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Calculus of Variations

Reinforcement problems for variational inequalities on fractal sets

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Abstract The aim of this paper is to study reinforcement problems for variational inequalities of the obstacle type on fractal sets.

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1 Introduction

The aim of this paper is to study reinforcement problems for variational inequalities on fractal sets.

The obstacle problem is a classic motivating example in the mathematical study of variational inequalities and free boundary problems. The problem entails finding the equilibrium position of an elastic membrane whose boundary is held fixed, and which is constrained to lie above (or under) a given obstacle. It is closely related to the study of minimal surfaces as well as to the capacity of a set in potential theory. Applications include the study of elasto-plasticity, optimal control, fluid filtration in porous media, constrained heating, and financial mathematics (see, for example, [8]).

The theory of variational inequalities starts from the paper [31] of Stampacchia: it was in fact in this paper that the name "variational inequalities" was introduced. The theory was subsequently further developed in the paper [22], in the book [20] and, later, in many papers and books (we refer to [8,32], and the references therein).

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In this paper, we state the existence, uniqueness and approximation results for variational solutions of obstacle problems on domains with a fractal boundary. The *fractal* solution can be approximated by solutions of obstacle problems related to the same operator in polygonal domains whose boundaries develop, at the limit, fractal geometry (extending the results of [9] and in [10]).

However, in this paper we prefer to adopt the approach of reinforcement by means of thin insulating layers according to the approach adopted in the celebrated paper by Brezis, Caffarelli and Friedman ([3]). We mention for related results the contributions [1,2,5–7,21, 30]. All these papers concern smooth domains (at least Lipschitz) and use tools and methods that can not be extended to domains with a fractal boundary.

The homogenization theory for domains with a fractal boundary have been developed in [26–28], and [29] for highly conductive layers and in [11,13], and [14] for insulating layers. We also wish to mention [4,15], where the reinforcement has a different structure.

A peculiar aspect of insulating layers is the loss of coerciveness of the energy functionals; moreover, in the case of fractals, this aspect is combined with the tricky geometry of the fibers. We overcome these difficulties by using some delicate tools such as extension theorems for (ϵ, δ) domains established by Jones (see [17]), sharp quantitative trace results (on polygonal curves) in terms of the increasing numbers of sides (see [12]), and establishing Poincaré type estimates adapted to the geometry (see Theorems 3.2, 7.1, 7.4). We study obstacle problems for both coercive energy forms and semi-coercive energy forms.

We wish to point out that Theorems 5.1 and 5.2 improve the results of [11] and [13] insofar as the hypotheses are weaker and the convergence of the approximating solutions is stated in a more precise way (see Remark 5.1). Theorems 5.3 and 7.5 concern the semicoercive case and the relative results are completely new to our knowledge. Moreover, simple examples in Sect. 6 show that our results are sharp.

We note that the fractal setting gives rise to a peculiar phenomenon. Owing to their tricky geometry, the reinforced domains have to be constructed starting from suitable *inner* polygonal domains Ω^n. Hence the forms a_n [see (3.3) and (3.5)] vanish in a part of the fractal domain (in $\Omega\backslash\Omega^n$) when the thickness of the layers goes to zero and, for fixed n, the forms a_n degenerate at the vertices of the polygonal curves $K^n_{j,\alpha}$, $j = 1, 2, 3$. Consequently, the *reinforced* solutions have gradients that are not uniformly bounded in L^2-norms (in $\Omega\backslash\Omega^n$), which is in contrast to the strong convergence of the gradients (in L^2) established in [3]; compare Theorem 5.3 with Theorem 9.3 in [3] (see Remark 5.4).

The plan of the paper is as follows. In Sect. 2, we introduce the fractal domains, we set up some obstacle problems and we state existence and uniqueness results of variational solutions. Section 3 is devoted to constructing suitable reinforced problems and to proving existence and uniqueness of the related solutions. In Sect. 4, we state Mosco-convergence of the related functionals. Section 5 concerns the asymptotic results. In Sect. 6, we comment on our results by discussing some simple examples. Finally, in Sect. 7 we deal with interior reinforcement.

2 Obstacle problems on fractal domains

First, we introduce the fractal domains in which we consider the obstacle problems. We recall the definition of the Koch curve with endpoints $A = (0, 0)$, and $B = (1, 0)$. We consider the family $\Psi^\alpha = \{\psi^\alpha_1, \dots, \psi^\alpha_4\}$ of contractive similitudes $\psi^\alpha_i : \mathbb{C} \to \mathbb{C}$, $i = 1, \dots, 4$, with contraction factor α^{-1}, $2 < \alpha < 4$,

$$\psi_1^\alpha(z) = \frac{z}{\alpha}, \quad \psi_2^\alpha(z) = \frac{z}{\alpha}e^{i\theta(\alpha)} + \frac{1}{\alpha},$$
$$\psi_3^\alpha(z) = \frac{z}{\alpha}e^{-i\theta(\alpha)} + \frac{1}{2} + i\sqrt{\frac{1}{\alpha} - \frac{1}{4}}, \quad \psi_4^\alpha(z) = \frac{z-1}{\alpha} + 1,$$

where $\theta(\alpha) = \arcsin(\frac{\sqrt{\alpha(4-\alpha)}}{2})$. According to the general theory of self-similar fractals (see [16,19]), there exists a unique closed bounded set K_α which is *invariant* with respect to Ψ^α, that is,

$$K_\alpha = \cup_{i=1}^4 \psi_i^\alpha(K_\alpha). \tag{2.1}$$

We recall that K_α supports a unique self-similar Borel measure μ_α, which is equivalent to the d_f-dimensional Hausdorff measure where $d_f = \frac{\log 4}{\log \alpha}$. Let K^0 be the line segment of unit length that has as endpoints $A = (0, 0)$ and $B = (1, 0)$. We set, for each n in $\mathbb{N}$,

$$K_\alpha^1 = \bigcup_{i=1}^4 \psi_i^\alpha(K^0), \quad K_\alpha^2 = \bigcup_{i=1}^4 \psi_i^\alpha(K_\alpha^1), \ldots, K_\alpha^{n+1} = \bigcup_{i=1}^4 \psi_i^\alpha(K_\alpha^n); \tag{2.2}$$

K_α^n is the so-called nth pre-fractal curve. Moreover, the iterates K_α^n converge to the self-similar set K_α in the Hausdorff metric, when n tends to infinity. Let Ω^0 be the triangle with vertices $A = (0, 0)$, $B = (1, 0)$, and $C = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$. We construct, on the side with endpoints A and B, the pre-fractal Koch curve defined above, which will be denoted by $K_{1,\alpha}^n$ and the Koch curve defined above, which will be denoted by $K_{1,\alpha}$. Similarly, we construct on the other sides the analogous pre-fractal Koch curves (the Koch curves) denoting by $K_{2,\alpha}^n$ and $K_{3,\alpha}^n$ (by $K_{2,\alpha}$ and $K_{3,\alpha}$) the curves with endpoints B and C, and C and A, respectively. We denote by Ω_α^n the pre-fractal domain that is the set bounded by the pre-fractal Koch curves $K_{j,\alpha}^n$, $j = 1, 2, 3$. Moreover, we denote by Ω_α the snowflake that is the set bounded by the Koch curves $K_{j,\alpha}$, $j = 1, 2, 3$. From now on, we omit α when it does not give rise to misunderstanding, by writing simply Ω instead of Ω_α and similar expressions.

In the following theorems, we state the existence and uniqueness of the variational solution of obstacle problems on the domain Ω. We consider the following bilinear forms

$$a_\infty(u, v) := \int_\Omega \sum_{i,j=1}^2 a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dxdy + \delta_0 \int_\Omega u\, v\, dxdy \tag{2.3}$$

with domain $H_0^1(\Omega)$ and

$$a_{c_0}(u, v) := \int_\Omega \sum_{i,j=1}^2 a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dxdy + \delta_0 \int_\Omega u\, v\, dxdy + c_0 \int_{\partial\Omega} u\, v\, d\mu \tag{2.4}$$

with domain $H^1(\Omega)$. In the last integral, μ is the measure on $\partial\Omega$ that coincides, on each K_j $j = 1, 2, 3$, with the Hausdorff measure defined in this section previously and u and v denote the traces of the functions u and v on the boundary of Ω. Here

$$\delta_0 \geq 0 \tag{2.5}$$

and the coefficients a_{ij} with $1 \leq i, j \leq 2$ satisfy

$$a_{ij} = a_{ji} \quad \forall i, j, \text{ and } \lambda|\xi|^2 \leq \sum_{i,j=1}^2 a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \tag{2.6}$$

for $\Lambda \geq \lambda > 0$. Furthermore, we assume the following condition

$$\mathcal{K}_0 = \{u \in H_0^1(\Omega), u \geq \varphi_1\}, \quad \varphi_1 \in C^1(\bar{\Omega}), \quad \varphi_1 \leq 0 \quad \text{on} \quad \partial\Omega. \tag{2.7}$$

We state the following result. Since the proof is similar to the proof of Theorem 2.2, we skip it. From now on, when it does not give rise to misunderstanding, we denote by C possibly different constants.

Theorem 2.1 *Let us assume (2.5)–(2.7). Then, for any $f \in L^2(\Omega)$, there exists one and only one solution u to the following problem*

$$\begin{cases} \textit{find} \quad u \in \mathcal{K}_0 \quad \textit{such that} \\ a_\infty(u, v-u) \geq \int_\Omega f\,(v-u)\,dxdy \quad \forall\, v \in \mathcal{K}_0 \end{cases} \tag{2.8}$$

where $a_\infty(\cdot,\cdot)$ is defined in (2.3). Moreover, u is the only function that realizes the minimum of the energy functional

$$\min_{v \in \mathcal{K}_0} \left\{ a_\infty(v,v) - 2\int_\Omega f\,v\,dxdy \right\} \tag{2.9}$$

and

$$||u||_{H^1(\Omega)} \leq C(||f||_{L^2(\Omega)} + ||\varphi_1||_{C^1(\overline{\Omega})}). \tag{2.10}$$

Remark 2.1 A similar result holds for the two obstacle problems where

$$\varphi_h \in C^1(\bar{\Omega}), \quad h = 1,2 \quad \varphi_1 \leq \varphi_2, \quad \varphi_1 \leq 0 \leq \varphi_2 \quad \text{on} \quad \partial\Omega \tag{2.11}$$

and

$$\mathcal{K}_0^* = \{u \in H_0^1(\Omega), \quad \varphi_1 \leq u \leq \varphi_2\}.$$

Moreover, the following estimates hold

$$||u||_{H^1(\Omega)} \leq C(||f||_{L^2(\Omega)} + \min(||\varphi_1||_{C^1(\overline{\Omega})}, ||\varphi_2||_{C^1(\overline{\Omega})})) \tag{2.12}$$

$$||u||_{L^\infty(\Omega)} \leq \max(||\varphi_1||_{C^0(\overline{\Omega})}, ||\varphi_2||_{C^0(\overline{\Omega})}). \tag{2.13}$$

Now we assume that

$$c_0 \geq 0, \; \delta_0 \geq 0, \quad \text{and} \quad \max(c_0, \delta_0) > 0. \tag{2.14}$$

Let

$$\mathcal{K} = \{u \in H^1(\Omega), \quad u \geq \varphi_1\}, \quad \varphi_1 \in C^1(\bar{\Omega}). \tag{2.15}$$

Theorem 2.2 *Let us assume (2.6), (2.14), (2.15) and $d \in \mathbb{R}$. Then, for any $f \in L^2(\Omega)$, there exists one and only one solution u to the following problem*

$$\begin{cases} \textit{find} \quad u \in \mathcal{K} \quad \textit{such that} \\ a_{c_0}(u, v-u) \geq \int_\Omega f\,(v-u)\,dxdy + d\int_{\partial\Omega}(v-u)\,d\mu \quad \forall\, v \in \mathcal{K} \end{cases} \tag{2.16}$$

where $a_{c_0}(\cdot,\cdot)$ is defined in (2.4). Moreover, u is the only function that realizes the minimum of the energy functional

$$\min_{v \in \mathcal{K}} \left\{ a_{c_0}(v,v) - 2\int_\Omega f\,v\,dxdy - 2d\int_{\partial\Omega} v\,d\mu \right\} \tag{2.17}$$

and

$$||u||_{H^1(\Omega)} \leq C(||f||_{L^2(\Omega)} + |d| + ||\varphi_1||_{C^1(\overline{\Omega})}). \tag{2.18}$$

Proof By Trace Theorem of Chapter V in [18] and Extension Theorem in [17], we have that

$$\left| d \int_{\partial\Omega} v \, d\mu \right| \leq C|d| \, ||v||_{H^1(\Omega)}$$

and

$$\left| \int_{\Omega} f \, v \, dxdy \right| \leq ||f||_{L^2(\Omega)} ||v||_{L^2(\Omega)}.$$

The bilinear form $a_{c_0}(u, v)$ is continuous. Indeed, by using Trace Theorem (see Chapter V in [18]) again

$$|a_{c_0}(u, v)| \leq (\max(\delta_0, \Lambda) + c_0 C)||u||_{H^1(\Omega)} ||v||_{H^1(\Omega)}.$$

Moreover, the form is coercive. In fact, if $\delta_0 > 0$ we obtain trivially

$$a_{c_0}(v, v) \geq \min(\delta_0, \lambda)||v||^2_{H^1(\Omega)}.$$

Instead, if $\delta_0 = 0$ and $c_0 > 0$ by using generalized Poincaré inequality (see Lemma 3.1.1 in [23]), we obtain that

$$a_{c_0}(v, v) \geq C \min(c_0, \lambda)||v||^2_{H^1(\Omega)}.$$

□

Remark 2.2 A similar result holds for the two obstacle problems where

$$\varphi_h \in C^1(\bar{\Omega}), \quad h = 1, 2 \quad \text{and} \quad \varphi_1 \leq \varphi_2 \tag{2.19}$$

and

$$\mathcal{K}^* = \{u \in H^1(\Omega), \quad \varphi_1 \leq u \leq \varphi_2\}. \tag{2.20}$$

