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Asymptotic analysis and sign-changing bubble towers for Lane–Emden problems

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Abstract. We consider the semilinear Lane–Emden problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{E}_p$}$$

where $p > 1$ and Ω is a smooth bounded domain of $\mathbb{R}^2$. The aim of the paper is to analyze the asymptotic behavior of sign-changing solutions of ($\mathcal{E}_p$) as $p \to \infty$. Among other results we show, under some symmetry assumptions on Ω, that the positive and negative parts of a family of symmetric solutions concentrate at the same point as $p \to \infty$, and the limit profile looks like a tower of two bubbles given by superposition of a regular and a singular solution of the Liouville problem in $\mathbb{R}^2$.

Keywords. Superlinear elliptic boundary value problems, sign-changing solutions, asymptotic analysis, bubble towers

1. Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^2$. We consider the Lane–Emden problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p > 1$.

The aim of this paper is to contribute to the analysis of the concentration phenomenon for sign-changing solutions of problem (1.1) as $p \to \infty$.

In order to explain properly the results and the difficulties related to this investigation let us make a short survey of known results and a comparison with the higher dimensional case when $\Omega \subseteq \mathbb{R}^N$, $N \geq 3$, $p < \frac{N+2}{N-2}$ and $p \to \frac{N+2}{N-2}$, i.e. p approaches the critical Sobolev exponent from below.

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In higher dimensions there is a large literature dealing with the asymptotic behavior of positive solutions, while very little is known for sign-changing ones. The reason is that there is a lack of understanding of the finite energy nodal solutions of the "limit" problem

$$-\Delta Z = |Z|^{4/(N-2)} Z \quad \text{in } \mathbb{R}^N,\ N \geq 3, \tag{1.2}$$

which naturally arises in the study of the asymptotic behavior of solutions of (1.1). We refer to [28] for further details.

The only completely understood case for sign-changing solutions, in higher dimensions, is when they have low energy, i.e. solutions (u_p) satisfy

$$\int_\Omega |\nabla u_p|^2\, dx \to 2S^{N/2} \quad \text{as } p \to \frac{N+2}{N-2}, \tag{1.3}$$

where S is the best Sobolev constant. In [3] a complete classification of such solutions is provided, showing that there are two possibilities. The first is that there exists a positive constant C such that

$$\frac{1}{C} \leq \frac{\|u_p^+\|_\infty}{\|u_p^-\|_\infty} \leq C \quad \text{as } p \to \frac{N+2}{N-2}. \tag{1.4}$$

Then u_p blows up and concentrates at two distinct points of Ω, and suitable scalings of u_p^+ and u_p^- (positive and negative part of u_p) converge, as $p \to \frac{N+2}{N-2}$, to a positive regular solution Z of (1.2). In other words, the limit profile of u_p is that of two separate bubbles carrying the same energy. Moreover the nodal set touches the boundary of Ω. The second case arises if (1.4) does not hold; then it is proved in [3] that u_p^+ and u_p^- blow up, they concentrate at the same point and they have the local limit profile, after scaling, of a positive regular solution Z of (1.2). Hence the solution u_p looks like a "tower" of two standard bubbles, each carrying the same energy. Moreover the nodal set does not touch $\partial\Omega$.

Let us now consider the case when $\Omega \subset \mathbb{R}^2$. The first papers where an asymptotic analysis of (1.1), as $p \to \infty$, has been carried out are [25, 26] where the authors considered the case of families (u_p) of least energy (hence positive) solutions and in some domains proved concentration results as well as some asymptotic estimates. Note that the solutions do not blow up as $p \to \infty$ (unlike in the higher dimensional case). Moreover the least energy solutions, for the 2-dimensional Lane–Emden problem, satisfy the condition

$$p \int_\Omega |\nabla u_p|^2\, dx \to 8\pi e \quad \text{as } p \to \infty. \tag{1.5}$$

Later, inspired by the paper [2] concerning 2-dimensional problems with critical exponential nonlinearities, Adimurthi and Grossi [1] (see also [18]) identified the "limit problem" by showing that suitable scalings of the least energy solutions u_p converge in $C^1_{\text{loc}}(\mathbb{R}^2)$ to a regular solution U of the Liouville problem

$$\begin{cases} -\Delta U = e^U & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^U\, dx = 8\pi. \end{cases} \tag{1.6}$$

They also considered general bounded domains and showed that $\|u_p\|_\infty$ converges to $\sqrt{e}$, thus confirming a previous conjecture of [7].

Recently in [27] the authors have analyzed the asymptotic behavior of solutions of some biharmonic equations and pointed out that the same analysis also applies to a family of positive solutions of (1.1) satisfying the condition

$$p \int_\Omega |\nabla u_p|^2 \, dx \to \beta < \infty \quad \text{as } p \to \infty, \tag{1.7}$$

for some positive constant $\beta \geq 8\pi e$. Their results show the concentration of the solutions at a finite number of distinct points in Ω, excluding the presence of nonsimple concentration points (i.e. bubble towers) and give a quantization of the energy.

Concerning sign-changing solutions, the asymptotic analysis was started in [19] by considering a family (u_p) of low-energy nodal solutions as for the higher dimensional case. Note that, for the 2-dimensional problem, this means

$$p \int_\Omega |\nabla u_p|^2 \, dx \to 16\pi e \quad \text{as } p \to \infty, \tag{1.8}$$

which is the analogue of (1.5) for low-energy positive solutions.

In [19] it was proved that if the minimum and the maximum of u_p are comparable, i.e. if there exists $K \geq 0$ such that

$$p(\|u_p^+\|_\infty - \|u_p^-\|_\infty) \to K \quad \text{as } p \to \infty \tag{1.9}$$

(which is the analogue of (1.4) when $N \geq 3$), then u_p concentrate at two distinct points of Ω and suitable scalings of u_p^+ and u_p^- converge to a regular solution U of (1.6). Hence the situation is the same as in the higher dimensional case when (1.4) holds. Moreover in [19] it was also proved that when u_p has Morse index 2 then the maximum and the minimum of u_p converge to $\pm\sqrt{e}$ and the nodal line touches the boundary of Ω.

Next, one would like to consider the case when (1.9) does not hold and would expect the presence of nonsimple concentration points or, in other words, the existence of solutions whose limit profile is given by superposition of two bubbles, as it happens when $N \geq 3$. The asymptotic analysis in this case looks difficult when $N = 2$. However, solutions with this limit profile do exist, as was first proved in [20] by analyzing the asymptotic behavior of least energy radial nodal solutions in the ball. More precisely, in [20] the authors proved the following result.

Theorem 1.1 (Grossi, Grumiau & Pacella [20]). *Let (u_p) be a family of least energy radial nodal solutions in the unit ball B centered at the origin with $u_p(0) > 0$. Then:*

(i) *a suitable scaling z_p^+ of u_p^+ converges in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$ to a regular solution U of (1.6);*

(ii) *a suitable scaling and translation z_p^- of u_p^- converges in $C^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$ to a singular radial solution V of*

$$\begin{cases} -\Delta V = e^V + H\delta_0 & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^V \, dx < \infty, \end{cases} \tag{1.10}$$

where H is a suitable negative constant and δ_0 is the Dirac measure centered at 0.

Moreover

$$\begin{aligned}
&\|u_p^+\|_\infty \xrightarrow[p\to\infty]{} \alpha^+ \quad (\simeq 2.46 > \sqrt{e}),\\
&\|u_p^-\|_\infty \xrightarrow[p\to\infty]{} \alpha^- \quad (\simeq 1.17 < \sqrt{e}),\\
&p\int_B |\nabla u_p|^2\,dx \xrightarrow[p\to\infty]{} C \quad (\simeq 332 > 16\pi e),\\
&pu_p(x) \xrightarrow[p\to\infty]{} 2\pi\gamma G(x,0) = \gamma \log|x|,
\end{aligned}$$

for some $\gamma > 0$, where G is the Green function on the ball B.

This result shows a substantial difference between the cases $N = 2$ and $N \geq 3$. For $N = 2$ there exist solutions which asymptotically look like the superposition of different bubbles given by a regular solution of (1.6) and a singular solution of (1.10). Moreover each bubble carries a different energy (unlike when $N \geq 3$).

One of the main results of this paper shows that the same phenomenon appears in other domains, different from balls, under some symmetry assumptions.

We obtain this through the asymptotic analysis of a family of sign-changing solutions found recently in [13]. A feature of these solutions is that they have an interior nodal line which does not meet the fixed point of the symmetry group of the domain.

To state our result precisely, we introduce some notation. For a given family (u_p) of sign-changing solutions of (1.1) we let

- NL_p be the nodal line of u_p,
- $x_p^\pm$ be a maximum/minimum point in Ω of u_p, i.e. $u_p(x_p^\pm) = \pm\|u_p^\pm\|_\infty$;
- $\mu_p^\pm := 1/\sqrt{p|u_p(x_p^\pm)|^{p-1}}$,
- $\widetilde{\Omega}_p^\pm := (\Omega - x_p^\pm)/\mu_p^\pm = \{x \in \mathbb{R}^2 : x_p^\pm + \mu_p^\pm x \in \Omega\}$.

We prove

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a connected bounded smooth domain, invariant under the action of a cyclic group G of rotations about the origin, with $|G| \geq 4e$ ($|G|$ is the order of G) and such that the origin O is in Ω. Let (u_p) be a family of sign-changing G-symmetric solutions of (1.1) with two nodal regions, $NL_p \cap \partial\Omega = \emptyset$, $O \notin NL_p$ and*

$$p\int_\Omega |\nabla u_p|^2\,dx \leq \alpha\, 8\pi e \tag{1.11}$$

for some $\alpha < 5$ and p large. Then, assuming without loss of generality that $\|u_p\|_\infty = \|u_p^+\|_\infty$, we have

(i) *$|x_p^\pm| \to O$ as $p \to \infty$;*
(ii) *NL_p shrinks to the origin as $p \to \infty$;*
(iii) *the rescaled functions $v_p^+(x) := p\frac{u_p(x_p^+ + \mu_p^+ x) - u_p(x_p^+)}{u_p(x_p^+)}$, about the maximum point, defined in $\widetilde{\Omega}_p^+$ converge (up to a subsequence) to the regular solution U of (1.6) with $U(0) = 0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$ as $p \to \infty$;*

(iv) *the rescaled functions* $v_p^-(x) := p\frac{u_p(x_p^- + \mu_p^- x) - u_p(x_p^-)}{u_p(x_p^-)}$, *about the minimum point, defined in* $\widetilde{\Omega}_p^-$ *converge (up to a subsequence) to* $V(x - x_\infty)$ *in* $C^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{x_\infty\})$ *as* $p \to \infty$, *where* V *is a singular radial solution of* (1.10) *for some suitable negative constant* H, *and* $x_\infty := -\lim_{p\to\infty} x_p^-/\mu_p^- \in \mathbb{R}^2 \setminus \{0\}$;

(v) $\sqrt{p}\, u_p \to 0$ *in* $C^1_{\mathrm{loc}}(\bar{\Omega} \setminus \{0\})$ *as* $p \to \infty$.

Remark 1.3. The existence of sign-changing solutions satisfying the hypotheses of Theorem 1.2 has been proved in [13] for any simply connected G-symmetric smooth bounded domain with $|G| \geq 4$.

The results of Theorem 1.2 show that both u^+ and u^- concentrate at the origin, and, after the above rescalings, they have the limit profile of a regular and a singular solution of the Liouville equation in $\mathbb{R}^2$.

As far as we know, this is the first result of this kind for problem (1.1) in domains different from the ball.

The starting point to prove Theorem 1.2 is an asymptotic analysis of a general family (u_p) of sign-changing solutions of (1.1) satisfying condition (1.7). This first results, inspired by the paper [16] (see also [17]) can be viewed as a first step towards the analysis of the asymptotic behavior of general sign-changing solutions of (1.1). This kind of profile decomposition results have been proved for several other problems and go back to the papers of Brezis–Coron [4, 5] whose proofs apply also to critical exponent problems (see for instance [21]).

Next we use the symmetry assumptions to prove that the maximum points x_p^+ converge to the origin as well as any other concentration points.

The hardest part of the asymptotic analysis is to prove that the rescaling about the minimum point x_p^- converges to a radial singular solution of a singular Liouville problem in $\mathbb{R}^2$. Indeed, while the rescaling of u_p about the maximum point x_p^+ can be studied in a "canonical" way, the analysis of the rescaling about x_p^- requires additional arguments. In particular the presence of the nodal line, with an unknown geometry, causes difficulties which, obviously, are not present when dealing with positive solutions or with radial sign-changing solutions. Also the proofs of the results for nodal radial solutions of [20] cannot be of any help since they strongly depend on 1-dimensional estimates.

We would like to point out that the analysis carried out in this paper also allows one to get the same asymptotic result of Theorem 1.2 if we replace the bound (1.11) on the energy with a bound on the Morse index of the solutions (see [14]).

