EFIMOV EFFECT FOR A THREE-PARTICLE SYSTEM WITH TWO IDENTICAL FERMIONS

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ABSTRACT. We consider a three-particle quantum system in dimension three composed of two identical fermions of mass one and a different particle of mass m. The particles interact via two-body short range potentials. We assume that the Hamiltonians of all the two-particle subsystems do not have bound states with negative energy and, moreover, that the Hamiltonians of the two subsystems made of a fermion and the different particle have a zero-energy resonance. Under these conditions and for $m < m_* = (13.607)^{-1}$, we give a rigorous proof of the occurrence of the Efimov effect, i.e., the existence of infinitely many negative eigenvalues for the three-particle Hamiltonian H. More precisely, we prove that for $m > m_*$ the number of negative eigenvalues of H is finite and for $m < m_*$ the number $N(z)$ of negative eigenvalues of H below $z < 0$ has the asymptotic behavior $N(z) \sim \mathcal{C}(m)|\log|z||$ for $z \to 0^-$. Moreover, we give an upper and a lower bound for the positive constant $\mathcal{C}(m)$.

1. INTRODUCTION

Efimov effect is a remarkable physical phenomenon occurring in three-particle quantum systems in dimension three. It was discovered by Efimov in 1970 ([7], [8]) and it consists in the following. Let us assume that the particles interact via two-body short range potentials, the two-particle subsystems do not have bound states and at least two of them exhibit a zero-energy resonance. Then the Hamiltonian describing the three-particle system has infinitely many negative eigenvalues E_n accumulating at zero. Moreover, the eigenvalues satisfy the asymptotic geometrical law

$$\frac{E_{n+1}}{E_n} \ \to \ e^{-\frac{2\pi}{s_0}} \,, \qquad \text{for} \quad n \to \infty \tag{1.1}$$

where the parameter $s_0 > 0$ depends only on the mass ratios and, possibly, on the statistics of the particles. The three-particle bound states (or trimers) associated to the eigenvalues E_n are characterized by a size much larger than the range of the two-body potentials. They are determined by a long range, attractive effective interaction of kinetic origin which is produced by the resonance condition and it is independent of the details of the two-body potentials. According to an intuitive physical picture, one can say that in a trimer the attraction between two particles is mediated by the third one, which is moving back and forth between the two. Note that the attraction disappears if the two-body potentials become more attractive causing the destruction of the zero-energy resonance.
We emphasize that Efimov effect describes a universal low-energy behavior of the three-particle system. As a consequence of this universality character, the effect can be realized and observed in various physical contexts (e.g., in atomic, molecular, nuclear or condensed matter physics) and this fact has motivated a large number of experimental and theoretical works published on the subject in recent years (see, e.g., the reviews [3], [17]).
The original Efimov's physical argument is based on the replacement of the two-body potential with a boundary condition, which is essentially equivalent to consider a two-body

zero-range interaction, and on the introduction of hyper-spherical coordinates. If the resonant condition is satisfied, in these coordinates the problem become separable and in the equation for the hyper-radius R the long range, attractive effective potential $-(s_0^2 + 1/4)/R^2$ appears. The behavior for small R of this potential is too singular and an extra boundary condition at short distance must be imposed to restore self-adjointness. After this ad hoc procedure, one obtains the infinite sequence of negative eigenvalues satisfying the law (1.1) as a consequence of the large R behavior of the effective potential.

The first mathematical result on the Efimov effect was obtained by Yafaev in 1974 ([24]). He studied a symmetrized form of the Faddeev equations for the bound states of the three-particle Hamiltonian and proved the existence of an infinite number of negative eigenvalues. In 1993 Sobolev ([20]) used a slightly different symmetrization of the equations and proved the asymptotics

$$\lim_{z \to 0^-} \frac{N(z)}{|\log|z||} = \frac{s_0}{2\pi} \tag{1.2}$$

where $N(z)$ denotes the number of eigenvalues smaller than $z < 0$. Note that (1.2) is consistent with the law (1.1). In the same year Tamura ([22]) obtained the same result under more general conditions on the two-body potentials. Other mathematical proofs of the effect were obtained by Ovchinnikov and Sigal in 1979 ([18]) and Tamura in 1991 ([21]) using a variational approach based on the Born-Oppenheimer approximation (see also the related result in [14]). For more results on the subject see [23], [12], [13]. It is worth mentioning that a mathematical proof of the geometric asymptotic law (1.1) is still lacking (see the conjecture discussed in [1]).

In this paper we study the case of a three-particle system in dimension three composed of two identical fermions with mass one and a different particle with mass m. In the physical literature such a system has been extensively studied (see, e.g., [4], [17] and references therein) and it is known that the Efimov effect can be present with some peculiar features. Indeed, the effect is present only for $m < m_* = (13.607)^{-1}$ and, considering a partial wave decomposition, it takes place only in the subspaces corresponding to the "odd waves" (contrary to the case of identical bosons or distinguishable particles where the effect takes place in the "s-wave" subspace). Following the approach based on the analysis of the Faddeev equations, we give a mathematical proof of these facts. More precisely, we assume that: a) the two-body potentials are short range, rotationally invariant, non positive; b) the Hamiltonians of all the two-particle subsystems are positive; c) the Hamiltonian of the subsystems composed of the particle with mass m and a fermion has a zero-energy resonance. Then we prove that the Hamiltonian of the three-particle system has a finite number of negative eigenvalues for $m > m_*$ and an infinite number of negative eigenvalues accumulating at zero for $m < m_*$. We also prove the asymptotic behavior (1.2), where the constant at the right hand side, which in our case is denoted by $\mathcal{C}(m)$, depends only on the mass m and it is estimated from below and from above.

We note that, under our assumptions, the interaction potential between the two fermions does not produce zero-energy resonance and therefore it plays no role in the occurrence of the Efimov effect.

The method of the proof follows the line of reasoning of [20], with the modifications required to take into account of the peculiarity of our system. In particular, we formulate the eigenvalue problem $H\psi = z\psi$, $z < 0$, for the three-particle Hamiltonian in terms of symmetrized Faddeev equations $\Psi = \boldsymbol{A}(z)\Psi$, where $\boldsymbol{A}(z)$ is a 2×2 matrix (compact) operator, and we prove that $N(z) = n(1, \boldsymbol{A}(z))$, where the right hand side is the number of eigenvalues of $\boldsymbol{A}(z)$

larger than one. Then the problem is reduced to the study of the asymptotics of $n(1, \boldsymbol{A}(z))$ for $z \to 0^-$. The presence of the zero-energy resonance for the subsystems composed of the particle with mass m and a fermion determines a singular behavior (and a lack of compactness) of $\boldsymbol{A}(z)$ for $z = 0$ and this is the reason for the possible divergence of $n(1, \boldsymbol{A}(z))$ for $z \to 0^-$. Through some successive steps, we single out such singular behavior neglecting operators which, for $z \le 0$, are compact and continuous in z. At the end we find that the asymptotics for $z \to 0^-$ of $n(1, \boldsymbol{A}(z))$ reduces to the asymptotics for $R \to \infty$ of an operator S_R which has an explicit form. By a direct analysis of such an operator, we conclude the proof of the main result in the two cases $m < m_*$ and $m > m_*$.

The paper is organized as follows.

In section 2 we describe the three-particle model, formulate our assumptions on the interaction potentials and state the main result. In section 3 we briefly recall some results on the low-energy behavior of a two-particle Hamiltonian in presence (or in absence) of a zero-energy resonance. In section 4 we introduce the symmetrized Faddeev equation for the bound states of our three-particle Hamiltonian and we prove that $N(z) = n(1, \boldsymbol{A}(z))$. In section 5 we characterize the leading term of $\boldsymbol{A}(z)$ for $z \to 0^-$, neglecting operators which, for $z \le 0$, are compact and continuous in z. In section 6 we prove the main result exploiting the asymptotic behavior of the leading term.

2. Notation and main result

We consider a quantum system composed by two identical fermions of unitary mass and a different particle of mass $m_3 = m$. Let $x_1, x_2 \in \mathbb{R}^3$ denote the coordinates of the fermions and $x_3 \in \mathbb{R}^3$ the coordinates of the third particle. The state of the system is then described by a wave function $\psi \in L^2(\mathbb{R}^9)$ which satisfies the symmetry condition $\psi(x_1, x_2, x_3) = -\psi(x_2, x_1, x_3)$ and the Hamiltonian is typically of the form

$$\tilde{H} = -\frac{1}{2}\Delta_{x_1} - \frac{1}{2}\Delta_{x_2} - \frac{1}{2m}\Delta_{x_3} + v_{12}(x_1 - x_2) + v_{23}(x_2 - x_3) + v_{31}(x_3 - x_1) \quad (2.1)$$

where Δ_{x_i} denotes the laplacian with respect to the coordinates of the i-th particle and v_α, $\alpha \in \{12, 23, 31\}$ is the two-body real-valued potential associated to the pair of particles α. Due to the symmetry constraint, we suppose v_α spherically symmetric and $v_{23} = v_{31} := v$. We introduce the coordinates (R, x_α, y_α), where R is the coordinate of the center of mass and (x_α, y_α) is any pair of Jacobi coordinates, e.g., for $\alpha = 12$ one has

$$x_{12} = x_1 - x_2, \qquad y_{12} = \frac{x_1 + x_2}{2} - x_3. \quad (2.2)$$

In such coordinates the Hamiltonian takes the form

$$\tilde{H} = -\frac{1}{2(m+2)}\Delta_R - \frac{1}{2\mu_\alpha}\Delta_{x_\alpha} - \frac{1}{2n_\alpha}\Delta_{y_\alpha} + \sum_\beta v_\beta(x_\beta) \quad (2.3)$$

where

$$\begin{aligned} \mu_{12} &= \frac{1}{2} & \mu_{23} = \mu_{31} = \mu = \frac{m}{m+1} \\ n_{12} &= \frac{2m}{m+2} & n_{23} = n_{31} = n = \frac{m+1}{m+2} \end{aligned} \quad (2.4)$$

and x_β, for $\beta \neq \alpha$, is expressed in terms of (x_α, y_α). Moreover, it is convenient to extract the center of mass motion and to study the problem in momentum space. Let (k_α, p_α) be the pair of variables conjugate with respect to the Jacobi coordinates (x_α, y_α). Denoting with k_i the conjugate variable of x_i, they are explicitely defined by

$$
\begin{aligned}
k_{12} &= \frac{k_1 - k_2}{2} & p_{12} &= \frac{m(k_1 + k_2) - 2k_3}{m + 2} \\
k_{23} &= \frac{mk_2 - k_3}{m + 1} & p_{23} &= \frac{(k_2 + k_3) - (m + 1)k_1}{m + 2} \\
k_{31} &= \frac{k_3 - mk_1}{m + 1} & p_{31} &= \frac{(k_1 + k_3) - (m + 1)k_2}{m + 2}.
\end{aligned}
\tag{2.5}
$$

Then, for any choice of pair α, the Hamiltonian of the system can be written as

$$
H = H_0 + \sum_\beta V_\beta
\tag{2.6}
$$

where

$$
H_0 = \frac{k_\alpha^2}{2\mu_\alpha} + \frac{p_\alpha^2}{2n_\alpha}
\tag{2.7}
$$

is the free Hamiltonian and

$$
(V_\beta \psi)(k_\beta, p_\beta) = \frac{1}{(2\pi)^{3/2}} \int dk \, \hat{v}_\beta(k - k_\beta) \psi(k, p_\beta)
\tag{2.8}
$$

describes each interaction term. In (2.8) and in the following we denote by $\hat{f}$ the Fourier transform of f. Taking into account of the definitions given in (2.5), the symmetry constraint reduces to $\psi(k_{23}, p_{23}) = -\psi(-k_{31}, p_{31})$ or equivalently $\psi(k_{12}, p_{12}) = -\psi(-k_{12}, p_{12})$. Therefore, choosing for instance the coordinates (k_{23}, p_{23}), the Hilbert space of the system is

$$
\mathcal{H} = \left\{ \psi \in L^2(\mathbb{R}^6) \,|\, \psi(k_{23}, p_{23}) = -\psi\left(\frac{k_{23}}{m+1} - \frac{m(m+2)}{(m+1)^2} p_{23}, -k_{23} - \frac{p_{23}}{m+1} \right) \right\}.
\tag{2.9}
$$

During the proof it will be useful to use also the system of coordinates (p_{23}, p_{31}). Using the relations

$$
\begin{aligned}
k_{23} &= -p_{31} - \frac{1}{m+1} p_{23} \\
k_{31} &= p_{23} + \frac{1}{m+1} p_{31}
\end{aligned}
\tag{2.10}
$$

one finds that the free Hamiltonian can be rewritten as

$$
H_0 = \frac{p_{23}^2}{2\mu} + \frac{p_{31}^2}{2\mu} + \frac{p_{23} \cdot p_{31}}{m}.
\tag{2.11}
$$

The exchange of the fermionic coordinates corresponds to the exchange of p_{23} and p_{31}. Thus, using the coordinates (p_{23}, p_{31}), the Hilbert space of the system can be equivalently written as

$$
\mathcal{H}_1 = \left\{ \psi \in L^2(\mathbb{R}^6) \,|\, \psi(p_{23}, p_{31}) = -\psi(p_{31}, p_{23}) \right\}.
\tag{2.12}
$$

Note that for $f \in \mathcal{H}$ we have $g \in \mathcal{H}_1$, where $g(p_{23}, p_{31}) = f\left(-p_{31} - \frac{p_{23}}{m+1}, p_{23}\right)$.

