0, 1, \infty$ and satisfies almost everywhere the Beltrami equation*

$$\mu \cdot \left(\frac{\partial f^\mu}{\partial z}\right) = \frac{\partial f^\mu}{\partial \overline{z}}.$$

The function f^μ is smooth wherever μ is and f^μ varies complex analytically with respect to μ.

We see that the uniqueness statement of this theorem implies that if $\mu(\overline{z}) = \overline{\mu(z)}$, then f^μ restricts to a self map of $\mathbb{H}$ and if μ is Λ-equivariant then also f^μ is Λ-equivariant. Consequently one obtains the following corollary.

Corollary 1.2.4. *Given any Riemann surface X of finite type and genus $g \geq 2$, for any $\mu \in \mathcal{B}(X)_1$, there is a Riemann surface X^μ and an homeomorphism $f^\mu : X \to X^\mu$ such that $\mu_{f^\mu} = \mu$. If there is another couple (Y, f) with the same properties then $f^\mu \circ f^{-1}$ is an isomorphism.*

1.2.2 Quadratic differentials

Definition 1.7. *A holomorphic quadratic differential on a compact Riemann surface X is a holomorphic section of the symmetric square of the holomorphic cotangent bundle of X.*

We denote by $\mathcal{Q}(X)$ the space of holomorphic quadratic differentials on X. Any holomorphic quadratic differential $q \in \mathcal{Q}(X)$ can be written locally in any holomorphic chart (U, z) as

$$q = f(z)dz^2,$$

where $f : U \to \mathbb{C}$ is a holomorphic function with a finite set of zeroes and if $g(w)dw^2$ is the expression of q on another chart (V, w) of X, then the function f and g are such that

$$g(w)\left(\frac{dw}{dz}\right)^2 = f(z).$$

We say that $p \in X$ is a zero of q if $f(p) = 0$, from the previous equality it follows that this definition does not depend from the chosen chart.
A holomorphic quadratic differential on a punctured Riemann surfaces is defined as a meromorphic quadratic differential on $\overline{X}$ with at most simple poles on the points corresponding to punctures. The space of holomorphic quadratic differentials on a punctured Riemann surface X is denoted by $\mathcal{Q}(X)$.
Sometimes it will be useful to consider punctured Riemann surface as *Riemann surfaces with marked points* (which should not be confused with marked Riemann surfaces, as defined in section 1.1.3), i.e. closed Riemann surfaces X together with a finite

set of points $P \subset X$ which correspond to the punctures. Given a marked Riemann surface (X, P), we denote by $\mathcal{Q}(\overline{X}, P)$ the space of meromorphic quadratic differentials on X having at most simple poles in the points of P.

Each holomorphic quadratic differential has a fixed finite number of zeroes: this number, counting multiplicities, minus the number of the poles, is equal to $4g - 4$.

Proposition 1.2.5. *Let X be a closed Riemann surface and q a meromorphic quadratic differential on X with at most simple poles.*
There exist local coordinates η on X, called natural coordinates, such that:

- *at any point where q has a zero or a pole of order $k \geq -1$, the local expression of q is $q = \eta^k d\eta^2$,*
- *at any other point the local expression of q is $d\eta^2$.*

Proof. Pick any point $p \in X$ and a local coordinate z on X such that $\{p\} = \{z = 0\}$. Suppose that q vanishes to order $k \geq 0$ at p, then we can write $q = f(z)dz^2$ with $f(z) = z^k g(z)$, where $g(z)$ is a holomorphic function with $g(0) \neq 0$.
Locally near p the function g admits a single valued branch of the square root, and we can consider the function

$$\Phi(z) := \int_0^z w^{k/2}\sqrt{g(w)}dw$$

which vanishes at order $\frac{k+2}{2}$ at p and admits a $\frac{k+2}{2}$-st root. We define the local coordinate $\eta(z)$ by

$$\frac{2}{k+2}\eta(z)^{\frac{k+2}{2}} = \Phi(z)$$

and after an easy computation it follows $\eta^k d\eta^2 = q$ as desired. □

The space $\mathcal{Q}(X)$ has complex dimension 1 in case of $g = 1$ and $3g-3+n$ otherwise (it follows from Riemann-Roch theorem). Furthermore, $\mathcal{Q}(X)$ can be made into a normed space with the L^1-norm:

$$||q||_1 := \int_X |q|$$

and with the topology induced by this norm $\mathcal{Q}(X)$ is homeomorphic to $\mathbb{R}^{6g-6+2n}$.

1.2.3 Complex structure

We now endow $\mathcal{T}_g$ with a complex structure which relies on Beltrami differentials. Using the result of corollary 1.2.4 it is possible to define a map

$$\Psi_X : \mathcal{B}(X) \to \mathcal{T}_g, \quad \Psi_X(\mu) := [(X^\mu, f^\mu)].$$

Denote by $\mathcal{B}(X)^{ban} \subset \mathcal{B}(X)$ the subset consisting of the Beltrami differentials which are mapped to $\mathcal{X} = [(X, Id)]$ by the function Ψ_X. Indicate by $\mathcal{B}^0$ the tangent space $T_0\mathcal{B}(X)^{ban}$, we claim that the quotient space $\mathcal{B}(X)/\mathcal{B}^0$ has dimension $3g-3$. Indeed, $\mathcal{B}(X)/\mathcal{B}^0$ can be interpreted as the Dolbeault cohomology space $H^{0,1}_{\overline{\partial}}(X, T_X^{1,0})$, since $\mathcal{B}(X)$ can be identified with the space of $\overline{\partial}$-closed forms $Z^{0,1}(X, T_X^{1,0})$ and $\mathcal{B}^0$ with the space of $\overline{\partial}$-exact forms $\overline{\partial}\mathcal{A}^{0,0}(X, T_X^{1,0}) = B^{0,1}(X, T_X^{1,0})$.
Since X is smooth, compact and connected, we can use Serre duality theorem to state that there is a bilinear, non degenerate pairing of $\mathcal{B}(X)/\mathcal{B}^0$ with the space of holomorphic quadratic differentials $\mathcal{Q}(X)$ on X, which has complex dimension $3g-3$. It follows that there exists a linear space Z_X of dimension $3g-3$ such that

$$T_0 Z_X \oplus \mathcal{B}^0 = T_0\mathcal{B}(X) = \mathcal{B}(X).$$

We state that there exists a neighborhood U_X of $0 \in \mathcal{B}(X)_1$ (the unit ball with respect to the L^∞-norm) such that the map

$$\Psi_X : U \cap Z \to \mathcal{T}_g \tag{1.1}$$

is injective on an open subset. The map (1.1) will be the local chart around $\mathcal{X}$ of the complex structure. Indeed, repeating the previous procedure around another point $\mathcal{X}' = [(X', f)] \in \mathcal{T}_g$, we see that the change of chart $\Psi_{X'} \circ \Psi_X$ is holomorphic, since $(\Psi_{X'} \circ \Psi_X)(\mu_0) = \mu(f^{\mu_0} \circ f) = \mu(z)\frac{d\overline{z}}{dz}$ with

$$\mu(z) = \frac{\mu_f(z) + \mu_0(f(z))\left(\overline{\frac{\partial f}{\partial z}(z)}/\frac{\partial f}{\partial z}(z)\right)}{1 + \overline{\mu_f(z)}\left(\overline{\frac{\partial f}{\partial z}(z)}/\frac{\partial f}{\partial z}(z)\right)\mu_0(f(z))}$$

and so depends holomorphically from μ_0.
This construction explains why Beltrami differentials are often interpreted as tangent vectors on $\mathcal{T}_g$ and holomorphic quadratic differentials as cotangent vectors.

We will now briefly explain another construction, firstly due to Bers ([$Be2$]), which endows the Teichmüller space with a complex structure.
A key role in this construction will be played by the *Schwarzian derivative*: the Schwarzian derivative $S(f)$ of any locally injective holomorphic function $f : U \to \mathbb{C}$ is defined by the following formula

$$S(f) := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

The intuition behind the Schwarzian derivative of f is that it measures the distortion of the cross-ratio of four points done by f: the Schwarzian derivative of a Möbius map is indeed zero (details about these facts can be found for example in [GL]).

Fix any Fuchsian group Λ, then we will construct an injective map Ω_Λ from $\mathcal{T}_g^n$ to a subset of the set of holomorphic maps on the lower half-plane $\mathbb{L}$.
To any $[(X,\phi)] \in \mathcal{T}_g^n$ associate the normalized lifting $\widetilde{\phi} : \mathbb{H} \to \mathbb{H}$ which fixes $0, 1, \infty$ and denote by $\widetilde{\mu}_\phi$ its Beltrami differential. The differential $\widetilde{\mu}_\phi$ is such that $||\widetilde{\mu}_\phi||_\infty < 1$ and

$$\widetilde{\mu}_\phi(z)\gamma'(z) = \widetilde{\mu}_\phi(\gamma(z))\overline{\gamma'(z)} \tag{1.2}$$

for every $z \in \mathbb{H}$ and $\gamma \in \Lambda$.
Denote by $L^\infty_\Lambda(\mathbb{H})_1$ the unit ball with respect to the L^∞-norm in the Banach space of measurable essentially bounded functions μ on $\mathbb{H}$ which satisfy condition (1.2). Each function $\widetilde{\mu} \in L^\infty_\Lambda(\mathbb{H})_1$ can be extended to a function $\hat{\mu}$ on $\hat{\mathbb{C}}$ which satisfies condition (1.2) for every $z \in \hat{\mathbb{C}}$ by imposing $\hat{\mu}(z) = \widetilde{\mu}(z)$ for $z \in \mathbb{H}$ and $\hat{\mu}(z) = 0$ for $z \in \mathbb{L}$. The Riemann mapping theorem then grants the existence of a quasi-conformal map $f^{\hat{\mu}} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, unique up to a composition with a Möbius map, whose Beltrami coefficient is $\hat{\mu}$.
We denote by $B_\mathbb{L}(\Lambda)$ the Banach space of holomorphic maps $h : \mathbb{L} \to \mathbb{C}$ which satisfy the following conditions

$$(S(h) \circ \gamma)(z)(\gamma'(z))^2 = S(h)(z), \quad \forall z \in \mathbb{L}, \gamma \in \Gamma$$

$$\sup_{z \in \mathbb{L}} |S(h)(z)\rho_\mathbb{L}^{-2}| < \infty,$$

where $\rho_\mathbb{L}$ is the Poincaré metric on $\mathbb{L}$ and $||h|| := \sup_{z \in \mathbb{L}} |S(h)(z)\rho_\mathbb{L}^{-2}|$ is the norm which gives the structure of Banach space.
The Schwartzian derivative then induces a map

$$\mathcal{S} : L^\infty_\Lambda(\mathbb{H})_1 \to B_\mathbb{L}(\Lambda), \quad \mathcal{S}(\widetilde{\mu}) := S(f^{\hat{\mu}}|_\mathbb{L})$$

which finally gives the announced map

$$\Omega_\Gamma : \mathcal{T}_g^n \to B_\mathbb{L}(\Lambda), \quad \Omega_\Lambda([(X,\phi)]) := \mathcal{S}(\widetilde{\mu}_\phi).$$

The following theorem unites the *Bers embedding* and the *Ahlfors-Weil section* and defines a structure of Banach complex manifold on the Teichmuller space. A proof can be found in $[GL]$ or $[Be]$.

Theorem 1.2.6. *The map Ω_Λ is an homeomorphism onto its image and defines a global holomorphic chart.*
Furthermore, for every $\mathcal{S}(\widetilde{\mu}) \in \Omega_\Lambda(\mathcal{T}_g^n)$ there is a neighborhood U of $\mathcal{S}(\widetilde{\mu})$ in $B_\mathbb{L}(\Lambda)$ and a holomorphic map $s : U \to L^\infty_\Lambda(\mathbb{H})_1$ such that $\mathcal{S} \circ s = Id$ and $s \circ \mathcal{S}(\widetilde{\mu}) = \widetilde{\mu}$.

1.3 Teichmüller's metric

1.3.1 Grötzsch and Teichmüller extremal problem

The following problem is known as *Grötzsch's extremal problem.*

Problem 1.3.1. *Fix any two rectangles $R, R' \subset \mathbb{R}^2$ and consider the set $QC(R, R')$ of quasi-conformal homeomorphisms between R and R' which send horizontal sides of R to horizontal sides of R' and vertical sides of R to vertical sides of R'.*
Define the set $\mathcal{K}$ of global dilatations K_f of functions $f \in QC(R, R')$.
Does the set $\mathcal{K}$ have a minimum? If so, is it obtained for a unique function?

The answer to the previous problem was given by Grötzsch in 1928 ([Gr]) by the following theorem.
For simplicity, fix the rectangles R, R' to be $R = [0, a] \times [0, b]$, $R' = [0, a'] \times [0, b']$ and define $m(R) = \frac{a}{b}, m(R') = \frac{a'}{b'}$.

Theorem 1.3.2. *Every function $f \in \mathcal{K}$ is such that*

$$K_f \geq \max \left\{ \frac{m(R)}{m(R')}, \frac{m(R')}{m(R)} \right\}$$

and the equality is realized only for the affine function.

Proof. For every $y_0 \in [0, b]$ denote by γ_{y_0} the arc

$$\gamma_{y_0} : [0, a] \to \mathbb{R}^2, \quad \gamma_{y_0}(t) = (t, y_0).$$

Then the following inequalities are satisfied:

$$a' \leq l(f(\gamma_{y_0})) = \int_0^a \left| \frac{\partial f}{\partial z}(x, y) \right| |1 + \mu_f(x, y)| \, dx,$$

$$a'b \leq \int_R \left| \frac{\partial f}{\partial z}(x, y) \right| |1 + \mu_f(x, y)| \, dx \wedge dy,$$

$$(a'b)^2 \leq \left(\int_R \left| \frac{\partial f}{\partial z}(x, y) \right|^2 (1 - |\mu_f(x, y)|^2) dx \wedge dy \right) \cdot \left(\int_R \frac{|1 + \mu_f(x, y)|^2}{1 - |\mu_f(x, y)|^2} dx \wedge dy \right)$$

and thus

$$\frac{m(R')}{m(R)} \leq \frac{1}{ab} \int_R K_f(x, y) dx \wedge dy \leq K_f.$$

The other inequality is obtained considering f^{-1}.
Finally, one can see that if the equality holds, then $\mu_f(x, y)$ must be constant and consequently f is affine.

□

Grötzsch extremal problem was then generalized by Teichmüller to the case of Riemann surfaces.

Problem 1.3.3. *Let $f : X \to Y$ be a homeomorphism between Riemann surfaces of finite type and consider the set of dilatations of quasi-conformal maps between X and Y in the homotopy class of f.*
Does this set have a minimum? If so, is it obtained for a unique function?

Teichmüller proved that both these questions have a positive answer: this is a consequence of two theorems we will review in section 1.3.3.

This extremal problem on Riemann surfaces is also closely related to a metric $d_{\mathcal{T}}$ on $\mathcal{T}_g^n$, called *Teichmüller metric.*
For any $\mathcal{X} = [(X, \phi)], \mathcal{X}' = [(X', \phi')] \in \mathcal{T}_g^n$, the Teichmüller metric $d_{\mathcal{T}}(\mathcal{X}, \mathcal{X}')$ is defined as

$$d_{\mathcal{T}}(\mathcal{X}, \mathcal{X}') := \inf_{f \approx (\phi' \circ \phi^{-1})} \frac{1}{2} \log(K_f).$$

All axioms of metrics are satisfied and $d_{\mathcal{T}}$ defines a metric also on $\mathcal{T}_1$, which is referred to as Teichmüller metric.

From Teichmüller theorems it will follow that there always is a quasi-conformal map $h : X \to X'$ such that

$$K_h = \inf_{f \approx (\phi' \circ \phi^{-1})} K_f$$

and consequently

$$d_{\mathcal{T}}(\mathcal{X}, \mathcal{X}') = \frac{1}{2} \log(K_h).$$

The quasi-conformal map $h : X \to Y$ is called a *Teichmüller map*: its definition will be given in section 1.3.3 and relies on quadratic differentials and measured foliations.

1.3.2 Measured foliations

We now define *measured foliations*, which are closely related to holomorphic quadratic differentials.

A *singular foliation* $\mathcal{F}$ on S_g^n is a decomposition of S_g^n into *leaves* and a finite set of *singular points* of $\mathcal{F}$, in such a way that the following conditions are satisfied.

(i) For any non-singular point $p \in S_g^n$, there is a smooth chart from a neighborhood of p to $\mathbb{R}^2$ which maps leaves of $\mathcal{F}$ to horizontal line segments. The transition maps between charts take horizontal lines to horizontal lines.

(ii) For any singular point $p \in S_g^n$, there is a smooth chart from a neighborhood of p to $\mathbb{R}^2$ which maps leaves of $\mathcal{F}$ to the level sets of a $k-$pronged singular point, $k \geq 3$.

(iii) At the punctures of S_g^n the foliation extends to a foliation of the unpunctured surface S_g in such a way that each puncture becomes a k-pronged singular point, $k \geq 1$.

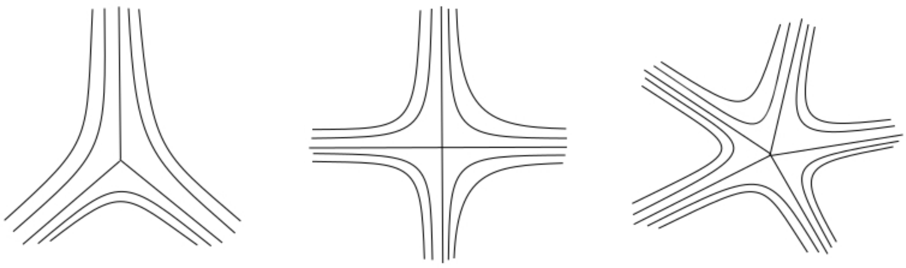

Figure 1.1: Examples of k-pronged singularities, with $k = 3, 4, 5$.

Any foliation $\mathcal{F}$ can be equipped with a transverse measure μ, which is a length function on arcs transverse to the foliation such that:

(i) it assigns a non negative real number to each arc,

(ii) it assigns 0 to an arc if and only if it lies inside a leaf of the foliation,

(iii) it is invariant for homotopies of arcs which keep arcs transverse and move the endpoints inside the same leaf.

Definition 1.8. *A measured foliation* $(\mathcal{F}, \mu)$ *on* S_g^n *is a singular foliation* $\mathcal{F}$ *equipped with a transverse measure* μ*.*

In sections 2.1 and 2.2 we will denote a measured foliation simply by $\mathcal{F}$ in order to lighten the notation and write $c\mathcal{F}$, $c \in \mathbb{R}_{>0}$ to indicate the measured foliation with the measure multiplied by the constant c.
There is an equivalence relation on the space of measured foliations called *Whitehead-equivalence*: it is generated by isotopies and Whitehead moves (which consist in collapsing to a point an arc which connects two singular points, see $[FLP]$ for details). We denote by $\mathcal{MF}$ (omitting to specify the genus and the number of punctures, since there will be no ambiguity on the topological type of the surface) the space of equivalence classes of measured foliations of S_g^n.

Each $q \in \mathcal{Q}(X)$ induces two foliations on X, called respectively horizontal and vertical foliation and denoted by $\mathcal{F}_q^+$ and $\mathcal{F}_q^-$.
Both foliations have singular set coinciding with the set of zeroes of q, while the leaves of $\mathcal{F}_q^+$ (respectively $\mathcal{F}_q^-$) are the smooth paths in X whose tangent vectors at each

point evaluate to positive (respectively negative) real numbers under q.
If, in a given chart, the local expression of q is $f(z)dz^2$, then the expressions of the measures μ_q^+ and μ_q^- respectively of $\mathcal{F}_q^+$ and $\mathcal{F}_q^-$ are

$$\mu_q^+(\gamma) := \int_\gamma |\Im(\sqrt{f}dz)|, \quad \mu_q^-(\gamma) := \int_\gamma |\Re(\sqrt{f}dz)|.$$

For any Riemann surface X of finite type it is thus possible to define a map

$$\Phi_X : \mathcal{Q}(X) \to \mathcal{MF}$$

which associates to each quadratic differential q its horizontal foliation $\mathcal{F}_q^+$.
Hubbard and Masur proved in 1979 that this map is in fact a bijection ([HM]).

1.3.3 Teichmüller's theorems

We now define what a *Teichmüller map* is.

Definition 1.9. *Given two Riemann surfaces with marked points (X, P) and (X', P'), a homeomorphism $h : (X, P) \to (X', P')$ is a Teichmüller map if there are meromorphic quadratic differentials $q_X \in \mathcal{Q}(X, P)$, $q_{X'} \in \mathcal{Q}(X', P')$ and a real number $K > 0$ such that:*

- *the homeomorphism h takes zeroes and poles of q_X respectively to zeroes and poles of $q_{X'}$,*
- *if p is not a zero or a pole of q, then the expression of h with respect to natural coordinates of q_X and $q_{X'}$ based at p and $h(p)$ is*

$$h(x + iy) = \sqrt{K}x + i\frac{1}{\sqrt{K}}y$$

or equivalently

$$h(z) = \frac{1}{2}\left(\left(\frac{K+1}{\sqrt{K}}\right)z + \left(\frac{K-1}{\sqrt{K}}\right)\overline{z}\right).$$

In other words the Teichmüller map h stretches the horizontal foliation of q_X by a factor of $\sqrt{K}$ and the vertical foliation of $q_{X'}$ by a factor of $\frac{1}{\sqrt{K}}$.
From the expression of h we get that its dilatation K_h is equal to K if $K \geq 1$ and to $1/K$ otherwise.
There is also a quite intuitive way of constructing a Teichmüller map from the initial data of a Riemann surface X, a holomorphic quadratic differential q_X on X and a constant $K > 0$: this is done by "stretching" the holomorphic charts of X by a factor of $\sqrt{K}$. To be more precise, consider $\hat{X}$ the complement in X of the zeroes and poles

of q_X and compose every chart of the natural coordinates for q_X on $\hat{X}$ with the affine function

$$h(x+iy) = \sqrt{K}x + i\frac{1}{\sqrt{K}}y.$$

This new set of charts defines a Riemann surface $\hat{Y}$, which can be made into a closed Riemann surface Y.

We thus obtain an induced Teichmüller map (the one which is the previous affine map in every chart), a Riemann surface Y and an induced quadratic differential q_Y on Y. Notice that, identifying X with S_g^n, we could consider the map $h : X \to Y$ as a marking and thus as a point in the Teichmüller space: in this way, varying the constant K in $\mathbb{R}$, we could obtain a one-parameter family in $\mathcal{T}_g^n$. Since the differential q_X specifies a unique ray in $\mathcal{T}_g^n$, it is possible to see again how the differential q_X can be thought of as a tangent direction in $\mathcal{T}_g$.

Going back to Teichmüller extremal problem, its answer is made of two parts, one dealing with existence of Teichmüller maps and one with uniqueness. We present the latter first.

Theorem 1.3.4. *(Uniqueness) Let $h : (X, P) \to (X', P')$ be a Teichmüller map between two Riemann surfaces with marked points. If $f : (X, P) \to (X', P')$ is another homeomorphism homotopic to h rel P, P', then it results*

$$K_f \geq K_h$$

and equality holds if and only if $f \circ h^{-1}$ is conformal.

For the proof of the theorem in case of punctured surfaces we refer the reader for example to $[Abi]$ or $[Hu]$.

We will instead briefly discuss the case of closed surfaces, since the proof has many similarities with the one of Grötzsch's theorem: the affine map is replaced with the Teichmüller map and the horizontal and vertical directions are replaced with the directions of $\mathcal{F}_q^+$ and $\mathcal{F}_q^-$.

Let h be a Teichmüller map with initial differential q on X and final differential q' on X'. The natural coordinates are $z = x + iy$ for q and $z' = x' + iy'$ for q', $x' = \sqrt{K}x$, $y' = \frac{y}{\sqrt{K}}$. We denote by dA_q the area form induced by q: it clearly results $dA_q = dA_{q'}$.

We will use the following lemma (see $[FM]$, lemma 11.11).

Lemma 1.3.5. *Let q be a holomorphic quadratic differential on a Riemann surface X and $\varphi : X \to X$ a homeomorphism homotopic to the identity.*

There exist a constant $C \geq 0$ depending only on φ such that, for every arc $\gamma : [0, 1] \to X$ embedded in a leaf of the horizontal foliation of q, it results

$$l_q(\varphi(\gamma)) \geq l_q(\gamma) - C.$$

We specify that $l_q(\gamma)$ denotes the length of the curve γ with respect to the metric $|q|$, and not the length of the geodesic representative of the homotopy class of γ.

Denote by γ_p^L the horizontal arc for $\mathcal{F}_q^+$ centered at $p \in X$ and of q-length $2L$. Since h takes γ_p^L to an arc of q'-length $2L\sqrt{K}$, from the lemma it follows

$$l_{q'}(f(\gamma_p^L)) \geq 2L\sqrt{K} - C$$

and consequently

$$\int_X l_{q'_X}(f(\gamma_p^L))dA_q \geq (2L\sqrt{K} - M)Area(q).$$

Applying Fubini's theorem one gets the equalities

$$\int_X l_{q'}(f(\gamma_p^L))dA_q = \int_X \left(\int_{-L}^{L} \left| \frac{\partial f}{\partial x} \right|_{q'} dx \right) dA_q = 2L \int_X \left| \frac{\partial f}{\partial x} \right|_{q'} dA_q$$

and combining them with the last inequality and dividing by $2L$ it results

$$\int_X \left| \frac{\partial f}{\partial x} \right|_{q'} dA_q \geq \left(\sqrt{K} - \frac{M}{2L} \right) Area(q)$$

which gives the following inequality allowing L to tend to infinity

$$\sqrt{K} \cdot Area(q) \leq \int_X \left| \frac{\partial f}{\partial x} \right|_{q'} dA_q.$$

Finally, one obtains

$$K \cdot Area(q)^2 \leq \left(\int_X \left| \frac{\partial f}{\partial x} \right|_{q'} dA_q \right)^2 \leq \left(\int_X \sqrt{K_f} \cdot \sqrt{J_f} dA_q) \right)^2 \leq K_f \cdot Area(q)^2,$$

where J_f is the Jacobian of f computed in the natural coordinates of q and q'.
If all the inequalities are equalities, then it follows that f must be an affine function in the natural coordinates of q and q'.

Here we present the answer to the existence part of Teichmüller's problem.

Theorem 1.3.6. *(Existence) Given any homeomorphism $f : X \to X'$ between Riemann surfaces of finite type, there exist a Teichmüller map h homotopic to f.*

Let $\mathcal{Q}(X)_1$ be the unit ball in $\mathcal{Q}(X)$ with respect to the L^1-norm. For each $q \in \mathcal{Q}(X)_1$ define

$$K(q) := \frac{1 + ||q||_1}{1 - ||q||_1}.$$

We have seen in the preceding section how it is possible to obtain a Teichmüller map $h : X \to Y$ starting only with X, an initial differential q on X and a horizontal stretch

factor $\sqrt{K(q)}$: identifying X with S_g this in fact gives a point $[(Y,h)] \in \mathcal{T}_g^n$.
In this way we can define a function

$$\Omega : \mathcal{Q}(X)_1 \to \mathcal{T}_g^n,$$

where $\Omega(q)$ is said point $[(Y,h)]$: note that Teichmüller's uniqueness theorem implies the injectivity of Ω, while the surjectivity of Ω would imply the claim of Teichmüller existence theorem.
The proof of the surjectivity of Ω can be reduced to the proof of its continuity and properness since any proper injective continuous map between real spaces of the same dimension is a homeomorphism.
It is possible to factor Ω as a composition of two continuous functions

$$\Phi_X : \mathcal{Q}(X)_1 \to \mathcal{B}(X)_1, \quad \Psi_X : \mathcal{B}(X)_1 \to \mathcal{T}_g^n,$$

where $\mathcal{B}(X)_1$ is the unit ball with respect to the L^∞-norm in $\mathcal{B}(X)$ and $\mathcal{T}_g^n$ is endowed with the Teichmüller metric.
The function Φ_X is defined imposing, for each $q \in \mathcal{Q}(X)_1$,

$$\Omega_1(q) := ||q||_1 \frac{\overline{q}}{|q|}.$$

Instead the function Ψ_X associates, to each $\mu \in \mathcal{B}(X)_1$, the point $[(f^\mu, X^\mu)] \in \mathcal{T}_g^n$ whose existence is granted by the corollary 1.2.4. The continiuty of Ψ_X descends directly from the fact that the dependence of f^μ from μ is analytic.

The properness of Ω can be deduced from the fact that, given any diverging sequence $\{q_n\}_{n\in\mathbb{N}} \subset \mathcal{Q}(X)_1$ (which means that the L^1-norm of q_n tends to 1), then, from the equalities

$$d_\mathcal{T}(\mathcal{X}, \Omega(q_n)) = \frac{1}{2}\log(K(q_n)), \quad K(q_n) = \frac{1+||q||_1}{1-||q||_1}$$

it follows

$$\lim_{n\to\infty} d_\mathcal{T}(\mathcal{X}, \Omega(q_n)) = \infty.$$

From Teichmüller's theorems one can easily deduce some important properties of Teichmüller metric.

Theorem 1.3.7. *The Teichmüller metric defines a complete and uniquely geodesic metric on $\mathcal{T}_g^n$. Every geodesic segment is a subset of some Teichmüller line.*

Proof. As we explained before, Teichmüller lines are geodesics for $d_\mathcal{T}$, the fact that they account for all geodesics for $d_\mathcal{T}$ descends from Teichmüller's uniqueness theorem. Since $\mathcal{T}_g^n$ is geodesic and hence intrinsic, it suffices to prove that closed balls for $\mathcal{T}_g^n$

are compact. This follows from the fact that Ω is a homeomorphism and the preimage through Ω of closed balls of $d_{\mathcal{T}}$ are closed balls of $\mathcal{Q}(X)_1$.

□

Turning back to the case of the torus, we can finally see why the bijection $\Phi_{\mathbb{H}} : \mathbb{H} \to \mathcal{T}_1$ of section 1.2 is in fact an isometry in case $\mathbb{H}$ is endowed with the hyperbolic metric and $\mathcal{T}_1$ is endowed with the Teichmüller metric.
Indeed, given $\xi, \xi' \in \mathbb{H}$, their distance with respect to the hyperbolic metric is

$$d_{\mathbb{H}}(\xi, \xi') = \frac{1}{2} \log \left(\frac{|\xi - \overline{\xi}'| + |\xi - \xi'|}{|\xi - \overline{\xi}'| - |\xi - \xi'|} \right)$$

which is equal to the halved complex dilatation of the affine map which sends the marked lattice $\mathbb{Z} + \xi\mathbb{Z}$ to the marked lattice $\mathbb{Z} + \xi'\mathbb{Z}$.

1.4 Moduli spaces

Given any Riemann surface S with boundary ∂S, we define the *Mapping class group* $\Gamma(S)$ as the discrete group obtained from the quotient

$$\Gamma(S) := Diff^+(S, \partial S)/Diff_0^+(S, \partial S),$$

where $Diff^+(S, \partial S)$ is the set of diffeomorphisms of S which fix the boundary pointwise. In order to maintain a coherent notation we denote by Γ_g^n the mapping class group of S_g^n.
In the simple case of the torus, one should notice that there is an isomorphism

$$\Theta_T : \Gamma_1 \to SL(2, \mathbb{Z})$$

given by the action on $H_1(T, \mathbb{Z}) \simeq \mathbb{Z}^2$ (see for example $[FM]$, theorem 2.5).
There is a natural action of Γ_g^n on $\mathcal{T}_g^n$: for each $[\psi] \in \Gamma_g^n$ and $[(X, \phi)] \in \mathcal{T}_g^n$ we set

$$[\psi] \cdot [(X, \phi)] := [(X, \phi \circ \psi^{-1})].$$

The *moduli space* $\mathcal{M}_g^n$ of S_g^n is a central object in many branches of geometry and can be defined as the quotient

$$\mathcal{M}_g^n := \mathcal{T}_g^n / \Gamma_g^n.$$

It is clear from the definition of the action of the mapping class group that $\mathcal{M}_g^n$ can be interpreted as the set of hyperbolic metrics on S_g^n up to isometry or the set of conformal structures on S_g^n up to isomorphism. Since intuitively the action of the mapping class group consists in removing the marking, we denote by $[X]$ the elements

of $\mathcal{M}_g^n$.
Again, the case of the torus is particularly enlightening: the moduli space $\mathcal{M}_1$ of flat tori is the quotient $\mathbb{H}/SL(2,\mathbb{Z})$, where the action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$ is thus defined:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \xi = \frac{a\xi + b}{c\xi + d}.$$

Indeed notice that any biholomorphism $\varphi : \mathbb{C}/(\mathbb{Z} + \xi\mathbb{Z}) \to \mathbb{C}/(\mathbb{Z} + \xi'\mathbb{Z})$ between flat tori can be lifted to an automorphism $\widetilde{\varphi}$ of $\mathbb{C}$ such that there exist $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ with the property

$$\widetilde{\varphi}(z+1) - \widetilde{\varphi}(z) = \delta + \gamma\xi,$$

$$\widetilde{\varphi}(z+\xi') - \widetilde{\varphi}(z) = \beta + \alpha\xi$$

which imply

$$\xi' = \frac{\alpha\xi + \beta}{\gamma\xi + \delta}, \quad \alpha\delta - \beta\gamma = \pm 1.$$

The fundamental domain of the action of Γ_1 on $\mathcal{T}_1$ is represented in figure 1.2.

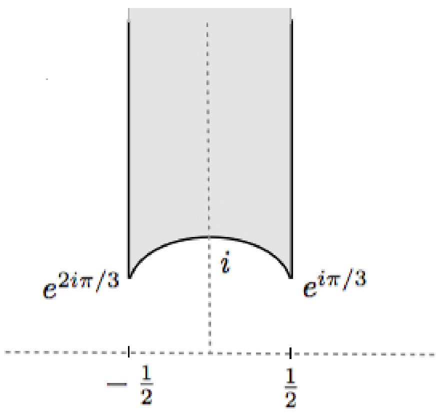

Figure 1.2: The fundamental domain of the action of Γ_1 on $\mathcal{T}_1$ is drawn in grey.

The moduli space $\mathcal{M}_g^n$ is a complex orbifold finitely covered by a complex manifold: it follows from the fact that the action of Γ_g^n is properly discontinuous and that points of $\mathcal{T}_g^n$ have finite stabilizers.
The original proof of the proper discontinuity of the action of Γ_g^n is often attributed to Fricke, but it also descends from the general fact that the action of $Diff^+(S_g^n)$ on the space of smooth Riemannian metrics on S_g^n is properly discontinuous (see for example [Eb]). Instead, the fact that points have finite stabilizers is a consequence of the finiteness of isometry groups of hyperbolic metrics on Riemann surfaces.

It can be easily seen that the mapping class group acts on $\mathcal{T}_g^n$ by isometries of $d_{\mathcal{T}}$: this fact, together with the proper discontinuity of the action of Γ_g^n, implies that the Teichmüller metric descends to a metric $\hat{d}_{\mathcal{T}}$ on $\mathcal{M}_g^n$:

$$\hat{d}_{\mathcal{T}}([X_1],[X_2]) = \inf_{[X_1,\phi_1],[X_2,\phi_2]\in\mathcal{T}_g^n} d_{\mathcal{T}}([(X_1,\phi_1)],[(X_2,\phi_2)]).$$

The metric $\hat{d}_{\mathcal{T}}$ can be used to prove that $\mathcal{M}_g^n$ is not compact: this fact is obvious for $\mathcal{M}_1$ and can be proved for $\mathcal{M}_g^n$ showing that the diameter of $\mathcal{M}_g^n$ with respect to $\hat{d}_{\mathcal{T}}$ is not finite.
For any $[X]\in\mathcal{M}_g^n$ denote by $l(X)$ the length of the shortest essential closed geodesic in X (notice that this definition is independent from the chosen representative), choose a lifting $\mathcal{X}$ of $[X]$ in $\mathcal{T}_g^n$ and suppose $l(X)=\hat{l}_{\mathcal{X}}(\gamma)$. Then for each $t\geq 1$ it is possible to find a marked surface $\mathcal{X}_t$ such that $l(X_t)\leq \hat{l}_{\mathcal{X}_t}(\gamma)=l(X)/t$, for example by varying the length of γ in a Fenchel-Nielsen coordinate system which contains γ.
A theorem by Wolpert ($[Wo]$) states that, given any K-quasi-conformal homeomorphism $\varphi: X\to X'$ between hyperbolic surfaces and any homotopy class of a simple closed curve γ, it results

$$\frac{l_X(\gamma)}{K}\leq l_{X'}(\varphi(\gamma))\leq K l_X(\gamma).$$

From these inequalities one concludes

$$\lim_{t\to\infty}\hat{d}_{\mathcal{T}}(X,X_t)=\infty.$$

Pinching a simple closed curve is indeed the only way a sequence in $\mathcal{M}_g^n$ can leave any compact set: this follows from *Mumford's compactness criterion* ($[Mu]$) for $g\geq 2$ and from a theorem of Mahler about lattices ($[Mah]$) for $g=1$.
In particular, every compact set of $\mathcal{M}_g^n$ is contained in one set $\mathcal{M}_\epsilon$:

$$\mathcal{M}_\epsilon := \{[X]\in\mathcal{M}_g^n \textit{ such that } l(X)\geq\epsilon\}$$

for a given $\epsilon>0$. The intuition behind this fact is that lifting any sequence of $\mathcal{M}_g^n$ to $\mathcal{T}_g^n$, up to changing the marking it is always possible to find an upper (but not lower) bound for the length of the closed curves of a Fenchel-Nielsen coordinate system.

