A Comparison between mathematical models of stationary configuration of an underwater towed system with experimental validations for Oceans '17 MTS/IEEE Aberdeen Conferences

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***Abstract*—The analysis of underwater towed systems attracted the interest of many researchers because of the recent years utilization of remotely-operated underwater vehicle (ROV) and towed array in offshore and military applications. The purpose of this work is to show, by experimental validation, that towed cable configurations may be computed effectively and accurately by discretizing the towing cable rather than using a continuous modeling approach. Two mathematical models have been developed to predict the stationary configuration of an underwater towed system loaded by hydrodynamic forces. The system is composed of a towed inextensible cable, with no bending stiffness, and a depressor that is fixed at the cable free end. This configuration is currently used for underwater remotely-operated vehicle. This work investigates the comparison between continuous and discrete models of the 2D static equations of the steady-state towing problem in a vertical plane at different towing speeds. The results of the models have been validating using experimental trials. In the first part of this paper, a continuous model is presented, which is based on geometric compatibility relations, equilibrium equation. A set of nonlinear differential equations has been derived and solved using Runge-Kutta iterative procedure. In the second part, a discrete rod model is proposed to determinate the cable shape, which is based on a system of nonlinear algebraic equations that are solved numerically. This two models are both suitable for analyzing an underwater towed system having a known top tension and inclination angle obtained from experiments. The third part of the paper describes the experiments, which have been in a towing tank basin (CNR-INSEAN). In the fourth and last part of this study it is demonstrated the effort and cost of numerically integrating the continuous model do not compare favorably with the relative ease and efficiency of solving the discrete model, which yields the same results.**

1. Introduction

The determination of cable configurations has long been an area of considerable research effort. Cables are extensively used for many ocean applications. Just a few examples of ocean systems that strongly depend on cables are the ocean mooring systems, towed array sonar systems and remotely operated vehicles. In the last few decades, towed cable has been widely employed in marine environments as an important tool for naval defense, ocean exploitation and ocean research. Static cable problems have traditionally found application in the design of buoy [1] and mooring systems [2], where the maximum cable tension is an important concern. In addition to the classical applications of mooring and buoy, the shape of a static cable is used to model other physical situations like towing system, turbine blades with zero bending moments and suspended pipelines subject to bending moments. The static cable shapes, especially the catenary, are often used as starting configurations for dynamic analyses [3], [4] and [5]. Numerous theories have been developed for the static response of submarine cable. A good survey can be found in Zajac [6], Irvine [7], Choo and Casarella [8]. The static cable configuration can be obtained from the solution of the system of nonlinear ordinary differential equations modeling the continuous problem. Due to the nonlinearity of the equations, a discretization of the continuous mathematical model must be used to obtain the configuration of the cable. For most of towed systems, the steady state problem can be resolved into two-point boundary-value problem (TPBVP), or initial value problem (IVP) in some special cases where the initial values are available directly. Many authors have solved this system with the classical approaches as finite element method [9] and finite differences method [10], [11]. Chucheepsakul and Huang [12] solved the problem utilizing the stationary condition of a functional coupled with an equilibrium equation with finite element method. Yang, Jeng and Zhou [13] presented a semi-analytical approximation of tension analysis of submarine cables during laying operation in order to found the cable configuration. In [14] a shooting method has been proposed to solve the system as a TPBV problem. A new bisection method was proposed in [15] to solve the TPBV problem rather than the conventional shooting method due to its algorithm complexity and low efficiency. An alternative discrete solution strategy is to discretize the physical cable into segments. Then a system of nonlinear algebraic equations is obtained from the equilibrium of forces on the segments. This physical discretization approach has been used extensively for mooring lines because the phenomena of varying cable properties like mass, elasticity, bending stiffness, etc. The chain of segments of the discretized cable could, for example, be modeled by lumped masses connected by string, or alternatively by hinged rods. The lumped mass model has been used extensively by researchers, [8], [15]. However two particularly attractive features of the rod model is that the forces on the segments are used directly in force balance equations on the rods, and that the characteristics of cables, as bending stiffness, can be incorporated into the model by using moments in a very natural way. The equivalence of these two modeling approaches for perfectly flexible cables was proved by Dreyer and Murray [16]. Following the Dreyer and Vuuren [17] work, in this paper, comparison between continuous and discrete models of the two-dimensional steady-state towing problem in a vertical plane at different towing speeds has been presented. The towed system in analysis is currently used for underwater remotely-operated vehicle. The towed system is composed of a cable and a depressor that is fixed at the cable free end. The propose of the depressor is to guarantee passively the reaching of the operative depth of the entire system. In order to found the cable configuration and the reached depth, two kind of mathematical models are compared: the continuous model and the discrete one. Both models solved the stationary towed problem as an initial values problem because initial values are available directly. In fact, the initial values, top tension and inclination angle at the towing point, have been founded by experimental trials at different towing speed. At the end the reached depth values at varying towing velocity will be compared with its experimental values. The paper is divided in the following sections. In Section 2 the two different mathematical models are presented. In Section 2.1 the continuous mathematical model is presented and numerically solved by Newton-Rapshon iterative procedure. Section 2.2 describes the discrete rigid-rod mathematical model, also solved numerically. In Section 3 the experimental setup and the trials are explained. In Section 4 the comparison between the two mathematical models, validated by measurements, and the discussion of the depth values have been done. The final Section 5 shows the conclusions. Intuitively better results from the continuous model than from discretizing the cable are expected. However, the solution of the continuous model is quite involved and costly in terms of computer time, while the solution of the discrete model is comparatively simple and time efficient. Moreover, the accuracy of the continuous model is not far superior to that of the discrete model, even for relatively few cable elements in the discrete case. Our main aim in this paper is therefore to show, by experimental validation, that cable configurations at varying towing speed may be computed effectively and accurately by discretizing the cable rather than using a continuous modeling approach.