Moreover, the following estimates hold

$$||u||_{H^1(\Omega)} \leq C(||f||_{L^2(\Omega)} + |d| + \min(||\varphi_1||_{C^1(\overline{\Omega})}, ||\varphi_2||_{C^1(\overline{\Omega})})) \tag{2.21}$$

$$||u||_{L^\infty(\Omega)} \leq \max(||\varphi_1||_{C^0(\overline{\Omega})}, ||\varphi_2||_{C^0(\overline{\Omega})}). \tag{2.22}$$

If $c_0 = \delta_0 = 0$, we can prove similar results by assuming further conditions on the data f, d and on the convex. For the sake of simplicity, we take

$$0 \in \mathcal{K} \tag{2.23}$$

and

$$\int_{\Omega} f \, dxdy + d \int_{\partial\Omega} d\mu < 0. \tag{2.24}$$

Theorem 2.3 *Let us assume $d \in \mathbb{R}$, (2.6), (2.15), (2.23), and (2.24). Then, for any $f \in L^2(\Omega)$, there exists one and only one solution u to the following problem*

$$\begin{cases} \text{find } \; u \in \mathcal{K} \; \text{ such that} \\ a(u, v - u) \geq \int_{\Omega} f \, (v - u) \, dxdy + d \int_{\partial\Omega} (v - u) \, d\mu \quad \forall \, v \in \mathcal{K} \end{cases} \tag{2.25}$$

where $a(u,v)=\int_\Omega \sum_{i,j=1}^2 a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dxdy$. *Moreover, u is the only function that realizes the minimum of the energy functional*

$$\min_{v\in\mathcal{K}}\left\{a(v,v)-2\int_\Omega f\,v\,dxdy-2d\int_{\partial\Omega} v\,d\mu\right\} \tag{2.26}$$

and

$$||\nabla u||_{L^2(\Omega)}\le C(||f||_{L^2(\Omega)}+|d|+||\varphi_1||_{C^1(\overline{\Omega})}). \tag{2.27}$$

Proof The existence can be proved as in Theorem 5.1 in [22] (see also Theorem 4.7 in [32]). We show the uniqueness by contradiction. Let u_1 and u_2 be two solutions of (2.25). As

$$a(u_1,u_1-u_2)\le\int_\Omega f\,(u_1-u_2)\,dxdy+d\int_{\partial\Omega}(u_1-u_2)\,d\mu \tag{2.28}$$

and

$$a(u_2,u_2-u_1)\le\int_\Omega f\,(u_2-u_1)\,dxdy+d\int_{\partial\Omega}(u_2-u_1)\,d\mu \tag{2.29}$$

we obtain

$$\lambda||\nabla(u_2-u_1)||^2_{L^2(\Omega)}\le a(u_1-u_2,u_1-u_2)\le 0 \tag{2.30}$$

and therefore $u_1=u_2+k_2$. From (2.28) we deduce

$$0=a(u_1,k_2)\le\int_\Omega f\,k_2\,dxdy+d\int_{\partial\Omega}k_2\,d\mu:$$

then, from (2.24),we obtain that $k_2\le 0$. From (2.29) we deduce

$$0=a(u_2,-k_2)\le-\int_\Omega f\,k_2\,dxdy-d\int_{\partial\Omega}k_2\,d\mu:$$

then, from (2.24), we obtain that $-k_2\le 0$: therefore, $k_2=0$. □

Remark 2.3 Similar results hold for the obstacle problem where

$$\mathcal{K}^\#=\{u\in H^1(\Omega),\quad u\le\varphi_2\},\quad \varphi_2\in C^1(\bar{\Omega}) \tag{2.31}$$

by assuming

$$0\in\mathcal{K}^\# \tag{2.32}$$

and

$$\int_\Omega f\,dxdy+d\int_{\partial\Omega}d\mu>0. \tag{2.33}$$

Remark 2.4 Analogous results hold for the two obstacle problems where $\mathcal{K}^*$ is defined in (2.20) assuming condition (2.19) and

$$0\in\mathcal{K}^*. \tag{2.34}$$

Moreover, the following estimates hold

$$||\nabla u||_{L^2(\Omega)}\le C(||f||_{L^2(\Omega)}+|d|+\min(||\varphi_1||_{C^1(\overline{\Omega})},||\varphi_2||_{C^1(\overline{\Omega})})) \tag{2.35}$$

$$||u||_{L^\infty(\Omega)}\le\max(||\varphi_1||_{C^0(\overline{\Omega})},||\varphi_2||_{C^0(\overline{\Omega})}). \tag{2.36}$$

3 Reinforcement for variational inequalities

We denote by Σ_1^0 the open triangle of vertices $A = (0, 0)$, $B = (1, 0)$ and $C = (1/2, b/2)$ where $b = \tan(\frac{\theta}{2})$. For each integer $n > 0$ we denote by

$$\psi_{i|n} = \psi_{i_1} \circ \psi_{i_2} \circ \cdots \circ \psi_{i_n}$$

the map associated with arbitrary $n-$tuple of indices $i|n = (i_1, i_2, \ldots, i_n) \in \{1, \ldots, 4\}^n$. If $n = 0$ we define $\psi_{i|n}$ to be the identity map in $\mathbb{R}^2$. For every set $\mathcal{O} \subseteq \mathbb{R}^2$, we define $\mathcal{O}^{i|n} = \psi_{i|n}(\mathcal{O})$, and, occasionally, we call $i|n$ the $n-$address of the set $\mathcal{O}^{i|n}$. With this notation, the polygonal curve K_α^n defined in (2.2) can be written $K_\alpha^n = \bigcup_{i|n} \psi_{i|n}(K^0)$. The triangle Σ_1^0 satisfies the *open set condition* with respect to the maps Ψ, that is, $\psi_{i|n}(\Sigma_1^0) \subset \Sigma_1^0$ for every $i|n$ and $\psi_{i|n}(\Sigma_1^0) \cap \psi_{j|n}(\Sigma_1^0) = \emptyset$ for every $i|n \neq j|n$ (see, for details, [16,19]).

For every $n \in \mathbb{N} \cup \{0\}$, we define the fiber $\Sigma_{1,\alpha}^n$, of $K_{1,\alpha}^n$ to be the (open) set

$$\Sigma_{1,\alpha}^n = \bigcup_{i|n} \Sigma_{1,\alpha}^{i|n},$$

where

$$\Sigma_{1,\alpha}^{i|n} = \psi_{i|n}(\Sigma_1^0).$$

We proceed in a similar way in order to construct the fiber $\Sigma_{j,\alpha}^n$, $(j = 2, 3)$, and we define the fiber Σ^n,

$$\Sigma^n = \Sigma_\alpha^n = \bigcup_{j=1}^{3} \Sigma_{j,\alpha}^n = \bigcup_{j=1}^{3} \Sigma_j^n$$

and

$$\widehat{\Omega}^n = \widehat{\Omega}_\alpha^n := \mathrm{int}\left(\overline{\Omega_\alpha^n} \bigcup \Sigma_\alpha^n\right).$$

We note that

$$\Omega^n \subset \Omega^{n+1} \subset \Omega \subset \widehat{\Omega}^{n+1} \subset \widehat{\Omega}^n.$$

We define a *weight* w^n as follows. Let P—for some $i|n$—belong to $\partial(\Sigma_1^{i|n}) \backslash (K^0)^{i|n}$ and let $P^\perp$ be the orthogonal projection of P on $(K^0)^{i|n}$. If (x, y) belongs to the segment with end-points P and $P^\perp$, we set, in our current notation,

$$w_1^n(x, y) = \frac{3|P - P^\perp|}{3 + b^2},$$

where $|P - P^\perp|$ is the (Euclidean) distance between P and $P^\perp$ in $\mathbb{R}^2$. We proceed in a similar way to construct the weights w_j^n on Σ_j^n $(j = 2, 3)$ and we define w^n on $\widehat{\Omega}^n$

$$w^n(x, y) = \begin{cases} w_j^n(x, y) & \text{if } (x, y) \in \Sigma_j^n \\ 1 & \text{if } (x, y) \in \overline{\Omega}^n. \end{cases} \tag{3.1}$$

Associated with the weight w^n, we consider the Sobolev spaces $H^1(\widehat{\Omega}^n; w^n)$ and $H_0^1(\widehat{\Omega}^n; w^n)$, defined as the completion of $C^\infty(\overline{\widehat{\Omega}^n})$ and $C_0^\infty(\widehat{\Omega}^n)$, respectively, in the norm

$$\|u\|_{H^1(\widehat{\Omega}^n; w^n)} = \left\{\int_{\widehat{\Omega}^n} u^2 dxdy + \int_{\widehat{\Omega}^n} |\nabla u|^2 w^n \, dxdy\right\}^{\frac{1}{2}}. \tag{3.2}$$

We define the coefficients

$$a^n_{ij}(x,y) = \begin{cases} \delta_{ij} c_n \sigma_n\, w^n(x,y) & \text{if } (x,y) \in \Sigma^n \\ a_{ij} & \text{if } (x,y) \in \overline{\Omega}^n, \end{cases} \tag{3.3}$$

where δ_{ij} denotes the Kroneker symbol, a_{ij} satisfy (2.6) and

$$c_n > 0, \quad \sigma_n = \frac{\alpha^n}{4^n}. \tag{3.4}$$

We consider the bilinear form associated with the reinforcement problem

$$a_n(u,v) := \int_{\widehat{\Omega}^n} \sum_{i,j=1}^{2} a^n_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dxdy + \delta_n \int_{\widehat{\Omega}^n} u\, v\, dxdy \tag{3.5}$$

where the coefficients a^n_{ij} are defined in (3.3), (3.4), and $\delta_n \geq 0$. For every n, for $h = 1, 2$, we define $(\varphi_h)_{1,n}$ on $\overline{\Sigma^n_1} = \overline{\bigcup_{i|n} \Sigma^{i|n}_1}$

$$(\varphi_h)_{1,n}(x,y) = G_1((\varphi_h) \circ \psi_{i|n}) \circ \psi^{-1}_{i|n}(x,y) \quad \text{if } (x,y) \in \overline{\Sigma^{i|n}_1} \tag{3.6}$$

where G_1 is the operator from $Lip(K^0)$ to $Lip(\overline{\Sigma^0_1})$ defined in the following way. For every $\zeta \in (0,1)$, we define $P_+ = P_+(\zeta) = (\zeta, \widehat{\eta}_+(\zeta)) \in \partial\Sigma^0_1$ to be the intersection of $\partial\Sigma^0_1 \backslash K^0$ with the vertical line through the point $(\zeta, 0) \in K^0$. Then, for a given $g \in Lip(K^0)$ we put

$$G_1(g)(\zeta,\eta) = \begin{cases} g(0,0) & \text{if } (\zeta,\eta) = (0,0) \\ g(\zeta,0)\frac{\widehat{\eta}_+ - \eta}{\widehat{\eta}_+} & \text{if } (\zeta,\eta) \in \overline{\Sigma^0_1} \backslash \{A, B\} \\ g(1,0) & \text{if } (\zeta,\eta) = (1,0). \end{cases} \tag{3.7}$$

We construct, in a similar way, G_j on $\overline{\Sigma^0_j}$ and $(\varphi_h)_{j,n}$ on $\overline{\Sigma^n_j}$ for $j = 2, 3$ with $h = 1, 2$. For every n, for $h = 1, 2$, we define

$$(\varphi_h)_n(x,y) = \begin{cases} \varphi_h(x,y) & \text{if } (x,y) \in \Omega^n \\ (\varphi_h)_{j,n}(x,y) & \text{if } (x,y) \in \overline{\Sigma^n_j}. \end{cases} \tag{3.8}$$

Let

$$\mathcal{K}_n = \{u \in H^1_0(\widehat{\Omega}^n; w^n),\ u \geq (\varphi_1)_n\} \tag{3.9}$$

and

$$\mathcal{K}^*_n = \{u \in H^1_0(\widehat{\Omega}^n; w^n), \quad (\varphi_1)_n \leq u \leq (\varphi_2)_n\}. \tag{3.10}$$

Theorem 3.1 *Let c_n and σ_n be as in (3.4). Then, for any $f_n \in L^2(\widehat{\Omega}^n)$, $d_n \in \mathbb{R}$ there exists one and only one solution u_n to the following problem*

$$\begin{cases} \text{find } u_n \in \mathcal{K}_n \text{ such that} \\ a_n(u_n, v - u_n) \geq \int_{\widehat{\Omega}^n} f_n\,(v - u_n)\, dxdy + \sigma_n d_n \int_{\partial\Omega^n} (v - u_n)\, ds \quad \forall\, v \in \mathcal{K}_n, \end{cases} \tag{3.11}$$

where $a_n(\cdot,\cdot)$ is defined in (3.5). Moreover, u_n is the only function that realizes the minimum of the energy functional

$$\min_{v \in \mathcal{K}_n} \left\{ a_n(v,v) - 2\int_{\widehat{\Omega}^n} f_n\, v\, dxdy - 2\sigma_n d_n \int_{\partial\Omega^n} v\, ds \right\}. \tag{3.12}$$

Before proving Theorem 3.1, we recall a Poincaré type inequality where the relevant fact is that the constant C_P is independent of n. We skip the proof because it is similar to the proof of Theorem 7.4 (following) (see also Theorem 6.1 in [13]).

Theorem 3.2 *For any function $u \in H_0^1(\widehat{\Omega}^n; w^n)$, the following estimate holds*

$$||u||^2_{L^2(\Sigma^n)} \leq \alpha^{-n} \int_{\Sigma^n} |\nabla u|^2 w^n dxdy. \tag{3.13}$$

Moreover, there exists a constant C_P independent of n, such that,

$$||u||_{L^2(\widehat{\Omega}^n)} \leq C_P \left(||\nabla u||^2_{L^2(\Omega^n)} + \sigma_n \int_{\Sigma^n} |\nabla u|^2 w^n dxdy \right)^{1/2} \tag{3.14}$$

for all $u \in H_0^1(\widehat{\Omega}^n; w^n)$.