We believe that our results could help to better understand the behavior of sign-changing solutions for other 2-dimensional nonlinear problems, just as the result in [22] for sinh-Poisson equations was inspired by [20].

Finally we would like to observe that the bubble-tower solutions of (1.1) are also interesting in the study of the associated heat flow because they induce a peculiar blow-up phenomenon (see [6, 15, 23] and in particular [12]).

The outline of the paper is the following. In Section 2 we show some results about the asymptotic analysis of sign-changing solutions of (1.1) in general, not necessarily symmetric, domains. In Section 3 we study the behavior of solutions around maximum points, while Section 4 is devoted to analyzing the scaling about minimum points. Finally, in Section 5 we prove some further properties of solutions and discuss some open questions.

2. Asymptotic analysis in general domains

In this section we study the asymptotic behavior of a family $(u_p)_{p>1}$ of sign-changing solutions of (1.1) satisfying the energy condition

$$p\int_\Omega |\nabla u_p|^2\,dx \to \beta, \quad \text{for some } \beta\in\mathbb{R}, \quad \text{as } p\to\infty. \tag{2.1}$$

We follow the approach of [16] where positive solutions of semilinear elliptic problems with critical exponential nonlinearities in dimension 2 are studied.

We denote by E_p the energy functional associated to (1.1), i.e.

$$E_p(u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \qquad u\in H_0^1(\Omega),$$

and recall that for a solution u of (1.1),

$$E_p(u) = \left(\frac{1}{2}-\frac{1}{p+1}\right)\|\nabla u\|_2^2 = \left(\frac{1}{2}-\frac{1}{p+1}\right)\|u\|_{p+1}^{p+1}. \tag{2.2}$$

Given a family (u_p) of solutions of (1.1) and assuming that there exist $n>0$ families of points $(x_{i,p})$, $i=1,\dots,n$, in Ω such that

$$p|u_p(x_{i,p})|^{p-1}\to\infty \qquad \text{as } p\to\infty, \tag{2.3}$$

we define the parameters $\mu_{i,p}$ by

$$\mu_{i,p}^{-2} = p|u_p(x_{i,p})|^{p-1} \qquad \text{for all } i=1,\dots,n. \tag{2.4}$$

By (2.3) it is clear that $\mu_{i,p}\to 0$ as $p\to\infty$ and

$$\forall\epsilon>0\ \exists p_{i,\epsilon}\ \forall p\ge p_{i,\epsilon} \qquad |u_p(x_{i,p})|\ge 1-\epsilon. \tag{2.5}$$

Then we define the concentration set

$$\mathcal{S} = \left\{\lim_{p\to\infty} x_{i,p} : i=1,\dots,n\right\}\subset\bar{\Omega} \tag{2.6}$$

and the function

$$R_{n,p}(x) = \min_{i=1,\dots,n}|x-x_{i,p}| \qquad \forall x\in\Omega. \tag{2.7}$$

Finally, we introduce the following properties:

$(\mathcal{P}_1^n)$ For any $i,j\in\{1,\dots,n\}$, $i\neq j$,

$$\lim_{p\to\infty}|x_{i,p}-x_{j,p}|/\mu_{i,p} = \infty.$$

($\mathcal{P}_2^n$) For any $i = 1, \dots, n$,

$$v_{i,p}(x) := \frac{p}{u_p(x_{i,p})}(u_p(x_{i,p} + \mu_{i,p}x) - u_p(x_{i,p})) \to U(x)$$

in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$ as $p \to \infty$, where

$$U(x) = \log\left(\frac{1}{1 + \frac{1}{8}|x|^2}\right)^2 \tag{2.8}$$

is the solution of $-\Delta U = e^U$ in $\mathbb{R}^2$, $U \le 0$, $U(0) = 0$ and $\int_{\mathbb{R}^2} e^U = 8\pi$.

($\mathcal{P}_3^n$) There exists $C > 0$ such that

$$pR_{n,p}(x)^2|u_p(x)|^{p-1} \le C$$

for all $p > 1$ and all $x \in \Omega$.

Lemma 2.1. *Let (u_p) be a family of solutions to (1.1) satisfying (2.1). Then:*

(i) *There exists $C > 0$ such that $\|u_p\|_{L^\infty(\Omega)} \le C$ for all $p > 1$.*

(ii) *If u_p changes sign, then $\|u_p^\pm\|^{p-1}_{L^\infty(\Omega)} \ge \lambda_1$ where $\lambda_1 := \lambda_1(\Omega)$ is the first eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$. In particular for the points $x_p^\pm$ where the maximum and the minimum are achieved, the analogues of (2.3) and (2.5) hold.*

(iii) *If there exists $n \in \mathbb{N} \setminus \{0\}$ such that the properties ($\mathcal{P}_1^n$) and ($\mathcal{P}_2^n$) hold for families $(x_{i,p})_{i=1,\dots,n}$ of points satisfying (2.3), then*

$$p\int_\Omega |\nabla u_p|^2\,dx \ge 8\pi \sum_{i=1}^n \alpha_i^2 + o_p(1) \quad \text{as } p \to \infty,$$

where $\alpha_i := \liminf_{p\to\infty} |u_p(x_{i,p})|$.

(iv) *$\sqrt{p}\,u_p \rightharpoonup 0$ in $H_0^1(\Omega)$ as $p \to \infty$.*

Proof. The proof of (i) is the same as in [19, Proposition 2.7], while the proof of (ii) is similar to that of [19, Proposition 2.5]. To prove (iii) let us write, for any $R > 0$,

$$\begin{aligned} p\int_{B_{R\mu_{i,p}}(x_{i,p})} |u_p|^{p+1}\,dx &= \int_{B_R(0)} \frac{|u_p(x_{i,p} + \mu_{i,p}y)|^{p+1}}{|u_p(x_{i,p})|^{p-1}}\,dy \\ &= \int_{B_R(0)} \left|1 + \frac{v_{i,p}(y)}{p}\right|^{p+1} u_p(x_{i,p})^2\,dy \end{aligned} \tag{2.9}$$

where $B_\rho(a)$ denotes the ball of center a and radius ρ. Thanks to ($\mathcal{P}_2^n$), we have

$$\|v_{i,p} - U\|_{L^\infty(B_R(0))} = o_p(1) \quad \text{as } p \to \infty. \tag{2.10}$$

Thus by (2.9), (2.10) and Fatou's lemma

$$\liminf_{p\to\infty} p\int_{B_{R\mu_{i,p}}(x_{i,p})} |u_p|^{p+1}\,dx \ge \alpha_i^2 \int_{B_R(0)} e^U\,dx. \tag{2.11}$$

Moreover by $(\mathcal{P}_1^n)$ it is not hard to see that $B_{R\mu_{i,p}}(x_{i,p}) \cap B_{R\mu_{j,p}}(x_{j,p}) = \emptyset$ for all $i \neq j$. Hence, in particular, thanks to (2.11),

$$\liminf_{p\to\infty} p \int_\Omega |u_p|^{p+1}\, dx \geq \sum_{i=1}^n \alpha_i^2 \int_{B_R(0)} e^U\, dx.$$

Finally, since this holds for any $R > 0$, we get

$$\begin{aligned} p\int_\Omega |\nabla u_p|^2\, dx = p\int_\Omega |u_p|^{p+1}\, dx &\geq \sum_{i=1}^n \alpha_i^2 \int_{\mathbb{R}^2} e^U\, dx + o(1) \\ &= 8\pi \sum_{i=1}^n \alpha_i^2 + o(1) \quad \text{as } p \to \infty. \end{aligned}$$

To prove (iv) note that, since (2.1) holds, there exists $w \in H_0^1(\Omega)$ such that, up to a subsequence, $\sqrt{p}\, u_p \rightharpoonup w$ in $H_0^1(\Omega)$. We want to show that $w = 0$ a.e. in Ω.

Using (1.1), for any test function $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \nabla(\sqrt{p}\, u_p)\nabla\varphi\, dx = \sqrt{p}\int_\Omega |u_p|^{p-1}u_p\varphi\, dx \leq \frac{\|\varphi\|_\infty}{\sqrt{p}}\, p\int_\Omega |u_p|^p\, dx \leq \frac{\|\varphi\|_\infty}{\sqrt{p}} C$$

for p large since, by (2.1) and (2.2), $\int_\Omega |u_p|^p\, dx \leq (\int_\Omega |u_p|^{p+1}\, dx)^{p/(p+1)}|\Omega|^{1/(p+1)} \leq C/p$. Hence

$$\int_\Omega \nabla w \nabla\varphi\, dx = 0 \quad \forall \varphi \in C_0^\infty(\Omega),$$

which implies that $w = 0$ a.e. in Ω. □

The next proposition gives the main result of this section.

Proposition 2.2. *Let (u_p) be a family of solutions to (1.1) and assume that (2.1) holds. Then there exist $k \in \mathbb{N} \setminus \{0\}$ and k families of points $(x_{i,p})$ in Ω, $i = 1, \ldots, k$ such that, after passing to a sequence, $(\mathcal{P}_1^k)$, $(\mathcal{P}_2^k)$, and $(\mathcal{P}_3^k)$ hold. Moreover, given any family of points $x_{k+1,p}$, it is impossible to extract a new sequence from the previous one such that $(\mathcal{P}_1^{k+1})$, $(\mathcal{P}_2^{k+1})$, and $(\mathcal{P}_3^{k+1})$ hold with the sequences $(x_{i,p})$, $i = 1, \ldots, k+1$. Finally,*

$$\sqrt{p}\, u_p \to 0 \quad \text{in } C^1_{\mathrm{loc}}(\bar{\Omega} \setminus \mathcal{S}) \text{ as } p \to \infty. \tag{2.12}$$

Proof. We mainly follow the proof of Proposition 1 of [16], but we have to deal with an extra difficulty because we allow the solutions u_p to be sign-changing. We divide the proof into several steps and all the claims are up to a subsequence.

Step 1. *There exists a family $(x_{i,p})$ of points in Ω such that, after passing to a sequence, $(\mathcal{P}_2^1)$ holds.*

Proof of Step 1. We let $x_{1,p}$ be a point in Ω where $|u_p|$ achieves its maximum. Without loss of generality we can assume that

$$u_p(x_{1,p}) = \max_\Omega u_p > 0. \tag{2.13}$$

By Lemma 2.1(ii) we have $pu_p(x_{1,p})^{p-1} \to \infty$ as $p \to \infty$, so that, defining (as in (2.4))

$$\mu_{1,p}^{-2} = pu_p^{p-1}(x_{1,p}),$$

we have $\mu_{1,p} \to 0$. Let

$$\widetilde{\Omega}_{1,p} = \frac{\Omega - x_{1,p}}{\mu_{1,p}} = \{x \in \mathbb{R}^2 : x_{1,p} + \mu_{1,p}x \in \Omega\}$$

and for $x \in \widetilde{\Omega}_{1,p}$,

$$v_{1,p}(x) = \frac{p}{u_p(x_{1,p})}(u_p(x_{1,p} + \mu_{1,p}x) - u_p(x_{1,p})).$$

By (2.13), we have

$$v_{1,p}(0) = 0 \quad \text{and} \quad v_{1,p} \le 0 \quad \text{in } \widetilde{\Omega}_{1,p}, \tag{2.14}$$

moreover $v_{1,p}$ solves

$$-\Delta v_{1,p} = \left|1 + \frac{v_{1,p}}{p}\right|^p \left(1 + \frac{v_{1,p}}{p}\right) \quad \text{in } \widetilde{\Omega}_{1,p}, \tag{2.15}$$

with $|1 + v_{1,p}/p| \le 1$ and $v_{1,p} = -p$ on $\partial\widetilde{\Omega}_{1,p}$. Then

$$|-\Delta v_{1,p}| \le 1 \quad \text{in } \widetilde{\Omega}_{1,p}. \tag{2.16}$$

Using (2.14) and (2.16) we now prove that

$$\widetilde{\Omega}_{1,p} \to \mathbb{R}^2 \quad \text{as } p \to \infty. \tag{2.17}$$

Indeed, since $\mu_{1,p} \to 0$ as $p \to \infty$, either $\widetilde{\Omega}_{1,p} \to \mathbb{R}^2$ or $\widetilde{\Omega}_{1,p} \to \mathbb{R} \times (-\infty, R)$ as $p \to \infty$ for some $R \ge 0$ (up to rotation). In the second case we let

$$v_{1,p} = \varphi_p + \psi_p \quad \text{in } \widetilde{\Omega}_{1,p} \cap B_{2R+1}(0)$$

with $-\Delta\varphi_p = -\Delta v_{1,p}$ in $\widetilde{\Omega}_{1,p} \cap B_{2R+1}(0)$ and $\psi_p = v_{1,p}$ in $\partial(\widetilde{\Omega}_{1,p} \cap B_{2R+1}(0))$.