2. Mathematical Models

The present theory is used to solve the steady-state configuration of an underwater towed system composed by a cable and a depressor fixed at the cable free end. The two mathematical models compared in this study are based on the following assumptions:

- The length of the cable is much larger than the diameter of its cross-section so that the cable can be modeled as a one-dimensional body.
- The constant cross-sectional area of the cable is chosen circular and with a small diameter in order to minimize the vortex shedding effect.
- The mass per unit length is constant.
- The cable is inextensible.
- The seawater is ideal fluid: inviscid and incompressible.

The cable is subjected to uniaxial tension without flexure, shear or torsion and the rotational motion has not been considered in the submarine cable motion balance. The gravity and hydrodynamic forces have been taking into account as external forces acting on the cable and the drag coefficients c_n and c_t are taken as constant values.

2.1. Continuous Model

A curvilinear axial coordinate s has been employed, which assumes a zero value at the towing point (Figure 1).

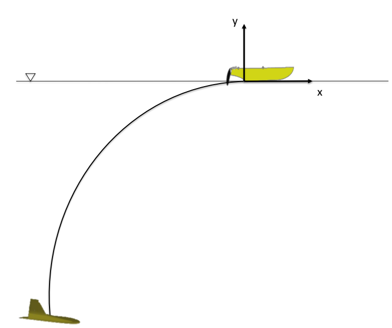

Figure 1: Towed system model.

When the system is towed by ship moving at constant speed, there are at least four components that contribute to the tensional force and configuration along the submarine cable:

- Net in water weight of the cable per unit length: W_n
- External normal force per unit length: $R_n(s)$
- External tangential force per unit length: $R_t(s)$
- Local tension: T(s)
- Local inclination angle: $\phi(s)$

point (x_e, y_e) where the depressor is fixed. A rod model has been used in order to discretize the cable, as Figure 3 shows.

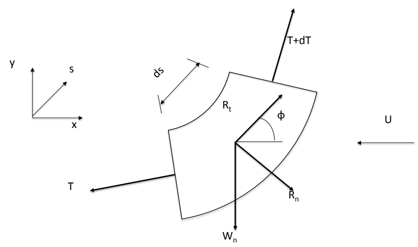

Figure 2: Force balance on continuos model.