Proof of Theorem 3.1 As $(\varphi_1)_n \in \mathcal{K}_n$, the convex $\mathcal{K}_n$ is not empty. By Theorems 5.3 and 5.8 in [9], we have

$$\left| \sigma_n d_n \int_{\partial\Omega^n} v \; ds \right| \leq C \, |d_n| \, ||v||_{H^1(\Omega^n)}$$

and

$$\left| \int_{\widehat{\Omega}^n} f_n \, v \, dxdy \right| \leq ||f_n||_{L^2(\widehat{\Omega}^n)} ||v||_{L^2(\widehat{\Omega}^n)}.$$

Moreover,

$$|a_n(u, v)| \leq \max(\delta_n, c_n\sigma_n, \Lambda) ||u||_{H_0^1(\widehat{\Omega}^n; w^n)} ||v||_{H_0^1(\widehat{\Omega}^n; w^n)}$$

and

$$||v||^2_{H_0^1(\widehat{\Omega}^n; w^n)} \leq (1 + C_P^2) \max \left(\frac{1}{\lambda}, \frac{1}{c_n \sigma_n} \right) a_n(v, v). \tag{3.15}$$

□

Remark 3.1 Similar results hold for the two obstacle problems by considering now the convex $\mathcal{K}_n^*$. Moreover, the following estimate holds

$$||u_n||_{L^\infty(\widehat{\Omega}^n)} \leq \max(||\varphi_1||_{C^0(\overline{\Omega})}, ||\varphi_2||_{C^0(\overline{\Omega})}). \tag{3.16}$$

4 Mosco convergence

We consider the sequence of weighted energy functionals in $L^2(\widehat{\Omega}^1)$

$$F^n[u] = \begin{cases} \int_{\widehat{\Omega}^n} \sum_{i,j=1}^2 a_{ij}^n \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \; dxdy + \delta_n \int_{\widehat{\Omega}^n} u^2 dxdy & \text{if } u|_{\widehat{\Omega}^n} \in H_0^1(\widehat{\Omega}^n; w^n) \\ +\infty & \text{otherwise in } L^2(\widehat{\Omega}^1) \end{cases} \tag{4.1}$$

[the coefficients a_{ij}^n are defined in (2.6), (3.3), (3.4), $\delta_n \geq 0$] and

$$F_{c_0}[u] = \begin{cases} \int_\Omega \sum_{i,j=1}^2 a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \; dxdy + \delta_0 \int_\Omega u^2 dxdy + c_0 \int_{\partial\Omega} u^2 d\mu & \text{if } u|_\Omega \in H^1(\Omega) \\ +\infty & \text{otherwise in } L^2(\widehat{\Omega}^1). \end{cases} \tag{4.2}$$

We recall the notion of *M-convergence* of functionals, introduced in [24] (see also [25]).

Definition 4.1 A sequence of functionals $F^n : H \to (-\infty, +\infty]$ is said to $M-$converge to a functional $F : H \to (-\infty, +\infty]$ in a Hilbert space H, if

(a) For every $u \in H$ there exists u_n converging strongly to u in H such that

$$\limsup F^n[u_n] \leq F[u], \quad as\ n \to +\infty. \tag{4.3}$$

(b) For every v_n converging weakly to u in H

$$\liminf F^n[v_n] \geq F[u], \quad as\ n \to +\infty. \tag{4.4}$$

In order to study the asymptotic behaviour, we fix the further assumptions

$$\delta_n \geq 0 \text{ and } \delta_n \to \delta_0 \text{ as } n \to +\infty, \tag{4.5}$$

$$c_n > 0 \text{ and } c_n \to c_0 \text{ as } n \to +\infty. \tag{4.6}$$

Theorem 4.1 *Let us assume* (4.5) *and* (4.6). *Then, the sequence of functionals F^n, defined in* (4.1)*, M-converges in $L^2(\widehat{\Omega}^1)$ to the functional F_{c_0} defined in* (4.2) *as $n \to +\infty$.*

Before proving Theorem 4.1 we recall the following convergence result that we shall use several times from now on (see Proposition 4.1 in [13]).

Proposition 4.1 *Let σ_n be as in* (3.4). *Then, for every sequence $g_n \in H^1(\Omega)$ weakly converging to g^* in $H^1(\Omega)$, we have*

$$\sigma_n \int_{\partial\Omega^n} g_n ds \to \int_{\partial\Omega} g^* d\mu, \quad as\ n \to +\infty. \tag{4.7}$$

Proof of Theorem 4.1 This theorem can be proved just as Theorem 4.1 was proved in [13]. The coefficients of the forms are different, as is the geometry of the layers: however, since these peculiarities do not change the basic proof, here we only highlight some crucial points and we refer the reader to the proof of Theorem 4.1 in [13] for details. First, we proceed with the proof of condition (a) in Definition 4.1. We consider a given function u as in condition (a) and we observe that, without loss of generality, we can assume that $u|_\Omega \in H^1(\Omega)$, otherwise the inequality (4.3) becomes trivial. We assume, in addition, that $u|_{\overline{\Omega}} \in Lip(\overline{\Omega})$. We construct, as in Sect. 3, G_j on $\overline{\Sigma_j^0}$ and $u_{j,n}$ on $\overline{\Sigma_j^n}$ for $j = 1, 2, 3$ [see (3.6) and (3.7)]. For every n, we define

$$u_n(x, y) = \begin{cases} u(x, y) & \text{if } (x, y) \in \Omega^n \\ u_{j,n}(x, y) & \text{if } (x, y) \in \overline{\Sigma_j^n}. \end{cases} \tag{4.8}$$

We denote by $\overline{u}$ the extension to zero of u outside Ω and by $\overline{u}_n$ the extension to zero of u_n outside $\widehat{\Omega}^n$. We note that $\overline{u}_n$ tend to $\overline{u}$ in $L^2(\widehat{\Omega}^1)$, $\sup_{\overline{\Sigma^n}} |\overline{u}_n| \leqslant \sup_{\overline{\Omega}} |u|$ and the functions u_n defined in (4.8) belong to $H_0^1(\widehat{\Omega}^n, w_n)$.

For each n, we split the integral $F^n[u_n]$ in three terms, taking into account the definitions of a_{ij}^n and u_n,

$$F^n[u_n] = \int_{\Omega^n} \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, dxdy + \delta_n \int_{\widehat{\Omega}^n} u_n^2 dxdy + \sigma_n c_n \int_{\Sigma^n} |\nabla u_n|^2 w^n dxdy.$$

Since the sets Ω^n tend to the set Ω as $n \to +\infty$, we get

$$\lim_{n\to+\infty} \int_{\Omega^n} \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\, dxdy = \int_{\Omega} \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\, dxdy, \tag{4.9}$$

$$\lim_{n\to+\infty} \delta_n \int_{\Omega^n} u^2 dxdy = \delta_0 \int_{\Omega} u^2 dxdy, \tag{4.10}$$

$$\lim_{n\to+\infty} \delta_n \int_{\Sigma^n} u_n^2 dxdy = 0. \tag{4.11}$$

Finally, as in Theorem 4.1 in [13] we can show

$$\lim_{n\to+\infty} c_n\sigma_n \int_{\Sigma^n} |\nabla u_n|^2 w^n dxdy = c_0 \int_{\partial\Omega} u^2 d\mu. \tag{4.12}$$

We complete the proof of part (a) of the Theorem by making use of the *diagonal* formula of Corollary 1.16 of [2].

Now we prove condition (b) of Definition 4.1. Let v_n be a sequence as in (b), that is,

$$v_n \rightharpoonup u \quad \text{in} \quad L^2(\widehat{\Omega}^1). \tag{4.13}$$

In order to prove the inequality (4.4), it is not restrictive to assume that

$$\liminf F^n[v_n] \leq C^* < +\infty. \tag{4.14}$$

Then, from (4.13) and (4.14), up to passing to a subsequence, we deduce that

$$||v_n||_{H^1(\Omega^n)} \leq C$$

where C is a constant independent of n. By Theorem 5.7 in [9], there exists a bounded linear extension operator $Ext_J : H^1(\Omega^n) \to H^1(\mathbb{R}^2)$, whose norm is independent of n, that is,

$$||Ext_J\, v_n||_{H^1(\mathbb{R}^2)} \leqslant C_J ||v_n||_{H^1(\Omega^n)} \tag{4.15}$$

with C_J independent of n. We put

$$\hat{v}_n = (Ext_J\, v_n|_{\Omega^n})|_{\Omega}, \tag{4.16}$$

then there exists $\hat{v} \in H^1(\Omega)$ and a subsequence of $\hat{v}_n$, denoted by $\hat{v}_n$ again, weakly converging to $\hat{v}$ in $H^1(\Omega)$. By a direct calculation, we can prove that the sequence $\hat{v}_n$ weakly converges to u in $L^2(\Omega)$ hence

$$\hat{v}_n \rightharpoonup u|_{\Omega} \quad \text{in} \quad H^1(\Omega), \tag{4.17}$$

$$\liminf_{n\to+\infty} \int_{\Omega^n} |\nabla v_n|^2\, dxdy \geq \int_{\Omega} |\nabla u|^2\, dxdy \tag{4.18}$$

and

$$\lim_{n\to+\infty} \delta_n \int_{\Omega^n} v_n^2 dxdy = \delta_0 \int_{\Omega} u^2 dxdy. \tag{4.19}$$

Finally, as in Theorem 4.1 in [13] we show that (if $c_0 > 0$)

$$\liminf_{n\to+\infty} c_n\sigma_n \int_{\Sigma^n} |\nabla v_n|^2 w^n dxdy \geq c_0 \int_{\partial\Omega} u^2 d\mu. \tag{4.20}$$

□

Remark 4.1 We note if $u \in Lip(\overline{\Omega}) \cap \mathcal{K}$ ($u \in Lip(\overline{\Omega}) \cap \mathcal{K}^*$), then the function u_n defined in (4.8) belongs to $\mathcal{K}_n$ ($\mathcal{K}_n^*$). Then, by making use of the *diagonal* formula of Corollary 1.16 of [2], we can deduce that for any $u \in \mathcal{K}$ ($u \in \mathcal{K}^*$) the corresponding u_n belongs to $\mathcal{K}_n$ ($\mathcal{K}_n^*$).

When the conductivity of the thin fibers vanishes more slowly than the thickness of the fiber, that is,

$$c_n w^n \to 0, \quad c_n \to +\infty \tag{4.21}$$

we introduce the limit functional (4.22) in $L^2(\widehat{\Omega}^1)$

$$F_\infty[u] = \begin{cases} \int_\Omega |\nabla u|^2 dxdy + \delta_0 \int_\Omega u^2 dxdy & \text{if } u|_\Omega \in H_0^1(\Omega) \\ +\infty & \text{otherwise in } \ L^2(\widehat{\Omega}^1) \end{cases}. \tag{4.22}$$

The following theorem can be proved just as Theorem 4.2 was proved in [13].

Theorem 4.2 *Let us assume* (4.21) *and* (4.5). *Then the sequence of functionals* F^n, *defined in* (4.1), M*-converges in* $L^2(\widehat{\Omega}^1)$ *as* $n \to +\infty$ *to the energy functional* $F_\infty[u]$ *defined in* (4.22).

5 Asymptotics

In order to study the asymptotic behaviour of the functions u_n, we assume that

$$f_n,\ f \in L^2(\widehat{\Omega}^1), \text{ and } f_n \to f \ \in L^2(\widehat{\Omega}^1), \quad \text{as } n \to +\infty; \tag{5.1}$$

$$d_n, d \in \mathbb{R}, \text{ and } d_n \to d \text{ as } n \to +\infty. \tag{5.2}$$

Our first result concerns the case of thin fibers whose conductivity vanishes more slowly than the thickness of the fiber. The cases in which the conductivity of the thin fibers vanishes at the same rate as the thickness of the fiber or more quickly will be taken into account in Theorems 5.2 (coercive case), and 5.3 (semicoercive).

5.1 Coercive case

When the conductivity of the thin fibers vanishes more slowly than the thickness of the fiber, we state the following Theorem 5.1 (see also Theorem 3.3 in [13]).

Theorem 5.1 *Let us assume* (4.5), (4.21), (5.1), *and* (5.2). *Then the sequence of the solutions* u_n [*defined in* (3.11)] *converges to the function* u [*defined in* (2.8)] *weakly in* $H^1_{loc}(\Omega)$ *and strongly in* $L^2(\Omega)$.

The following theorem deals with thin fibers whose conductivity vanishes at the same rate as the thickness of the fiber or more quickly.

Theorem 5.2 *Let us assume conditions* (2.14), (4.5), (4.6), (5.1), *and* (5.2). *Then the sequence of the solutions* u_n [*defined in* (3.11)] *converges to the function* u [*defined in* (2.16)] *weakly in* $H^1_{loc}(\Omega)$, *and weakly in* $L^2(\Omega)$. *Moreover, if*

$$\lim_{n\to+\infty} \sigma_n c_n \alpha^n = +\infty \tag{5.3}$$

then the sequence of the solutions u_n *converges to the function* u *strongly in* $L^2(\Omega)$.