It is also useful to introduce the two-particle subsystems of our three-particle system, described by the following Hamiltonians in $L^2(\mathbb{R}^3)$

$$
h_\alpha = -\frac{1}{2\mu_\alpha} \Delta_x + v_\alpha(x)
\tag{2.13}
$$

In order to formulate our main result we introduce below our assumptions on the two-body potentials, on the zero-energy properties of the two-particle subsystems and on the mass ratio.

Concerning the two-body potentials v_α and the Hamiltonians h_α, we assume that the following conditions hold for any pair α

(A_1): $|v_\alpha(x)| \leq C(1+|x|)^{-b}$, with $b > 3$

(A_2): v_α is spherically symmetric, that is $v_\alpha(x) = v_\alpha(|x|)$

(A_3): $v_\alpha \leq 0$

(A_4): $h_\alpha \geq 0$

A further important assumption is the presence of a zero-energy resonance for the two-particle Hamiltonian $h_{23} = h_{31}$. Here we recall the definition of a zero-energy resonance, while further comments and some useful results will be given in the next section.

Let us consider an Hamiltonian in $L^2(\mathbb{R}^3)$

$$h = -\frac{1}{2\mathfrak{m}}\Delta_x + u(x) \tag{2.14}$$

with $\mathfrak{m} > 0$ and u a generic potential satisfying (A_1) and (A_4). We denote by g_0 the integral operator with kernel

$$g_0(x, x') = \frac{\mathfrak{m}}{2\pi|x - x'|}. \tag{2.15}$$

Definition 1. Zero is a (simple) resonance for h if there exists $\varphi \in L^2(\mathbb{R}^3)$ such that

$$\mathrm{sgn}(u)|u|^{1/2}g_0|u|^{1/2}\varphi = -\varphi, \qquad (|u|^{1/2}, \varphi) \neq 0, \tag{2.16}$$

$\mathrm{sgn}(u)$ denoting the multiplication operator by the function sign of u.

Let us comment on the above assumptions. We remark that if each v_α satisfies condition (A_1) then, via Kato-Rellich Theorem, we have self-adjointness and lower boundedness of H on the same domain of H_0.

Moreover, using the HVZ Theorem, we know that the essential spectrum of H is of the form $[l, +\infty)$, where l is the lowest point of the spectra of the operators h_α describing the two-particle subsystems. Thus the assumption (A_4) implies that on the left of zero the spectrum of H consists of isolated eigenvalues with finite multiplicities.

The assumptions (A_2), (A_3) are introduced to simplify the analysis. Note that under the assumption (A_3) the first equation in (2.16) reduces to

$$|u|^{1/2}g_0|u|^{1/2}\varphi = \varphi. \tag{2.17}$$

Moreover (A_2) prevents the presence of zero-energy resonance in the subsystem composed by the identical fermions (see Remark 2 at the end of the next section).

As we already pointed out in the introduction, a peculiar aspect of the Efimov effect in our fermionic system is that it only occurs for a certain range of values of the mass ratio $(0, m_*)$, which can be defined as follows. Let $\Lambda(m)$ be the following function of the mass

$$\Lambda(m) = \frac{2(m+1)^2}{\pi}\left(\frac{1}{\sqrt{m(m+2)}} - \arcsin\left(\frac{1}{m+1}\right)\right). \tag{2.18}$$

It is easy to check that $\Lambda(m)$ is decreasing and

$$\lim_{m \to 0} \Lambda(m) = +\infty, \qquad \lim_{m \to +\infty} \Lambda(m) = 0. \tag{2.19}$$

Thus the following definition makes sense

Definition 2. The critical mass m_* is the unique solution of the equation $\Lambda(m) = 1$.

Note that for $m < m_*$ ($m > m_*$) we have $\Lambda(m) > 1$ ($\Lambda(m) < 1$). Moreover, we stress that m_* is the same mass threshold obtained in the study of the corresponding system with point interactions (see, e.g., [11], [5], [15]).
We are now ready to formulate our main result.

Theorem 1. *Let us assume that conditions* (A_1), (A_2), (A_3), (A_4) *hold for any pair* α *and that the two-particle Hamiltonian* $h_{23} = h_{31}$ *has a zero-energy resonance. Then the following holds.*

  i) For $m < m_*$ *there exists a positive constant* $\mathcal{C}(m)$ *such that*

$$\lim_{z \to 0^-} \frac{N(z)}{|\log|z||} = \mathcal{C}(m). \tag{2.20}$$

  Moreover, $\mathcal{C}(m)$ *satisfies*

$$\mathcal{C}_1(m) \leq \mathcal{C}(m) \leq \mathcal{C}_2(m) \tag{2.21}$$

  where $\mathcal{C}_1(m)$ *is the unique positive solution of the equation* $F_m(x) = 1$, *with*

$$F_m(x) := \frac{m+1}{\sqrt{m(m+2)}} \int_0^1 dy\, y\, \frac{\sinh\left(\frac{2\pi}{3} x \arcsin\left(\frac{y}{m+1}\right)\right)}{\sinh\left(\frac{\pi^2}{3} x\right) \cos\left(\arcsin\left(\frac{y}{m+1}\right)\right)}, \tag{2.22}$$

  and

$$\mathcal{C}_2(m) = \frac{1}{4\pi\beta(m)} \log\left(\sqrt{\frac{2\pi}{3}} \alpha(m)\right) (l_0^2(m) + 3l_0(m) + 2) \tag{2.23}$$

  with

$$\alpha(m) := \frac{(m+1)^{3/2}}{2\sqrt{\pi}\sqrt{m(m+2)}} \log^{1/2}\left(1 + \frac{2}{m}\right), \quad \beta(m) := \frac{\pi}{2} - \arcsin\left(\frac{1}{m+1}\right) \tag{2.24}$$

  and $l_0(m)$ *is the largest odd integer smaller than* $\pi\alpha(m)^2 - 1/2$.
  ii) For $m > m_*$ *the number of negative eigenvalues of* H *is finite.*

3. TWO-PARTICLE SUBSYSTEMS

Here we recall some properties of the two-body Hamiltonian operator h in the position space defined by (2.14), under the hypothesis that the assumptions $(A_1),(A_2),(A_3)$ and (A_4) are satisfied. In particular, we are interested in the low energy behavior connected with the presence of zero-energy resonance (see Definition 1). We first observe that if φ is a square integrable solution of $|u|^{1/2} g_0 |u|^{1/2}\varphi = \varphi$ then $\psi = g_0 |u|^{1/2}\varphi$ satisfies $h\psi = 0$ in the sense of distributions. Furthermore, ψ is an eigenfunction of h with eigenvalue zero if and only if $(|u|^{1/2}, \varphi) = 0$ (see, e.g., Section 1 in [22]). On the other hand, if zero is a resonance for h then there exists ψ solution of $h\psi = 0$ in the sense of distribution with $\psi \in L^2_{\text{loc}}(\mathbb{R}^3)$ but $\psi \notin L^2(\mathbb{R}^3)$.

Remark 1. The spherical symmetry of the potential u implies that zero-energy resonance can only occur in *s-wave* subspace. Indeed, if φ belongs to the subspace with angular momentum $l \geq 1$ then obviously $(|u|^{1/2}, \varphi) = 0$.

Let us introduce the resolvent of h

$$r(z) = (h - z)^{-1} \tag{3.1}$$

which is a bounded operator in $L^2(\mathbb{R}^3)$ for any $z < 0$ due to condition (A_4), the free resolvent

$$r_0(z) = \left(-\frac{1}{2\mathfrak{m}}\Delta - z \right)^{-1} \tag{3.2}$$

(note that $r(0) = g_0$) and the operator

$$w(z) = I + |u|^{1/2}r(z)|u|^{1/2}. \tag{3.3}$$

Using the resolvent identity $r(z) = r_0(z) - r(z)\, u\, r_0(z)$ one verifies that

$$w(z) = (I - |u|^{1/2}r_0(z)|u|^{1/2})^{-1} \tag{3.4}$$

The following Lemma describes the behavior of the operator $w(z)$ in the case 1 is not an eigenvalue of $|u|^{1/2}g_0|u|^{1/2}$.

Lemma 1. *Let us suppose that* 1 *is not an eigenvalue of the operator* $|u|^{1/2}g_0|u|^{1/2}$. *Then* $w(z)$ *is continuous in* $z \leq 0$.

Let us assume that h has a zero-energy resonance. Then we fix the following normalization condition on the eigenvector φ

$$(|u|^{1/2}, \varphi) = (2\pi)^{3/2}(\widehat{\varphi|u|^{1/2}})(0) = 2^{1/4}\pi^{1/2}\mathfrak{m}^{-3/4} \tag{3.5}$$

and we characterize the behavior of the operator $w(z)$ when $z \to 0$.

Lemma 2. *Suppose that* h *has a zero-energy resonance. If* $z < 0$ *is small enough and* $\delta < min\{1, b-3\}$ *then*

$$w(z) = \frac{(\cdot, \varphi)\varphi}{|z|^{1/2}} + |z|^{-\frac{1-\delta}{2}}w^{(\delta)}(z) \tag{3.6}$$

where the operator $w^{(\delta)}(z)$ *is continuous in* $z \leq 0$. *Moreover*

$$(w(z))^{1/2} = \frac{(\cdot, \varphi)\varphi}{\|\varphi\||z|^{1/4}} + |z|^{-\frac{1-\delta}{4}}\tilde{w}^{(\delta)}(z) \tag{3.7}$$

where the operator $\tilde{w}^{(\delta)}(z)$ *is continuous in* $z \leq 0$.

For the proof of the previous Lemmas we refer the reader to [20].

Remark 2. Let us consider the Hamiltonians of the two-body subsystems defined in (2.13). We note that the operator h_{12}, due to the symmetry constraint, acts on

$$L^2_{\mathrm{asym}}(\mathbb{R}^3) = \{\psi \in L^2(\mathbb{R}^3) \,|\, \psi(-x) = -\psi(x)\}. \tag{3.8}$$

Hence, by Remark 1, it cannot exhibit a zero-energy resonance because its domain does not include s-wave functions. Moreover, using the assumptions (A_2) and (A_4), h_{12} cannot have zero as an eigenvalue (see e.g. [25]). Hence we conclude that 1 is not an eigenvalue for the operator $|v_{12}|^{1/2}g_0|v_{12}|^{1/2}$. Then we can apply Lemma 1 and, as we will see, this implies that

the potential between the two fermions does not play any role in the proof of the Efimov effect.