Chapter 2

Thurston's metric

In the present chapter we summarize the properties of the metric which inspired the work of this thesis: the Thurston's metric on Teichmüller space. In order to do so, we first introduce and study measured laminations.

2.1 Measured laminations

Definition 2.1. *Let X be an hyperbolic surface and $\pi : \mathbb{H} \to X$ its metric universal cover. A geodesic lamination on X is a closed subset of X which is the union of disjoint images by π of bi-infinite geodesics γ of $\mathbb{H}$ such that either $\pi(\gamma)$ is a simple closed geodesic (referred to as closed leaves) or the restriction of π to γ is injective (referred to as bi-infinite leaves).*

We say that a geodesic lamination λ is *complete* if there is no other geodesic lamination which strictly contains it, or equivalently if every connected component of $X \setminus \lambda$ is isometric to the interior of a hyperbolic triangle.

Geodesic laminations are a central object in hyperbolic geometry, their importance will be underlined throughout this section.
Although being defined fixing an hyperbolic metric, geodesic laminations are in fact a topological object: indeed, there is a bijective correspondence between geodesic laminations associated to two hyperbolic metrics on surfaces of the same topological type (see for example $[Le]$) and correspondent laminations are in fact isotopic as topological objects, by a global isotopy of the surface which fixes the punctures ($[Th2]$).
We define the *stump* of a geodesic lamination λ as the support of any maximal (with respect to inclusion) geodesic sublamination of λ with compact support. The space of geodesic laminations $\mathcal{L}$ of a surface clearly contains the space of multicurves $\mathcal{M}$ and

can be endowed with the Hausforff metric on closed sets: each element of the closure of $\mathcal{M}$ in $\mathcal{L}$ with respect to the Hausdorff metric is said to be a *chain-recurrent lamination*.

As for measured foliations, it is possible to define *measured geodesic laminations*, which are geodesic laminations equipped with a nonnegative Radon measure on transverse arcs. The measure must be invariant under homotopies of the arc respecting the lamination and the support of the measure must be equal to the intersection of the arc with the support of the lamination (which is the union of the leaves).
We denote by $\mathcal{ML}$ the set of measured geodesic laminations on S_g^n and by $\mathcal{ML}_0$ the subset corresponding to measured geodesic laminations with compact support. An element of $\mathcal{ML}$ will be indicated simply by λ, without specifying the measure.
As for geodesic lamination, the *stump* of a measured geodesic lamination λ is the support of any maximal (with respect to inclusion) measured geodesic sublamination of λ with compact support. Notice that if the surface has punctures then any measured geodesic lamination with compact support can not be complete, since otherwise it would have leaves converging to punctures. Moreover, the stump of a geodesic lamination λ is empty if and only if each end of every leaf of λ converges towards a puncture.
The set $\mathcal{S}$ of simple closed curves on S_g^n is naturally embedded in $\mathcal{ML}$: each element $\alpha \in \mathcal{S}$ is naturally a measured lamination, once it is provided of the Dirac measure of mass 1.
It is also possible to extend the length function

$$l_{\mathcal{S}} : \mathcal{T}_g^n \times \mathcal{S} \to \mathbb{R}, \quad l(\mathcal{X}, \alpha) := \hat{l}_{\mathcal{X}}(\alpha),$$

where $\hat{l}_{\mathcal{X}}(\alpha)$ is the length of the geodesic representative for the hyperbolic metric of X of the homotopy class of $\phi(\alpha)$, continuously to

$$l_{\mathcal{ML}_0} : \mathcal{T}_g^n \times \mathcal{ML}_0 \to \mathbb{R}.$$

For a proof of this face see $[PT2]$.

There is an obvious action of $\mathbb{R}_{>0}$ on $\mathcal{ML}_0$ and $\mathcal{ML}$ which is realized multiplying the measure by a constant: the quotients of this action are respectively $\mathcal{PML}_0$ and $\mathcal{PML}$ and are referred to as *projective measured geodesic laminations*.

We recall the embedding $l_* : \mathcal{T}_g^n \to \mathbb{R}^{\mathcal{S}}$ of section 1.1, since it is possible to define another two similar embeddings

$$\iota_{\mathcal{F}} : \mathcal{MF} \to \mathbb{R}^{\mathcal{S}}, \quad \iota_{\mathcal{L}} : \mathcal{ML} \to \mathbb{R}^{\mathcal{S}}$$

which play a central role in a compactification of $\mathcal{T}_g^n$ first proposed by Thurston: we refer the reader to $[FLP]$ for details of the following discussion.
The map $\iota_{\mathcal{F}}$ sends a measured foliation $\mathcal{F}$ to the element of $\mathbb{R}^{\mathcal{S}}$ which associates to $\alpha \in \mathcal{S}$ the quantity $i(\mathcal{F}, \alpha)$, defined as the infimum of the total mass of simple closed curves in the homotopy class of α with respect to the measure of $\mathcal{F}$. This map is well-defined since, by a result of Thurston, two measured foliations are White-head-equivalent if and only if they have the same image in $\mathbb{R}^{\mathcal{S}}$.
The map $\iota_{\mathcal{L}}$ is defined in the same way.
The two subsets $\iota_{\mathcal{F}}(\mathcal{MF})$ and $\iota_{\mathcal{L}}(\mathcal{ML})$ of $\mathbb{R}^{\mathcal{S}}$ actually coincide and are homeomorphic to $\mathbb{R}^{6g-6+2n} \setminus \{0\}$: this fact can be used to establish a bijective correspondence between $\mathcal{MF}$ and $\mathcal{ML}$. What is more, the set $\mathcal{S} \times \mathbb{R}_{>0}$ is a dense subset of $\mathcal{ML}$.
Finally, through these embeddings the space $\mathcal{PML}$ can be seen as the boundary of $\mathcal{T}_g^n$, showing that the closure of the Teichmüller space is homeomorphic to a closed ball of dimension $6g - 6 + 2n$.

2.2 Thurston's metric on $\mathcal{T}_g^n$

W. Thurston in $[Th]$ introduced a metric L on $\mathcal{T}_g^n$ defined computing the infimum of Lipschitz constants of maps. The main result of $[Th]$ is that this metric L is in fact equal to another metric K on $\mathcal{T}_g^n$ whose value equals the supremum of the ratio of lengths of simple closed curves.

Coherently with the notation of the preceding chapter, we denote by $Diff_0^+(S_g^n)$ the set of diffeomorphisms of S_g^n which are homotopic to the identity.
In order to be as close as possible to Thurston's original approach, we will now represent points of $\mathcal{T}_g^n$ as equivalence classes of hyperbolic surfaces for the action of $Diff_0^+(S_g^n)$ and not as marked surfaces. As before, we will denote by X a Riemann surface and the corresponding hyperbolic surface. Points of $\mathcal{T}_g^n$ will be simply denoted by X: one should keep in mind that X represents an equivalence class.
For every $\varphi \in Diff_0^+(S_g^n)$, define the Lipschitz constant $Lip(\varphi)_{X_1}^{X_2}$ of φ with respect to any pair $X_1, X_2 \in \mathcal{T}_g^n$ as

$$Lip(\varphi)_{X_1}^{X_2} := \sup_{x \neq y \in S_g^n} \frac{d_{X_2}(\varphi(x), \varphi(y))}{d_{X_1}(x, y)},$$

where d_{X_1} (resp. d_{X_2}) is the metric on S_g^n induced by the hyperbolic metric of X_1 (resp. X_2).
It is possible to define a map $L : \mathcal{T}_g^n \times \mathcal{T}_g^n \to \mathbb{R}$ as

$$L(X_1, X_2) := \inf_{\varphi \in Diff_0^+(S_g^n)} \log(Lip(\varphi)_{X_1}^{X_2}).$$

Proposition 2.1 of $[Th]$ states that L is in fact an asymmetric metric.
In the same paper Thurston introduced another asymmetric metric K on $\mathcal{T}_g^n$ defined as follows

$$K(X_1, X_2) := \sup_{\alpha \in \mathcal{S}} \log(r_{X_1,X_2}(\alpha)), \quad r_{X_1,X_2}(\alpha) := \frac{\hat{l}_{X_2}(\alpha)}{\hat{l}_{X_1}(\alpha)},$$

where as before $\mathcal{S}$ is the set of simple closed curves of S_g^n and $\hat{l}_X(\alpha)$ is the length of the geodesic representative for the hyperbolic metric of X of the homotopy class of α.
The fact $K(h_1, h_2) > 0$ for $X_1 \neq X_2$ is not at all trivial and is proved through the results of lemma 3.2, 3.3 and 3.4 of $[Th]$.
Notice that using the extension of the hyperbolic length function to $\mathcal{ML}_0$, the supremum of the definition of K can be equivalently taken over $\mathcal{ML}_0$ and thus on the compact space $\mathcal{PML}_0$: this means that there is a measured lamination which maximizes the ratio of lengths.
As we announced, the main result of $[Th]$ is the equality of the following theorem.

Theorem 2.2.1. *For every $X_1, X_2 \in \mathcal{T}_g^n$ it results*

$$L(X_1, X_2) = K(X_1, X_2).$$

Notice that the inequality $L \geq K$ can be easily proved, since for every $\varphi \in Diff_0^+(S_g^n)$ and every curve γ on S_g^n it results

$$l_{X_2}(\varphi(\gamma)) \leq Lip(\varphi)_{X_1}^{X_2} l_{X_1}(\gamma).$$

Conversely, the other inequality $L \leq K$ is not at all trivial.

Thurston proved theorem 2.2.1 first showing that each couple of points $X_1, X_2 \in \mathcal{T}_g^n$ can be connected by a geodesic for L, which can be constructed as a concatenation of a finite number of *stretch lines*, along which the equality $L = K$ is realized.
Given the geometric importance of stretch lines, we consider necessary to give a brief explanation of their construction, which is based on *horocyclic measured foliations*.

The horocyclic measured foliation of the hyperbolic ideal triangle is a partial foliation obtained foliating the triangle by horocycles starting from the vertices. The leafs are perpendicular to the sides and the unfoliated central region is a triangle bounded by three horocycles (see figure 2.1). The transverse measure is such that it assigns to any arc contained in an edge of the triangle the Lebesgue measure induced by the hyperbolic metric.

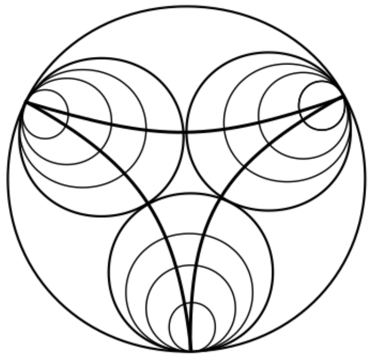

Figure 2.1: Horocyclic of the ideal triangle in the hyperbolic disk.

Using the horocyclic foliation of the ideal triangle one can prove the following proposition, which corresponds to proposition 2.2 of $[Th]$ and states a fundamental property about hyperbolic triangles.

Proposition 2.2.2. *For any $K > 1$ there is a K-Lipschitz homeomorphism of a filled hyperbolic triangle to itself which maps each side to itself, multiplying arc length on the side by K.*

The K-Lipschitz homeomorphism can indeed be obtained fixing the central region and mapping horocycles at distance t from the central region to horocycles at distance Kt.

Given any complete geodesic lamination λ on a Riemann surface of finite type X, one can endow every connected component of $X \setminus \lambda$ (since it is an ideal hyperbolic triangle) with the horocyclic measured foliation obtained from the hyperbolic metric of X. The foliations of each triangle fit together smoothly and thus define a partial measured foliation of S_g^n: collapsing each non-foliated region into a tripod one obtains a measured foliation $\mathcal{F}_X(\lambda)$ which we define as the *horocyclic measured foliation of X associated to λ*.
Notice that from the fact that each hyperbolic metric of $\mathcal{T}_g^n$ is complete and of finite area, it follows that $\mathcal{F}_X(\lambda)$ is *trivial around the punctures* in the sense of definition 3.6 of $[PT2]$: each puncture has a neighborhood on which the induced foliation is a cylinder foliated by homotopic closed leaves and any segment transverse the the foliation and converging to a puncture has infinite total mass with respect to the transverse measure.

Given any complete geodesic lamination λ of X, we denote by $\mathcal{MF}(\lambda) \subset \mathcal{MF}$

the subset corresponding to measured foliation classes which can be represented by measured foliations transverse to λ and trivial around the punctures.
Using horocyclic foliations Thurston defined a new type of coordinates on $\mathcal{T}_g^n$, called *cataclysm coordinates*, provided by the map Φ_λ of the following theorem (see sections 4 and 9 of $[Th]$ or theorem 3.10 of $[PT2]$)

Theorem 2.2.3. *The map*

$$\Phi_\lambda : \mathcal{T}_g^n \to \mathcal{MF}(\lambda), \quad \Phi_\lambda(X) = \mathcal{F}_X(\lambda)$$

is a homeomorphism.

Finally we can define what a *stretch line* is. Given any $\mathcal{F} \in \mathcal{MF}(\lambda)$, the stretch line directed by the complete geodesic lamination λ and starting at $\Phi_\lambda^{-1}(\mathcal{F})$ is the map from $\mathbb{R}$ to $\mathcal{T}_g^n$ defined by $t \mapsto \Phi_\lambda^{-1}(e^t\mathcal{F})$.
Notice that if λ is an ideal triangulation (which means that both ends of each leaf of λ converge to a cusp) then the stretch line directed by λ is constant: for this reason from now on we will consider complete laminations whose stump is not empty.
Denote by $\lambda_0 \subset \lambda$ the stump of λ, along the stretch line directed by λ it results

$$r_{\Phi_\lambda^{-1}(\mathcal{F}),\Phi_\lambda^{-1}(e^t\mathcal{F})}(\lambda_0) = e^t.$$

Since the Lipschitz constant of the identity map Id of S_g^n with respect to $\Phi_\lambda^{-1}(\mathcal{F})$ and $\Phi_\lambda^{-1}(e^t\mathcal{F})$ is also equal to e^t, from the inequality $L \geq K$ it follows the equality

$$L(\Phi_\lambda^{-1}(\mathcal{F}), \Phi_\lambda^{-1}(e^t\mathcal{F})) = K(\Phi_\lambda^{-1}(\mathcal{F}), \Phi_\lambda^{-1}(e^t\mathcal{F})) = t.$$

One can also deduce that the stretch line directed by λ is a geodesic for Thurston's asymmetric metric L:

$$L(\Phi_\lambda^{-1}(e^{t_0}\mathcal{F}), \Phi_\lambda^{-1}(e^{t_1}\mathcal{F})) = t_1 - t_0.$$

In order to prove theorem 2.2.1 Thurston showed the existence, for any couple of points $X_1, X_2 \in \mathcal{T}_g^n$, of a *maximally stretched chain-recurrent geodesic lamination* from X_1 to X_2. The union of all maximally stretched geodesic laminations from X_1 to X_2 is a geodesic lamination which is denoted by $\lambda(X_1, X_2)$.
The claim of theorem 2.2.1 is then proved showing that it is always possible to join X_1 to X_2 in $\mathcal{T}_g^n$ with a geodesic for L which is a finite concatenation of pieces of stretch lines along complete geodesic laminations $\lambda_1, \dots, \lambda_k$ containing $\lambda(X_1, X_2)$, where the number k depends only on the topological type of the surface.
The geodesic lamination λ_1 is arbitrarily chosen and there exists a $t_0 > 0$ minimum such that $\hat{\lambda}_1 := \lambda(\Phi_{\lambda_1}^{-1}(e^{t_0}\mathcal{F}_{X_1}(\lambda_1)), X_2) \neq \lambda(X_1, X_2)$. Notice that it necessarily follows $\lambda(X_1, X_2) \subset \hat{\lambda}_1$. Choosing λ_2 arbitrarily between the set of complete geodesic

laminations which contain $\hat{\lambda}_1$ and continuing in the same manner it is possible to reach X_2 after a finite number of steps, since there is a bound on the length of a strictly increasing sequence of geodesic laminations.

In particular it follows that $\mathcal{T}_g^n$, endowed with the metric L, is a geodesic space. A.Papadopoulos and G.Théret proved that it is a complete asymmetric space ([PT]), showing that left closed balls for L are compact and then using a generalization of the Hodf-Rinow theorem to asymmetric metric spaces ([Bu], theorem 8).

2.3 Thurston's metric on $\mathcal{T}_1$

A. Belkhirat, A. Papadopoulos and M. Troyanov studied in [BPT] two asymmetric pseudometrics on $\mathcal{T}_1$ which are similar to the ones defined by Thurston on $\mathcal{T}_g^n$. In order to maintain coherence of notation with [BPT] we denote them by λ and κ. Throughout this section we will define them and give a quick summary of their properties.

The correspondent of the Thurston metric L given in [BPT] for the torus is the map λ,

$$\lambda : \mathcal{T}_1 \times \mathcal{T}_1 \to \mathbb{R},$$

$$\lambda([(\sigma_1,\phi_1)],[(\sigma_2,\phi_2)]) := \inf_{\varphi \in Diff^+(T), \varphi_* = \phi_2 \circ \phi_1^{-1}} \left(\log \left(Lip(\varphi)_{[(\sigma_1,\phi_1)]}^{[(\sigma_2,\phi_2)]} \right) \right),$$

$$Lip(\varphi)_{[(\sigma_1,\phi_1)]}^{[(\sigma_2,\phi_2)]} := \sup_{x \neq y} \left(\left(\frac{d_{\sigma_2}(\varphi(x),\varphi(y))}{d_{\sigma_1}(x,y))} \right) \left(\frac{\hat{l}_{\sigma_1}(\phi_1(1,0))}{\hat{l}_{\sigma_2}(\phi_2(1,0))} \right) \right),$$

where as before $\hat{l}_\sigma(\phi(1,0))$ is the length of the geodesic representative of the homotopy class of $\phi(1,0)$ for the metric σ.

The correspondent of the Thurston metric K for the torus is the map κ:

$$\kappa : \mathcal{T}_1 \times \mathcal{T}_1 \to \mathbb{R},$$

$$\kappa([(\sigma_1,\phi_1)],[(\sigma_2,\phi_2)]) := \sup_{(m,n) \in \mathbb{Z}^2} \log \left(\left(\frac{\hat{l}_{\sigma_2}(\phi_2(m,n))}{\hat{l}_{\sigma_1}(\phi_1(m,n))} \right) \left(\frac{\hat{l}_{\sigma_1}(\phi_1(1,0))}{\hat{l}_{\sigma_2}(\phi_2(1,0))} \right) \right).$$

The main result of [BPT] is the following theorem, which establishes an equality similar to the one of [Th].

Theorem 2.3.1. *The functions κ, λ are asymmetric pseudo-metrics and coincide on $\mathcal{T}_1$.*

First notice that the inequality $\kappa \leq \lambda$ is easily proved, since for every curve $\gamma : [0,1] \to T$ and every homeomorphism $\varphi \in Diff^+(T)$ one gets

$$l_{\sigma_2}(\varphi(\gamma)) \leq Lip(\varphi)_{\sigma_1}^{\sigma_2} l_{\sigma_1}(\gamma).$$

In order to prove the other inequality the authors introduced an asymmetric pseudo-metric δ on $\mathbb{H}$,

$$\delta(\xi, \xi') := \log M(\xi, \xi'), \quad M(\xi, \xi') := \sup_{x \in \mathbb{R}} \left| \frac{\xi' - x}{\xi - x} \right|$$

which could be equivalently defined as

$$\delta(\xi, \xi') = \log \left(\frac{|\xi' - \bar{\xi}| + |\xi' - \xi|}{|\xi - \bar{\xi}|} \right)$$

and proved that the map $\Phi_{\mathbb{H}}$ of section 1.2 is in fact an isometry between $\mathbb{H}$ endowed with the pseudo-metric δ and $\mathcal{T}_1$ endowed with κ. In particular it results

$$\delta(\xi, \xi') = \kappa([(\sigma_\xi, \phi_\xi)], [(\sigma_{\xi'}, \phi_{\xi'})]).$$

The isotopy class of the curve $\phi_\xi(m,n)$ is given in $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\xi)$ by an arbitrary path connecting 0 to $m + n\xi$.
Using such identification, the following equalities are clear:

$$\hat{l}_{\sigma_\xi}(\phi_\xi(m,n)) = |m + n\xi|, \quad \hat{l}_{\sigma_\xi}(\phi_\xi(1,0)) = 1$$

and consequently one gets

$$\kappa([(\sigma_\xi, \phi_\xi)], [(\sigma_{\xi'}, \phi_{\xi'})]) = \log \sup_{m,n \in \mathbb{Z}} \left(\frac{m + n\xi'}{m + n\xi} \right) = \log \sup_{q \in \mathbb{Q}} \left(\frac{q + \xi'}{q + \xi} \right) = \delta(\xi, \xi').$$

Finally the inequality $\kappa \geq \lambda$ can be proved finding a map $f \in Diff_0^+(T)$, $f_* = \phi_2 \circ \phi_1^{-1}$, such that

$$Lip(f)_{[(\sigma_\xi, \phi_\xi)]}^{[(\sigma_{\xi'}, \phi_{\xi'})]} \leq \delta(\xi, \xi').$$

Indeed, the affine map f which fixes $(1,0)$ and sends ξ to ξ' has this property, since it results

$$Lip(f)_{[(\sigma_\xi, \phi_\xi)]}^{[(\sigma_{\xi'}, \phi_{\xi'})]} = \frac{|\xi' - \bar{\xi}| + |\xi' - \xi|}{|\xi - \bar{\xi}|}.$$

As a final observation, notice that since the symmetrization $S\delta$ of δ,

$$S\delta(\xi, \xi') := \frac{1}{2}(\delta(\xi, \xi') + \delta(\xi', \xi))$$

coincides with the Poincaré metric on $\mathbb{H}$, it is also true that the symmetrizations of λ and κ coincide with the Teichmüller metric on $\mathcal{T}_1$. The affine map f defined above is extremal for both λ and the Teichmüller metric.

Chapter 3

Semi-translation surfaces

In the present chapter we define translation and semi-translation surfaces, underlining some of their fundamental properties.

Definition 3.1. *A semi-translation surface is a closed topological surface S_g endowed with a semi-translation structure, that is:*

(i) a finite set of points $\Sigma \subset S_g$ and an atlas of charts on $S_g \setminus \Sigma$ to $\mathbb{C}$ such that transition maps are of the form $z \mapsto \pm z + c$ with $c \in \mathbb{C}$,

(ii) a flat singular metric on S_g such that for each point $p \in \Sigma$ there is a homeomorphism of a neighborhood of p with a neighborhood of a cone angle of $\pi(k+2)$ for some $k > 0$, which is an isometry away from p (we call such point a singular point of order k). Furthermore, charts of the atlas of (i) are isometries for the flat singular metric.

Equivalently, a semi-translation surface can be defined as a closed Riemann surface X endowed with a non-vanishing holomorphic quadratic differential q. Indeed, it follows directly from the proof of proposition 1.2.5 that natural coordinates for q and the metric $|q|$ endow S_g with a semi-translation structure. Conversely, given a semi-translation structure one can obtain a quadratic differential by setting $q = dz^2$ on $S_g \setminus \Sigma$ (where z is a coordinate of the charts of the semi-translation structure) and $q = z^k dz^2$ in a neighborhood of a singular point. It is clear then that the sum of the orders of singular points is $4g - 4$.
A semi-translation surface is naturally endowed with a locally $Cat(0)$ metric. Actually, one can extend the definition to allow the quadratic differentials to have at most simple poles (and consequently cone angles of π), but then the resulting metric will not be locally $Cat(0)$ anymore.

In a similar way one can define *translation surfaces*: the only differences are that transition maps must be translations and that the cone angles should be of $2\pi(k+1)$. The definition of natural coordinates for holomorphic quadratic differentials we gave in proposition 1.2.5 can be adapted to Abelian differentials ω on X: indeed they are local coordinates z on X such that $\omega = dz$ near a point which is not a zero of ω and $\omega = z^k dz$ near a point which is a zero of ω of order $k \geq 1$.
It follows that translation surfaces can be equivalently defined as closed Riemann surfaces X equipped with Abelian differentials ω: the natural coordinates of ω and the metric $|\omega|^2$ indeed give this correspondence and one should notice that a zero of ω corresponds to a cone angle of $2\pi(k+1)$. The sum of orders of singular points is then equal to $2g-2$.

While translation surfaces are naturally semi-translation surface, the inverse is not true: this is because the holonomy of (X, ω) is trivial, while the holonomy of (X, q) could be $\pm Id$.
Notice that the surfaces (X, q) and $(X, e^{i\theta}q)$ induce the same metric: this reflects the fact that semi-translation surfaces have a fixed *north direction*, which is forgotten when considering only the metric structure.

The flat singular metric $|q|$ can be nicely characterized (see $[St]$) stating that its local geodesics are continuous maps $\gamma : \mathbb{R} \to S_g$ such that for every $t \in \mathbb{R}$:

- if $\gamma(t) \notin \Sigma$, then there is a neighborhood U of t in $\mathbb{R}$ such that $\gamma|_U$ is an Euclidean segment,
- if $\gamma(t) \in \Sigma$, then there is a small neighborhood V of $\gamma(t)$ in S_g and an $\epsilon > 0$ small enough such that the angles defined by $\gamma([t, t+\epsilon))$ and $\gamma((t-\epsilon, t])$ in V are both at least π.

We say that a *saddle connection* on (X, q) is a geodesic for the flat metric going from a singularity to a singularity, without any singularities in the interior of the segment. Since the metric $|q|$ is locally $Cat(0)$, for any arc γ with endpoints in Σ there always is a unique geodesic representative in the homotopy class of γ with fixed endpoints. This geodesic representative is a concatenation of saddle connections.
Finally, we define the *systole* of a semi-translation surface (X, q), and denote it with $sys(q)$, to be the length of the shortest saddle connection.

A first remarkable fact about semi-translation surfaces is that they always admit a triangulation by saddle connections (see $[Tr]$). Using this property we can state another equivalent definition of semi-translation surfaces.

Definition 3.2. *A semi-translation surface is an equivalence class of a finite union of planar polygons with edge identifications: each edge must be paired with exactly one other edge, which is parallel and of the same length. Two such collections of polygons are the same semi-translation surface if one can be cut into pieces along straight lines and these pieces can be re-glued identifying sides by semi-translations to form the other collection of polygons.*

The alternative definition of translation surfaces is the same, but with sides identified by translations.
Notice that, since every semi-translation surface can be triangulated, the collection of polygons can be obtained cutting along the saddle connections of the triangulation.
On the other side, given a collection of polygons, the paired edges can be identified and in the interior of each edge and polygon a natural coordinate $z \in \mathbb{C}$ can be used. Some vertices will become singular points of the flat metric, since the total angle around these singularities will always be an integer multiple of π.

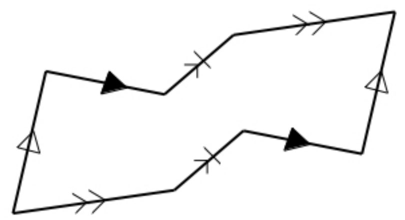

Figure 3.1: An example of translation surface with one singular point of cone angle 6π.

We now introduce the Teichmüller and moduli spaces of translation surfaces.

Given any $g \geq 2$ and $m \geq 1$, fix a finite set of points $\Sigma = \{p_1, \ldots, p_m\} \subset S_g$ and an m-ple $\underline{k} = (k_1, k_2, \ldots, k_m) \in \mathbb{N}^m$ such that $\sum_{l=1}^{m} k_l = 2g - 2$.
We denote by $\Omega_g(\underline{k}, \Sigma)$ the set of translation surfaces on S_g which have singularities prescribed by $\underline{k}$ on the points of Σ (i.e. it has a zero of order k_i on p_i, $i = 1, \ldots, m$).
Consider the subgroups $Diff_0^+(S_g, \Sigma)$ and $Diff^+(S_g, \Sigma)$ respectively of $Diff_0^+(S_g)$ and $Diff^+(S_g)$ and which consist of diffeomorphisms which fix the points of Σ.
We define the Teichmüller space of translation surfaces with singularities prescribed by $\underline{k}$ on the points of Σ as the quotient

$$\mathcal{TH}_g(\underline{k}, \Sigma) := \Omega_g(\underline{k}, \Sigma)/Diff_0^+(S_g, \Sigma)$$

and the moduli space of translation surfaces with singularities prescribed by $\underline{k}$ on the points of Σ as the quotient

$$\mathcal{H}_g(\underline{k}, \Sigma) := \Omega_g(\underline{k}, \Sigma)/Diff^+(S_g, \Sigma).$$

In order to lighten the notation, from now on we will denote these spaces simply as $\Omega_g(\underline{k}), \mathcal{TH}_g(\underline{k}), \mathcal{H}_g(\underline{k})$: one should keep in mind that in the definition is implicit the choice of Σ.
We will denote simply by ω an element of $\mathcal{TH}_g(\underline{k})$ and $\mathcal{H}_g(\underline{k})$: the fact that it is an equivalence class will be clear from the context.

The space $\mathcal{TH}_g(\underline{k})$ can be endowed with a topology using developing maps. For a given $\omega \in \mathcal{TH}_g(\underline{k})$ fix $p \in \Sigma$, a universal cover $\pi : \widetilde{S}_g \to S_g$ and a point $\widetilde{p} \in \widetilde{S}_g$ over p. Then the developing map

$$Dev_\omega : (\widetilde{S}_g, \widetilde{p}) \to (\mathbb{C}, 0), \quad \widetilde{p}_1 \mapsto \int_{\widetilde{p}}^{\widetilde{p}_1} \pi^*\omega$$

is such that the association

$$\Theta : \Omega_g(\underline{k}) \to C^0(\widetilde{S}_g, \mathbb{C}), \quad \omega \mapsto Dev_\omega$$

is injective: endowing $C^0(\widetilde{S}_g, \mathbb{C})$ with the compact-open topology one gets naturally a topology on $\Omega_g(\underline{k})$ and $\mathcal{TH}_g(\underline{k})$.

Definition 3.3. *For every point $\omega_0 \in \mathcal{TH}_g(\underline{k})$ and a sufficiently small open neighborhood U of ω_0 in $\mathcal{TH}_g(\underline{k})$, it is possible to define the local period map*

$$\mathcal{P} : U \to H^1(S_g, \Sigma, \mathbb{C})$$

$$\mathcal{P}(\omega) := \left(\gamma \mapsto \int_\gamma \omega\right) \in Hom(H_1(S_g, \Sigma, \mathbb{Z}), \mathbb{C}) \simeq H^1(S_g, \Sigma, \mathbb{C}).$$

Notice that by definition the local period map is continuous with respect to the topology on $\mathcal{TH}_g(\underline{k})$ induced by the developing map.

Theorem 3.0.1. *The local period maps endow $\mathcal{TH}_g(\underline{k})$ with the structure of a complex manifold of dimension $2g + m - 1$.*

The main point of the proof is showing the existence of an isotopy between any two forms ω_0 and ω_1 close to each other in $\mathcal{TH}_g(\underline{k})$, with transverse real and imaginary parts and such that $\mathcal{P}(\omega_0) = \mathcal{P}(\omega_1)$. This can be done in many ways, for example using *Veech's zipped rectangles construction* (as in $[Yo]$) or a variant of the so called *Moser's homotopy trick* (as in $[FMa]$).
Having done that, it is just left to notice that composing the isomorphisms

$$H^1(S_g, \Sigma, \mathbb{C}) \simeq \mathbb{C}^{2g+m-1}$$

with the period maps, one obtains affine transition maps given by the change of basis of the relative homology.

It is possible to define a natural measure $\lambda_{\underline{k}}$ on $\mathcal{TH}_g(\underline{k})$ pulling back the Lebesgue measure on $\mathbb{C}^{2g+m-1}$ and normalizing in such a way that the integral lattice $H^1(S_g, \Sigma, \mathbb{Z} \oplus i\mathbb{Z})$ has covolume 1.

At this point one could wonder wether the same nice structure of $\mathcal{TH}_g(\underline{k})$ transfers to $\mathcal{H}_g(\underline{k})$. The answer is no, because, similarly to the case of the moduli space of Riemann surfaces, one should be aware of the issues which arise when X has automorphisms preserving ω. For this reason each moduli space $\mathcal{H}_g(\underline{k})$ is only an $orbifold$ of dimension $2g + m - 1$.

The construction we made on the space of translation surfaces can be adapted to the space of semi-translation surfaces: one should notice that this time, besides the m-ple $\underline{k} = (k_1, k_2, \ldots, k_m)$ with $\sum_{l=1}^{m} k_l = 4g - 4$ indicating the multiplicity of the zeroes on the points of Σ, one needs an additional index ϵ, which is set to 1 in case of trivial holonomy and to -1 otherwise. We denote by $\mathcal{S}\Omega(\underline{k}, \epsilon, \Sigma)$ the set of semi-translation surfaces indexed by $\underline{k}$ and ϵ.
We define the Teichmüller and moduli space of semi-translation surfaces with singularities prescribed by $\underline{k}$ on Σ and holonomy defined by ϵ in the following way:

$$\mathcal{TQ}(\underline{k}, \epsilon, \Sigma) := \mathcal{S}\Omega(\underline{k}, \epsilon, \Sigma)/Diff_0^+(S_g, \Sigma), \quad \mathcal{Q}(\underline{k}, \epsilon, \Sigma) := \mathcal{S}\Omega(\underline{k}, \epsilon, \Sigma)/Diff^+(S_g, \Sigma).$$

As we already did for translation surfaces, we will denote $\mathcal{TQ}_g(\underline{k}, \epsilon, \Sigma)$ and $\mathcal{Q}_g(\underline{k}, \epsilon, \Sigma)$ simply as $\mathcal{TQ}_g(\underline{k}, \epsilon)$ and $\mathcal{Q}_g(\underline{k}, \epsilon)$.

All Teichmüller spaces $\mathcal{TQ}(\underline{k}, \epsilon)$ have the same nice structure of the Teichmüller spaces of translation surfaces.

Theorem 3.0.2. *Each space $\mathcal{TQ}_g(\underline{k}, 1)$ has the structure of a complex manifold of dimension $2g + m - 1$, while $\mathcal{TQ}_g(\underline{k}, -1)$ has the structure of a complex manifold of dimension $2g + m - 2$.*

Proof. If we consider a quadratic differential q in a Teichmülelr space with trivial holonomy then there is an Abelian differential ω such that $q = \omega^2$. In fact the squaring defines an homeomorphism between a neighborhood of ω in $\mathcal{TH}_g(\underline{k})$ and a neighborhood of q in $\mathcal{TQ}_g(\underline{k}, 1)$. The claim then follows from theorem 3.0.1.
In case $q \in \mathcal{TQ}_g(\underline{k}, -1)$, one should consider the fact that there exists a canonical non trivial ramified double covering $\pi : \hat{X} \to X$ such that $\pi^* q$ is the square of an Abelian differential $\hat{\omega}$ on $\hat{X}$ (see for example $[La]$). Let

$$\mathcal{P} : \mathcal{TH}_{\hat{g}}(\hat{\underline{k}}) \to H^1(\hat{X}, \Sigma(\hat{\omega}), \mathbb{C})$$

(where $\Sigma(\hat{\omega})$ is the set of zeroes of $\hat{\omega}$) be a period coordinate near $\hat{\omega}$: at this point one should note that the pullback by π and the squaring of Abelian differentials induce an homeomorphism between a neighborhood of q in $\mathcal{TQ}_g(\underline{k}, -1)$ and a neighborhood of $\mathcal{P}(\hat{\omega})$ in the (-1)-eigenspace of the covering involution of π in $H^1(\hat{X}, \Sigma(\hat{\omega}), \mathbb{C})$. The claim then follows from the computation of the dimension of this eigenspace.
Let m_{odd} be the number of indexes l such that k_l is odd and similarly let m_{even} be the number of indexes i such that k_i is even. Then

$$dim_{\mathbb{C}} H^1(\hat{X}, \Sigma(\hat{\omega}), \mathbb{C}) = 2\hat{g} + m_{odd} + 2m_{even} - 1$$

and using Riemann-Hurwitz formula one can compute

$$\hat{g} = 2g + \frac{m_{odd}}{2} - 1.$$

Finally the computation of the dimension of the (-1)-eigenspace can be completed considering that the dimension of the $(+1)$-eigenspace equals the dimension of $H^1(X, \Sigma(q), \mathbb{C})$. □

For the same reasons explained before, each moduli space $\mathcal{Q}(\underline{k}, \epsilon)$ is an orbifold of the same dimension of $\mathcal{TQ}(\underline{k}, \epsilon)$.