Proof Let u_n be the solution to the problem (3.11). Then

$$a_n(u_n,u_n) \le ||f_n||_{L^2(\widehat{\Omega}^n)}||u_n-v_n||_{L^2(\widehat{\Omega}^n)} + C|d_n|||u_n-v_n||_{H^1(\Omega^n)} + a_n(u_n,v_n) \tag{5.4}$$

where we have used Theorems 5.3 and 5.7 in [9]. Suppose first that $\delta_0 > 0$. Then,

$$\begin{aligned} &\int_{\Omega^n} |\nabla u_n|^2 dxdy + c_n\sigma_n \int_{\Sigma^n} |\nabla u_n|^2 w_n dxdy + \int_{\widehat{\Omega}^n} |u_n|^2 dxdy \\ &\le \frac{1}{\lambda}\int_{\Omega^n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\, dxdy + c_n\sigma_n \int_{\Sigma^n} |\nabla u_n|^2 w_n dxdy + \frac{2\delta_n}{\delta_0}\int_{\widehat{\Omega}^n} |u_n|^2 dxdy \\ &\le \max\left(\frac{1}{\lambda}, \frac{2}{\delta_0}, 1\right) a_n(u_n,u_n). \end{aligned} \tag{5.5}$$

The right end side of inequality (5.4) can be estimated as follows

$$\begin{aligned} &||f_n||_{L^2(\widehat{\Omega}^n)}||u_n-v_n||_{L^2(\widehat{\Omega}^n)} + C|d_n|||u_n-v_n||_{H^1(\Omega^n)} + a_n(u_n,v_n) \\ &\le ||f_n||_{L^2(\widehat{\Omega}^n)}(||u_n|| + ||v_n||_{L^2(\widehat{\Omega}^n)}) + C|d_n|(||u_n|| + ||v_n||_{H^1(\Omega^n)}) \\ &+ \Lambda||\nabla u_n||_{L^2(\Omega^n)}||\nabla v_n||_{L^2(\Omega^n)} + c_n\sigma_n\left(\int_{\Sigma^n}|\nabla u_n|^2 w_n\, dxdy\right)^{\frac{1}{2}} \\ &\times\left(\int_{\Sigma^n}|\nabla v_n|^2 w_n\, dxdy\right)^{\frac{1}{2}}. \end{aligned} \tag{5.6}$$

We choose as test function $v_n = (\varphi_1)_n$ and we obtain

$$||v_n||^2_{L^2(\widehat{\Omega}^n)} \le ||\varphi_1||^2_{L^2(\Omega)} + \sup_{\overline{\Omega}} |\varphi_1|^2|\Sigma_n|, \tag{5.7}$$

$$||\nabla v_n||^2_{L^2(\Omega^n)} \le ||\nabla\varphi_1||^2_{L^2(\Omega)}. \tag{5.8}$$

Moreover, by (4.12), we deduce

$$c_n\sigma_n\int_{\Sigma^n}|\nabla v_n|^2 w^n dxdy \le \max\left(1, 2c_0\int_{\partial\Omega}|\varphi_1|^2 d\mu\right). \tag{5.9}$$

By using the previous inequalities (5.4)–(5.9) we obtain

$$\begin{aligned} &\int_{\Omega^n}|\nabla u_n|^2 dxdy + c_n\sigma_n\int_{\Sigma^n}|\nabla u_n|^2 w_n dxdy \\ &\leqslant C_1(\delta_0, \lambda, \Lambda, |d|, ||f||_{L^2(\widehat{\Omega}^1)}, ||\varphi_1||_{C^1(\overline{\Omega})}), \end{aligned} \tag{5.10}$$

and

$$||u_n||^2_{L^2(\widehat{\Omega}^n)} \leqslant C_1(\delta_0, \lambda, \Lambda, |d|, ||f||_{L^2(\widehat{\Omega}^1)}, ||\varphi_1||_{C^1(\overline{\Omega})}), \tag{5.11}$$

where the constant C_1 does not depend on n.

By assumption (2.14), if $\delta_0 = 0$, then $c_0 > 0$: in this case, by using Theorem 3.2, we have

$$\begin{aligned}
&\int_{\Omega^n} |\nabla u_n|^2 dxdy + c_n\sigma_n \int_{\Sigma^n} |\nabla u_n|^2 w_n dxdy + \int_{\widehat{\Omega}^n} |u_n|^2 dxdy \\
&\leq \frac{1+C_p^2}{\lambda} \int_{\Omega^n} a_{ij} \frac{\partial u_n}{\partial x_i}\frac{\partial u_n}{\partial x_j}\, dxdy + c_n\sigma_n \frac{2C_p^2+c_0}{c_0} \int_{\Sigma^n} |\nabla u_n|^2 w_n dxdy \\
&\leq \max\left(\frac{1+C_p^2}{\lambda}, \frac{2C_p^2+c_0}{c_0}\right) a_n(u_n, u_n). \qquad (5.12)
\end{aligned}$$

Proceeding as before,

$$\int_{\Omega^n} |\nabla u_n|^2 dxdy + c_n\sigma_n \int_{\Sigma^n} |\nabla u_n|^2 w_n dxdy \leq C_2(c_0, \lambda, \Lambda, |d|, ||f||_{L^2(\widehat{\Omega}^1)}, ||\varphi_1||_{C^1(\overline{\Omega})}), \qquad (5.13)$$

and

$$||u_n||^2_{L^2(\widehat{\Omega}^n)} \leqslant C_2(c_0, \lambda, \Lambda, |d|, ||f||_{L^2(\widehat{\Omega}^1)}, ||\varphi_1||_{C^1(\overline{\Omega})}), \qquad (5.14)$$

where the constant C_2 does not depend on n. We consider the function u_n^*, which is a suitable extension of the function u_n from the set Ω^n to the set $\widehat{\Omega}^1$ (we use an extension operator whose norm is independent of the increasing number of sides, see Theorem 5.7 in [9]) and for every n, from either (5.10) and (5.11) or (5.13) and (5.14), we derive

$$||u_n^*||_{H^1(\widehat{\Omega}^1)} \leq C_J ||u_n||_{H^1(\Omega^n)} \leq C. \qquad (5.15)$$

Therefore, there exists a subsequence still denoted by u_n^* that weakly converges to a function u^* in $H^1(\widehat{\Omega}^1)$. Now we prove that $u^*|_\Omega = u$. By using condition (b) of M-convergence and Proposition 4.1, we obtain that

$$\begin{aligned}
&F_{c_0}[u^*] - 2\int_\Omega f\, u^*\, dxdy - 2d \int_{\partial\Omega} u^*\, d\mu \\
&\leq \liminf \left(F^n[u_n^*] - 2\int_{\widehat{\Omega}^n} f_n\, u_n^*\, dxdy - 2\sigma_n d_n \int_{\partial\Omega^n} u_n^*\, ds\right). \qquad (5.16)
\end{aligned}$$

By using condition (a) of M-convergence there exists $v_n \in L^2(\widehat{\Omega}^1)$ [defined as in (4.8) in the proof of Theorem 4.1] converging strongly in $L^2(\widehat{\Omega}^1)$ to $\overline{u}$ such that

$$\lim F^n[v_n] = F_{c_0}[\overline{u}] = F_{c_0}[u],$$

as $n \to +\infty$. We recall that we denote by $\overline{u}$ the extension to zero of u outside Ω. Moreover, by Remark 4.1 $v_n|_{\widehat{\Omega}^n} \in \mathcal{K}_n$. Then by Proposition 4.1 (where g_n is a suitable extension of v_n) we obtain

$$\begin{aligned}
&\lim \left(F^n[v_n] - 2\sigma_n d_n \int_{\partial\Omega^n} v_n\, ds - 2\int_{\widehat{\Omega}^n} f_n\, v_n\, dxdy\right) \\
&= F_{c_0}[u] - 2d \int_{\partial\Omega} u\, d\mu - 2\int_\Omega f\, u\, dxdy. \qquad (5.17)
\end{aligned}$$

Then

$$\begin{aligned}
&F^n[u_n] - 2\sigma_n d_n \int_{\partial\Omega^n} u_n\, ds - 2\int_{\widehat{\Omega}^n} f_n\, u_n\, dxdy \\
&= \min_{v\in\mathcal{K}_n} \left(F^n[v] - 2\sigma_n d_n \int_{\partial\Omega^n} v\, ds - 2\int_{\widehat{\Omega}^n} f_n\, v\, dxdy \right) \\
&\le F^n[v_n] - 2\sigma_n d_n \int_{\partial\Omega^n} v_n\, ds - 2\int_{\widehat{\Omega}^n} f_n\, v_n\, dxdy. \qquad (5.18)
\end{aligned}$$

Just as we obtain, by direct calculations,

$$\begin{aligned}
&\liminf \left(F^n[u_n^*] - 2\sigma_n d_n \int_{\partial\Omega^n} u_n^*\, ds - 2\int_{\widehat{\Omega}^n} f_n\, u_n^*\, dxdy \right) \\
&\le \liminf \left(F^n[u_n] - 2\sigma_n d_n \int_{\partial\Omega^n} u_n\, ds - 2\int_{\widehat{\Omega}^n} f_n\, u_n\, dxdy \right) \qquad (5.19)
\end{aligned}$$

by combining (5.16)–(5.19) we obtain that

$$F_{c_0}[u^*] - 2\int_{\Omega} f\, u^*\, dxdy - 2d\int_{\partial\Omega} u^*\, d\mu \le F_{c_0}[u] - 2\int_{\Omega} f\, u\, dxdy - 2d\int_{\partial\Omega} u\, d\mu.$$

By the uniqueness of the solution (2.16), we conclude that $u^*|_{\Omega} = u$, and $u_n^*|_{\Omega}$ converges to u weakly in $H^1(\Omega)$. As the $u_n^* = u_n$ in Ω_N with $n \ge N$ then u_n converges to u weakly in $H^1_{loc}(\Omega)$ and from (5.11) or (5.14) we deduce that u_n converge to u weakly in $L^2(\Omega)$. Moreover, if we assume condition (5.3), then the strong convergence holds in $L^2(\Omega)$: in fact,

$$\begin{aligned}
&\limsup \int_{\Omega} u_n^2\, dxdy \le \limsup \left(\int_{\Omega^n} u_n^2\, dxdy + \int_{\Sigma^n} u_n^2\, dxdy \right) \\
&\le \limsup \left(\int_{\Omega} (u_n^*)^2\, dxdy + \int_{\Sigma^n} u_n^2\, dxdy \right) = \int_{\Omega} u^2\, dxdy
\end{aligned}$$

where we have used the strong convergence of the sequence u_n^* to u in the space $L^2(\Omega)$ and estimates (3.13), (5.3), and (5.10) or (5.13). □

Remark 5.1 The results of Theorems 5.1 and 5.2 apply also to the case of equations by the same proof and they improve the results of Theorems 3.1 and 3.3 in [13]. In fact, in the present paper, the assumptions are weaker than the assumptions of [13]. Moreover, we prove the convergence in $L^2(\Omega)$ directly for the solutions rather than for suitable extensions of the solutions.

Remark 5.2 The results of Theorems 5.1 and 5.2 hold for the two obstacle problems where $\mathcal{K}^*$ and $\mathcal{K}_n^*$ are defined respectively in (2.20) and (3.10). Moreover, in this case, from estimates (3.16) and (5.15) we obtain strong convergence in $L^p(\Omega)$ with $p < +\infty$.

In the next subsection, we establish results without requiring condition (2.14).

5.2 Semicoercive case

We note that, in the assumptions and notation of Sect. 2, conditions (2.23) and (2.24) (as well as analogous conditions in Remarks 2.3, 2.4) guarantee the existence and uniqueness of the solution of obstacle problems (see Theorem 2.3). In this subsection, we discuss assumptions on the approximation data f_n and on the coefficients c_n that guarantee asymptotic results. In particular, we can extend f to zero outside Ω_n or, according to the classical setting (see

Theorem 9.3 in [3]), we can impose some conditions on the vanishing rate of the sequence c_n (for a complete discussion, see Remarks 5.3 and 5.4 following).

Theorem 5.3 *Let us assume conditions* (4.6) *with* $c_0 = 0$, (4.5) *with* $\delta_0 = 0$, (2.23), (2.24), (5.1), (5.2), *and*

$$\frac{||f_n||^2_{L^2(\Sigma^n)}}{c_n \sigma_n \alpha^n} \le c^* \tag{5.20}$$

with $c^* > 0$.

Then the sequence of the solutions u_n [*defined in* (3.11)] *converges to the function* u [*defined in* (2.25)] *weakly in* $H^1_{loc}(\Omega)$. *If*

$$c_n \sigma_n \alpha^n \ge c^{**} \tag{5.21}$$

for some $c^{**} > 0$ *and thus*

$$\frac{||f_n||^2_{L^2(\Sigma^n)}}{c_n \sigma_n \alpha^n} \to 0, \tag{5.22}$$

then the solutions u_n [*defined in* (3.11)] *converge weakly in* $L^2(\Omega)$. *Moreover, if assumption* (5.3) *holds, then the solutions* u_n *converge strongly in* $L^2(\Omega)$.

Proof As δ_n tends to zero, we assume, for the sake of simplicity, that $\delta_n = 0$. From (2.23) we deduce, by taking into account (3.8) and (3.9), that $0 \in \mathcal{K}_n$ and

$$a_n(u_n, u_n) \le \int_{\widehat{\Omega}^n} f_n\, u_n\, dxdy + d_n\, \sigma_n \int_{\partial\Omega^n} u_n\, ds. \tag{5.23}$$

Bearing in mind the proof of Theorem 5.2, we show that there exists a constant C such that

$$||u_n||_{H^1(\Omega^n)} \le C \tag{5.24}$$

where the functions u_n are the solutions defined in (3.11). Suppose the statement to be proved is false: for every m in $\mathbb{N}$, there exists u_{n_m} that we shall denote, from now on, simply by u_m such that

$$A_m^2 := ||u_m||^2_{H^1(\Omega^m)} \ge m^2 \tag{5.25}$$

and

$$a_m(u_m, u_m) \le \int_{\widehat{\Omega}^m} f_m\, u_m\, dxdy + d_m\, \sigma_m \int_{\partial\Omega^m} u_m\, ds. \tag{5.26}$$

Set

$$v_m = \frac{u_m}{A_m}:$$

we have

$$||v_m||^2_{H^1(\Omega^m)} = 1. \tag{5.27}$$

We denote by v_m^* an extension of $v_{m|\Omega^n}$ to $\widehat{\Omega}^1$ such that

$$||v_m^*||^2_{H^1(\widehat{\Omega}^1)} \le C \tag{5.28}$$

with C independent to m (see Theorem 5.7 in [9]). Then, there exists a function $v^* \in H^1(\widehat{\Omega}^1)$ and a subsequence (still denoted by v^*_m) that converges to v^* weakly in $H^1(\widehat{\Omega}^1)$ (and strongly in $H^s(\widehat{\Omega}^1)$ for $0 \le s < 1$). From (5.23) we deduce

$$\lambda ||\nabla v_m||^2_{L^2(\Omega^m)} + c_m\sigma_m \int_{\Sigma^m} |\nabla v_m|^2 w^m dx\, dy \\ \le \frac{1}{A_m}\left(\int_{\widehat{\Omega}^m} f_m\, v_m\, dxdy + d_m\, \sigma_m \int_{\partial\Omega^m} v_m ds\right). \tag{5.29}$$

