PHYSICAL REVIEW A **95**, 011802(R) (2017)

Squeezing of X waves with orbital angular momentum

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(Received 7 July 2016; published 20 January 2017)

Multilevel quantum protocols may potentially supersede standard quantum optical polarization-encoded protocols in terms of amount of information transmission and security. However, for free-space telecommunications, we do not have tools for limiting loss due to diffraction and perturbations, as, for example, turbulence in air. Here we study propagation invariant quantum X waves with angular momentum; this representation expresses the electromagnetic field as a quantum gas of weakly interacting bosons. The resulting spatiotemporal quantized light pulses are not subject to diffraction and dispersion, and are intrinsically resilient to disturbances in propagation. We show that spontaneous down-conversion generates squeezed X waves useful for quantum protocols. Surprisingly, the orbital angular momentum affects the squeezing angle, and we predict the existence of a characteristic axicon aperture for maximal squeezing. These results may boost the applications in free space of quantum optical transmission and multilevel quantum protocols, and may also be relevant for novel kinds of interferometers, such as satellite-based gravitational wave detectors.

DOI: 10.1103/PhysRevA.95.011802

Standard protocols of quantum communication encode information into the polarization degrees of freedom of photons [1,2]. As a result, only one bit of information can be imprinted onto each photon. In recent years, there has been great interest in the development of a free-space system for quantum communications based on the use of modes that carry orbital angular momentum (OAM) [3–9]. When using OAM there is no limit to the number of bits of information that can be carried by a single photon, as the OAM states span an infinite-dimensional space. Correspondingly, the rate of information increases drastically. In addition, the security of the considered protocol is increased by a multilevel basis [3]. A key problem related with multimode quantum communications is the diffraction and dispersion of the wave packet. Diffraction and dispersion create an inhomogeneous transmission loss for different spatial frequencies that results in mixing of spatial modes [4]. Moreover, it has been shown that OAM states are strongly affected by perturbations. A great deal of work has been done in studying the effect of atmospheric turbulence in free-space communication [5,10–13]. A promising solution is the use of nondiffracting or localized waves such as X waves that are naturally resilient against perturbation [14].

Localized waves, i.e., linear solutions of Maxwell's equations that propagate without diffracting in both space and time, have been the subject of extensive research in the past years [14]. In particular, X waves, first introduced in acoustics in 1992 by Lu and Greenleaf [15], have been studied in different areas of physics [16,17]. Despite the great amount of literature concerning X waves, however, the investigations of their quantum properties are very few and they are limited to the case of traditional X waves, without OAM [18,19]. Very recently, X waves carrying OAM have been proposed [20] and they can constitute a new possible platform for free-space quantum communication.

In this work, we present a quantum theory of X waves carrying OAM based on the quantization of the motion of an optical pulse propagating in a normally dispersive medium. Although there is already some literature dealing with the quantization of waves carrying angular momentum, from both a quantum field theory [21] and an atomic physics [22] perspective, our approach is more general, as it also accounts for dispersion and nonlinearities. Moreover, our formalism is rather simple and provides a very useful theoretical toolkit for the study of quantum waves carrying AM in complex nonlinear media. In addition, the use of X waves carrying OAM as quantization basis allows one to easily generalize the results obtained for Bessel beams in the monochromatic [23] as well as in the polychromatic domain [24–28]. In particular, we study the case of spontaneous parametric down-conversion (SPDC) in a quadratic medium, with particular attention to the effect of the OAM carried by quantum X waves on the squeezing properties of the down-converted states generated by the nonlinear process. We find that squeezing is strongly affected by OAM. Changing the parity of the OAM, in fact, rotates the squeezing angle. This effect has a direct experimental signature and may be employed for novel quantum protocols. The results presented here are limited to the case of a scalar pulse in a dispersive medium. The full vectorial case will be the subject of future investigations.