On the other hand, as we have already underlined, the presence of a zero-energy resonance for $h_{23} = h_{31}$ is a crucial ingredient of the proof.

4. FADDEEV EQUATIONS

Our proof of the occurrence of the Efimov effect is based on the analysis of the Faddeev equations ([10]), which are the three-body analogous of the Birman-Schwinger equation for the one body problem. For the convenience of the reader, we recall here the derivation following the clear and simple presentation contained in [16].

Let us consider the eigenvalue equation for H

$$\left(H_0 + \sum_\alpha V_\alpha\right)\psi = z\,\psi, \qquad z < 0 \tag{4.1}$$

Note that the free resolvent

$$R_0(z) = (H_0 - z)^{-1} \tag{4.2}$$

is bounded in $\mathcal{H}$ for any $z < 0$, so we can equivalently write

$$\psi = -R_0(z) \sum_\alpha V_\alpha \psi \tag{4.3}$$

We decompose ψ in the Faddeev components

$$\psi = \sum_\alpha \eta_\alpha \tag{4.4}$$

where, for each pair α, from (4.3) we have

$$\eta_\alpha = -R_0(z)V_\alpha\psi \tag{4.5}$$

In order to find the equations for η_α, we introduce the following operators acting in the Hilbert space $L^2(\mathbb{R}^6)$

$$H_\alpha = H_0 + V_\alpha \tag{4.6}$$

i.e., the Hamiltonian of the three-particle system where the two-body interactions V_β, with $\beta \neq \alpha$, are removed, and its resolvent

$$R_\alpha(z) = (H_\alpha - z)^{-1} \tag{4.7}$$

which is bounded in $L^2(\mathbb{R}^6)$ for any $z < 0$ by our assumptions on the potentials. Then we rewrite (4.5) in the form

$$\eta_\alpha = -R_0(z)V_\alpha \sum_\beta \eta_\beta \tag{4.8}$$

or

$$\eta_\alpha + R_0(z)V_\alpha\eta_\alpha = -R_0(z)V_\alpha \sum_{\beta \neq \alpha} \eta_\beta \tag{4.9}$$

Applying the operator $R_\alpha(z)(H_0 - z)$ to both sides of the above equation, we obtain the Faddeev equations

$$\eta_\alpha = -R_\alpha(z)V_\alpha \sum_{\beta \neq \alpha} \eta_\beta \tag{4.10}$$

Thus, we conclude that if ψ is a solution of (4.1), then $\psi = \sum_\alpha \eta_\alpha$ and η_α are solutions of (4.10). The converse is also true, i.e., if η_α are solutions of (4.10), then $\psi = \sum_\alpha \eta_\alpha$ is a solution of (4.1). As it is well known, a suitable iterated form of Faddeev equations is characterized by a compact operator and this is the main advantage of Faddeev equations with respect to Equation (4.3).

In the above derivation we have not used the symmetry property of our system and therefore it is valid for a generic three-particle system. In order to take into account of the fermionic symmetry we proceed as follows.

For notational convenience we describe the symmetry by an operator T on $L^2(\mathbb{R}^6)$. In the coordinates (k_{23}, p_{23}), T is defined by

$$(T\psi)(k_{23}, p_{23}) = -\psi\left(\frac{k_{23}}{m+1} - \frac{m(m+2)}{(m+1)^2}p_{23}, -k_{23} - \frac{p_{23}}{m+1}\right) \tag{4.11}$$

and then we rewrite

$$\mathcal{H} = \left\{\psi \in L^2(\mathbb{R}^6) \mid T\psi = \psi\right\}. \tag{4.12}$$

In the coordinates (p_{23}, p_{31}) and (k_{12}, p_{12}) we have

$$(T\psi)(p_{23}, p_{31}) = -\psi(p_{31}, p_{23}), \qquad (T\psi)(k_{12}, p_{12}) = -\psi(-k_{12}, p_{12}). \tag{4.13}$$

A direct computation shows that T commutes with H_0 and V_{12} and it satisfies $TV_{23} = V_{31}T$ and $TV_{31} = V_{23}T$. Indeed, recalling the expression of H_0 in the coordinates (p_{23}, p_{31}) given in (2.11), we immediately get $TH_0 = H_0T$. By (2.8) and assumption (A_2), one also obtains $TV_{12} = V_{12}T$. Finally, using the Equations (2.10) and assumption (A_2) we write

$$(V_{23}\psi)(p_{23}, p_{31}) = \frac{1}{(2\pi)^{3/2}} \int dp\, \hat{v}(p - p_{31})\psi(p_{23}, p)$$

$$(V_{31}\psi)(p_{23}, p_{31}) = \frac{1}{(2\pi)^{3/2}} \int dp\, \hat{v}(p - p_{23})\psi(p, p_{31})$$

which imply $TV_{23} = V_{31}T$ and $TV_{31} = V_{23}T$.

Using the above properties, from the definition of η_α given in (4.5) we obtain $T\eta_{12} = \eta_{12}$, $\eta_{31} = T\eta_{23}$ and the system (4.10) reduces to

$$\begin{cases} \eta_{23} = -R_{23}(z)V_{23}(T\eta_{23} + \eta_{12}) \\ \\ \eta_{12} = -R_{12}(z)V_{12}(I + T)\eta_{23} \end{cases} \tag{4.14}$$

where $\eta_{23} \in L^2(\mathbb{R}^6)$ and $\eta_{12} \in \mathcal{H}$. Consequently, in our fermionic system the solution of the eigenvalue equation (4.1) reads $\psi = \eta_{12} + (I + T)\eta_{23}$.

In this paper we find convenient to use a symmetrized form of the Faddeev equations similar to the one used in [20]. In order to derive such equations, we first introduce the following bounded and positive operators on $L^2(\mathbb{R}^6)$ for $z < 0$

$$W_\alpha(z) = I + |V_\alpha|^{1/2}R_\alpha(z)|V_\alpha|^{1/2}. \tag{4.15}$$

Using the resolvent identity $R_\alpha(z) = R_0(z) - R_0(z)V_\alpha R_\alpha(z)$, we find

$$W_\alpha(z) = \left(I - |V_\alpha|^{1/2}R_0(z)|V_\alpha|^{1/2}\right)^{-1} \tag{4.16}$$

Moreover, we define the resolvent of h_α

$$r_\alpha(z) = (h_\alpha - z)^{-1} \tag{4.17}$$

which is a bounded operator in $L^2(\mathbb{R}^3)$ for any $z < 0$, the bounded and positive operator

$$w_\alpha(z) = I + |v_\alpha|^{1/2} r_\alpha(z) |v_\alpha|^{1/2} \tag{4.18}$$

and the Fourier transform $\mathcal{F}_\alpha$ with respect to x_α

$$(\mathcal{F}_\alpha f)(k_\alpha, p_\alpha) = \frac{1}{(2\pi)^{3/2}} \int dx_\alpha\, e^{-ik_\alpha \cdot x_\alpha}\, f(x_\alpha, p_\alpha). \tag{4.19}$$

Then one verifies that

$$W_\alpha(z) = \mathcal{F}_\alpha w_\alpha \left(z - \frac{p_\alpha^2}{2n_\alpha} \right) \mathcal{F}_\alpha^*. \tag{4.20}$$

Let us reconsider the first equation in (4.14). Using the resolvent identity and taking into account that $V_\alpha = -|V_\alpha|$, we have

$$\begin{aligned}
\eta_{23} &= -\big(R_0(z) - R_0(z)V_{23}R_{23}(z)\big)V_{23}(T\eta_{23} + \eta_{12}) \\
&= \big(R_0(z) + R_0(z)|V_{23}|R_{23}(z)\big)|V_{23}|(T\eta_{23} + \eta_{12}) \\
&= R_0(z)|V_{23}|^{1/2}W_{23}(z)^{1/2}W_{23}(z)^{1/2}|V_{23}|^{1/2}(T\eta_{23} + \eta_{12})
\end{aligned} \tag{4.21}$$

Analogously, the second equation in (4.14) can be rewritten as

$$\eta_{12} = R_0(z)|V_{12}|^{1/2}W_{12}(z)^{1/2}W_{12}(z)^{1/2}|V_{12}|^{1/2}(I + T)\eta_{23}. \tag{4.22}$$

Now we apply T to both sides of (4.21) and sum it with (4.22). On the resulting equation we apply $W_{23}^{1/2}|V_{23}|^{1/2}$. Denoting

$$\begin{aligned}
\psi_{23} &= W_{23}(z)^{1/2}|V_{23}|^{1/2}(T\eta_{23} + \eta_{12}) \\
\psi_{12} &= W_{12}(z)^{1/2}|V_{12}|^{1/2}(I + T)\eta_{23}
\end{aligned} \tag{4.23}$$

we find

$$\psi_{23} = W_{23}(z)^{1/2}|V_{23}|^{1/2}R_0(z)\left(T|V_{23}|^{1/2}W_{23}(z)^{1/2}\psi_{23} + |V_{12}|^{1/2}W_{12}(z)^{1/2}\psi_{12}\right). \tag{4.24}$$

On the other hand, applying $W_{12}(z)^{1/2}|V_{12}|^{1/2}(I + T)$ to both sides of (4.21) we find

$$\psi_{12} = W_{12}(z)^{1/2}|V_{12}|^{1/2}R_0(z)(I + T)|V_{23}|^{1/2}W_{23}(z)^{1/2}\psi_{23}. \tag{4.25}$$

Hence we have the following symmetrized form of the Faddeev equations for our model of two identical fermions and a different particle

$$\Psi = \boldsymbol{A}(z)\Psi \tag{4.26}$$

where $\Psi = (\psi_{23}, \psi_{12})$ and $\boldsymbol{A}(z)$ is a 2×2 matrix operator acting on the space

$$\mathcal{K} = L^2(\mathbb{R}^6) \times \mathcal{H} \tag{4.27}$$

defined by

$$\boldsymbol{A}(z) = \boldsymbol{W}(z)^{1/2}\boldsymbol{U}(z)\boldsymbol{W}(z)^{1/2}, \tag{4.28}$$

with

$$\boldsymbol{W}(z)^{1/2} = \mathrm{diag}\{W_{23}(z)^{1/2}, W_{12}(z)^{1/2}\} \tag{4.29}$$

and

$$\boldsymbol{U}(z) = \begin{pmatrix} |V_{23}|^{1/2}R_0(z)T|V_{23}|^{1/2} & |V_{23}|^{1/2}R_0(z)|V_{12}|^{1/2} \\ |V_{12}|^{1/2}R_0(z)(I+T)|V_{23}|^{1/2} & 0 \end{pmatrix}. \tag{4.30}$$

The advantage of such symmetrized form of the equations is the fact that $\boldsymbol{A}(z)$ is compact for $z < 0$.

Theorem 2. *For $z < 0$ the operator $\boldsymbol{A}(z)$ is compact and it is continuous in z.*

The proof of Theorem 2 goes exactly as that of Theorem 4.1 in [20] and it is omitted.
It turns out that the number of eigenvalues of H smaller than $z < 0$ equals the number of eigenvalues of $\boldsymbol{A}(z)$ larger than 1. In order to prove this fact it is useful to introduce the following definition.

Definition 3. Let B be a selfadjoint operator on the Hilbert space $\mathfrak{h}$ and let $\lambda \in \mathbb{R}$. We set

$$n(\lambda, B) = \sup_{\mathfrak{h}_B(\lambda)} \dim \mathfrak{h}_B(\lambda) \tag{4.31}$$

where $\mathfrak{h}_B(\lambda)$ denotes any subspace of $\mathcal{D}(B)$ such that if f belongs to $\mathfrak{h}_B(\lambda)$ then $(Bf, f) > \lambda \|f\|^2$.