The force balance in the tangential and normal coordinates, shown in Figure 2, gives two coupled equations depending on T and ϕ:

$$\frac{dT}{ds} = W_n sin\phi - R_t \quad (1)$$

$$T\frac{d\phi}{ds} = W_n cos\phi + R_n \quad (2)$$

The external forces are the fluid drag. The tangential component of drag is proportional to the drag coefficient C_t and the normal drag is controlled by a crossflow drag coefficient C_n. The value of C_t has been considered proportional to C_n, as usual for underwater cylindrical body [15]. The fluid velocity vector, U horizontal in both cases toward the left, has been projected onto the global axes, obtaining:

$$R_t = -\frac{1}{2}\rho C_t d U^2 cos\phi^2 \quad (3)$$

$$R_n = -\frac{1}{2}\rho C_n d U^2 sin\phi^2 \quad (4)$$

The drag law has been simplified from the usual form |u|u to u^2 since the configuration angle ϕ of the cable can change between 0 and $\pi/2$ values. With regard to the global cartesian coordinates x and y, the cable configuration follows

$$\frac{dx}{ds} = cos\phi \quad (5)$$

$$\frac{dy}{ds} = sin\phi \quad (6)$$

The simultaneous integration of all four equations (1), (2), (5) and (6) defines the cable configuration and tension. The obtained system of Ordinary Differential Equations (ODE) has been solved as an initial values problem with the fourth order Runge-Kutta iterative procedure. The initial values, T_0, ϕ_0, x_0 and y_0, are available directly from the experiments which are explained in Section 3.

2.2. Discrete Model

A two-dimensional cable of length L is divided in n segments with l_i length. The discretization of the cable has been developed between the towing point (0,0) and the end point (x_e, y_e) where the depressor is fixed. A rod model has been used in order to discretize the cable, as Figure 3 shows.

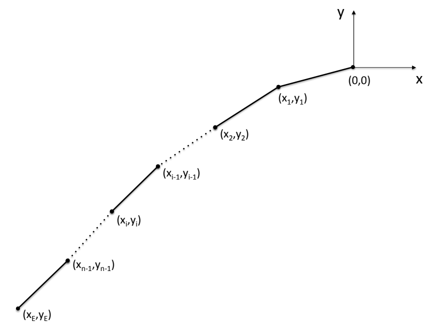

Figure 3: Two-dimension cable discretization.

Consider a general segment i-th of the cable (Figure 4). The i-th segment presents an inclination angle ϕ_i and the (x_{i+1}, y_{i+1}) and (x_i, y_i) indicates, respectively, the position of the left (A) and right (B) endpoint. The components of the tension force at the endpoints of the i-th rod is directly referred to the global system as continuity forces. Let ($F_{x,i+1}$, $F_{y,i+1}$, ϕ_{i+1}) and ($F_{x,i}$, $F_{y,i}$, ϕ_i) be the continuity forces acting on the left and right endpoint.

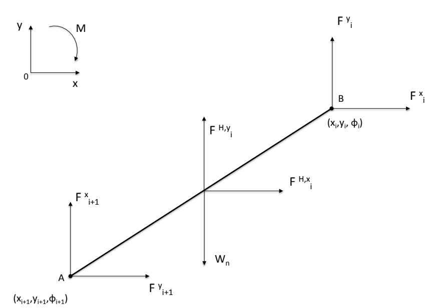

Figure 4: Two-dimension cable discretization.

