# Estimation and Inference of Skew–Stable distributions using the Multivariate Method of Simulated Quantiles

## *Stima e inferenza per i parametri delle distribuzioni Stabili asimmetriche utilizzando il metodo dei quantili simulati*

Stolfi Paola and Bernardi Mauro and Petrella Lea

**Abstract** The multivariate method of simulated quantiles (MMSQ) is proposed as a likelihood–free alternative to indirect inference procedures that does not rely on an auxiliary model specification and its asymptotic properties are established. As a further improvement we introduce the Smoothly clipped absolute deviation (SCAD) $\ell_1$–penalty into the MMSQ objective function in order to achieve sparse estimation of the scaling matrix. We extend the asymptotic theory and we show that the sparse–MMSQ estimator enjoys the oracle properties under mild regularity conditions. The method is applied to estimate the parameters of the Skew Elliptical Stable distribution.

**Abstract** *In questo lavoro viene proposto il metodo dei quantili simulati multivariati che rappresenta una valida alterativa alle procedure di inferenza indiretta e che non richiede la specificazione di un modello ausiliario e vengono dimostrate le proprietà asintotiche dello stimatore. Allo scopo di indurre una stima sparsa della matrice di scala introduciamo inoltre la funzione di penalita SCAD all'interno della funzione obiettivo del metodo. Un ulteriore contributo e rappresentato dall'estensione della teoria asintotica nel caso sparso e dalla dimostrazione che lo stimatore soddisfa le proprieta ORACLE. Il metodo e applicato alla stima dei parametri della distribuzione Stabile asimmetrica.*

**Key words:** Directional quantiles; Sparse regularisation; Skew Elliptical Stable distribution.

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## 1 Introduction

In this paper we extend the method of simulated quantiles (MSQ) of Dominicy and Veredas (2013) to a multivariate framework (MMSQ). The method of simulated quantiles like alternative likelihood–free procedures is based on the minimisation of the distance between appropriate quantile–based statistics evaluated on the true and simulated data. The MMSQ effectively deals with distributions that do not admit moments of any order, like the $\alpha$–Stable or the Tukey lambda, without relying on the choice of a misspecified auxiliary model. The lack of a natural ordering in the multivariate setting requires a careful definition of the concept of quantile. Here, we rely on the notion of projectional quantile recently introduced by Hallin et al. (2010) and Kong and Mizera (2012). This notion of multivariate quantile makes the estimator flexible and it allows us to deal with non–elliptically contoured distributions. As a further improvement we introduce the smoothly clipped absolute deviation (SCAD) $\ell_1$–penalty of Fan and Li (2001) into the MMSQ objective function in order to achieve sparse estimation of the scaling matrix. The method is illustrated using several synthetic datasets from distributions for which alternative procedures are recognised to perform poorly, such as the Skew Elliptical Stable distribution (SESD) firstly mentioned by Branco and Dey (2001).

The remainder of the paper is structured as follows. Section 2 introduces the sparse MMSQ estimator. Section 3 defines the Skew–Elliptical distribution of Branco and Dey (2001) while Section 4 presents simulated–data experiments to assess the effectiveness of the proposed method. Section 5 concludes.

## 2 The Multivariate Method of Simulated Quantiles

Let:

*(i)* $\mathbf{Y} \in \mathbb{R}^d$ be a random variable with distribution function $F_{\mathbf{Y}}(\cdot, \vartheta)$, which depends on a vector of unknown parameters $\vartheta \subset \Theta \in \mathbb{R}^k$, and $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n)'$ be a vector of $n$ independent realisations of $\mathbf{Y}$;

*(ii)* $\mathbf{q}_{\vartheta}^{\tau,\mathbf{u}} = \left(q_{\vartheta}^{\tau_1\mathbf{u}}, q_{\vartheta}^{\tau_2\mathbf{u}}, \dots, q_{\vartheta}^{\tau_s\mathbf{u}}\right)$ be a $s \times 1$ vector of projectional quantiles at given confidence levels $\tau_i \in (0,1)$ with $i = 1, 2, \dots, s$, and $\mathbf{u} \in \mathbb{S}^{d-1}$;