We stress that if the spectrum of the operator B on the right of λ is purely discrete then $n(\lambda, B)$ coincides with the number of eigenvalues (with multiplicities) on the right of λ. Thus in particular

$$N(z) = n(-z, -H). \tag{4.32}$$

Due to the compactness of $\boldsymbol{A}(z)$ stated in Theorem 2, we also have that $n(1, \boldsymbol{A}(z))$ equals the number of eigenvalues of $\boldsymbol{A}(z)$ larger than 1.
In the next Theorem we prove a "Birman-Schwinger Principle" for our three-particle system, which is crucial for our analysis.

Theorem 3. *For $z < 0$ we have*

$$N(z) = n(1, \boldsymbol{A}(z)). \tag{4.33}$$

Proof. We adapt the proof of Theorem 3.1 in [20] to our case. First we show that

$$N(z) = n\big(1, R_0(z)^{1/2}|V|R_0(z)^{1/2}\big) \tag{4.34}$$

where $|V| = |V_{23}| + T|V_{23}| + |V_{12}|$. Indeed, let $x \in \mathfrak{h}_{-H}(-z)$ then

$$((H_0 - z)x, x) < (|V|x, x). \tag{4.35}$$

Setting $y = (H_0 - z)^{1/2}x$ and using selfadjointness of $(H_0 - z)^{1/2}$ and $R_0(z)^{1/2}$ we have

$$(R_0(z)^{1/2}|V|R_0(z)^{1/2}y, y) > (y, y) \tag{4.36}$$

that is $y \in \mathfrak{h}_{R_0(z)^{1/2}|V|R_0(z)^{1/2}}(1)$. This implies $n(1, R_0(z)^{1/2}|V|R_0(z)^{1/2}) \geq N(z)$. Reversing the argument we get the opposite inequality.
Next we introduce the matrix operator $\boldsymbol{L}(z)$ on $\mathcal{H}^2$ defined by

$$\boldsymbol{L}(z) = \frac{1}{2}\begin{pmatrix} R_0(z)^{1/2}|V|R_0(z)^{1/2} & R_0(z)^{1/2}|V|R_0(z)^{1/2} \\ R_0(z)^{1/2}|V|R_0(z)^{1/2} & R_0(z)^{1/2}|V|R_0(z)^{1/2} \end{pmatrix} \tag{4.37}$$

and we note that

$$n\big(1, R_0(z)^{1/2}|V|R_0(z)^{1/2}\big) = n(1, \boldsymbol{L}(z)). \tag{4.38}$$

The operator $\boldsymbol{L}(z)$ can be written as

$$\boldsymbol{L}(z) = \boldsymbol{S}(z)\boldsymbol{S}^*(z) \tag{4.39}$$

where

$$\boldsymbol{S}(z) = \begin{pmatrix} \frac{1}{2}R_0(z)^{1/2}(I+T)|V_{23}|^{1/2} & \frac{1}{\sqrt{2}}R_0(z)^{1/2}|V_{12}|^{1/2} \\ \frac{1}{2}R_0(z)^{1/2}(I+T)|V_{23}|^{1/2} & \frac{1}{\sqrt{2}}R_0(z)^{1/2}|V_{12}|^{1/2} \end{pmatrix} \tag{4.40}$$

is an operator acting from $\mathcal{K}$ to $\mathcal{H}^2$ and its adjoint $\boldsymbol{S}^*(z)$, acting from $\mathcal{H}^2$ to $\mathcal{K}$, is given by

$$\boldsymbol{S}^*(z) = \begin{pmatrix} |V_{23}|^{1/2}R_0(z)^{1/2} & |V_{23}|^{1/2}R_0(z)^{1/2} \\ \frac{1}{\sqrt{2}}|V_{12}|^{1/2}R_0(z)^{1/2} & \frac{1}{\sqrt{2}}|V_{12}|^{1/2}R_0(z)^{1/2} \end{pmatrix} \tag{4.41}$$

where we have used $T = I$ on the space $\mathcal{H}$. Since $n(\lambda, BB^*) = n(\lambda, B^*B)$ for any bounded operator B (see e.g., Lemma 4.2 in [20]), we have

$$n(1, \boldsymbol{L}(z)) = n(1, \boldsymbol{S}^*(z)\boldsymbol{S}(z)) \tag{4.42}$$

where $\boldsymbol{S}^*(z)\boldsymbol{S}(z)$ is an operator on $\mathcal{K}$ explicitly given by

$$\boldsymbol{S}^*(z)\boldsymbol{S}(z) = \begin{pmatrix} |V_{23}|^{1/2}R_0(z)(I+T)|V_{23}|^{1/2} & \sqrt{2}|V_{23}|^{1/2}R_0(z)|V_{12}|^{1/2} \\ \frac{\sqrt{2}}{2}|V_{12}|^{1/2}R_0(z)(I+T)|V_{23}|^{1/2} & |V_{12}|^{1/2}R_0(z)|V_{12}|^{1/2} \end{pmatrix}. \tag{4.43}$$

Let us decompose the above operator as follows

$$\boldsymbol{S}^*(z)\boldsymbol{S}(z) = \boldsymbol{D}_1(z) + \boldsymbol{D}_2(z) \tag{4.44}$$

where

$$\boldsymbol{D}_1(z) = \begin{pmatrix} |V_{23}|^{1/2}R_0(z)|V_{23}|^{1/2} & 0 \\ 0 & |V_{12}|^{1/2}R_0(z)|V_{12}|^{1/2} \end{pmatrix} \tag{4.45}$$

and

$$\boldsymbol{D}_2(z) = \begin{pmatrix} |V_{23}|^{1/2}R_0(z)T|V_{23}|^{1/2} & \sqrt{2}|V_{23}|^{1/2}R_0(z)|V_{12}|^{1/2} \\ \frac{\sqrt{2}}{2}|V_{12}|^{1/2}R_0(z)(I+T)|V_{23}|^{1/2} & 0 \end{pmatrix}. \tag{4.46}$$

Moreover, let us define

$$\tilde{\boldsymbol{A}}(z) = (I - \boldsymbol{D}_1(z))^{-1/2}\boldsymbol{D}_2(z)(I - \boldsymbol{D}_1(z))^{-1/2} \tag{4.47}$$

and note that, from the Definition (4.16), it follows $(I - \boldsymbol{D}_1(z))^{-1/2} = \boldsymbol{W}(z)^{1/2}$.
Let us prove that

$$n(1, \boldsymbol{S}^*(z)\boldsymbol{S}(z)) := n(1, \boldsymbol{D}_1(z) + \boldsymbol{D}_2(z)) = n(1, \tilde{\boldsymbol{A}}(z)). \tag{4.48}$$

Assume $x \in \mathfrak{h}_{\tilde{\boldsymbol{L}}(z)}(1)$, i.e.,

$$((\boldsymbol{D}_1(z) + \boldsymbol{D}_2(z))x, x) > (x, x) \tag{4.49}$$

then

$$(\boldsymbol{D}_2(z)x, x) > ((I - \boldsymbol{D}_1(z))x, x). \tag{4.50}$$

Defining $y = (I - \boldsymbol{D}_1(z))^{1/2}x$ and using (4.47) we get

$$(\tilde{\boldsymbol{A}}(z)y, y) > (y, y) \tag{4.51}$$

which means $y \in \mathfrak{h}_{\tilde{\boldsymbol{A}}(z)}(1)$. This proves $n(1, \boldsymbol{S}^*(z)\boldsymbol{S}(z)) \leq n(1, \tilde{\boldsymbol{A}}(z))$. To get the opposite inequality it is sufficient to reverse the argument.
We also note that for $z < 0$ the operator $\tilde{\boldsymbol{A}}(z)$ is compact and it is continuous in z.
Finally, by a direct computation one verifies that

$$\tilde{\boldsymbol{A}}(z) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix} \boldsymbol{A}(z) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}. \tag{4.52}$$

This implies that $\tilde{\boldsymbol{A}}(z)$ and $\boldsymbol{A}(z)$ have the same eigenvalues and if $\tilde{\Psi} = (\psi_{23}, \psi_{12})$ is an eigenfunction of $\tilde{\boldsymbol{A}}(z)$ then $\Psi = (\psi_{23}, \sqrt{2}\psi_{12})$ is an eigenfunction of $\boldsymbol{A}(z)$ with the same eigenvalue. Thus, in particular

$$n(1, \tilde{\boldsymbol{A}}(z)) = n(1, \boldsymbol{A}(z)) \tag{4.53}$$

Taking into account of (4.34), (4.38), (4.42), (4.48), (4.53), we conclude the proof.

$\square$

We conclude this section describing the behavior of the operators $W_\alpha(z)$ when $z < 0$ is small. Let us introduce the multiplication operator in $L^2(\mathbb{R}^6)$

$$(\Gamma_\alpha(z)f)(k_\alpha, p_\alpha) = \gamma\left(\frac{p_\alpha^2}{2n_\alpha} - z\right)f(k_\alpha, p_\alpha) \tag{4.54}$$

where $\gamma \in C^\infty(\mathbb{R}_+)$ is such that $\gamma(t) > 0$ for all t, $\gamma(t) = t$ if $t \leq 1$ and $\gamma(t) = 1$ if $t \geq 2$. Moreover, for the resonant pair 23, we define the operator in $L^2(\mathbb{R}^6)$

$$(\Pi_{23}f)(k_{23}, p_{23}) = \frac{1}{\|\varphi\|}\hat{\varphi}(k_{23})\int dk\, f(k, p_{23})\overline{\hat{\varphi}(k)} \tag{4.55}$$

where φ is the eigenfunction of $|v|^{1/2}g_0|v|^{1/2}$ with eigenvalue 1 (see Definition 1). Using the relation (4.20) and Lemma 2, we find

$$W_{23}(z)^{1/2} = \Gamma_{23}(z)^{-1/4}\Pi_{23} + \Gamma_{23}(z)^{-\frac{1-\delta}{4}}\tilde{W}_{23}^{(\delta)}(z) \tag{4.56}$$

where $\delta < \min\{1, b - 3\}$ and $\tilde{W}_{23}^{(\delta)}(z)$ is continuous in $z \leq 0$.
On the other hand, Lemma 1 implies that $W_{12}(z)^{1/2}$ is continuous in $z \leq 0$.

5. Leading term of $\mathbf{A}(z)$ for $z \to 0^-$

The proof of our main result expressed in Theorem 1 requires, via Theorem 3, an asymptotic analysis of $n(1, \mathbf{A}(z))$ for $z \to 0^-$. From Theorem 2, we know that for $z < 0$ the operator $\mathbf{A}(z)$ is compact but there is a lack of compactness for $z = 0$ and this is the reason why we find that $N(z)$ diverges for $z \to 0^-$. In this section we shall prove various intermediate results where, at each step, we single out the leading term of $\mathbf{A}(z))$ for $z \to 0^-$, neglecting operators which are compact for $z \leq 0$. At the end, we shall obtain the following integral operator acting in $L^2((0, R) \times \mathbb{S}^2, dr \otimes d\Omega)$

$$(S_R f)(r, \omega) = \int_0^R d\rho \int_{\mathbb{S}^2} d\Omega(\zeta)\, S(r - \rho, \omega \cdot \zeta) f(\rho, \zeta) \tag{5.1}$$

where

$$S(x, y) = -b(m)\frac{1}{\cosh x + \frac{y}{m+1}}, \qquad x \in \mathbb{R}, \quad y \in [-1, 1] \tag{5.2}$$

$$R = R(z) = \frac{1}{2}|\log|z||, \tag{5.3}$$

$$b(m) = \frac{1}{4\pi^2}\frac{m+1}{\sqrt{m(m+2)}}. \tag{5.4}$$

In section 6 we shall prove that the asymptotic behavior of $n(1, \mathbf{A}(z))$ for $z \to 0^-$ coincides with that of $n(1, S_R)$ for $R \to +\infty$.