There is a natural action of $GL(2, \mathbb{R})^+$ on $\mathcal{TQ}_g(\underline{k}, \epsilon)$ and $\mathcal{Q}_g(\underline{k}, \epsilon)$ (and consequently on $\mathcal{TH}_g(\underline{k})$ and $\mathcal{H}_g(\underline{k})$): for each $A \in GL(2, \mathbb{R})^+$ and each quadratic differential q, the element $A \cdot q$ is the quadratic differential obtained post-composing the natural charts of q with A. Another way of understanding the action of $GL(2, \mathbb{R})^+$ is considering q as a collection of polygons. Then $A \cdot q$ is obtained applying A on the polygons.

We denote by $\mathcal{TH}_g^{(1)}(\underline{k})$ the subset of $\mathcal{TH}_g(\underline{k})$ corresponding to Abelian differentials whose associated area form has total area equal to 1. Note that, since the area form $Area(\omega)$ of ω can be expressed as

$$Area(\omega) = \frac{i}{2} \int_{S_g} \omega \wedge \overline{\omega} = \frac{i}{2} \sum_{j=1}^{g} (A_j \overline{B}_j - \overline{A}_j B_j),$$

where $A_j = \int_{\alpha_j} \omega$, $B_j = \int_{\beta_j} \omega$ and $\{\alpha_j, \beta_j\}_{j=1}^g$ is a symplectic basis of $H_1(S_g, \mathbb{R})$, we see that $\mathcal{TH}_g^{(1)}(\underline{k})$ can be considered as a unit hyperboloid.
Note that there is a natural $SL(2, \mathbb{R})$-action on $\mathcal{TH}_g^{(1)}(\underline{k})$ for which the Lebesgue measure $\lambda_{\underline{k}}^{(1)}$ is $SL(2, \mathbb{R})$-invariant.

All the previous properties apply also for $\mathcal{H}_g^{(1)}(\underline{k}), \mathcal{TQ}_g^{(1)}(\underline{k}, \epsilon), \mathcal{Q}_g^{(1)}(\underline{k}, \epsilon)$, but the following result (proved by H.Masur in $[Ma]$ and by W. Veech in $[Ve]$) only applies to $\mathcal{H}_g^{(1)}(\underline{k})$ and consequently to $\mathcal{Q}_g^{(1)}(\underline{k}, \epsilon)$, using the ramified double cover we referred to in the proof of theorem 3.0.2.

Theorem 3.0.3. *The total mass of* $\lambda_{\underline{k}}^{(1)}$ *is finite.*

Chapter 4

Thurston's metric on $\mathcal{TQ}_g(\underline{k}, \epsilon)$

As it is explained in the previous chapter, every element of $\mathcal{TQ}_g(\underline{k}, \epsilon)$ defines a singular flat metric on S_g: the idea behind the work of the present thesis is to investigate how the definition of the Thurston's metrics L and K on $\mathcal{T}_g^n$ could be adapted to the case of flat singular metrics.

W.A. Veech already did something similar in $[Ve2]$ defining a complex-valued distance map D_0 on the Teichmuller space $\mathcal{TQ}_g^n(\underline{k}, \epsilon)$ of semi-translation structures on S_g^n (which can be defined in the same way of $\mathcal{TQ}_g(\underline{k}, \epsilon)$).
We copy the definition of D_0 maintaining the original notation of Veech:

$$D_0(q_1, q_2) := \inf_{\varphi \in Diff_0^+(S_g^n)} \alpha(\varphi^* q_1, q_2),$$

$$\alpha(\varphi^* q_1, q_2) := \sup_{x \in S_g^n} \left(\sup_{(U_i, f_i) \in q_i, x \in U_1 \cap U_2} \left(\limsup_{x' \to x} Log \left(\left(\frac{f_1(\varphi(x')) - f_1(\varphi(x))}{f_2(x') - f_2(x)} \right)^2 \right) \right) \right),$$

where q_i, $i = 1, 2$ is regarded as a semi-translation structures and $f_i : U_i \to \mathbb{C}$, $U_i \subset S_g^n$, are natural charts of q_i. The map Log is a branch of the complex logarithm. The real part of $\alpha(\varphi^* q_1, q_2)$ is the Lipschitz constant of φ computed with respect to the metrics $|q_1|$ and $|q_2|$ and consequently the real part of the distance function D_0 is asymmetric.
Veech claimed that the map D_0 is a complete pseudo-metric on $\mathcal{TQ}_g^n(\underline{k}, \epsilon)$ (the proof should be contained in unpublished preprints $[Ve3]$).

We modified the definition of Veech's distance map to make it a symmetric pseudo-metric L_F on $\mathcal{TQ}_g(\underline{k}, \epsilon)$ (and thus L_F is different from the real part of the distance map defined by Veech), which in fact becomes non degenerate if considered as a function on spaces of flat singular metrics: in the present chapter we study its properties.

As a second introductory observation, one should notice that C.T.McMullen in $[Mc]$ defined a metric on leaves of the absolute period foliation on strata of $\overline{\mathcal{H}_g}$, the moduli space of stable forms (see $[CDF]$ for precise definitions of these objects), which has some similarities with the metric K defined by Thurston on $\mathcal{T}_g^n$.
The absolute period foliation $\mathcal{A}$ on $\mathcal{H}_g$ is defined in such a way that its leaves are the fibers of the map

$$\Phi : \mathcal{H}_g \to H^1(S_g), \quad \Phi(\omega) = [\omega]$$

and restricts to a foliation $\mathcal{A}(\underline{k})$ on each stratum $\mathcal{H}_g(\underline{k})$, $\underline{k} \in \mathbb{N}^m$.
For any sufficiently small neighborhood U in a leaf of $\mathcal{A}(\underline{k})$, one can define the relative period map

$$\mathcal{P} : U \to \mathbb{C}^m/\mathbb{C}\cdot(1,\ldots,1), \quad \mathcal{P}(\omega) = \left(\int_{p_0}^{p_1}\omega,\ldots,\int_{p_0}^{p_m}\omega\right)$$

defined choosing paths between the zeroes $p_1,\ldots,p_m$ of ω and a point p_0 of S_g: the map $\mathcal{P}$ provides each leaf with local coordinates.
The foliation $\mathcal{A}(\underline{k})$ can be extended to a foliation $\overline{\mathcal{A}(\underline{k})}$ on strata $\overline{\mathcal{H}_g(\underline{k})}$ of $\overline{\mathcal{H}_g}$ (by abuse of notation we will still denote its leaves by $\Phi^{-1}(\varphi)$), and the relative period map can be extended in such a way to provide local coordinates.
On each leaf of $\overline{\mathcal{A}(\underline{k})}$ which does not contain any stable form which vanishes identically on any irreducible component of the underlying stable curve, McMullen defined a natural path metric coming from the norm $||\cdot||_M$ on $\mathbb{C}^m/\mathbb{C}\cdot(1,\ldots,1)$:

$$||(z_1,\ldots,z_m)||_M := \max_{i,j=1,\ldots,m}|z_i - z_j|$$

and proved in $[Mc]$ that this metric is complete on each leaf $\Phi^{-1}(\phi)$ of $\overline{\mathcal{A}(\underline{k})}$ corresponding to a form ϕ such that $\phi(H_1(S_g,\mathbb{Z})) \simeq \mathbb{Z}^{2g}$.

In the present chapter we define and study a complete symmetric pseudo-metric K_F on $\mathcal{TQ}_g(\underline{k},\epsilon)$ which is quite similar to the Thurston's metric K on $\mathcal{T}_g^n$.

4.1 Definitions and comparison with $d_{\mathcal{T}}$

Fix any genus $g \geq 2$ and consider the Teichmüller space of semi-translation surfaces $\mathcal{TQ}_g(\underline{k},\epsilon)$ with singularities on $\Sigma \subset S_g$ prescribed by the m-ple $\underline{k} = (k_1,\ldots,k_m) \in \mathbb{N}^m$ such that $\sum_{i=1}^m k_i = 4g-4$ and holonomy determined by $\epsilon \in \{+1,-1\}$.
We will introduce now all flat analogues to Thurston's metrics.

First we define the following function L_F, which is a symmetric analogue to Thurston's metric L.

$$L_F : \mathcal{TQ}_g(\underline{k},\epsilon) \times \mathcal{TQ}_g(\underline{k},\epsilon) \to \mathbb{R},$$

$$L_F(q_1,q_2) := \inf_{\varphi \in Diff_0^+(S_g,\Sigma)} \mathcal{L}_{q_1}^{q_2}(\varphi),$$

$$\mathcal{L}_{q_1}^{q_2}(\varphi) := \sup_{p \in S_g \setminus \Sigma} \left(\sup_{v \in T_pS_g, ||v||_{q_1}=1} |\log(||d\varphi_p v||_{q_2})| \right).$$

The quantity $\mathcal{L}_{q_1}^{q_2}(\varphi)$ can be rewritten as

$$\mathcal{L}_{q_1}^{q_2}(\varphi) = \max\{\log(Lip_{q_1}^{q_2}(\varphi)), -\log(lip_{q_1}^{q_2}(\varphi))\},$$

with $Lip_{q_1}^{q_2}(\varphi)$ being the *upper Lipschitz constant* of φ:

$$Lip_{q_1}^{q_2}(\varphi) := \sup_{p \in S_g \setminus \Sigma} \left(\sup_{v \in T_pS_g, ||v||_{q_1}=1} ||d\varphi_p v||_{q_2} \right)$$

and $lip_{q_1}^{q_2}(\varphi)$ being the *lower Lipschitz constant* of φ:

$$lip_{q_1}^{q_2}(\varphi) := \inf_{p \in S_g \setminus \Sigma} \left(\inf_{w \in T_pS_g, ||w||_{q_1}=1} ||d\varphi_p w||_{q_2} \right).$$

We define also an asymmetric analogue to L on $\mathcal{TQ}_g^{(1)}(\underline{k},\epsilon)$

$$L_F^a : \mathcal{TQ}_g^{(1)}(\underline{k},\epsilon) \times \mathcal{TQ}_g^{(1)}(\underline{k},\epsilon) \to \mathbb{R}$$

associating to any pair $q_1, q_2 \in \mathcal{TQ}_g^{(1)}(\underline{k},\epsilon)$ of semi-translation surfaces of unitary area the quantity

$$L_F^a(q_1,q_2) := \inf_{\varphi \in \mathcal{D}} \log(Lip(\varphi)_{q_1}^{q_2}),$$

$$Lip(\varphi)_{q_1}^{q_2} = \sup_{p \in S_g \setminus \Sigma} \left(\sup_{v \in T_pS_g, ||v||_{q_1}=1} ||d\varphi_p v||_{q_2} \right),$$

where $\mathcal{D}$ is the set of functions $\varphi : S_g \to S_g$ which are homotopic to the identity, differentiable almost everywhere and which fix the points of Σ.
Since $Diff_0^+(S_g,\Sigma) \subset \mathcal{D}$, one can immediately deduce $L_F(q_1,q_2) \geq L_F^a(q_1,q_2)$ for every $q_1, q_2 \in \mathcal{TQ}_g^{(1)}(\underline{k},\epsilon)$.

We define two flat counterparts to the metric K, which are K_F^a and K_F. The first one is asymmetric and the second one is its symmetrization.
In particular, for every $q_1, q_2 \in \mathcal{TQ}_g(\underline{k},\epsilon)$, we set

$$K_F^a(q_1,q_2) := \sup_{\gamma \in SC(q_1)} \log\left(\frac{\hat{l}_{q_2}(\gamma)}{\hat{l}_{q_1}(\gamma)}\right),$$

where $SC(q_1)$ is the set of saddle connections of q_1, and $\hat{l}_{q_i}(\gamma)$ is the length of the geodesic representative for $|q_i|$ in the homotopy class of γ with fixed endpoints.
Finally the symmetric analogue to K is defined as

$$K_F(q_1, q_2) := \max\{K_F^a(q_1, q_2), K_F^a(q_2, q_1)\}$$

for every $q_1, q_2 \in \mathcal{TQ}_g(\underline{k}, \epsilon)$.

In this chapter we will study the properties of L_F and K_F and explain the difficulties in trying to prove $L_F = K_F$ on $\mathcal{TQ}_g(\underline{k}, \epsilon)$.
These difficulties can be solved considering L_F^a instead of L_F: the fact that L_F^a is asymmetric and the infimum is taken over functions in $\mathcal{D}$ will play a crucial role.
Indeed, L_F^a is defined specifically to get $L_F^a = K_F^a$: the next chapter will be completely devoted to the proof of such equality.

We now begin the study of the properties of L_F.

Proposition 4.1.1. *The function L_F is a symmetric pseudo-metric on $\mathcal{TQ}_g(\underline{k}, \epsilon)$.*

Proof. It is clear that $L(q, q) = |\log(Lip_q^q(Id))| = 0$ for all $q \in \mathcal{TQ}_g(\underline{k}, \epsilon)$.
The equality

$$lip_{q_1}^{q_2}(\varphi) = \frac{1}{Lip_{q_2}^{q_1}(\varphi^{-1})}$$

grants

$$\mathcal{L}_{q_1}^{q_2}(\varphi) = \mathcal{L}_{q_2}^{q_1}(\varphi^{-1})$$

and thus the symmetry of L_F.
The triangular inequality follows from the inequality

$$\mathcal{L}_{q_1}^{q_3}(\varphi \circ \psi) \leq \mathcal{L}_{q_2}^{q_3}(\varphi) + \mathcal{L}_{q_1}^{q_2}(\psi).$$

Finally, one could easily note that, given any $q_1 \in \mathcal{TQ}_g(\underline{k}, \epsilon)$, it results $L_F(q_1, q_2) = 0$ exactly for all $q_2 \in \mathcal{TQ}_g(\underline{k}, \epsilon)$ such that $q_2 = e^{i\theta} q_1$. □

Since it results $L_F(q_1, q_2) = 0$ if and only if q_1 and q_2 are in the same orbit of the action of the unitary group $U(1) \subset \mathbb{C}^*$, it follows that L_F can be considered as a metric on the space of flat singular metrics with singularities prescribed by $\underline{k}$ and holonomy prescribed by ϵ.
For the same reason, L_F descends to a metric $\mathbb{P}L_F$ on the projectivization $\mathbb{P}\mathcal{TQ}_g(\underline{k}, \epsilon) = \mathcal{TQ}_g(\underline{k}, \epsilon)/\mathbb{C}^* = \mathcal{TQ}_g^{(1)}(\underline{k}, \epsilon)/U(1)$ by setting

$$\mathbb{P}L_F([q_1], [q_2]) := L_F\left(\frac{q_1}{Area(q_1)}, \frac{q_2}{Area(q_2)}\right).$$

The first result we present on the pseudo-metric L_F is an inequality concerning the Teichmüller metric $d_{\mathcal{T}}$.

Proposition 4.1.2. *For any $q_1, q_2 \in \mathcal{TQ}_g(\underline{k}, \epsilon)$, denote by $\mathcal{X}_1, \mathcal{X}_2 \in \mathcal{T}_g$ the points in the Teichmüller space relative to the corresponding conformal structure. It results:*

$$L_F(q_1, q_2) \geq d_{\mathcal{T}}(\mathcal{X}_1, \mathcal{X}_2)$$

In case there is a Teichmüller map between $\mathcal{X}_1$ and $\mathcal{X}_2$ with respect to the differentials q_1 and q_2 the last inequality is an equality.

Proof. For every $\varphi \in Diff_0^+(S_g, \Sigma)$ and $p \in S_g \setminus \Sigma$ we define the quantities

$$Lip_{q_1}^{q_2}(\varphi)_p := \sup_{v \in T_p S_g, ||v||_{q_1}=1} ||d\varphi_p v||_{q_2},$$

$$lip_{q_1}^{q_2}(\varphi)_p := \inf_{w \in T_p S_g, ||w||_{q_1}=1} ||d\varphi_p w||_{q_2}.$$

Then, since the global dilatation $K(\varphi)$ is independent of the holomorphic charts and thus can be computed in the natural coordinates respectively of q_1 and q_2, we get the inequality

$$K(\varphi) = \sup_{p \in S_g \setminus \Sigma} \frac{Lip_{q_1}^{q_2}(\varphi)_p}{lip_{q_1}^{q_2}(\varphi)_p} \leq \frac{Lip_{q_1}^{q_2}(\varphi)}{lip_{q_1}^{q_2}(\varphi)}.$$

Since for every $\varphi \in Diff_0^+(S_g, \Sigma)$ it also results

$$K(h) \leq K(\varphi),$$

where $K(h)$ is the global dilatation of a Teichmüller map h such that $d_{\mathcal{T}}(\mathcal{X}_1, \mathcal{X}_2) = \frac{1}{2}\log(K(h))$, combining the last two inequalities we get that it can not be at the same time

$$Lip_{q_1}^{q_2}(\varphi) < \sqrt{K(h)} \text{ and } lip_{q_1}^{q_2}(\varphi) > \frac{1}{\sqrt{K(h)}}$$

and this implies the inequality $L(q_1, q_2) \geq d_{\mathcal{T}}(\mathcal{X}_1, \mathcal{X}_2)$.
Finally, in case h is a Teichmüller map with respect to the quadratic differentials q_1 and q_2, then, since h can be written in local coordinates as

$$h(x + iy) = \sqrt{K(h)}x + \frac{i}{\sqrt{K(h)}}y$$

it follows

$$Lip_{q_1}^{q_2}(h) = \sqrt{K(h)}, \quad lip_{q_1}^{q_2}(h) = \frac{1}{\sqrt{K(h)}}$$

and thus the equality of the claim.

□

Observation 4.1.3. *Notice that in the proof of proposition 4.1.2 the fact that the metric induced by the quadratic differential is locally $Cat(0)$ is never used. For this reason, one could allow the quadratic differentials to have simple poles on the marked points and define L_F in the same way.*
Then the same inequality $L_F(q_1, q_2) \geq d_{\mathcal{T}}(\mathcal{X}_1, \mathcal{X}_2)$ will be true for $\mathcal{X}_1, \mathcal{X}_2 \in \mathcal{T}_g^n$.

4.2 Induced topology

We define *standard topology* on $\mathcal{TQ}_g(\underline{k}, \epsilon)$, and denote it by $\mathbb{T}_{std}$, the topology induced by the structure of complex manifold, that is, the topology induced by the period maps. Given a sequence $\{q_n\}_{n\in\mathbb{N}} \subset \mathcal{TQ}_g(\underline{k}, \epsilon)$, we write $q_n \to q$ to denote its convergence to $q \in \mathcal{TQ}_g(\underline{k}, \epsilon)$ with respect to the standard topology.
Similarly, we denote by $\mathbb{T}_{L_F}$ the topology on $\mathcal{TQ}_g(\underline{k}, \epsilon)$ induced by L_F.

Proposition 4.2.1. *The topology $\mathbb{T}_{std}$ is finer than $\mathbb{T}_{L_F}$.*

We will prove the equivalent claim that for every sequence $\{q_n\}_{n\in\mathbb{N}} \subset \mathcal{TQ}_g(\underline{k}, \epsilon)$ the convergence $q_n \to q$ implies $\lim_{n\to\infty} L_F(q_n, q) = 0$.

To this end, we need to first make an observation concerning Euclidean triangles. Denote by Ξ the set of non-degenerate Euclidean triangles $T \subset \mathbb{R}^2$ with one vertex in the origin of $\mathbb{R}^2$: since every triangle $T \in \Xi$ can be identified by the coordinates of its two vertices different from the origin, Ξ can be considered as a subset of $\mathbb{R}^4$.
Given any sequence $\{T_n\}_{n\in\mathbb{N}}$ in Ξ, we say that it converges to $T \in \Xi$, and write $T_n \to T$, if $\{T_n\}_{n\in\mathbb{N}}$ converges to T as a sequence of $\mathbb{R}^4$ with respect to the standard Euclidean metric. For every $n \in \mathbb{N}$ consider the affine map A_n which sends T_n to T and denote by $\sigma_1(A_n), \sigma_2(A_n)$ its eigenvalues. It is easy to verify that if $T_n \to T$ then $\lim_{n\to\infty} \sigma_1(A_n) = 1$, $\lim_{n\to\infty} \sigma_2(A_n) = 1$.

Proof. In order to prove the proposition, given any sequence $\{q_n\}_{n\in\mathbb{N}} \subset \mathcal{TQ}_g(\underline{k}, \epsilon)$ such that $q_n \to q$, we will find a sequence of maps $A_n \in Diff_0^+(S_g, \Sigma)$ with the property $\mathcal{L}_q^{q_n}(A_n) \to 0$. The claim then will follow from the inequality $L(q_n, q) \leq \mathcal{L}_q^{q_n}(A_n)$.
If $q_n \to q$ then one could find a collection of arcs $\Gamma = \{\gamma_j\}_{j=1}^{3(m+2g-2)}$ with endpoints in Σ which triangulate S_g and an $n_0 > 0$ such that the geodesic representative of the homotopy class of every γ_j for $|q|$ and $|q_n|$, $n > n_0$, is a saddle connection.
The geodesic representatives of the homotopy classes of the arcs in Γ for $|q|$ (resp. $|q_n|$), provide us of a set of Euclidean triangles $\Xi_q = \{T_l\}_{l=1}^{2(k+2g-2)}$ (resp $\Xi_{q_n} = \{T_l^n\}_{l=1}^{2(k+2g-2)}$) which cover S_g. Using period coordinates of $\mathcal{TQ}_g(\underline{k}, \epsilon)$ one can indeed observe that $q_n \to q$ implies that every triangle T_l^n converges to T_l in the sense explained in the observation preceding this proof.

For every $n \in \mathbb{N}$, we define by $A_n \in Diff_0^+(S_g, \Sigma)$ the map which is piecewise affine in natural coordinates respectively of q_n and q, and which on every triangle T_l^n of Ξ_{q_n} is the affine map A_n^l which sends T_l^n to the corresponding triangle T_l of Ξ_q.
As before, we denote by $\sigma_1(A_n^l), \sigma_2(A_n^l)$ the eigenvalues of A_n^l. Since it results

$$Lip_q^{q_n}(A_n^l) = \max_{l=1,\dots,2(m+2g-2)} \left(\max\{\sigma_1(A_n^l), \sigma_2(A_n^l)\}\right)$$

$$lip_q^{q_n}(A_n^l) = \min_{l=1,\dots,2(m+2g-2)} \left(\min\{\sigma_1(A_n^l), \sigma_2(A_n^l)\}\right)$$

the claim of the proposition follows from the preceding observation about Euclidean triangles. □

From proposition 4.2.1, it follows that compact sets of $\mathbb{T}_{std}$ are also compact sets of $\mathbb{T}_{L_F}$. It is thus useful to characterize them in a way which is similar to the statement of Mumford's compactness criterion.
Before doing so, let us fix once and for all some notation: for any arc γ in S_g with endpoints in Σ and any quadratic differential $q \in \mathcal{TQ}_g(\underline{k}, \epsilon)$, we denote by $l_q(\gamma)$ the length of γ with respect to the metric $|q|$ and by $\hat{l}_q(\gamma)$ the length of the geodesic representative for $|q|$ in the homotopy class of γ with fixed endpoints.

The following proposition about compact sets of $\mathbb{T}_{std}$ is a consequence of proposition 1, section 3, of $[KMS]$, which establishes the compactness of subsets of quadratic differentials with lower bound on the area.

Proposition 4.2.2. *Fix $\epsilon, L > 0$ and a collection of arcs $\Gamma = \{\gamma_i\}_{i=1}^{3(m+2g-2)}$ with endpoints in Σ which triangulates S_g.*
Define the subset $\mathcal{K}_{\epsilon,L} \subset \mathcal{TQ}_g(\underline{k}, \epsilon)$ as the set of quadratic differentials q which satisfy the following two conditions.

(i) $sys(q) \geq \epsilon$,

(ii) $\sum_{i=1}^{3(m+2g-2)} \hat{l}_q(\gamma_i) \leq L$.

The set $\mathcal{K}_{\epsilon,L}$ is a compact set of $\mathbb{T}_{std}$.

Using this characterization of compact sets we can prove the following proposition.

Proposition 4.2.3. *Each Teichmüller space $\mathcal{TQ}(\underline{k}, \epsilon)$ endowed with the pseudometric L_F is a proper topological space.*

Proof. We prove that closed balls $B_{L_F}^R(q)$ of L_F,

$$B_{L_F}^R(q) := \{q' \in \mathcal{TQ}_g(\underline{k}, \epsilon) | L_F(q, q') \leq R\}$$

are contained in a compact subset of $\mathbb{T}_{std}$: thanks to the result of proposition 4.2.1 they will be contained also in a compact set of $\mathbb{T}_{L_F}$.
Let γ be any geodesic arc for $|q|$ with endpoints in Σ, and γ_n the geodesic representative of its homotopy class for the metric $|q_n|$. Then it follows

$$\frac{l_q(\gamma)}{l_{q_n}(\gamma_n)} \leq \frac{l_q(\varphi(\gamma_n))}{l_{q_n}(\gamma_n)} \leq \sup_{p\in S_g\setminus\Sigma}\left(\sup_{v\in T_pS_g,||v||_{q_n}=1}||d\varphi_p v||_q\right),$$

$$\frac{l_{q_n}(\gamma_n)}{l_q(\gamma)} \leq \frac{l_{q_n}(\varphi(\gamma))}{l_q(\gamma)} \leq \sup_{p\in S_g\setminus\Sigma}\left(\sup_{v\in T_pS_g,||v||_q=1}||d\varphi_p v||_{q_n}\right)$$

and from the fact that $L_F(q,q_n)$ is bounded it follows that it can not happen

$$\lim_{n\to\infty}\hat{l}_{q_n}(\gamma)=0 \text{ or } \lim_{n\to\infty}\hat{l}_{q_n}(\gamma)=\infty.$$

□

By abuse of notation we will denote by $\mathbb{T}_{std}$ and $\mathbb{T}_{L_F}$ the induced topologies on $\mathbb{P}\mathcal{TQ}_g(\underline{k},\epsilon)$.

Proposition 4.2.4. *$\mathbb{T}_{std}$ and $\mathbb{T}_{L_F}$ are the same topology on $\mathbb{P}\mathcal{TQ}_g(\underline{k},\epsilon)$.*

Proof. It will be sufficient to prove that $\mathbb{T}_{L_F}$ is finer than $\mathbb{T}_{std}$ and thus that for every sequence $\{q_n\}_{n\in\mathbb{N}}\subset\mathcal{TQ}_g^{(1)}(\underline{k},\epsilon)$ such that $\lim_{n\to\infty}L_F(q_n,q)=0$ it follows that there exists $c\in U(1)$ with the property $q_n\to cq$.
Since $\lim_{n\to\infty}L_F(q_n,q)=0$ it follows that $\{q_n\}_{n\in\mathbb{N}}$ is contained in a closed ball of L_F and thus in a compact set. Up to passing to a subsequence we can state that there is $q'\in\mathcal{TQ}_g(\underline{k},\epsilon)$ such that $q_n\to q'$. Since

$$L_F(q,q')\leq L_F(q,q_n)+L_F(q_n,q')$$

it follows $q'=e^{i\theta}q$. □

In the following theorem we establish another similarity between L_F and Thurston's asymmetric metric L: L_F is a complete pseudo-metric.
The notion of completeness makes sense also for pseudo-metrics: a pseudo-metric d on a topological space X is complete if every Cauchy sequence for d admits at least one limit point for d. Thus in the proof of the following theorem we will prove that every Cauchy sequence for L_F admits at least one limit point for L_F.

Theorem 4.2.5. *Every Teichmüller space $\mathcal{TQ}_g(\underline{k},\epsilon)$ and its quotient $\mathbb{P}\mathcal{TQ}_g(\underline{k},\epsilon)$, endowed respectively with the metrics L_F and $\mathbb{P}L_F$, are complete pseudo-metric spaces.*

Proof. We prove that any Cauchy sequence $\{q_n\}_{n\in\mathbb{N}}$ for L_F on $\mathcal{TQ}_g(\underline{k},\epsilon)$ is contained in a compact set of $\mathbb{T}_{std}$: from proposition 4.2.1 it will follow that $\{q_n\}_{n\in\mathbb{N}}$ is contained in a compact set of $\mathbb{T}_{L_F}$ and therefore is convergent. We will use the same inequalities of the proof of proposition 4.2.3.
Consider any Cauchy sequence $\{q_n\}_{n\in\mathbb{N}}$ for L_F on $\mathcal{TQ}_g(\underline{k},\epsilon)$, given any arc γ on S_g with endpoints in Σ denote by γ_n the geodesic representative of the homotopy class of γ for the metric $|q_n|$. Then for every $\varphi \in Diff_0^+(S_g,\Sigma)$ it results:

$$\frac{l_{q_m}(\gamma_m)}{l_{q_n}(\gamma_n)} \leq \frac{l_{q_m}(\varphi(\gamma_n))}{l_{q_n}(\gamma_n)} \leq Lip_{q_n}^{q_m}(\varphi)$$

and thus the sequence $\{\log(l_{q_n}(\gamma_n))\}_{n\in\mathbb{N}}$ is a Cauchy sequence and consequently bounded: this means that $\{q_n\}_{n\in\mathbb{N}}$ is contained in a set of the form described in proposition 4.2.2.
The completeness of $(\mathbb{P}\mathcal{TQ}_g(\underline{k},\epsilon),\mathbb{P}L_F)$ follows from the same reasoning considering a Cauchy sequence $\{q_n\}_{n\in\mathbb{N}} \subset \mathcal{TQ}_g^{(1)}(\underline{k},\epsilon)$. □

Finally, it is worth mentioning that the mapping class group $\Gamma(S_g,\Sigma)$ acts on $\mathcal{TQ}_g(\underline{k},\epsilon)$ by isometries of L_F: in particular for every $q_1,q_2 \in \mathcal{TQ}_g(\underline{k},\epsilon)$ and $\psi \in \Gamma(S_g,\Sigma)$ it results

$$L_F(q_1,q_2) = L_F(\psi\cdot q_1, \psi\cdot q_2),$$

where $\psi\cdot q$ is the pullback by ψ^{-1} of the quadratic differential q. This result follows from the equality

$$\mathcal{L}_{q_1}^{q_2}(\varphi) = \mathcal{L}_{\psi\cdot q_1}^{\psi\cdot q_2}(\psi\circ\varphi\circ\psi^{-1})$$

for every $\varphi \in Diff_0^+(S_g,\Sigma)$ and the fact that the conjugation of $Diff_0^+(S_g,\Sigma)$ by any element of $\Gamma(S_g,\Sigma)$ is an isomorphism of $Diff_0^+(S_g,\Sigma)$.
Since the action of the mapping class group $\Gamma(S_g,\Sigma)$ on $\mathcal{TQ}_g(\underline{k},\epsilon)$ is also properly discontinuous, one gets that the metric L_F descends also to a metric $\hat{L}_F$ on $\mathcal{Q}(\underline{k},\epsilon)$,

$$\hat{L}_F(\hat{q}_1,\hat{q}_2) = \inf L_F(q_1,q_2),$$

where the infimum is taken over all liftings q_1,q_2 to $\mathcal{TQ}_g(\underline{k},\epsilon)$ of $\hat{q}_1,\hat{q}_2 \in \mathcal{Q}_g(\underline{k},\epsilon)$.

Proposition 4.2.6. *The space $\mathcal{Q}_g(\underline{k},\epsilon)$ endowed with the metric $\hat{L}_F$ is a complete pseudo-metric space.*

Proof. The proof is identical to the one of proposition 4.2.5. □

4.3 Relation with pseudo-metric $\mathbf{K}_F$

A first analogy with the metric L_F is given by the fact that K_F has all the properties we just proved for L_F and in particular it follows:

Theorem 4.3.1. *The function K_F is a complete and proper symmetric pseudo-metric on $\mathcal{TQ}_g(\underline{k}, \epsilon)$.*

Proof. All the previous proofs for L_F adapt to K_F (in particular, the fact that $q_n \to q$ implies $K_F(q_n, q) \to q$ is a direct consequence of the definition of period maps), except for

$$K_F(q_1, q_2) = 0 \text{ if and only if } q_1 = e^{i\theta} q_2$$

which can be proved as we now explain.
If $q_1 = e^{i\theta} q_2$ then q_1 and q_2 induce the same flat metric on S_g and consequently $K_F(q_1, q_2) = 0$, so let us prove the other implication.
If $K_F(q_1, q_2) = 0$, then consider any saddle connection σ of q_1 and let τ be the geodesic representative for $|q_2|$ in the homotopy class of σ. The curve τ is a concatenation of saddle connections $\tau_1, \ldots, \tau_k$ of q_2 and since $K_F^a(q_1, q_2) \leq 0$ it results

$$l_{q_1}(\sigma) \geq l_{q_2}(\tau_1) + \ldots l_{q_2}(\tau_k).$$

For each $i = 1, \ldots, k$ let σ_i be the geodesic representative for the metric $|q_1|$ in the homotopy class of τ_i. Since $K_F^a(q_2, q_1) \leq 0$ it follows

$$l_{q_2}(\tau_1) + \ldots l_{q_2}(\tau_k) \geq l_{q_1}(\sigma_1) + \cdots + l_{q_1}(\sigma_k)$$

and, since the concatenation $\sigma_1 * \cdots * \sigma_k$ is in the same homotopy class of σ, it also results

$$l_{q_1}(\sigma_1) + \cdots + l_{q_1}(\sigma_k) \geq l_{q_1}(\sigma).$$

These inequalities can be realized at the same time only if they are equalities, and since σ is the only geodesic representative in its homotopy class it follows that τ must be a saddle connection of q_2: we have thus proved that if $K_F(q_1, q_2) = 0$ then the geodesic representative for $|q_2|$ (resp. for $|q_1|$) of any saddle connection of q_1 (resp. of q_2) must be a saddle connection of the same length.
At this point the claim is basically already proved, since q_1 and q_2 give triangulations of S_g by saddle connections of the same length.

□

We can define on $\mathbb{P}\mathcal{TQ}_g(\underline{k}, \epsilon)$ the metric $\mathbb{P}K_F$ in the same way we defined $\mathbb{P}L_F$ and prove that its induced topology $\mathbb{T}_{K_F}$ coincides with the standard topology $\mathbb{T}_{std}$.

As for the metrics L and K on $\mathcal{T}_g$, the inequality

$$L_F(q_1, q_2) \geq K_F(q_1, q_2), \quad \forall q_1, q_2 \in \mathcal{TQ}_g(\underline{k}, \epsilon)$$

is straightforward, while proving the inverse inequality is a much harder problem, which could be solved finding a function $\varphi \in Diff_0^+(S_g, \Sigma)$ such that

$$\mathcal{L}_{q_1}^{q_2}(\varphi) \leq K_F(q_1, q_2).$$

Before studying the general case, let us first state a much simpler fact.

Proposition 4.3.2. *Given any $q \in \mathcal{TQ}_g(\underline{k}, \epsilon)$ and any $A \in GL(2, \mathbb{R})^+$, it results*

$$L_F(q, A \cdot q) = K_F(q, A \cdot q) = \log(\sigma),$$

where $\sigma := \max\{\sigma_1(A), \sigma_1(A)^{-1}, \sigma_2(A), \sigma_2(A)^{-1}\}$ and $\sigma_1(A), \sigma_2(A)$ are the two eigenvalues of A.

Proof. Without loss of generality, we can suppose $\sigma_1(A)$ is realized in the horizontal direction of q and $\sigma_2(A)$ in the vertical direction. Notice furthermore that it results

$$\log(\sigma) = \mathcal{L}_q^{A \cdot q}(Id).$$

If $\sigma = \sigma_1(A)$ or $\sigma = \sigma_1(A)^{-1}$, then a saddle connection in the horizontal direction will have stretch factor σ: although it is not always possible to suppose the existence of such geodesic, it is a consequence of theorem 2 of $[Ma2]$ that the directions of saddle connections of a quadratic differential are dense in S^1. Consequently, we can always consider a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ of saddle connections of q asymptotic in the horizontal direction: this means that it results $\lim_{n \to \infty} \theta(\gamma_n) = 0$, where $\theta(\gamma_n)$ is the difference between the direction of γ_n and the horizontal direction.
Then it follows

$$K_F(q, A \cdot q) \geq \lim_{n \to \infty} \left| \log\left(\frac{\hat{l}_{A \cdot q}(\gamma_n)}{\hat{l}_q(\gamma_n)} \right) \right| = \log(\sigma) \geq L_F(q, A \cdot q)$$

and from $K_F(q, A \cdot q) \leq L_F(q, A \cdot q)$ one gets $K_F(q, A \cdot q) = L_F(q, A \cdot q) = \log(\sigma)$. If $\sigma = \sigma_2(A)$ or $\sigma = \sigma_2(A)^{-1}$, one can repeat the same reasoning for the vertical direction. □

Considering the general case, one could be tempted to adapt the ideas behind Thurston's proof in $[Th]$ to the case of L_F and K_F. Specifically, one could try to build a flat analogue to Thurston's stretch maps.