*(iii)* $\Phi_{\mathbf{u},\vartheta} = \Phi\left(\mathbf{q}_{\vartheta}^{\tau,\mathbf{u}}\right)$ be a $b \times 1$ vector of quantile functions assumed to be continuously differentiable with respect to $\vartheta$ for all $\mathbf{Y}$ and measurable for $\mathbf{Y}$ and for all $\vartheta \subset \Theta$;

*(iv)* $\hat{\mathbf{q}}^{\tau,\mathbf{u}} = (\hat{q}^{\tau_1\mathbf{u}}, \hat{q}^{\tau_2\mathbf{u}}, \dots, \hat{q}^{\tau_s\mathbf{u}})$ and $\hat{\Phi}_{\mathbf{u}} = \Phi(\hat{\mathbf{q}}^{\tau,\mathbf{u}})$ be the corresponding sample counterparts;

and assume that $\Phi_{\mathbf{u},\vartheta}$ cannot be computed analytically but it can be empirically calculated on simulated data. At each iteration $j = 1, 2, \dots$ the MMSQ compute $\tilde{\Phi}_{\mathbf{u},\vartheta_j}^{R} = \frac{1}{R}\sum_{r=1}^{R} \tilde{\Phi}_{\mathbf{u},\vartheta_j}^{r}$, where $\tilde{\Phi}_{\mathbf{u},\vartheta_j}^{r}$ is the function $\Phi_{\mathbf{u},\vartheta}$ computed at the $r$–th simulation path from $F_{\mathbf{Y}}\left(\cdot, \vartheta^{(j)}\right)$. The parameters are subsequently updated by minimising the distance between the vector of quantile measures calculated on the true observations $\hat{\Phi}_{\mathbf{u}}$ and that calculated on simulated realisations $\tilde{\Phi}^R_{\mathbf{u},\vartheta_j}$. The subscript $\mathbf{u}$ denotes that those quantities depend on a set of directions that should be properly chosen. We establish consistency and asymptotic normality of the proposed estimator. The MMSQ estimator is then extended in order to achieve sparse estimation of a scaling matrix $\Sigma$. Specifically, the SCAD $\ell_1$–penalty of Fan end Li (2001) is introduced into the MMSQ objective function as follows

$$\hat{\vartheta} = \arg\min_{\vartheta} \left(\hat{\Phi}_{\mathbf{u}} - \tilde{\Phi}^R_{\mathbf{u},\vartheta}\right)' \mathbf{W}_{\vartheta} \left(\hat{\Phi}_{\mathbf{u}} - \tilde{\Phi}^R_{\mathbf{u},\vartheta}\right) + n \sum_{i<j} p_\lambda \left(|\sigma_{i,j}|\right), \tag{1}$$

where $\mathbf{W}_{\vartheta}$ is a $b \times b$ symmetric positive definite weighting matrix, $\Sigma = (\sigma_{i,j})^n_{i,j=1}$ is the scale matrix and $p_\lambda(\cdot)$ is the SCAD $\ell_1$–penalty. By setting the tuning parameter $\lambda = 0$, equation (1) reduces to non sparse MMSQ estimator. We extend the asymptotic theory and we show that the sparse–MMSQ estimator enjoys the oracle properties under mild regularity conditions.

## 3 Skew Elliptical Stable distribution

In this Section we define the quantile–based measures and the optimal directions $\mathbf{u} \in \mathbb{S}^{m-1}$ for the parameters of the SESD distribution $\mathbf{Y} \sim \mathsf{SESD}_m(\alpha, \xi, \Omega, \delta)$ introduced by Branco and Dey (2001). For the shape parameter $\alpha$, the locations $\xi_i$, the skewness parameters $\delta_i$ and scale parameters $\omega_{ii}$, $i = 1, 2, \ldots, m$ we consider