The other distributed loads acting on the i-th rod, which act through rod center l_i/2, are also directly represented in the global system. The components of the hydrostatic loads are denoted by the vectors F_{Hxi} and F_{Hyi} respectively along the x and y global axis direction.

$$F_{Hxi} = R_t sin\phi_i - R_n cos\phi_i \quad (7)$$

$$F_{Hyi} = R_t cos\phi_i + R_n sin\phi_i \quad (8)$$

The expression of R_t and R_n, which are respectively the values of the tangential and normal component of the hydrodynamics force referred to the local body system, have been already shown in (3) and (4). The weight per unit length W_n of the i-th rod has been also apply in the segment center l_i/2. The inclination angle ϕ_i of the i-th rod has been calculated from the moment balance respect to the left endpoint A which is the only unknown:

$$F_{y,i}l_icos\phi_i - F_{x,i}l_isin\phi_i + F_{Hyi}\frac{l_i}{2}cos\phi_i - F_{Hxi}\frac{l_i}{2}sin\phi_i + \\ + W_n\frac{l_i^2}{2}cos\phi_i = 0 \quad (9)$$

Once ϕ_i has been calculated, the coordinates of the left endpoint A can be computed:

$$x_{i+1} = x_i - l_icos\phi_i \quad (10)$$

$$y_{i+1} = y_i - l_isin\phi_i \quad (11)$$

The continuity forces acting on the A endpoint have been also compute:

$$F_{x,i+1} = F_{x,i} + F_{Hxi} \quad (12)$$

$$F_{y,i+1} = F_{y,i} + F_{Hyi} - W_nl_i \quad (13)$$

Repenting this iterative scheme of solution the stationary configuration of the cable has been estimate starting from the known experimental values of x_0, y_0, F_{x0} and F_{y0} solving the problem as an initial values problem.

3. Experimental Set-up

The physical system under investigation comprises a towing cable and a depressor connected at its free end. The tests was performed at the CNR-INSEAN towing tank basin (470 m long, 9 m wide and 6,5 m deep) and the entire system has been towed by moving a carriage along the tank. The experimental measures have been performed in order to compute the values of the top tension and inclination angle at the towing point which have been used as initial values of both the mathematical models and also the reached depth value in order to validate the models. The trials have been conducted for different operative towing speeds from 1m/s up to 4m/s. The geometry and the dimensions of the depressor are respectively shown in Figure 5 and Table 1 in which W_d is the weight in water of the depressor.

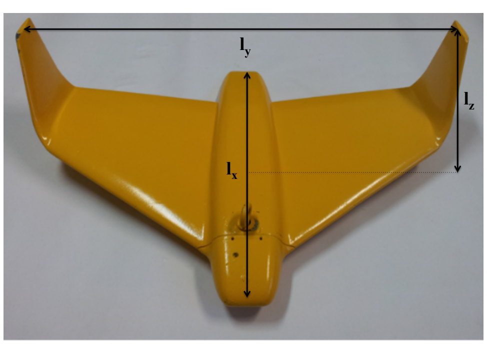

Figure 5: Shape of depressor.

Inside the body of the depressor sensors of pressure have been inserted in order to be able to estimate the value of the depth reached from the entire system. The reached depth is an important value for the validation of the models. In order to reduce normal drag and thus achieve more depth and also to eliminate cable vibration caused by vortex shedding, commonly known as cable strum, a towing cable has been chosen which presents a very small diameter. The geometry and dimensions of the towing cable are shown respectively in Figure 6 and Table 2. In Table 2 the nondimensional hydrodynamic coefficients are also listed.

TABLE 1: Depressor dimension

l_x[m]	l_y[m]	l_z[m]	W_d[N]
0,45	0,58	0,15	5

Figure 6: Shape of towing cable

TABLE 2: Towing cable dimension

[m]	Towing cable
l_c	4
d	0,005
c_n	0,9
c_t	0,014

As already explained, the experimental trials have been also developed in order to reach the initial values of the mathematical models, the top tension and the inclination angle at the towing point at different towing speeds. The towing cable has been attached to the carriage by using a load cell in order to measure the value of the top tension T_0 and in the same point an inclinometer has been used to compute the inclination angle ϕ_0 referred to the y axis direction. For the continuos model this experimental values are directly utilized as initial values instead for the discrete model it is useful to remember that the initial values, referred to the global system, F_{x0} and F_{y0} are correlated to the experimental values as (14) and (15) show.

$$F_{x0} = T_0sin\phi_0 \quad (14)$$

$$F_{y0} = T_0cos\phi_0 \quad (15)$$

4. Experimental results and comparison between mathematical models

In this section the experimental results at different towing speeds are shown and a comparison between the mathematical models for each towing speed are done. In Table 3 the experimental values of the top tension and angle of inclination at varying towing speed have been shown. The values of F_{x0} and F_{y0} have been also list in Table 3 in order to have directly the initial values in the better form for the evaluation of the discrete model.