Thanks to (2.16), by standard elliptic theory, we see that φ_p is uniformly bounded in $\widetilde{\Omega}_{1,p} \cap B_{2R+1}(0)$. The function ψ_p is harmonic in $\widetilde{\Omega}_{1,p} \cap B_{2R+1}(0)$, bounded from above by (2.14) and satisfies $\psi_p = -p \to -\infty$ on $\partial\widetilde{\Omega}_{1,p} \cap B_{2R+1}(0)$. Since $\partial\widetilde{\Omega}_{1,p} \cap B_{2R+1}(0) \to (\mathbb{R} \times \{R\}) \cap B_{2R+1}(0)$ as $p \to \infty$ one easily sees that $\psi_p(0) \to -\infty$ as $p \to \infty$ (if $R = 0$ this is trivial, if $R > 0$ it follows by the Harnack inequality). This is a contradiction since $\psi_p(0) = -\varphi_p(0)$ and φ_p is bounded, hence (2.17) is proved.

Then for any $R > 0$, $B_R(0) \subset \widetilde{\Omega}_{1,p}$ for p sufficiently large. So $(v_{1,p})$ is a family of nonpositive functions with uniformly bounded Laplacian in $B_R(0)$ and with $v_{1,p}(0) = 0$.

Thus, arguing as before, we write $v_{1,p} = \varphi_p + \psi_p$ where φ_p is uniformly bounded in $B_R(0)$ and ψ_p is a harmonic function which is uniformly bounded from above. By the Harnack inequality, either ψ_p is uniformly bounded in $B_R(0)$, or it tends to $-\infty$ on each compact set of $B_R(0)$. The second alternative cannot happen because, by definition, $\psi_p(0) = v_{1,p}(0) - \varphi_p(0) = -\varphi_p(0) \ge -C$. Hence $v_{1,p}$ is uniformly bounded in $B_R(0)$, for all $R > 0$. After passing to a subsequence, standard elliptic theory implies that $v_{1,p}$ is bounded in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ and, on each ball, $1 + v_{1,p}/p > 0$ for p large. Thus

$$v_{1,p} \to U \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^2) \text{ as } p \to \infty, \tag{2.18}$$

with $U \in C^1(\mathbb{R}^2)$, $U \le 0$ and $U(0) = 0$. Thanks to (2.15) and (2.18) we see that

$$-\Delta U = e^U \quad \text{in } \mathbb{R}^2.$$

Moreover for any $R > 0$, by (2.5), we have

$$\begin{aligned}
\int_{B_R(0)} e^U\,dx &\overset{(2.18)+\text{Fatou}}{\le} \int_{B_R(0)} \frac{|u_p(x_{1,p} + \mu_{1,p}x)|^{p+1}}{u_p(x_{1,p})^{p+1}}\,dx + o_p(1) \\
&= \frac{p}{u_p(x_{1,p})^2} \int_{B_{R\mu_{1,p}}(x_{1,p})} |u_p(y)|^{p+1} dy + o_p(1) \\
&\overset{(2.5)}{\le} \frac{p}{(1-\varepsilon)^2} \int_{B_{R\mu_{1,p}}(x_{1,p})} |u_p(y)|^{p+1} dy + o_p(1) \\
&\overset{(2.1)}{\le} \frac{p}{(1-\varepsilon)^2} \int_{\Omega} |u_p(y)|^{p+1} dy + o_p(1) < C < \infty,
\end{aligned}$$

so that $e^U \in L^1(\mathbb{R}^2)$. Thus, since $U(0) = 0$, by the classification due to Chen and Li [8] we obtain

$$U(x) = \log\left(\frac{1}{1 + \frac{1}{8}|x|^2}\right)^2.$$

Then an easy computation shows that $\int_{\mathbb{R}^2} e^U = 8\pi$. This ends the proof of Step 1.

Step 2. *Assume that* $(\mathcal{P}_1^n)$ *and* $(\mathcal{P}_2^n)$ *hold for some* $n \in \mathbb{N} \setminus \{0\}$. *Then either* $(\mathcal{P}_1^{n+1})$ *and* $(\mathcal{P}_2^{n+1})$ *hold, or* $(\mathcal{P}_3^n)$ *holds, namely there exists* $C > 0$ *such that*

$$pR_{n,p}(x)^2|u_p(x)|^{p-1} \le C$$

for all $x \in \Omega$, *with* $R_{n,p}$ *defined as in* (2.7).

Proof of Step 2. Let $n \in \mathbb{N} \setminus \{0\}$ and assume that $(\mathcal{P}_1^n)$ and $(\mathcal{P}_2^n)$ hold, while

$$\sup_{x\in\Omega} pR_{n,p}(x)^2|u_p(x)|^{p-1} \to \infty \quad \text{as } p \to \infty. \tag{2.19}$$

We will prove that $(\mathcal{P}_1^{n+1})$ and $(\mathcal{P}_2^{n+1})$ hold.

We let $x_{n+1,p} \in \bar{\Omega}$ be such that

$$pR_{n,p}(x_{n+1,p})^2|u_p(x_{n+1,p})|^{p-1} = \sup_{x\in\Omega} pR_{n,p}(x)^2|u_p(x)|^{p-1}. \tag{2.20}$$

Clearly $x_{n+1,p} \in \Omega$ because $u_p = 0$ on $\partial\Omega$. By (2.20) and since Ω is bounded it is clear that

$$p|u_p(x_{n+1,p})|^{p-1} \to \infty \quad \text{as } p \to \infty.$$

We claim that

$$|x_{i,p}-x_{n+1,p}|/\mu_{i,p}\to\infty \quad \text{as } p\to\infty \tag{2.21}$$

for all $i=1,\dots,n$ and $\mu_{i,p}$ as in (2.4). In fact, assuming for contradiction that there exists $i\in\{1,\dots,n\}$ such that $|x_{i,p}-x_{n+1,p}|/\mu_{i,p}\to R$ as $p\to\infty$ for some $R\geq 0$, thanks to $(\mathcal{P}_2^n)$, we get

$$\lim_{p\to\infty} p|x_{i,p}-x_{n+1,p}|^2|u_p(x_{n+1,p})|^{p-1}=R^2\left(\frac{1}{1+\frac{1}{8}R^2}\right)^2<\infty,$$

contrary to (2.20).

Setting

$$\mu_{n+1,p}^{-2}:=p|u_p(x_{n+1,p})|^{p-1}, \tag{2.22}$$

by (2.19) and (2.20) we deduce that

$$R_{n,p}(x_{n+1,p})/\mu_{n+1,p}\to\infty \quad \text{as } p\to\infty. \tag{2.23}$$

Then (2.22), (2.23) and $(\mathcal{P}_1^n)$ imply that $(\mathcal{P}_1^{n+1})$ holds with the added sequence $(x_{n+1,p})$.

Next we show that also $(\mathcal{P}_2^{n+1})$ holds with the sequence $(x_{n+1,p})$. Let us define the scaled domain

$$\widetilde{\Omega}_{n+1,p}=\{x\in\mathbb{R}^2 : x_{n+1,p}+\mu_{n+1,p}x\in\Omega\},$$

and, for $x\in\widetilde{\Omega}_{n+1,p}$, the rescaled functions

$$v_{n+1,p}(x)=\frac{p}{u_p(x_{n+1,p})}(u_p(x_{n+1,p}+\mu_{n+1,p}x)-u_p(x_{n+1,p})), \tag{2.24}$$

which, by (1.1), satisfy

$$-\Delta v_{n+1,p}(x)=\frac{|u_p(x_{n+1,p}+\mu_{n+1,p}x)|^{p-1}u_p(x_{n+1,p}+\mu_{n+1,p}x)}{|u_p(x_{n+1,p})|^{p-1}u_p(x_{n+1,p})} \quad \text{in } \widetilde{\Omega}_{n+1,p}, \tag{2.25}$$

or equivalently

$$-\Delta v_{n+1,p}(x)=\left|1+\frac{v_{n+1,p}(x)}{p}\right|^{p-1}\left(1+\frac{v_{n+1,p}(x)}{p}\right) \quad \text{in } \widetilde{\Omega}_{n+1,p}. \tag{2.26}$$

Fix $R>0$ and let (z_p) be any point in $\widetilde{\Omega}_{n+1,p}\cap B_R(0)$, whose corresponding point in Ω is

$$x_p=x_{n+1,p}+\mu_{n+1,p}z_p.$$

Thanks to the definition of $x_{n+1,p}$ we have

$$pR_{n,p}(x_p)^2|u_p(x_p)|^{p-1}\leq pR_{n,p}(x_{n+1,p})^2|u_p(x_{n+1,p})|^{p-1}. \tag{2.27}$$

Since $|x_p-x_{n+1,p}|\leq R\mu_{n+1,p}$ we have

$$R_{n,p}(x_p)\geq \min_{i=1,\dots,n}|x_{n+1,p}-x_{i,p}|-|x_p-x_{n+1,p}|\geq R_{n,p}(x_{n+1,p})-R\mu_{n+1,p},$$

and analogously $R_{n,p}(x_p) \leq R_{n,p}(x_{n+1,p}) + R\mu_{n+1,p}$. Thus, by (2.23) we get

$$R_{n,p}(x_p) = (1 + o(1))R_{n,p}(x_{n+1,p}),$$

and in turn from (2.27),

$$|u_p(x_p)|^{p-1} \leq (1 + o(1))|u_p(x_{n+1,p})|^{p-1}. \tag{2.28}$$

In the following we show that for any compact subset K of $\mathbb{R}^2$,

$$-1 + o(1) \leq -\Delta v_{n+1,p} \leq 1 + o(1) \quad \text{in } \widetilde{\Omega}_{n+1,p} \cap K, \tag{2.29}$$

$$\limsup_{p\to\infty} \sup_{\widetilde{\Omega}_{n+1,p}\cap K} v_{n+1,p} \leq 0. \tag{2.30}$$

To do so we will distinguish several cases.

(i) *Assume that* $v_{n+1,p}(z_p) \geq 0$. If $u_p(x_{n+1,p}) > 0$ then $u_p(x_p) = \frac{u_p(x_{n+1,p})}{p} v_{n+1,p}(z_p) + u_p(x_{n+1,p}) \geq u_p(x_{n+1,p}) > 0$, while if $u_p(x_{n+1,p}) < 0$ then analogously $u_p(x_p) \leq u_p(x_{n+1,p}) < 0$. So in both cases

$$u_p(x_p)/u_p(x_{n+1,p}) = |u_p(x_p)|/|u_p(x_{n+1,p})|.$$

By (2.28) we get $|u_p(x_p)|^p \leq (1 + o(1))|u_p(x_{n+1,p})|^p$, and so by (2.25),

$$(0 \leq)\ -\Delta v_{n+1,p}(z_p) = |u_p(x_p)|^p/|u_p(x_{n+1,p})|^p \leq 1 + o(1). \tag{2.31}$$

Moreover, since (2.26) implies $-\Delta v_{n+1,p}(z_p) = e^{v_{n+1,p}(z_p)} + o(1)$, we get

$$\limsup_{p\to\infty} v_{n+1,p}(z_p) \leq 0.$$

By the arbitrariness of z_p we obtain (2.30).

(ii) *Assume that* $v_{n+1,p}(z_p) \leq 0$. We distinguish two cases:

Case 1: $u_p(x_{n+1,p}) > 0$. Then $u_p(x_p) = \frac{u_p(x_{n+1,p})}{p} v_{n+1,p}(z_p) + u_p(x_{n+1,p}) \leq u_p(x_{n+1,p})$. So either $u_p(x_p) \geq 0$ and then $(0 \leq) - \Delta v_{n+1,p}(z_p) \leq 1$, or $u_p(x_p) < 0$ and then by (2.28),

$$0 \geq -\Delta v_{n+1,p}(z_p) = -\frac{|u_p(x_p)|^p}{u_p(x_{n+1,p})^p} \geq -1 + o(1).$$

Case 2: $u_p(x_{n+1,p}) < 0$. Then analogously $u_p(x_p) \geq u_p(x_{n+1,p})$. So either $u_p(x_p) \leq 0$ and then $(0 \leq)\ -\Delta v_{n+1,p}(z_p) \leq 1$, or $u_p(x_p) > 0$ and then by (2.28),

$$0 \geq -\Delta v_{n+1,p}(z_p) = -\frac{u_p(x_p)^p}{|u_p(x_{n+1,p})|^p} \geq -1 + o(1).$$

In the end, in both Case 1 and Case 2 we have proved that, as $p \to \infty$,

$$-1 + o(1) \leq -\Delta v_{n+1,p}(z_p) \leq 1 + o(1). \tag{2.32}$$

Putting together (2.31) and (2.32) it follows that (2.29) holds.