We thought the more natural approach to try to do so was to triangulate S_g by saddle connections: clearly this could work only locally on $\mathcal{TQ}_g(\underline{k}, \epsilon)$, since for quadratic differentials q_1, q_2 too far apart there will not be any triangulation $\Gamma = \{\gamma_i\}_{i=1}^{3(m+2g-2)}$ of S_g by arcs and a continuous path $t \mapsto q_t$ which connects q_1 and q_2 and is such that the geodesic representative of the homotopy class of each γ_i is a

saddle connection for all q_t.
Another possibility concerned the use of a flat counterpart to geodesic laminations, called *flat lamination* (for definitions and properties we refer the reader to $[Mo]$) in order to obtain a triangulation of S_g.
Unfortunately, both approaches suffered of the same problem: instead of hyperbolic triangles, singular flat metrics require the use of Euclidean triangles. Indeed, one can triangulate a semi-translation surface (X, q) with Euclidean triangles and stretch each side of each Euclidean triangle by the same factor $K > 1$ as in proposition 2.2.2, but then the resulting semi-translation surface will simply be $K \cdot q$.
The point is that in this case the sides of the triangles of the triangulation should be stretched by different factors. When trying to do so, one should notice that there are plenty of couples of Euclidean triangles T_1, T_2 with each side stretched by a factor lower or equal to $c > 1$, and such that there could be no homeomorphism $f : T_1 \to T_2$ which sends sides to corresponding sides and with $Lip(f) \leq c$.

Example 4.1. *Consider the equilateral triangle T_1 with sides of length 1 and the isosceles triangle T_2 with base side of length 1 and height $\sqrt{3}$. Then clearly the maximal stretching of the sides of T_1 and T_2 is $\frac{\sqrt{13}}{2}$, while each homeomorphism $f : T_1 \to T_2$ which sends sides to corresponding sides must also send the arc parametrizing the height of T_1 to an arc of length at least $\sqrt{3}$. This implies that the Lipschitz constant of such f must be at least $2 > \frac{\sqrt{13}}{2}$.*

The fundamental fact enlightened by the previous conter-example is that, if one tries to obtain a diffeomorphism $\varphi \in Diff_0^+(S_g, \Sigma)$ with $\mathcal{L}_{q_1}^{q_2}(\varphi) = K_F(q_1, q_2)$ by defining it first on the Euclidean triangles of a triangulation of S_g, then $\mathcal{L}_{q_1}^{q_2}(\varphi)$ should be attained along a curve of the triangulation. As a consequence, when searching for flat analogues to Thurston's stretch maps, one should impose strict conditions on the triangles considered.
As we made clear before, for q_1 and q_2 sufficiently close in $\mathcal{TQ}_g(\underline{k}, \epsilon)$, there is a triangulation $\Gamma = \{\gamma_i\}_{i=1}^{3(m+2g-2)}$ of S_g by arcs with endpoints in Σ such that the geodesic representative of each γ_i for $|q_1|$ and $|q_2|$ is a saddle connection. This procedure provides us of a collection $\Xi^1 = \{T_j^1\}_{j=1}^{2(m+2g-2)}$ of Euclidean triangles in the natural coordinates of q_1 and a collection $\Xi^2 = \{T_j^2\}_{j=1}^{2(m+2g-2)}$ of Euclidean triangles in the natural coordinates of q_2.
Our problem is now to establish if there is a triangulation Γ of S_g such that it is possible to obtain a function $\varphi \in Diff_0^+(S_g, \Sigma)$ with $\mathcal{L}_{q_1}^{q_2}(\varphi) = K_F(q_1, q_2)$ by defining it first on each couple of corresponding triangles of Ξ^1 and Ξ^2.

To this end one should consider the following fact:

Given two Euclidean triangles T_1, T_2 with sides labeled, consider the set $L(T_1, T_2)$ of Lipschitz constants of diffeomorphisms $f : T_1 \to T_2$ which send sides to corresponding sides in a linear way.
The minimum of $L(T_1, T_2)$ is the Lipschitz constant of the affine map A which maps T_1 in T_2.

Note that we considered functions which are linear on the sides of the triangles since we want the Lipschitz constant to be equal to ratio of lengths of a side. This suggests the fact that the function φ we are trying to obtain should be affine on each triangle T_j^1 and that its greater eigenvalue should be attained on the most stretched side of Γ.

Finally, we see that this last condition imposes a very strong constrain on the collections Ξ_1 and Ξ_2 and consequently on the triangulation Γ. Since this problem is related to the nature of Euclidean triangles, it does not seem likely to be solved using flat laminations.

For the reasons we just explained, we were not able to prove the local equality $L_F = K_F$ trying to adapt Thurston's approach. In the succeeding chapter we will explain another approach we used to prove that the equality of two asymmetric pseudo-metrics L_F^a and K_F^a on $\mathcal{TQ}_g^{(1)}(\underline{k}, \epsilon)$ depends on two statements about 1-Lipschitz maps between polygons.

4.4 Geodesics of L_F

In the previous discussion we explained why we are not able to produce a flat counterpart to Thurston's stretch lines, but it is interesting nonetheless to investigate what do geodesics of L_F look like.
We could only find geodesics of L_F which are also geodesics of K_F: this is because the only feasible strategy to find geodesics $t \mapsto q_t$ of L_F we could think of was to find functions $\varphi_t \in Diff_0^+(S_g, \Sigma)$ such that $\mathcal{L}_q^{q_t}(\varphi_t) = t = K_F(q, q_t)$ and then conclude from $K_F(q, q_t) \leq L_F(q, q_t)$.

As one can easily notice, these geodesics of L_F are very particular: as soon as some hypothesis are lighten, one can no longer be sure to find functions φ_t such that $\mathcal{L}_q^{q_t}(\varphi_t) = t = K_F(q, q_t)$.

Let us explain first how to obtain geodesics of L_F and K_F entirely contained in one orbit of $GL(2,\mathbb{R})^+$.

Proposition 4.4.1. *Consider any $q \in \mathcal{TQ}_g(\underline{k},\epsilon)$, and any pair of continuous functions*

$$\theta : [0,1] \to [0,2\pi), \quad f : [0,1] \to \mathbb{R}^+$$

such that for every $t_0, t_1 \in [0,1]$, $t_0 < t_1$ it results

$$e^{t_1/t_0} \geq \max\left\{\frac{f(t_1)}{f(t_0)}, \frac{f(t_0)}{f(t_1)}\right\}.$$

Using these data one can produce four geodesics for K_F and L_F starting at q of the following form

$$\Phi^j : [0,1] \to \mathcal{TQ}_g(\underline{k},\epsilon), \quad t \mapsto q_t^j := e^{i\theta(t)} \cdot \Sigma_t^j \cdot q, \quad j = 1,2,3,4,$$

where Σ_t^j is one of the following four diagonal matrices

$$\Sigma_t^1 := \begin{pmatrix} e^t & 0 \\ 0 & f(t) \end{pmatrix} \quad \Sigma_t^2 := \begin{pmatrix} e^{-t} & 0 \\ 0 & f(t) \end{pmatrix} \quad \Sigma_t^3 := \begin{pmatrix} f(t) & 0 \\ 0 & e^t \end{pmatrix} \quad \Sigma_t^4 := \begin{pmatrix} f(t) & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Proof. The proof is identical for all four geodesics, so we will just prove it for Φ^1. For any $t_0, t_1 \in [0,1]$, $t_0 < t_1$ it results $q_{t_1}^1 = A \cdot q_{t_0}^1$, where $A = e^{i\theta(t_1)} \cdot \Sigma \cdot e^{-i\theta(t_0)}$ and Σ is the following diagonal matrix

$$\Sigma := \begin{pmatrix} e^{t_1 - t_0} & 0 \\ 0 & \frac{f(t_1)}{f(t_0)} \end{pmatrix}.$$

Since Φ^1 is contained in a $GL(2,\mathbb{R})^+$-orbit, one can apply previous proposition 4.3.2 and get $K_F(q_{t_0}^1, q_{t_1}^1) = L_F(q_{t_0}^1, q_{t_1}^1) = t_1 - t_0$.

□

Given any Teichmüller geodesic

$$\Psi : [0,1] \to \mathcal{T}_g, \quad t \mapsto [(X_t, h_t)]$$

with initial differential q on X, we define its lifting on $\mathcal{TQ}_g(\underline{k},\epsilon)$ to be

$$\widetilde{\Psi} : [0,1] \to \mathcal{TQ}_g(\underline{k},\epsilon), \quad t \mapsto q_t$$

where q_t is the holomorphic quadratic differential on X_t such that $h_t : X \to X_t$ is a Teichmüller map with respect to q and q_t and with dilatation e^{2t}.

Proposition 4.4.2. *Liftings to $\mathcal{TQ}_g(\underline{k},\epsilon)$ of Teichmüller geodesics are geodesics for L_F and K_F.*

Proof. The claim follows immediately from the previous proposition: one just has to notice that the Teichmüller map h_t can be locally written in natural coordinates of q and q_t as

$$h_t(x+iy) = e^t x + ie^{-t} y.$$

□

At this point, one could be tempted to try to obtain other geodesics using the result of proposition 4.3.2. In particular, considering functions θ and f as in proposition 4.4.1, one may wonder if it could be possible to impose for example $q_t := \Sigma_t^1 \cdot e^{i\theta(t)} \cdot q$. The answer is no: since the direction where the stretching e^t is obtained varies, there is no hope to get $L_F(q_{t_0}, q_{t_1}) = t_1 - t_0$ or $K_F(q_{t_0}, q_{t_1}) = t_1 - t_0$.

It is possible however to obtain other kinds of geodesics modifying only one part of the semi-translation surface, as we will now explain.

Proposition 4.4.3. *Let $q \in \mathcal{TQ}_g(\underline{k}, \epsilon)$ be a semi-translation surface which contains a flat cylinder C of height $h > 0$ such that there is at least one saddle connection entirely contained in C which realizes the height of the cylinder.*
The arc $\Phi : [0,1] \to \mathcal{TQ}_g(\underline{k}, \epsilon)$, $t \mapsto q_t$, where q_t is the semi-translation surface obtained from q changing the height of the flat cylinder to $e^t h$, is a geodesic for L_F and K_F.

Proof. Denote by $\gamma_1, \dots, \gamma_2$ the saddle connections entirely contained in C which realize the height of the cylinder. Clearly, if h is stretched by e^t then the length of $\gamma_1, \dots, \gamma_k$ is stretched by the same factor.
For any $t_0, t_1 \in [0,1]$, $t_0 < t_1$, the semi-translation surface q_{t_1} is obtained from q_{t_0} stretching the height of the cylinder by the factor $e^{t_1 - t_0}$. All saddle connections of q_{t_0} different from $\gamma_1, \dots, \gamma_k$ are stretched by a factor which is smaller than $e^{t_1-t_0}$, and consequently one can conclude

$$K_F(q_{t_1}, q_{t_0}) = \log\left(\frac{\hat{l}_{q_{t_1}}(\gamma_i)}{\hat{l}_{q_{t_0}}(\gamma_i)}\right) = t_1 - t_0.$$

Without loss of generality, we can suppose the direction of the saddle connection $\gamma_1, \dots, \gamma_k$ is the vertical one. Consequently there is a function $\varphi \in Diff_0^+(S_g, \Sigma)$ which, in natural coordinates of q_{t_0} and q_{t_1}, can be written as the affine function $\begin{pmatrix} 1 & 0 \\ 0 & e^{t_1-t_0} \end{pmatrix}$ on the cylinder and as the identity on the complement of the cylinder.
From $L_F(q_{t_0}, q_{t_1}) \le \mathcal{L}_{q_{t_0}}^{q_{t_1}}(\varphi) = t_1 - t_0 = K_F(q_{t_0}, q_{t_1})$ and $L_F(q_{t_0}, q_{t_1}) \ge K_F(q_{t_0}, q_{t_1})$ one gets the last desired equality $L_F(q_{t_1}, q_{t_0}) = t_1 - t_0$.

□

The idea behind the previous proposition can be applied also to the case of a semi-translation surface q obtained *gluing two semi-translation surfaces* q_1, q_2 *along a slit in the horizontal direction.* This means that one cuts two slits of the same length, one in q_1 and one in q_2, both in the horizontal direction. Each q_i will then have boundary consisting of two segments: each segment of the boundary of q_1 will be glued with a segment of the boundary of q_2 and the resulting surface q will have two singularities of total angle 4π at the extremities of the slit.

Proposition 4.4.4. *Let $q \in \mathcal{TQ}_g(\underline{k}, \epsilon)$ be a semi-translation surface obtained gluing two semi-translation surfaces q_1, q_2 along a slit in the horizontal direction. Furthermore, suppose that q_1 is such that it contains a sequence of saddle connections $\{\gamma_n\}_{n\in\mathbb{N}}$ asymptotic in the vertical direction (i.e. the limit of the differences of their directions with the vertical direction is zero) such that no γ_n intersects the slit.*
Then one obtains the geodesic $\Phi : [0,1] \to \mathcal{TQ}_g(\underline{k}, \epsilon)$, $t \mapsto q_t$, where q_t is the semi-translation surface obtained gluing $\begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} \cdot q_1$ and q_2 along the same slit.

Proof. The idea of the proof is very similar to the one of the previous proposition. First of all notice that q_t is a well-defined semi-translation surface since the slit is horizontal and q_1 is stretched only in the vertical direction.
Then, for every $t_0, t_1 \in [0,1]$, $t_0 < t_1$, from the fact that no γ_n intersects the slit it follows

$$K_F(q_{t_0}, q_{t_1}) = \lim_{n\to\infty} \log\left(\frac{\hat{l}_{q_{t_1}}(\gamma_n)}{\hat{l}_{q_{t_0}}(\gamma_n)}\right) = t_1 - t_0.$$

One can then conclude noting $L_F(q_{t_0}, q_{t_1}) \leq \mathcal{L}_{q_{t_0}}^{q_{t_1}}(Id) = t_1 - t_0$. □

Chapter 5

Equality of Thurston's metrics

In this chapter we investigate the equality of two asymmetric pseudo-metrics L_F^a and K_F^a on each Teichmüller space $\mathcal{TQ}_g^{(1)}(\underline{k},\epsilon)$ of holomorphic quadratic differentials of unitary area without simple poles.
In particular, using the method we develop in this chapter, the equality of L_F^a and K_F^a on whole $\mathcal{TQ}_g^{(1)}(\underline{k},\epsilon)$ can be proved if two statements about 1-Lipschitz maps between planar polygons are true. We are able to prove the first statement, but the second one remains a conjecture: nonetheless, we explain why we believe it is true.

5.1 Definitions of the metrics

For any $g \geq 2$ and any Teichmüller space $\mathcal{TQ}_g^{(1)}(\underline{k},\epsilon)$ of holomorphic quadratic differentials of unitary area without simple poles, we define the function

$$L_F^a : \mathcal{TQ}_g^{(1)}(\underline{k},\epsilon) \times \mathcal{TQ}_g^{(1)}(\underline{k},\epsilon) \to \mathbb{R}$$

associating to any pair $q_1, q_2 \in \mathcal{TQ}_g^{(1)}(\underline{k},\epsilon)$ of semi-translation surfaces of unitary area the quantity

$$L_F^a(q_1,q_2) := \inf_{\varphi\in\mathcal{D}} \log(Lip(\varphi)_{q_1}^{q_2}),$$

$$Lip(\varphi)_{q_1}^{q_2} = \sup_{p\in S_g\setminus\Sigma}\left(\sup_{v\in T_pS_g,||v||_{q_1}=1} ||d\varphi_p v||_{q_2}\right),$$

where $\mathcal{D}$ is the set of functions $\varphi : S_g \to S_g$ which are homotopic to the identity, differentiable almost everywhere and which fix the points of Σ.

Proposition 5.1.1. *The function L^a_F is an asymmetric pseudo-metric on $\mathcal{TQ}^{(1)}_g(\underline{k},\epsilon)$.*

Proof. It is clear that $L^a_F(q,q)=0$ for every $q\in\mathcal{TQ}_g(\underline{k},\epsilon)$ and that L^a_F is not symmetric.
Note that every function $\varphi\in\mathcal{D}$ must be surjective, since it has degree 1: from this fact it follows $Lip(\varphi)^{q_2}_{q_1}\geq 1$ and $Lip(\varphi)^{q_2}_{q_1}=1$ if and only if $q_2=e^{i\theta}q_1$.
Finally, L^a_F satisfies the triangular inequality since for every couple of functions $\varphi,\phi\in\mathcal{D}$ it follows

$$Lip(\varphi\circ\phi)^{q_3}_{q_1}\leq Lip(\varphi)^{q_3}_{q_2}Lip(\phi)^{q_2}_{q_1}.$$

□

The other pseudo-metric we consider in the present chapter is K^a_F: for every $q_1,q_2\in\mathcal{TQ}^{(1)}_g(\underline{k},\epsilon)$, $K^a_F(q_1,q_2)$ is defined as

$$K^a_F(q_1,q_2):=\sup_{\gamma\in SC(q_1)}\log\left(\frac{\hat{l}_{q_2}(\gamma)}{\hat{l}_{q_1}(\gamma)}\right).$$

For every $q_1,q_2\in\mathcal{TQ}^{(1)}_g(\underline{k},\epsilon)$ it clearly results

$$L^a_F(q_1,q_2)\geq K^a_F(q_1,q_2).$$

With the techniques exposed in the present chapter we are able to reduce the proof of the equality of L^a_F and K^a_F on the whole $\mathcal{TQ}^{(1)}_g(\underline{k},\epsilon)$ to the proof of two statements about 1-Lischitz maps between planar polygons. Given their importance, we feel it is necessary to briefly anticipate them now in a slightly simplified version.

Consider two planar polygons Δ and Δ' such that there is an injective function

$$\iota: Vertices(\Delta)\to Vertices(\Delta')$$

which to every vertex v associates a unique vertex $\iota(v)=v'$. Suppose both Δ and Δ' have exactly three vertices with strictly convex internal angle, which we denote x_i and x'_i, $i=1,2,3$ respectively.
Suppose furthermore that for every $x,y\in Vertices(\Delta)$ it results

$$d_\Delta(x,y)\geq d_{\Delta'}(x',y'),$$

where d_Δ (resp. $d_{\Delta'}$) is the intrinsic Euclidean metric inside Δ (resp. Δ'): $d_\Delta(x,y)$ (resp. $d_{\Delta'}(x',y')$) is defined as the infimum of the lengths, computed with respect to the Euclidean metric, of all paths from x to y (resp. from x' to y') entirely contained in Δ (resp. in Δ').
We say that vertices of Δ and of $\iota(Vertices(\Delta))$ are *disposed in the same order* if it

is possible to choose two parametrizations $\gamma : [0,1] \to \partial\Delta$ and $\gamma_1 : [0,1] \to \partial\Delta'$ such that $\gamma(0) = x_1$, $\gamma_1(0) = x'_1$ and γ, γ_1 meet respectively vertices of Δ and of Δ' in the same order.

Statement 5.1. *If $Vertices(\Delta)$ and $\iota(Vertices(\Delta))$ are disposed in the same order, then there is a 1-Lipschitz map $f : \Delta \to \Delta'$ (with respect to the intrinsic Euclidean metrics of the polygons) which sends vertices to corresponding vertices.*

Statement 5.2. *If $Vertices(\Delta)$ and $\iota(Vertices(\Delta))$ are not disposed in the same order, then for every point $p \in \Delta$ there is a point $p' \in \Delta'$ such that*

$$d_\Delta(p, x_i) \geq d_{\Delta'}(p', x'_i), \quad i = 1, 2, 3.$$

We were able to prove the first statement, which corresponds to following theorem 5.3.3, but not the second one, which from now on will be referred to as conjecture 5.1: we will still explain why we believe it must be true.
We state the following theorem, which is the main result of this thesis.

Theorem 5.1.2. *If conjecture 5.1 is true, then for every $q_1, q_2 \in \mathcal{TQ}_g^{(1)}(\underline{k}, \epsilon)$, it results*

$$L_F^a(q_1, q_2) = K_F^a(q_1, q_2).$$

We proved theorem 5.1.2 using an approach similar to a proof by F.A. Valentine (which can be found in $[Va]$) of Kirszbraun's theorem for $\mathbb{R}^2$ (firstly proved by M.D. Kirszbraun in $[Ki]$).

Theorem 5.1.3. *(Kirszbraun)*
Let $S \subset \mathbb{R}^2$ be any subset and $f : S \to \mathbb{R}^2$ a 1-Lipschitz map.
Given any set T which contains S, it is possible to extend f to a 1-Lipschitz map $\hat{f} : T \to \mathbb{R}^2$ such that $\hat{f}(T)$ is contained in the convex hull of $f(S)$.

The key ingredients of Valentine's proof of Kirszbraun theorem are the following two lemmas.

Lemma 5.1.4. *Fix two Euclidean triangles $\Delta(x_1, x_2, x_3)$ and $\Delta(x'_1, x'_2, x'_3)$ in $\mathbb{R}^2$ such that*

$$|x'_i - x'_j| \leq |x_i - x_j| \quad \textit{for every } i, j = 1, 2, 3.$$

Then for any $x_4 \in \mathbb{R}^4$ there is a point x'_4 contained in $\Delta(x'_1, x'_2, x'_3)$ such that

$$|x'_4 - x'_i| \leq |x_4 - x_i| \quad \textit{for every } i = 1, 2, 3.$$

The second lemma is often referred to as *Helly's theorem* (firstly proved by E.Helly in $[He]$).

Lemma 5.1.5. *(Helly)*
Let F be any family of compact and convex subsets of $\mathbb{R}^n$. Suppose that for every $C_1, \ldots, C_{n+1} \in F$ it results

$$\bigcap_{i=1}^{n+1} C_i \neq \emptyset$$

then it also results

$$\bigcap_{C \in F} C \neq \emptyset.$$

Together, these two lemmas imply the ensuing proposition, from which one easily deduces theorem 5.1.3.

Proposition 5.1.6. *Given any two collections $\{B_{r_j}(x_j)\}_{j\in J}$ and $\{B_{r_j}(x'_j)\}_{j\in J}$ of closed disks in $\mathbb{R}^2$ with the same radii and with centers such that*

$$|x'_i - x'_j| \leq |x_i - x_j|.$$

Then, if

$$\bigcap_{j\in J} B_{r_j}(x_j) \neq \emptyset$$

it follows

$$\bigcap_{j\in J} B_{r_j}(x'_j) \neq \emptyset.$$

We performed a similar reasoning in order to find a function $\phi \in \mathcal{D}$ such that

$$Lip(\phi)_{q_1}^{\sigma_2} = 1,$$

where σ_2 is the rescaled differential

$$\sigma_2 := \frac{q_2}{e^{K_F^a(q_1,q_2)}}.$$

The existence of such function ϕ proves the equality

$$e^{L_F^a(q_1,\sigma_2)} = e^{K_F^a(q_1,\sigma_2)} = 1 \tag{5.1}$$

and consequently, since for every $c > 0$ it follows

$$ce^{L_F^a(q_1,q_2)} = e^{L_F^a(q_1,cq_2)}, \quad ce^{K_F^a(q_1,q_2)} = e^{K_F^a(q_1,cq_2)}$$

multiplying both termes of equation (5.1) by $e^{K_F^a(q_1,q_2)}$ and then composing with the logarithm, one gets the desired result

$$L_F^a(q_1,q_2) = K_F^a(q_1,q_2).$$

It is important to specify that in our proof we used the following version of Helly's lemma, which can be found in $[Iv]$.

Lemma 5.1.7. *Let X be a uniquely geodesic space of compact topological dimension $n < \infty$. If $\{A_j\}_{j \in J}$ is any finite collection of convex sets in X such that every subcollection of cardinality at most $n + 1$ has a nonempty intersection, then*

$$\bigcap_{j \in J} A_j \neq \emptyset.$$

If q is a holomorphic quadratic differential on a closed Riemann surface of genus $g \geq 2$ one can consider a universal cover $\pi : \widetilde{S}_g \to S_g$ and the pullback $\widetilde{q}$ of q on $\widetilde{S}_g$. Then $|\widetilde{q}|$ induces a metric which is $Cat(0)$ and consequently uniquely geodesic. But if q has poles then $|\widetilde{q}|$ does not induce an uniquely geodesic metric space: this is the reason why our proof could not be adapted to the Teichmüller space of quadratic differentials with poles.

One should notice that the equality $L_F^a = K_F^a$ could be implied by a version of Kirszbraun theorem which suits semi-translation surfaces (without simple poles). The generalization of theorem 5.1.3 which could be considered closer to semi-translation surfaces was proved by S.Alexander, V.Kapovitch and A.Petrunin in $[AKP]$ and applies to the case of functions from complete $CBB(k)$ spaces (spaces with curvature bounded below by k) to complete $Cat(k)$ spaces (spaces with curvature bounded above by k). Since semi-translation surfaces are only locally $Cat(0)$ spaces, unfortunately the theorem of $[AKP]$ does not apply to our case.

At this point it should be more clear why we decided to prove the equality of the two pseudo-metrics L_F^a, K_F^a instead of the equality of the two pseudo-metrics L_F, K_F studied in the preceding chapter.
Indeed, one reason is that it is more convenient to study asymmetric pseudo-metrics, since it is more complicated to control both Lipschitz constants (the lower and the upper one) at once: for an attempt in this direction in the simple case of the unit square see $[DP]$.
The other reason is that using this kind of *Kirszbraun approach* there is no hope to obtain an injective 1-Lipschitz function. This is the reason why we defined L_F^a as the infimum of Lipschitz constants of functions in $\mathcal{D}$.
Finally one should notice that the condition of unitary area of the two semi-translation surfaces q_1 and q_2 will never be used in the proof. We could actually prove the equality of L_F^a and K_F^a on the whole $\mathcal{TQ}_g(\underline{k}, \epsilon)$, where the two pseudo-metrics are much more degenerate.

The next section is devoted to the explanation of our proof of the construction of the function $\phi \in \mathcal{D}$ such that $Lip(\phi)_{q_1}^{\sigma_2} = 1$.

5.2 Proof of the equality

Let $\pi : \widetilde{S}_g \to S_g$ be a universal cover. Lifting through π the complex structure of X_1 and the differential q_1 one obtains the metric universal cover $\pi : (\widetilde{X}_1, |\widetilde{q}_1|) \to (X_1, |q_1|)$ and doing the same thing to X_2 and q_2 one obtains the metric universal cover $\pi : (\widetilde{X}_2, |\widetilde{\sigma}_2|) \to (X_2, |\sigma_2|)$.
Denote by $d_{\widetilde{q_1}}$ the $Cat(0)$ metric induced by $|\widetilde{q}_1|$ and by $d_{\widetilde{\sigma_2}}$ the $Cat(0)$ metric induced by $|\widetilde{\sigma}_2|$. In order to avoid confusion, when we will want to underline that a point of $\widetilde{S}_g$ is regarded as a point of $\widetilde{X}_2$, we will denote it with an additional prime symbol: for example a point $\widetilde{x} \in \pi^{-1}(\Sigma)$ will be denoted as $\widetilde{x}$ if regarded as a point of $\widetilde{X}_1$ and $\widetilde{x}'$ if regarded as a point of $\widetilde{X}_2$.
For every couple of points $\widetilde{x}, \widetilde{y} \in \widetilde{X}_1$, $\overline{\widetilde{x}\widetilde{y}}$ is the $d_{\widetilde{q}_1}$-geodesic from $\widetilde{x}$ to $\widetilde{y}$. Since there will be no ambiguity, we will denote geodesics of $d_{\widetilde{\sigma}_2}$ in the same way: for every couple of points $\widetilde{x}', \widetilde{y}' \in \widetilde{X}_2$, $\overline{\widetilde{x}'\widetilde{y}'}$ is the $d_{\widetilde{\sigma}_2}$-geodesic from $\widetilde{x}'$ to $\widetilde{y}'$.

Fix a point $x_0 \in \Sigma \subset S_g$ and $\widetilde{x}_0 \in \pi^{-1}(x_0)$: as it is well known, the group $\pi_1(S_g, x_0)$ acts on $\widetilde{S}_g$ and for every $\gamma \in \pi_1(S_g, x_0)$, $\widetilde{x} \in \widetilde{S}_g$ it results

$$\gamma \cdot \widetilde{x} = \widetilde{\tau}(1),$$

where $\widetilde{\tau}$ is the lifting of $\gamma * \pi(\widetilde{\sigma})$ ($\widetilde{\sigma}$ is any path in $\widetilde{S}_g$ from $\widetilde{x}_0$ to $\widetilde{x}$) such that $\widetilde{\tau}(0) = \widetilde{x}_0$.

Fix a fundamental domain $P \subset (\widetilde{X}_1, d_{\widetilde{q}_1})$ for the action of $\pi_1(S_g, x_0)$, suppose $\widetilde{x}_0 \in P$.
We want to build a map $\hat{\phi} : \hat{U} \to (\widetilde{X}_2, d_{\widetilde{\sigma}_2})$ (where $\hat{U}$ is a dense countable subset of P which includes the zeroes of $\widetilde{q}_1$ contained in P), such that for every couple of points $\widetilde{x}, \widetilde{y} \in \hat{U}$ (eventually equal) and every $\gamma \in \pi_1(S_g, x_0)$, it results

$$d_{\widetilde{\sigma}_2}(\hat{\phi}(\widetilde{x}), \gamma \cdot \hat{\phi}(\widetilde{y})) \leq d_{\widetilde{q}_1}(\widetilde{x}, \gamma \cdot \widetilde{y}) \tag{5.2}$$

and for every zero $\widetilde{z}$ of $\widetilde{q}_1$ contained in P it results $\hat{\phi}(\widetilde{z}) = \widetilde{z}'$ (notice that $\widetilde{q}_1$ and $\widetilde{q}_2$ have zeroes in the same points, which are the points of $\pi^{-1}(\Sigma)$).

Having done so, we define the dense subset $\widetilde{U} := \pi_1(S_g, x_0) \cdot \hat{U}$ of $\widetilde{X}_1$ and extend the function $\hat{\phi}$ by equivariance to a function $\widetilde{\phi}^U : \widetilde{U} \to \widetilde{X}_2$, imposing

$$\widetilde{\phi}^U(\gamma \cdot \widetilde{x}) := \gamma \cdot \hat{\phi}(\widetilde{x})$$

for every $\gamma \in \pi_1(X_1, x_0), \widetilde{x} \in \hat{U}$.
Notice that for every $\gamma_1 \cdot \widetilde{x}_1, \gamma_2 \cdot \widetilde{x}_2 \in \widetilde{U}$ it results:

$$d_{\widetilde{\sigma}_2}(\widetilde{\phi}^U(\gamma_1 \cdot \widetilde{x}_1), \widetilde{\phi}^U(\gamma_2 \cdot \widetilde{x}_2)) = d_{\widetilde{\sigma}_2}(\gamma_1 \cdot \hat{\phi}(\widetilde{x}_1), \gamma_2 \cdot \hat{\phi}(\widetilde{x}_2)) = d_{\widetilde{\sigma}_2}(\hat{\phi}(\widetilde{x}_1), (\gamma_1^{-1} * \gamma_2) \cdot \hat{\phi}(\widetilde{x}_2)) \leq$$

$$\leq d_{\widetilde{q}_1}(\widetilde{x}_1, (\gamma_1^{-1} * \gamma_2) \cdot \widetilde{x}_2)) = d_{\widetilde{q}_1}(\gamma_1 \cdot \widetilde{x}_1, \gamma_2 \cdot \widetilde{x}_2)$$

and consequently $\widetilde{\phi}^U$ can be extended to a function $\widetilde{\phi} : (\widetilde{X}_1, |\widetilde{q}_1|) \to (\widetilde{X}_2, |\widetilde{\sigma}_2|)$ which has Lipschitz constant 1.
In particular, for every point $\widetilde{x} \in \widetilde{X} \setminus \widetilde{U}$ we define $\widetilde{\phi}(\widetilde{x})$ as

$$\widetilde{\phi}(\widetilde{x}) := \lim_{n\to\infty} \widetilde{\phi}^U(\widetilde{x}_n),$$

where $\{\widetilde{x}_n\}_{n\in\mathbb{N}} \subset \widetilde{U}$ is a sequence such that $\lim_{n\to\infty} \widetilde{x}_n = x$: since $\widetilde{\phi}^U$ is 1-Lipschitz on $\widetilde{U}$, the limit in the definition of $\widetilde{\phi}(\widetilde{x})$ exists and does not depend from the chosen sequence $\{\widetilde{x}_n\}_{n\in\mathbb{N}}$.
Notice furthermore that $\widetilde{\phi}$ is equivariant for the action of $\pi_1(S_g, x_0)$: for every $\widetilde{x} \in \widetilde{X} \setminus \widetilde{U}$ and $\gamma \in \pi_1(S_g, x_0)$ consider a sequence $\{\widetilde{x}_n\}_{n\in\mathbb{N}} \subset \widetilde{U}$ such that $\lim_{n\to\infty} \widetilde{x}_n = \widetilde{x}$, then it results $\lim_{n\to\infty} \gamma \cdot \widetilde{x}_n = \gamma \cdot \widetilde{x}$ and consequently

$$\widetilde{\phi}(\gamma \cdot \widetilde{x}) = \lim_{n\to\infty} \widetilde{\phi}^U(\gamma \cdot \widetilde{x}_n) = \lim_{n\to\infty} \gamma \cdot \widetilde{\phi}^U(\widetilde{x}_n) = \gamma \cdot \lim_{n\to\infty} \widetilde{\phi}^U(\widetilde{x}_n) = \gamma \cdot \widetilde{\phi}(\widetilde{x}).$$

We have proved that $\widetilde{\phi}$ descends to a function $\phi : (X_1, q_1) \to (X_2, \sigma_2)$ which is 1−Lipschitz and such that

$$(\phi)_* = Id : \pi_1(S_g, x_0) \to \pi_1(S_g, x_0)$$

which implies that ϕ is homotopic to the identity.
In the rest of the chapter we will explain how to obtain a function $\hat{\phi}$ which satisfies previous inequality (5.2).

We have imposed $\hat{\phi}(\widetilde{z}) = \widetilde{z}'$ for every zero $\widetilde{z}$ of $\widetilde{q}_1$ which is contained in P, so we have to verify

$$d_{\widetilde{\sigma}_2}(\widetilde{z}_1', \gamma \cdot \widetilde{z}_2') \leq d_{\widetilde{q}_1}(\widetilde{z}_1, \gamma \cdot \widetilde{z}_2)$$

for every pair of zeroes $\widetilde{z}_1, \widetilde{z}_2$ of $\widetilde{q}_1$ contained in P and every $\gamma \in \pi_1(S_g, x_0)$.
Notice that it results $d_{\widetilde{\sigma}_2}(\widetilde{z}_1', \gamma \cdot \widetilde{z}_2') = \hat{l}_{\sigma_2}(\tau)$, where $\hat{l}_{\sigma_2}(\tau)$ is the length of the geodesic representative for $|\sigma_2|$ of the homotopy class (with fixed endpoints) of $\pi(\widetilde{\tau})$ and $\widetilde{\tau}$ is any arc in $\widetilde{X}_2$ from $\widetilde{z}_1'$ to $\gamma \cdot \widetilde{z}_2'$. In the same way it results $d_{q_1}(\widetilde{z}_1, \gamma \cdot \widetilde{z}_2) = \hat{l}_{q_1}(\tau)$.
Let τ^{q_1} be the geodesic representative for $|q_1|$ of the homotopy class (with fixed endpoints) of τ and suppose τ^{q_1} is a concatenation of $k \geq 1$ saddle connections $\tau_1^{q_1}, \ldots, \tau_k^{q_1}$.
From the definition of σ_2 it follows

$$l_{q_1}(\tau_i^{q_1}) \geq \hat{l}_{\sigma_2}(\tau_i^{q_1})$$

for every $i = 1, \ldots, k$. We thus obtain the following inequalities:

$$d_{q_1}(\widetilde{z}_1, \gamma \cdot \widetilde{z}_2) = l_{q_1}(\tau^{q_1}) = \sum_{i=1,\ldots,k} l_{q_1}(\tau_i^{q_1}) \geq \sum_{i=1,\ldots,k} \hat{l}_{\sigma_2}(\tau_i^{q_1}) \geq \hat{l}_{\sigma_2}(\tau) = d_{\sigma_2}(\widetilde{z}_1', \gamma \cdot \widetilde{z}_2').$$

Now we are going to define the function $\hat{\phi}$ on $\hat{U}$ one point at a time.
Let $\widetilde{p}_1 \in P \setminus \pi^{-1}(\Sigma)$ be the first point (besides the zeroes of $\widetilde{q}_1$) on which we want to define $\hat{\phi}$: we have to find $\hat{\phi}(\widetilde{p}_1) \in \widetilde{X}_2$ such that

$$d_{\widetilde{\sigma}_2}(\hat{\phi}(\widetilde{p}_1), \gamma \cdot \widetilde{x}') \leq d_{\widetilde{q}_1}(\widetilde{p}_1, \gamma \cdot \widetilde{x}) \tag{5.3}$$

for every zero $\widetilde{x}$ of $\widetilde{q}_1$ contained in P and for every $\gamma \in \pi_1(S_g, x_0)$. The point $\hat{\phi}(\widetilde{p}_1)$ should also satisfy the condition

$$d_{\widetilde{\sigma}_2}(\hat{\phi}(\widetilde{p}_1), \theta \cdot \hat{\phi}(\widetilde{p}_1)) \leq d_{\widetilde{q}_1}(\widetilde{p}_1, \theta \cdot \widetilde{p}_1) \tag{5.4}$$

for every $\theta \in \pi_1(S_g, x_0)$.
Notice that, in order for equation (5.3) to be always satisfied, it is sufficient to check only the distances of $\widetilde{p}_1$ from the zeroes $\gamma \cdot \widetilde{x}$ such that $\overline{\widetilde{p}_1(\gamma \cdot \widetilde{x})}$ is smooth and does not contain other zeroes. Indeed, suppose $\overline{\widetilde{p}_1(\gamma \cdot \widetilde{x})}$ is the concatenation of the following segments

$$\overline{\widetilde{p}_1(\gamma \cdot \widetilde{x})} = \overline{\widetilde{p}_1(\gamma_1 \cdot \widetilde{x}_1)} * \widetilde{\tau}_1^{q_1} * \cdots * \widetilde{\tau}_l^{q_1},$$

where:

- $\gamma_1 \in \pi_1(S_g, x_0)$,
- $\widetilde{x}_1$ is a zero of $\widetilde{q}_1$ contained in P,
- $\widetilde{\tau}_i^{q_1}$ are saddle connections for $\widetilde{q}_1$,

then from the inequality

$$d_{\widetilde{\sigma}_2}(\hat{\phi}(\widetilde{p}_1), \gamma_1 \cdot \widetilde{x}_1') \leq d_{\widetilde{q}_1}(\widetilde{p}_1, \gamma_1 \cdot \widetilde{x}_1)$$

and the definition of σ_2 it will follow

$$d_{\widetilde{q}_1}(\widetilde{p}_1, \gamma \cdot \widetilde{x}) = d_{\widetilde{q}_1}(\widetilde{p}_1, \gamma_1 \cdot \widetilde{x}_1) + \sum_{i=1,\dots,l} l_{\widetilde{q}_1}(\widetilde{\tau}_i^{q_1}) \geq$$

$$\geq d_{\widetilde{\sigma}_2}(\hat{\phi}(\widetilde{p}_1), \gamma_1 \cdot \widetilde{x}_1') + \sum_{i=1,\dots,l} \hat{l}_{\widetilde{\sigma}_2}(\widetilde{\tau}_i^{q_1}) \geq d_{\widetilde{\sigma}_2}(\hat{\phi}(\widetilde{p}_1), \gamma \cdot \widetilde{x}').$$

For the same reason it suffices to verify equation (5.4) only for $\theta \in \pi_1(S_g, x_0)$ such that $\overline{\widetilde{p}_1(\theta \cdot \widetilde{p}_1)}$ is smooth and does not contain zeroes of $\widetilde{q}_1$.

