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Bessel processes and hyperbolic Brownian motions stopped at different random times

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Abstract

Iterated Bessel processes $R^\gamma(t), t > 0, \gamma > 0$ and their counterparts on hyperbolic spaces, i.e. hyperbolic Brownian motions $B^{hp}(t), t > 0$ are examined and their probability laws derived. The higher-order partial differential equations governing the distributions of $I_R(t) = R_1^{\gamma_1}(R_2^{\gamma_2}(t)), t > 0$ and $J_R(t) = R_1^{\gamma_1}(R_2^{\gamma_2}(t)^2), t > 0$ are obtained and discussed. Processes of the form $R^\gamma(T_t), t > 0, B^{hp}(T_t)$, $t > 0$ where $T_t = \inf\{s \geq 0 : B(s) = t\}$ are examined and numerous probability laws derived, including the Student law, the arcsine laws (also their asymmetric versions), the Lamperti distribution of the ratio of independent positively skewed stable random variables and others. For the random variable $R^\gamma(T_t^\mu), t > 0$ (where $T_t^\mu = \inf\{s \geq 0 : B^\mu(s) = t\}$ and B^μ is a Brownian motion with drift μ), the explicit probability law and the governing equation are obtained. For the hyperbolic Brownian motions on the Poincaré half-spaces H_2^+, H_3^+ (of respective dimensions 2, 3) we study $B^{hp}(T_t), t > 0$ and the corresponding governing equation. Iterated processes are useful in modelling motions of particles on fractures idealized as Bessel processes (in Euclidean spaces) or as hyperbolic Brownian motions (in non-Euclidean spaces).

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1. Introduction

The analysis of the composition of different types of stochastic processes has recently received a certain attention with the publication of a series of papers (see for example [1–3,12]).

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The prototype of these compound processes is the iterated Brownian motion whose investigation was started in the middle of the 90s (see [6,7]). Beside the distributional properties of the compound processes much work was done in order to derive the equations governing their probability laws. It was found that these processes are related both to fractional equations and to higher-order equations as is the case of iterated Brownian motion (see [1,11,32]). Iterated processes $X(T(t)),\ t > 0$ have been considered to model the wear of equipment because the inner process, $T(t)$, represents the effective time during the $(0, t)$ interval where a machine is working while $X(t)$ (external process) represents the wear to which the equipment is submitted (see e.g. [27]). A similar interpretation of subordinated processes has been suggested by several authors in an economic context (see [27]).

The core of this paper considers Bessel processes stopped at different random times. We first study for all $t > 0$ the random variable $I_R(t) = R_1^{\gamma_1}(R_2^{\gamma_2}(t))$ where $R_1^{\gamma_1}$, $R_2^{\gamma_2}$ are independent Bessel processes with parameters γ_1, γ_2. This is equivalent to studying a Bessel process at a random time which is represented by an independent Bessel process. Iterated processes of this form have proved to be suitable for describing the motion of gas particles in cracks (or fractures). For the iterated Brownian motion this is considered in the papers by Burdzy and Khoshnevisan [7] and DeBlassie [11] but a similar interpretation can be given to iterated processes obtained by composing Bessel processes (this is the case here) or fractional Brownian motions (see [12]). The r.v. $I_R(t)$ can be related to the motion of a Brownian particle on a fractal medium possessing a Brownian structure. The law of $I_R(t)$ for a fixed time $t > 0$ is expressed in terms of Fox functions (see, for example [28,29]) and possesses a Mellin transform equal to

$$E\left(I_R(t)\right)^{\eta-1} = (8t)^{\frac{\eta-1}{4}} \frac{\Gamma\left(\frac{\gamma_1+\eta-1}{2}\right)\Gamma\left(\frac{2\gamma_2+\eta-1}{4}\right)}{\Gamma\left(\frac{\gamma_1}{2}\right)\Gamma\left(\frac{\gamma_2}{2}\right)}.$$