As a first step, we show that the terms in $\mathbf{A}(z)$ depending on the interaction between the two fermions give a compact contribution and can be neglected.

Lemma 3. *For $z \leq 0$ the operator $\mathbf{A}(z) - \mathbf{A}_0(z)$ is compact and it is continuous in z, where*

$$\mathbf{A}_0(z) = \begin{pmatrix} A_0(z) & 0 \\ 0 & 0 \end{pmatrix} \tag{5.5}$$

and

$$A_0(z) = \Pi_{23}\Gamma_{23}^{-1/4}(z)|V_{23}|^{1/2}R_0(z)T|V_{23}|^{1/2}\Gamma_{23}^{-1/4}(z)\Pi_{23}. \tag{5.6}$$

Proof. Let us introduce the operators

$$\mathbf{\Gamma}(z) = \text{diag}\{\Gamma_{23}(z), \Gamma_{12}(z)\}, \tag{5.7}$$

$$\tilde{\mathbf{W}}^{(\delta)}(z) = \text{diag}\{\tilde{W}_{23}^{(\delta)}(z), \tilde{W}_{12}^{(\delta)}(z)\}, \tag{5.8}$$

$$\mathbf{\Pi} = \begin{pmatrix} \Pi_{23} & 0 \\ 0 & 0 \end{pmatrix} \tag{5.9}$$

where $\delta < \min\{1, b - 3\}$, $\tilde{W}_{12}^{(\delta)}(z) = \Gamma_{12}(z)^{\frac{1-\delta}{4}}W_{12}(z)$ is continuous in $z \leq 0$ and the other terms have been defined in (4.54), (4.55), (4.56).
Using the above notation we write

$$\mathbf{A}(z) = \mathbf{A}_0(z) + \mathbf{R}(z) \tag{5.10}$$

where

$$\begin{aligned} \mathbf{R}(z) &= \mathbf{\Pi}\, \mathbf{U}^{(1/4, (1-\delta)/4)}(z)\tilde{\mathbf{W}}^{(\delta)}(z) + \tilde{\mathbf{W}}^{(\delta)}(z)\mathbf{U}^{((1-\delta)/4, (1-\delta)/4)}(z)\tilde{\mathbf{W}}^{(\delta)}(z) \\ &\quad + \tilde{\mathbf{W}}^{(\delta)}(z)\mathbf{U}^{((1-\delta)/4, 1/4)}(z)\mathbf{\Pi} \end{aligned} \tag{5.11}$$

and

$$\boldsymbol{U}^{(\mu,\nu)}(z) = \boldsymbol{\Gamma}(z)^{-\mu}\boldsymbol{U}(z)\boldsymbol{\Gamma}(z)^{-\nu}, \qquad 0 \le \mu, \nu \le \frac{1}{4}, \quad \mu + \nu < \frac{1}{2}. \tag{5.12}$$

By Lemma 4.4 in [20], for $z \le 0$ the operator $\boldsymbol{U}^{(\mu,\nu)}(z)$ is compact and it is continuous in z. Since $\tilde{W}_\alpha^{(\delta)}(z)$, $z \le 0$, and Π_{23} are bounded, we conclude that for $z \le 0$ the operator $\boldsymbol{R}(z)$ is compact and it is continuous in z.

$\square$

In the next step we reduce the problem to the analysis of an operator in $L^2(\mathbb{R}^3)$. Such an operator is better analysed using the coordinates (p_{23}, p_{31}).

Lemma 4. *For $\lambda > 0$ and $z < 0$ we have*

$$n(\lambda, \boldsymbol{A}_0(z)) = n(\lambda, \mathcal{B}(z)) \tag{5.13}$$

where $\mathcal{B}(z)$ is the integral operator in $L^2(\mathbb{R}^3)$ with kernel

$$\mathcal{B}(p, q; z) = -\frac{\widehat{|v|^{1/2}\varphi}\left(p + \frac{q}{m+1}\right)\overline{\widehat{|v|^{1/2}\varphi}\left(q + \frac{p}{m+1}\right)}}{\gamma\left(\frac{p^2}{2n} - z\right)^{1/4}(H_0 - z)\gamma\left(\frac{q^2}{2n} - z\right)^{1/4}}. \tag{5.14}$$

Proof. We first observe that $n(\lambda, \boldsymbol{A}_0(z)) = n(\lambda, A_0(z))$. Moreover, $A_0(z)$ is compact for $z < 0$ and then $n(\lambda, A_0(z))$ is the number of its eigenvalues larger than λ. By (5.6), recalling (4.55) and (4.54) we can write explicitly the eigenvalue equation for A_0 corresponding to the eigenvalue $\tilde{\lambda} > \lambda > 0$

$$\tilde{\lambda}\psi(p_1, p_2) = -\frac{1}{(2\pi)^{3/2}\|\varphi\|^2}\hat{\varphi}\left(p_2 + \frac{p_1}{m+1}\right)\int dp\,\overline{\hat{\varphi}\left(p + \frac{p_1}{m+1}\right)}$$

$$\times \int dp_2'\frac{\widehat{|v|^{1/2}}(p - p_2')\widehat{|v|^{1/2}\varphi}\left(p_1 + \frac{p_2'}{m+1}\right)}{\gamma\left(\frac{p_1^2}{2n} - z\right)^{1/4}(H_0(p_1, p_2') - z)\gamma\left(\frac{p_2'^2}{2n} - z\right)^{1/4}}\int dq\,\overline{\hat{\varphi}\left(q + +\frac{p_2'}{m+1}\right)}\psi(p_2', q). \tag{5.15}$$

Let us define

$$\xi(p) = \int dq\,\overline{\hat{\varphi}\left(q + \frac{p}{m+1}\right)}\psi(p, q). \tag{5.16}$$

Then $\xi \in L^2(\mathbb{R}^3)$ and, by (5.15), it satisfies the equation

$$\tilde{\lambda}\xi(p_1) = -\frac{1}{(2\pi)^{3/2}}\int dp_2\,\overline{\hat{\varphi}\left(p_2 + \frac{p_1}{m+1}\right)}$$

$$\times \int dp_2'\frac{\widehat{|v|^{1/2}}(p_2 - p_2')\widehat{|v|^{1/2}\varphi}\left(p_1 + \frac{p_2'}{m+1}\right)}{\gamma\left(\frac{p_1^2}{2n} - z\right)^{1/4}(H_0(p_1, p_2') - z)\gamma\left(\frac{p_2'^2}{2n} - z\right)^{1/4}}\xi(p_2')$$

$$= -\int dq\,\frac{\widehat{|v|^{1/2}\varphi}\left(p_1 + \frac{q}{m+1}\right)\overline{\widehat{|v|^{1/2}\varphi}\left(q + \frac{p_1}{m+1}\right)}}{\gamma\left(\frac{p_1^2}{2n} - z\right)^{1/4}(H_0(p_1, q) - z)\gamma\left(\frac{q^2}{2n} - z\right)^{1/4}}\xi(q) \tag{5.17}$$

that is $\mathcal{B}(z)\xi = \tilde{\lambda}\xi$. On the other hand, if $\xi \in L^2(\mathbb{R}^3)$ is such that $\mathcal{B}(z)\xi = \tilde{\lambda}\xi$ then

$$\psi(p_1, p_2) = -\frac{1}{(2\pi)^{3/2}} \int dq \frac{\widehat{|v|^{1/2}}(p_2 - q)\widehat{|v|^{1/2}\varphi}\left(p_1 + \frac{q}{m+1}\right)}{\gamma\left(\frac{p_1^2}{2n} - z\right)^{1/4}(H_0(p_1, q) - z)\gamma\left(\frac{q^2}{2n} - z\right)^{1/4}} \xi(q) \tag{5.18}$$

satisfies the equation $A_0(z)\psi = \tilde{\lambda}\psi$ and therefore the Lemma is proved.

$\square$

The lack of compactness of $\mathcal{B}(z)$ for $z = 0$ is clearly due to the behavior of its integral kernel near the origin. Indeed, in the following Lemma we show that the difference of $\mathcal{B}(z)$ with an operator whose kernel is different from zero only in a ball of radius one is compact and continuous in $z \leq 0$. Denoted by χ_a the characteristic function of the ball of radius $a > 0$, we have

Lemma 5. *For $z \leq 0$ the operator $\mathcal{B}(z) - \tilde{\mathcal{B}}(z)$ is compact and it is continuous in z, where $\tilde{\mathcal{B}}(z)$ is the operator in $L^2(\mathbb{R}^3)$ with kernel*

$$\tilde{\mathcal{B}}(p, q; z) = -\frac{1}{c(m)} \frac{\chi_1(p)\chi_1(q)}{\left(\frac{p^2}{2n} - z\right)^{1/4}(H_0(p, q) - z)\left(\frac{q^2}{2n} - z\right)^{1/4}} \tag{5.19}$$

and $c(m) = 2^{5/2}\pi^2 \left(\frac{m}{m+1}\right)^{3/2}$. Moreover

$$n(\lambda, \tilde{\mathcal{B}}(z)) = n(\lambda, \mathcal{B}_0(z)) \tag{5.20}$$

where $\mathcal{B}_0(z)$ is the integral operator in $L^2(\mathbb{R}^3)$ with kernel

$$\mathcal{B}_0(p, q; z) = -\frac{1}{c(m)} \frac{\chi_{|z|^{-1/2}}(p)\chi_{|z|^{-1/2}}(q)}{\left(\frac{p^2}{2n} + 1\right)^{1/4}(H_0(p, q) + 1)\left(\frac{q^2}{2n} + 1\right)^{1/4}}. \tag{5.21}$$

Proof. The proof is divided into three steps. We first introduce the operator $\mathcal{E}_0(z)$ with integral kernel

$$\mathcal{E}_0(p, q; z) = -\chi_1(p)\frac{\widehat{|v|^{1/2}\varphi}\left(p + \frac{q}{m+1}\right)\overline{\widehat{|v|^{1/2}\varphi}\left(q + \frac{p}{m+1}\right)}}{\gamma\left(\frac{p^2}{2n} - z\right)^{1/4}(H_0(p, q) - z)\gamma\left(\frac{q^2}{2n} - z\right)^{1/4}}\chi_1(q) \tag{5.22}$$

and note that the integral kernel of the difference $\mathcal{B}(z) - \mathcal{E}_0(z)$ can be written as

$$(\mathrm{I} + \mathrm{II} + \mathrm{III})(p, q; z) = (1 - \chi_1(p))\frac{\widehat{|v|^{1/2}\varphi}\left(p + \frac{q}{m+1}\right)\overline{\widehat{|v|^{1/2}\varphi}\left(q + \frac{p}{m+1}\right)}}{\gamma\left(\frac{p^2}{2n} - z\right)^{1/4}(H_0(p, q) - z)\gamma\left(\frac{q^2}{2n} - z\right)^{1/4}}\chi_1(q)$$

$$+ \chi_1(p)\frac{\widehat{|v|^{1/2}\varphi}\left(p + \frac{q}{m+1}\right)\overline{\widehat{|v|^{1/2}\varphi}\left(q + \frac{p}{m+1}\right)}}{\gamma\left(\frac{p^2}{2n} - z\right)^{1/4}(H_0(p, q) - z)\gamma\left(\frac{q^2}{2n} - z\right)^{1/4}}(1 - \chi_1(q))$$

$$+ (1 - \chi_1(p))\frac{\widehat{|v|^{1/2}\varphi}\left(p + \frac{q}{m+1}\right)\overline{\widehat{|v|^{1/2}\varphi}\left(q + \frac{p}{m+1}\right)}}{\gamma\left(\frac{p^2}{2n} - z\right)^{1/4}(H_0(p, q) - z)\gamma\left(\frac{q^2}{2n} - z\right)^{1/4}}(1 - \chi_1(q)). \tag{5.23}$$