We start our analysis considering an electromagnetic field propagating in a medium with refractive index $n = n(\omega)$. Under the paraxial and slowly varying envelope approximation, the field envelope $A(\mathbf{r},t)$ satisfies the following equation:

$$i\frac{\partial A}{\partial t} + i\omega'\frac{\partial A}{\partial z} - \frac{\omega''}{2}\frac{\partial^2 A}{\partial z^2} + \frac{\omega'}{2k}\nabla_\perp^2 A = 0. \tag{1}$$

We use (1) to study the propagation of an electromagnetic field in a dispersive medium characterized by a refractive index n, first-order dispersion $\omega' = d\omega/dk$, and second-order dispersion $\omega'' = d^2\omega/dk^2$ [29]. For a field propagating in vacuum one has $\omega' = c$ and $\omega'' = c^2/\omega$. The general solution of Eq. (1) can be written as a polychromatic superposition of

Bessel beams as follows [20]:

$$A(\mathbf{r},t) = \sum_m d_m e^{im\theta} \times \int dk_z \int_0^\infty dk_\perp k_\perp S(k_\perp,k_z) J_m(k_\perp R) e^{i(k_z Z - \Omega t)}, \tag{2}$$

where $Z = z - \omega' t$ and $\Omega = -\omega'' k_z^2/2 + \omega' k_\perp^2/2k$. In the above equation the cylindrical coordinates $\{R,\theta,Z\}$ and $\{k_\perp,\varphi,k_z\}$ have been used. This integral furnishes the field at instant t, given its spectrum $S(k_\perp,k_z)$ at $t = 0$, with transversal and longitudinal wave numbers $k_\perp$ and k_z, respectively. If we introduce the change of variables $\{k_\perp,k_z\} \to \{\alpha,v\}$ such that $k_\perp = \sqrt{\omega'' k/\omega'}\alpha$ and $k_z = \alpha - v\omega''$, with $v = c/\cos\vartheta$, being ϑ the Bessel cone angle, after some manipulation, Eq. (2) can be rewritten as a superposition of OAM-carrying X waves as follows:

$$A(\mathbf{r},t) = \sum_{m,p} \int dv\, C_{m,p}(v) e^{-(iv^2 t/2\omega'')} \psi_{m,p}^{(v)}(R,\zeta) \tag{3}$$

where $\psi_{m,p}^{(v)}(R,Z - vt)$ is the OAM-carrying X wave of order p and velocity v [30],

$$\psi_{m,p}^{(v)}(R,\zeta) \equiv \int_0^\infty d\alpha \sqrt{\frac{k}{\pi^2\omega'(1+p)}}(\alpha\Delta) L_p^{(1)}(2\alpha\Delta) e^{-\alpha\Delta} J_m \times \left(\sqrt{\frac{\omega'' k}{\omega'}}\alpha R\right) e^{i[\alpha-(v/\omega'')]\zeta} e^{im\theta}. \tag{4}$$

Here, $L_p(x)$ are the generalized Laguerre polynomials of the first kind, $\zeta = Z - vt$ is the comoving reference frame associated to the X wave [14], and Δ is a reference length related to the spatial extension of the beam. Following the orthogonality relation

$$\left\langle \psi_{l,q}^{(u)}(R,\zeta) \middle| \psi_{m,p}^{(v)}(R,\zeta) \right\rangle = \delta_{m,l}\delta_{p,q}\delta(u - v) \tag{5}$$

we find that OAM-carrying X waves have an infinite norm, like the plane waves typically adopted for field quantization. To quantize the field given by Eq. (3) we employ the standard technique of expressing the total energy of the field as a collection of harmonic oscillators [18,31]. From Eq. (2) we find for the total energy carried by $A(\mathbf{r},t)$,

$$\mathcal{E} = \int d^3 r |A(\mathbf{r},t)|^2 = \sum_{p,m} \int dv |C_{m,p}(v,t)|^2 \tag{6}$$

with $C_{m,p}(v,t) = C_{m,p}(v) e^{-(iv^2/2\omega'')t}$. As can be seen, the above equation can be interpreted as a collection of harmonic oscillators with complex amplitude $C_{mp}(v)$ and frequency $\omega_m(v) = \frac{v^2}{2\omega''}$. Without loss of generality we perform the quantization of the fundamental X wave ($p = 0$). The generalization to $p \neq 0$ is straightforward. We introduce a pair of real canonical variables $Q_m(v,t)$ and $P_m(v,t)$ defined by

$$C_m(v,t) = \frac{1}{\sqrt{2}}[\omega_m(v,t) Q_{m,p}(v) + i P_m(v,t)], \tag{7}$$

where $Q_m(v,t)$ and $P_m(v,t)$ oscillate sinusoidally in time at a frequency $\omega_m(v)$. We then obtain

$$H = \frac{1}{2}\sum_m \int dv \left[P_m^2(v,t) + \omega_m^2(v) Q_m^2(v,t)\right]. \tag{8}$$