We define the following two sets:

$$\mathcal{X}(\widetilde{p}_1) := \{\widetilde{z} \in \pi^{-1}(\Sigma) \text{ such that } \overline{\widetilde{z}\widetilde{p}_1} \text{ is smooth and does not contain other zeroes of } \widetilde{q}_1\},$$

$$\Theta(\widetilde{p}_1) := \{\gamma \in \pi_1(S_g, x_0) \text{ such that } \overline{\widetilde{p}_1(\theta \cdot \widetilde{p}_1)} \text{ is smooth and does not contain zeroes of } \widetilde{q}_1\}.$$

For every $\theta \in \Theta(\widetilde{p}_1)$ we define the set

$$V_\theta := \{\widetilde{p}' \in \widetilde{X}_2 \mid d_{\widetilde{\sigma}_2}(\widetilde{p}', \theta \cdot \widetilde{p}') \leq d_{\widetilde{q}_1}(\widetilde{p}_1, \theta \cdot \widetilde{p}_1)\}.$$

Lemma 5.2.1. *For every $\theta \in \Theta(\widetilde{p}_1)$, the set V_θ is convex in $(\widetilde{X}_2, d_{\widetilde{\sigma}_2})$.*

Proof. Consider any two points $\widetilde{x}', \widetilde{y}' \in V_\theta$. Since $\widetilde{\sigma}_2$ is invariant by covering transformations, it is possible to obtain two parametrizations $\widetilde{\tau}, \widetilde{\tau}_\theta : [0,1] \to \widetilde{X}_2$ respectively of $\overline{\widetilde{x}'\widetilde{y}'}$ and of $\overline{(\theta \cdot \widetilde{x}')(\theta \cdot \widetilde{y}')}$ such that $\widetilde{\tau}_\theta(s) = \theta \cdot \widetilde{\tau}(s)$.
The space $(\widetilde{X}_2, d_{\widetilde{\sigma}_2})$ is $Cat(0)$ and consequently Busemann-convex: this means that the function

$$s \mapsto d_{\widetilde{\sigma}_2}(\tau(s), \tau_\theta(s)) = d_{\widetilde{\sigma}_2}(\tau(s), \theta \cdot \tau(s))$$

is convex. From this fact we get $\tau(s) \in V_\theta$ for every $s \in [0,1]$. □

For every $\widetilde{x} \in \mathcal{X}(\widetilde{p}_1)$ we define the following closed ball

$$B^2_{d_{\widetilde{q}_1}(\widetilde{x},\widetilde{p}_1)}(\widetilde{x}') := \{\widetilde{p}' \in \widetilde{X}_2 | d_{\widetilde{\sigma}_2}(\widetilde{p}', \widetilde{x}') \leq d_{\widetilde{q}_1}(\widetilde{x}, \widetilde{p}_1)\}.$$

Clearly, our goal is to prove that the set $\Pi(\widetilde{p}_1)$,

$$\Pi(\widetilde{p}_1) := \left(\bigcap_{\widetilde{x} \in \mathcal{X}(\widetilde{p}_1)} B^2_{d_{\widetilde{q}_1}(\widetilde{x},\widetilde{p}_1)}(\widetilde{x}') \right) \bigcap \left(\bigcap_{\theta \in \Theta(\widetilde{p}_1)} V_\theta \right)$$

is not empty, in order to being able to choose $\hat{\phi}(\widetilde{p}_1) \in \Pi(\widetilde{p}_1)$.
Since the sets $B^2_{d_{\widetilde{q}_1}(\widetilde{x},\widetilde{p}_1)}(\widetilde{x}')$ and V_θ are convex, we can use Helly's lemma 5.1.7 for uniquely geodesic spaces to prove $\Pi(\widetilde{p}_1) \neq \emptyset$. There are four cases:

1. $B^2_{d_{\widetilde{q}_1}(\widetilde{x}_1,\widetilde{p}_1)}(\widetilde{x}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x}_2,\widetilde{p}_1)}(\widetilde{x}'_2) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x}_3,\widetilde{p}_1)}(\widetilde{x}'_3) \neq \emptyset$,
2. $V_{\theta_1} \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x}_1,\widetilde{p}_1)}(\widetilde{x}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x}_2,\widetilde{p}_1)}(\widetilde{x}'_2) \neq \emptyset$,
3. $V_{\theta_1} \cap V_{\theta_2} \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x},\widetilde{p}_1)}(\widetilde{x}') \neq \emptyset$,
4. $V_{\theta_1} \cap V_{\theta_2} \cap V_{\theta_3} \neq \emptyset$.

The proofs of the four cases will be presented later, since we feel it is now best to conclude the procedure of the definition of $\hat{\phi}$.
So suppose we have proved each of the four preceding cases and we have chosen $\hat{\phi}(\widetilde{p}_1) \in \Pi(\widetilde{p}_1)$, we now have to find the image of a second point $\widetilde{p}_2 \in P \setminus \pi^{-1}(\Sigma)$ in such a way that it results:

(i)

$$d_{\widetilde{\sigma}_2}(\hat{\phi}(\widetilde{p}_2), \gamma \cdot \widetilde{x}') \leq d_{\widetilde{q}_1}(\widetilde{p}_2, \gamma \cdot \widetilde{x})$$

for every zero $\widetilde{x}$ of $\widetilde{q}_1$ contained in P and $\gamma \in \pi_1(S_g, x_0)$ such that $\overline{\widetilde{p}_2(\gamma \cdot \widetilde{x})}$ is smooth and does not contain other zeroes of $\widetilde{q}_1$,

(ii)

$$d_{\widetilde{\sigma}_2}(\hat{\phi}(\widetilde{p}_2), \gamma \cdot \hat{\phi}(\widetilde{p}_1)) \leq d_{\widetilde{q}_1}(\widetilde{p}_2, \gamma \cdot \widetilde{p}_1)$$

for every $\gamma \in \pi_1(S_g, x_0)$ such that $\overline{\widetilde{p}_2(\gamma \cdot \widetilde{p}_1)}$ is smooth and does not contain zeroes of $\widetilde{q}_1$,

(iii)

$$d_{\widetilde{\sigma}_2}(\hat{\phi}(\widetilde{p}_2), \theta \cdot \hat{\phi}(\widetilde{p}_2)) \leq d_{\widetilde{q}_1}(\widetilde{p}_2, \theta \cdot \widetilde{p}_2)$$

for every $\theta \in \pi_1(S_g, x_0)$ such that $\overline{\widetilde{p}_2(\theta \cdot \widetilde{p}_2)}$ is smooth and does not contain zeroes of $\widetilde{q}_1$.

As we did for $\widetilde{p}_1$, we now define the sets $\mathcal{X}(\widetilde{p}_2)_\Sigma, \mathcal{X}(\widetilde{p}_2)_{\widetilde{p}_1}$ and $\Theta(\widetilde{p}_2)$:

$$\mathcal{X}(\widetilde{p}_2)_\Sigma := \{\widetilde{x} \in \pi^{-1}(\Sigma) \mid \overline{\widetilde{p}_2\widetilde{x}} \text{ is smooth and does not contain other zeroes of } \widetilde{q}_1\},$$

$$\mathcal{X}(\widetilde{p}_2)_{\widetilde{p}_1} := \{\gamma{\cdot}\widetilde{p}_1 \mid \gamma \in \pi_1(S_g, x_0) \text{ and } \overline{\widetilde{p}_2(\gamma \cdot \widetilde{p}_1)} \text{ is smooth and does not contain zeroes of } \widetilde{q}_1\},$$

$$\Theta(\widetilde{p}_2) := \{\theta \in \pi_1(S_g, x_0) \mid \overline{\widetilde{p}_2(\theta \cdot \widetilde{p}_2)} \text{ is smooth and does not contain zeroes of } \widetilde{q}_1\}.$$

We define the following intersections:

$$B_\Sigma := \bigcap_{\widetilde{x} \in \mathcal{X}(\widetilde{p}_2)_\Sigma} B^2_{d_{\widetilde{q}_1}(\widetilde{x}, \widetilde{p}_2)}(\widetilde{x}'),$$

$$B_{\widetilde{p}_1} := \bigcap_{\gamma \cdot \widetilde{p}_1 \in \mathcal{X}(\widetilde{p}_2)_{\widetilde{p}_1}} B^2_{d_{\widetilde{q}_1}(\gamma \cdot \widetilde{p}_1, \widetilde{p}_2)}(\gamma \cdot \hat{\phi}(\widetilde{p}_1)),$$

$$V_{\widetilde{p}_2} := \bigcap_{\theta \in \Theta(\widetilde{p}_2)} V_\theta.$$

Again, we want to prove

$$\Pi(\widetilde{p}_2) := B_\Sigma \cap B_{\widetilde{p}_1} \cap V_{\widetilde{p}_2} \neq \emptyset$$

in order to pick $\hat{\phi}(\widetilde{p}_2) \in \Pi(\widetilde{p}_2)$. One can consider the four cases we previously deduced for $\Pi(\widetilde{p}_1)$, noting that this time the closed balls can also be centered in points $\gamma \cdot \hat{\phi}(\widetilde{p}_1)$.

We now proceed in the same way, defining $\hat{\phi}$ on P one point at a time. Suppose $\hat{\phi}$ is already defined on the points $\widetilde{p}_1, \ldots, \widetilde{p}_n \in P \setminus \pi^{-1}(\Sigma)$ and that we wish to determine its value at $\widetilde{p}_{n+1}$. In order to do so we define the following sets:

$$\mathcal{X}(\widetilde{p}_{n+1})_\Sigma := \{\widetilde{x} \in \pi^{-1}(\Sigma) \mid \overline{\widetilde{p}_{n+1}\widetilde{x}} \text{ is smooth and does not contain any other zero of } \widetilde{q}_1\},$$

$$\Theta(\widetilde{p}_{n+1}) := \{\theta \in \pi_1(S_g, x_0) \mid \overline{\widetilde{p}_{n+1}(\theta \cdot \widetilde{p}_{n+1})} \text{ is smooth and does not contain zeroes of } \widetilde{q}_1\},$$

$$\mathcal{X}(\widetilde{p}_{n+1})_{\widetilde{p}_i} := \{\gamma{\cdot}\widetilde{p}_i \mid \gamma \in \pi_1(S_g, x_0) \text{ and } \overline{\widetilde{p}_{n+1}(\gamma \cdot \widetilde{p}_i)} \text{ is smooth and does not contain zeroes of } \widetilde{q}_1\},$$

for every $i = 1, \ldots, n$.
Again, we want to prove

$$\Pi(\widetilde{p}_{n+1}) := B_\Sigma \cap \left(\bigcap_{i=1,\ldots,n} B_{\widetilde{p}_i} \right) \cap V_{\widetilde{p}_{n+1}} \neq \emptyset,$$

where the sets $B_\Sigma, B_{\widetilde{p}_i}, V_{\widetilde{p}_{n+1}}$ are defined as follows:

$$B_\Sigma := \bigcap_{\widetilde{x} \in \mathcal{X}(\widetilde{p}_{n+1})_\Sigma} B^2_{d_{\widetilde{q}_1}(\widetilde{x}, \widetilde{p}_{n+1})}(\widetilde{x}'),$$

$$B_{\widetilde{p}_i} := \bigcap_{\gamma \cdot \widetilde{p}_i \in \mathcal{X}(\widetilde{p}_{n+1})_{\widetilde{p}_i}} B^2_{d_{\widetilde{q}_1}(\gamma \cdot \widetilde{p}_i, \widetilde{p}_{n+1})}(\gamma \cdot \hat{\phi}(\widetilde{p}_i)),$$

$$V_{\widetilde{p}_{n+1}} := \bigcap_{\theta \in \Theta(\widetilde{p}_{n+1})} V_\theta.$$

Then we will pick $\hat{\phi}(\widetilde{p}_{n+1}) \in \Pi(\widetilde{p}_{n+1})$: notice that even in this case there are only the same four types of intersections we pointed out for $\widetilde{p}_1$.

Since we have now fully explained our method to define $\hat{\phi}$ on a dense countable subset of P, we can now concentrate on the four types of intersections which appear in the sets $\Pi(\widetilde{p}_i)$ (we will prove it for $\Pi(\widetilde{p}_{n+1})$, the reasoning will be the same for the other sets $\Pi(\widetilde{p}_i)$).
The following procedure will not vary in case closed balls are centered in zeroes of $\widetilde{q}_1$ or in points outside $\pi^{-1}(\Sigma)$: in order to lighten the notation, given any point $\widetilde{x} = \gamma \cdot \widetilde{p}_i \in \mathcal{X}(\widetilde{p}_{n+1})_{\widetilde{p}_i}$, we will denote the corresponding point $\gamma \cdot \hat{\phi}(\widetilde{p}_i)$ simply as $\widetilde{x}'$.
From now on we will also denote the set $\mathcal{X}(\widetilde{p}_{n+1})_\Sigma \cup (\cup_{i=1}^n \mathcal{X}(\widetilde{p}_{n+1})_{\widetilde{p}_i})$ simply as $\mathcal{X}(\widetilde{p}_{n+1})$.

The first case concerns the intersection of three closed balls and is the most important, since it will imply all other three cases. Its proof is quite long and involves the two statements about 1-Lipschitz maps between polygons we introduced at the beginning of this chapter: for these reasons we feel it is best to postpone it and dedicate to it the whole next section.
We will thus state the following theorem and take it for granted.

Theorem 5.2.2. *If following conjecture 5.1 is true, for every $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3 \in \mathcal{X}(\widetilde{p}_{n+1})$ it results*

$$B^2_{d_{\widetilde{q}_1}(\widetilde{x}_1, \widetilde{p}_{n+1})}(\widetilde{x}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x}_2, \widetilde{p}_{n+1})}(\widetilde{x}'_2) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x}_3, \widetilde{p}_{n+1})}(\widetilde{x}'_3) \neq \emptyset.$$

It is important to notice that all next results will be implied by theorem 5.2.2: the reader is advised to keep in mind that they consequently depend on conjecture 5.1.

We state the following corollary, which is a consequence of theorem 5.2.2, Helly's lemma and some observations we already made.

Corollary 5.2.3. *Consider any finite number of points* $\widetilde{y}_1, \dots, \widetilde{y}_n \in \widetilde{X}_1 \setminus \pi^{-1}(\Sigma)$ *and* $\widetilde{y}'_1, \dots, \widetilde{y}'_n \in \widetilde{X}_2 \setminus \pi^{-1}(\Sigma)$ *such that*

$$d_{\widetilde{\sigma}_2}(\widetilde{y}'_i, \widetilde{y}'_j) \le d_{\widetilde{q}_1}(\widetilde{y}_i, \widetilde{y}_j) \quad \forall i, j = 1, \dots, n$$

and

$$d_{\widetilde{\sigma}_2}(\widetilde{y}'_i, \widetilde{z}') \le d_{\widetilde{q}_1}(\widetilde{y}_i, \widetilde{z})$$

for every $\widetilde{z} \in \pi^{-1}(\Sigma)$ *and* $i = 1, \dots, n$.
Then for every finite set of zeroes $\widetilde{x}_1, \dots, \widetilde{x}_m \in \pi^{-1}(\Sigma)$ *and for every* $\widetilde{p} \in \widetilde{X}_1$ *it results*

$$\left(\bigcap_{i=1,\dots,n} B^2_{d_{\widetilde{q}_1}(\widetilde{y}_i, \widetilde{p})}(\widetilde{y}'_i) \right) \bigcap \left(\bigcap_{i=1,\dots,m} B^2_{d_{\widetilde{q}_1}(\widetilde{x}_i, \widetilde{p})}(\widetilde{x}'_i) \right) \ne \emptyset.$$

Proof. Closed balls of $d_{\widetilde{\sigma}_2}$ are convex, so one can use Helly's lemma 5.1.7 and prove that the intersection of every triple of closed balls is not empty.
As we have already seen, given a point $\widetilde{y}_i$, if it results

$$\overline{\widetilde{y}_i \widetilde{p}} = \overline{\widetilde{p}\widetilde{z}} * \widetilde{\tau}_1^{q_1} * \cdots * \widetilde{\tau}_r^{q_1} * \overline{\widetilde{w}\widetilde{y}_i}$$

with $\widetilde{w}, \widetilde{z} \in \pi^{-1}(\Sigma)$, $\widetilde{\tau}_i^{q_1}$ saddle connections and $\overline{\widetilde{p}\widetilde{z}}, \overline{\widetilde{w}\widetilde{y}_i}$ smooth, one can replace the ball $B^2_{d_{\widetilde{q}_1}(\widetilde{y}_i, \widetilde{p})}(\widetilde{y}'_i)$ in the intersection with the ball $B^2_{d_{\widetilde{q}_1}(\widetilde{z}, \widetilde{p})}(\widetilde{z}')$. The same is true for all points $\widetilde{x}_i$.
The result then follows directly from theorem 5.2.2. □

We now want to focus ourselves on the remaining three cases. In order to do so we first need to characterize closed geodesics and flat cylinders of a semi-translation surface (X, q). A proof of the following lemma can be found in [*St*].

Lemma 5.2.4. *Let* θ *be a simple closed geodesic for* $|q|$ *on* X. *Then* θ *is a cylinder curve of a flat cylinder* C *of* (X, q). *This means that* C *is foliated by simple closed geodesics all parallel to* θ *and of the same length. The border of* C *is composed by two components, both consisting of saddle connections of* q *parallel to* θ. *The length of both components equals the length of* θ.

Lemma 5.2.5. *Consider any* $\theta \in \Theta(\widetilde{p}_{n+1})$ *and let* $\widetilde{C}$ *be the lifting to* $\widetilde{X}_1$ *of the flat cylinder of* (X_1, q_1) *corresponding to* θ.
Let $\widetilde{y}$ *be any point of* $\partial\widetilde{C}$ *and* $\widetilde{z}_1, \widetilde{z}_2$ *the two zeroes on* $\partial\widetilde{C}$ *such that* $\overline{\widetilde{z}_1\widetilde{z}_2}$ *is a saddle connection containing* $\widetilde{y}$. *Then it results*

$$B^2_{d_{\widetilde{q}_1}(\widetilde{y}, \widetilde{z}_1)}(\widetilde{z}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{y}, \widetilde{z}_2)}(\widetilde{z}'_2) \subset V_\theta.$$

Proof. Let $\widetilde{\tau}_1^{q_1}, \dots, \widetilde{\tau}_k^{q_1}$ be the saddle connections such that

$$\overline{\widetilde{y}(\theta \cdot \widetilde{y})} = \overline{\widetilde{y}\widetilde{z}_1} * \widetilde{\tau}_1^{q_1} * \cdots * \widetilde{\tau}_k^{q_1} * \overline{(\theta \cdot \widetilde{z}_2)(\theta \cdot \widetilde{y})}.$$

Then for every point $\widetilde{y}' \in B^2_{d_{\widetilde{q}_1}(\widetilde{y},\widetilde{z}_1)}(\widetilde{z}_1') \cap B^2_{d_{\widetilde{q}_1}(\widetilde{y},\widetilde{z}_2)}(\widetilde{z}_2')$ it results

$$d_{\widetilde{\sigma}_2}(\widetilde{y}', \theta \cdot \widetilde{y}') \le d_{\widetilde{\sigma}_2}(\widetilde{y}', \widetilde{z}_1') + \sum_{i=1,\dots,k} \hat{l}_{\widetilde{\sigma}_2}(\tau_i^q) + d_{\widetilde{\sigma}_2}(\theta \cdot \widetilde{y}', \theta \cdot \widetilde{z}_2') =$$

$$= d_{\widetilde{\sigma}_2}(\widetilde{y}', \widetilde{z}_1') + \sum_{i=1,\dots,k} \hat{l}_{\widetilde{\sigma}_2}(\tau_i^{q_1}) + d_{\widetilde{\sigma}_2}(\widetilde{y}', \widetilde{z}_2') \le d_{\widetilde{q}_1}(\widetilde{y}, \widetilde{z}_1) + \sum_{i=1,\dots,k} l_{\widetilde{q}_1}(\tau_i^{q_1}) + d_{\widetilde{q}_1}(\widetilde{y}, \widetilde{z}_2) =$$

$$= d_{\widetilde{q}_1}(\widetilde{y}, \widetilde{z}_1) + \sum_{i=1,\dots,k} l_{\widetilde{q}_1}(\tau_i^{q_1}) + d_{\widetilde{q}_1}(\theta \cdot \widetilde{y}, \theta \cdot \widetilde{z}_2) = d_{\widetilde{q}_1}(\widetilde{y}, \theta \cdot \widetilde{y})$$

and consequently $B^2_{d_{\widetilde{q}_1}(\widetilde{y},\widetilde{z}_1)}(\widetilde{z}_1') \cap B^2_{d_{\widetilde{q}_1}(\widetilde{y},\widetilde{z}_2)}(\widetilde{z}_2') \subset V_\theta$. □

We are now ready to prove the case of the second type of intersections.

Proposition 5.2.6. *For every* $\theta \in \Theta(\widetilde{p}_{n+1})$ *and* $\widetilde{x}_1, \widetilde{x}_2 \in \mathcal{X}(\widetilde{p}_{n+1})$ *it results*

$$V_\theta \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x}_1,\widetilde{p}_{n+1})}(\widetilde{x}_1') \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x}_2,\widetilde{p}_{n+1})}(\widetilde{x}_2') \neq \emptyset.$$

Proof. Let $\widetilde{C}$ be the lifting to $\widetilde{X}_1$ of the flat cylinder corresponding to θ.
We will first consider the case $\widetilde{x}_1 \notin \widetilde{C}$ and $\widetilde{x}_2 \notin \widetilde{C}$, since it is the more complicated one.
We define the following points $\widetilde{z}_1, \widetilde{z}_2 \in \widetilde{X}_1$:

$$\widetilde{z}_1 := \overline{\widetilde{p}_{n+1}\widetilde{x}_1} \cap \partial\widetilde{C}, \qquad \widetilde{z}_2 := \overline{\widetilde{p}_{n+1}\widetilde{x}_2} \cap \partial\widetilde{C}.$$

Consider the following two cases:

- $\overline{\widetilde{x}_1\widetilde{x}_2}$ does not traverse $\widetilde{C}$.

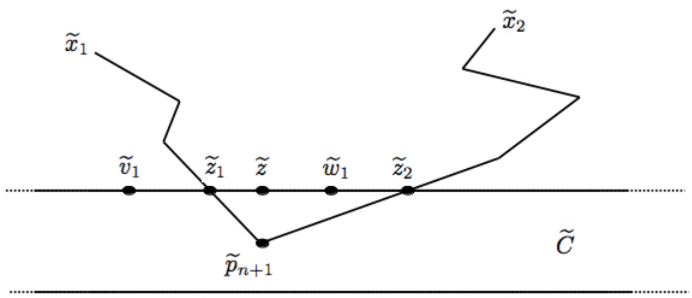

Figure 5.1: The case $\overline{\widetilde{x}_1\widetilde{x}_2}$ does not traverse $\widetilde{C}$

There is a point $\widetilde{z} \in \overline{\widetilde{z}_1\widetilde{z}_2}$ (eventually equal to $\widetilde{z}_1$ or $\widetilde{z}_2$) such that $\widetilde{z} \in \partial\widetilde{C}$ and

$$d_{\widetilde{q}_1}(\widetilde{z}, \widetilde{z}_i) \le d_{\widetilde{q}_1}(\widetilde{z}_i, \widetilde{p}_{n+1}), \quad i = 1, 2$$

and consequently

$$d_{\widetilde{q}_1}(\widetilde{z}, \widetilde{x}_i) \leq d_{\widetilde{q}_1}(\widetilde{x}_i, \widetilde{p}_{n+1}).$$

Let $\widetilde{v}_1$ and $\widetilde{w}_1$ be the two zeroes on $\partial\widetilde{C}$ such that $\overline{\widetilde{v}_1\widetilde{w}_1}$ is a saddle connection containing $\widetilde{z}$.
From corollary 5.2.3 it follows

$$\Lambda := B^2_{d_{\widetilde{q}_1}(\widetilde{v}_1,\widetilde{z})}(\widetilde{v}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{w}_1,\widetilde{z})}(\widetilde{w}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x}_1,\widetilde{z})}(\widetilde{x}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x}_2,\widetilde{z})}(\widetilde{x}'_2) \neq \emptyset.$$

The inequality $d_{\widetilde{q}_1}(\widetilde{z}, \widetilde{x}_i) \leq d_{\widetilde{q}_1}(\widetilde{x}_i, \widetilde{p}_{n+1})$ grants

$$\Lambda \subset B^2_{d_{\widetilde{q}_1}(\widetilde{v}_1,\widetilde{z})}(\widetilde{v}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{w}_1,\widetilde{z})}(\widetilde{w}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x}_1,\widetilde{p}_{n+1})}(\widetilde{x}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x}_2,\widetilde{p}_{n+1})}(\widetilde{x}'_2)$$

and applying the preceding lemma we can finally get

$$\Lambda \subset V_\theta \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x}_1,\widetilde{p}_{n+1})}(\widetilde{x}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x}_2,\widetilde{p}_{n+1})}(\widetilde{x}'_2).$$

- $\overline{\widetilde{x}_1\widetilde{x}_2}$ traverses $\widetilde{C}$.

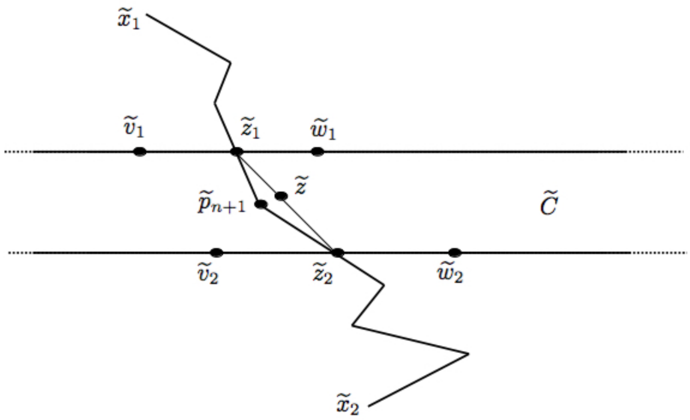

Figure 5.2: The case $\overline{\widetilde{x}_1\widetilde{x}_2}$ traverses $\widetilde{C}$.

There is a point $\widetilde{z} \in \overline{\widetilde{z}_1\widetilde{z}_2}$ such that $d_{\widetilde{q}_1}(\widetilde{z}, \widetilde{z}_i) \leq d_{\widetilde{q}_1}(\widetilde{p}_{n+1}, \widetilde{z}_i)$, $i = 1, 2$.
For $i = 1, 2$, let $\widetilde{v}_i$ and $\widetilde{w}_i$ the two zeroes on $\partial\widetilde{C}$ such that $\overline{\widetilde{v}_i\widetilde{w}_i}$ is a saddle connection and $\widetilde{z}_i \in \overline{\widetilde{v}_i\widetilde{w}_i}$.
Denote by $\mathcal{X}(\widetilde{z}_1)$ the set of the zeroes of $\widetilde{q}_1$ joined to $\widetilde{z}_1$ by a smooth geodesic of $|\widetilde{q}_1|$: clearly $\widetilde{v}_1, \widetilde{w}_1 \in \mathcal{X}(\widetilde{z}_1)$.
Corollary 5.2.3 and the previous lemma grant the existence of the following points $\widetilde{z}'_i \in \widetilde{X}_2$:

$$\widetilde{z}'_1 \in \left(\bigcap_{\widetilde{x} \in \mathcal{X}(\widetilde{z}_1)} B^2_{d_{\widetilde{q}_1}(\widetilde{z}_1,\widetilde{x})}(\widetilde{x}') \right) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{z}_1,\widetilde{x}_1)}(\widetilde{x}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{z}_1,\widetilde{x}_2)}(\widetilde{x}'_2) \subset$$

$$\subset B^2_{d_{\widetilde{q}_1}(\widetilde{z}_1,\widetilde{x}_1)}(\widetilde{x}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{z}_1,\widetilde{x}_2)}(\widetilde{x}'_2) \cap V_\theta,$$

$$\widetilde{z}'_2 \in B^2_{d_{\widetilde{q}_1}(\widetilde{z}_2,\widetilde{v}_2)}(\widetilde{v}'_2) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{z}_2,\widetilde{w}_2)}(\widetilde{w}'_2) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{z}_2,\widetilde{z}_1)}(z'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{z}_2,\widetilde{x}_2)}(\widetilde{x}'_2) \subset$$

$$\subset B^2_{d_{\widetilde{q}_1}(\widetilde{z}_2,\widetilde{z}_1)}(\widetilde{z}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{z}_2,\widetilde{x}_2)}(\widetilde{x}'_2) \cap V_\theta.$$

The set V_θ is convex, so it follows $\overline{\widetilde{z}'_1\widetilde{z}'_2} \subset V_\theta$ and since $d_{\widetilde{\sigma}_2}(\widetilde{z}'_1,\widetilde{z}'_2) \leq d_{\widetilde{q}_1}(\widetilde{z}_1,\widetilde{z}_2)$, we can choose $\widetilde{z}' \in \overline{\widetilde{z}'_1\widetilde{z}'_2}$ such that $d_{\widetilde{\sigma}_2}(\widetilde{z}',\widetilde{z}'_i) \leq d_{\widetilde{q}_1}(\widetilde{z},\widetilde{z}_i)$, $i = 1, 2$.

In this way one finally gets the following inequalities:

$$d_{\widetilde{\sigma}_2}(\widetilde{z}',\widetilde{x}'_i) \leq d_{\widetilde{\sigma}_2}(\widetilde{z}',\widetilde{z}'_i) + d_{\widetilde{\sigma}_2}(\widetilde{z}'_i,\widetilde{x}'_i) \leq$$

$$\leq d_{\widetilde{q}_1}(\widetilde{z},\widetilde{z}_i) + d_{\widetilde{q}_1}(\widetilde{z}_i,\widetilde{x}_i) \leq d_{\widetilde{q}_1}(\widetilde{p}_{n+1},\widetilde{z}_i) + d_{\widetilde{q}_1}(\widetilde{z}_i,\widetilde{x}_i) = d_{\widetilde{q}_1}(\widetilde{p}_{n+1},\widetilde{x}_i).$$

The case $\widetilde{x}_1 \in \widetilde{C}$ and $\widetilde{x}_2 \notin \widetilde{C}$ can be solved in the same way. Define as before $\widetilde{z}_2 := \overline{\widetilde{p}_{n+1}\widetilde{x}_2} \cap \partial\widetilde{C}$, then one just has to notice that there always is a point $\widetilde{z} \in \overline{\widetilde{z}_2\widetilde{x}_1}$ such that $d_{\widetilde{q}_1}(\widetilde{z},\widetilde{x}_1) \leq d_{\widetilde{q}_1}(\widetilde{p}_{n+1},\widetilde{x}_1)$ and $d_{\widetilde{q}_1}(\widetilde{z},\widetilde{z}_2) \leq d_{\widetilde{q}_1}(\widetilde{p}_{n+1},\widetilde{z}_2)$.

Finally, if $\widetilde{x}_1 \in \widetilde{C}$ and $\widetilde{x}_2 \in \widetilde{C}$ one could notice that it results $\overline{\widetilde{x}'_1\widetilde{x}'_2} \subset V_\theta$. Since $d_{\widetilde{\sigma}_2}(\widetilde{x}'_1,\widetilde{x}'_2) \leq d_{\widetilde{q}_1}(\widetilde{x}_1,\widetilde{x}_2)$, there is a point $\widetilde{p}'_{n+1} \in \overline{\widetilde{x}'_1\widetilde{x}'_2}$ such that

$$d_{\widetilde{\sigma}_2}(\widetilde{p}'_{n+1},\widetilde{x}'_i) \leq d_{\widetilde{q}_1}(\widetilde{p}_{n+1},\widetilde{x}_i), \quad i = 1, 2.$$

□

Corollary 5.2.7. *For every $\theta \in \Theta(\widetilde{p}_{n+1})$ and $\widetilde{x}_i \in \mathcal{X}(\widetilde{p}_{n+1})$, $i = 1, \ldots, n$, it results:*

$$V_\theta \cap \bigcap_{i=1,\ldots,n} B^2_{d_{\widetilde{q}_1}(\widetilde{x}_i,\widetilde{p}_{n+1})}(\widetilde{x}'_i) \neq \emptyset.$$

Proof. It is a consequence of previous results and Helly's lemma for uniquely geodesic spaces. □

Finally, we can prove that the intersection is not empty also in the last two cases.