As

$$\left| d_m\, \sigma_m \int_{\partial\Omega^m} v_m\, ds \right| \le C\, |d|\, ,$$

we have to estimate

$$\int_{\widehat{\Omega}^m} f_m\, v_m\, dxdy.$$

We start by noting that

$$\left| \int_{\Omega^m} f_m\, v_m\, dxdy \right| \le C||f||_{L^2(\Omega)},$$

hence, we have to estimate the part on Σ^m; by taking into account statement (3.13) we obtain

$$\left| \int_{\Sigma^m} f_m\, v_m\, dxdy \right| \le ||f_m||_{L^2(\Sigma^m)} \left(\int_{\Sigma^m} v_m^2\, dxdy \right)^{1/2} \\ \le ||f_m||_{L^2(\Sigma^m)} \left(\frac{c_m\sigma_m}{c_m\sigma_m\alpha^m} \int_{\Sigma^m} |\nabla v_m|^2\, w^m\, dxdy \right)^{1/2}$$

and by (5.29)

$$\lambda ||\nabla v_m||^2_{L^2(\Omega^m)} + c_m\sigma_m \int_{\Sigma^m} |\nabla v_m|^2 w^m dx\, dy \\ \le \frac{1}{A_m}\left(\int_{\Omega^m} f_m\, v_m\, dxdy + C + \frac{||f_m||^2_{L^2(\Sigma^m)}}{c_m\sigma_m\alpha^m} c_m\sigma_m \int_{\Sigma^m} |\nabla v_m|^2 w^m dx\, dy \right).$$

Hence, by (5.20) and (5.25),

$$\lambda ||\nabla v_m||^2_{L^2(\Omega^m)} + c_m\sigma_m \int_{\Sigma^m} |\nabla v_m|^2 w^m dx\, dy \le \frac{C}{A_m}. \tag{5.30}$$

From (5.30), by the weak lower semicontinuity of the norm, we obtain, for any fixed N_0,

$$||\nabla v^*||^2_{L^2(\Omega^{N_0})} \le \liminf_m ||\nabla v_m||^2_{L^2(\Omega^{N_0})} = 0 \tag{5.31}$$

then $v^* = k_1$ a. e. in Ω^{N_0}. We observe that $k_1 \ge 0$ in fact by construction $v_m \ge \frac{1}{A_m} \min_{\bar{\Omega}}(\varphi_1)$. As N_0 is arbitrary, we deduce that $v^* = k_1$ a. e. in Ω and ∇v^*_m weakly converges to 0 in $L^2(\Omega)$. We show that $k_1 = 0$: if $k_1 > 0$ we obtain a contradiction with (2.24). In fact, from (5.29) we obtain

$$0 \le \int_{\widehat{\Omega}^m} f_m\, v_m\, dxdy + d_m\sigma_m \int_{\partial\Omega^m} v_m ds \tag{5.32}$$

and

$$\int_{\widehat{\Omega}^m} f_m\, v_m\, dxdy = \int_{\Omega^m} f_m\, v_m\, dxdy + \int_{\Sigma^m} f_m\, v_m\, dxdy. \tag{5.33}$$

We estimate $\int_{\Sigma^m} f_m\, v_m\, dxdy$. As previously, we obtain

$$\left|\int_{\Sigma^m} f_m\, v_m\, dxdy\right| \leq ||f_m||_{L^2(\Sigma^m)}\left(\frac{c_m\sigma_m}{c_m\sigma_m\alpha^m}\int_{\Sigma^m}|\nabla v_m|^2\, w^m\, dxdy\right)^{1/2} \tag{5.34}$$

and hence, by (5.20) and (5.30),

$$\int_{\Sigma^m} f_m\, v_m\, dxdy \to 0. \tag{5.35}$$

As the term $\int_{\Sigma^m} f_m\, v_m^*\, dxdy$ tends to 0 by the strong convergence of the functions v_m^* in $L^p(\Omega),\ p > 2$, and by (4.7), (5.1), and (5.2), from (5.32) we deduce

$$0 \leq k_1\left(\int_\Omega f\, dxdy + d\int_{\partial\Omega} d\mu\right) \tag{5.36}$$

that is a contradiction with (2.24). On the other hand, if $k_1 = 0$, we have a contradiction with (5.27). In fact, by (5.27)

$$1 = ||v_m||^2_{L^2(\Omega^m)} + ||\nabla v_m||^2_{L^2(\Omega^m)} \leq ||v_m^*||^2_{L^2(\Omega)} + C\frac{1}{A_m\lambda} :$$

by taking into account (5.25) and the strong convergence of v_m^* to zero in $L^2(\Omega)$ we have a contradiction. Then estimate (5.24) is proved and we can repeat the proof of Theorem 5.2 in order to prove that the sequence of the solutions u_n [defined in (3.11)] converges to the function u [defined in (2.25)] weakly in $H^1_{loc}(\Omega)$. We show that assumptions (5.21) and (5.22) imply the weak convergence in $L^2(\Omega)$. In fact, by (3.13), (5.2) and (5.23),

$$\lambda||\nabla u_n||^2_{L^2(\Omega^n)} + c_n\sigma_n\int_{\Sigma^n}|\nabla u_n|^2 w^n dx\, dy$$
$$\leq \int_{\Omega^n} f_n\, u_n\, dxdy + C + \frac{||f_n||^2_{L^2(\Sigma^n)}}{c_n\sigma_n\alpha^n}c_n\sigma_n\int_{\Sigma^n}|\nabla u_n|^2 w^n dx\, dy.$$

Hence, by (5.22)

$$\lambda||\nabla u_n||^2_{L^2(\Omega^n)} + c_n\sigma_n\int_{\Sigma^n}|\nabla u_n|^2 w^n dx\, dy \leq C \tag{5.37}$$

and by (3.13)

$$||u_n||^2_{L^2(\Sigma^n)} \leq \frac{C}{c_n\sigma_n\alpha^n}. \tag{5.38}$$

From (5.21), (5.24), (5.38) we deduce the uniform boundedness in $L^2(\Omega)$. Finally, if we require assumption (5.3), we can show the strong convergence as in the proof of Theorem 5.2 by using (5.38). □

Remark 5.3 We note that if

$$f_n = f \quad \text{on } \Omega^n, \quad f_n = 0 \quad \text{on } \widehat{\Omega}^1\backslash\Omega^n \tag{5.39}$$

then assumptions (5.1) and (5.20) are obviously fulfilled; therefore, for the weak convergence in $H^1_{loc}(\Omega)$, no conditions on the vanishing rate of the coefficients c_n are required. If we do not assume (5.39), we can link the vanishing rate of the sequence c_n to the vanishing rate of the L^2-norms of the data f_n in the *reinforcement sets* Σ^n.

Remark 5.4 We note that the fractal setting gives rise to a peculiar phenomenon. Owing to the tricky geometry, the reinforced domains have to be constructed starting from suitable *inner* polygonal domains Ω^n. Then the functions u_n are solutions to equations with *reinforced* coefficients in a (small) part of the limit domain $(\Omega \backslash \Omega^n)$. Consequently, the *reinforced* solutions have gradients that are not uniformly bounded in L^2-norms (in the limit domain), which is in contrast to the strong convergence of the gradients (in L^2) established in [3]. We give simple examples where the assumptions of Theorem 9.3 in [3] are fulfilled but the gradients are not strongly convergent (see Remarks 6.2 and 6.3 following). Moreover, in order to compare Theorem 5.3 with Theorem 9.3 in [3], we note that in our setting we recover the smooth case with $\alpha = 4$ ($\sigma_n = 1$); then condition (5.20)—that guarantees the weak convergence in $H^1_{loc}(\Omega)$—is weaker than the assumptions of Theorem 9.3 in [3].

Remark 5.5 If we assume conditions (2.31)–(2.33), then the results of Theorem 5.3 hold for the obstacle problem where

$$\mathcal{K}^{\#}_n = \{u \in H^1_0(\Omega, w^n), \quad u \leq (\varphi_2)_n\}. \tag{5.40}$$

Similar results hold for the two obstacle problems in the assumptions and notation of Remarks 2.4 and 5.2. Moreover, for the two obstacle problems we also obtain strong convergence in $L^p(\Omega)$ with $p < +\infty$.

6 Comments

In this section we discuss some simple examples. Our first example shows that the results of Theorems 5.1 and 5.2 are sharp.

Example 1 Consider the reinforced obstacle problem in the 1-dimensional case where $\Omega = [-1, 1]$, $\Omega^n = [-1 + \varepsilon, 1 - \varepsilon]$, $\widehat{\Omega}^n = [-1 - \varepsilon, 1 + \varepsilon]$ where

$$\varphi = 0, \; f_n = 1 \quad \text{in } \widehat{\Omega}^n, \; a_n = 1 \quad \text{in } \Omega^n, \; a_n = \lambda \quad \text{in } \widehat{\Omega}^n \backslash \Omega^n. \tag{6.1}$$

The solution to the reinforced obstacle problem is

$$u_n = \begin{cases} -\frac{x^2}{2} + \frac{4\varepsilon + \lambda(1-\varepsilon)^2}{2\lambda} & \text{in } \Omega^n \\ -\frac{x^2}{2\lambda} + \frac{(1+\varepsilon)^2}{2\lambda} & \text{in } \widehat{\Omega}^n \backslash \Omega^n. \end{cases} \tag{6.2}$$

The $L^2(\Omega \backslash \Omega^n)$-norms of the derivatives of the functions u_n defined in (6.2) are

$$||u_n'||^2_{L^2(\Omega \backslash \Omega^n)} = 2\frac{3\varepsilon - 3\varepsilon^2 + \varepsilon^3}{3\lambda^2} \tag{6.3}$$

and the $L^2(\Omega \backslash \Omega^n)$-norms of the functions u_n defined in (6.2) are

$$||u_n||^2_{L^2(\Omega \backslash \Omega^n)} = 2\frac{7/3\varepsilon^3 + 11/12\varepsilon^4 + 2/15\varepsilon^5}{\lambda^2}. \tag{6.4}$$

Remark 6.1 We note that we can not expect weak convergence in the space $H^1(\Omega)$. In fact, if we choose $\lambda = \varepsilon$ then the limit problem is the obstacle problem with Robin condition where

$$\varphi = 0,\ f = 1 \quad \text{in } \Omega, \quad u'(1) + 1/2u(1) = -u'(-1) + 1/2u(-1) = 0 \tag{6.5}$$

and the solution is

$$u = -\frac{x^2}{2} + \frac{5}{2} \tag{6.6}$$

(see Theorem 5.2) but the sequence of the derivatives of the functions u_n is not uniformly bounded in the space $L^2(\Omega)$. If we choose $\lambda = (\varepsilon)^{1/2}$ then the limit problem is the obstacle problem with Dirichlet condition where

$$\varphi = 0,\ \ f = 1 \quad \text{in } \Omega, \quad u(1) = u(-1) = 0. \tag{6.7}$$

All the assumptions of Theorem 5.1 are fulfilled and the solution is

$$u = -\frac{x^2}{2} + \frac{1}{2} \tag{6.8}$$

but the sequence of the derivatives of the functions u_n is not strongly convergent in the space $L^2(\Omega)$. If we choose $\lambda = (\varepsilon)^{1/2+\eta}$, $\eta \in (0, 1/2)$ then all the assumptions of Theorem 5.1 are still fulfilled, the limit solution is (6.8) but the sequence of the derivatives of the functions u_n is not uniformly bounded in the space $L^2(\Omega)$.

Remark 6.2 We discuss the behaviour of $L^2(\Omega)$-norms of the functions u_n defined in (6.2). Formula (6.4) shows that condition (5.3) does not assure that the sequence of norms in $L^2(\Omega)$ of the functions u_n is uniformly bounded if the coerciveness assumption (2.14) is not fulfilled (see Theorem 5.2). In fact, if we choose $\lambda = \varepsilon^{2-\eta}$, $\eta \in (0, 1/2)$ condition (5.3) is satisfied but the sequence of norms in $L^2(\Omega)$ of the functions u_n diverges like $\varepsilon^{\eta-\frac{1}{2}}$. Now we discuss the previous example in relation to the semicoercive case. Formula (6.4) shows that conditions (5.21) and (5.22) do not assure that the sequence of norms in $L^2(\Omega)$ of the functions u_n is uniformly bounded if the sign condition (2.24) is not fulfilled (see Theorem 5.3). If we choose $\lambda = \varepsilon^{5/3}$ condition (5.3) is satisfied but the sequence of norms in $L^2(\Omega)$ of the functions u_n diverges like $\varepsilon^{-1/6}$. If instead $\lambda = \varepsilon^2$ conditions (5.21) and (5.22) are satisfied but the sequence of norms in $L^2(\Omega)$ of the functions u_n diverges like $\varepsilon^{-1/2}$. Finally we note that with this choice the condition of Remark 1 (page 243) in [3] is satisfied whereas the uniform boundedness of the $L^2(\Omega)$-norms (see (9.9) in [3]) fails. We point out that sign condition (2.24) is not fulfilled (see also Remark 2 (page 243) in [3]).

In the following examples, the sign condition (2.24) is satisfied. Example 2 shows that we cannot expect weak convergence in the space $H^1(\Omega)$ under condition (5.39) in Theorem 5.3 (see Remark 5.3).