The total energy of the field can be, therefore, expressed as an integral sum of harmonic oscillators characterized by the frequency $\omega_{m,p}(v)$, and $Q_{m,p}(v)$ and $P_{m,p}(v)$ play the role of position and momentum of the field, respectively. We promote these quantities to operators and introduce the creation and annihilation operators in the usual way as

$$\hat{Q}_m(v,t) = \sqrt{\frac{\hbar}{2\omega_m(v)}}[\hat{a}_m^\dagger(v,t) + \hat{a}_m(v,t)], \tag{9a}$$

$$\hat{P}_m(v,t) = i\sqrt{\frac{\hbar\omega_m(v)}{2}}[\hat{a}_m^\dagger(v,t) - \hat{a}_m(v,t)], \tag{9b}$$

where $\hat{a}_m(v,t) = e^{i\omega_m(v)t}\hat{a}_m(v)$ and the standard canonical bosonic commutation relations are understood [31]. Using the relations above, and remembering that $\omega_m(v) = v^2/2\omega''$, we write the Hamilton operator for the field from Eq. (8) in the following form:

$$\hat{H} = \sum_m \int dv \frac{Mv^2}{2}\left[\hat{a}_m^\dagger(v)\hat{a}_m(v) + \frac{1}{2}\right], \tag{10}$$

where $M = \hbar/\omega''$. Hereafter we drop the zero point energy as a standard renormalization procedure [32]. This is the first result of our work. Written in the above form, the dynamics of $(3+1)$-dimensional quantum X waves in a dispersive media can be seen as the ones of a one-dimensional quantum gas of weakly interacting bosons with mass M and velocity v [33], with the mass essentially accounting for the material dispersion, and the velocity containing the characteristic parameter of the X wave, namely, its opening angle. The formulation given by Eq. (10), moreover, allows one to study the evolution of nondiffracting waves in a dispersive medium using the techniques and concepts typical of condensed matter physics. This is very important, as it opens new possibilities for the study of spatiotemporal dynamics of single and few photons in dispersive media and, more in general, in more complicated structures, such as metamaterials and plasmonic structures.

We now substitute the expression of $C_m(v)$ in terms of $\hat{a}_m^\dagger(v)$ and $\hat{a}_m(v)$ into Eq. (3) to obtain the field operator

$$\hat{A}(\mathbf{r},t) = \sum_m \int dv\, e^{-i\omega_m(v)t}\sqrt{\hbar\omega_m(v)}\psi_m^{(v)}(\mathbf{r},\zeta)\hat{a}_m(v). \tag{11}$$

The above expression for the field operator can be intuitively understood as the result of the quantization of the electromagnetic field in a cavity, where the normal modes of the cavity are represented by OAM-carrying X waves. We remark that this quantization approach is rigorous, and similar results can be obtained by using a standard Lagrangian approach, as we will report elsewhere.

As an example of application of the formalism developed above, we now consider the case of phase-matched, collinear SPDC. In particular, we assume that a beam of frequency ω impinges upon a dielectric crystal with second-order nonlinearity and we call it a pump beam. As a result of the nonlinear interaction, a photon from the pump beam can be annihilated to create two new photons having lower frequencies ω_1 and ω_2 with $\omega = \omega_1 + \omega_2$ [31]. Moreover, we assume that the nondepleted pump approximation holds and that the pump beam can therefore be represented by a bright coherent state and treated like a classical beam. This allows us to consider its action on the Hamiltonian of the system as only a constant term, which can therefore be incorporated into the nonlinear coefficient χ associated to the process itself. Under these assumptions, the Hamiltonian describing such a system is then given by

$$\hat{H} = \sum_m \int dv\, \hbar\omega_m(v)[\hat{a}_m^\dagger(v)\hat{a}_m(v) + \hat{b}_m^\dagger(v)\hat{b}_m(v)] + \hat{H}_I(t), \tag{12}$$

where the interaction Hamiltonian $\hat{H}_I(t)$ is obtained by quantizing its classical counterpart [34]