Proposition 5.2.8. *For every $\theta_1, \theta_2 \in \Theta(\widetilde{p}_{n+1})$ and $\widetilde{x} \in \mathcal{X}(\widetilde{p}_{n+1})$, it follows*

$$V_{\theta_1} \cap V_{\theta_2} \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x},\widetilde{p}_{n+1})}(\widetilde{x}') \neq \emptyset.$$

Proof. Let $\widetilde{C}_i$ be the lifting to $\widetilde{X}_1$ of the flat cylinder corresponding to θ_i, $i = 1, 2$. We first consider the case $\widetilde{x} \notin \widetilde{C}_1 \cup \widetilde{C}_2$, since it is the more complicated one.
We choose the point $\widetilde{z}$:

$$\widetilde{z} := \overline{\widetilde{x}\widetilde{p}_{n+1}} \cap \partial(\widetilde{C}_1 \cap \widetilde{C}_2).$$

Notice that it results $d_{\widetilde{q}_1}(\widetilde{p}_{n+1},\widetilde{x}) \geq d_{\widetilde{q}_1}(\widetilde{z},\widetilde{x})$ and suppose $\widetilde{z} \in \partial\widetilde{C}_1$.
Let $\widetilde{v}_1$ and $\widetilde{w}_1$ be the two zeroes of $\widetilde{q}_1$ such that $\overline{\widetilde{v}_1\widetilde{w}_1}$ is the saddle connection of $\partial\widetilde{C}_1$

containing $\widetilde{z}$.
Then one gets the following inclusion of sets:

$$B^2_{d_{\widetilde{q}_1}(\widetilde{z},\widetilde{v}_1)}(\widetilde{v}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{z},\widetilde{w}_1)}(\widetilde{w}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x},\widetilde{z})}(\widetilde{x}') \cap V_{\theta_2} \subset V_{\theta_1} \cap B^2_{d_{\widetilde{q}_1}(\widetilde{x},\widetilde{p}_{n+1})}(\widetilde{x}') \cap V_{\theta_2}$$

and we can conclude applying corollary 5.2.7.
The case $\widetilde{x} \in \widetilde{C}_1$ and $\widetilde{x} \notin \widetilde{C}_2$ can be solved in the same way, choosing

$$\widetilde{z} := \overline{\widetilde{x}\widetilde{p}_{n+1}} \cap \partial\widetilde{C}_1.$$

Finally, the case $\widetilde{x} \in \widetilde{C}_1 \cap \widetilde{C}_2$ is trivial since $\widetilde{x}' \in V_{\theta_1} \cap V_{\theta_2}$. □

Proposition 5.2.9. *For every $\theta_1, \theta_2, \theta_3 \in \Theta(\widetilde{p}_{n+1})$ it follows*

$$V_{\theta_1} \cap V_{\theta_2} \cap V_{\theta_3} \neq \emptyset.$$

Proof. As before, denote by $\widetilde{C}_i$ the lifting to $\widetilde{X}_1$ of the flat cylinder corresponding to θ_i, $i = 1, 2, 3$. Up to renumbering the indexes, we can suppose there is a point $\widetilde{z} \in \partial(\widetilde{C}_1 \cap \widetilde{C}_2) \cap \widetilde{C}_3$.
Then, for $i = 1, 2$ let $\widetilde{v}_i, \widetilde{w}_i$ be the zeroes of $\widetilde{q}_1$ on the border of $\widetilde{C}_i$ such that $\overline{\widetilde{v}_i\widetilde{w}_i}$ is a saddle connection and $\widetilde{z} \in \overline{\widetilde{v}_i\widetilde{w}_i}$.
Using lemma 5.2.5 we just have to prove

$$B^2_{d_{\widetilde{q}_1}(\widetilde{z},\widetilde{v}_1)}(\widetilde{v}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{z},\widetilde{w}_1)}(\widetilde{w}'_1) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{z},\widetilde{v}_2)}(\widetilde{v}'_2) \cap B^2_{d_{\widetilde{q}_1}(\widetilde{z},\widetilde{w}_2)}(\widetilde{w}'_2) \cap \widetilde{V}_{\theta_3} \neq \emptyset$$

which is granted by corollary 5.2.7. □

This ends the proof of the existence of the desired function ϕ: if conjecture 5.1 is true, we have described how to obtain the equality $L^a_F(q_1, q_2) = K^a_F(q_1, q_2)$.

5.3 Proof of theorem 5.2.2

The first step towards the proof of theorem 5.2.2 consists in the characterization of geodesic triangles in $(\widetilde{X}, d_{\widetilde{q}})$ (where as before (X, q) is a semi-translation surface and $\pi : (\widetilde{X}, |\widetilde{q}|) \to (X, |q|)$ is a metric universal cover). We will use the following lemma, the proof of which can be found in [*St*], theorem 16.1.

Lemma 5.3.1. *Let $\widetilde{\gamma} : [0, 1] \to \widetilde{X}$ be a locally minimizing geodesic for $d_{\widetilde{q}}$. It follows*

$$d_{\widetilde{q}}(\widetilde{\gamma}(0), \widetilde{\gamma}(1)) = l_{\widetilde{q}}(\widetilde{\gamma})$$

that is, $\widetilde{\gamma}$ is also globally minimizing. Furthermore, $\widetilde{\gamma}$ is the unique geodesic with these properties.

Given any triple of points $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3 \in \widetilde{X}$ denote by T the corresponding geodesic triangle for $d_{\widetilde{q}}$, which is the subset of $\widetilde{X}$ composed by the three geodesics $\overline{\widetilde{x}_i\widetilde{x}_j}$.
Since $\widetilde{X} \simeq \mathbb{H}$, it makes sense to define the internal part $\overset{\circ}{\Delta}$ of T: we call *filled geodesic triangle* the set $T \cup \overset{\circ}{\Delta}$ and we denote it by Δ.

Given any planar polygon P, we denote by d_P its *intrinsic Euclidean metric*: for every $x_1, x_2 \in P$, we define $d_P(x_1, x_2)$ as the infimum of the lengths, computed with respect to the Euclidean metric, of all paths from x_1 to x_2 entirely contained in P. Every polygon used in the following proofs will be endowed with such intrinsic Euclidean metric.

Proposition 5.3.2. *Filled geodesic triangles of $d_{\widetilde{q}}$ are convex and do not contain zeroes of $\widetilde{q}$ in their internal part, which is connected.*
Given a triple of points $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3 \in \widetilde{X}$, the corresponding filled geodesic triangle Δ can have one dimensional components. For every $i = 1, 2, 3$ we define $\widetilde{v}_i$ as the point on $\overline{\widetilde{x}_i\widetilde{x}_j} \cap \overline{\widetilde{x}_i\widetilde{x}_k}$, $i \neq j \neq k$ which has maximum distance with $\widetilde{x}_i$.
If $\overset{\circ}{\Delta}$ is not empty, then its border is exactly $\overline{\widetilde{v}_1\widetilde{v}_2} \cup \overline{\widetilde{v}_2\widetilde{v}_3} \cup \overline{\widetilde{v}_1\widetilde{v}_3}$ and for every $i = 1, 2, 3$, if $\widetilde{x}_i \neq \widetilde{v}_i$, then $\overline{\widetilde{x}_i\widetilde{v}_i}$ is the only one dimensional component of Δ starting from $\widetilde{x}_i$.
The internal angles of $\overline{\overset{\circ}{\Delta}}$ in the three points $\widetilde{v}_i$ are strictly convex, while all other internal angles are concave and less than 2π.
Finally, every filled geodesic triangle for $d_{\widetilde{q}}$ is isometric to a planar polygon, which eventually could be degenerate (one dimensional) or with at most three one dimensional components.

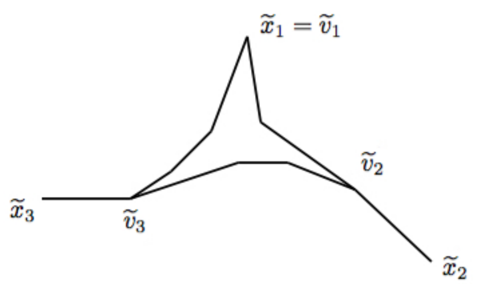

Figure 5.3: An example of a filled geodesic triangle Δ.

Proof. By lemma 5.3.1, if $\overline{\widetilde{x}_i\widetilde{x}_j}$ and $\overline{\widetilde{x}_i\widetilde{x}_k}$ intersect in a point $\widetilde{p} \neq \widetilde{x}_i$, then they must coincide over all $\overline{\widetilde{p}\widetilde{x}_i}$. It follows that $\overset{\circ}{\Delta}$ is connected and its border is $\overline{\widetilde{v}_1\widetilde{v}_2} \cup \overline{\widetilde{v}_1\widetilde{v}_3} \cup \overline{\widetilde{v}_2\widetilde{v}_3}$.

Suppose $\overset{\circ}{\Delta} \neq \emptyset$ and denote by α_1 the internal angle of $\overline{\overset{\circ}{\Delta}}$ in $\widetilde{v}_1$: we prove $\alpha_1 < \pi$. Let α_{12} be the angle in $\widetilde{v}_1$ determined by $\overline{\widetilde{x}_1\widetilde{v}_1}$ and $\overline{\widetilde{v}_1\widetilde{v}_2}$ completely outside Δ and let

α_{13} be the angle in $\widetilde{v}_1$ determined by $\overline{\widetilde{x}_1\widetilde{v}_1}$ and $\overline{\widetilde{v}_1\widetilde{v}_3}$ completely outside Δ. Clearly it results $\alpha_{12} \geq \pi$ and $\alpha_{13} \geq \pi$: if $\alpha_1 \geq \pi$ then lemma 5.3.1 would imply $\widetilde{v}_1 \in \overline{\widetilde{v}_2\widetilde{v}_3}$ and consequently $\overset{\circ}{\Delta} = \emptyset$. In the same way one proves that the internal angles of $\overline{\overset{\circ}{\Delta}}$ in $\widetilde{v}_2, \widetilde{v}_3$ must be strictly convex.

If $\widetilde{v}$ is a zero of $\widetilde{q}$ in the border of $\overset{\circ}{\Delta}$, $\widetilde{v} \neq \widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3$, the internal angle $\beta_{\widetilde{v}}$ of $\overline{\overset{\circ}{\Delta}}$ in $\widetilde{v}$ must be concave, and we now also prove $\beta_{\widetilde{v}} < 2\pi$.
Let $\widetilde{v} \in \overline{\widetilde{v}_1\widetilde{v}_2}$, and suppose by contradiction $\beta_{\widetilde{v}} \geq 2\pi$. Let τ_i, $i = 1, 2$, be the angle in $\widetilde{v}$ determined by $\overline{\widetilde{v}_3\widetilde{v}}$ and $\overline{\widetilde{v}\widetilde{v}_i}$ inside Δ. Since $\beta_{\widetilde{v}} \geq 2\pi$, it must follow $\tau_1 \geq \pi$ or $\tau_2 \geq \pi$. Suppose $\tau_1 \geq 1$, then $\overline{\widetilde{v}_3\widetilde{v}_1}$ would be a concatenation of $\overline{\widetilde{v}_1\widetilde{v}}$ and $\overline{\widetilde{v}\widetilde{v}_3}$, implying $\widetilde{v} = \widetilde{v}_1$. This last equality contradicts the previous assumption $\widetilde{v} \neq \widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3$.

Finally, suppose by contradiction that one or more zeroes $\widetilde{z}_j$ of $\widetilde{q}$ are contained in $\overset{\circ}{\Delta}$. Denote by $\alpha_i, \theta_j, \beta_k$ the internal angles of $\overline{\overset{\circ}{\Delta}}$ respectively in $\widetilde{v}_i, \widetilde{z}_j, \widetilde{w}_k$, where $\widetilde{w}_k$ is a zero of $\widetilde{q}$ on the border of $\overset{\circ}{\Delta}$.
Applying Gauss-Bonnet formula on $\overline{\overset{\circ}{\Delta}}$ one gets:

$$\sum_i (\pi - \beta_i) + (\pi - \alpha_1) + (\pi - \alpha_2) + (\pi - \alpha_3) = 2\pi + \sum_j (\theta_j - 2\pi).$$

From what we have proved it follows

$$\sum_k (\pi - \beta_k) \leq 0, \quad (\pi - \alpha_1) + (\pi - \alpha_2) + (\pi - \alpha_3) < 3\pi$$

and consequently we now get

$$2\pi + \sum_j (\theta_j - 2\pi) < 3\pi.$$

The total angle in $\widetilde{z}_j \in \overset{\circ}{\Delta}$ must be greater than or equal to 3π, but this contradicts the last inequality.

In order to prove that Δ is isometric to a planar polygon endowed with its intrinsic Euclidean metric it is clearly sufficient to prove that $\overline{\overset{\circ}{\Delta}}$ is isometric to a planar polygon.
Let $Dev : (\widetilde{X}, |\widetilde{q}|) \to \mathbb{R}^2$ be the developing map (for a precise definition see for example [$Tr2$]), notice that, if Dev is injective on a point $\widetilde{v}$ on the border of $\overline{\overset{\circ}{\Delta}}$, then the internal angle of $\overline{\overset{\circ}{\Delta}}$ in $\widetilde{v}$ coincides with the internal angle of $Dev(\overline{\overset{\circ}{\Delta}})$ in $Dev(\widetilde{v})$.
We will prove that $Dev : \overline{\overset{\circ}{\Delta}} \to \mathbb{R}^2$ is injective, or equivalently that $Dev(\overline{\overset{\circ}{\Delta}})$ is a simple polygon (not self-intersecting).
Suppose by contradiction that Dev is not injective. We divide two cases:

1. Dev is not injective on any of the points $\widetilde{v}_i$. Then denote by P_1 be the simple polygon identified by the external border of $Dev(\overline{\mathring{\Delta}})$. Notice that internal angles of P_1 can correspond to internal angles of $\overline{\mathring{\Delta}}$ or can be originated by overlays on points where Dev fails to be injective. Internal angles of the latter kind must be strictly concave and consequently convex internal angles of P_1 must correspond to convex internal angles of $\overline{\mathring{\Delta}}$. Since P_1 is simple, it must have at least three strictly convex internal angles. It would follow that $\overline{\mathring{\Delta}}$ must have at least six strictly convex internal angles: the three angles α_i plus the angles which correspond to the three strictly convex internal angles of P_1. This fact clearly contradicts the hypothesis.

2. Dev is injective on $\widetilde{v}_1$. Then there is a polygon $P_2 \subset Dev(\overline{\mathring{\Delta}})$ which is maximal with respect to inclusion on the set of polygons $\{Q\}$ such that

   - $Q \subset Dev(\overline{\mathring{\Delta}})$,
   - $Dev(\widetilde{v}_1)$ is a vertex of Q,
   - Dev is injective on $Dev^{-1}(Q)$.

   Let P_0 be the simple polygon identified by the external border of $Dev(\overline{\mathring{\Delta}})$ and define $P_1 := \overline{P_0 \setminus P_2}$ (see figure 5.4 for an example.). As before, convex internal angles of P_1 must correspond to convex internal angles of $\overline{\mathring{\Delta}}$.

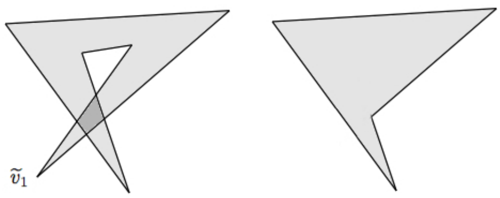

Figure 5.4: On the left there is an example of $Dev(\overline{\mathring{\Delta}})$ we want to exclude. On the right there is the corresponding polygon P_1.

   It would follow that $\overline{\mathring{\Delta}}$ must have at least four strictly convex internal angles: α_1 plus the angles which correspond to the three strictly convex internal angles of P_1. This fact clearly contradicts the hypothesis.

Finally, convexity of Δ follows from the fact that, given any pair $\widetilde{x}, \widetilde{y} \in \Delta$, the geodesic for the intrinsic Euclidean metric connecting them is also a locally minimizing geodesic for $d_{\widetilde{q}}$ and consequently also globally minimizing.

□

We now go back to consider the fundamental domain P defined in the preceding section.
Given the point $\widetilde{p}_{n+1} \in P$ and the three points $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3 \in \mathcal{X}(\widetilde{p}_n)$ corresponding to the centers of the closed balls, we consider the filled geodesic triangle Δ for $(\widetilde{X}_1, d_{\widetilde{q}_1})$ with vertices $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3$. Following the characterization of the previous proposition, we divide two cases:

1. $\widetilde{p}_{n+1} \in \Delta$, then, since the three geodesics $\overline{\widetilde{p}_{n+1}\widetilde{x}_i}$ are smooth and do not contain other zeroes of $\widetilde{q}_1$, it follows that Δ can not have one dimensional components.

2. $\widetilde{p}_{n+1} \notin \Delta$, then Δ can have one dimensional components and even be a degenerate polygon (one dimensional).

Denote by Δ' the filled geodesic triangle for $(\widetilde{X}_2, d_{\widetilde{\sigma}_2})$ with vertices $\widetilde{x}'_1, \widetilde{x}'_2, \widetilde{x}'_3$. Again, we divide three cases:

(i) Δ' is not one dimensional, but can have at most three one dimensional components,

(ii) Δ' is one dimensional and it is not possible to renumber the vertices in order to obtain $x'_3 \in \overline{x'_1 x'_2}$,

(iii) Δ' is one dimensional and it is possible to renumber the vertices in order to obtain $x'_3 \in \overline{x'_1 x'_2}$.

Combining them, we have a total of six cases we need to care care of.
In cases (1,i),(1,ii),(1,iii) our goal is to find a point $\widetilde{p}'_{n+1} \in \Delta'$ such that

$$d_{\widetilde{q}_1}(\widetilde{x}_i, \widetilde{p}_{n+1}) \geq d_{\widetilde{\sigma}_2}(\widetilde{x}'_i, \widetilde{p}'_{n+1}) \text{ for } i = 1, 2, 3.$$

In the remaining cases (2,i),(2,ii),(2,iii) we will use the orthogonal projection on convex sets in $Cat(0)$ spaces:

$$pr : (\widetilde{X}_1, d_{\widetilde{q}_1}) \to \Delta,$$

where the image $pr(\widetilde{x})$ of every point $\widetilde{x} \in \widetilde{X}_1$ is defined as the unique point such that

$$d_{\widetilde{q}_1}(\widetilde{x}, pr(\widetilde{x})) = \inf_{\widetilde{y} \in \Delta} d_{\widetilde{q}_1}(\widetilde{x}, \widetilde{y}).$$

The projection pr does not increase distances (for a proof and a list of other properties of pr one could see $[BH]$, proposition 2.4, page 176) and in particular it results

$$d_{\widetilde{q}_1}(\widetilde{x}_i, \widetilde{p}_{n+1}) \geq d_{\widetilde{q}_1}(\widetilde{x}_i, pr(\widetilde{p}_{n+1})) \text{ for } i = 1, 2, 3.$$

Then, we will look for a point $\widetilde{p}'_{n+1} \in \Delta'$ such that

$$d_{\widetilde{q}_1}(\widetilde{x}_i, pr(\widetilde{p}_{n+1})) \geq d_{\widetilde{\sigma}_2}(\widetilde{x}'_i, \widetilde{p}'_{n+1}) \text{ for } i = 1, 2, 3.$$

We chose to confront distances with $pr(\widetilde{p}_{n+1})$ instead of $\widetilde{p}_{n+1}$ because in the following procedures it will be crucial to always consider points inside Δ.

Cases (1,ii),(1,iii),(2,ii) and (2,iii) are easily solvable. We will prove only case (1,ii), since the others are almost identical.
If Δ' is one dimensional, then it always contains a vertex $\widetilde{v}'$ such that

$$\widetilde{v}' \in \overline{\widetilde{x}'_1\widetilde{x}'_2} \cap \overline{\widetilde{x}'_1\widetilde{x}'_3} \cap \overline{\widetilde{x}'_2\widetilde{x}'_3}.$$

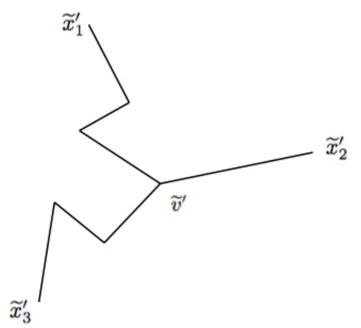

Figure 5.5: An example of vertex $\widetilde{v}' \in \Delta'$.

If, for every index $i = 1,2,3$, it results $d_{\widetilde{\sigma}_2}(\widetilde{x}'_i, \widetilde{v}') \leq d_{\widetilde{q}_1}(\widetilde{x}_i, \widetilde{p}_{n+1})$, then we can choose $\widetilde{p}'_{n+1} = \widetilde{v}'$.
If, up to renumbering the indexes, it results $d_{\widetilde{\sigma}_2}(\widetilde{x}'_1, \widetilde{v}') > d_{\widetilde{q}_1}(\widetilde{x}_1, \widetilde{p}_{n+1})$, we choose $\widetilde{p}'_{n+1}$ to be the point on $\overline{\widetilde{x}'_1\widetilde{v}'}$ such that $d_{\widetilde{q}_1}(\widetilde{x}_1, \widetilde{p}_{n+1}) = d_{\widetilde{\sigma}_2}(\widetilde{x}'_1, \widetilde{p}'_{n+1})$.
Then it will follow $d_{\widetilde{\sigma}_2}(\widetilde{x}'_i, \widetilde{p}'_{n+1}) \leq d_{\widetilde{q}_1}(\widetilde{x}_i, \widetilde{p}_{n+1})$ for $i = 2,3$, since

$$d_{\widetilde{\sigma}_2}(\widetilde{x}'_i, \widetilde{p}'_{n+1}) = d_{\widetilde{\sigma}_2}(\widetilde{x}'_1, \widetilde{x}'_i) - d_{\widetilde{q}_1}(\widetilde{x}_1, \widetilde{p}_{n+1}) \leq d_{\widetilde{q}_1}(\widetilde{x}_1, \widetilde{x}_i) - d_{\widetilde{q}_1}(\widetilde{x}_1, \widetilde{p}_{n+1}) \leq d_{\widetilde{q}_1}(\widetilde{p}_{n+1}, \widetilde{x}_i).$$

In case (2,i) it will always be possible to suppose Δ does not have one dimensional components, since

- if $pr(\widetilde{p}_{n+1})$ is on a one dimensional component $\overline{\widetilde{x}_1\widetilde{v}_1}$ of Δ then it suffices to choose $\widetilde{p}'_{n+1} \in \overline{\widetilde{x}'_1\widetilde{v}'_1}$ such that

$$d_{\widetilde{\sigma}_2}(\widetilde{p}'_{n+1}, \widetilde{x}'_1) \leq d_{\widetilde{q}_1}(pr(\widetilde{p}_{n+1}), \widetilde{x}_1) \text{ and } d_{\widetilde{\sigma}_2}(\widetilde{p}'_{n+1}, \widetilde{v}'_1) \leq d_{\widetilde{q}_1}(pr(\widetilde{p}_{n+1}), \widetilde{v}_1).$$

- Otherwise, $pr(\widetilde{p}_{n+1}) \in \overline{\overset{\circ}{\Delta}}$, where $\overline{\overset{\circ}{\Delta}}$ corresponds to the filled geodesic triangle of vertices $\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3$ (which are the vertices with strictly convex internal angle as in proposition 5.3.2).

In this case one can choose $\widetilde{p}'_{n+1}$ such that $d_\Delta(\widetilde{v}_i, \widetilde{p}_{n+1}) \geq d_{\Delta'}(\widetilde{v}'_i, \widetilde{p}'_{n+1})$.
In this way one obtains

$$d_{\widetilde{\sigma}_2}(\widetilde{p}'_{n+1}, \widetilde{x}'_i) \leq d_{\widetilde{\sigma}_2}(\widetilde{p}'_{n+1}, \widetilde{v}'_i) + d_{\widetilde{\sigma}_2}(\widetilde{v}'_i, \widetilde{x}'_i) \leq$$

$$\leq d_{\widetilde{q}_1}(pr(\widetilde{p}_{n+1}), \widetilde{v}_i) + d_{\widetilde{q}_1}(\widetilde{v}_i, \widetilde{x}_i) = d_{\widetilde{q}_1}(pr(\widetilde{p}_{n+1}), \widetilde{x}_i)$$

for $i = 1, 2, 3$ as desired.

The rest of the chapter will be devoted to the explanation of our method to find $\widetilde{p}_{n+1}$ in cases (1,i) and (2,i). As we anticipated it will depend on following theorem 5.3.3 and conjecture 5.1.

Consider the two previously defined filled geodesic triangles of vertices respectively $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3$ and $\widetilde{x}'_1, \widetilde{x}'_2, \widetilde{x}'_3$. Zeroes on the border of Δ can change position in Δ' and in particular the following things can happen:

(i) if $\widetilde{z} \in \overline{\widetilde{x}_i\widetilde{x}_j}$, then it can happen $\widetilde{z}' \in \overline{\widetilde{x}'_i\widetilde{x}'_k}$,

(ii) a zero $\widetilde{z}$ on the border of Δ can be such that $\widetilde{z}' \notin \Delta'$,

(iii) a zero $\widetilde{z}'$ on the border of Δ' can be such that $\widetilde{z} \notin \Delta$.

Every time case (ii) is verified, we consider the previously defined orthogonal projection on convex sets in $Cat(0)$ spaces

$$pr : (\widetilde{X}_2, d_{\widetilde{\sigma}_2}) \to \Delta'$$

and take into account the point $pr(\widetilde{z}') \in \partial\Delta'$. Then it will follow

$$d_{\widetilde{\sigma}_2}(pr(\widetilde{z}'), \widetilde{x}'_i) \leq d_{\widetilde{\sigma}_2}(\widetilde{z}', \widetilde{x}'_i) \leq d_{\widetilde{q}_1}(\widetilde{z}, \widetilde{x}_i)$$

for $i = 1, 2, 3$ and

$$d_{\widetilde{\sigma}_2}(pr(\widetilde{z}'), \widetilde{w}') \leq d_{\widetilde{\sigma}_2}(\widetilde{z}', \widetilde{w}') \leq d_{\widetilde{q}_1}(\widetilde{z}, \widetilde{w})$$

for every zero $\widetilde{w}'$ on the border of Δ'.
In the following construction we will need to consider, for every point on the border of Δ, a corresponding point on the border of Δ'. For this reason, by abuse of notation, every time previous case (ii) is verified we will denote the point $pr(\widetilde{z}')$ simply by $\widetilde{z}'$ and consider it the point on the border of Δ' corresponding to $\widetilde{z}$.
Notice that proceeding in this way Δ' could end up having two or more coinciding vertices: this will not be a problem.

From now on it will be more convenient to consider filled geodesic triangles Δ and Δ' exclusively as planar polygons endowed respectively with the intrinsic Euclidean

metrics d_Δ and $d_{\Delta'}$. For this reason we will consider zeroes on the border of Δ simply as vertices of the polygon. Furthermore, in order to lighten up the notation, vertices will be denoted without the overlying tilde.
For every couple of points $u, v \in \Delta$ we will denote by $\overline{uv}$ the geodesic for d_Δ connecting them. Given any two points $u', v' \in \Delta'$, we will denote by $\overline{u'v'}$ the geodesic for $d_{\Delta'}$ connecting them.

We will initially consider the case there is a function

$$\iota : Vertices(\Delta) \to Vertices(\Delta')$$

which to every vertex z of Δ associates a vertex $\iota(z) = z'$ of Δ' in such a way that vertices of Δ and of $\iota(Vertices(\Delta))$ are *disposed in the same order*. This means that:

- for every vertex z of Δ, if $z \in \overline{x_i x_j}$, then $z' \in \overline{x_i x_j}$,
- for every couple of vertices $z_1, z_2 \in \overline{x_i x_j}$, if $d_\Delta(x_i, z_1) < d_\Delta(x_i, z_2)$, then $d_{\Delta'}(x'_i, z'_1) \leq d_{\Delta'}(x'_i, z'_2)$.

We will summarize this condition on the vertices of Δ and Δ' saying that the common vertex of Δ and Δ' have the same order.
We noticed that, given two vertices v_1, v_2 of Δ, it can happen that their corresponding vertices of Δ' coincide as points on $\partial\Delta'$. For a reason which will be clear in the following proofs, we will consider v'_1 and v'_2 as distinct vertices of Δ' which are at distance zero on $\partial\Delta'$: we will refer to them as multiple vertices.
We can thus suppose the function ι is always injective and the number of vertices of Δ' is always greater than or equal to the number of vertices of Δ.

We underline again an important hypothesis on distances between vertices of Δ and Δ': for every pair of vertices u, v of Δ such that $\overline{uv}$ is smooth it results

$$d_\Delta(u, v) \geq d_{\Delta'}(u', v').$$

This fact clearly implies the same inequality also in case $\overline{uv}$ is a concatenation of smooth segments.

The following theorem is our fundamental tool to find the desired point $\widetilde{p}'_{n+1}$.

Theorem 5.3.3. *Suppose the number of vertices of Δ' is greater than or equal to the number of vertices of Δ and that the common vertices have the same order, in the sense we explained earlier. Suppose furthermore that Δ' can have one dimensional components.*

Then there is a 1-Lipschitz map $f : \Delta \to \Delta'$ *(with respect to the intrinsic Euclidean metrics of the polygons) such that:*

$$f(z) = z'$$

for every vertex z *of* Δ.

Clearly, given any point $\widetilde{p}_{n+1} \in \Delta$, we will set the point $\widetilde{p}'_{n+1}$ to be $f(\widetilde{p}_{n+1})$. Instead of proving theorem 5.3.3 directly, we will prove the following theorem 5.3.4 which will then imply theorem 5.3.3. The reason for this choice will be made clear in the proof of theorem 5.3.4 and in particular by the example of figure 5.11.

Given any planar polygon P with $n \geq 3$ vertices, we will say that P' is a *degenerate polygon comparable with* P if P' is obtained connecting planar polygons through common vertices or one dimensional components and furthermore all the following conditions are satisfied.

(i) P' is connected, simply connected, can be embedded in $\mathbb{R}^2$ and contains at least one planar polygon.

(ii) Every planar polygon of P' is linked (by shared vertices or one dimensional components) to at most other two planar polygons of P'. The degenerate polygon P' can have one dimensional components which are linked to just one planar polygon of P' (as polygons Δ' corresponding to geodesic triangles of $d_{\widetilde{q}}$ do).

(iii) There is an injective function $\iota : Vertices(P) \to Vertices(P')$, which to every vertex z of P associates a unique vertex z' of P'.
Given two vertices z_1, z_2 of P, their corresponding vertices of P' can coincide as points on $\partial P'$: we will consider z'_1, z'_2 as distinct vertices of P' at distance zero on $\partial P'$ and refer to them as multiple vertices.
Consequently, the total number of vertices of P' is $m \geq n$.

(iv) For every pair of vertices z_1, z_2 of P it results

$$d_P(z_1, z_2) \geq d_{P'}(z'_1, z'_2).$$

(v) If y' is a vertex of P' which does not correspond to any vertex of P and y' does not lie on a one dimensional component, then the internal angle at y' is:

- convex, if y' is a shared vertex of two planar polygons of P' or from y' starts a one dimensional component,
- concave, otherwise.

A vertex y' which does not correspond to any vertex of P can also lie on a one dimensional component, but it can not be at the extremity which is not connected to a planar polygon.

(vi) The vertices of P and of $\iota(Vertices(P))$ *are disposed in the same order* in the following sense. There is a continuous, surjective function $\tau : [0,1] \to \partial P'$ such that $\tau(0) = z' \in \iota(Vertices(P))$ and for every $x' \in \partial P'$ the cardinality of $\tau^{-1}(x')$ is:

- two, if x' is a shared vertex of two planar polygons of P' or x' in on a one dimensional component,
- one, otherwise.

Then one can choose a parametrization $\gamma : [0,1] \to \partial P$ of ∂P such that $\gamma(0) = z$ and γ and τ meet respectively the vertices of P and of $\iota(Verices(P))$ in the same order (up to removing one copy of the vertices which τ meets twice).

In figure 5.6 there are some example which will clarify our definition of degenerate polygons comparable with P and of condition (vi).

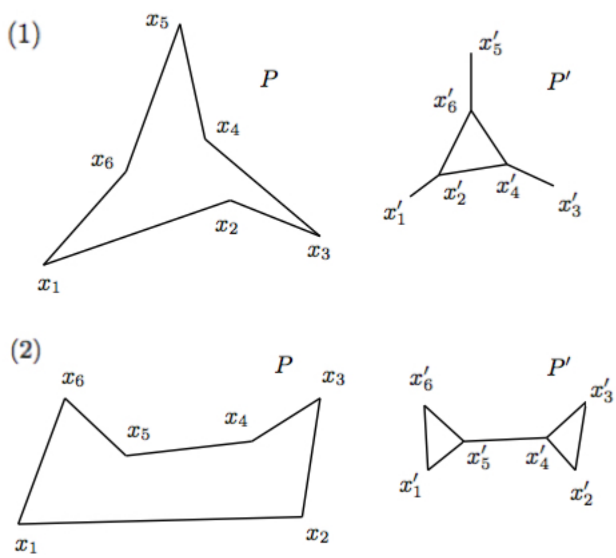

Figure 5.6: In example (1) one can find τ such that it encounters the vertices of P' in the order $x'_1, x'_2, x'_4, x'_3, x'_4, x'_6, x'_5, x'_6, x'_2$. One then discards the first copy of x'_4 and x'_6 and the last copy of x'_2. In example (2) one can find τ such that it encounters the vertices of P' in the order $x'_1, x'_5, x'_4, x'_2, x'_3, x'_4, x'_5, x'_6$. One then discards the first copy of x'_5 and x'_4.

As it is easily verifiable, polygons Δ and Δ' as in the hypothesis of theorem 5.3.3 satisfy all previous conditions.

If u, v are vertices of P, $\overline{uv}$ is smooth and lies entirely on the border of P then we will call $\overline{uv}$ a *side* of P and sometimes denote it simply by γ. If w, z are vertices of P, $\overline{wz}$ is smooth and $\overline{wz} \cap \partial P = \{w, z\}$, then we will call $\overline{uv}$ a *smooth diagonal* of P and sometimes denote it simply by d. If $\overline{wz}$ is a concatenation of segments and is not entirely contained in the border of P we call $\overline{wz}$ a diagonal of P and sometimes denote it with the same symbol d.
We define sides γ' of P' in the same way. A diagonal d' of P' is a geodesic $\overline{u'v'}$ such that $\overline{uv}$ is a diagonal of P. In particular one should notice that:

- a diagonal d' of P' can be entirely contained in a one dimensional component,
- $\overline{u'v'}$ can be a diagonal of P' only if $u', v' \in \iota(Vertices(P))$.

Given sides γ, γ' and diagonals d, d', we will denote by $l(\gamma), l(\gamma'), l(d), l(d')$ their lengths (of which the first and the third are computed with respect to d_P and the second and the fourth with respect to $d_{P'}$).

Before starting the proof, we feel it is necessary to anticipate why we decided to consider such a complicated set of degenerate polygons. The short answer is that the set of degenerate polygons P' comparable to P is closed with respect to the operation of *cutting along a diagonal d' of P'*, operation which is crucial in the proof of theorem 5.3.4. We will further clarify this concept in the proof.

Theorem 5.3.4. *Let P be a planar polygon with $n \geq 3$ vertices and P' a degenerate polygon which is comparable with P in the sense we just explained. Then there is a 1-Lipschitz map $f : P \to P'$ (with respect to the intrinsic Euclidean metrics of the polygons) such that*

$$f(z) = z'$$

for every other vertex z of P.

The idea of the proof will be to turn P' into the polygon P through a finite number of steps, called *elementary steps*, which will modify lengths of sides and diagonals of P'. Each elementary step will provide us of a 1-Lipschitz map: the final 1-Lipschitz map f will be the composition of all intermediate 1-Lipschitz maps. Of course, all intermediate polygons will be endowed with the corresponding intrinsic Euclidean metric and the intermediate maps will have Lipschitz coefficient 1 with respect to those metrics.
We specify that intermediate polygons obtained through elementary steps can fail to

be planar and just be *generalized polygons*: a generalized polygon is a polygon which is obtained gluing planar polygons along sides of the same length and which can not be embedded in $\mathbb{R}^2$. For any generalized polygon it still makes sense to define the intrinsic Euclidean metric.
Given any generalized polygon Q and a vertex v of Q, in the following proofs we will denote by α_v the internal angle of Q at v.

We will now define the two types of elementary steps we will use. In order to make the definition easier, we will first make the assumption P' does not have one dimensional components.

Elementary step of type one:
If γ' is a side of P' such that $l(\gamma') < l(\gamma)$ then though an elementary step of type one on the side γ' of P' it is possible to obtain a polygon $\hat{P}$ and a 1-Lipschitz map $\phi : \hat{P} \to P'$ such that:

- $l(\gamma) \geq l(\hat{\gamma}) > l(\gamma')$ (where $\hat{\gamma}$ denotes the side of $\hat{P}$ corresponding to γ') and all other sides of $\hat{P}$ are of the same length of the corresponding sides of P',
- all the diagonals $\hat{d}$ of $\hat{P}$ are such that $l(d) \geq l(\hat{d}) \geq l(d')$.

We presently explain how the elementary step of type one on the side $\gamma' = \overline{x'y'}$ of P' is performed.
Let z' be another vertex of P' such that $\overline{x'z'}$ and $\overline{y'z'}$ are sides or smooth diagonals of P' (it is always possible to suppose the existence of such z'). Let $P(x', y', z') \subset P'$ be the triangle of vertices x', y', z', the set $\overline{P' \setminus P(x', y', z')}$ consists of a number of polygons Q'_i which varies between zero and two.
Denote by $P(\hat{x}, \hat{y}, \hat{z})$ the triangle obtained from $P(x', y', z')$ increasing $d_{P'}(x', y')$ and without changing $d_{P'}(x', z')$ and $d_{P'}(y', z')$.

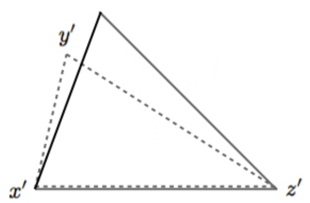

Figure 5.7: An example of an elementary step of type one: $P(x', y', z')$ is the triangle drawn with a dashed line, while $P(\hat{x}, \hat{y}, \hat{z})$ is the triangle drawn with a continuous line.

The polygon $\hat{P}$ is then obtained gluing back on the sides of $P(\hat{x}, \hat{y}, \hat{z})$ the corresponding polygons Q'_i.
There is a 1-Lipschitz map $\phi_1 : P(\hat{x}, \hat{y}, \hat{z}) \to P(x', y', z')$ which is the identity on the two sides whose length is not increased. The map ϕ_1 can then be extended to a 1-Lipschitz map $\phi : \hat{P} \to P'$ by defining it as the identity on Q'_i.