$$\begin{aligned}
\kappa_{\mathbf{u}} &= \frac{q_{0.95\mathbf{u}} - q_{0.05\mathbf{u}}}{q_{0.75\mathbf{u}} - q_{0.25\mathbf{u}}} \\
m_{\mathbf{u}} &= q_{0.5\mathbf{u}} \\
\gamma_{\mathbf{u}} &= \frac{q_{0.95\mathbf{u}} + q_{0.05,\mathbf{u}} - 2q_{0.5\mathbf{u}}}{q_{0.95\mathbf{u}} - q_{0.05\mathbf{u}}} \\
\varsigma_{\mathbf{u}} &= q_{0.75\mathbf{u}} - q_{0.25\mathbf{u}},
\end{aligned}$$

where $\mathbf{u} \in \mathbb{S}^{m-1}$ defines a relevant direction. Once the quantile–based measures have been selected, we need to identify the optimal directions for each parameter. Let us consider the locations first. Because of the presence of skewness, the median computed along the canonical directions is not a good quantile measure for the locations. Therefore, we consider a transformation of the data in order to remove the skewness. The properties of the Skew Elliptical Stable distribution imply that $\mathbf{Y}^- = -\mathbf{Y} \sim \mathsf{SESD}_m(\alpha, \xi, \Omega, -\delta)$ independent of $\mathbf{Y}$, therefore it holds

$$\mathbf{Z} = \frac{\mathbf{Y} + \mathbf{Y}^-}{\sqrt{2}} \sim \mathsf{SESD}_m\left(\alpha, \sqrt{2}\xi, \Omega, \mathbf{0}\right), \tag{2}$$

which means that the variable $\mathbf{Z}$ is symmetric and, up to a constant, it has the same location parameter of $\mathbf{Y}$. Therefore, we choose, as informative measure for the locations, the median of the transformed variable $\mathbf{Z}$ in equation (2). In order to estimate the remaining parameters, we consider univariate marginals that have Skew Elliptical Stable distribution, i.e., $Y_i \sim \mathsf{SESD}_1(\alpha, \xi_i, \omega_{ii}, \delta_i)$, by construction. The quantile–based measures for the shape, skewness and for the diagonal elements of the scale matrix $\omega_{ii}$ are then computed along the canonical directions.

Now we need to identify the optimal directions for the off–diagonal elements of the scale matrix $\Omega$. To this end, we consider the bivariate marginal variables $\mathbf{Y}_{ij} = (Y_i, Y_j)'$ for $1 \leq i < j \leq m$. It holds $\mathbf{Y}_{ij} \sim \mathsf{SESD}_2(\alpha, \xi_{ij}, \Omega_{ij}, \delta_{ij})$, where $\xi_{ij} = (\xi_i, \xi_j)'$ and $\Omega_{ij} = \begin{bmatrix} \omega_{ii} & \omega_{ij} \\ \omega_{ij} & \omega_{jj} \end{bmatrix}$, while $\delta_{ij} = (\delta_i, \delta_j)'$. Moreover, let $\mathbf{Y}_{ij}^{-} \sim \mathsf{SESD}_2(\alpha, \xi_{ij}, \Omega_{ij}, -\delta_{ij})$ independent of $\mathbf{Y}_{ij}$ and let us consider the same construction introduced for the locations, that is the random variable $\mathbf{Z}_{ij} = \frac{\mathbf{Y}_{ij} + \mathbf{Y}_{ij}^{-}}{\sqrt{2}}$, having distribution $\mathbf{Z}_{ij} \sim \mathsf{SESD}_2\left(\alpha, \sqrt{2}\xi_{ij}, \Omega_{ij}, \mathbf{0}\right)$. Since $\mathbf{Z}_{ij}$ is a symmetric variable we choose the optimal direction $\mathbf{u}^* \in \mathbb{S}^1$ such that

$$\mathbf{u}^* = \arg\max_{\mathbf{u} \in \mathbb{S}^1} \sqrt{\mathbf{u}' \Omega_{ij} \mathbf{u}}. \tag{3}$$