Using (2.29) and (2.30) we can prove, as in Step 1, that

$$\widetilde{\Omega}_{n+1,p} \to \mathbb{R}^2 \quad \text{as } p \to \infty. \tag{2.33}$$

Indeed, since $\mu_{n+1,p} \to 0$ as $p \to \infty$, either $\widetilde{\Omega}_{n+1,p} \to \mathbb{R}^2$ or $\widetilde{\Omega}_{n+1,p} \to \mathbb{R} \times (-\infty, R)$ as $p \to \infty$ for some $R \geq 0$ (up to rotation). In the second case we let

$$v_{n+1,p} = \varphi_p + \psi_p \quad \text{in } \widetilde{\Omega}_{n+1,p} \cap B_{2R+1}(0)$$

with $-\Delta\varphi_p = -\Delta v_{n+1,p}$ in $\widetilde{\Omega}_{n+1,p} \cap B_{2R+1}(0)$ and $\psi_p = v_{n+1,p}$ in $\partial(\widetilde{\Omega}_{n+1,p} \cap B_{2R+1}(0))$.

Thanks to (2.29), since $\varphi_p = v_{n+1,p}$ in $\partial(\widetilde{\Omega}_{n+1,p} \cap B_{2R+1}(0))$, by standard elliptic theory φ_p is uniformly bounded in $\widetilde{\Omega}_{n+1,p} \cap B_{2R+1}(0)$. The function ψ_p is harmonic in $\widetilde{\Omega}_{n+1,p} \cap B_{2R+1}(0)$, bounded from above by (2.30) and satisfies $\psi_p = -p \to -\infty$ on $\partial\widetilde{\Omega}_{n+1,p} \cap B_{2R+1}(0)$. Since $\partial\widetilde{\Omega}_{n+1,p} \cap B_{2R+1}(0) \to (\mathbb{R} \times \{R\}) \cap B_{2R+1}(0)$ as $p \to \infty$, one easily sees that $\psi_p(0) \to -\infty$ as $p \to \infty$ (if $R = 0$ this is trivial, if $R > 0$ it follows from the Harnack inequality). This is a contradiction since $\psi_p(0) = -\varphi_p(0)$ and φ_p is bounded. Therefore the limit domain of $\widetilde{\Omega}_{n+1,p}$ is the whole $\mathbb{R}^2$.

Then for any $R > 0$, $B_R(0) \subset \widetilde{\Omega}_{n+1,p}$ for p large enough and the $v_{n+1,p}$ are functions with uniformly bounded Laplacian in $B_R(0)$ and with $v_{n+1,p}(0) = 0$. So, by the Harnack inequality, $v_{n+1,p}$ is uniformly bounded in $B_R(0)$ for all $R > 0$ and then $v_{n+1,p} \to U$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$ as $p \to \infty$ with $U \in C^1(\mathbb{R}^2)$, $U(0) = 0$ and, by (2.30), $U \leq 0$. Passing to the limit in (2.26) we get

$$-\Delta v_{n+1,p}(x) \to e^{U(x)} \quad \text{as } p \to \infty,$$

and so $-\Delta U = e^U$ in $\mathbb{R}^2$. Next, for any $R > 0$,

$$\int_{B_R(0)} e^U \, dx \leq p \int_{B_{R\mu_{n+1,p}}(x_{n+1,p})} u_p \Delta u_p \, dx + o_p(1) \leq p \int_\Omega |\nabla u_p|^2 \, dx + o_p(1),$$

so that $e^U \in L^1(\mathbb{R}^2)$. By [8] and $v_{n+1,p}(0) = 0$ we have $U(x) = \log\left(\frac{1}{1+\frac{1}{8}x^2}\right)^2$.

This proves that $(\mathcal{P}_2^{n+1})$ holds with the added points $(x_{n+1,p})$, thus Step 2 is proved.

Step 3. Completion of the proof of Proposition 2.2. From Step 1 we know that $(\mathcal{P}_1^1)$ and $(\mathcal{P}_2^1)$ hold. Then, by Step 2, either $(\mathcal{P}_1^2)$ and $(\mathcal{P}_2^2)$ hold, or $(\mathcal{P}_3^1)$ holds. In the latter case the assertion is proved with $k = 1$. In the former case we go on and proceed with the same alternative until we reach a number $k \in \mathbb{N} \setminus \{0\}$ for which $(\mathcal{P}_1^k)$, $(\mathcal{P}_2^k)$ and $(\mathcal{P}_3^k)$ hold up to a sequence. Note that this is possible because the solutions u_p satisfy (2.1) and Lemma 2.1 holds and hence the maximal number k of families of points for which $(\mathcal{P}_1^k)$, $(\mathcal{P}_2^k)$ hold must be finite.

Moreover, given any other family of points $x_{k+1,p}$, it is impossible to extract a new sequence from it such that $(\mathcal{P}_1^{k+1})$, $(\mathcal{P}_2^{k+1})$ and $(\mathcal{P}_3^{k+1})$ hold together with the points $(x_{i,p})_{i=1,..,k+1}$. Indeed, if $(\mathcal{P}_1^{k+1})$ held then

$$|x_{k+1,p} - x_{i,p}|/\mu_{k+1,p} \to \infty \quad \text{as } p \to \infty, \text{ for any } i \in \{1, \dots, k\},$$

but this would contradict $(\mathcal{P}_3^k)$.

Finally, the proof of (2.12) is a direct consequence of $(\mathcal{P}_3^k)$. Indeed, if K is a compact subset of $\bar{\Omega} \setminus \mathcal{S}$, by $(\mathcal{P}_3^k)$ there exists $C_K > 0$ such that

$$p|u_p(x)|^{p-1} \le C_K \quad \text{for all } x \in K.$$

Then by (1.1), $\|\Delta(\sqrt{p}\, u_p)\|_{L^\infty(K)} \le C_K \|u_p\|_{L^\infty(K)}/\sqrt{p} \to 0$ as $p \to \infty$. Hence standard elliptic theory shows that $\sqrt{p}\, u_p \to w$ in $C^1(K)$ for some w. But by Lemma 2.1(iv) we know that $\sqrt{p}\, u \rightharpoonup 0$, so $w = 0$. This ends the proof. □

We conclude this section by showing some consequences of Proposition 2.2.

Remark 2.3. Under the assumptions of Proposition 2.2 we have

$$\operatorname{dist}(x_{i,p}, \partial\Omega)/\mu_{i,p} \xrightarrow[p\to\infty]{} \infty \quad \text{for all } i \in \{1, \dots, k\}.$$

Corollary 2.4. *Under the assumptions of Proposition 2.2, if u_p is sign-changing then*

$$\operatorname{dist}(x_{i,p}, NL_p)/\mu_{i,p} \xrightarrow[p\to\infty]{} \infty \quad \textit{for all } i \in \{1, \dots, k\}$$

where NL_p denotes the nodal line of u_p. As a consequence, for any $i \in \{1, \dots, k\}$, letting $\mathcal{N}_{i,p} \subset \Omega$ be the nodal domain of u_p containing $x_{i,p}$ and setting $u_p^i := u_p \chi_{\mathcal{N}_{i,p}}$ (χ_A is the characteristic function of the set A), then the scaling of u_p^i around $x_{i,p}$:

$$z_{i,p}(x) := \frac{p}{u_p(x_{i,p})}(u_p^i(x_{i,p} + \mu_{i,p}x) - u_p(x_{i,p})),$$

defined on $\widetilde{\mathcal{N}}_{i,p} := (\mathcal{N}_{i,p} - x_{i,p})/\mu_{i,p}$, converges to U in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$, where U is the function defined in $(\mathcal{P}_2^k)$.

Proof. Suppose for contradiction that

$$\operatorname{dist}(x_{i,p}, NL_p)/\mu_{i,p} \xrightarrow[p\to\infty]{} \ell \ge 0;$$

then there exist $y_p \in NL_p$ such that $|x_{i,p} - y_p|/\mu_{i,p} \to \ell$ as $p \to \infty$. Setting

$$v_{i,p}(x) := \frac{p}{u_p(x_{i,p})}(u_p(x_{i,p} + \mu_{i,p}x) - u_p(x_{i,p})),$$

on the one hand

$$v_{i,p}\left(\frac{y_p - x_{i,p}}{\mu_{i,p}}\right) = -p \xrightarrow[p\to\infty]{} -\infty,$$

on the other hand by $(\mathcal{P}_2^k)$ and up to subsequences

$$v_{i,p}\left(\frac{y_p - x_{i,p}}{\mu_{i,p}}\right) \xrightarrow[p\to\infty]{} U(x_\infty) > -\infty,$$

where $x_\infty = \lim_{p\to\infty}(y_p - x_{i,p})/\mu_{i,p} \in \mathbb{R}^2$ and so $|x_\infty| = \ell$. This contradiction completes the proof. ◻

From now on, for any family of points $(x_p)_p \subset \Omega$ we denote by $\mu(x_p)$ the numbers defined by

$$\mu(x_p)^{-2} := p|u_p(x_p)|^{p-1}. \tag{2.34}$$

Proposition 2.5. *Let $(x_p)_p \subset \Omega$ be a family of points such that $p|u_p(x_p)|^{p-1} \to \infty$ and let $\mu(x_p)$ be as in (2.34). By $(\mathcal{P}_3^k)$, up to a sequence, $R_{k,p}(x_p) = |x_{i,p} - x_p|$ for a certain $i \in \{1, \ldots, k\}$. Then*

$$\limsup_{p\to\infty} \frac{\mu_{i,p}}{\mu(x_p)} \le 1.$$

Proof. To shorten notation we write $\mu(x_p)$ simply as μ_p. Let us start by proving that $\mu_{i,p}/\mu_p$ is bounded. Assume for contradiction that there exists a sequence $p_n \to \infty$ as $n \to \infty$ such that

$$\mu_{i,p_n}/\mu_{p_n} \to \infty \quad \text{as } n \to \infty. \tag{2.35}$$

By $(\mathcal{P}_3^k)$ and (2.35) we then have

$$\frac{|x_{p_n} - x_{i,p_n}|}{\mu_{i,p_n}} = \frac{|x_{p_n} - x_{i,p_n}|}{\mu_{p_n}} \frac{\mu_{p_n}}{\mu_{i,p_n}} \to 0 \quad \text{as } n \to \infty,$$

so that by $(\mathcal{P}_2^k)$,

$$v_{i,p_n}\left(\frac{x_{p_n} - x_{i,p_n}}{\mu_{i,p_n}}\right) \to U(0) = 0 \quad \text{as } n \to \infty.$$

As a consequence,

$$\frac{\mu_{i,p_n}}{\mu_{p_n}} = \left(\frac{u_{p_n}(x_{p_n})}{u_{p_n}(x_{i,p_n})}\right)^{p_n-1} = \left(1 + \frac{v_{i,p_n}\left(\frac{x_{p_n}-x_{i,p_n}}{\mu_{i,p_n}}\right)}{p_n}\right)^{p_n-1} \to e^{U(0)} = 1 \quad \text{as } n \to \infty,$$

which contradicts (2.35). Hence we have proved that $\mu_{i,p}/\mu_p$ is bounded.

Next we show that $\mu_{i,p}/\mu_p \le 1$. Assume for contradiction that there exists $\ell > 1$ and a sequence $p_n \to \infty$ as $n \to \infty$ such that

$$\mu_{i,p_n}/\mu_{p_n} \to \ell \quad \text{as } n \to \infty. \tag{2.36}$$

By $(\mathcal{P}_3^k)$ and (2.36) we have

$$\frac{|x_{p_n} - x_{i,p_n}|}{\mu_{i,p_n}} = \frac{|x_{p_n} - x_{i,p_n}|}{\mu_{p_n}} \frac{\mu_{p_n}}{\mu_{i,p_n}} \le \frac{2\sqrt{C}}{\ell}$$

for n large, so that by $(\mathcal{P}_2^k)$ there exists $x_\infty \in \mathbb{R}^2$ with $|x_\infty| \le 2\sqrt{C}/\ell$ such that, up to a subsequence,

$$v_{i,p_n}\left(\frac{x_{p_n} - x_{i,p_n}}{\mu_{i,p_n}}\right) \to U(x_\infty) \le 0 \quad \text{as } n \to \infty.$$

As a consequence,

$$\frac{\mu_{i,p_n}}{\mu_{p_n}} = \left(\frac{u_{p_n}(x_{p_n})}{u_{p_n}(x_{i,p_n})}\right)^{p_n-1} = \left(1 + \frac{v_{i,p_n}\left(\frac{x_{p_n}-x_{i,p_n}}{\mu_{i,p_n}}\right)}{p_n}\right)^{p_n-1} \to e^{U(x_\infty)} \quad \text{as } n \to \infty.$$

By (2.36) and the assumption $\ell > 1$ we deduce $U(x_\infty) = \log \ell + o_n(1) > 0$, reaching a contradiction. □

Proposition 2.6. *Let x_p and $x_{i,p}$ be as in the statement of Proposition 2.5. If*

$$|x_p - x_{i,p}|/\mu_{i,p} \to \infty \quad \text{as } p \to \infty, \tag{2.37}$$

then $\mu_{i,p}/\mu(x_p) \to 0$ as $p \to \infty$, where $\mu(x_p)$ is defined in (2.34).