$$\mathcal{E}_I = \chi\, \langle A_1|A_2^*\rangle + \text{c.c.} \tag{13}$$

Here, the expression c.c. denotes the complex conjugation, χ is proportional to the second-order nonlinearity $\chi^{(2)}$, and $\hat{a}_m(v)$ and $\hat{b}_m(v)$ are the annihilation operators related to the quantized fields $\hat{A}_1(\mathbf{r},t)$ and $\hat{A}_2(\mathbf{r},t)$, respectively. We assume that the two X waves travel with the same velocity v, such that the the two Bessel angles satisfy $\vartheta_2 = \vartheta_1 + 2n\pi$. After a lengthy but straightforward calculation, we can write the quantized interaction Hamiltonian as follows:

$$\hat{H}_I(t) = \hbar \sum_m \int dv\, \chi_m(2v)\omega_m(v)\hat{a}_m^\dagger(v,t)b_{-m}^\dagger(v,t) + \text{H.c.} \tag{14}$$

In the expression above, H.c. denotes the Hermitian conjugate, while $\chi_m(x)$ is the interaction function, whose explicit expression reads

$$\chi_m(x) = (-1)^m 4\pi^2\chi x e^{-2x}. \tag{15}$$

Moreover, in Eq. (14) we used $\hat{a}_m(v,t) = e^{iF(v)t}\hat{a}_m(v)$, with

$$F(v) = \frac{v^2}{\omega''} + \frac{v(1-\rho)(\omega_1' - \omega_2')}{\omega''(1+\rho)}, \tag{16}$$

where $\rho = \sqrt{k_1\omega_1'/k_2\omega_2'}$, $\omega_{1,2}' = d\omega_{1,2}/dk$, and $k_{1,2} = \omega_{1,2}n_{1,2}/c$.

To determine the electromagnetic field after the evolution driven by the interaction Hamiltonian $\hat{H}_I$, we employ perturbation theory. Writing the total (time-dependent) Hamiltonian as

$$\hat{H}(t) = \hat{H}_0(t) + \lambda\hat{H}_I(t), \tag{17}$$

then, using the Schwinger-Dyson expansion truncated at the first order, we get the following result for the state of the system [32]:

$$|\psi^{(1)}(t)\rangle = -\frac{i}{\hbar}\int_0^t d\tau\, \hat{H}_I(\tau)\,|0\rangle\,, \tag{18}$$

where $|\psi(0)\rangle = |0\rangle$ has been assumed. If we now introduce the quantities $K(v) = [v(1-\rho)(\omega_1' - \omega_2')]/[2\omega''(1+\rho)]$ and $G(v,t) = -[2i/F(v)]\sin[F(v)t/2]$, and we define the function

$$\mathcal{G}_m(v,t) = \sqrt{\omega_m(v)\omega_{-m}(v)}G(v,t)e^{iK(v)t}\chi_m(2v), \tag{19}$$

after some algebra the final expression for the state after the interaction is

$$|\psi^{(1)}(t)\rangle = \sum_m \int dv\, \mathcal{G}_m(v,t)|m,v;-m,v\rangle, \tag{20}$$

where $|m,v;-m,v\rangle \equiv \hat{a}_m^\dagger(v)\hat{b}_{-m}^\dagger(v)\,|0\rangle$. The state given by Eq. (20) represents a superposition of two particles, corresponding to the two modes ω_1 and ω_2 traveling with the same velocity v. Notice, moreover, that due to angular momentum conservation, the photon pairs generated by SPDC are constrained to possess the same amount of OAM but with opposite sign. This is possible since we have made no particular assumption about the OAM content of the pump beam. In the general case, in fact, the conservation of OAM implies that $m_s + m_i = m_p$, where the subscripts s, i, and p stand for signal, idler, and pump, respectively.