We are able to derive the PDE satisfied by the pdf q of $I_R(t)$ for a fixed $t > 0$ which reads

$$\begin{aligned}\frac{\partial q}{\partial t}(r,t) &= \frac{1}{8}\left(\frac{\partial^2}{\partial r^2} - \frac{\gamma_1 - 1}{r}\frac{\partial}{\partial r} + \frac{\gamma_1 - 1}{r^2}\right)\\ &\quad \times \left(\frac{\partial^2}{\partial r^2} - \frac{(\gamma_1 + 2\gamma_2 - 3)}{r}\frac{\partial}{\partial r} + \frac{(\gamma_1 - 1)(2\gamma_2 - 1)}{r^2}\right) q(r,t). \end{aligned} \tag{1.1}$$

In the case $\gamma_1 = \gamma_2 = 1$, Eq. (1.1) reduces to the fourth-order equation

$$\frac{\partial q}{\partial t} = \frac{1}{8}\frac{\partial^4 q}{\partial r^4}, \quad r, t > 0 \tag{1.2}$$

obtained by Funaki in [13] where the space variable r varies on the whole real line. Equations of the form (1.2) like

$$\frac{\partial u}{\partial t}(x,t) = \kappa_N \frac{\partial^N u}{\partial x^N}(x,t), \quad x \in \mathbb{R},\ t > 0,\ N \geq 2$$

where

$$\kappa_N = \begin{cases} (-1)^{N/2+1} & \text{if } N \text{ is even} \\ \pm 1 & \text{if } N \text{ is odd} \end{cases}$$

emerge in the study of pseudo-processes analyzed in several papers; see [21] where the arcsine law of the sojourn time of the even-order pseudo-processes has been obtained. More recently, Lachal [22–24] and Cammarota and Lachal [8], have obtained general results on the maximum and hitting location of pseudo-processes of any order generalizing and extending (employing different approaches) special results that appeared in the literature (for example in [4,31]). A related process, considered in Section 4, is $J_R(t) = R_1^{\gamma_1}(R_2^{\gamma_2}(t)^2)$ where $R_1^{\gamma_1}, R_2^{\gamma_2}$ are independent Bessel processes starting at the origin. The probability density q of $J_R(t)$ for a fixed $t > 0$ can be expressed in closed form as

$$q(r,t) = \Pr\{J_R(t) \in \mathrm{d}r\}/\mathrm{d}r = \frac{2^{2-\frac{\gamma_1+\gamma_2}{2}}}{\Gamma\left(\frac{\gamma_1}{2}\right)\Gamma\left(\frac{\gamma_2}{2}\right)} \frac{r^{\gamma_1-1}}{t^{\gamma_2/2}} K_0\left(\frac{r}{\sqrt{t}}\right), \quad r,t > 0 \tag{1.3}$$

where K_0 is the modified Bessel function of order zero. The equation satisfied by (1.3) has the form

$$\frac{\partial q}{\partial t} = -\frac{r}{2}\frac{\partial^3 q}{\partial r^3}(r,t) + \frac{(\gamma_1+\gamma_2-4)}{2}\frac{\partial^2 q}{\partial r^2}(r,t) - \frac{(\gamma_1-1)(\gamma_2-1)}{2}\frac{\partial}{\partial r}\frac{q(r,t)}{r}$$

and includes the equations governing the process $|B_1(B_2(t)^2)|,\ t > 0$, for $\gamma_1 = \gamma_2 = 1$ (and coincides with 3.16 of [12] for $H = 1/2$) where B_1 and B_2 are independent Brownian motions.