## 4 Simulated–data experiment

To illustrate the effectiveness of the MMSQ in dealing with parameters estimation of the SESD we consider a simulation example where we fix the dimension $m = 5$ and $\alpha = 1.70$, while the location, shape and scale parameters are $\xi = \mathbf{0}$, $\delta = (0, 0, 0, 0.9, 0.9)$ and

$$\Sigma_5^{\delta} = \begin{pmatrix} 0.25 & 0.25 & 0.4 & 0 & 0 \\ 0.25 & 0.5 & 0.4 & 0 & 0 \\ 0.4 & 0.4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2.55 \\ 0 & 0 & 0 & 2.55 & 4 \end{pmatrix}, \tag{4}$$

respectively. We consider two different sample sizes $n = 500, 2000$ and we fix the number of simulated paths $R = 5$. Simulation results over 100 replications are reported in Table 1. Table 1 reports the bias (BIAS), the standard deviation (SSD) and the empirical coverage probabilities (ECP) obtained over 100 replications of the simulation experiment. Our results show that the MMSQ estimator is always unbiased, indeed the BIAS is always less 0.15. The SSDs are always small, in particular for $n = 500$ it is always less then 0.5. Moreover, the empirical coverages are always in line with their expected values. In order to apply the sparse–MMSQ we consider a simulation example of dimension $m = 12$, with $n = 200$ and $R = 5$ where the location parameters are equal to zero, the shape parameters $\delta = (0, 0, 0.6, 0, 0, 0, 0, 0, 0, 0.6, 0.6, 0)'$, while we consider the same scale matrix as

| | | $n=500$ | | | $n=2000$ | | |
|---|---|---|---|---|---|---|---|
| Par. | True | BIAS | SSD | ECP | BIAS | SSD | ECP |
| $\alpha$ | **1.70** | -0.0068 | 0.0690 | 0.9500 | 0.0013 | 0.0320 | 0.9500 |
| $\delta_1$ | 0.00 | 0.0048 | 0.0067 | 0.9400 | 0.0022 | 0.0010 | 0.1100 |
| $\delta_2$ | 0.00 | 0.0048 | 0.0063 | 0.9500 | 0.0082 | 0.0016 | 0.0100 |
| $\delta_3$ | 0.00 | 0.0040 | 0.0038 | 0.9100 | 0.0013 | 0.0005 | 0.0600 |
| $\delta_4$ | 0.90 | -0.0116 | 0.1648 | 0.9700 | 0.0180 | 0.0185 | 0.8100 |
| $\delta_5$ | 0.90 | -0.0179 | 0.1649 | 0.9700 | 0.0167 | 0.0234 | 0.9200 |
| $\xi_1$ | 0.00 | 0.0016 | 0.0365 | 0.9600 | 0.0032 | 0.0218 | 0.9400 |
| $\xi_2$ | 0.00 | -0.0029 | 0.0534 | 0.9700 | 0.0023 | 0.0286 | 0.9400 |
| $\xi_3$ | 0.00 | 0.0093 | 0.0757 | 0.9400 | 0.0065 | 0.0393 | 0.9500 |
| $\xi_4$ | 0.00 | -0.0051 | 0.0703 | 0.9700 | 0.0041 | 0.0356 | 0.9500 |
| $\xi_5$ | 0.00 | -0.0059 | 0.1089 | 0.9200 | -0.0040 | 0.0618 | 0.9400 |
| $\omega_{11}^2$ | 0.2500 | -0.0126 | 0.0259 | 0.9400 | -0.0027 | 0.0140 | 0.9800 |
| $\omega_{22}^2$ | 0.5000 | 0.0184 | 0.0596 | 0.9200 | 0.0003 | 0.0261 | 0.9400 |
| $\omega_{33}^2$ | 1.0000 | 0.0038 | 0.0998 | 0.9700 | 0.0166 | 0.0538 | 0.9500 |
| $\omega_{44}^2$ | 2.0000 | -0.1397 | 0.3571 | 0.9300 | -0.1571 | 0.1700 | 0.8800 |
| $\omega_{55}^2$ | 4.0000 | -0.4342 | 0.6637 | 0.9100 | -0.1142 | 0.3980 | 0.9600 |
| $\omega_{12}$ | 0.7071 | -0.0438 | 0.1336 | 0.9400 | -0.0345 | 0.1055 | 0.9100 |
| $\omega_{13}$ | 0.8000 | -0.1043 | 0.1487 | 0.9200 | -0.0173 | 0.1050 | 0.9800 |
| $\omega_{14}$ | 0.00 | 0.0075 | 0.0256 | 0.9300 | 0.0018 | 0.0148 | 0.9400 |
| $\omega_{15}$ | 0.00 | 0.0085 | 0.0445 | 0.9700 | 0.0040 | 0.0170 | 0.9400 |
| $\omega_{23}$ | 0.5657 | -0.0851 | 0.1680 | 0.9300 | -0.0323 | 0.1255 | 0.9700 |
| $\omega_{24}$ | 0.00 | 0.0049 | 0.0306 | 0.9600 | 0.0032 | 0.0154 | 0.9200 |
| $\omega_{25}$ | 0.00 | 0.0076 | 0.0414 | 0.9300 | 0.0053 | 0.0172 | 0.9600 |
| $\omega_{34}$ | 0.00 | 0.0047 | 0.0277 | 0.9100 | 0.0022 | 0.0151 | 0.9300 |
| $\omega_{35}$ | 0.00 | 0.0100 | 0.0332 | 0.9500 | 0.0032 | 0.0151 | 0.9300 |
| $\omega_{45}$ | 0.9016 | -0.0727 | 0.0785 | 0.8200 | -0.0552 | 0.0573 | 0.9300 |