We can now study the squeezing effect in the case of degenerate down-conversion, corresponding to $\omega_1 = \omega_2 = \omega/2$. The quantized field operator associated to the mode $\omega/2$ is then given as follows [35]:

$$\hat{A}(\mathbf{r},t) = \sum_{m,p}\int dv\sqrt{\hbar\omega_{m,p}(v)}\psi^{(v)}(R,\zeta)\hat{a}_{m,p}(v,t). \tag{21}$$

In this case the Hamiltonian of the system is the same as the one presented in Eq. (14) with $b_m(v) = a_m(v)$ and $\rho = 1$, since we are considering the degenerate down-conversion in which $\omega_1' = \omega_2'$ and $|k_1| = |k_2|$. In the interaction picture we consider only the time evolution controlled by H_I [32]. Thus the two equations of motion for $\hat{a}_m(v,t)$ and $\hat{a}_{-m}(v,t)$ are then

$$\frac{d}{dt}\hat{a}_j(v,t) = \omega_j(v)[\chi_j(2v) + \chi_{-j}(2v)]\hat{a}_{-j}^\dagger(v,t), \tag{22}$$

where $j \in \{m,-m\}$. A general solution of these equations is [31] $\hat{a}_j(v,t) = \mathcal{A}_j(v,t)\hat{a}_j(v) + \mathcal{B}_j(v,t)\hat{a}_j^\dagger(v)$, where $j \in \{m,-m\}$, $\mathcal{A}_m(v,t) = \cosh[\xi_m(v)t]$, $\mathcal{B}_m(v,t) = e^{i(\phi+m\pi)}\sinh[\xi_m(v)t]$ and the squeezing parameter $\xi_m(v)$ is given, for a fundamental X wave, as follows:

$$\xi_m(v) = (-1)^m\frac{\pi\chi c}{2\Delta}e^{-\frac{\tilde{v}}{\omega''}}\frac{\tilde{v}^3}{(\omega'')^3}, \tag{23}$$

being $\tilde{v} = v\Delta$. Therefore, the state after the SPDC is a squeezed state with a squeezing parameter $\xi_m(v)$ depending on both velocity and OAM (see Fig. 1 below). To further elaborate on that, we can introduce the quadrature operators: $\hat{X}_j(v,t) = \hat{a}_j(v,t) + \hat{a}_j^\dagger(v,t)$ and $\hat{Y}_j(v,t) = i[\hat{a}_j^\dagger(v,t) - \hat{a}_j(v,t)]$, where $j = \{-m,m\}$. For $\phi = 0$ we have $\hat{X}_j(v,t) = e^{\xi_j(v)t}\hat{X}_j(v,0)$ and $\hat{Y}_j(v,t) = e^{-\xi_j(v)t}\hat{Y}_j(v,0)$, and the variance of such quadrature operators is then given by $\Delta X_j(v,t) = e^{\xi_m(v)t}\Delta X_j(v,0)$ and $\Delta Y_j(v,t) = e^{-\xi_m(v)t}\Delta Y_j(v,0)$. This shows that the down-conversion interaction Hamiltonian for OAM-carrying X waves acts like a two mode squeezing operator. Remarkably, we find that OAM changes the sign of the squeezing parameter $\xi_m(v)$, i.e., the squeezed quadrature changes depending on the parity of the angular momentum number m. In particular, if

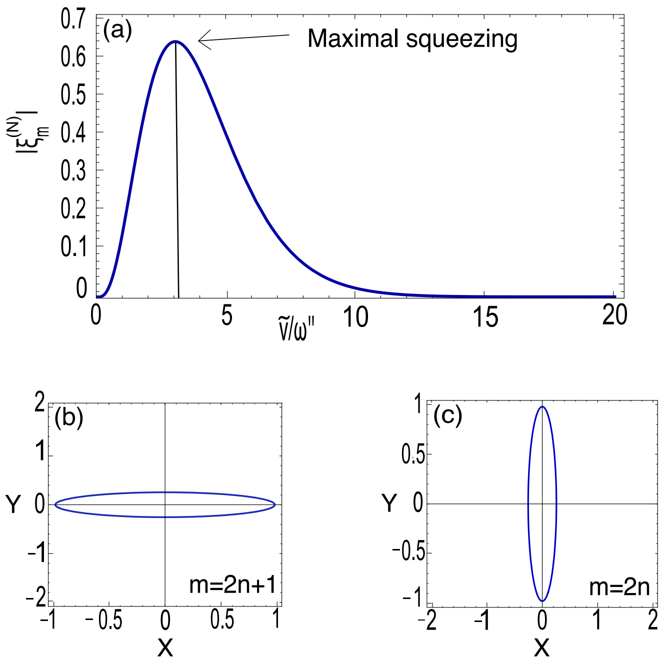

FIG. 1. (a) Plot of the normalized squeezing parameter modulus $|\xi_m^{(N)}| = \Delta|\xi_m|/(\pi\chi c)$ in function of the normalized velocity $\tilde{v}/\omega''$. (b),(c) Quadrature space representation of the squeezed down-converted state in the case of odd [panel (b)] and even [panel (c)] values of the OAM parameter m with a normalized velocity $\tilde{v}/\omega'' = 3$ and fixing the time t so that $\pi\chi ct/\Delta = 1$.