Interesting results can be obtained by considering the Bessel process $R^\gamma(t), t > 0$ stopped at the first-passage time process $T_t, t > 0$ of an independent Brownian motion. Processes stopped at different types of random times can be viewed as processes with a new clock which is regulated by an independent Brownian motion B. The r.v. $T_t = \inf\{s \geq 0 : B(s) = t\}$ tells the time at which the Bessel process must be examined. This means that the clock considered below is timed by an independent Brownian motion. Therefore $R^\gamma(T_t), t > 0$ represents a motion where accelerations and decelerations of time occur randomly and continuously. We show that the pdf of $R^\gamma(T_t)$ for a fixed $t > 0$ reads

$$\Pr\{R^\gamma(T_t) \in \mathrm{d}r\}/\mathrm{d}r = \frac{2}{\sqrt{\pi}} \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)} \frac{t r^{\gamma-1}}{(r^2+t^2)^{\frac{\gamma+1}{2}}}, \quad r,t > 0.$$

Bessel processes $R^\gamma(T_t^\mu), t > 0$ stopped at first-passage time $T_t^\mu = \inf\{s \geq 0 : B^\mu(s) = t\}$ where B^μ is a Brownian motion with drift μ are examined in Section 3. In particular we prove that when $\gamma > 0$ and $\mu > 0$

$$\Pr\left\{R^\gamma(T_t^\mu) \in \mathrm{d}r\right\}/\mathrm{d}r = \frac{4t\, \mathrm{e}^{\mu t} r^{\gamma-1}}{2^{\frac{\gamma}{2}}\Gamma\left(\frac{\gamma}{2}\right)\sqrt{2\pi}} \left(\frac{\mu^2}{r^2+t^2}\right)^{\frac{\gamma+1}{4}} K_{\frac{\gamma+1}{2}}\left(\mu\sqrt{r^2+t^2}\right)$$

for $r > 0, t > 0$.

The last section is devoted to compositions involving the hyperbolic Brownian motion, that is a diffusion on the Poincaré upper half-space

$$H_n^+ = \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_n > 0\}$$

with particular attention to the planar case H_2^+ and the three-dimensional Poincaré half-space H_3^+. The hyperbolic Brownian motion in H_2^+ was introduced by Gertsenshtein and Vasiliev [14] and in H_3^+ by Karpelevich et al. [20]. Applications and extensions of Hyperbolic Brownian motion can be found in [25,17,34] and [30] where a scholar review on this topic is presented. In the space H_2^+ we study the hyperbolic distance from the origin of a hyperbolic Brownian motion stopped at the first-passage time process $T_t,\ t > 0$ of the standard Brownian motion whose probability law can be explicitly written for $\eta > 0, t > 0$ as

$$p_{J_2}(\eta, t) = \frac{\sinh \eta}{\pi \sqrt{2^3}} \int_\eta^\infty \frac{\varphi}{\sqrt{\cosh \varphi - \cosh \eta}} \frac{t}{t^2 + \varphi^2} K_2 \left(\frac{1}{2} \sqrt{t^2 + \varphi^2} \right) \mathrm{d}\varphi. \tag{1.4}$$

In H_3^+ the corresponding distribution of $J_3(t), t > 0$ reads

$$p_{J_3}(\eta, t) = \frac{2\sqrt{2}}{\pi} \frac{\eta t \sinh \eta}{(\eta^2 + t^2)} K_2 \left(\sqrt{\eta^2 + 2t^2} \right), \quad \eta > 0, \ t > 0. \tag{1.5}$$

The equations governing (1.4) and (1.5) are respectively

$$-\frac{\partial^2 p_{J_2}}{\partial t^2} = \frac{\partial^2 p_{J_2}}{\partial \eta^2} - \frac{\partial}{\partial \eta} \left(\coth \eta p_{J_2} \right), \quad \eta > 0, \ t > 0$$
$$-\frac{\partial^2 p_{J_3}}{\partial t^2} = \frac{\partial^2 p_{J_3}}{\partial \eta^2} - 2\frac{\partial}{\partial \eta} \left(\coth \eta p_{J_3} \right), \quad \eta > 0, \ t > 0.$$

The hyperbolic distance of a hyperbolic Brownian motion plays in the non-Euclidean spaces $H_n^+, \ n = 2, 3, \ldots$ the same role of Bessel processes in the Euclidean spaces. The structure of the probability law of two-dimensional hyperbolic Brownian motion is rather complicated (see formulas (6.4) and (6.5)) and therefore we have restricted ourselves only to compositions involving first-passage times. Much more flexibility is allowed by three-dimensional hyperbolic Brownian motion. Millson formula (see [16]), in principle, permits us to examine compositions of higher-dimensional hyperbolic Brownian motions stopped at random times.