**Table 1** Bias (BIAS), sample standard deviation (SSD), and empirical coverage probability (ECP) at the 95% confidence level for the locations $\mu = (\mu_1, \mu_2, \ldots, \mu_d)$, scale matrix $\Omega = \{\omega_{ij}\}$, with $i, j = 1, 2, \ldots, d$ and $i \le j$, tail parameter $\alpha = 1.70$ and skewness parameter $\delta_i$, $i = 1, 2, \ldots, d$ of the Skew Elliptical Stable distribution in dimension 5. The results reported above are obtained using 100 replications.

in Wang (2015) and reported below

$$\Sigma_{12}^s =$$
$$\begin{pmatrix}
0.239 & 0.117 & 0 & 0 & 0 & 0 & 0 & 0.031 & 0 & 0 & 0 & 0 \\
0.117 & 1.554 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.362 & 0.002 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.002 & 0.199 & 0.094 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.094 & 0.349 & 0 & 0 & 0 & 0 & 0 & 0 & -0.036 \\
0 & 0 & 0 & 0 & 0 & 0.295 & -0.229 & 0.002 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.229 & 0.715 & 0 & 0 & 0 & 0 & 0 \\
0.031 & 0 & 0 & 0 & 0 & 0.002 & 0 & 0.164 & 0.112 & -0.028 & -0.008 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.112 & 0.518 & -0.193 & -0.09 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.028 & -0.193 & 0.379 & 0.167 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.008 & -0.09 & 0.167 & 0.159 & 0 \\
0 & 0 & 0 & 0 & -0.036 & 0 & 0 & 0 & 0 & 0 & 0 & 0.207
\end{pmatrix}. \tag{5}$$

As regards the simulated example in dimension $m = 12$ we plot in Figure 1 the images displaying the band structure of the true estimated scale matrices are very close.

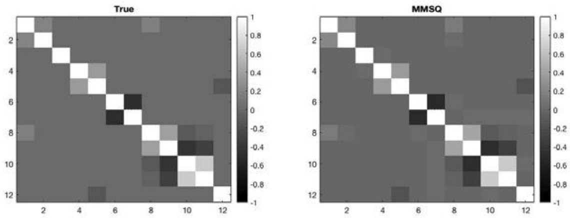

**Fig. 1** Images displaying the band structure of the true *(left)* and estimated *(right)* scale matrices of the simulated example in dimension $m = 12$.

## 5 Conclusion

In this paper the problem of parameter estimation and inference of Skew–Stable distributions has been approached using the multivariate method of simulated quantiles. Moreover, since as the number of dimensions increases the course of dimensionality problem prevents any effective inferential procedure we introduce the sparse–MMSQ estimator and we prove that the estimator enjoys the oracle properties under mild regularity conditions. The MMSQ and the sparse–MMSQ have been applies to the problem of estimating the parameters of the multivariate Skew–Stable distribution introduced by Branco and Dey (2001). Our simulation results show that the proposed methodology effectively achieve sparse estimation of the scale parameter.

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