m is an even number, $\xi_m(v) > 0$ and the squeezing occurs in the Y quadrature. On the other hand, if m is an odd number, $\xi_m(v) < 0$ and the X quadrature will result squeezed as we can observe in Figs. 1(b) and 1(c). This is our second result.

In addition, Eq. (23) reveals a dependence of the squeezing parameter from the X wave velocity. Therefore, there exists an optimal value of the velocity $v_{\mathrm{opt}} = 3\omega/\Delta$ that maximizes the amount of squeezing produced by the nonlinear process [Fig. 1(a)]. This corresponds to the optimal axion angle $\cos\vartheta_0^{\mathrm{opt}} = \Delta/3\lambda$. If we, for example, assume a nondiffracting pulse with a duration of $\Delta t = 8$ fs and a carrier wavelength of $\lambda = 850$ nm, the optimal axicon angle that maximizes the squeezing is given by $\vartheta_0^{\mathrm{opt}} \simeq 20^\circ$. Using these values and assuming for the second-order nonlinearity $\chi^{(2)} \cong 10^{-12}\frac{m}{V}$ [34], we can evaluate the maximal squeezing parameter to be $\xi_m \simeq 100\ \mathrm{s}^{-1}$.

We remark that the experimental generation of the proposed quantum states of light may be implemented by using a spiral phase plate and a system of cylindrical lenses to control the OAM carried by the pump beam [36,37]. The spiral phase plate transforms a TEM_{00} mode in a spiral mode with fixed OAM [36]. The cylindrical lenses transform an input mode with a fixed OAM number m (e.g., a Laguerre-Gauss mode) into one with number $-m$ [37]. In this way we can generate two input beams, one with OAM per photon $\hbar m$ and one with OAM $-\hbar m$ per photon, that are sent to the nonlinear crystal for SPDC. Another way to realize X waves carrying OAM is the use of metasurfaces to convert the spin angular momentum (SAM) in OAM [38,39]. Suppose we have an input X wave with $m = 0$ and uniform circular polarization; after the interaction with the metasurface the output beam switches handedness with a SAM variation $\pm 2\hbar$ per photon. Since the total angular momentum must be conserved an OAM $m = \pm 2\hbar$ per photon is generated. The result for the field amplitude is an X wave carrying OAM. Alternatively, the same result can be achieved by using total internal reflection in an isotropic medium [40].

In conclusion, we have presented a quantized theory of optical pulses propagating in a normally dispersive medium as a collection of harmonic oscillators associated to traveling modes represented by X waves carrying OAM. This allows us to describe the dynamics of the quantized field as those of a one-dimensional quantum gas of weakly interacting bosons with velocity $v = c/\cos\vartheta$ and mass $M = \hbar/\omega''$. Moreover, we have shown that it is possible to select the quadrature squeezed state generated by SPDC [Figs. 1(b) and 1(c)] and that there exists an optimal velocity (i.e., axicon angle) that maximizes the amount of squeezing generated. Moreover, in comparison with already existing results concerning parametric down-conversion of Bessel beams [41], the result of our theory, although limited to the scalar case, provides a way to find the optimum Bessel cone angle, thus maximizing the squeezing effect.

We believe that these results are helpful for future multilevel, free-space quantum communication protocols that are potentially free of diffraction and dispersion and not affected from external perturbations, in particular from atmospheric turbulence. Further applications include the use of the proposed diffraction-free OAM states in free-space interferometric setups for high-sensitivity interferometers for gravitational wave detection.

A.S. gratefully acknowledges financial support from the Deutsche Forschungsgemeinschaft (Grants No. SZ 276/7-1, No. SZ 276/9-1, No. BL 574/13-1, and No. GRK 2101/1) and the German Ministry for Science and Education (Grant No. 03Z1HN31). C.C. gratefully acknowledges financial support from the Templeton Foundation (Grant No. 58277).

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