We will also consider degenerate elementary steps of type one, in which $P(x', y', z')$ is one dimensional and is then turned into a triangle. This will happen for example in case of coinciding vertices, which correspond to sides of length zero.

Elementary step of type two:
If d' is a smooth diagonal of P' such that $l(d') < l(d)$ then through an elementary step of type two on the diagonal d' of P' it is possible to obtain a polygon $\hat{P}$ and a 1-Lipschitz map $\psi : \hat{P} \to P'$ such that:

- all sides of $\hat{P}$ have the same length of the corresponding sides of P',
- all diagonals $\hat{d}$ of $\hat{P}$ are such that $l(d) \geq l(\hat{d}) \geq l(d')$.

We presently explain how the elementary step of type two on the smooth diagonal $d' = \overline{x'y'}$ of P' is performed.

Unlike elementary steps of type one, it is possible to perform an elementary step on a smooth diagonal $d' = \overline{x'y'}$ of P' only if there are other vertices u', v' of P' such that:

- all four geodesics $\overline{u'x'}, \overline{x'v'}, \overline{v'y'}, \overline{y'u'}$ are smooth and thus define a quadrilateral $P(x', y', u', v') \subset P'$,
- $\overline{x'y'}$ is a smooth diagonal of $P(x', y', u', v')$,
- $P(x', y', u', v')$ has only one strictly concave internal angle, which is in x' or y'. Consequently all other three internal angles of $P(x', y', u', v')$ are strictly convex.

We allow the quadrilateral $P(x', y', u', v')$ to be *degenerate* in the sense that one of the internal angles of $P(x', y', u', v')$ in u' or v' can be zero.
The set $\overline{P' \setminus P(x', y', u', v')}$ consists of a number of polygons Q'_i which varies between zero and four. It is possible to obtain another quadrilateral $P(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ from $P(x', y', u', v')$ increasing $d_{P'}(x', y')$, decreasing the strictly concave angle of the quadrilateral $P(x', y', u', v')$ and leaving unchanged the lengths of its sides. The polygon $\hat{P}$ is then obtained gluing back the polygons Q'_i on the corresponding sides of $P(\hat{x}, \hat{y}, \hat{u}, \hat{v})$.

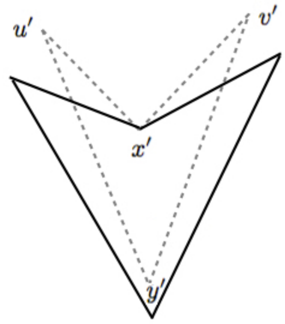

Figure 5.8: An example of an elementary step of type two: $P(x', y', u', v')$ is the quadrilateral drawn with a dashed line, while $P(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ is the quadrilateral drawn with a continuous line.

There is a 1-Lipschitz map $\psi_1 : P(\hat{x}, \hat{y}, \hat{u}, \hat{v}) \to P(x', y', u', v')$, with respect to the intrinsic Euclidean metrics of the polygons, which is the identity on the sides of the quadrilaterals. It can be extended to a 1-Lipschitz map $\psi : \hat{P} \to P'$ defining it as the identity on the polygons Q_i'.

Notice that both types of elementary steps do not change the sum of the internal angles of polygons Q' on which they are performed.

Now that we have defined the two types of elementary steps, we can go back to explaining how to obtain the desired 1-Lipschitz map $f : P \to P'$.
We will use the following lemma regarding generalized polygons. Notice that any generalized polygon Q' with m vertices has sum of internal angles equal to $\pi(m-2)$. Indeed, suppose Q is obtained gluing two planar polygons Q_1 and Q_2 along a side $\overline{vw}$: denote by m_1, m_2 the number of vertices of Q_1 and Q_2, then it must follow $m_1 + m_2 = m + 2$ (since gluing Q_1 and Q_2 we lose two vertexes which are identified together). Consequently the sum of internal angles of Q is equal to $\pi(m_1 - 2) + \pi(m_2 - 2) = \pi(m - 2)$.

Lemma 5.3.5. *Let Q' be a generalized polygon with n vertices. Then it is possible to apply a finite sequence of elementary steps of type two on Q' turning it into a convex polygon $\hat{Q}$ such that all sides of $\hat{Q}$ are of the same length of the corresponding sides of Q'.*

Proof. We proceed by induction on the number m of vertices of Q'.
If $m = 4$ then the result is trivial. Suppose the thesis is true for all polygons Q' with number of vertices between 4 and $m > 4$, then we will prove it for polygons Q' with

$m+1$ vertices.
We cut Q' along a smooth diagonal $\overline{v'w'}$ obtaining two generalized polygons Q_i on which we can apply the inductive hypothesis thus turning them into two convex polygons $\hat{Q}_i$: we glue $\hat{Q}_1, \hat{Q}_2$ back together along $\overline{\hat{v}\hat{w}}$ obtaining a polygon $\hat{Q}$ which can have strictly concave internal angles only in $\hat{v}$ and $\hat{w}$. If $\alpha_{\hat{v}} > \pi$ then one performs an elementary step of type two on $P(\hat{v}_1, \hat{v}, \hat{v}_2, \hat{w})$ (where $\hat{v}_1, \hat{v}_2$ are the vertices next to $\hat{v}$) stretching $\overline{\hat{v}\hat{w}}$ until $\alpha_{\hat{v}} = \pi$. Finally, only the angle $\alpha_{\hat{w}}$ can be strictly concave. Notice that all diagonals $\overline{\hat{w}\hat{z}}$ must be smooth, where $\hat{z}$ is any vertex of $\hat{Q}$ not adjacent to $\hat{v}$: using this fact and the hypothesis on the internal angles of Q' one gets that it is always possible to flatten the angle $\alpha_{\hat{w}}$ performing elementary steps of type 2 stretching $\overline{\hat{w}\hat{x}}$, where $\hat{x}$ is a vertex of $\hat{Q}$ such that $\alpha_{\hat{x}} < \pi$, without making any angle $\alpha_{\hat{x}}$ strictly concave. □

Observation 5.3.6. *Notice that, given polygons Q' and $\hat{Q}$ as in the previous lemma, if x' is a vertex of Q' such that $\alpha_{x'} \geq \pi$, then it can not result $\alpha_{\hat{x}} < \pi$.*
To see this, denote by x'_1, x'_2 the two vertices of Q' adjacent to x' and by $\hat{x}_1, \hat{x}_2$ the two corresponding vertices of $\hat{Q}$. If $\alpha_{\hat{x}} < \pi$ then $\overline{\hat{x}_1\hat{x}_2}$ is a segment of length strictly smaller than $d_{Q'}(x'_1, x') + d_{Q'}(x', x'_2) = d_{Q'}(x'_1, x'_2)$ and consequently it would result $d_{Q'}(x'_1, x'_2) > d_{\hat{Q}}(\hat{x}_1, \hat{x}_2)$.
This inequality would contradict the fact that, since $\hat{Q}$ is obtained from Q' through a sequence of elementary steps of type two, there is a 1-Lipschitz map $f : \hat{Q} \to Q'$ which sends vertices to corresponding vertices.

We can now start the proof of theorem 5.3.4, using induction on the number n of vertices of P. In order to make the proof more easily readable, we will divide the following arguments in succeeding lemmas.
Suppose $n = 3$, then P is an Euclidean triangle, while P' can have many more vertices than P. In order to satisfy condition (v) of the definition of degenerate polygons comparable to P, P' can have only one planar subpolygon and at most three one dimensional components ending in points of $\iota(Vertices(\Delta))$.
Consequently, for $n = 3$, P and P' will be polygons of the type described in theorem 5.3.3: for this reason we will denote them by Δ, Δ'.

Lemma 5.3.7. *If $n = 3$, it is possible to turn Δ' into Δ using only elementary steps of type one and two and consequently get a 1-Lipschitz map $f : \Delta \to \Delta'$ obtained composing all intermediate 1-Lipschitz maps between intermediate polygons.*

Proof. We first get rid of the one dimensional components of Δ', turning them into part of $\overline{\mathring{\Delta}}$ using elementary steps as thus explained (we will explain the procedure only for the one dimensional component starting at x'_1, the other two will be treated

in the same way).
Suppose there is a one dimensional component $\overline{x'_1v'_1}$ starting at x'_1 and $v'_1 \in \overline{\overset{\circ}{\Delta}}$ is the vertex such that the corresponding internal angle of $\overline{\overset{\circ}{\Delta}}$ must be strictly convex. Let v'_2 be the vertex of $\overline{x'_1v'_1}$ closer to v'_1 and let w'_1, u'_1 be the two vertices of $\overline{\overset{\circ}{\Delta}}$ adjacent to v'_1. We perform an elementary step of type two on $P(u'_1, v'_1, v'_2, w'_1)$ (which is a degenerate polygon, since the internal angle in v'_2 is zero) until $P(u'_1, v'_1, v'_2, w'_1)$ is no longer degenerate. Notice that there are two ways of performing an elementary step of type two on a degenerate quadrilateral $P(u'_1, v'_1, v'_2, w'_1)$ (as it is showed in figure 5.9): in one way $\overline{u'_1v'_1}$ is stretched and it will result $\hat{v}_1 \in \overline{\hat{w}_1\hat{v}_2}$ and in the other way $\overline{v'_1w'_1}$ is stretched and it will result $\hat{v}_1 \in \overline{\hat{u}_1\hat{v}_2}$. Since we do not care on which side the vertex $\hat{v}_1$ will end up, we can choose either way.

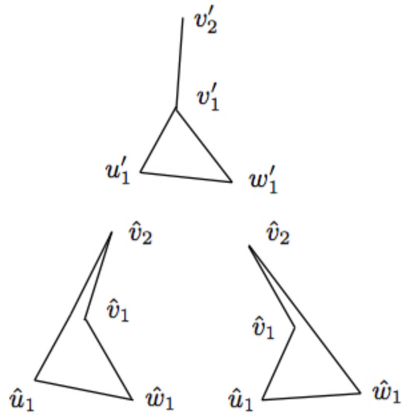

Figure 5.9: Two ways of performing an elementary step of type two on $P(u'_1, v'_1, v'_2, w'_1)$

We then proceed in the same way considering the vertex v'_3 of the one dimensional component $\overline{v'_2x'_1}$ closer to v'_2.
Having done so, we obtain a polygon without one dimensional components and with concave internal angles in all vertices which are not in $\iota(Vertices(\Delta))$: we turn it into an Euclidean triangle $\widehat{\Delta}$ of vertices $\hat{x}_i$, $i = 1, 2, 3$ using lemma 5.3.5 through elementary steps of type two. Finally, we perform a finite sequence of elementary steps of type one on the three sides $\overline{\hat{x}_i\hat{x}_j}$ of $\widehat{\Delta}$ in order to make them of the same length of the corresponding sides of Δ.
Suppose all three sides of $\widehat{\Delta}$ are such that $d_{\widehat{\Delta}}(\hat{x}_i, \hat{x}_j) < d_\Delta(x_i, x_j)$. We start by stretching the length of $\overline{\hat{x}_1\hat{x}_2}$ until the angle in $\hat{x}_3$ is equal to $\pi - \epsilon$, with $\epsilon > 0$ very small: in this way it results $l(\overline{\hat{x}_1\hat{x}_3})^2 + l(\overline{\hat{x}_2\hat{x}_3})^2 = l(\overline{\hat{x}_1\hat{x}_2})^2 + \psi(\epsilon)$ with $\lim_{\epsilon \to 0} \frac{\psi(\epsilon)}{\epsilon} = 0$.
Performing again an elementary step of type one stretching $\overline{\hat{x}_2\hat{x}_3}$ until the angle in $\hat{x}_1$

is equal to $\pi - \epsilon$ one gets $2l(\overline{\hat{x}_1\hat{x}_3})^2 + l(\overline{\hat{x}_2\hat{x}_3})^2 = l(\overline{\hat{x}_2\hat{x}_3})^2 + \psi_1(\epsilon)$ with $\lim_{\epsilon\to 0} \frac{\psi_1(\epsilon)}{\epsilon} = 0$. Consequently, proceeding in this way, after a finite number of steps one side must reach its maximum length. Then we proceed in the same way until all sides of $\widehat{\Delta}$ are of the same length of the corresponding sides of Δ. □

Now suppose the inductive hypothesis is verified if the number of vertices of P is not greater than n, then we shall find the 1-Lipschitz map if P has $n+1$ vertices.

Lemma 5.3.8. *If there is a diagonal $\overline{v'w'}$ of P' such that $l(\overline{v'w'}) = l(\overline{vw})$, then it is possible to apply the inductive hypothesis to obtain the 1-Lipschitz map $f : P \to P'$.*

Proof. We divide two cases.

- if $\overline{vw}$ is smooth then we cut the polygons P and P' in the following way:
  - we cut P along $\overline{vw}$ obtaining P_1 and P_2,
  - we cut P' along $\overline{v'w'}$ obtaining P'_1 and P'_2.

  Notice that if $\overline{v'w'}$ is not smooth then the operation of *cutting along* $\overline{v'w'}$ must be further clarified. If d' passes through a side of P' (resp. a one dimensional component), then such side (resp. one dimensional component) will appear on both polygons P'_i. Notice that in this way the polygons P'_i could acquire new one dimensional components and new vertices. We will follow this rule to name the new vertices: if u' is a vertex of $\iota(Vertices(P))$ on $\overline{v'w'}$ and $u \in \Delta_1$ (resp. $u \in \Delta_2$), then u' will be a vertex only of Δ'_1 (resp. Δ'_2).

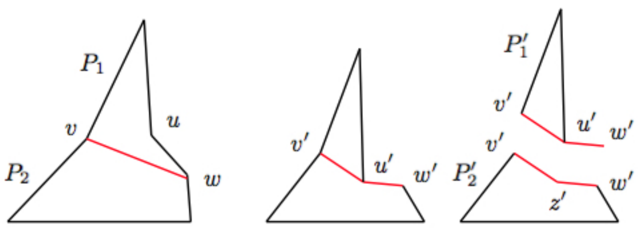

Figure 5.10: An example of cutting in case d' is not smooth: notice that the points v', w' appear on both P'_1 and P'_2, while u' appears only on P'_1, since $u \in P_1$. On P'_2 there is a new vertex $z' \notin \iota(Vertices(P_2))$.

Sometimes a polygon P'_i could be entirely degenerate (i.e. one dimensional): in that case we perform a degenerate elementary step of type one on P'_i turning it into a degenerate polygon which includes at least one planar polygon.

We can thus suppose both newly obtained polygons P'_1 and P'_2 are degenerate polygons comparable respectively with P_1 and P_2. Indeed, condition (v) of the definition is verified since, if $z' \in \overline{v'w'}$, $z' \notin \iota(Vertices(P))$ is a vertex of a planar polygon of P', a corresponding vertex $z'_i \in P'_i$ can have strictly convex internal angle only if from z'_i starts a one dimensional component.

This is the crucial property we were looking for: we can now apply the inductive hypothesis and obtain two 1-Lipschitz maps $f_i : \Delta_i \to \Delta'_i$ which must agree on $\overline{vw}$: we will define $f : \Delta \to \Delta'$ to be such that $f|_{\Delta_i} := f_i$.
Notice that the same reasoning could not have been done considering polygons Δ and Δ' of the hypothesis of theorem 5.3.3, as explained in picture 5.11.

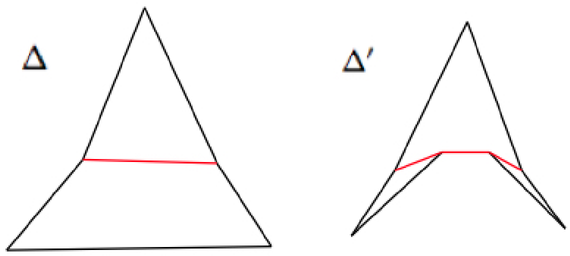

Figure 5.11: The diagonals d and d' are drawn in red. One clearly sees that the bottom half of Δ has four strictly convex angles, while the bottom half of Δ' is composed by two triangles connected by a one dimensional component.

- if $\overline{vw}$ is not smooth, then suppose $\overline{vw}$ is the concatenation of segments $\overline{vv_1} * \overline{v_1v_2} * \cdots * \overline{v_mw}$: at least one of them must be a smooth diagonal, so suppose $\overline{vv_1}$ is. Notice that if $l(\overline{vw}) = l(\overline{v'w'})$ then it must follow $\overline{v'w'} = \overline{v'v'_1} * \overline{v'_1v'_2} * \cdots * \overline{v'_mw'}$, otherwise one would get

$$l(\overline{vv_1}) + l(\overline{v_1v_2}) + \cdots + l(\overline{v_mw}) < l(\overline{v'v'_1}) + l(\overline{v'_1v'_2}) + \cdots + l(\overline{v'_mw'})$$

which contradicts the hypothesis on the distances in P and P'.
Now one can just consider the diagonals $\overline{vv_1}$ (which is smooth) and $\overline{v'v'_1}$ and fall into the previous case.

□

After these considerations we can always suppose all diagonals of P' are strictly shorter than the corresponding diagonals of P. We will now deal with one dimensional components of P'.

Lemma 5.3.9. *Suppose all diagonals of P' are strictly shorter than the corresponding diagonals of P. Then, using elementary steps, it is possible to turn P' in a degenerate polygon comparable to P without one dimensional components. If in doing so one diagonal of P' reaches its maximum length (i.e. the length of the corresponding diagonal of P) it is possible to apply the inductive hypothesis to obtain the desired 1-Lipschitz map $f : P \to P'$.*

Proof. We will proceed in a way which is almost identical to the one applied in the previous case $n = 3$. Let $\overline{v_1'x_1'}$ be a one dimensional component of P' and v_2' the vertex of $\overline{v_1'x_1'}$ closer to v_1', then we will apply an elementary step of type two on $P(u_1', v_1', v_2', w_1')$ (where, as before, u_1' and w_1' are vertices of a planar subpolygon of P' adjacent to v_1') in such a way that the newly obtained degenerate polygon $\hat{P}$ satisfies axiom (vi). In particular, if v_1', u_1', w_1', v_2' are all vertices of $\iota(Vertices(P))$ then we will apply the elementary step which gives $\hat{v}_1 \in \overline{\hat{w}_1\hat{v}_2}$ (resp. $\hat{v}_1 \in \overline{\hat{u}_1\hat{v}_2}$) if $v_1 \in \overline{w_1v_2}$ (resp. $v_1 \in \overline{u_1v_2}$). If v_1' is not a vertex of $\iota(Vertices(P'))$, then it is possible to perform both types of elementary step of type one.
Proceeding in this way one could end up with a vertex x_1' which connects two planar polygons of P': it is possible to get rid of this "pathology" with another elementary step of type two as it is explained in figure 5.12.

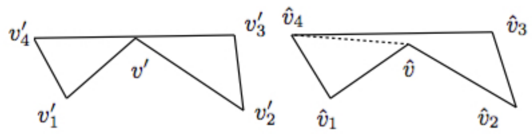

Figure 5.12: If $v \in \overline{v_1v_2}$ then one performs an elementary step of type two on $P(v_4', v', v_2', v_3')$.

In this way we explained also how to get rid of vertices of P' which link two different planar polygons. Clearly, if at any point during this procedure of elimination of one dimensional components, one ends up with a diagonal d' of P' such that $l(d') = l(d)$ then the 1-Lipschitz map $f : P \to P'$ is obtained as explained before. □

At this point, we can suppose P' does not have one dimensional components, but it can stil have more vertices than P. Notice that a straightforward consequence of the definition of elementary steps of type two and of condition (v) of the definition of degenerate polygons comparable to P is that all internal angles in vertices of P' which are not in $\iota(Vertices(P))$ will have concave internal angle.

Lemma 5.3.10. *Suppose all diagonals of P' are strictly shorter than the corresponding diagonals of P and P' does not have one dimensional components. Then, using* *elementary steps, it is possible to turn P' in a degenerate polygon comparable to P with the same vertices of P. If in doing so one diagonal of P' reaches its maximum length (i.e. the length of the corresponding diagonal of P) it is possible to apply the inductive hypothesis to obtain the desired 1-Lipschitz map $f : P \to P'$.*

Proof. One just has to apply lemma 5.3.5 (and observation 5.3.6), turning P' into a convex polygon $\hat{P}$. The polygon $\hat{P}$ will have flat internal angles at vertices $\hat{z}$ such that the corresponding vertex z' of P' is not in $\iota(Vertices(\Delta))$. At this point one simply "forgets" about $\hat{z}$ and removes it from the set of vertices of $\hat{P}$.
Again, if, performing any of the elementary steps of type two of lemma 5.3.5, one diagonal d' of P' is stretched until $l(\hat{d}) = l(d)$, then the procedure is finished as we already explained. □

We will now stretch all sides of P' until they become of the same length of the corresponding sides of P.

Lemma 5.3.11. *Suppose all diagonals of P' are strictly shorter than the corresponding diagonals of P, P' has the same vertices of P and P' does not have one dimensional components. Then, using elementary steps of type one, it is possible to stretch all sides of P' until they become of the same length of the corresponding sides of P. If in doing so one diagonal of P' reaches its maximum length (i.e. the length of the corresponding diagonal of P) it is possible to apply the inductive hypothesis to obtain the desired 1-Lipschitz map $f : P \to P'$.*

Proof. First, notice that it is not always possible to stretch a side of Δ' with just one elementary step of type one until it reaches its maximum length. Indeed, let $\overline{x_1'x_2'}$ be a side of P' such that $l(\overline{x_1'x_2'}) < l(\overline{x_1x_2})$ and $P(x_1', x_2', z')$ a triangle as in the definition of elementary step of type one. Notice that the upper limit of the length of the side $\overline{x_1'x_2'}$ obtainable through an elementary step of type one on $P(x_1', x_2', z')$ is $l(\overline{x_1'z'}) + l(\overline{x_2'z'})$ (at whose length $P(x_1', x_2', z')$ becomes a segment).

In order to overcome this difficulty, we number the vertices of P' in an increasing order starting from from x_1', in such a way that its adjacent vertices are x_2' and x_m'. Then we will explain how to turn P' into a triangle with convex angles in x_1', x_2' and internal angle in x_m' equal to $\pi - \epsilon$. In this way it will result that $l(\overline{x_1'x_2'})^2$ is equal to the sum of the squares of the lengths of all other sides of P' minus a term $\psi(\epsilon)$ such that $\lim_{\epsilon \to 0} \frac{\psi(\epsilon)}{\epsilon} = 0$: the conclusion will follow in the same way of the case of the proof of lemma 5.3.7. Clearly, if doing so one diagonal d' of P' is stretched until $l(d') = l(d)$ then the procedure is finished as explained before. One should notice that coinciding vertices do not constitute a problem, since they just correspond to sides of length zero (and will now be stretched by degenerate elementary steps of type one).

We now explain how to turn P' into a triangle with convex angles in x'_1, x'_2 and internal angle in x'_m equal to $\pi - \epsilon$: first of all we apply lemma 5.3.5 and turn P' into a convex polygon $\hat{P}$.
Denote by $\alpha_{\hat{x}_i}$ the internal angle of $\hat{P}$ in $\hat{x}_i$. If $\alpha_{\hat{x}_2} < \pi$ and $\alpha_{\hat{x}_j} = \pi$ for $j = 3, \dots, l-1$, we perform an elementary step of type one on $P(\hat{x}_1, \hat{x}_2, \hat{x}_l)$ until $\alpha_{\hat{x}_j} = \pi$.
If $\alpha_{\hat{x}_i} = \pi$ for $i = 2, \dots, k-1$, we perform an elementary step of type one on $P(\hat{x}_1, \hat{x}_2, \hat{x}_k)$ until $\alpha_{\hat{x}_k} = \pi$ and then perform an elementary step of type one on $P(\hat{x}_1, \hat{x}_2, \hat{x}_{k-1})$ until $\alpha_{\hat{x}_{k-1}} = \pi$.
Proceeding in this way one can flatten all angles $\alpha_{\hat{x}_i}$, $i = 3, \dots, m-1$, until $\hat{P}$ becomes a triangle with convex angles only in $\hat{x}_1, \hat{x}_2, \hat{x}_m$. Finally, one performs an elementary step of type one until $\alpha_{\hat{x}_m} = \pi - \epsilon$. □

The following lemma concludes the proof of theorem 5.3.4.

Lemma 5.3.12. *Suppose all diagonals of P' are strictly shorter than the corresponding diagonals of P, P' does not have one dimensional components, P' has the same vertices of P and all sides of P' have the same length of the corresponding sides of P. Then it is possible to obtain the desired 1-Lipschitz map $f : P \to P'$.*

Proof. We will prove that it will always be possible to obtain a diagonal d' of P' of maximum length, applying a finite number of elementary steps of type two on P': then the conclusion will follow as in the previous lemmas.
Once again, we turn P' into a convex polygon $\hat{P}$ using lemma 5.3.5. In case $\hat{P} \neq P$, there must be a vertex $\hat{x}$ of $\hat{P}$ such that $\alpha_{\hat{x}} > \alpha_x$. If we can prove that this implies the existence of a diagonal $\hat{d}$ of $\hat{P}$ such that $l(\hat{d}) \geq l(d)$ then the proof is finished, since this means that at some point during the sequence of elementary steps of type two which turns P' into $\hat{P}$ one gets $l(\hat{d}) = l(d)$.
We prove the equivalent statement that if all diagonals of $\hat{P}$ are strictly shorter than the corresponding diagonals of P, then all convex angles of P must be greater than the corresponding angles of $\hat{P}$.
Denote by y and z the vertices of P next to x: suppose $\overline{yz}$ is the concatenation of the smooth segments $\overline{yx_1} * \overline{x_1x_2} * \cdots * \overline{x_kz}$ for $k \geq 0$.
Denote by Q the polygon delimited by $\overline{xy}, \overline{xz}$ and $\overline{yz}$: all internal angles of Q are concave except for the ones in x, y, z and $\overline{xx_i}$, $i = 1, \dots, k$ are smooth diagonals contained in Q. We claim that decreasing the length of all diagonals $\overline{xx_i}$ without increasing the length of the sides of Q and without changing the lengths of $\overline{xy}$ and $\overline{xz}$, the angle α_x will decrease: this can be proved modifying the lengths of sides of Q one at a time.

Indeed, if only $d_Q(x_i, x_{i+1})$ decreases, then α_x must decrease: this can be easily seen shortening the side $\overline{x_i x_{i+1}}$ of the triangle $P(x, x_i, x_{i+1})$ of vertices x, x_i, x_{i+1} without changing the lengths of the other two sides of $P(x, x_i, x_{i+1})$. In the same way, if only $d_Q(x, x_i)$ decreases, then α_x must decrease: this can be easily seen shortening the diagonal $\overline{x x_i}$ of the quadrilateral $P(x, x_{i-1}, x_i, x_{i+1})$ of vertices x, x_{i-1}, x_i, x_{i+1} without changing the lengths of the sides of $P(x, x_{i-1}, x_i, x_{i+1})$. □

As we said, this ends the proof of theorem 5.3.4 and consequently also theorem 5.3.3 is proved.

We are now left with the case common vertices of Δ and of Δ' are not disposed in the same order.
From now on, we will denote the vertices of Δ with concave internal angle in the following way, which will be useful in the succeeding reasonings.

- Denote by w_j the vertices on $\overline{x_1 x_2}$, ordered in increasing order from x_1 to x_2.
- Denote by u_k the vertices on $\overline{x_2 x_3}$, ordered in increasing order from x_3 to x_2.
- Denote by v_l the vertices on $\overline{x_1 x_3}$, ordered in increasing order from x_3 to x_1.

As before, we denote by w'_j, u'_k, v'_l the corresponding vertices of Δ'.
We say a vertex w_j has *changed side* on Δ' if $w'_j \notin \overline{x'_1 x'_2}$.
Two vertices $w'_m, w'_n \in \overline{x'_1 x'_2}$ have *changed their order* if $m < n$ and it results $d_{\Delta'}(w'_n, x'_1) < d_{\Delta'}(w'_m, x'_1)$.
Changes of side and order of vertices u_k and v_l are defined in the same way.
Common vertices of Δ and of Δ' are not disposed in the same order if there is at least one change of side or one change of order.
As we anticipated, we are not able to prove a statement similar to the one of theorem 5.3.3 in case common vertices of Δ and Δ' are not disposed in the same order. So we can only state the following conjecture.

Conjecture 5.1. *Suppose the number of vertices of Δ' can be greater than the number of vertices of Δ, Δ' can have one dimensional components and the common vertices of Δ and Δ' are not disposed in the same order.*
Then for every $p \in \Delta$ there is a corresponding point $p' \in \Delta'$ such that

$$d_{\Delta'}(p', x'_i) \leq d_{\Delta}(p, x_i), \quad i = 1, 2, 3.$$

Clearly, it is not possible to adapt the proof of theorem 5.3.4 to prove conjecture 5.1, since the method consisting of elementary steps would only work if common vertices of Δ and Δ' have the same order.

Nonetheless, we are quite confident conjecture 5.1 must be true: this is because changes of side or order of vertices force the polygon Δ' to become smaller.
Indeed, if two vertices w_m, w_n of $\overline{x_1x_2}$ change order in Δ', then it must result $d_{\Delta'}(x_1', x_2') \leq d_\Delta(x_1, x_2) - d_\Delta(w_m, w_n)$. Since each change of order of the vertices contributes to the shortening of $\overline{x_1'x_2'}$, as the number of changes of order of vertices of $\overline{x_1x_2}$ increases, the shortening of $\overline{x_1'x_2'}$ also increases.
In a similar way, if a vertex of Δ changes side and for example it is $u'_{k_0} \in \overline{x_1'x_3'}$, then, since the distances $d_{\Delta'}(u'_k, u'_{k_0})$ can not be greater than the corresponding distances $d_\Delta(u_k, u_{k_0})$, all other vertices u'_k are forced to "follow" u'_{k_0} and become closer to vertices of $\overline{x_1'x_3'}$. This fact will force some distances inside Δ' to become smaller than the corresponding distances in Δ.
In light of these observations, one could even consider the case common vertices of Δ and Δ' are disposed in the same order as the worst one to prove the existence of p', since no distance inside Δ' is forced to decrease.

The following two propositions should support our intuition. Indeed, they show some cases where the change of side of one or more vertices of Δ' directly implies the existence of p'.

Proposition 5.3.13. *Suppose there is at least one vertex of Δ' which changes side, for example $u' \in \overline{x_1'x_3'}$. Then for every $p \in \Delta$ such that $d_\Delta(p, x_3) \leq d_{\Delta'}(u', x_3')$ there is a point $p' \in \Delta'$ such that*

$$d_{\Delta'}(p', x_i') \leq d_\Delta(p, x_i), \quad i = 1, 2, 3.$$

Proof. Choose $p' \in \overline{u'x_3'}$ at distance $d_\Delta(p, x_3)$ from x_3'. Then it results

$$d_{\Delta'}(p', x_1') = d_{\Delta'}(x_1', x_3') - d_{\Delta'}(p', x_3') \leq d_\Delta(x_1, x_3) - d_\Delta(p, x_3) \leq d_\Delta(p, x_1),$$

$$d_{\Delta'}(p', x_2') \leq d_{\Delta'}(p', u') + d_{\Delta'}(u', x_2') \leq d_\Delta(x_2, x_3) - d_\Delta(p, x_3) \leq d_\Delta(p, x_2).$$

□

Proposition 5.3.14. *Suppose one of the following three conditions is satisfied:*

(i) there are vertices $u', v' \in \overline{x_1'x_2'}$ such that $d_{\Delta'}(x_1', v') > d_{\Delta'}(x_1', u')$,

(ii) there are vertices $u', w' \in \overline{x_1'x_3'}$ such that $d_{\Delta'}(x_1', w') > d_{\Delta'}(x_1', u')$,

(iii) there are vertices $v', w' \in \overline{x_2'x_3'}$ such that $d_{\Delta'}(x_2', w') > d_{\Delta'}(x_2', v')$.

Then for every $p \in \Delta$ there is a corresponding point $p' \in \Delta'$ such that

$$d_{\Delta'}(p', x_i') \leq d_\Delta(p, x_i), \quad i = 1, 2, 3.$$

Proof. We will prove the proposition only for case (i), since the proof is identical for the other two cases.
One can find p' as follows.

1. If $d_\Delta(p, x_2) \le d_{\Delta'}(u', x_2')$ let p' be the point on $\overline{u'x_2'}$ at distance $d_\Delta(p, x_2)$ from x_2. It then results

$$d_{\Delta'}(p', x_1') = d_{\Delta'}(x_1', x_2') - d_{\Delta'}(p', x_2') \le d_\Delta(x_1, x_2) - d_\Delta(p, x_2) \le d_\Delta(p, x_1),$$

$$d_{\Delta'}(p', x_3') \le d_{\Delta'}(x_3', u') + d_{\Delta'}(u', x_2') - d_{\Delta'}(p', x_2') \le d_\Delta(x_2, x_3) - d_\Delta(p, x_2) \le d_\Delta(p, x_3).$$

2. If $d_\Delta(p, x_3) \le d_{\Delta'}(u', x_3')$ let p' be the point on $\overline{x_3'u'}$ at distance $d_\Delta(p, x_3)$ from x_3'. It results

$$d_{\Delta'}(p', x_2') \le d_{\Delta'}(x_3', u') + d_{\Delta'}(u', x_2') - d_{\Delta'}(p', x_3') \le d_\Delta(x_2, x_3) - d_\Delta(p, x_3) \le d_\Delta(p, x_2),$$

$$d_{\Delta'}(p', x_3') + d_{\Delta'}(p', x_1') \le d_{\Delta'}(v', x_3') + d_{\Delta'}(v', x_1') \le d_\Delta(x_1, x_3),$$
$$d_{\Delta'}(p', x_1') \le d_\Delta(x_1, x_3) - d_{\Delta'}(p', x_3') = d_\Delta(x_1, x_3) - d_\Delta(p, x_3) \le d_\Delta(p, x_1)$$

3. If $d_\Delta(p, x_2) > d_{\Delta'}(u', x_2')$ and $d_\Delta(p, x_3) > d_{\Delta'}(u', x_3')$ then there is always a point $p' \in \overline{x_1'u'}$ such that one of the following two conditions is satisfied:
   - $d_{\Delta'}(p', x_2') = d_\Delta(p, x_2)$ and $d_{\Delta'}(p', x_3') \le d_\Delta(p, x_3)$, then one can proceed as in previous case (1),
   - $d_{\Delta'}(p', x_3') = d_\Delta(p, x_3)$ and $d_{\Delta'}(p', x_2') \le d_\Delta(p, x_2)$, then one can proceed as in previous case (2).

□

One could try to prove conjecture 5.1 using the following approach.
Consider a subpolygon $\widehat{\Delta} \subset \Delta'$ such that to every vertex x_i, w_j, u_k, v_l of Δ there is a unique corresponding vertex $\hat{x}_i, \hat{w}_j, \hat{u}_k, \hat{v}_l$ of $\widehat{\Delta}$.
We say that $\widehat{\Delta}$ is a *subpolygon of* Δ' *comparable to* Δ if $\hat{x}_i = x_i'$, $i = 1, 2, 3$ and $\Delta, \widehat{\Delta}$ satisfy the hypothesis of theorem 5.3.4. In particular, this condition implies that :

- common vertices of $\widehat{\Delta}$ and of Δ are disposed in the same order,
- the distance between any two vertices of Δ is greater than or equal to the distance between the corresponding two points of $\widehat{\Delta}$.

If such polygon $\widehat{\Delta}$ exists, then preceding theorem 5.3.4 will grant the existence of a 1-Lipschitz map $\phi : \Delta \to \widehat{\Delta}$ which sends vertices of Δ to corresponding vertices of $\widehat{\Delta}$. Since for every couple of points $x', y' \in \widehat{\Delta}$ it results $d_{\widehat{\Delta}}(x', y') \ge d_{\Delta'}(x', y')$, one will conclude that ϕ is also a 1-Lipschitz map from Δ to Δ' such that $\phi(x_i) = x_i'$,

$i = 1, 2, 3$.
Notice that there is no need to require the polygon $\widehat{\Delta}$ to have exactly three strictly convex internal angles, since it is not required in the hypothesis of theorem 5.3.4.
Unfortunately we were not able to develop a method which always produces such polygon $\widehat{\Delta}$ for every Δ, Δ'. Indeed, we can only make the following conjecture.

Conjecture 5.2. *Suppose the number of vertices of Δ' can be greater than the number of vertices of Δ, Δ' can have one dimensional components and the common vertices of Δ and Δ' are not disposed in the same order.*
Then there always is a subpolygon $\widehat{\Delta}$ of Δ' comparable to Δ.

As we just explained, conjecture 5.2 implies conjecture 5.1.
We feel conjecture 5.2 must be true for the same reasons we explained to justify conjecture 5.1: each vertex which changes side or order in Δ' forces some distances to decrease.
We are only able to prove conjecture 5.2 in two simple cases, which we now illustrate.