Let us consider $I(p, q; z)$. We note that $|v|^{1/2}\varphi \in L^1(\mathbb{R}^3)$ which implies $\widehat{|v|^{1/2}\varphi} \leq c$. Moreover, $H_0(p, q) - z \geq c\, p^2$ and therefore we obtain

$$I(p, q; z) \leq c\, \frac{(1 - \chi_1(p))\, \chi_1(q)}{p^2\, q^{1/2}} \tag{5.24}$$

which is a square integrable function. Analogously, one can prove square integrability of $II(p, q; z)$. In order to estimate $III(p, q; z)$, we note that $\widehat{|v|^{1/2}\varphi} \in L^2(\mathbb{R}^3)$ and therefore

$$III(p, q; z) \leq c(1 - \chi_1(p))\frac{\widehat{|v|^{1/2}\varphi}\left(p + \frac{q}{m+1}\right)}{q^2}(1 - \chi_1(q)) \tag{5.25}$$

is square integrable. Hence we conclude that for $z \leq 0$ the operator $\mathcal{B}(z) - \mathcal{E}_0(z)$ is Hilbert-Schmidt and it is continuous in z. Now we consider the operator $\mathcal{E}_1(z)$ with integral kernel

$$\mathcal{E}_1(p, q; z) = -\frac{1}{c(m)} \frac{\chi_1(p)\chi_1(q)}{\gamma\left(\frac{p^2}{2n} - z\right)^{1/4}(H_0(p, q) - z)\, \gamma\left(\frac{q^2}{2n} - z\right)^{1/4}}. \tag{5.26}$$

We note that

$$|\,\widehat{|v|^{1/2}\varphi}(k) - \widehat{|v|^{1/2}\varphi}(0)| \leq c\,|k|^\nu, \qquad 0 < \nu < \frac{b-3}{2}. \tag{5.27}$$

Indeed, using $|e^{-ik \cdot x} - 1| \leq c\,|k|^\nu |x|^\nu$, we have

$$|\,\widehat{|v|^{1/2}\varphi}(k) - \widehat{|v|^{1/2}\varphi}(0)| \leq c\,|k|^\nu \int dx\, |x|^\nu |v|^{1/2}(x)|\varphi|(x)$$

$$\leq c\,\|\varphi\|\,|k|^\nu \left(\int dx\, |x|^{2\nu}|v|(x)\right)^{1/2}$$

and the last integral is finite by assumption (A_1). Moreover, using Young's inequality, we get

$$H_0(p, q) - z = \frac{p^2}{2\mu} + \frac{q^2}{2\mu} + \frac{p \cdot q}{m} \geq c\left(\frac{(p^{2\kappa})^{1/\kappa}}{\kappa} + \frac{(q^{2\kappa'})^{1/\kappa'}}{\kappa'}\right) \geq c\, p^{2\kappa}q^{2\kappa'} \tag{5.28}$$

for any $\kappa, \kappa' > 0$ such that $\kappa + \kappa' = 1$.

By (5.27), (5.28) and condition (3.5), we find

$$|\mathcal{E}_0(p, q; z) - \mathcal{E}_1(p, q; z)| \leq c\,\chi_1(p)\frac{|p|^\nu + |q|^\nu}{\left(\frac{p^2}{2n} - z\right)^{1/4}(H_0(p, q) - z)\left(\frac{q^2}{2n} - z\right)^{1/4}}\chi_1(q) \tag{5.29}$$

$$\leq c\,\frac{\chi_1(p)\chi_1(q)}{|p|^{-\nu+2\kappa+1/2}|q|^{2\kappa'+1/2}} + c\,\frac{\chi_1(p)\chi_1(q)}{|q|^{-\nu+2\kappa+1/2}|p|^{2\kappa'+1/2}} \tag{5.30}$$

which is square integrable choosing $\kappa \in (\frac{1}{2}, \frac{1+\delta}{2})$. In order to obtain the operator $\tilde{\mathcal{B}}(z)$ from $\mathcal{E}_1(z)$, it remains to replace $\gamma\left(\frac{p^2}{2n} - z\right)$ and $\gamma\left(\frac{q^2}{2n} - z\right)$ with $\frac{p^2}{2n} - z$ and $\frac{q^2}{2n} - z$, respectively. One can easily see that the difference $\mathcal{E}_1(z) - \tilde{\mathcal{B}}(z)$ is compact up to $z = 0$. This concludes the proof that for $z \leq 0$ the operator $\mathcal{B}(z) - \tilde{\mathcal{B}}(z)$ is compact and it is continuous in z.

In order to prove (5.20) it is sufficient to observe that $\tilde{\mathcal{B}}(z)$ is unitarily equivalent to $\mathcal{B}_0(z)$ via the unitary operator U_z defined by $U_z\xi(p) = |z|^{3/4}\xi(|z|^{1/2}p)$.

$\square$

In the following Lemma we finally arrive at the operator S_R defined in (5.1), (5.2).

Lemma 6. *For $z \leq 0$ the operator $\mathcal{B}_0(z) - \mathcal{S}(z)$ is compact and it is continuous in z, where $\mathcal{S}(z)$ is the integral operator in $L^2(\mathbb{R}^3)$ with kernel*

$$\mathcal{S}(p, q; z) = -\frac{(2n)^{1/2}}{c(m)} \frac{\left(\chi_{|z|^{-1/2}} - \chi_1\right)(p)\left(\chi_{|z|^{-1/2}} - \chi_1\right)(q)}{|p|^{1/2}\left(\frac{p^2}{2\mu} + \frac{q^2}{2\mu} + \frac{p \cdot q}{m}\right)|q|^{1/2}}. \tag{5.31}$$

Moreover

$$n(\lambda, \mathcal{S}(z)) = n(\lambda, S_R). \tag{5.32}$$

Proof. The first point is easy to check. Let us prove the second statement. Using the unitary operator $M : L^2(\mathbb{R}^3) \to L^2(\mathbb{R} \times \mathbb{S}^2, dr \otimes d\Omega)$ defined by

$$(M\xi)(r, \omega) = e^{3r/2}\xi(e^r, \omega) \tag{5.33}$$

we see that $\mathcal{S}(z)$ is unitarily equivalent to the operator on $L^2(\mathbb{R} \times \mathbb{S}^2, dr \otimes d\Omega)$ with integral kernel

$$-b(m)\frac{\chi_{(0,R)}(x)\chi_{(0,R)}(x')}{\cosh(x - x') + \frac{\zeta \cdot \omega}{m+1}}. \tag{5.34}$$

where $\chi_{(a,b)}$ is the characteristic function of the interval (a, b). Indeed,

$$(M\mathcal{S}(z)\xi)(r, \omega) = -\frac{(m+1)^2}{4\pi^2\sqrt{m^3(m+2)}}e^{3r/2}\int_0^{+\infty} d\rho\, \rho^{3/2}$$

$$\times \int_{\mathbb{S}^2} d\Omega(\zeta) \frac{\left(\chi_{|z|^{-1/2}}(e^r, \omega) - \chi_1(e^r, \omega)\right)\left(\chi_{|z|^{-1/2}}(\rho, \zeta) - \chi_1(\rho, \zeta)\right)}{e^{r/2}\left(\frac{e^{2r}}{2\mu} + \frac{\rho^2}{2\mu} + e^r\rho\frac{\omega \cdot \zeta}{m}\right)}\xi(\rho, \zeta)$$

$$= -\frac{(m+1)^2}{4\pi^2\sqrt{m^3(m+2)}}e^r\int_{-\infty}^{+\infty} dx\, e^{5x/2}\int_{\mathbb{S}^2} d\Omega(\zeta) \frac{\chi_{(0,R)}(r)\chi_{(0,R)}(x)}{\frac{e^{2r}}{2\mu} + \frac{e^{2x}}{2\mu} + \frac{\omega \cdot \zeta}{m}e^{r+x}}\xi(e^x, \zeta)$$

$$= -\frac{(m+1)^2}{4\pi^2\sqrt{m^3(m+2)}}e^r\int_{-\infty}^{+\infty} dx\, e^x\int_{\mathbb{S}^2} d\Omega(\zeta) \frac{\chi_{(0,R)}(r)\chi_{(0,R)}(x)}{\frac{e^{r+x}}{\mu}\left(\frac{e^{r-x}+e^{x-r}}{2} + \frac{\omega \cdot \zeta}{m+1}\right)}e^{3x/2}\xi(e^x, \zeta)$$

$$= -b(m)\int_{-\infty}^{+\infty} dx\int_{\mathbb{S}^2} d\Omega(\zeta) \frac{\chi_{(0,R)}(r)\chi_{(0,R)}(x)}{\cosh(r - x) + \frac{\omega \cdot \zeta}{m+1}}(M\xi)(x, \zeta). \tag{5.35}$$

Since the operator with the kernel given by (5.34) can be considered as an operator on $L^2((0, R) \times \mathbb{S}^2, d\rho \otimes d\Omega)$, we find S_R.

$\square$

6. Proof of Theorem 1

In this section we give the proof of Theorem 1. Taking into account of Theorem 3, the result for $m < m_*$ is obtained in two steps. We first show that

$$\frac{n(1, S_R)}{2R} \tag{6.1}$$

converges for $R \to \infty$ (see Proposition 1 below). Then we prove that

$$\frac{n(1, \boldsymbol{A}(z))}{|\log|z||} \tag{6.2}$$

converges to the same limit for $z \to 0^-$ (see Proposition 2 below).

In order to study the asymptotic behavior of (6.1) for $R \to \infty$, it is convenient to decompose the integral kernel of S_R in spherical harmonics. Indeed, denoted by P_l the Legendre polynomial of order l and by Y_l^ν the spherical harmonic of order l, ν, we write

$$S(x, \omega \cdot \zeta) = \sum_{l=0}^{+\infty} \frac{2l+1}{2} P_l(\omega \cdot \zeta) \int_{-1}^{1} dy\, P_l(y) S(x, y) \tag{6.3}$$

and, using the addition formula

$$P_l(\omega \cdot \zeta) = \frac{4\pi}{2l+1} \sum_{\nu=-l}^{l} \overline{Y_l^\nu(\zeta)} Y_l^\nu(\omega), \tag{6.4}$$

we find (see (5.1), (5.2))

$$(S_R f)(r, \omega) = 2\pi \sum_{l=0}^{+\infty} \sum_{\nu=-l}^{l} Y_l^\nu(\omega) \int_0^R d\rho \int_{-1}^1 dy\, S(r-\rho, y) P_l(y) \int_{\mathbb{S}^2} d\Omega(\zeta)\, f(\rho, \zeta) \overline{Y_l^\nu(\zeta)}$$

$$= -2\pi b(m) \sum_{l=0}^{+\infty} \sum_{\nu=-l}^{l} Y_l^\nu(\omega) \int_0^R d\rho\, f_{l\nu}(\rho) \int_{-1}^1 dy\, \frac{P_l(y)}{\cosh(r-\rho) + \frac{y}{m+1}} \tag{6.5}$$

$$:= \sum_{l=0}^{+\infty} \sum_{\nu=-l}^{l} (S_R^{(l)} f_{l\nu})(\rho) Y_l^\nu(\omega)$$

where

$$f_{l\nu}(\rho) = \int_{\mathbb{S}^2} d\Omega(\zeta)\, f(\rho, \zeta) \overline{Y_l^\nu(\zeta)} \tag{6.6}$$

and $S_R^{(l)}$ is the integral operator in $L^2((0, R))$, for any $R > 0$, with kernel defined by

$$S^{(l)}(x - x') = -2\pi b(m) \int_{-1}^1 dy\, \frac{P_l(y)}{\cosh(x-x') + \frac{y}{m+1}}, \qquad x, x' \in \mathbb{R}. \tag{6.7}$$

In particular, this decomposition implies

$$n(\lambda, S_R) = \sum_{l=0}^{+\infty} (2l+1) n(\lambda, S_R^{(l)}). \tag{6.8}$$

We are now in position to characterize the asymptotics of (6.1).