2. Notations

For the convenience of the reader we list here some of the symbols appearing in the paper:

- $B(t), \ t > 0$ is a Brownian motion
- $B_\mu(t), \ t > 0$ is a Brownian motion with drift $\mu \in \mathbb{R}$,
- $R^\gamma(t), t > 0$ is a Bessel process of parameter $\gamma > 0$,
- $T_t = \inf\{s \geq 0 : B(s) = t\}$ is the first-passage time of B,
- $T_t^\mu = \inf\{s \geq 0 : B_\mu(s) = t\}$ is the first-passage time of B_μ,
- $B_n^{hp}(t), t > 0$ is a hyperbolic Brownian motion on the n-dimensional space H_n^+,
- $S^\gamma(t) = R^\gamma(t)^2, t > 0$ is a squared Bessel process,
- $\mathcal{A}_r^\gamma, (\mathcal{A}_r^\gamma)^*$ are the infinitesimal generator of R^γ and its adjoint,
- $\tilde{\mathcal{A}}_r^\gamma, (\tilde{\mathcal{A}}_r^\gamma)^*$ are the infinitesimal generator of S^γ and its adjoint,
- $\mathcal{H}_n$ is the generator of $B_n^{hp}(t), t > 0$.

3. Compound processes and PDEs

Let $X_1(t), t > 0$ and $X_2(t), t > 0$ be two independent stochastic processes such that $X_1(t) \in \mathbb{R}$ and $X_2(t) \geq 0$ for all $t > 0$ and set $p_1(x, s) = \Pr\{X_1(s) \in \mathrm{d}x\}/\mathrm{d}x, s > 0, x \in \mathbb{R}$ and $p_2(s, t) = \Pr\{X_2(t) \in \mathrm{d}s\}/\mathrm{d}s$. Let us introduce the compound process $I_X(t) = X_1(X_2(t)), t > 0$ and set $q(x, t) = \Pr\{I_X(t) \in \mathrm{d}x\}/\mathrm{d}x$. We readily have

$$q(x, t) = \int_0^\infty p_1(x, s) p_2(s, t) \, \mathrm{d}s, \quad x \in \mathbb{R}, \ t > 0.$$

Note that in $p_1(x, s)$, x stands for the space variable and s for the time variable related to X_1 while in $p_2(s, t)$, s stands for the space variable and t for the time variable related to X_2. The main result for the compound process I_X is the following one.

Theorem 3.1. *Assume that p_2 satisfies the PDE*

$$\frac{\partial p_2}{\partial t}(s,t) = \frac{\partial^2}{\partial s^2}\left(a_2(s)p_2(s,t)\right) - \frac{\partial}{\partial s}\left(a_1(s)p_2(s,t)\right) + a_0(s)p_2(s,t) \tag{3.1}$$

where a_0, a_1, a_2 are some functions and that p_1 satisfies the PDEs

$$\begin{aligned} \mathcal{D}_{0,x}p_1(x,s) &= a_0(s)p_1(x,s), \\ \mathcal{D}_{1,x}p_1(x,s) &= a_1(s)\frac{\partial p_1}{\partial s}(x,s), \\ \mathcal{D}_{2,x}p_1(x,s) &= a_2(s)\frac{\partial^2 p_1}{\partial s^2}(x,s), \end{aligned} \tag{3.2}$$

where $\mathcal{D}_{0,x}, \mathcal{D}_{1,x}, \mathcal{D}_{2,x}$ are some differential operators related to the variable x. Moreover assume that

$$\lim_{s\to+\infty} p_1(x,s) = \lim_{s\to+\infty} p_2(s,t) = 0, \quad \text{for any } x \in \mathbb{R},\ t > 0. \tag{3.3}$$