Proof. By Proposition 2.5 we know that $\mu_{i,p}/\mu(x_p) \le 1 + o(1)$. Assume for contradiction that (2.37) holds but there exists $0 < \ell \le 1$ and a sequence $p_n \to \infty$ such that

$$\mu_{i,p_n}/\mu(x_{p_n}) \to \ell \quad \text{as } n \to \infty. \tag{2.38}$$

Then (2.38) and $(\mathcal{P}_3^k)$ imply

$$\frac{|x_{p_n} - x_{i,p_n}|}{\mu_{i,p_n}} = \frac{|x_{p_n} - x_{i,p_n}|}{\ell\mu(x_{p_n})} + o_n(1) \le \frac{C}{\ell} + o_n(1) \quad \text{as } n \to \infty,$$

which contradicts (2.37). □

Remark 2.7. If $u_p(x_p)$ and $u_p(x_{i,p})$ have opposite sign, i.e. $u_p(x_p)u_p(x_{i,p}) < 0$, then, by Corollary 2.4, necessarily (2.37) holds. Hence in this case $\mu_{i,p}/\mu(x_p) \to 0$ as $p \to \infty$.

3. G-symmetric case: asymptotic analysis about the maximum points

In this section we start the asymptotic analysis which leads to the proof of Theorem 1.2. So we assume that $\Omega \subset \mathbb{R}^2$ is a G-symmetric domain as in the statement of Theorem 1.2. In particular we recall the hypothesis

$$|G| \ge 4e. \tag{3.1}$$

Then we consider a family (u_p) of sign-changing G-symmetric solutions of (1.1) with all the properties listed in Theorem 1.2.

In particular u_p satisfies

$$p \int_\Omega |\nabla u_p|^2 \le \alpha\, 8\pi e \tag{3.2}$$

for some $\alpha < 5$ and p large.

We keep all the notation introduced in Sections 1 and 2 and add the following:

- $\mathcal{N}_p^{\pm} \subset \Omega$ denotes the positive/negative nodal domain of u_p.
- $\widetilde{\mathcal{N}}_p^{\pm}$ are the rescaled nodal domains about the points $x_p^{\pm}$ by the parameters $\mu_p^{\pm}$ defined in the introduction, i.e.

$$\widetilde{\mathcal{N}}_p^{\pm} := \frac{\mathcal{N}_p^{\pm} - x_p^{\pm}}{\mu_p^{\pm}} = \{x \in \mathbb{R}^2 : x_p^{\pm} + \mu_p^{\pm} x \in \mathcal{N}_p^{\pm}\}.$$

We recall an energy lower bound (see for example [13]) and some obvious properties deriving from (3.2).

Lemma 3.1. *For any $\epsilon > 0$ there exists $p_\epsilon > 1$ such that*

$$pE_p(u_p^{\pm}) \geq 4\pi e - \epsilon \quad \forall p \geq p_\epsilon. \tag{3.3}$$

Moreover

$$E_p(u_p) \to 0, \quad E_p(u_p^{\pm}) \to 0, \quad \|\nabla u_p\|_2 \to 0, \quad \|\nabla u_p^{\pm}\|_2 \to 0$$

as $p \to \infty$.

From now on we assume without loss of generality, as in Section 1, that the L^∞-norm of u_p is achieved at the maximum point x_p^+, i.e.

$$u_p(x_p^+) = \|u_p\|_\infty \geq -u_p(x_p^-).$$

Thanks to (3.2) we can apply Proposition 2.2 to the solutions (u_p).

For the scaling about x_p^+ we then have

Proposition 3.2. *The rescaled function*

$$v_p^+(x) := \frac{p}{u_p(x_p^+)}(u_p(x_p^+ + \mu_p^+ x) - u_p(x_p^+)) \tag{3.4}$$

defined on $\widetilde{\Omega}_p^+$ (see Section 1 for the definition) converges to U in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$ as $p \to \infty$, where U is the function introduced in (2.8). Moreover, the scaling of u_p^+ around x_p^+,

$$z_p^+(x) := z_{1,p}(x) = \frac{p}{u_p(x_p^+)}(u_p^+(x_p^+ + \mu_p^+ x) - u_p(x_p^+)), \tag{3.5}$$

defined on $\widetilde{\mathcal{N}}_p^+$ converges to U in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$ as $p \to \infty$.

Proof. Since at x_p^+ the L^∞-norm of u_p is achieved, the proof of the convergence of v_p^+ is the same as that of Step 1 of Proposition 2.2. The convergence of z_p^+ then comes from Corollary 2.4. □

The previous Lemma 3.1 and Proposition 3.2 hold regardless of the symmetry of Ω. Now using the symmetry assumptions on Ω and on the solutions we derive more specific and precise results.

Let k and $(x_{i,p})$, $i = 1, \ldots, k$, be as in Proposition 2.2. Then, defining $\mu_{i,p}$ as in (2.4), we get

Proposition 3.3. *For $i = 1, \dots, k$, $|x_{i,p}|/\mu_{i,p}$ is bounded. So in particular $|x_{i,p}| \to 0$, $i = 1, \dots, k$, as $p \to \infty$, so that the set $\mathcal{S}$ of concentration points is $\{O\}$.*

Proof. We can assume that either $(x_{i,p})_p \subset \mathcal{N}_p^+$ or $(x_{i,p})_p \subset \mathcal{N}_p^-$. We prove the result in the former case, the latter being similar. To simplify notation we drop the dependence on i and set $x_p := x_{i,p}$ and $\mu_p := \mu_{i,p}$.

Let $h := |G|$ and denote by g^j, $j = 0, \dots, h-1$, the elements of G. We consider the rescaled nodal domains

$$\widetilde{\mathcal{N}_p}^{j,+} := \{x \in \mathbb{R}^2 : \mu_p x + g^j x_p \in \mathcal{N}_p^+\}, \qquad j = 0, \dots, h-1,$$

and the rescaled functions $z_p^{j,+} : \widetilde{\mathcal{N}_p}^{j,+} \to \mathbb{R}$ defined by

$$z_p^{j,+}(x) := \frac{p}{u_p^+(x_p)}(u_p^+(\mu_p x + g^j x_p) - u_p^+(x_p)), \qquad j = 0, \dots, h-1. \tag{3.6}$$

It is not difficult to see (as in Corollary 2.4) that each $z_p^{j,+}$ converges to $U(x) = \log\left(\frac{1}{(1+\frac{1}{8}|x|^2)^2}\right)$ in $C^1_{\text{loc}}(\mathbb{R}^2)$ as $p \to \infty$ and $8\pi = \int_{\mathbb{R}^2} e^U \, dx$.

Assume for contradiction that there exists a sequence $p_n \to \infty$ as $n \to \infty$ such that $|x_{p_n}|/\mu_{p_n} \to \infty$. Then, since the h distinct points $g^j x_{p_n}$, $j = 0, \dots, h-1$, are the vertices of a regular polygon centered at O, we have $d_n := |g^j x_{p_n} - g^{j+1} x_{p_n}| = 2\widetilde{d}_n \sin(\pi/h)$, where $\widetilde{d}_n := |g^j x_{p_n}|$, $j = 0, \dots, h-1$, and so $d_n/\mu_{p_n} \to \infty$ as $n \to \infty$. Let

$$R_n := \min\{d_n/3, \operatorname{dist}(x_{p_n}, \partial\Omega)/2, \operatorname{dist}(x_{p_n}, NL_{p_n})/2\}. \tag{3.7}$$

Then by construction $B_{R_n}(g^j x_{p_n}) \subseteq \mathcal{N}_{p_n}^+$ for $j = 0, \dots, h-1$,

$$B_{R_n}(g^j x_{p_n}) \cap B_{R_n}(g^l x_{p_n}) = \emptyset \qquad \text{for } j \neq l, \tag{3.8}$$

and

$$R_n/\mu_{p_n} \to \infty \qquad \text{as } n \to \infty. \tag{3.9}$$

Using (3.9), the convergence of $z_{p_n}^{j,+}$ to U, (2.5) and Fatou's lemma, we have

$$\begin{aligned}
8\pi &= \int_{\mathbb{R}^2} e^U \, dx \overset{\text{Fatou + conv.+(3.9)}}{\leq} \lim_n \int_{B_{R_n/\mu_{p_n}}(0)} e^{z_{p_n}^{j,+} + (p_n+1)(\log|1+\frac{z_{p_n}^{j,+}}{p_n}| - \frac{z_{p_n}^{j,+}}{p_n+1})} \, dx \\
&= \lim_n \int_{B_{R_n/\mu_{p_n}}(0)} \left|1 + \frac{z_{p_n}^{j,+}}{p_n}\right|^{p_n+1} dx = \lim_n \int_{B_{R_n/\mu_{p_n}}(0)} \left|\frac{u_{p_n}^+(\mu_{p_n} x + g^j x_{p_n})}{u_{p_n}^+(x_{p_n})} \, dx\right|^{p_n+1} dx \\
&= \lim_n \int_{B_{R_n}(g^j x_{p_n})} \frac{|u_{p_n}^+|^{p_n+1}}{(\mu_{p_n})^2 |u_{p_n}^+(x_{p_n})|^{p_n+1}} \, dx \\
&= \lim_n \frac{p_n}{|u_{p_n}^+(x_{p_n})|^2} \int_{B_{R_n}(g^j x_{p_n})} |u_{p_n}^+|^{p_n+1} \, dx \overset{(2.5)}{\leq} \lim_n p_n \int_{B_{R_n}(g^j x_{p_n})} |u_{p_n}^+|^{p_n+1} \, dx.
\end{aligned} \tag{3.10}$$

Summing over $j = 0, \ldots, h-1$, using (3.8), (3.2), (3.3) and (2.2) we get

$$
\begin{aligned}
h \cdot 8\pi &\leq \lim_n p_n \sum_{j=0}^{h-1} \int_{B_{R_n}(g^j x_{p_n})} |u_{p_n}^+|^{p_n+1}\, dx \overset{(3.8)}{\leq} \lim_n p_n \int_{\mathcal{N}_{p_n}^+} |u_{p_n}^+|^{p_n+1}\, dx \\
&= \lim_n \left(p_n \int_\Omega |u_{p_n}|^{p_n+1}\, dx - p_n \int_{\mathcal{N}_{p_n}^-} |u_{p_n}^-|^{p_n+1}\, dx \right) \\
&\overset{(3.2)+(3.3)}{\leq} (\alpha - 1) \cdot 8\pi e \overset{\alpha<5}{<} 4 \cdot 8\pi e,
\end{aligned}
$$

which contradicts our assumption (3.1) on $|G|$. □

Remark 3.4. If we knew that $\|u_p\|_\infty \geq \sqrt{e}$, then we would obtain a better estimate in (3.10), and so Proposition 3.3 would hold under the weaker symmetry assumption $|G| \geq 4$ (recall that $|G| \geq 4$ is the assumption under which one can prove existence, see [13]).

Corollary 3.5. *We have:*

(i) *$O \in \mathcal{N}_p^+$ for p large.*
(ii) *Let $i \in \{1, \ldots, k\}$. Then $x_{i,p} \in \mathcal{N}_p^+$ for p large.*

Proof. By the properties of the solutions (u_p) we know that the nodal line NL_p is the boundary of a domain containing O in his interior. Hence if $O \notin \mathcal{N}_{p_n}^+$ for a sequence $p_n \to \infty$ as $n \to \infty$, it would follow that

$$\operatorname{dist}(x_{p_n}^+, NL_{p_n}) \leq |x_{p_n}^+|. \tag{3.11}$$

Dividing by $\mu_{p_n}^+$ and passing to the limit, from Corollary 2.4 (remember that $x_{p_n}^+$ has the role of x_{1,p_n} in the general Proposition 2.2) we get $|x_{p_n}^+|/\mu_{p_n}^+ \to \infty$ as $n \to \infty$, contrary to Proposition 3.3. So (i) holds.

To prove (ii) assume for contradiction that for a sequence $p_n \to \infty$ as $n \to \infty$, we have $u_{p_n}(x_{i,p_n}) < 0$ for some $i \in \{1, \ldots, k\}$. Then $\operatorname{dist}(x_{i,p_n}, NL_{p_n}) \leq |x_{i,p_n}|$, so exactly as in (i) we reach a contradiction with Proposition 3.3. □

Proposition 3.6. *The maximal number k of families of points $(x_{i,p})$, $i = 1, \ldots, k$, for which $(\mathcal{P}_1^k)$, $(\mathcal{P}_2^k)$ and $(\mathcal{P}_3^k)$ hold is* 1.

Proof. Assume for contradiction that $k > 1$ and set $x_p^+ = x_{1,p}$. For a family $(x_{j,p})$, $j \in \{2, \ldots, k\}$, by Proposition 3.3 there exists $C > 0$ such that

$$|x_{1,p}|/\mu_{1,p} \leq C \quad \text{and} \quad |x_{j,p}|/\mu_{j,p} \leq C.$$

Thus, since by definition $\mu_p^+ = \mu_{1,p} \leq \mu_{j,p}$, also $|x_{1,p}|/\mu_{j,p} \leq C$. Hence

$$\frac{|x_{1,p} - x_{j,p}|}{\mu_{j,p}} \leq \frac{|x_{1,p}| + |x_{j,p}|}{\mu_{j,p}} \leq C,$$

which contradicts $(\mathcal{P}_1^k)$ when $p \to \infty$. □

Then we easily get

Corollary 3.7. *There exists $C > 0$ such that for any family $(x_p)_p \subset \Omega$, one has*

$$|x_p|/\mu(x_p) \leq C \tag{3.12}$$

where $\mu(x_p)$ is defined in (2.34).