TABLE 3: Experimental results

v[m/s]	$\mathbf{T_0}$[N]	ϕ_0[grad]	$\mathbf{F_{x0}}$[N]	$\mathbf{F_{y0}}$[N]
1	12	31	6	10
2	51	48	38	34
3	114	51	89	72
4	197	52	155	121

The experimental results seem to be in a good accordance with the theoretical predictions. In fact the values of the drag and lift of the entire towed system are directly proportional to the square of the towing velocity value. This proportion is due to the fact that the fluid has constant density and the drag and lift coefficients have been consider constant. The last assumption is off course an approximation because the dependency between Reynolds number, ϕ_0 and drag and lift coefficients. The experimental initial values have been insert in both the mathematical models and the configuration of the towing system has been plot for each mathematical models at different operative towing speed. In the next figures the configurations of the towing system has been show for the continuos model (Figure 7) and discrete one (Figure 8) for each towing speed (1-4 m/s). The origin of the system coordinates has been fixed at the towing point, the x axis direction represents the distance from the ship and y axis direction is the depth from the free surface.

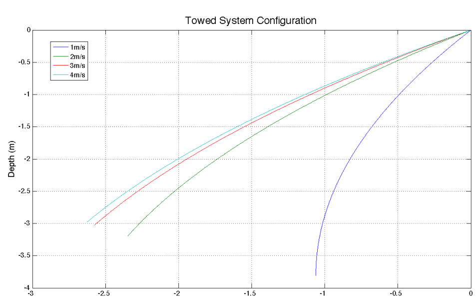

Figure 7: Configuration of towing system for the continuos mathematical model at different towing speeds

Figure 7 shows the different continuos model configuration of the entire towing system at varying towing speed. From the graphics it is possible to notice how the reached depth from the system increases for low value of the towing speed and, on the contrary, it decreases for high towing velocity values, as we expect. Furthermore, the configuration of the towing cable becomes flatter with the increasing of the towing speed.

In the Table 4, the values of the reached depth for the continuos mathematical model have been compared with the experimental values of depth reached by the sensors of pressure. The experimental and mathematical values of the depth present a percentage difference around 2%.

TABLE 4: Towing system reached depth: Continuos model

v [m/s]	Depth [m]	Experimental Depth [m]
1	3,651	3,691
2	2,893	2,933
3	2,728	2,768
4	2,676	2,716

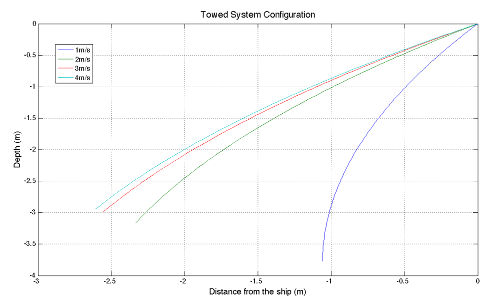

Figure 8: Configuration of towing system for the discrete mathematical model at different towing speeds

In Figure 8 the different configurations of the towing system at varying towing speed for the discrete model have been report. Even for the discrete model case, the reached total depth of the system presents a trend inversely proportional to the speed as for the continuos model. The configuration of the towing cable has the same behavior already explain in the continuos case: it is flatter when the towing speed increases. In Table 5, the values of the reached depth have been report in the case of the discrete mathematical model and also in this case, they have been compared with the experimental values. The experimental and mathematical values of the depth present a percentage difference around 2%.