Then q satisfies the PDE

$$\frac{\partial q}{\partial t}(x,t) = \mathcal{D}_x q(x,t) + c(x,t), \quad x \in \mathbb{R},\ t > 0 \tag{3.4}$$

where $\mathcal{D}_x$ is the differential operator $\mathcal{D}_x = \mathcal{D}_{0,x} + \mathcal{D}_{1,x} + \mathcal{D}_{2,x}$ and c is the function (or possibly the distribution)

$$c(x,t) = \lim_{s\to 0^+}\left[a_1(s)p_1(x,s)p_2(s,t) + a_2(s)p_2(s,t)\frac{\partial p_1}{\partial s}(x,s) - p_1(x,s)\frac{\partial}{\partial s}\left(a_2(s)p_2(s,t)\right)\right].$$

Proof. We have by (3.1), integration by parts and by assuming that the following interchanges of derivatives and integrals are valid

$$\begin{aligned} \frac{\partial q}{\partial t}(x,t) &= \int_0^\infty p_1(x,s)\frac{\partial p_2}{\partial t}(s,t)\mathrm{d}s \\ &= \int_0^\infty p_1(x,s)\frac{\partial^2}{\partial s^2}\left(a_2(s)p_2(s,t)\right)\mathrm{d}s - \int_0^\infty p_1(x,s)\frac{\partial}{\partial s}\left(a_1(s)p_2(s,t)\right)\mathrm{d}s \\ &\quad + \int_0^\infty p_1(x,s)a_0(s)p_2(s,t)\mathrm{d}s \\ &= \left[p_1(x,s)\frac{\partial}{\partial s}\left(a_2(s)p_2(s,t)\right) - a_2(s)p_2(s,t)\frac{\partial p_1}{\partial s}(x,s)\right]_{s=0}^{s=\infty} \\ &\quad + \int_0^\infty a_2(s)p_2(s,t)\frac{\partial^2 p_1}{\partial s^2}(x,s)\mathrm{d}s - \left[a_1(s)p_1(x,s)p_2(s,t)\right]_{s=0}^{s=\infty} \\ &\quad + \int_0^\infty a_1(s)p_2(s,t)\frac{\partial p_1}{\partial s}(x,s)\mathrm{d}s + \int_0^\infty a_0(s)p_1(x,s)p_2(s,t)\mathrm{d}s. \end{aligned}$$

From conditions (3.2) and (3.3) we get

$$\begin{aligned}\frac{\partial q}{\partial t}(x,t) &= c(x,t) + \int_0^\infty p_2(s,t)\bigg[a_2(s)\frac{\partial^2 p_1}{\partial s^2}(x,s) \\ &\quad + a_1(s)\frac{\partial p_1}{\partial s}(x,s) + a_0(s)p_1(x,s)\bigg]\mathrm{d}s \\ &= c(x,t) + \int_0^\infty p_2(s,t)\mathcal{D}_x p_1(x,s)\mathrm{d}s \\ &= c(x,t) + \mathcal{D}_x q(x,t)\end{aligned}$$

which proves (3.4). □

Example. If X_2 is a non-negative diffusion process with infinitesimal generator $\mathcal{A}_{1,s} = a_2(s)\frac{\partial^2}{\partial s^2} + a_1(s)\frac{\partial}{\partial s} + a_0(s)$ (where s is the space variable) then the pdf $p_2(s,t)$ satisfies the Fokker–Planck equation $\frac{\partial p_2}{\partial t}(s,t) = \mathcal{A}^*_{1,s}p_2(s,t)$. Moreover, if the functions a_0, a_1, a_2 are constant and if X_1 is a diffusion process with infinitesimal generator $\mathcal{A}_{2,x}$ (a second-order differential operator where x is the space variable), then conditions (3.2) are fulfilled with $\mathcal{D}_{0,x} = a_0, \mathcal{D}_{1,x} = a_1\mathcal{A}_{2,x}$, $\mathcal{D}_{2,x} = a_2\mathcal{A}^2_{2,x}$ and then $\mathcal{D}_x$ is the fourth-order differential operator $a_2\mathcal{A}^2_{2,x} + a_1\mathcal{A}_{2,x} + a_0$.