Proposition 1. *For any $\lambda > 0$ we have*

$$\lim_{R \to +\infty} \frac{n(\lambda, S_R)}{2R} = \sum_{l=0}^{+\infty} \frac{2l+1}{4\pi} \left| \left\{ k \in \mathbb{R} \mid \hat{S}^{(l)}(k) > \frac{\lambda}{\sqrt{2\pi}} \right\} \right| \tag{6.9}$$

and the limit is continuous in $\lambda > 0$.

Proof. The kernel of the operator $S_R^{(l)}$ is an even function and satisfies the estimate

$$|S^{(l)}(x)| \le \frac{c}{\cosh x - \frac{1}{m+1}}. \tag{6.10}$$

By (6.10) we have $S^{(l)} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $e^{\varepsilon|x|} S^{(l)} \in L^2(\mathbb{R})$ for any $\varepsilon \in [0, 1)$. These properties imply that $|\hat{S}^{(l)}(k)| \to 0$ for $|k| \to \infty$, by Riemann-Lebesgue Theorem, and that $\hat{S}^{(l)}(k)$ has an analytic continuation to a neighborhood of the real axis (see, e.g., Theorem

IX.13 of [19]). Then, for any $\lambda > 0$ the set $\mathfrak{M}(\lambda) = \left\{ k \in \mathbb{R} \mid \hat{S}^{(l)}(k) = \frac{\lambda}{\sqrt{2\pi}} \right\}$ consists of a finite number of points and, in particular, it has measure zero. Thus, the hypotheses of Lemma 4.6 in [20] are satisfied and we have

$$\lim_{R \to +\infty} \frac{n(\lambda, S_R^{(l)})}{R} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \, \chi_{\left(\frac{\lambda}{\sqrt{2\pi}}, +\infty\right)}(\hat{S}^{(l)}(k)) = \frac{1}{2\pi} \left| \left\{ k \in \mathbb{R} \mid \hat{S}^{(l)}(k) > \frac{\lambda}{\sqrt{2\pi}} \right\} \right| \quad (6.11)$$

where $|A|$ denotes the Lebesgue measure of the set A. Taking into account of (6.8), we obtain (6.9). Concerning the continuity of the limit, we observe that $\lim_{\lambda' \to \lambda} \chi_{(\lambda', +\infty)}(\hat{S}^{(l)}(k)) = \chi_{(\lambda, +\infty)}(\hat{S}^{(l)}(k))$ for any k such that $\hat{S}^{(l)}(k) \neq \lambda$. Using the Dominated Convergence Theorem we conclude the proof.

$\square$

We collect here some properties of $\hat{S}^{(l)}(k)$ which will be useful in the sequel. By definition, we have

$$\begin{aligned}
\sqrt{2\pi} \hat{S}^{(l)}(k) &= -2\pi \, b(m) \int_{-\infty}^{+\infty} dx \, e^{-ikx} \int_{-1}^{1} dy \, \frac{P_l(y)}{\cosh x + \frac{y}{m+1}} \\
&= -\frac{1}{2\pi} \frac{m+1}{\sqrt{m(m+2)}} \int_{-1}^{1} dy \, P_l(y) \int_{-\infty}^{+\infty} dx \, \frac{e^{-ikx}}{\cosh x + \frac{y}{m+1}} \\
&= -\frac{1}{\pi} \frac{m+1}{\sqrt{m(m+2)}} \int_{-1}^{1} dy \, P_l(y) \int_{0}^{+\infty} dx \, \frac{1}{\cosh x + \frac{y}{m+1}} \cos(kx) \\
&= -\frac{m+1}{\sqrt{m(m+2)}} \int_{-1}^{1} dy \, P_l(y) \frac{\sinh\left(k \arccos\left(\frac{y}{m+1}\right)\right)}{\sinh(k\pi) \sin\left(\arccos\left(\frac{y}{m+1}\right)\right)}
\end{aligned} \quad (6.12)$$

where, for the computation of the last integral, we refer the reader to [9, p. 30]. Moreover, by the elementary relations $\arccos(\alpha) = \frac{\pi}{2} - \arcsin(\alpha)$ and $\sinh(\alpha \pm \beta) = \sinh(\alpha)\cosh(\beta) \pm \sinh(\beta)\cosh(\alpha)$, we find

$$\begin{aligned}
\hat{S}^{(l)}(k) = -\frac{1}{\sqrt{2\pi}} \frac{m+1}{\sqrt{m(m+2)}} \int_{-1}^{1} dy \, P_l(y) &\left[\frac{\cosh\left(k \arcsin\left(\frac{y}{m+1}\right)\right)}{2\cosh\left(k\frac{\pi}{2}\right) \cos\left(\arcsin\left(\frac{y}{m+1}\right)\right)} \right. \\
&\left. - \frac{\sinh\left(k \arcsin\left(\frac{y}{m+1}\right)\right)}{2\sinh\left(k\frac{\pi}{2}\right) \cos\left(\arcsin\left(\frac{y}{m+1}\right)\right)} \right]
\end{aligned} \quad (6.13)$$

and using the parity of the Legendre polynomials in the equation above we obtain

$$\hat{S}^{(l)}(k) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \dfrac{m+1}{\sqrt{m(m+2)}} \displaystyle\int_{0}^{1} dy \, P_l(y) \dfrac{\sinh\left(k \arcsin\left(\frac{y}{m+1}\right)\right)}{\sinh\left(k\frac{\pi}{2}\right) \cos\left(\arcsin\left(\frac{y}{m+1}\right)\right)} & l \text{ odd} \\[6mm] -\dfrac{1}{\sqrt{2\pi}} \dfrac{m+1}{\sqrt{m(m+2)}} \displaystyle\int_{0}^{1} dy \, P_l(y) \dfrac{\cosh\left(k \arcsin\left(\frac{y}{m+1}\right)\right)}{\cosh\left(k\frac{\pi}{2}\right) \cos\left(\arcsin\left(\frac{y}{m+1}\right)\right)} & l \text{ even.} \end{cases} \quad (6.14)$$

Note that $\hat{S}^{(l)}(k) = \hat{S}^{(l)}(-k)$. Moreover, in [5] the following properties of $\hat{S}^{(l)}(k)$ are proved

$$\hat{S}^{(l)}(k) \geq 0 \qquad l \text{ odd}$$
$$\hat{S}^{(l)}(k) \leq 0 \qquad l \text{ even,}$$
$$(6.15)$$

$$\hat{S}^{(l+2)}(k) \leq \hat{S}^{(l)}(k) \qquad l \text{ odd}$$
$$\hat{S}^{(l+2)}(k) \geq \hat{S}^{(l)}(k) \qquad l \text{ even}$$
$$(6.16)$$

and

$$\max_{k \in \mathbb{R}} \hat{S}^{(1)}(k) = \hat{S}^{(1)}(0) = \frac{1}{\sqrt{2\pi}}\Lambda(m) \tag{6.17}$$

where $\Lambda(m)$ is defined in (2.18).

For the proof of the last step we make repeatedly use of the following technical Lemma (for the proof see Lemma 4.9 in [20]).

Lemma 7. *Let $B(z) = B_0(z) + K(z)$, where for $z < 0$ ($z \leq 0$) the operator $B_0(z)$ ($K(z)$) is compact and continuous in z. Suppose that for a function f such that $f(z) \to 0$ when $z \to 0^-$ there exists the limit*

$$\lim_{z \to 0^-} f(z) n(\lambda, B_0(z)) = l(\lambda) \tag{6.18}$$

and $l(\lambda)$ is continuous in $\lambda > 0$. Then the following holds

$$\lim_{z \to 0^-} f(z) n(\lambda, B(z)) = l(\lambda). \tag{6.19}$$

Then we have

Proposition 2.

$$\lim_{R \to +\infty} \frac{n(1, S_R)}{2R} = \lim_{z \to 0^-} \frac{n(1, \boldsymbol{A}(z))}{|\log|z||}. \tag{6.20}$$

Proof. The proof is obtained using Proposition 1, Lemma 7 and Lemmas 3, 4, 5, 6.

□

Let us prove our main result in the case $m < m_*$.

Proof of Theorem 1 (case $m < m_*$). By Theorem 3 and Propositions 1, 2 we find that the limit relation (2.20) holds with

$$\mathcal{C}(m) := \sum_{\substack{l=1, \\ l \text{ odd}}}^{+\infty} \frac{2l+1}{2\pi} \left| \left\{ k \in [0, +\infty) \,|\, \hat{S}^{(l)}(k) > \frac{1}{\sqrt{2\pi}} \right\} \right| \tag{6.21}$$

where we have used the parity of $\hat{S}^{(l)}(k)$ and the sum is only for l odd due to the property (6.15). It remains to show that $\mathcal{C}(m)$ is finite and strictly positive.

Let us prove the upper bound for $\mathcal{C}(m)$. We first look for an estimate of $\hat{S}^{(l)}(k)$ for l odd and $k \geq 0$. By Cauchy-Schwarz inequality we have

$$\hat{S}^{(l)}(k) \leq \frac{m+1}{\sqrt{2\pi}\sqrt{m(m+2)}} \int_0^1 dy \, |P_l(y)| \frac{\sinh\left(k \arcsin\left(\frac{y}{m+1}\right)\right)}{\sinh\left(k\frac{\pi}{2}\right) \cos\left(\arcsin\left(\frac{y}{m+1}\right)\right)}$$
$$\leq \frac{(m+1)^{3/2}}{\sqrt{2\pi}\sqrt{m(m+2)}} \frac{1}{\sqrt{2l+1}} \left[\int_0^{z_0} dz \frac{\sinh^2(kz)}{\sinh^2\left(k\frac{\pi}{2}\right)\cos z} \right]^{1/2}$$
$$(6.22)$$

where $z_0 = \arcsin\left(\frac{1}{m+1}\right)$. Using the estimate $\dfrac{\sinh(kz)}{\sinh(k\pi/2)} \le e^{-k(\pi/2 - z_0)}$ for $z \in (0, z_0)$, we find

$$
\begin{aligned}
\hat{S}^{(l)}(k) &\le \frac{(m+1)^{3/2}}{\sqrt{2\pi}\sqrt{m(m+2)}} \frac{1}{\sqrt{2l+1}} e^{-k(\pi/2 - z_0)} \left[\int_0^{z_0} dz \frac{1}{\cos z} \right]^{1/2} \\
&\le \frac{(m+1)^{3/2}}{\sqrt{2\pi}\sqrt{m(m+2)}} \frac{1}{\sqrt{2l+1}} e^{-k(\pi/2 - z_0)} \left[\log\left(\tan\left(\frac{\pi}{4} + \frac{z_0}{2} \right) \right) \right]^{1/2} \\
&\le \frac{(m+1)^{3/2}}{\sqrt{2\pi}\sqrt{m(m+2)}} \frac{1}{\sqrt{2l+1}} e^{-k(\pi/2 - z_0)} \left[\log\left(\sqrt{1 + \frac{2}{m}} \right) \right]^{1/2}
\end{aligned}
\tag{6.23}
$$

where in the last step we have used the elementary formula $\tan\frac{x}{2} = \frac{\sin x}{1 + \cos x}$ and the definition of z_0. Taking into account of the definition of $\alpha(m)$ and $\beta(m)$ given in (2.24), we have shown

$$
\hat{S}^{(l)}(k) \le \frac{\alpha(m)}{\sqrt{2l+1}} e^{-\beta(m)k}.
\tag{6.24}
$$

By Equation (6.21) and the above inequality we obtain

$$
\mathcal{C}(m) \le \sum_{\substack{l=1,\\ l \text{ odd}}}^{+\infty} \frac{2l+1}{2\pi} \left| \left\{ k \in [0, +\infty) \,|\, \frac{\alpha(m)}{\sqrt{2l+1}} e^{-\beta(m)k} > \frac{1}{\sqrt{2\pi}} \right\} \right|.
\tag{6.25}
$$