Example 2 Let us consider the obstacle problem in the 1-dimensional case where $\Omega = [-1, 1]$, $\varphi_1 = 0$ and $f = \begin{cases} -2 & \text{if } |x| \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < |x| \le 1 \end{cases}$ with Neumann condition $u'(1) = -u'(-1) = 0$: the solution is

$$u = \begin{cases} 0 & \text{if } |x| \le \frac{1}{2} \\ -\frac{x^2}{2} + |x| - \frac{3}{8} & \text{if } \frac{1}{2} < |x| \le 1. \end{cases} \tag{6.9}$$

Set $\Omega^n = [-1+\varepsilon, 1-\varepsilon]$, $\widehat{\Omega}^n = [-1-\varepsilon, 1+\varepsilon]$,

$$f_n = \begin{cases} -2 & \text{if } |x| \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < |x| \le 1-\varepsilon \\ 0 & \text{if } 1-\varepsilon < |x| \le 1+\varepsilon \end{cases},$$

and

$$a_n = 1 \quad \text{in } \Omega^n, \quad a_n = \lambda \quad \text{in } \widehat{\Omega}^n \backslash \Omega^n. \tag{6.10}$$

The reinforced problem has the solution

$$u_n = \begin{cases} 0 & \text{if } |x| \le \frac{1}{2} \\ -\frac{x^2}{2} + b|x| + \frac{1}{8} - \frac{b}{2} & \text{if } \frac{1}{2} < |x| \le 1-\varepsilon \\ \frac{b-1+\varepsilon}{\lambda}|x| - \frac{b-1+\varepsilon}{\lambda}(1+\varepsilon) & \text{if } 1-\varepsilon < |x| \le 1+\varepsilon \end{cases} \tag{6.11}$$

with

$$b = \frac{16 - 16\varepsilon + 3\frac{\lambda}{\varepsilon} + 4\varepsilon\lambda - 8\lambda}{4\left(4 + \frac{\lambda}{\varepsilon} - 2\lambda\right)}.$$

The $L^2(\Omega\backslash\Omega^n)$—norms of the derivatives of the functions u_n defined in (6.11) are

$$||u_n'||^2_{L^2(\Omega\backslash\Omega^n)} = 2\frac{\varepsilon(b-1+\varepsilon)^2}{\lambda^2}. \tag{6.12}$$

Remark 6.3 It is easy to verify that the sequence of the derivatives of the functions u_n is not uniformly bounded in the space $L^2(\Omega)$. In fact formula (6.12) shows that if $\frac{\lambda}{\varepsilon} \to 0$ then $||u_n'||^2_{L^2(\Omega\backslash\Omega^n)}$ behaves like $\frac{2}{16^2\varepsilon}$. In particular, if we choose $\lambda = \varepsilon^2$ then all the conditions of Theorem 9.3 in [3] are satisfied while the uniform boundedness of the $L^2(\Omega)$-norms of the gradients [see (9.5)] fails. We think that this discrepancy is due to the fact that the reinforcement goes inside the domain Ω, a choice that is *imposed* by the tricky geometry of the fractal (see also Remark 5.4).

Example 3 shows that we cannot expect weak convergence in the space $H^1(\Omega)$ in the assumptions of Theorem 5.3.

Example 3 Let us consider the obstacle problem of Example 2. In the reinforced problems, we choose the approximating data f_n according to Theorem 5.3, the coefficients of the operators being as before. Then

$$f_n = \begin{cases} -2 & \text{if } |x| \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < |x| \le 1+\varepsilon, \end{cases} \tag{6.13}$$

the coefficients are

$$a_n = 1 \quad \text{in } \Omega^n, \; a_n = \lambda \quad \text{in } \widehat{\Omega}^n \backslash \Omega^n \tag{6.14}$$

and the solution is

$$u_n = \begin{cases} 0 & \text{if } |x| \le \frac{1}{2} \\ -\frac{x^2}{2} + b|x| + \frac{1}{8} - \frac{b}{2} & \text{if } \frac{1}{2} < |x| \le 1-\varepsilon \\ -\frac{x^2}{2\lambda} + \frac{b|x|}{\lambda} - \frac{b(1+\varepsilon)}{\lambda} + \frac{(1+\varepsilon)^2}{2\lambda} & \text{if } 1-\varepsilon < |x| \le 1+\varepsilon \end{cases} \tag{6.15}$$

with

$$b = \frac{16 + 3\frac{\lambda}{\varepsilon} + 4\varepsilon\lambda - 8\lambda}{4\left(4 + \frac{\lambda}{\varepsilon} - 2\lambda\right)}.$$

The $L^2(\Omega\backslash\Omega^n)$-norms of the derivatives of the functions u_n defined in (6.15) are

$$||u_n'||^2_{L^2(\Omega\backslash\Omega^n)} = 2\frac{\varepsilon(b-1)^2 + \varepsilon^2(b-1) + \varepsilon^3 1/3}{\lambda^2} \tag{6.16}$$

The $L^2(\Omega\backslash\Omega^n)$-norms of the functions u_n defined in (6.15) are

$$||u_n||^2_{L^2(\Omega\backslash\Omega^n)} = \frac{7/3\varepsilon^3(b-1)^2 + 11/3\varepsilon^4(1-b) + 8/15\varepsilon^5}{2\lambda^2}. \tag{6.17}$$

Remark 6.4 It is easy to verify that the sequence of the derivatives of the functions u_n is not uniformly bounded in the space $L^2(\Omega)$. In fact, formula (6.16) shows that if $\frac{\lambda}{\varepsilon} \to 0$ then $||u_n'||^2_{L^2(\Omega\backslash\Omega^n)}$ behaves like $\frac{2}{16^2\varepsilon} - \frac{2\varepsilon}{16\lambda} + \frac{2\varepsilon^3}{3\lambda^2}$. If we choose $\lambda = \varepsilon^{5/3}$, condition (5.20) is satisfied but the sequence of norms in $L^2(\Omega)$ of the derivatives of the functions u_n diverges. As regards the behaviour of the functions, formula (6.17) shows that $||u_n||^2_{L^2(\Omega\backslash\Omega^n)}$ behaves like $\max(\frac{\varepsilon^5}{\lambda^2}, \frac{\varepsilon^3}{\lambda})$. If we choose $\lambda = \varepsilon^{5/2+\eta}$, $0 < \eta < \frac{1}{2}$, then condition (5.21) does not hold [only condition (5.22) is satisfied] and the sequence of $L^2(\Omega)$-norms of the functions u_n diverges like $\varepsilon^{-\eta}$.

7 Interior reinforcement

In [14] we established homogenization results for an insulating fractal surface S of Koch type which is approximated by 3-dimensional insulating layers with both vanishing conductivity and thickness. Also in this case we consider the corresponding obstacle problems.

We consider a 3-dimensional Euclidean domain Q containing a fractal subset S, the *layer*. Our basic model refers to the geometry illustrated in Fig. 1. Here the layer is of the type

$$S = K \times I, \tag{7.1}$$

where K is the so-called *Koch type curve* defined in Sect. 2 and

$$Q = \mathcal{O} \times (0,1) = (0,1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times (0,1);$$

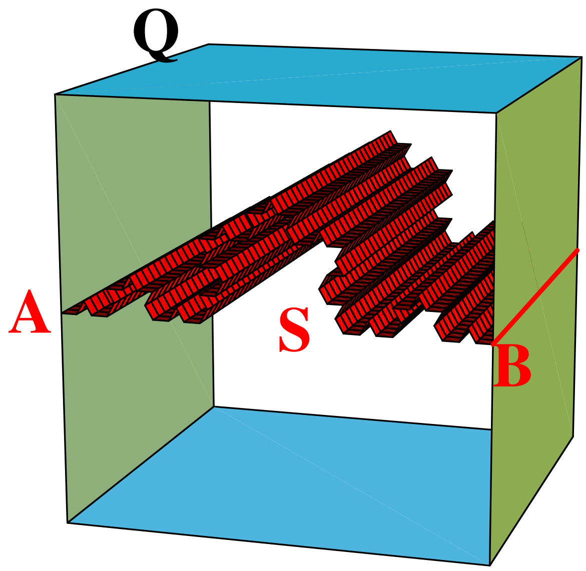
every point $P \in Q$ has coordinates (x_1, x_2, y) and the boundary of S belongs to the boundary of Q. We note that the layer S is a d-set in $\mathbb{R}^3$ with respect to the measure $d\mu dy$ with $d = d_S = d_f + 1$ (see [18]). The fractal layer S divides the domain Q in two adjacent subdomains $Q_i, i = 1, 2$, where Q_1 denotes the domain above the layer S.

We denote by Σ^0 the *open set condition* triangle of vertices $A = (0,0)$, $B = (1,0)$ and $C = (1/2, b/2)$ where $b = \tan(\frac{\theta}{2})$ and by K^0 the line segment that has as endpoints A and B.

For every n, we define the (open) polygonal fiber Σ^n in the cross-section

$$\Sigma^n = \bigcup_{i|n} \Sigma^{i|n} \quad \text{where } \Sigma^{i|n} = \psi_{i|n}(\Sigma^0), \tag{7.2}$$

Fig. 1 Basic model



and the auxiliary curve

$$\check{K}^n = \bigcup_{i|n} \check{K}^{i|n} \quad \text{where } \check{K}^{i|n} = \partial\Sigma^{i|n} \setminus \overset{\circ}{K}{}^{i|n}, \quad K^{i|n} = \psi_{i|n}(K^0). \tag{7.3}$$

We set $\Gamma^0 = \Sigma^0 \times \overset{\circ}{I}$, $\Gamma^{i|n} = \Sigma^{i|n} \times \overset{\circ}{I}$, $\Gamma^n = \Sigma^n \times \overset{\circ}{I}$, $S^0 = K^0 \times I$, $S^{i|n} = K^{i|n} \times I$, $S^n = K^n \times I$, $\check{K}^0 = \partial\Sigma^0 \setminus \overset{\circ}{K}{}^0$, $\mathcal{G}^0 = \check{K}^0 \times I$, $\mathcal{G}^{i|n} = \check{K}^{i|n} \times I$ and $\mathcal{G}^n = \check{K}^n \times I$. The polyhedral surfaces S^n and $\mathcal{G}^n$ divide the domain Q into three subdomains. We denote by $\hat{Q}_n^2$ the domain below S^n and by $\hat{Q}_n^1$ the domain above $\mathcal{G}^n$, that is

$$Q = \overset{\circ}{\overbrace{\hat{Q}_n^1 \cup \bar{\Gamma}^n \cup \hat{Q}_n^2}}. \tag{7.4}$$

We note that if $P = (x, y) \in \Gamma^n$, then $x \in \Sigma^n$. Each $x \in \Sigma^n$ belongs to a segment of endpoints X and $X^\perp$ where $X \in \check{K}^{i|n}$ and $X^\perp$ is the orthogonal projection of X on $K^{i|n}$ for some index $i|n$. By $|X - X^\perp|$ we denote the (Euclidean) distance between X and $X^\perp$ (in $\mathbb{R}^2$).

In the domain Q, for any n, we define a *weight* w^n as follows. Let $P = (x, y) \in Q$

$$w^n(P) = w^n(x) = \begin{cases} \frac{3|X-X^\perp|}{3+b^2} & \text{if } (x, y) \in \Gamma^{i|n} \\ 1 & \text{if } (x, y) \notin \Gamma^n. \end{cases} \tag{7.5}$$

Associated with the weight w^n, we consider the Sobolev spaces $H^1(Q; w^n)$ and $H_0^1(Q; w^n)$, defined as the completion of $C^\infty(\overline{Q})$ and $C_0^\infty(Q)$, respectively, in the norm

$$\|u\|_{H^1(Q;w^n)} = \left\{ \int_Q u^2 dxdy + \int_Q |\nabla u|^2 w^n dxdy \right\}^{\frac{1}{2}}. \tag{7.6}$$

We consider the sequence of weighted energy functionals in $L^2(Q)$

$$a_n(u, v) := \int_Q \sum_{i,j=1}^{3} a_{ij}^n \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dxdy \tag{7.7}$$

where we define the coefficients

$$a_{ij}^n(x,y)=\begin{cases}\sigma_n c_n \delta_{ij}\, w^n(x) & \text{if } (x,y)\in\Gamma^n\\ a_{ij} & \text{if } (x,y)\in Q\setminus\Gamma^n\end{cases} \tag{7.8}$$

where

$$c_n>0,\quad \sigma_n=\frac{\alpha^n}{4^n} \tag{7.9}$$

and the coefficients a_{ij} with $1\le i,j\le 3$ satisfy (2.6) and

$$\lambda|\xi|^2\le\sum_{i,j=1}^{3}a_{ij}\xi_i\xi_j\le\Lambda|\xi|^2 \tag{7.10}$$

for $\Lambda\ge\lambda>0$.

We define the set

$$\mathcal{V}(Q)=\{g\in L^2(Q):\exists g_j^*\in Lip(\overline{Q}_j): g_j^*|_{Q_j}=g|_{Q_j}\}. \tag{7.11}$$

From now on, when it does not create ambiguity, we drop the superscript $*$ and we simply write u_j, g_j and similar expressions.
We recall that Σ^0 is the *open set condition* triangle of vertices $A=(0,0)$, $B=(1,0)$ and $C=(1/2,b/2)$ where $b=\tan(\frac{\theta}{2})$, we then divide Σ^0 into two triangles $\mathcal{T}_h$, $h=1,2$ with vertices A,H,C and H,B,C respectively, where $H=(1/2,0)$.
We denote by G the operator from $\mathcal{V}(Q)$ to $H^1(\Gamma_0)$ defined as follows: for $(\xi_1,\xi_2,y)\in\overline{\Gamma}_0$

$$G(g)(\xi_1,\xi_2,y)=\begin{cases}g_2(0,0,y) & \text{if } (\xi_1,\xi_2)=(0,0)\\ \frac{\xi_2}{b\xi_1}g_1(\xi_1,b\xi_1,y)+(1-\frac{\xi_2}{b\xi_1})g_2(\xi_1,0,y) & \text{if } (\xi_1,\xi_2)\in\overline{\mathcal{T}}_1\setminus\{A\}\\ \frac{\xi_2}{b(1-\xi_1)}g_1(\xi_1,b(1-\xi_1),y)+(1-\frac{\xi_2}{b(1-\xi_1)})g_2(\xi_1,0,y) & \text{if } (\xi_1,\xi_2)\in\overline{\mathcal{T}}_2\setminus\{B\}\\ g_2(1,0,y) & \text{if } (\xi_1,\xi_2)=(1,0).\end{cases} \tag{7.12}$$

For every $u\in\mathcal{V}(Q)$ and n we define in $\overline{\Gamma^{i|n}}$ the function $v_{i|n}$

$$v_{i|n}(x,y)=G(u\circ\Psi_{i|n})\circ\Psi_{i|n}^{-1}(x,y)\qquad \text{if } (x,y)\in\overline{\Gamma^{i|n}} \tag{7.13}$$

where $\Psi_{i|n}(\xi_1,\xi_2,y)=(\psi_{i|n}(\xi_1,\xi_2),y)$. For every $u\in\mathcal{V}(Q)$ and n we define

$$v_n(x,y)=\begin{cases}u_j(x,y) & \text{if } (x,y)\in\hat{Q}_j^n,\ j=1,2\\ v_{i|n}(x,y) & \text{if } (x,y)\in\overline{\Gamma^{i|n}}.\end{cases} \tag{7.14}$$

We assume the following condition on the obstacle

$$\varphi_1\in C^1(\bar{Q}),\ \varphi_1\le 0\quad \text{on } \partial Q, \tag{7.15}$$

and we first study the *coercive case*, that is, we choose the convex sets

$$\mathcal{K}_\infty=\{u\in H_0^1(Q),\ u\ge\varphi_1\}$$

and

$$\mathcal{K}=\{u\in D_0(Q),\ u\ge\varphi_1\}$$

where $D_0(Q)=\{u\in L^2(Q): u_j{:=}u|_{Q_j}\in H^1(Q_j), u_j=0 \text{ on } \partial Q_j\setminus S\}$.