Proof. By Proposition 3.3, (3.12) holds for x_p^+. Moreover, since by Proposition 3.6 we have $k = 1$, applying $(\mathcal{P}_3^1)$ to the points (x_p), for $x_p \neq x_p^+$, we obtain

$$|x_p - x_p^+|/\mu(x_p) \leq C.$$

By definition, $\mu_p^+ \leq \mu(x_p)$, hence

$$\frac{|x_p|}{\mu(x_p)} \leq \frac{|x_p - x_p^+|}{\mu(x_p)} + \frac{|x_p^+|}{\mu(x_p)} \leq \frac{|x_p - x_p^+|}{\mu(x_p)} + \frac{|x_p^+|}{\mu_p^+} \leq C. \qquad \square$$

Proposition 3.8. *Let $(x_p) \subset \Omega$ be such that $p|u_p(x_p)|^{p-1} \to \infty$ and let $\mu(x_p)$ be as in (2.34). Assume that the rescaled functions $v_p(x) := \frac{p}{u_p(x_p)}(u_p(x_p + \mu(x_p)x) - u_p(x_p))$ converge to U in $C^1_{\mathrm{loc}}\big(\mathbb{R}^2 \setminus \big\{-\lim_p \frac{x_p}{\mu(x_p)}\big\}\big)$ as $p \to \infty$ (with U as in (2.8)). Then*

$$|x_p|/\mu(x_p) \to 0 \quad \text{as } p \to \infty. \tag{3.13}$$

As a byproduct, $v_p \to U$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$ as $p \to \infty$.

Proof. By Corollary 3.7 we know that $|x_p|/\mu(x_p) \leq C$. Assume for contradiction that $|x_p|/\mu(x_p) \to \ell > 0$. Let $g \in G$ be such that $|x_p - gx_p| = C_g|x_p|$ with a constant $C_g > 1$ (such a g exists because G is a group of rotations about the origin). Hence

$$|x_p - gx_p|/\mu(x_p) = C_g|x_p|/\mu(x_p) \to C_g\ell > \ell.$$

Then $x_0 := \lim_{p\to\infty} \frac{gx_p - x_p}{\mu(x_p)} \in \mathbb{R}^2 \setminus \big\{-\lim_p \frac{x_p}{\mu(x_p)}\big\}$ and so by C^1_{loc} convergence we get

$$v_p\left(\frac{gx_p - x_p}{\mu(x_p)}\right) \to U(x_0) < 0 \quad \text{as } p \to \infty.$$

On the other hand, for any $g \in G$, one also has

$$v_p\left(\frac{gx_p - x_p}{\mu(x_p)}\right) = \frac{p}{u_p(x_p)}(u_p(gx_p) - u_p(x_p)) = 0,$$

by the symmetry of u_p, and this gives a contradiction. $\square$

Proposition 3.9. *Let (x_p) be as in Proposition 3.8. Then*

$$\textit{either} \quad \mathrm{dist}(x_p, NL_p)/\mu(x_p) \to \infty \quad \textit{or} \quad \mathrm{dist}(x_p, NL_p)/\mu(x_p) \to 0 \quad \textit{as } p \to \infty.$$

Moreover if $u_p(x_p) > 0$ then the first alternative holds.

Proof. By Proposition 3.8 the rescaled functions v_p converge to U in $C^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$. Therefore in order to prove the first assertion we can argue exactly as in the proof of Corollary 2.4 but now we cannot exclude $\ell = 0$ because we do not have the convergence of v_p in the whole $\mathbb{R}^2$.

If instead we know that $u_p(x_p) > 0$, then we will show that the second alternative cannot occur. Indeed, assume for contradiction that there exists $z_p \in NL_p$ such that $|x_p - z_p|/\mu(x_p) \to 0$. Let $y_p \in \partial\Omega$ be such that $|x_p - y_p|/\mu(x_p) \to \infty$ and define a continuous curve $\gamma_p : [0,1] \to \mathcal{N}_p^-$ such that $\gamma_p(0) = z_p$, $\gamma_p(1) = y_p$. Then, by continuity, there exists $t_p \in [0,1]$ such that $|x_p - s_p|/\mu(x_p) \to 1$ for $s_p := \gamma_p(t_p)$. Therefore $v_p\big(\frac{s_p - x_p}{\mu(x_p)}\big) \to U(x_0) < 0$ as $p \to \infty$ for a point x_0 such that $|x_0| = 1$. On the other hand, since $u_p(x_p) > 0$, it follows that $v_p\big(\frac{s_p - x_p}{\mu(x_p)}\big) \le -p \to -\infty$, giving a contradiction. □

4. G-symmetric case: asymptotic analysis about the minimum points and proof of Theorem 1.2

As defined in the introduction, we consider a family (x_p^-) of minimum points of u_p. By Lemma 2.1 we have $p|u_p(x_p^-)|^{p-1} \to \infty$ as $p \to \infty$. So defining μ_p^- by $(\mu_p^-)^{-2} := p|u_p(x_p^-)|^{p-1}$, we see by $(\mathcal{P}_3^1)$ that

$$|x_p^+ - x_p^-|/\mu_p^- \le C. \tag{4.1}$$

Moreover, since we already know that $\mathrm{dist}(x_p^+, NL_p)/\mu_p^+ \to \infty$ as $p \to \infty$, we deduce that $|x_p^+ - x_p^-|/\mu_p^+ \to \infty$ as $p \to \infty$, and in turn by (4.1) we get

$$\mu_p^+/\mu_p^- \to 0 \quad \text{as } p \to \infty. \tag{4.2}$$

Note that (4.1) and (4.2) more generally hold for any family (x_p) of points such that $u_p(x_p) < 0$ and $p|u_p(x_p)|^{p-1} \to \infty$.

By Lemma 2.1 and Corollary 3.7 we have

$$|x_p^-|/\mu_p^- \le C, \tag{4.3}$$

so there are two possibilities: either $|x_p^-|/\mu_p^- \to \ell > 0$ or $|x_p^-|/\mu_p^- \to 0$ as $p \to \infty$, up to subsequences. We will exclude the latter case.

We start with a preliminary result:

Lemma 4.1. *For $x \in \Omega/|x_p^-| := \{y \in \mathbb{R}^2 : y|x_p^-| \in \Omega\}$ define the rescaled function*

$$w_p^-(x) := \frac{p}{u_p(x_p^-)}(u_p(|x_p^-|x) - u_p(x_p^-)).$$

Then

$$w_p^- \to \gamma \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\}) \text{ as } p \to \infty, \tag{4.4}$$

where $\gamma \in C^1(\mathbb{R}^2 \setminus \{0\})$, $\gamma \le 0$, $\gamma(x_\infty) = 0$ for a point $x_\infty \in \partial B_1(0)$ and it is a solution to

$$-\Delta\gamma = \ell^2 e^\gamma \quad \text{in } \mathbb{R}^2 \setminus \{0\}.$$

In particular $\gamma \equiv 0$ when $\ell = 0$.

Proof. (4.3) implies that $|x_p^-| \to 0$ as $p \to \infty$, so it follows that $\Omega/|x_p^-| \to \mathbb{R}^2$ as $p \to \infty$.

By definition we have

$$w_p^- \leq 0 \quad \text{and} \quad w_p(x_p^-/|x_p^-|) = 0 \tag{4.5}$$

and $w_p^- = -p$ on $\partial(\Omega/|x_p^-|)$. Moreover, for $x \in \Omega/|x_p^-|$ we define $\xi_p := |x_p^-|x$ and μ_{ξ_p} via $\mu_{\xi_p}^{-2} := p|u_p(\xi_p)|^{p-1}$. Thanks to (1.1) we then have

$$|-\Delta w_p^-(x)| = \frac{p|x_p^-|^2|u_p(\xi_p)|^p}{|u_p(x_p^-)|} = \frac{|u_p(\xi_p)|}{|u_p(x_p^-)|}\frac{|x_p^-|^2}{\mu_{\xi_p}^2} \leq c_\infty \frac{|x_p^-|^2}{\mu_{\xi_p}^2}, \tag{4.6}$$

where $c_\infty := \lim_p \|u_p\|_\infty$. Then, observing that $|x_p^-|/\mu_{\xi_p} \leq C/|x|$ by Corollary 3.7 applied to ξ_p, we have

$$|-\Delta w_p^-(x)| \leq c_\infty C^2/|x|^2.$$

Hence for any $R > 0$,

$$|-\Delta w_p^-| \leq c_\infty C^2 R^2 \quad \text{in } \Omega/|x_p^-| \setminus B_{1/R}(0). \tag{4.7}$$

So, similarly to Step 1 of the proof of Proposition 2.2 (using now $w_p^-(x_p^-/|x_p^-|) = 0$), it follows that for any $R > 1$ ($x_p^-/|x_p^-| \in \partial B_1(0) \subset B_R(0) \setminus B_{1/R}(0)$ for $R > 1$), w_p^- is uniformly bounded in $B_R(0) \setminus B_{1/R}(0)$.

After passing to a subsequence, standard elliptic theory applied to the equation

$$-\Delta w_p^-(x) = \frac{|x_p^-|^2}{(\mu_p^-)^2}\left(1 + \frac{w_p^-(x)}{p}\right)\left|1 + \frac{w_p^-(x)}{p}\right|^{p-1} \tag{4.8}$$

gives that w_p^- is bounded in $C^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$. Hence (4.4) and the properties of γ follow.

In particular when $\ell = 0$ it follows that γ is harmonic in $\mathbb{R}^2 \setminus \{0\}$ and $\gamma(x_\infty) = 0$ for some $x_\infty \in \partial B_1(0)$, therefore by the maximum principle we obtain $\gamma \equiv 0$. □

Proposition 4.2. *There exists $\ell > 0$ such that $|x_p^-|/\mu_p^- \to \ell$ as $p \to \infty$.*

Proof. By Corollary 3.7 we know that $|x_p^-|/\mu_p^- \to \ell \in [0, \infty)$ as $p \to \infty$. Suppose for contradiction that $\ell = 0$. Then Lemma 4.1 implies that

$$w_p^- \to 0 \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\}) \text{ as } p \to \infty. \tag{4.9}$$

By (1.1), applying the divergence theorem in $B_{|x_p^-|}(0)$ we get

$$\begin{aligned} p\int_{\partial B_{|x_p^-|}(0)} \nabla u_p(y) \cdot \frac{y}{|y|}\, d\sigma(y) \\ = p\int_{B_{|x_p^-|}(0)\cap\mathcal{N}_p^-} |u_p(x)|^p\, dx - p\int_{B_{|x_p^-|}(0)\cap\mathcal{N}_p^+} |u_p(x)|^p\, dx. \end{aligned} \tag{4.10}$$

Scaling u_p with respect to $|x_p^-|$ as in Lemma 4.1, by (4.9) we obtain

$$\begin{aligned}\left|p\int_{\partial B_{|x_p^-|}(0)} \nabla u_p(y)\cdot\frac{y}{|y|}\,d\sigma(y)\right| &= \left|p\int_{\partial B_1(0)} |x_p^-|\nabla u_p(|x_p^-|x)\cdot\frac{x}{|x|}\,d\sigma(x)\right| \\ &= \left|\int_{\partial B_1(0)} u_p(x_p^-)\,\nabla w_p^-(x)\cdot\frac{x}{|x|}\,d\sigma(x)\right| \\ &\le |u_p(x_p^-)|\cdot 2\pi\sup_{|x|=1}|\nabla w_p^-(x)| = o_p(1). \qquad (4.11)\end{aligned}$$

Now we want to estimate the right hand side in (4.10). We first observe that scaling around $|x_p^-|$ with respect to μ_p^- we get

$$\begin{aligned}p\int_{B_{|x_p^-|}(0)\cap\mathcal{N}_p^-}|u_p(x)|^p\,dx &= p\int_{B_1(0)\cap\mathcal{N}_p^-/|x_p^-|}|u_p(|x_p^-|y)|^p|x_p^-|^2\,dy \\ &\le c_\infty\int_{B_1(0)\cap\mathcal{N}_p^-/|x_p^-|}\frac{|u_p(|x_p^-|y)|^{p-1}}{|u_p(x_p^-)|^{p-1}}\,\frac{|x_p^-|^2}{(\mu_p^-)^2}\,dx = o_p(1), \qquad (4.12)\end{aligned}$$

where in the last equality we have used the fact that $|u_p(|x_p^-|y)|^{p-1}/|u_p(x_p^-)|^{p-1}\le 1$, since $|x_p^-|y\in\mathcal{N}_p^-$, and the assumption $|x_p^-|/\mu_p^-\to 0$ as $p\to\infty$.