The measure of the set in the r.h.s. of (6.25) is different from zero only if

$$
\frac{\alpha(m)}{\sqrt{2l+1}} > \frac{1}{\sqrt{2\pi}}
\tag{6.26}
$$

i.e., only if $l \le l_0(m)$, where $l_0(m)$ is the largest odd integer smaller that $\pi\alpha(m)^2 - \frac{1}{2}$. Therefore we have

$$
\mathcal{C}(m) \le \sum_{\substack{l=1,\\ l \text{ odd}}}^{l_0(m)} \frac{2l+1}{2\pi} \left| \left\{ k \in [0, +\infty) \,|\, \frac{\alpha(m)}{\sqrt{2l+1}} e^{-\beta(m)k} > \frac{1}{\sqrt{2\pi}} \right\} \right|.
\tag{6.27}
$$

For any $l \le l_0(m)$, let

$$
K_l(m) := \frac{1}{\beta(m)} \log \frac{\sqrt{2\pi}\,\alpha(m)}{\sqrt{2l+1}}
\tag{6.28}
$$

be the unique positive solution of the equation $\dfrac{\alpha(m)}{\sqrt{2l+1}} e^{-\beta(m)k} = \dfrac{1}{\sqrt{2\pi}}$. Then

$$
\begin{aligned}
\mathcal{C}(m) &\le \sum_{\substack{l=1,\\ l \text{ odd}}}^{l_0(m)} \frac{2l+1}{2\pi\beta(m)} \log\left(\frac{\sqrt{2\pi}\,\alpha(m)}{\sqrt{2l+1}} \right) \\
&\le \frac{1}{2\pi\beta(m)} \log\left(\frac{\sqrt{2\pi}\,\alpha(m)}{\sqrt{3}} \right) \sum_{\substack{l=1,\\ l \text{ odd}}}^{l_0(m)} (2l+1) \\
&= \frac{1}{4\pi\beta(m)} \log\left(\frac{\sqrt{2\pi}\,\alpha(m)}{\sqrt{3}} \right) (l_0(m)^2 + 3l_0(m) + 2)
\end{aligned}
\tag{6.29}
$$

and this concludes the proof of (2.23).

Let us prove the lower bound for $\mathcal{C}(m)$. By (6.21) we immediately get

$$\mathcal{C}(m) \geq \frac{3}{2\pi} \left| \left\{ k \in [0, +\infty) \, | \, \hat{S}^{(1)}(k) > \frac{1}{\sqrt{2\pi}} \right\} \right|. \tag{6.30}$$

Using (6.17) and Definition 2, we find that

$$\hat{S}^{(1)}(0) = \frac{\Lambda(m)}{\sqrt{2\pi}} > \frac{1}{\sqrt{2\pi}} \quad \text{for} \ \ m < m_* \tag{6.31}$$

and this implies strictly positivity of the right hand side of (6.30). Furthermore, by monotonicity of $\hat{S}^{(1)}(k)$, we have

$$\left| \left\{ k \in [0, \infty) | \hat{S}^{(1)}(k) > \frac{1}{\sqrt{2\pi}} \right\} \right| = k_1(m) \tag{6.32}$$

where $k_1(m)$ is the unique positive solution of the equation $\hat{S}^{(1)}(k) = \frac{1}{\sqrt{2\pi}}$. From (6.30) and (6.32) the lower bound (2.21) follows.

$\square$

Remark 3. Let us check that $l_0(m) \geq 1$ for $m < m_*$. Recall that $l_0(m)$ is the largest odd integer smaller than $g(m) := \pi \alpha(m)^2 - 1/2$. It is easy to see that the function g is decreasing, $g(m) \to +\infty$ for $m \to 0$ and $g(m) \to 0$ for $m \to +\infty$. Then the assertion follows if one observes that $g(m_*) > 1$ (indeed, $g(m_*) \simeq 6.65$).

Remark 4. The lower bound of $\mathcal{C}(m)$ can be improved. Recall that $m_* := m_1^*$ is defined as the solution of $\sqrt{2\pi} \, \hat{S}^{(1)}(0) = \Lambda(m) = 1$. By analogy, for each l odd we can define a critical mass m_l^* as the solution of $\sqrt{2\pi} \, \hat{S}^{(l)}(0) = 1$. By (6.14), m_l^* solves the equation

$$\frac{2(m+1)}{\pi \sqrt{m(m+2)}} \int_0^1 dy \, P_l(y) \frac{\arcsin\left(\frac{y}{m+1}\right)}{\cos\left(\arcsin\left(\frac{y}{m+1}\right)\right)} = 1. \tag{6.33}$$

One can show (see [6, Appendix A]) that m_l^* is uniquely defined, it is decreasing in l and $\sqrt{2\pi} \, \hat{S}^{(l)}(0) > 1 \ \left(\sqrt{2\pi} \, \hat{S}^{(l)}(0) < 1 \right)$ if $m < m_l^*$ ($m > m_l^*$). Hence, if $m_{L+2}^* < m < m_L^*$ for some L then in (6.21) each term with $l \leq L$ is certainly nonzero and this implies

$$\mathcal{C}(m) \geq \sum_{\substack{l=1 \\ l \ \text{odd}}}^{L} \frac{2l+1}{2\pi} \left| \left\{ k \in [0, +\infty) | \hat{S}^{(l)}(k) > \frac{1}{\sqrt{2\pi}} \right\} \right|. \tag{6.34}$$

Obviously, for $L = 1$ we have only the first term and the inequality above reduces to (6.30).

Remark 5. It is worth noticing that, by Definition 2, we have $\sqrt{2\pi} \, \hat{S}^{(1)}(0) < 1$ for $m > m_*$ which, via (6.16) and (6.17), implies $\mathcal{C}(m) = \lim_{z \to 0^-} \frac{N(z)}{|\log|z||} = 0$. In fact, we conclude this section showing that for $m > m_*$ the discrete spectrum of H is finite, i.e., there is no Efimov effect.

Proof of Theorem 1 (case $m > m_*$). For $z < 0$ and $\varepsilon \in (0, 1)$ the following inequality holds

$$\begin{aligned} n(1, \boldsymbol{A}(z)) &\leq n(1 - \varepsilon, \mathcal{S}(z)) + n\left(\frac{\varepsilon}{3}, \boldsymbol{A}(z) - \boldsymbol{A}_0(z)\right) + n\left(\frac{\varepsilon}{3}, \mathcal{B}(z) - \tilde{\mathcal{B}}(z)\right) \\ &\quad + n\left(\frac{\varepsilon}{3}, \mathcal{B}_0(z) - \mathcal{S}(z)\right) \end{aligned} \tag{6.35}$$

where the operators $\boldsymbol{A}_0(z), \mathcal{B}(z), \tilde{\mathcal{B}}(z), \mathcal{S}(z)$ have been defined in the previous section. Such an inequality is a direct consequence of Lemmas 3, 4, 5, 6 and of the following technical result (see, e.g., [2]): if A, B are compact selfadjoint operators and $\lambda_i > 0$, $i = 1, 2$ then

$$n(\lambda_1 + \lambda_2, A + B) \leq n(\lambda_1, A) + n(\lambda_2, B). \tag{6.36}$$

Note that for $z \leq 0$ the operators appearing in the last three terms of (6.35) are compact and continuous in z. Therefore, the last three terms of (6.35) remain finite for $z \to 0^-$. Let us consider the first term in the right hand side of (6.35), i.e., $n(1 - \varepsilon, \mathcal{S}(z))$. We notice that

$$(\mathcal{S}(z)\xi, \xi) = (\mathcal{S}\xi_z, \xi_z) = (SM\xi_z, M\xi_z) \tag{6.37}$$

where $\xi_z(p) = (\chi_{|z|^{-1/2}} - \chi_1)(p)\xi(p)$, $\mathcal{S}$ is the operator in $L^2(\mathbb{R}^3)$ defined by

$$(\mathcal{S}f)(p) = -\frac{1}{4\pi^2} \frac{(m+1)^2}{\sqrt{m^3(m+2)}} \int dq \, \frac{f(q)}{|p|^{1/2} \left(\frac{p^2}{2\mu} + \frac{q^2}{2\mu} + \frac{p \cdot q}{m}\right) |q|^{1/2}}, \tag{6.38}$$

M is the unitary operator defined in (5.33) and S acts on $L^2(\mathbb{R}_+ \times \mathbb{S}^2, dr \otimes d\Omega)$ as follows

$$(Sf)(r, \omega) = -\frac{1}{4\pi^2} \frac{m+1}{\sqrt{m(m+2)}} \int_{-\infty}^{+\infty} d\rho \int_{\mathbb{S}^2} d\Omega(\zeta) \frac{f(\rho, \zeta)}{\cosh(r - \rho) + \frac{\omega \cdot \zeta}{m+1}}. \tag{6.39}$$

Let us estimate (Sf, f). By (6.6) and (6.7) we get

$$
\begin{aligned}
(Sf, f) &= -\frac{1}{2\pi} \frac{m+1}{\sqrt{m(m+2)}} \sum_{l=0}^{+\infty} \sum_{\nu=-l}^{l} \int_{-\infty}^{+\infty} dr \, \overline{f_{l\nu}(r)} \int_{-\infty}^{+\infty} d\rho \, f_{l\nu}(\rho) \int_{-1}^{1} dy \, \frac{P_l(y)}{\cosh(r - \rho) + \frac{y}{m+1}} \\
&= \sum_{l=0}^{+\infty} \sum_{\nu=-l}^{l} \sqrt{2\pi} (\hat{f}_{l\nu}, \hat{f}_{l\nu} \hat{S}^{(l)}) \\
&= \sqrt{2\pi} \sum_{l=0}^{+\infty} \sum_{\nu=-l}^{l} \int_{-\infty}^{+\infty} dk \, |\hat{f}_{l\nu}(k)|^2 \hat{S}^{(l)}(k).
\end{aligned}
\tag{6.40}
$$

Taking into account of (6.15), (6.16) and (6.17) we deduce

$$
\begin{aligned}
(Sf, f) &\leq \sqrt{2\pi} \sum_{\substack{l=1 \\ l \text{ odd}}}^{+\infty} \sum_{\nu=-l}^{l} \int_{-\infty}^{+\infty} dk \, |\hat{f}_{l\nu}(k)|^2 \hat{S}^{(1)}(k) \\
&\leq \Lambda(m) \sum_{\substack{l=1 \\ l \text{ odd}}}^{+\infty} \sum_{\nu=-l}^{l} \|f_{l\nu}\|^2 \\
&\leq \Lambda(m) \|f\|^2.
\end{aligned}
\tag{6.41}
$$

By (6.37) and (6.41) we have

$$(\mathcal{S}(z)\xi_z, \xi_z) \leq \Lambda(m)\|M\xi_z\|^2 \leq \Lambda(m)\|\xi\|^2 \tag{6.42}$$

where $0 < \Lambda(m) < 1$ for $m > m_*$ (see Definition 2). Therefore, the operator $\mathcal{S}(z)$ does not have eigenvalues larger than $\Lambda(m)$, i.e.,

$$n(1 - \varepsilon, \mathcal{S}(z)) = 0 \tag{6.43}$$

for any $\varepsilon \in (0, 1 - \Lambda(m))$. By Theorem 3 and (6.35), (6.43) we find

$$N(z) \le n\left(\frac{\varepsilon}{3}, \boldsymbol{A}(z) - \boldsymbol{A}_0(z)\right) + n\left(\frac{\varepsilon}{3}, \mathcal{B}(z) - \tilde{\mathcal{B}}(z)\right) + n\left(\frac{\varepsilon}{3}, \mathcal{B}_0(z) - \mathcal{S}(z)\right) \qquad (6.44)$$

for any $z < 0$ and $\varepsilon \in (0, 1 - \Lambda(m))$. Taking the limit $z \to 0^-$ and using compactness and continuity in $z \le 0$ of the operators in the right hand side of (6.44), we obtain the finiteness of the number of negative eigenvalues of H.

<div align="right">□</div>

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