We define the limit forms as follows

$$a_{c_0}(u,v)=\int_{Q_1}\sum_{i,j=1}^{3}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dxdy+\int_{Q_2}\sum_{i,j=1}^{3}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dxdy$$
$$+\int_S c_0(u_1-u_2)(v_1-v_2)d\mu dy \tag{7.16}$$

and

$$a_\infty(u,v)=\int_Q\sum_{i,j=1}^{3}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dxdy. \tag{7.17}$$

We construct a suitable obstacle $(\varphi_1)_n$ according to the formula (7.14) and we consider

$$\mathcal{K}_n=\{u\in H_0^1(Q;w^n),\ u\geq(\varphi_1)_n\}.$$

Before stating our results in this framework, we specify, in this geometry, an inequality Poincaré type where, as previously in Sect. 3, the relevant fact is that the constant C_P is independent of n. The proof is similar to the proof of Theorem 7.4 (following), hence we skip it (see also Theorem 6.1 in [13]).

Theorem 7.1 *For any function $u\in H^1(Q;w^n)$ the following estimate holds*

$$||u||^2_{L^2(\Gamma^n)}\leq\alpha^{-n}\left(\int_{\Gamma^n}|\nabla u|^2w^n dxdy+\int_{S^n}u^2dsdy\right). \tag{7.18}$$

Moreover, there exists a constant C_P independent of n, such that,

$$||u||_{L^2(Q)}\leq C_P\left(||\nabla u||^2_{L^2(Q\setminus\Gamma^n)}+\alpha^{-n}\int_{\Gamma^n}|\nabla u|^2w^n dxdy\right)^{1/2} \tag{7.19}$$

for all $u\in H_0^1(Q;w^n)$.

Taking into account Poincaré inequality (7.19) and the classical Poincaré inequality (see e.g. [23]), we can prove existence and uniqueness results. More precisely

Proposition 7.1 *For any $f_n\in L^2(Q)$, there exists one and only one solution u_n to the following problem*

$$\begin{cases}\text{find } u_n\in\mathcal{K}_n \text{ such that}\\ a_n(u_n,v-u_n)\geq\int_Q f_n\,(v-u_n)\,dxdy\quad\forall\, v\in\mathcal{K}_n\end{cases} \tag{7.20}$$

where a_n is defined in (7.7). For any $f\in L^2(Q)$, there exists one and only one solution u to the following problem

$$\begin{cases}\text{find } u\in\mathcal{K}_\infty \text{ such that}\\ a_\infty(u,v-u)\geq\int_Q f\,(v-u)\,dxdy\quad\forall\, v\in\mathcal{K}_\infty\end{cases} \tag{7.21}$$

where a_∞ is defined in (7.17). Moreover, for any $f\in L^2(Q)$, $c_0\geq 0$, there exists one and only one solution u to the following problem

$$\begin{cases}\text{find } u\in\mathcal{K} \text{ such that}\\ a_{c_0}(u,v-u)\geq\int_Q f\,(v-u)\,dxdy\quad\forall\, v\in\mathcal{K}\end{cases} \tag{7.22}$$

where a_{c_0} is defined in (7.16).

We assume

$$f_n,\ f \in L^2(Q),\ \text{ and } f_n \to f \quad \text{in } L^2(Q); \tag{7.23}$$

by taking into account the results of Theorems 1.2 and 1.1 of [14], we obtain the following results that we can prove by proceeding as in Sect. 5.

Theorem 7.2 *Let us assume* (4.21) *and* (7.23). *Then the sequence of the solutions u_n [defined in* (7.20)*] converges to the function u [defined in* (7.21)*] weakly in $H^1_{loc}(Q_i)$, $i = 1, 2$, and strongly in $L^2(Q)$.*

Theorem 7.3 *Let us assume* (4.6), (5.21), *and* (7.23). *Then the sequence of the solutions u_n [defined in* (7.20)*] converges to the function u [defined in* (7.22)*] weakly in $H^1_{loc}(Q_i)$, $i = 1, 2$, and weakly in $L^2(Q)$. Moreover, if we assume condition* (5.3)*, then the sequence of the solutions u_n converges to the function u strongly in $L^2(Q)$.*

We will now consider the *semi-coercive case*. Let

$$\mathcal{F}_n = \{u \in H^1(Q; w^n),\ u \geq (\varphi_1)_n\}$$

and

$$\mathcal{F} = \{u \in D(Q),\ u \geq \varphi_1\}$$

where $D(Q) = \{u \in L^2(Q) : u_j{:=}u|_{Q_j} \in H^1(Q_j)\}$, the obstacle $\varphi_1 \in C^1(\bar{Q})$ and the obstacle $(\varphi_1)_n$ is as in (7.14). We recall that in the semi-coercive case, we have $c_0 = 0$, hence we assume suitable conditions on the data f, f_n and on the convex [see (2.23) and (2.24)]. More precisely, we assume

$$0 \in \mathcal{F}, \tag{7.24}$$

$$\int_{Q_i} f\, dxdy < 0 \quad i = 1, 2 \tag{7.25}$$

$$\int_{Q} f_n\, dxdy < 0. \tag{7.26}$$

As in [22] and [32], see also the proof of Theorem 2.3, we can show the following proposition

Proposition 7.2 *Assume $f_n \in L^2(Q)$,* (7.24) *and* (7.26). *Then there exists one and only one solution u_n to the following problem*

$$\begin{cases} \textit{find} \ \ u_n \in \mathcal{F}_n \ \ \textit{such that} \\ a_n(u_n, v - u_n) \geq \int_Q f_n\,(v - u_n)\, dxdy \quad \forall\, v \in \mathcal{F}_n \end{cases} \tag{7.27}$$

where a_n is defined in (7.7). *Assume $f \in L^2(Q)$,* (7.24) *and* (7.25). *Then there exists one and only one solution u to the following problem*

$$\begin{cases} \textit{find} \ \ u \in \mathcal{F} \ \ \textit{such that} \\ a(u, v - u) \geq \int_Q f\,(v - u)\, dxdy \quad \forall\, v \in \mathcal{F} \end{cases} \tag{7.28}$$

where

$$a(u, v) = \int_{Q_1} \sum_{i,j=1}^{3} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dxdy + \int_{Q_2} \sum_{i,j=1}^{3} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dxdy. \tag{7.29}$$

Before stating our results in this framework, we specify, in this geometry, an inequality of Poincaré type, where the relevant fact is that the constant C_P is independent of n.

Theorem 7.4 *For any function $u \in H^1(Q; w^n)$, the following estimate holds*

$$||u||^2_{L^2(\Gamma^n)} \leq \alpha^{-n}\left(\int_{\Gamma^n} |\nabla u|^2 w^n dxdy + \int_{S^n} u^2 dsdy\right). \tag{7.30}$$

Moreover, there exists a constant C_P independent of n, such that,

$$||u||_{L^2(Q)} \leq C_P\left(||\nabla u||^2_{L^2(Q\setminus\Gamma^n)} + \sigma_n \int_{\Gamma^n} |\nabla u|^2 w^n dxdy + \sigma_n \int_{S^n} u^2 dsdy\right)^{1/2} \tag{7.31}$$

for all $u \in H^1(Q; w^n)$.

Proof We start by proving estimate (7.30). For every n, we split the corresponding integrals

$$\int_{\Sigma^n} u^2 dx = \sum_{i|n} \int_{\Sigma^{i|n}} u^2 dx.$$

It suffices to give the proof for $\Sigma^{i|n}$, as the other integrals can be evaluated similarly. By the change of coordinates $x = (x_1, x_2) = \psi_{i|n}(\xi_1, \xi_2)$, with $(\xi_1, \xi_2) \in \Sigma^0$, we define

$$g(\xi_1, \xi_2) = (u \circ \psi_{i|n})(\xi_1, \xi_2) \tag{7.32}$$

and we have

$$\begin{aligned}\int_{\Sigma^{i|n}} u^2 dx &= \alpha^{-2n}\left\{\int_0^{1/2} d\xi_1 \int_0^{b\xi_1} g^2(\xi_1, \xi_2) d\xi_2 + \int_{1/2}^1 d\xi_1 \int_0^{b(1-\xi_1)} g^2(\xi_1, \xi_2) d\xi_2\right\} \\ &:= \alpha^{-2n}\{I_1 + I_2\}.\end{aligned} \tag{7.33}$$

As

$$g^2(\xi_1, \xi_2) \leq 2((g(\xi_1, \xi_2) - g(\xi_1, 0))^2 + g^2(\xi_1, 0))$$

then

$$I_1 \leq 2\left\{\frac{1}{2}\int_0^{1/2} d\xi_1 b^2 \xi_1^2 \int_0^{b\xi_1} |\nabla g|^2 d\xi_2 + \frac{b}{2}\int_0^{1/2} g^2(\xi_1, 0) d\xi_1\right\}. \tag{7.34}$$

Analogously

$$I_2 \leq 2\left\{\frac{1}{2}\int_{1/2}^1 d\xi_1 b^2 (1-\xi_1)^2 \int_0^{b(1-\xi_1)} |\nabla g|^2 d\xi_2 + \frac{b}{2}\int_{1/2}^1 g^2(\xi_1, 0) d\xi_1\right\}. \tag{7.35}$$

Then by definition of the weight [see (7.5)]

$$\alpha^{-2n}\{I_1 + I_2\} \leq 2\alpha^{-2n}\left\{\frac{b(3+b^2)}{12}\int_{\Sigma^0} |\nabla g|^2 w^0(\xi_1, \xi_2) d\xi_1 d\xi_2 + \frac{b}{2}\int_{K^0} g^2(\xi_1, 0) d\xi_1\right\}. \tag{7.36}$$

By the change of coordinates $(\xi_1, \xi_2) = \psi^{-1}_{i|n}(x_1, x_2)$, and taking into account that $\alpha^{-n} w^0(\psi^{-1}_{i|n}(x_1, x_2)) = w^n(x_1, x_2)$ we derive

$$\int_{\Sigma^{i|n}} u^2 dx \le \alpha^{-n}\left(\int_{\Sigma^{i|n}} |\nabla u|^2 w^n dx + \int_{K^{i|n}} u^2 ds\right). \tag{7.37}$$

By summing up on $i|n$ and integrating on the interval I we conclude the proof of (7.30).

Now we prove statement (7.31). Suppose (7.31) is false: then, there exists an increasing sequence of indices $n_m \in \mathbb{N}$ and there exists a sequence of functions $u_m \in H^1(Q; w^{n_m})$, such that, for every $m \in \mathbb{N}$,

$$||u_m||^2_{L^2(Q)} > m^2\left(||\nabla u_m||^2_{L^2(Q\setminus\Gamma^{n_m})} + \sigma_{n_m}\int_{\Gamma^{n_m}} |\nabla u_m|^2 w^{n_m} dxdy + \sigma_{n_m}\int_{S^{n_m}} u_m^2 dsdy\right). \tag{7.38}$$

In fact, if such a sequence of indices does not exist we can repeat, for a fixed $\bar{n}$, the argument below to obtain a contradiction again. Set

$$v_m := \frac{u_m}{||u_m||_{L^2(Q)}}.$$

Therefore, there exist a function v^* in $L^2(Q)$ and a subsequence (still denoted by v_m) that converges to v^* weakly in $L^2(Q)$. From (7.38), we obtain

$$||\nabla v_m||^2_{L^2(Q\setminus\Gamma^{n_m})} + \sigma_{n_m}\int_{\Gamma^{n_m}} |\nabla v_m|^2 w^{n_m} dxdy + \sigma_{n_m}\int_{S^{n_m}} v_m^2 dsdy < \frac{1}{m^2} \tag{7.39}$$

and, in particular,

$$||v_m||^2_{H^1(Q\setminus\Gamma^{n_m})} \leqslant 1 + \frac{1}{m^2}.$$

For $i = 1, 2$, we consider the restriction to $\hat{Q}_i^n$ and we denote by $v^*_{m,i}$ the extension to Q and we have

$$||v^*_{m,i}||_{H^1(Q)} \le C \tag{7.40}$$

with C independent of n and m (see Theorem 5.7 in [9]).