Next we claim that there exists $\bar p>1$ such that for any $p\ge\bar p$,

$$B_{\mu_p^+}(x_p^+)\subset B_{|x_p^-|}(0). \qquad (4.13)$$

Indeed, Corollary 2.4 implies that

$$\infty=\lim_p\frac{\mathrm{dist}(x_p^+,NL_p)}{\mu_p^+}\le\lim_p\frac{|x_p^+-x_p^-|}{\mu_p^+}\le\lim_p\frac{|x_p^+|}{\mu_p^+}+\lim_p\frac{|x_p^-|}{\mu_p^+}=\lim_p\frac{|x_p^-|}{\mu_p^+},$$

where the last equality follows from Proposition 3.8 (i.e. $|x_p^+|/\mu_p^+\to 0$). Hence for any $x\in B_1(0)$ we have

$$\frac{|x_p^++\mu_p^+x|}{|x_p^-|}\le\frac{|x_p^+|}{|x_p^-|}+\frac{\mu_p^+}{|x_p^-|}\le\frac{2\mu_p^+}{|x_p^-|}\to 0 \quad \text{as } p\to\infty,$$

and so (4.13) is proved.

Thus by (4.13) and scaling around x_p^+ with respect to μ_p^+ we obtain

$$p\int_{B_{|x_p^-|}(0)\cap\mathcal{N}_p^+}|u_p(x)|^p\,dx\ge p\int_{B_{\mu_p^+}(x_p^+)}|u_p(x)|^p\,dx=c_\infty\int_{B_1(0)}e^U\,dx+o_p(1). \qquad (4.14)$$

Collecting (4.10), (4.11), (4.12) and (4.14) we clearly get a contradiction. □

Next we show that the nodal line shrinks to the origin faster than μ_p^- as $p\to\infty$.

Proposition 4.3. *We have*

$$\frac{\max_{y_p \in NL_p} |y_p|}{\mu_p^-} \to 0 \quad \text{as } p \to \infty.$$

Proof. By Proposition 4.2 it is enough to prove that

$$\frac{\max_{y_p \in NL_p} |y_p|}{|x_p^-|} \to 0 \quad \text{as } p \to \infty.$$

First we show that, for any $y_p \in NL_p$, the following alternative holds:

$$\text{either} \quad |y_p|/|x_p^-| \to 0 \quad \text{or} \quad |y_p|/|x_p^-| \to \infty \quad \text{as } p \to \infty. \tag{4.15}$$

Indeed, assume for contradiction that $|y_p|/|x_p^-| \to m \in (0, \infty)$ as $p \to \infty$. Then $w_p^-(y_p/|x_p^-|) = -p \to -\infty$ as $p \to \infty$. But we have proved in Lemma 4.1 that $w_p^-(y_p/|x_p^-|) \to \gamma(y_m) \in \mathbb{R}$, where y_m is such that $|y_m| = m > 0$, and this gives a contradiction.

To conclude the proof we have to exclude the second alternative in (4.15). For $y_p \in NL_p$, assume for contradiction that $|y_p|/|x_p^-| \to \infty$ as $p \to \infty$ and observe that

$$\exists z_p \in NL_p, \quad |z_p|/|x_p^-| \to 0 \quad \text{as } p \to \infty. \tag{4.16}$$

Indeed, in the previous section we have shown that $O \in \mathcal{N}_p^+$, hence there exists t_p in $(0, 1)$ such that $z_p := t_p x_p^- \in NL_p$. Since $|z_p|/|x_p^-| < 1$, by (4.15) we get (4.16).

Hence for any $M > 0$ there exists $\alpha_p^M \in NL_p$ such that $|\alpha_p^M|/|x_p^-| \to M$ as $p \to \infty$, and this contradicts (4.15). □

Finally, we can analyze the local behavior of u_p around the minimum point x_p^-. Note that by Propositions 3.8 and 4.2 we can already claim that the rescaling v_p^- about x_p^- (see (4.17) below) cannot converge to the regular solution U of the Liouville problem (1.6) such that $U(0) = 0$ in $\mathbb{R}^2 \setminus \{0\}$.

Proposition 4.4. *The scaling of u_p around x_p^- defined by*

$$v_p^-(x) := \frac{p}{u_p(x_p^-)}\left(u_p(\mu_p^- x + x_p^-) - u_p(x_p^-)\right) \tag{4.17}$$

for $x \in \widetilde{\Omega}_p^-$ converges (passing to a subsequence) in $C^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{x_\infty\})$ as $p \to \infty$ to the function

$$V_\ell(x) := \log\left(\frac{2\alpha^2 \beta^\alpha |x - x_\infty|^{\alpha-2}}{(\beta^\alpha + |x - x_\infty|^\alpha)^2}\right),$$

where $\alpha = \alpha(\ell) = \sqrt{2\ell^2 + 4}$, $\beta = \beta(\ell) = \ell\left(\frac{\alpha+2}{\alpha-2}\right)^{1/\alpha}$, $x_\infty \in \mathbb{R}^2$, $|x_\infty| = \ell$ and $\ell = \lim_p |x_p^-|/\mu_p^- > 0$. The function $V(x) := V_\ell(x + x_\infty)$ is a radial singular solution of (1.10) for $H = H(\ell) < 0$.

Proof. Consider the translations of (4.17):

$$s_p^-(x) := v_p^-(x - x_p^-/\mu_p^-) = \frac{p}{u_p(x_p^-)}(u_p(\mu_p^- x) - u_p(x_p^-)), \qquad x \in \Omega/\mu_p^-,$$

which solve

$$\begin{aligned} -\Delta s_p^-(x) &= \left|1 + \frac{s_p^-(x)}{p}\right|^{p-1}\left(1 + \frac{s_p^-(x)}{p}\right), \\ s_p^-(x_p^-/\mu_p^-) &= 0, \qquad s_p^- \le 0. \end{aligned}$$

Observe that $\Omega/\mu_p^- \to \mathbb{R}^2$ as $p \to \infty$.

We claim that for any fixed $r > 0$, $|-\Delta s_p^-|$ is bounded in $\Omega/\mu_p^- \setminus B_r(0)$. Indeed, Proposition 4.3 implies that if $x \in \mathcal{N}_p^+/\mu_p^-$, then $|x| \le (\max_{z_p \in NL_p} |z_p|)/\mu_p^- < r$ for p large, hence

$$\Omega/\mu_p^- \setminus B_r(0) \subset \mathcal{N}_p^-/\mu_p^- \qquad \text{for } p \text{ large},$$

and so the claim follows by observing that if $x \in \mathcal{N}_p^-/\mu_p^-$, then $|-\Delta s_p^-(x)| \le 1$.

Hence, by the arbitrariness of $r > 0$, $s_p^- \to V$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$ as $p \to \infty$ where V is a solution of

$$-\Delta V = e^V \quad \text{in } \mathbb{R}^2 \setminus \{0\}, \qquad V \le 0, \qquad V(x_\ell) = 0,$$

where $x_\ell := \lim_p x_p^-/\mu_p^-$ and $|x_\ell| = \ell$ by Proposition 4.2. Moreover $e^V \in L^1(\mathbb{R}^2)$: indeed, for any $r > 0$ and any $\varepsilon \in (0,1)$,

$$\begin{aligned} \int_{B_{1/r}(0)\setminus B_r(0)} e^V\,dx &\le \int_{B_{1/r}(0)\setminus B_r(0)} \frac{|u_p(\mu_p^- x)|^{p+1}}{|u_p(x_p^-)|^{p+1}}\,dx + o_p(1) \\ &= \frac{p}{|u_p(x_p^-)|^2} \int_{B_{\mu_p^-/r}(0)\setminus B_{r\mu_p^-}(0)} |u_p(y)|^{p+1} dy + o_p(1) \\ &\overset{\text{Lemma 2.1(ii)}}{\le} \frac{p}{(1-\varepsilon)^2}\int_\Omega |u_p(y)|^{p+1} dy + o_p(1) \overset{(3.2)}{<} \infty. \end{aligned}$$

Observe that if V were a classical solution of $-\Delta V = e^V$ in the whole $\mathbb{R}^2$ then necessarily $V(x) = U(x - x_\ell)$. As a consequence $v_p^-(x) = s_p^-(x + x_p^-/\mu_p^-) \to V(x + x_\ell) = U(x)$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{-x_\ell\})$ as $p \to \infty$. Observe that $x_\ell = \lim_p x_p^-/\mu_p^-$ and so Proposition 3.8 would imply that $|x_p^-|/\mu_p^- \to 0$ as $p \to \infty$, contrary to Proposition 4.2. Thus, by [9, 10, 11] and the classification in [8], V solves, for some $\eta > 0$, the entire equation

$$\begin{cases} -\Delta V = e^V - 4\pi\eta\delta_0 & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^V\,dx = 8\pi(1+\eta), \end{cases} \tag{4.18}$$

where δ_0 denotes the Dirac measure centered at the origin.

We claim that V is radial. Indeed, by the classification in [24], either V is radial, or $\eta \in \mathbb{N}$ and V is $(\eta+1)$-symmetric. Suppose the latter; then, since V is the limit of s_p^- (which is G-symmetric with $|G| \geq 4e$) we get $\eta + 1 \geq 4e$ and so

$$\int_{\mathbb{R}^2} e^V\, dx \geq 4e \cdot 8\pi. \tag{4.19}$$

On the other hand, for any $R > 0$,

$$\begin{aligned}
\int_{B_R(0)\setminus B_{1/R}(0)} e^V\, dx &\leq \lim_{p\to\infty} \int_{B_R(0)\setminus B_{1/R}(0)} \left|\frac{u_p(\mu_p^- x)}{u_p(x_p^-)}\right|^{p+1} dx \\
&= \lim_{p\to\infty} \frac{p}{u_p(x_p^-)^2} \int_{B_{R\mu_p^-}(0)\setminus B_{\mu_p^-/R}(0)} |u_p(y)|^{p+1} dy \\
&\overset{(*)}{\leq} \lim_{p\to\infty} \frac{p}{u_p(x_p^-)^2} \left(\int_\Omega |u_p(y)|^{p+1} dy - \int_{\mathcal{N}_p^+} |u_p(y)|^{p+1} dy \right) \\
&\overset{(\sharp)}{\leq} (\alpha - 1)\cdot 8\pi e
\end{aligned} \tag{4.20}$$

where in $(*)$ we have used the fact that, by Proposition 4.3, $\mathcal{N}_p^+ \subset B_{\mu_p^-/R}(0)$, and in $(\sharp)$ we have applied (2.5), Lemma 3.1 and (3.2). By the arbitrariness of R from (4.20) we then get

$$\int_{\mathbb{R}^2} e^V\, dx \leq (\alpha - 1)e \cdot 8\pi. \tag{4.21}$$

Lastly, using the assumption $\alpha < 5$ in (4.21) we get a contradiction with (4.19).

Thus V is radial and $V(r)$ satisfies

$$\begin{cases} -V'' - \frac{1}{r}V' = e^V & \text{in } (0,\infty), \\ V \leq 0, \\ V(\ell) = V'(\ell) = 0. \end{cases}$$

The solutions of this problem are

$$V(r) = \log\left(\frac{4}{\delta^2} \frac{e^{\frac{\sqrt{2}}{\delta}(\log r - y)}}{(1 + e^{\frac{\sqrt{2}}{\delta}(\log r - y)})^2} \right) - 2\log r \tag{4.22}$$

for $\delta > 0$ and $y \in \mathbb{R}$. Observe that from $V'(\ell) = 0$ we get $\frac{1-\sqrt{2}\delta}{1+\sqrt{2}\delta} = e^{\frac{\sqrt{2}}{\delta}(\log \ell - y)}$ and moreover $V(\ell) = 0$ for $\ell = \sqrt{1-2\delta^2}/\delta$. Hence from $V(\ell) = V'(\ell) = 0$ it follows that $\ell^2 = (1-2\delta^2)/\delta^2$, which implies that $\delta = 1/\sqrt{2+\ell^2}$. Inserting this estimate into (4.22) we get

$$V(r) = \log\left(\frac{2\alpha^2 \beta^\alpha r^{\alpha-2}}{(\beta^\alpha + r^\alpha)^2} \right),$$

where $\alpha = \sqrt{2\ell^2 + 4}$ and $\beta = \ell\left(\frac{\alpha+2}{\alpha-2}\right)^{1/\alpha}$. The conclusion follows by observing that $v_p^-(x) = s_p^-(x + x_p^-/\mu_p^-)$. $\square$

Proof of Theorem 1.2. It follows from all previous results. More precisely, (i) follows from (3.12) and Lemma 2.1; (ii) is from Proposition 4.3; and the asymptotic behavior of the rescaled functions v_p^+ and v_p^- is shown in Propositions 3.2 and 4.4. □

Remark 4.5. By (4.2) applied to any $(x_p)_p \subset \Omega$ such that $u_p(x_p) < 0$ and $p|u_p(x_p)|^{p-1} \to \infty$ as $p \to \infty$, we easily derive

$$p\big(u_p(x_p^+) + u_p(x_p)\big) \to \infty \quad \text{as } p \to \infty. \tag{4.23}$$

Indeed, if

$$p_n\big(u_{p_n}(x_{p_n}^+) + u_{p_n}(x_{p_n})\big) \to K \geq 0 \quad \text{as } p \to \infty,$$

then, recalling the definition of $\mu(x_p)$ in (2.34) and setting $c_\infty := \lim_p u_p(x_p^+) > 0$, we would have

$$\frac{\mu_p^{+2}}{\mu(x_p)^2} = \left(\frac{|u_p(x_p)|}{u_p(x_p^+)}\right)^{p-1} = \left(1 - \frac{\frac{p(u_p(x_p^+)+u_p(x_p))}{u_p(x_p^+)}}{p}\right)^{p-1} \xrightarrow[p\to\infty]{} e^{-K/c_\infty} \neq 0,$$

which contradicts (4.2).