TABLE 5: Towing system reached depth: Discrete model

v [m/s]	Depth [m]	Experimental Depth [m]
1	3,614	3,654
2	2,862	2,902
3	2,699	2,739
4	2,647	2,687

The specific comparisons between the two mathematical models have been developed in the next figures for each towing speed. The red line represents the results in the case of the continuos model while the blue crossed line the discrete one. In Figures 9,10,11 and 12 the comparison between the results obtained from the two mathematical models at each towing velocity values have been show. The towing system configuration is perfectly the same in the two models, in fact the percentage difference between the two models is lower than 1% for each speed. In Figure 9 it possible to notice that the value of the angle at the attack point between the depressor and the end of the towing cable is around $\pi/2$ this is due to the fact that at 1 m/s the drag of the depressor is lower than the drag of the cable and so the value of the tension at the cable-depressor connection is closed to the weight of the depressor. For the other values of towing speed this consideration is not valid because the angle is smaller than $\pi/2$ given different proportion between the drag and the lift of the depressor.

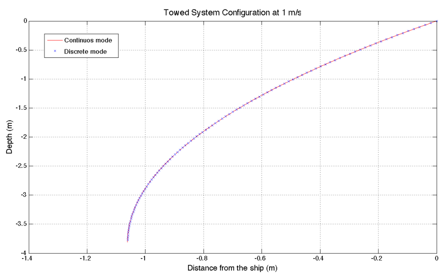

Figure 9: Comparison between continuos and discrete mathematical model at 1m/s

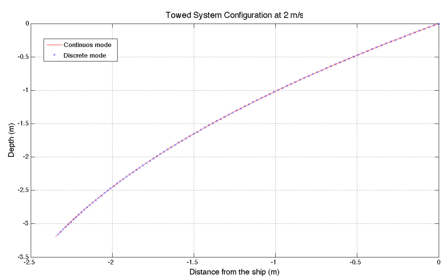

Figure 10: Comparison between continuos and discrete mathematical model at 2m/s

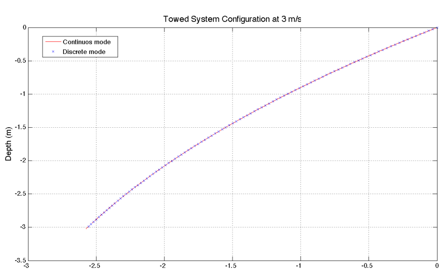

Figure 11: Comparison between continuos and discrete mathematical model at 3m/s

With this final results it is demonstrated that the effort and cost of numerically integrating the continuous model do not compare favorably with the relative ease and efficiency of solving the discrete model, which yields perfectly the same results of the continuous one. Obviously, this is due to the fact that the cable has been discretize in n=1000 element and off course the accuracy of the result depends on the increasing number of elements.