Then, there exist a function v_1^* in $H^1(Q)$ and a subsequence (still denoted by $v^*_{m,1}$) that converges to v_1^* weakly in $H^1(Q)$ (and strongly in $H^s(Q)$ for $0 \le s < 1$). Analogously, there exist a function v_2^* in $H^1(Q)$ and a subsequence (still denoted by $v^*_{m,2}$) that converges to v_2^* weakly in $H^1(Q)$ (and strongly in $H^s(Q)$ for $0 \le s < 1$). We observe that $v_i^*|_{Q_i} = v^*|_{Q_i}$. We fix n_0 : we deduce

$$||\nabla v_i^*||_{L^2(\hat{Q}_i^{n_0})} \le \liminf_m ||\nabla v_m|_{\hat{Q}_i^{n_0}}||_{L^2(\hat{Q}_i^{n_0})} = 0, \tag{7.41}$$

i.e. the function v_i^* is constant on $\hat{Q}_i^{n_0}$ and we denote this constant by k_i; passing to the limit in n_0, we obtain $v_i^* = k_i$ in Q_i. From (7.39) we obtain that

$$k_2^2 = \int_S (v_2^*)^2 d\mu dy = \lim_m \sigma_{n_m}\int_{S^{n_m}} v_m^2 dsdy = 0.$$

We prove that $k_1 = 0$ too. We first prove that

$$\sigma_{n_m}\left|\int_{S^{n_m}} v^*_{m,2} dsdy - \frac{1}{\sqrt{1+b^2}}\int_{\mathcal{G}^{n_m}} v^*_{m,1} dsdy\right|$$
$$\le \left(\frac{(3+b^2)}{3}\sigma_{n_m}\int_{\Gamma^{n_m}} |\nabla v_m|^2 w^{n_m} dxdy\right)^{\frac{1}{2}}. \tag{7.42}$$

In fact by Hölder inequality we obtain

$$
\begin{aligned}
&\left|\int_{K^{n_m}} v^*_{m,2}ds - \frac{1}{\sqrt{1+b^2}}\int_{\check{K}^{n_m}} v^*_{m,1}ds\right| \\
&= \left|\sum_{i|n_m}\left\{\int_{K^{i|n_m}} v_m ds - \frac{1}{\sqrt{1+b^2}}\int_{\check{K}^{i|n_m}} v_m ds\right\}\right| \\
&\le 4^{\frac{n_m}{2}}\left\{\sum_{i|n_m}\left|\int_{K^{i|n_m}} v_m ds - \frac{1}{\sqrt{1+b^2}}\int_{\check{K}^{i|n_m}} v_m ds\right|^2\right\}^{\frac{1}{2}}.
\end{aligned}
$$

By the change of coordinates $x = (x_1, x_2) = \psi_{i|n_m}(\xi_1, \xi_2)$, we have

$$
\begin{aligned}
&\left|\int_{K^{i|n_m}} v_m ds - \frac{1}{\sqrt{1+b^2}}\int_{\check{K}^{i|n_m}} v_m ds\right|^2 \\
&= \alpha^{-2n_m}\left|\int_0^1 v_m(\psi_{i|n}(\xi_1, 0))d\xi_1 - \int_0^{1/2} v_m(\psi_{i|n_m}(\xi_1, b\xi_1))d\xi_1\right. \\
&\quad \left. - \int_{1/2}^1 v_m(\psi_{i|n_m}(\xi_1, b(1-\xi_1))d\xi_1\right|^2 \\
&= \alpha^{-2n_m}\left|\int_0^{1/2}(v_m(\psi_{i|n}(\xi_1, 0)) - v_m(\psi_{i|n_m}(\xi_1, b\xi_1)))d\xi_1\right. \\
&\quad \left. + \int_{1/2}^1 (v_m(\psi_{i|n}(\xi_1, 0)) - v_m(\psi_{i|n_m}(\xi_1, b(1-\xi_1)))d\xi_1\right|^2 \\
&:=\alpha^{-2n_m}|X_1 + X_2|^2.
\end{aligned}
$$

We proceed as previously in proving estimate (7.30) and we obtain [see definition of the weight (7.5)]

$$
\begin{aligned}
|X_1 + X_2|^2 &\le 2(|X_1|^2 + |X_2|^2) \\
&\le 2\frac{1}{2}\frac{(3+b^2)}{3}\left(\int_0^{1/2} d\xi_1 \int_0^{b\xi_1} |\nabla(v_m \circ \psi_{i|n_m})|^2 w^0(\xi_1, \xi_2)d\xi_2\right. \\
&\quad \left. + \int_{1/2}^1 d\xi_1 \int_0^{b(1-\xi_1)} |\nabla(v_m \circ \psi_{i|n_m})|^2 w^0(\xi_1, \xi_2)d\xi_2\right) \\
&= \frac{(3+b^2)}{3}\int_{\Sigma^0} \nabla(v_m \circ \psi_{i|n_m})|^2 w^0(\xi_1, \xi_2)d\xi_2.
\end{aligned}
\tag{7.43}
$$

By the change of coordinates $(\xi_1, \xi_2) = \psi^{-1}_{i|n_m}(x_1, x_2)$, and taking into account that $\alpha^{-n_m} w^0(\psi^{-1}_{i|n_m}(x_1, x_2)) = w^{n_m}(x_1, x_2)$ we derive

$$
\left|\int_{K^{i|n_m}} v_m ds - \frac{1}{\sqrt{1+b^2}}\int_{\check{K}^{i|n_m}} v_m ds\right|^2 \le \frac{3+b^2}{3}\alpha^{-n_m}\int_{\Sigma^{i|n_m}} |\nabla v_m|^2 w^{n_m} dx. \tag{7.44}
$$

By summing up on $i|n_m$ and integrating on the interval I, we conclude the proof of (7.42). From (7.42), by using (7.39), we obtain

$$
\sigma_{n_m}\left|\int_{S^{n_m}} v^*_{m,2}dsdy - \frac{1}{\sqrt{1+b^2}}\int_{\mathcal{G}^{n_m}} v^*_{m,1}dsdy\right| \le \sqrt{\frac{3+b^2}{3}\frac{1}{m}}. \tag{7.45}
$$

Passing to the limit in (7.45) as $m \to 0$ we obtain $k_1 = 0$. By (7.30) we deduce

$$||v_m||^2_{L^2(\Gamma^{n_m})} \le \alpha^{-n_m}\left(\int_{\Gamma^{n_m}} |\nabla v_m|^2 w^{n_m} dxdy + \int_{S^{n_m}} v_m^2 dsdy\right) \le \frac{1}{m^2} \tag{7.46}$$

therefore, as

$$\begin{aligned} 1 = ||v_m||^2_{L^2(Q)} &= ||v_m||^2_{L^2(\hat{Q}_1^{n_m})} + ||v_m||^2_{L^2(\hat{Q}_2^{n_m})} + ||v_m||^2_{L^2(\Gamma^{n_m})} \\ &\le ||v^*_{m,1}||^2_{L^2(Q_1)} + ||v^*_{m,2}||^2_{L^2(Q_2)} + \frac{1}{m^2} \end{aligned}$$

passing to the limit, we obtain a contradiction. □

Theorem 7.5 *Let us assume conditions* (4.6) *with* $c_0 = 0$, (7.23)–(7.26)*, and,*

$$\frac{||f_n||^2_{L^2(\Gamma^n)}}{c_n \sigma_n \alpha^n} \le c^* \tag{7.47}$$

with $c^* > 0$. *Then the sequence of the solutions* u_n [*defined in* (7.27)] *converges to the function* u [*defined in* (7.28)] *weakly in* $H^1_{loc}(Q_i)$, $i = 1, 2$. *If assumption* (5.21) *holds and thus*

$$\frac{||f_n||^2_{L^2(\Gamma^n)}}{c_n \sigma_n \alpha^n} \to 0 \tag{7.48}$$

then the solutions u_n *converge weakly in* $L^2(Q)$. *Moreover, if assumption* (5.3) *holds then the solutions* u_n *converge strongly in* $L^2(Q)$.

Remark 7.1 We note that if

$$f_n = f \text{ on } \hat{Q}_i^n, \quad i = 1, 2 \quad \text{and} \quad f_n = 0 \text{ on } \Gamma^n \tag{7.49}$$

then assumptions (7.23) and (7.47) are obviously fulfilled; hence, we have weak convergence in $H^1_{loc}(Q_i)$ without assuming any condition on the vanishing rate of the coefficients c_n. Alternatively, we can link the vanishing rate of the sequence c_n to the vanishing rate of the L^2-norms of the data f_n in the *reinforcement sets* Γ^n.

Proof The proof is achieved when we prove

$$||u_n||^2_{L^2(Q\setminus\Gamma^n)} + ||\nabla u_n||^2_{L^2(Q\setminus\Gamma^n)} \leqslant C \tag{7.50}$$

where the constant C does not depend on n. In order to show this estimate, we proceed by contradiction as in the proof of Theorem 5.3. Suppose that for every m in $\mathbb{N}$, there exists an increasing sequence of indices n_m and a sequence u_{n_m}, that we shall denote, from now on, simply by u_m such that

$$A_m := (||u_m||^2_{L^2(Q\setminus\Gamma^m)} + ||\nabla u_m||^2_{L^2(Q\setminus\Gamma^m)})^{\frac{1}{2}} \ge m. \tag{7.51}$$

Set

$$v_m := \frac{u_m}{A_m}$$

we have

$$||v_m||^2_{L^2(Q\setminus\Gamma^m)} + ||\nabla v_m||^2_{L^2(Q\setminus\Gamma^m)} = 1, \tag{7.52}$$

that is,

$$\|v_m\|^2_{H^1(Q\setminus\Gamma^m)} = 1. \tag{7.53}$$

We denote by $v^*_{m,i}$ an extension of $(v_m)|_{\hat{Q}^m_i}$ to Q such that $v^*_{m,i} \in H^1(Q)$ and

$$\|v^*_{m,i}\|^2_{H^1(Q)} \leq C \tag{7.54}$$

with C independent of m, $i = 1, 2$ (see Theorem 5.7 in [9]). Then, there exist a function v^*_1 in $H^1(Q)$ and a subsequence (still denoted by $v^*_{m,1}$) that converges to v^*_1 weakly in $H^1(Q)$ (and strongly in $H^s(Q)$ for $0 \leq s < 1$). Analogously there exist a function v^*_2 in $H^1(Q)$ and a subsequence (still denoted by $v^*_{m,2}$) that converges to v^*_2 weakly in $H^1(Q)$(and strongly in $H^s(Q)$ for $0 \leq s < 1$). From (7.24) and the construction of $(\varphi_1)_m$ [see (7.14)] we deduce that $0 \in \mathcal{F}_m$, hence

$$a_m(u_m, u_m) \leq \int_Q f_m\, u_m\, dxdy \tag{7.55}$$

and

$$\lambda\|\nabla v_m\|^2_{L^2(Q\setminus\Gamma^m)} + c_m\sigma_m \int_{\Gamma^m} |\nabla v_m|^2 w^m\, dxdy \leq \frac{1}{A_m}\int_Q f_m\, v_m\, dxdy;$$

by using (7.23), (7.30), (7.47), (7.52), and Theorem 3.1 in [14], we obtain

$$\lambda\|\nabla v_m\|^2_{L^2(Q\setminus\Gamma^m)} + (1 - \frac{1}{A_m})c_m\sigma_m \int_{\Gamma^m} |\nabla v_m|^2 w^m\, dxdy \leq C\frac{1}{A_m}. \tag{7.56}$$

Hence, by the weak lower semi-continuity of the norm, we obtain, for any fixed m_0,

$$\|\nabla v^*_i\|^2_{L^2(\hat{Q}^{m_0}_i)} \leq \liminf_m \|\nabla v_{m,i}\|^2_{L^2(\hat{Q}^{m_0}_i)} = 0 \tag{7.57}$$

then $v^*_i = k_i$ a. e. in $\hat{Q}^{m_0}_i$. We observe that $k_i \geq 0$: in fact, by construction $v_m \geq \frac{1}{A_m}\min_{\bar{Q}}(\varphi_1)$. As m_0 is arbitrary, we deduce that $v^*_i = k_i$ a.e. in Q_i and $\nabla v^*_{m,i}$ weakly converges to 0 in $L^2(Q_i)$, $i = 1, 2$. We show that $k_i = 0$: if $k_i > 0$ we obtain a contradiction with (7.25). In fact, from (7.55), we obtain

$$0 \leq \int_Q f_m\, v_m\, dxdy. \tag{7.58}$$

We have that

$$\begin{aligned}\int_Q f_m\, v_m\, dxdy &= \int_{\hat{Q}^m_1} f_m\, v_m\, dxdy + \int_{\hat{Q}^m_2} f_m\, v_m\, dxdy + \int_{\Gamma^m} f_m\, v_m\, dxdy \\ &= \int_{Q_1} f_m\, v^*_{m,1}\, dxdy + \int_{Q_2} f_m\, v^*_{m,2}\, dxdy + \int_{\Gamma^m} f_m\, v_m\, dxdy \\ &\quad - \sum_{j=1}^{2}\int_{\Gamma^m\cap Q_j} f_m\, v^*_{m,j}\, dxdy.\end{aligned} \tag{7.59}$$

We estimate $\int_{\Gamma^m} f_m\, v_m\, dxdy$ by (7.30) and (7.56)

$$\|v_m\|^2_{L^2(\Gamma^m)} \leq C\left(\frac{1}{A_m c_m \sigma_m \alpha^m} + \frac{1}{\sigma_m\alpha^m}\|v^*_{m,2}\|^2_{H^1(Q_2)}\right). \tag{7.60}$$

By using (7.23), (7.47), and (7.54) we have that $\int_{\Gamma^m} f_m\, v_m\, dxdy \to 0$. On the other hand, the term $\sum_{j=1}^{2} \int_{\Gamma^m \cap Q_j} f_m\, v^*_{m,j}\, dxdy$ tends to 0 by the strong convergence of the functions $v^*_{m,j}$ in $L^p(Q),\ p > 2$, and by (7.23). From (7.25), (7.58), and (7.59) we deduce

$$0 \le \lim \int_{Q} f_m\, v_m\, dxdy = k_1 \int_{Q_1} f\, dxdy + k_2 \int_{Q_2} f\, dxdy \le 0 \tag{7.61}$$

and hence by (7.25) we have $k_i = 0$. On the other hand, if $k_i = 0$ we have a contradiction with (7.52). In fact, by (7.30), (7.52), (7.56)

$$\begin{aligned} 1 &= ||v_m||^2_{L^2(\hat{Q}^m_1)} + ||v_m||^2_{L^2(\hat{Q}^m_2)} + ||\nabla v_m||^2_{L^2(\hat{Q}^m_1)} + ||\nabla v_m||^2_{L^2(\hat{Q}^m_2)} \\ &\le ||v^*_{m,1}||^2_{L^2(Q_1)} + ||v^*_{m,2}||^2_{L^2(Q_2)} + C\frac{1}{A_m \lambda}. \end{aligned}$$

By taking into account (7.51) and the strong convergence of $v^*_{m_k,i}$ to zero in $L^2(Q_i)$, we obtain a contradiction. Then estimate (7.50) is proved and we can repeat the proof of Theorem 5.2 in order to show that the sequence of the solutions u_n [defined in (7.27)] converges to the function u [defined in (7.28)] weakly in $H^1_{loc}(Q_i)$, $i = 1, 2$. By proceeding as in the proof of Theorem 5.3, we can show that assumptions (7.48) and (5.21) provide the weak convergence in $L^2(Q)$ and assumption (5.3) provides the strong convergence in $L^2(Q)$. □

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