In particular by (4.23) we get

$$p\big(u_p(x_p^+) + u_p(x_p^-)\big) \to \infty,$$

which means, in the notation of [19], that u_p is of type B'.

Remark 4.6. It is not difficult to prove an analogue of Theorem 1.2 for higher energy solutions, under stronger symmetry assumptions. Precisely for any choice of $m \in \mathbb{N} \setminus \{0\}$ one could replace the assumptions (3.1) and (3.2) by

$$|G| \geq me, \tag{4.24}$$

$$p\int_\Omega |\nabla u_p|^2 \leq \alpha\, 8\pi e \quad \text{for some } \alpha < m+1 \text{ and } p \text{ large.} \tag{4.25}$$

5. Further results and open questions

The asymptotic result of Theorem 1.2 together with the existence result of [13] shows the presence of sign-changing G-symmetric solutions of (1.1) whose limit profile, as $p \to \infty$, looks like the superposition of (at least) two different signed bubbles coming, roughly speaking, from a regular and a singular solution of (1.6) and (1.10).

The two bumps could carry different energies but we cannot precisely estimate them and deduce that they “exhaust” all the energy of the solutions u_p which is bounded by (1.11). This means that “a priori” one could think that other bumps could develop as $p \to \infty$. We believe that this is not the case, as confirmed by the radial setting studied in [20].

A partial result in this direction is obtained in the next proposition which excludes the presence of other positive bumps having the limit profile of a regular solution of (1.6).

Proposition 5.1. *Under the assumptions of Theorem 1.2, let* $(x_p) \subset \Omega$ *be such that* $\mu(x_p)^{-2} := p|u_p(x_p)|^{p-1} \to \infty$ *as* $p \to \infty$ *and assume that* $u_p(x_p) > 0$ *and that the rescaled functions* $v_p(x) := \frac{p}{u_p(x_p)}(u_p(x_p + \mu(x_p)x) - u_p(x_p))$ *converge to* U *in* $C^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$ *as* $p \to \infty$. *Then*

$$
\begin{aligned}
&x_p = x_p^+ + o_p(1)\mu_p^+,\\
&\mu_p^+/\mu(x_p) \to 1 \quad \text{as } p \to \infty,\\
&u_p(x_p) \to c_\infty \quad\;\; \text{as } p \to \infty,
\end{aligned}
$$

where $c_\infty := \lim_p \|u_p^+\|_\infty$. *So, roughly speaking, scaling about* x_p *with respect to its parameter we obtain the same bubble appearing from the scaling about* x_p^+ *with respect to* μ_p^+.

Proof. Denote $\mu(x_p)$ simply by μ_p.

Step 1. *The following alternative holds:*

$$
\text{either} \quad |x_p^+ - x_p|/\mu_p^+ \to 0 \quad \text{or} \quad |x_p^+ - x_p|/\mu_p^+ \to \infty \quad \text{as } p \to \infty. \tag{5.1}
$$

Indeed, if for contradiction there exists $C > 0$ such that $|x_p^+ - x_p|/\mu_p^+ \to C$, then by Proposition 3.2 we get, for $x_C := \lim_p (x_p - x_p^+)/\mu_p^+$, $x_C \in \partial B_C(0)$,

$$
v_p^+\left(\frac{x_p - x_p^+}{\mu_p^+}\right) \to U(x_C) \in (-\infty, 0) \quad \text{as } p \to \infty
$$

and so

$$
\frac{|x_p^+ - x_p|}{\mu_p} = \frac{|x_p^+ - x_p|}{\mu_p^+}\left(1 + \frac{v_p^+\left(\frac{x_p - x_p^+}{\mu_p^+}\right)}{p}\right)^{(p-1)/2} \to Ce^{U(x_\infty)/2} > 0 \quad \text{as } p \to \infty.
$$

This leads to a contradiction because by Proposition 3.8,

$$
|x_p^+ - x_p|/\mu_p \le |x_p|/\mu_p + |x_p^+|/\mu_p^+ \to 0 \quad \text{as } p \to \infty.
$$

Step 2. *The first alternative in* (5.1) *holds, that is,*

$$
|x_p^+ - x_p|/\mu_p^+ \to 0 \quad \text{as } p \to \infty. \tag{5.2}
$$

Suppose that $|x_p^+ - x_p|/\mu_p^+ \to \infty$ as $p \to \infty$. As a consequence, by Proposition 2.6,

$$
\mu_p^+/\mu_p \to 0 \quad \text{as } p \to \infty. \tag{5.3}
$$

By the divergence theorem, for any $r > 0$ and $p \ge p_r$ we also have

$$
\begin{aligned}
-p\int_{\partial B_{r\mu_p}(x_p)} \nabla u_p(y) \cdot \frac{y - x_p}{|y - x_p|}\, d\sigma(y) &= -p\int_{B_{r\mu_p}(x_p)} \Delta u_p(x)\, dx\\
&= p\int_{B_{r\mu_p}(x_p)} |u_p(x)|^p\, dx,
\end{aligned} \tag{5.4}
$$

where for the last equality we have used (1.1), the assumption $u_p(x_p) > 0$ and Proposition 3.9 to deduce that $B_{r\mu_p}(x_p) \subset \mathcal{N}_p^+$ for $p \geq p_r$.

Now, since the function U introduced in (2.8) is in $C^\infty(\mathbb{R}^2)$, one can find $r > 0$ such that

$$2\pi r \sup_{|x|=r} |\nabla U(x)| \leq \frac{2}{3} \int_{B_1(0)} e^U \, dx. \tag{5.5}$$

With this choice of r we estimate the two terms of (5.4).

By Proposition 3.8 and (5.3), there exists $p'_r > 1$ such that $B_{\mu_p^+}(x_p^+) \subset B_{r\mu_p}(x_p)$ for any $p \geq p'_r$; moreover, using the convergence of v_p^+ to U in $C^1_{\text{loc}}(\mathbb{R}^2)$, we get

$$\begin{aligned} p \int_{B_{r\mu_p}(x_p)} |u_p(x)|^p \, dx &\geq p \int_{B_{\mu_p^+}(x_p^+)} |u_p(x)|^p \, dx = \int_{B_1(0)} \frac{|u_p(x_p^+ + \mu_p^+ y)|^p}{|u_p(x_p^+)|^{p-1}} \, dy \\ &= u_p(x_p^+) \int_{B_1(0)} \left| 1 + \frac{u_p(x_p^+ + \mu_p^+ y) - u_p(x_p^+)}{u_p(x_p^+)} \right|^p dy = u_p(x_p^+) \int_{B_1(0)} \left| 1 + \frac{v_p^+(y)}{p} \right|^p dy \\ &= c_\infty \int_{B_1(0)} e^U + o_p(1), \end{aligned} \tag{5.6}$$

where $c_\infty := \lim_{p\to\infty} \|u_p\|_\infty$. Finally, scaling u_p around x_p with respect to μ_p, by the convergence of v_p to U in $C^1_{\text{loc}}(\mathbb{R}^2 \setminus \{0\})$ we obtain, for $p \geq p''_r$,

$$\begin{aligned} \left| p \int_{\partial B_{r\mu_p}(x_p)} \nabla u_p(y) \cdot \frac{y - x_p}{|y - x_p|} \, d\sigma(y) \right| &= \left| \int_{\partial B_r(0)} u_p(x_p) \nabla v_p(x) \cdot \frac{x}{|x|} \, d\sigma(x) \right| \\ &= u_p(x_p) \left| \int_{\partial B_r(0)} \nabla v_p(x) \cdot \frac{x}{|x|} \, d\sigma(x) \right| \leq u_p(x_p) \cdot 2\pi r \sup_{|x|=r} |\nabla v_p(x)| \\ &\leq c_\infty \cdot 2\pi r \sup_{|x|=r} |\nabla U(x)| + o_p(1). \end{aligned} \tag{5.7}$$

In conclusion, by our choice of r, collecting (5.6) and (5.7) we derive, for $p \geq \max\{p_r, p'_r, p''_r\}$,

$$\begin{aligned} 0 < c_\infty \int_{B_1(0)} e^U \, dx + o_p(1) &\leq p \int_{B_{r\mu_p}(x_p)} |u_p(x)|^p \, dx \\ &= \left| p \int_{\partial B_{r\mu_p}(x_p)} \nabla u_p(y) \cdot \frac{y - x_p}{|y - x_p|} \, d\sigma(y) \right| \\ &\leq c_\infty \cdot 2\pi r \sup_{|x|=r} |\nabla U(x)| + o_p(1) \overset{(5.5)}{\leq} c_\infty \cdot \frac{2}{3} \int_{B_1(0)} e^U \, dx + o_p(1), \end{aligned}$$

which is clearly a contradiction.

Step 3. *Conclusion of the proof.* By (5.2) and Proposition 3.2 we get

$$v_p^+ \left(\frac{x_p - x_p^+}{\mu_p^+} \right) \to U(0) = 0 \quad \text{as } p \to \infty,$$

and so

$$\left(\frac{\mu_p^+}{\mu_p}\right)^2 = \left(\frac{u_p(x_p)}{u_p(x_p^+)}\right)^{p-1} = \left(1 + \frac{v_p^+\left(\frac{x_p - x_p^+}{\mu_p^+}\right)}{p}\right)^{p-1} \to 1 \quad \text{as } p \to \infty,$$

$$\frac{u_p(x_p)}{u_p(x_p^+)} - 1 = \frac{1}{p}\, v_p^+\left(\frac{x_p - x_p^+}{\mu_p^+}\right) \to 0 \quad \text{as } p \to \infty. \qquad \square$$

Remark 5.2. We are not able to get a similar result in the negative nodal region, i.e. for $(x_p) \subset \Omega$ such that $u_p(x_p) < 0$ and $\mu(x_p)^{-2} := p|u_p(x_p)|^{p-1} \to \infty$ as $p \to \infty$. In this case, using Proposition 4.3, Corollary 3.7 and Proposition 4.2 it is easy to get

$$\max_{y_p \in NL_p} |x_p - y_p|/\mu(x_p) \le C \quad \text{and} \quad |x_p - x_p^-|/\mu(x_p) \le C \tag{5.8}$$

for p large, which seems to indicate that there are no other negative bumps other than the one previously found.

As previously said, the main reason why we cannot exclude the presence of other bubbles, under the hypothesis of Theorem 1.2, is that we cannot precisely estimate the energy carried by each bubble so as to use the bound (1.11) to say that the two bubbles given by rescaling about x_p^+ and x_p^- use all the available energy. Let us point out that the energy carried by each of these bubbles depends on two quantities:

(i) the energy of the solution of the limit problem (related to the bubble),
(ii) the limit values of $u_p(x_p^+)$ or $u_p(x_p^-)$.

In the case of the positive bubble, obtained by rescaling about x_p^+, we know (i) but we lack a good estimate of $u_p(x_p^+)$ in (ii). Motivated by the results concerning the radial situation [20], we conjecture that

$$\lim_p u_p(x_p^+) = A^+ > \sqrt{e}. \tag{C1}$$

Note that if we knew this, we could reduce the assumption on the symmetry group G, by just requiring $|G| \ge 4$, as in [13] (see the proof of Proposition 3.3 and Remark 3.4).

In the case of the negative bubble, obtained by rescaling about x_p^-, we have neither a good estimate of the energy of the singular solution of the limit problem (since it depends on the constant $\ell = \lim_{p\to\infty} |x_p^-|/\mu_p^- > 0$), nor a good estimate of $u_p(x_p^-)$. Thinking again of the radial solution [20] we conjecture that

$$\lim_p p \int_\Omega |\nabla u_p^-| = B^- > 8\pi e, \tag{C2}$$

$$\lim_p u_p(x_p^-) = A^-, \qquad 1 < A^- < \sqrt{e}. \tag{C3}$$

More generally we believe that estimates analogous to **(C1)**, **(C2)** and **(C3)** should hold for bubble tower solutions of (1.1) in general domains.

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