Proposition 5.3.15. *If there is only one point u'_{k_0} of Δ' such that $u'_k \in \overline{x'_1 x'_3}$ and no other vertex changes order or side, then there is a subpolygon $\widehat{\Delta}$ of Δ' comparable to Δ.*

Proof. In this case it is possible to obtain $\widehat{\Delta}$ in the following way.
One replaces $\overline{x'_2 x'_3}$ with $\overline{x'_3 u'_{k_0}} * \overline{u'_{k_0} x'_2}$ and evaluates if it is possible to find points $\hat{u}_1, \dots, \hat{u}_{k_0-1}, \hat{u}_{k_0+1}, \dots, \hat{u}_k \in \overline{x'_3 u'_{k_0}} * \overline{u'_{k_0} x'_2}$ corresponding to $u'_1, \dots, u'_{k_0-1}, u'_{k_0+1}, \dots, u'_k$ such that the polygon identified by $\overline{x'_1 x'_2}, \overline{x'_3 u'_{k_0}} * \overline{u'_{k_0} x'_2}, \overline{x'_1 x'_3}$ is a subpolygon of Δ' comparable to Δ.
If so, the proof is concluded, since we have found the desired polygon $\widehat{\Delta}$.
If not, consider the orthogonal projection $pr : \Delta' \to \overline{x'_2 x'_3}$: one moves the point u_{k_0} in $\hat{u}_{k_0}$ on $\overline{u'_{k_0} pr(u'_{k_0})}$ towards $pr(u'_{k_0})$ until one of the following events happens.

(i) Replacing $\overline{x'_2 x'_3}$ with $\overline{x'_3 \hat{u}_{k_0}} * \overline{\hat{u}_{k_0} x'_2}$ it is possible to find points $\hat{u}_1, \dots, \hat{u}_{k_0-1}$, $\hat{u}_{k_0+1}, \dots, \hat{u}_k \in \overline{x'_3 \hat{u}_{k_0}} * \overline{\hat{u}_{k_0} x'_2}$ corresponding to $u'_1, \dots, u'_{k_0-1}, u'_{k_0+1}, \dots, u'_k$ such that the polygon identified by $\overline{x'_1 x'_2}, \overline{x'_3 \hat{u}_{k_0}} * \overline{\hat{u}_{k_0} x'_2}, \overline{x'_1 x'_3}$ is a subpolygon of Δ' comparable to Δ. Then the polygon $\widehat{\Delta}$ is found.

(ii) The distance of $\hat{u}_{k_0}$ with one of the points x'_1, v'_j, w'_l becomes equal to the distance between corresponding points of Δ (suppose for example $d_{\Delta'}(\hat{u}_{k_0}, v'_j) = d_\Delta(u_{k_0}, v_j)$).
Define $\widehat{\Delta}$ as the polygon obtained from Δ' replacing $\overline{x'_2 x'_3}$ with $\overline{x'_3 \hat{u}_{k_0}} * \overline{\hat{u}_{k_0} x'_2}$. The vertices $\hat{u}_1, \dots, \hat{u}_{k_0-1}$ of $\widehat{\Delta}$ corresponding to $u'_1, \dots, u'_{k_0-1}$ will be their orthogonal projection on $\overline{x'_3 \hat{u}_{k_0}}$, while the vertices $\hat{u}_{k_0+1}, \dots, \hat{u}_k$ of $\widehat{\Delta}$ corresponding to $u'_{k_0+1}, \dots, u'_k$ will be their orthogonal projection on $\overline{\hat{u}_{k_0} x'_2}$. One then cuts $\widehat{\Delta}$

along $\overline{\hat{u}_{k_0}v'_j}$ obtaining $\widehat{\Delta}_1, \widehat{\Delta}_2$ and cuts Δ along $\overline{u_{k_0}v_j}$ obtaining Δ_1, Δ_2.
Since both $\Delta_1, \widehat{\Delta}_1$ and $\Delta_2, \widehat{\Delta}_2$ satisfy the hypothesis of theorem 5.3.4, one can conclude there are two 1-Lipschitz maps $\phi_i : \Delta_i \to \widehat{\Delta}_i$, $i = 1, 2$. From the equality $d_{\Delta'}(\hat{u}_{k_0}, v'_j) = d_{\Delta}(u_{k_0}, v_j)$ one gets that it is possible to obtain a 1-Lipschitz map $\phi : \Delta \to \widehat{\Delta}$ sending vertices to corresponding vertices and such that $\phi(p) := \phi_i(p)$ if $p \in \Delta_i$. This proves that the distance between any two vertices of Δ is greater than or equal to the distance between the two corresponding vertices of $\widehat{\Delta}$.

□

Proposition 5.3.16. *If only two adjacent vertices w_m, w_{m+1} of Δ change order in Δ' and no vertex of Δ changes side, then there is a subpolygon $\widehat{\Delta}$ of Δ' comparable to Δ.*

Proof. In figure 5.13 is represented an example of this situation with $m = 1$.

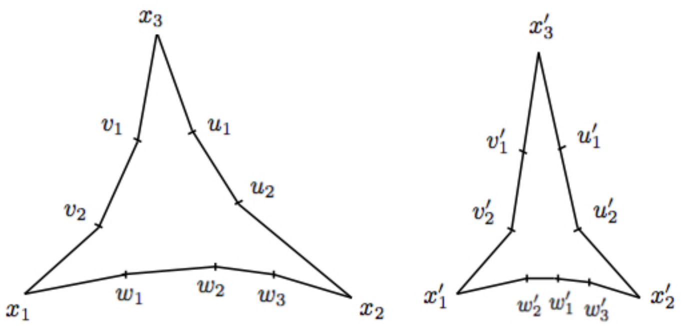

Figure 5.13: The order of w'_1 and w'_2 is changed.

We will move only one vertex between w'_m and w'_{m+1} and prove it will always be possible to set $\hat{w}_m = \hat{w}_{m+1} := w'_{m+1}$ or $\hat{w}_m = \hat{w}_{m+1} := w'_m$.
Clearly, in case one sets $\hat{w}_m = \hat{w}_{m+1} := w'_{m+1}$ then w'_{m+1} will become a multiple vertex of $\widehat{\Delta}$ and w'_m will be a vertex of $\widehat{\Delta}$ not in $\iota(Vertices(\Delta))$.
All other vertices of $\widehat{\Delta}$ will coincide with the corresponding vertices of Δ'. As it will be clear, all our considerations will not change in case Δ' has one dimensional components, more vertices than Δ or multiple vertices.
We define the following subsets of the sets of vertices of Δ and Δ':

$$T_{w_m} := \{p \text{ is a vertex of } \Delta \text{ such that } d_{\Delta}(p, w_m) \leq d_{\Delta}(p, w_{m+1})\},$$

$$T_{w_{m+1}} := \{p \text{ is a vertex of } \Delta \text{ such that } d_{\Delta}(p, w_{m+1}) \leq d_{\Delta}(p, w_m)\},$$

$$T'_{w_m} := \{p' \text{ is a vertex of } \Delta' \text{ such that } p \in T_{w_m}\},$$

$$T'_{w_{m+1}} := \{p' \text{ is a vertex of } \Delta' \text{ such that } p \in T_{w_{m+1}}\}.$$

The meaning of these sets is that in order to being able to impose $\hat{w}_m := w'_{m+1}$ one has to check only the distances of w'_{m+1} with the points of T'_{w_m}, since for every $p' \in T'_{w_{m+1}}$ it results

$$d_{\Delta'}(p', w'_{m+1}) \leq d_{\Delta}(p, w_{m+1}) \leq d_{\Delta}(p, w_m).$$

For the same reason, in order to set $\hat{w}_{m+1} := w'_m$ one has to check only the distances of w_m with the points of $T'_{w_{m+1}}$.
It is possible to further develop this reasoning defining the following two sets:

$$\hat{T}_{w_m} := \{p' \in T'_{w_m} \text{ such that } d_{\Delta'}(p', w'_m) \leq d_{\Delta'}(p', w'_{m+1})\},$$

$$\hat{T}_{w_{m+1}} := \{p' \in T'_{w_{m+1}} \text{ such that } d_{\Delta'}(p', w'_{m+1}) \leq d_{\Delta'}(p', w'_m)\}.$$

Following our previous idea, only points of $\hat{T}_{w_m}$ (resp. of $\hat{T}_{w_{m+1}}$) can prevent one from imposing $\hat{w}_m := w'_{m+1}$ (resp. $\hat{w}_{m+1} := w'_m$).
The example of figure 5.13 is particularly simple, since the sets $\hat{T}_{w_1}, \hat{T}_{w_2}$ are empty and consequently it is possible to impose both $\hat{w}_1 = \hat{w}_2 := w'_2$ and $\hat{w}_1 = \hat{w}_2 := w'_1$.

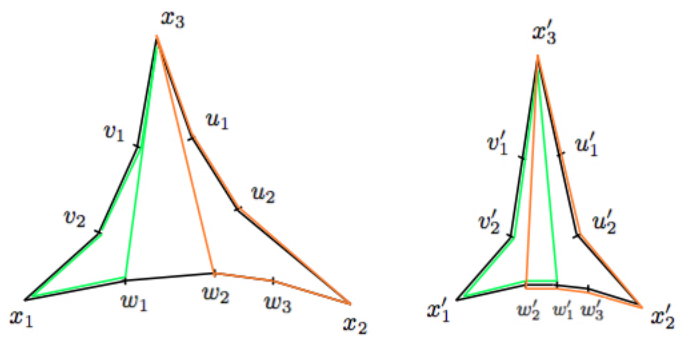

Figure 5.14: The convex envelope of T_{w_1} and T'_{w_1} is drawn in green and the convex envelope of T_{w_2} and T'_{w_2} is drawn in orange.

One will not always be so lucky: we will use the following simple lemma to study the general case.

Lemma 5.3.17. *Consider $w'_m, w'_{m+1} \in \overline{x'_1 x'_2}$ such that $d_{\Delta'}(x'_1, x'_{m+1}) < d_{\Delta'}(x'_1, x'_m)$. If there is a vertex $v' \in \overline{x'_1 x'_3}$ such that $d_{\Delta'}(v', w'_m) \leq d_{\Delta'}(v', w'_{m+1})$, then for every $p' \in \overline{v' x'_3}, \overline{w'_m x'_2}, \overline{x'_2 x'_3}$ it will also follow $d_{\Delta'}(p', w'_m) \leq d_{\Delta'}(p', w'_{m+1})$.*

Proof. We will first prove $d_{\Delta'}(p'_1, w'_m) \leq d_{\Delta'}(p'_1, w'_{m+1})$ for every $p'_1 \in \overline{v' w'_m}$. Suppose by contradiction there is a point $p'_2 \in \overline{v' w'_m}$ such that $d_{\Delta'}(p'_2, w'_m) > d_{\Delta'}(p'_2, w'_{m+1})$

and denote by $pr : \Delta' \to \overline{v'w'_{m+1}}$ the orthogonal projection on $\overline{v'w'_{m+1}}$. Then it would follow:

$$d_{\Delta'}(v', w'_{m+1}) = d_{\Delta'}(v', pr(p'_2)) + d_{\Delta'}(pr(p'_2), w'_{m+1}) \leq d_{\Delta'}(v', p'_2) + d_{\Delta'}(p'_2, w'_{m+1}) <$$

$$< d_{\Delta'}(v', p'_2) + d_{\Delta'}(p'_2, w'_m) = d_{\Delta'}(v', w'_m)$$

which contradicts the hypothesis.
For every point $p' \in \overline{v'x'_3}, \overline{w'_m x'_2}, \overline{x'_2 x'_3}$ then define $\widetilde{p} := \overline{p'w'_{m+1}} \cap \overline{v'w'_m}$ (notice that if $v'w'_m$ is not smooth then it can happen $p' = \widetilde{p}$ if $p' \in \overline{x'_2 x'_3}$). It results:

$$d_{\Delta'}(p', w'_{m+1}) = d_{\Delta'}(p', \widetilde{p}) + d_{\Delta'}(\widetilde{p}, w'_{m+1}) \geq d_{\Delta'}(p', \widetilde{p}) + d_{\Delta'}(\widetilde{p}, w'_m) \geq d_{\Delta'}(p', w'_m).$$

□

Notice that there can not be points w'_j in $\hat{T}_{w_m}$ or $\hat{T}_{w_{m+1}}$, otherwise there would be another change of order of the vertices.
One can apply the preceding lemma to make the following inferences.

- There can not be a point $u'_j \in \hat{T}_{w_{m+1}}$ and a point $v'_i \in \hat{T}_{w_m}$, since if it results $d_{\Delta'}(v'_i, w'_m) \leq d_{\Delta'}(v'_i, w'_{m+1})$ then it must follow $d_{\Delta'}(u'_j, w'_m) \leq d_{\Delta'}(u'_j, w'_{m+1})$.
- There can not be a point $v'_i \in \hat{T}_{w_{m+1}}$ and a point $u'_j \in \hat{T}_{w_m}$, since if it results $d_{\Delta}(v_i, w_{m+1}) \leq d_{\Delta}(v_i, w_m)$ then it must follow $d_{\Delta}(u_j, w_{m+1}) \leq d_{\Delta}(u_j, w_m)$ (since clearly there is an analogue version of the preceding lemma on Δ).
- There can not be a point $v'_i \in \hat{T}_{w_{m+1}}$ and a point $v'_j \in \hat{T}_{w_m}$, since they would be forced to have inverted order.

One can conclude that the two sets $\hat{T}_{w_{m+1}}, \hat{T}_{w_m}$ can not be both non-empty and consequently it is always possible to set $\hat{w}_m := w'_{m+1}$ or $\hat{w}_{m+1} := w'_m$.

□

Chapter 6

An hermitian metric on $\mathcal{H}_g(2g-2)$

In this last chapter we present a result which has some significance on the theory of moduli spaces of translation surfaces, although not being related to Thurston's metric.

Fix any genus $g \geq 2$ and consider the moduli space $\mathcal{H}_g(2g-2)$ of Abelian differentials with one zero.
For any $\varphi \in \mathcal{H}_g(2g-2)$ we denote by X_φ the complex structure on S_g with respect to which φ is an Abelian differential.
Every tangent space $T_\varphi \mathcal{H}_g(2g-2) \simeq H^1(X, \mathbb{C})$ can be endowed with the hermitian product h_φ of signature $(2g, 0)$, defined in the following way:

$$h_\varphi(\dot{\varphi}, \dot{\psi}) := \frac{1}{2} \int_{X_\varphi} \dot{\varphi} \wedge (\overline{*\dot{\psi}}) = \frac{i}{2} \left(\int_{X_\varphi} \dot{\varphi}^{1,0} \wedge \overline{\dot{\psi}^{1,0}} - \int_{X_\varphi} \dot{\varphi}^{0,1} \wedge \overline{\dot{\psi}^{0,1}} \right)$$

for every $\dot{\phi}, \dot{\psi} \in T_\varphi \mathcal{H}_g(2g-2)$.
In the preceding expression the decomposition in holomorphic and anti-holomorphic forms is done with respect to the complex structure X_φ and $*$ is the Hodge star operator of X_φ on complex differential forms:

$$* : \mathcal{A}^1_{\mathbb{C}}(X_\varphi) \to \mathcal{A}^1_{\mathbb{C}}(X_\varphi)$$

which in local coordinates $z = x + iy$ is defined by

$$*dx = dy \quad \textit{and} \quad *dy = -dx$$

and for every complex 1-form $\sigma = f dz + g d\overline{z}$ on X_φ it results

$$*\sigma := -if dz + ig d\overline{z}.$$

Endowing every tangent space $T_\varphi \mathcal{H}_g(2g-2)$ with the hermitian product h_φ, one obtains a well defined hermitian form h on $\mathcal{H}^1_g(2g-2)$.
We denote by Ω the corresponding skew symmetric 2-form

$$\Omega := -Im(h).$$

The present chapter is devoted to the proof of the following theorem.

Theorem 6.0.1. *The hermitian form h on $\mathcal{H}^1_g(2g-2)$ is not a Kähler form, since*

$$d\Omega \neq 0.$$

Proof. In order to being able to compute $d\Omega$ it is clearly necessary to first write down the hermitian product h in local coordinates on $\mathcal{H}_g(2g-2)$ and thus to consider the variation of the Hodge star operator to nearby surfaces.
We will perform the computations at first order.

Let $\{\varphi_1, \varphi_2, \dots, \varphi_{2g}\}$ be any basis of $H^1(X, \Sigma(\varphi), \mathbb{C}) \simeq H^1(X, \mathbb{C})$ (where $\Sigma(\varphi)$ is the zero of φ) made of complex differential forms which are zero in a neighborhood of $\Sigma(\varphi)$ in X_φ. For any $t = (t_1, \dots, t_{2g}) \in D \subset \mathbb{C}^{2g}$, where D is an open neighborhood of the origin, one obtains a first order deformation $\varphi(t)$ of φ in $\mathcal{H}_g(2g-2)$ as follows:

$$\varphi(t) = \varphi + \epsilon\dot{\varphi}(t) + o(\epsilon), \quad \dot{\varphi}(t) = \sum_{l=1}^{2g} t_l \varphi_l$$

and the coefficients t can be considered as local coordinates around φ.
We will be consistent with the previous notation and write $X_{\varphi(t)}$ to denote the complex structure with respect to which $\varphi(t)$ is an Abelian differential.
The hermitian form h can be written down explicitly in the point $\varphi(t)$ as:

$$h_{\varphi(t)} = \sum_{j,k=1}^{2g} h_{jk}(t) dt_i d\overline{t_j}$$

and, thanks to the fact that the forms φ_i vanish in a neighborhood of $\Sigma(\varphi)$, the coefficients $h_{jk}(t)$ are:

$$h_{jk}(t) = h_{\varphi(t)}(\varphi_j, \varphi_k) = \frac{1}{2} \int_{C_{\varphi(t)}} \varphi_j \wedge (*_t \overline{\varphi_k}),$$

where we wrote $*_t$ instead of $*_{X_{\varphi(t)}}$ to linghten the notation.
In order to being able to compute the derivatives of the coefficients $h_{jk}(t)$, we first have to find an explicit way to compute the Hodge operator $*_t$.

Given any Riemann surface X and a Beltrami differential μ on it with local coefficient ν, the following formulas, which can be found in $[Ro]$, allow one to compute the Hodge operator $*_{X_\mu}$ of the complex structure X_μ obtained deforming X with the Beltrami differential μ:

$$*_{C_\mu}(dz + \nu d\overline{z}) = -i(dz + \nu d\overline{z}), \tag{6.1}$$

$$*_{C_\mu} dz = \frac{-i(1+|\nu|^2)}{1-|\nu|^2} dz - \frac{2i\nu}{1-|\nu|^2} d\overline{z}. \tag{6.2}$$

In case of the first order deformation $\varphi(t)$, the complex structure $X_{\varphi(t)}$ can be obtained deforming X_φ with the Beltrami differential $\mu(t)$:

$$\mu(t) = \epsilon\dot{\mu}(t) + o(\epsilon), \quad \dot{\mu}(t) = \frac{\sum_{l=1}^{2g} t_l \varphi_l^{0,1}}{\varphi},$$

where $\dot{\mu}(t)$ has local coefficient $\dot{\nu}(t)$:

$$\dot{\nu}(t) = \frac{\sum_{i=1}^{2g} t_l f_l^{0,1}}{f}$$

and $f, f_l^{0,1}$ are the local coefficients of φ and $\varphi_l^{0,1}$:

$$\varphi = f dz, \quad \varphi_l = \varphi_l^{1,0} + \varphi_l^{0,1} = f_l^{1,0} dz + f_l^{0,1} d\overline{z}.$$

Applying formulas (6.1) and (6.2) to this case we obtain:

$$*_t dz = -idz - 2i\dot{\nu}(t) d\overline{z}\epsilon + o(\epsilon),$$

$$*_t d\overline{z} = id\overline{z} + 2i\overline{\dot{\nu}(t)} dz\epsilon + o(\epsilon)$$

and thus the following local expression for $h_{jk}(t)$:

$$h_{jk}(t) = h_{jk}^0(t) + \epsilon i h_{jk}^\epsilon(t) + o(\epsilon)$$

with

$$h_{jk}^0(t) = \frac{1}{2}\left(\int_{X_{\varphi(t)}} \varphi_j^{1,0} \wedge \overline{\varphi_k^{1,0}} - \int_{X_{\varphi(t)}} \varphi_j^{0,1} \wedge \overline{\varphi_k^{0,1}}\right),$$

$$h_{jk}^\epsilon(t) = \int_{X_{\varphi(t)}} \varphi_j \wedge \overline{f_k^{1,0}} \sum_{l=1}^{2g} \frac{\overline{t}_l \overline{f_l^{0,1}}}{\overline{f}} dz - \int_{X_{\varphi(t)}} \varphi_j \wedge \overline{f_k^{0,1}} \sum_{m=1}^{2g} \frac{t_m f_m^{0,1}}{f} d\overline{z}$$

which can be rewritten as

$$h_{jk}(t) = \frac{1}{2}\int_{X_{\varphi(t)}} \varphi_j \wedge (*\overline{\varphi_k}) + i\epsilon \sum_{l=1}^{2g} \left(\overline{t}_l \int_{X_{\varphi(t)}} \frac{\overline{f_k^{1,0}}}{\overline{f}} \varphi_j^{0,1} \wedge \overline{\varphi_l^{0,1}} - t_l \int_{X_{\varphi(t)}} \frac{\overline{f_k^{0,1}}}{f} \varphi_j^{1,0} \wedge \varphi_l^{0,1}\right) + o(\epsilon).$$

It is now possible to compute the derivatives of $h_{jk}(t)$ with respect to t_i and $\overline{t}_i$:

$$\frac{\partial}{\partial t_i} h_{jk}(t) = -i\epsilon \int_{X_{\varphi(t)}} \frac{\overline{f_k^{0,1}}}{f} \varphi_j^{1,0} \wedge \varphi_i^{0,1}$$

$$\frac{\partial}{\partial \bar{t}_i} h_{jk}(t) = i\epsilon \int_{X_{\varphi(t)}} \frac{\overline{f_k^{1,0}}}{\overline{f}} \varphi_j^{0,1} \wedge \overline{\varphi_i^{0,1}} = i\epsilon \int_{X_{\varphi(t)}} \frac{f_j^{0,1}}{\overline{f}} \overline{\varphi_k^{1,0}} \wedge \overline{\varphi_i^{0,1}}.$$

The skew symmetric 2-form $\Omega = -Im(h)$ has the following local expression in coordinates $t_1, \dots, t_{2g}$:

$$\Omega = \frac{i}{2} \sum_{j,k=1}^{2g} h_{jk}(t) dt_j \wedge d\bar{t}_k$$

and so it follows

$$d\Omega = \frac{i}{2} \sum_{i,j,k=1}^{2g} \frac{\partial}{\partial t_i} h_{jk}(t) dt_i \wedge dt_j \wedge d\bar{t}_k + \frac{\partial}{\partial \bar{t}_i} h_{jk}(t) d\bar{t}_i \wedge dt_j \wedge d\bar{t}_k.$$

It is clear that the existence of a triple i, j, k such that

$$\frac{\partial}{\partial t_i} h_{jk}(t) \neq \frac{\partial}{\partial t_j} h_{ik}(t)$$

will imply $d\Omega \neq 0$.
To this end we define

$$\Theta(i,j,k) := i\Big(\int_{X_{\varphi(t)}} \frac{\overline{f_k^{0,1}}}{f} \varphi_j^{1,0} \wedge \varphi_i^{0,1} - \int_{X_{\varphi(t)}} \frac{\overline{f_k^{0,1}}}{f} \varphi_i^{1,0} \wedge \varphi_j^{0,1} \Big) = i \int_{X_{\varphi(t)}} \frac{\overline{f_k^{0,1}}}{f} \varphi_j \wedge \varphi_i$$

and we will display a basis $\{\varphi_1, \dots, \varphi_{2g}\}$ of $H^1(X_\varphi, \mathbb{C})$ made of complex differential forms which vanish in a neighborhood of $\Sigma(\varphi)$ such that there is a triple i, j, k with $\Theta(i,j,k) \neq 0$.

Let $\{\sigma_1, \dots, \sigma_g\}$ be any basis of $H^0(X, \Omega^1_C)$ such that $\sigma_1 = \varphi$.
Choose a small neighborhood U of $\Sigma(\varphi)$ in X_φ and let $\iota : U \to X_\varphi$ be the inclusion map. Up to shrinking U, we can assume the existence, for every $i = 1, \dots, g$, of an holomorphic function $F_i : U \to \mathbb{C}$ such that $\iota^* \sigma_i = dF_i$.
Let V be another neighborhood of $\Sigma(\varphi)$ such that $V \subset U$. Pick a smooth function $\psi : X_\varphi \to \mathbb{R}$ such that $\psi|_V \equiv 1$ and $\psi|_{X \setminus U} \equiv 0$.
We define the basis $\{\varphi_1, \dots, \varphi_{2g}\}$ of $H^1(X, \mathbb{C})$ in the following way:

$$\varphi_i := \sigma_i - d(\psi F_i), \quad \varphi_{i+g} = \overline{\sigma_i} - d(\psi \overline{F}_i) \; for \; i = 1, \dots, g.$$

In this basis it will follow $\Theta(g+1, 1, g+1) \neq 0$. Indeed, proceeding with the computations we obtain

$$\Theta(g+1, 1, g+1) = i \int_{X_{\varphi(t)}} \frac{\overline{f_{g+1}^{0,1}}}{f} \varphi_1 \wedge \varphi_{g+1} = I_1 + I_2$$

with

$$I_1 = i\int_{X_{\varphi(t)}\setminus U} \varphi\wedge\overline{\varphi},$$

$$I_2 = i\int_{U\setminus V}\left(1-\psi-\frac{\partial_z(\psi)F_1}{f}\right)((\varphi-d(\psi F_1))\wedge(\overline{\varphi}-d(\psi\overline{F}_1))).$$

Since

$$i\int_{X_{\varphi(t)}\setminus U}\varphi\wedge\overline{\varphi}>0$$

we just have to evaluate I_2.
In a natural coordinate $z = x+iy$ near $\Sigma(\varphi)$ the differential φ and the function F_1 have the local expressions

$$\varphi = z^{2g-2}dz,\quad F_1 = \frac{z^{2g-1}}{2g-1}.$$

Substituting these expressions in I_2 we obtain

$$I_2 = i\int_{U\setminus V}\left(1-\psi-\frac{\partial_z(\psi)z}{2g-1}\right)\left(\left(1-2\psi+\psi^2+\frac{(\psi-1)}{2g-1}\left((\partial_z\psi)z+(\partial_{\overline{z}}\psi)\overline{z}\right)\right)|z|^{4g-4}dz\wedge d\overline{z}\right)$$

and consequently

$$I_2 = J_1+J_2+J_3$$

with

$$J_1 = i\int_{U\setminus V}|z|^{4g-4}(1-\psi)^3dz\wedge d\overline{z},$$

$$J_2 = \frac{-i}{2g-1}\int_{U\setminus V}|z|^{4g-4}(1-\psi)^2(2z(\partial_z\psi)+\overline{z}(\partial_{\overline{z}}\psi))dz\wedge d\overline{z},$$

$$J_3 = \frac{i}{(2g-1)^2}\int_{U\setminus V}|z|^{4g-4}(z^2(\partial_z\psi)^2+|z|^2(\partial_z\psi)(\partial_{\overline{z}}\psi))dz\wedge d\overline{z}.$$

We can choose U to be the open disk of center $\Sigma(\varphi)$ and radius r in the natural coordinates for φ, V to be the open disk of center $\Sigma(\varphi)$ and radius $r'<r$ and choose ψ to be

$$\psi = e^{\frac{-(|z|-r')^2}{(|z|-r)^2}}\text{ on } U\setminus V.$$

With these choices it will follow that all three summands J_1, J_2, J_3 are real and greater than zero.
In particular, since $1-\psi>0$ for $r'<|z|<r$ and $1-\psi=0$ for $|z|=r'$, it results

$$i\int_{U\setminus V}|z|^{4g-4}(1-\psi)^3dz\wedge d\overline{z}>0.$$

One could also simplify the integrand of J_2 as follows

$$2z(\partial_z\psi)+\overline{z}(\partial_{\overline{z}}\psi) = \frac{3}{2}(x(\partial_x\psi)+y(\partial_y\psi))+\frac{i}{2}(y(\partial_x\psi)-x(\partial_y\psi))$$

and since on $U \setminus V$ it results

$$x(\partial_x\psi) + y(\partial_y\psi) = -2(r'-r)|z|\psi\frac{(|z|-r')}{(|z|-r)^3} < 0, \quad y(\partial_x\psi) - x(\partial_y\psi) = 0$$

one obtains

$$\frac{-i}{2g-1}\int_{U\setminus V}|z|^{4g-4}(1-\psi)^2(2z(\partial_z\psi)+\overline{z}(\partial_{\overline{z}}\psi))dz\wedge d\overline{z} > 0.$$

Finally, since $z^2(\partial_z\psi)^2 + |z|^2(\partial_z\psi)(\partial_{\overline{z}}\psi)$ can be rewritten as

$$2(x(\partial_x\psi)+y(\partial_y\psi))^2 + 2i((y^2-x^2)(\partial_x\psi)(\partial_y\psi) + xy((\partial_x\psi)^2-(\partial_y\psi)^2))$$

and on $U \setminus V$ it results

$$x(\partial_x\psi)+y(\partial_y\psi) < 0, \qquad ((y^2-x^2)(\partial_x\psi)(\partial_y\psi) + xy((\partial_x\psi)^2-(\partial_y\psi)^2)) = 0$$

it finally follows

$$\frac{i}{(2g-1)^2}\int_{U\setminus V}|z|^{4g-4}(z^2(\partial_z\psi)^2+|z|^2(\partial_z\psi)(\partial_{\overline{z}}\psi))dz\wedge d\overline{z} > 0.$$

We have thus proved $\Theta(g+1,1,g+1) > 0$ and $d\Omega \neq 0$.

□

Bibliography

[AB] L.Ahlfors and L.Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. (2), 72:385-404,1960.

[Abi] W.Abikoff, *The real analytic theory of Techmüller space*, AMS, Lecture note in mathematics, Springer-Verlag,1980.

[Ah] L. Ahlfors, *Complex analysis*, McGraw-Hill.

[AKP] S.Alexander, V.Kapovitch and A.Petrunin, *Alexandrov meets Kirszbraun*, Proceedings of 17^{th} Gökova Geomtry-Topology Conference pp.88-109.

[Be] L.Bers, *Correction to "Spaces of Riemann surfaces as bounded domains"*, Bull. Amer. Math. Soc. 67 1961- 465-466.

[Be2] L.Bers, *Simultaneus uniformization*, Bulletin of the American Mathematical Society 66 (1960), 94-97.

[BH] M.R. Bridson, A. Haefliger, *Metric spaces of non positive-curvature*, Springer-Verlag, Berlin (1999).

[BPT] A. Belkhirat, A. Papadopoulos and M. Troyanov, *Thurston's weak metric on the Teichmüller space of the torus*, Transactions of the mathematical society, Volume 357, Number 8, Pages 3311-3324.

[Bu] H. Busemann, *Recent synthetic differential geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 54, Springer-Verlag, 1970.

[CDF] G.Casamiglia, B.Deroin and S.Francaviglia, *A transfer principle: from periods to isoperiodic foliations*, arxiv:1511.07635, 2015.

[DP] S.Daneri and A.Pratelli, *A planar bi-Lipschitz extension theorem*, Adv. Calc. Var. 8,3 (2015), 221-266.

[Eb] D.G.Ebin, *The manifold of Riemannian metrics*, Global Analysis, Proc. Sympos. Pure Math. 15, Amer. Math. Soc., Providence, R.I., 1970, 11-40.

[FLP] A.Fathi, F.Laundnbach and V.Poénaru, *Thurston's work on surfaces*, Mathematical Notes, vol. 48, Princeton University Press, Princeton, NJ, 2012.

[FM] B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton university press.

[FMa] G.Forni, C.Matheus, *Introduction to Teichmüller theory and its applications to dynamics of interval exchange transformations, flows on surfaces and billiards*, arXiv:1311.2758v1, 2013.

[GL] F.Gardiner and N.Lakic, *Quasiconformal Teichmüller theory*, Mathematical Surveys and Monographs, 76. American Mathematical Society, Providence, RI, 2000.

[Gr] H.Grötzsch, *Über die Verzerrung bei schlichten nichtkonformen Abbildungen und über eine damit zusammenhängende Erweiterung des Picardschen Satzes*, Ber. Verh. Sächs. Akad. Wiss. Math. Nat. Kl Leipzig 80 (1928), 503-507.

[He] E. Helly, *Über Mengen konvexer Kürper mit gemeinschaftlichen Punkten*, Jber. Deutschen Math. Verein vol.32 (1923).

[HM] J.Hubbard and H.Masur, *Quadratic differentials and foliations*, Acta Math., 142:221-274, 1979.

[Hu] J.H.Hubbard, *Teichmüller theory and applications to geometry, topology and dynamics*, Matrix editions.

[Iv] S.Ivanov, *On Helly's theorem in geodesic spaces*, Electron. Res. Announc. Math. Sci. 21 (2014), 109-112.

[Ki] M.D. Kirszbraun, *Über die zusammenziehende und Lipschitzsche Transformationen*, Fundamenta Math. 22 (1934), 77-108.

[KMS] S.Kerckhoff, H.Masur and J.Smillie, *Ergodicity of Billiard Flows and Quadratic Differentials*, Annals of Mathematics, second series, Vol.124, No2 (sep.1986) pp. 293,311.

[La] E.Lanneau, *Parity of the spin structure defined by a quadratic differential*, Geometry and Topology, Volume 8 (2004) 511-538.

[Le] G.Levitt, *Foliations and laminations on hyperbolic surfaces*, Topology 22 (1983) 119-135.

[Lei] C.J.Leininger, *Degenerations of hyperbolic structures on surfaces*, Geometry, topology and dynamics of character varieties, 2012, pp. 95-138.

[Ma] H.Masur, *Interval exchange transformations and measured foliations*, Annals of Math. 115 (1982), 169-200.

[Ma2] H.Masur, *Closed trajectories for quadratic differentials with an application to billiards*, Duke Math. J. 53 (1986), no.2, 307-314.

[Mah] K.Mahler, *On lattice points in n-dimensional star bodies. I. Existence theorems*, Proc. Roy. Soc. London. Ser. A., 187:151-187, 1946.

[Mc] C.T.McMullen, *Moduli spaces of isoperiodic forms on Riemann surfaces*, Duke Math. J., Volume 163, Number 12 (2014), 2271-2323.

[Mo] T.Morzadec, *Measured flat geodesic laminations*, arXiv:1501.04057, 2014.

[Mu] D.Mumford, *A remark on Mahler's compactness theorem*, Proc. Amer. Math. Soc. 28 (1971), 289-294.

[PT] A.Papadopoulos, G.Théret, *On the topology defined by Thuston's asymmetric metric*, Mathematical Proceedings, Cambridge University Press (CUP), 2007, 142 (3), pp.487-496.

[PT2] A.Papadopoulos, G.Theret, *On Theichmüller's metric and Thurston's asymmetric metric on Teichmüller space*, Handbook of Teichmüller Theory, Volume 1, 11, European Mathematical Society Publishing House, 2007, IRMA Lectures in Mathematics and Theoretical Physics.

[Ro] H.L.Royden, *The variation of harmonic differentials and their periods*, Complex analysis, Birkhäuser Verlag Basel 1988.

[St] K.Strebel, *Quadratic differentials*, Springer Verlag, 1984.

[Th] W.P.Thurston, *Minimal stretch maps between hyperbolic surfaces*, preprint, arXiv:math/9801039, 1986.

[Th2] W.P.Thurston, *The geometry and topology of Three-manifolds*, Mimeographed notes, Princeton University, 1976.

[Tr] M. Troyanov, *Les surfaces euclidiennes á singularités coniques*, Enseign. Math. (2) (1986) vol. 32 (1-2) pp. 79-94.

[Tr2] M. Troyanov, *On the Moduli Space of Singular Euclidean Surfaces*, chapter 12 of Handbook of Teichmüller theory, volume 1.

[Va] F.A. Valentine, *On the extension of a vector function so as to preserve a Lipschitz condition*, Bull. Amer. Math. Soc., vol. 49, 1943, pp 100-108.

[Ve] W.A.Veech, *Gauss measures for transformations on the space of interval exchange maps*, Annals of Math. 115 (1982), 201-242.

[Ve2] W.A.Veech, *Moduli spaces of quadratic differentials*, Journal d'Analyse Mathematique, december 1990, Volume 55, Issue 1, pp 117-171.

[Ve3] W.A.Veech, *Dynamical systems on analytic manifolds of quadratic differentials. I. F-strctures. H. Analytic manifolds of quadratic differentials*, Preprints, 1984.

[Wo] S.Wolpert, *The length spectra as moduli for compact Riemann surfaces*, Ann. of Math. (2), 109(2):323-351, 1979.

[Yo] J.C. Yoccoz, *Interval exchange maps and translation surfaces*, Clay Math. Proc. 10 (2010), 1-69, Amer.Math.Soc. Providence, RI.