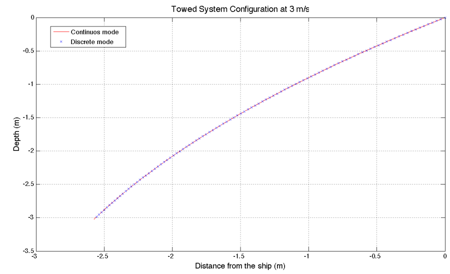

Figure 12: Comparison between continuos and discrete mathematical model at 4m/s

5. Conclusion

In this study the 2D stationary configuration of an underwater towed system has been developed by experimental trials. The system under analysis is composed of a towed inextensible cable, with no bending stiffness, and a depressor that is fixed at the cable free end. This configuration is currently used for underwater remotely-operated vehicle. Two different mathematical models have been compare at different towing speeds: the continuos and discrete model. The purpose of this work is to show, by experimental validation that configurations of this system can be computed effectively and accurately by discretizing the towing cable rather than using a continuous modeling approach. The results of the models have been validating using experimental trials. A continuous model has been present, which is based on geometric compatibility relations, equilibrium equation. A set of nonlinear differential equations has been derived and solved using Runge-Kutta fourth order iterative procedure. A discrete rod model has been also propose to determinate the cable shape, which is based on a system of nonlinear algebraic equations that are solved numerically. This two models are both suitable for analyzing an underwater towed system having a known initial values, top tension and inclination angle, obtained from experiments. Also the values of the reached depth has been experimentally evaluated by using pressure sensors inside the depressor in order to validate the mathematical models.The experimental trials done in a towing tank basin (CNR-INSEAN), have been illustrated. In order to compute the initial values to solve both the mathematical models the top tension and the inclination angle have been measured respectively with a load cell and an inclinometer. In the last part of the paper the results have been illustrated. The experimental values of top tension and angle at the attack towing point are listed in Table 3. After inserting the experimental initial values into the mathematical models, the configuration of the cable has been plotted at varying speed for both the mathematical models. In this part of the study the dependency between the towing speed and the reached depth from the entire system has been discuss. Both the models have been validated by comparing the experimental and numerical values of the depth that seem to be in good accordance presenting a percentage difference around 2%. Finally the comparison between the models for each towing has been done. So it is demonstrated that the discrete model presents the same results of the continuos models (difference lower than 1%), so the effort and cost of numerically integrating the continuous model do not compare favorably with the relative ease and efficiency of solving the discrete model.

References

[1] H.O. Berteaux *Buoy Engineering*, Wiley, New York, 1976.

[2] M.S. Triantafyllou *Preliminary design of mooring systems*, Journal of Ship Res. 26, 1982, pp 25-35.

[3] A. Culla, A. Carcaterra *Statistical linearization of cable equations: an application to a guyed tower offshore structure*, Hydroelasticity in Marine Technology, 1998.

[4] A. Culla, A. Carcaterra *Statistical moments predictions for a moored floating body oscillating in random waves*, Journal of Sound and Vibration 308, 2007, pp 44-66

[5] F.S. Hover, M.A. Grosenbaugh, M.S. Triantafyllou *Calculation of Dynamic Motions and Tension in Towed Underwater Cables*, Journal of Oceanic Engineering, vol.19, NO.3, July 1994, p. 449.

[6] E.E. Zajac *Dynamics and Kinematics of the Laying and Recovery of Submarine Cable*, 1954.

[7] H.M. Irvine *Dynamics of a Suspended Cables*, MIT Press, 1981.

[8] Y.I. Choo, M.J. Casarella *A Survey of Analytical Methods for Dynamic Simulation of Cable-Body System*, Journal of Hydronautics, Oct 1973.

[9] R.L. Webster *Non Linear Static and Dynamic Response of Underwater Cable Structures Using the Finite Element Method*, Offshore Technology Conference, 1975.

[10] H.I. Park, D.H. Jung, W. Koterayama *A numerical and experimental study on dynamics of a towed low tension cable*, Applied Ocean Research 25, 2003, pp 289-299.

[11] T. Huang, S. Chucheepsakul *Large Displacement Analysis of a Marine Riser*

[12] N. Yang, D.S. Jeng, X.L. Zhou *Tension Analysis of Submarine Cables During Laying Operations*, The Open Civil Engineering Journal, 2013, 7, pp 282-291.

[13] M.I. Friswell *Steady-State Analysis of Underwater Cables*, Journal of Waterway Port Coastal and Ocean Engineering, March 1995.

[14] M.D Masciola, M. Nahon, F.R. Driscoli *Static Analysis Of The Lumped Mass Cable Model Using A Shooting Algorithm*, ASCE Journal of Waterway, Port, Coastal, and Ocean Engineering, 2011.

[15] T.P Dreyer, D.M. Murray *A pre-elimination approach to the modelling of segmented representations of cable ?shape with application to the positioning of a towed sensor*, Appl. Math. Modelling 9, 1985, pp 163-169.

[16] T.P Dreyer, J.H. Van Vuuren *A comparison between continuous and discrete modelling of cables with bending stiffness*, Applied Mathematical Modelling 23, 1999, pp 527-541.