4. Composition of Bessel processes with different types of processes

4.1. Preliminaries

Let $R^\gamma(t), t > 0$ be a Bessel process of parameter $\gamma > 0$ starting at 0 and set $p^\gamma(r,t) = \Pr\{R^\gamma(t) \in \mathrm{d}r\}/\mathrm{d}r$ where r is the space variable. This process is a non-negative diffusion process with infinitesimal generator $\mathcal{A}^\gamma_r = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{\gamma-1}{r}\frac{\partial}{\partial r}\right)$. The pdf $p^\gamma(r,t)$ solves the Fokker–Planck equation

$$\frac{\partial p^\gamma}{\partial t}(r,t) = \left(\mathcal{A}^\gamma_r\right)^* p^\gamma(r,t) = \frac{1}{2}\left[\frac{\partial^2 p^\gamma}{\partial r^2}(r,t) - (\gamma-1)\frac{\partial}{\partial r}\left(\frac{p^\gamma(r,t)}{r}\right)\right], \quad r,t>0$$

with the initial condition $p^\gamma(r,0) = \delta(r)$. The pdf $p^\gamma(r,t)$ admits the following explicit expression

$$p^\gamma(r,t) = r^{\gamma-1}k^\gamma(r,t), \quad r,t>0$$

where $k^\gamma(r,t) = \frac{2}{\Gamma(\gamma/2)}\frac{\mathrm{e}^{-\frac{r^2}{2t}}}{(2t)^{\gamma/2}}$ is the heat kernel of the differential operator $\mathcal{A}^\gamma_r$

$$\frac{\partial k^\gamma}{\partial t}(r,t) = \mathcal{A}^\gamma_r k^\gamma(r,t).$$

An important feature of the kernel k^γ for our analysis is the relationship

$$\frac{\partial k^\gamma}{\partial r}(r,t) = -\frac{r}{t}k^\gamma(r,t).$$

The Mellin transform of the function $r \mapsto p^\gamma(r,t)$ is given by

$$\mathcal{M}[p^\gamma(\cdot,t)](\eta) = \int_0^\infty r^{\eta-1}p^\gamma(r,t)\mathrm{d}r = \frac{\Gamma\left(\frac{\gamma+\eta-1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)}(2t)^{\frac{\eta-1}{2}}, \quad \Re\{\eta\} > 1-\gamma. \tag{4.1}$$

Concerning the squared Bessel process $S(t) = R^{\gamma}(t)^2$, this process is a non-negative diffusion process satisfying the stochastic differential equation

$$\mathrm{d}S^{\gamma}(t) = \gamma \mathrm{d}t + 2\sqrt{S^{\gamma}(t)}\mathrm{d}B(t).$$

The corresponding infinitesimal generator is $\tilde{\mathcal{A}}_r^{\gamma} = 2r\frac{\partial^2}{\partial r^2} + \gamma\frac{\partial}{\partial r}$ and the pdf $\tilde{p}^{\gamma}(r,t) = \Pr\{S^{\gamma}(t)\mathrm{d}r\}/\mathrm{d}r$ solves the Fokker–Planck equation

$$\frac{\partial \tilde{p}^{\gamma}}{\partial t}(r,t) = \left(\tilde{\mathcal{A}}_r^{\gamma}\right)^* \tilde{p}^{\gamma}(r,t) = 2\frac{\partial^2}{\partial r^2}\left(r\tilde{p}^{\gamma}(r,t)\right) - \gamma\frac{\partial \tilde{p}^{\gamma}}{\partial r}(r,t), \quad r,t > 0.$$

The pdf $\tilde{p}^{\gamma}(r,t)$ admits the following expression

$$\tilde{p}^{\gamma}(r,t) = \frac{p^{\gamma}(\sqrt{r},t)}{2\sqrt{r}} = \frac{1}{\Gamma(\gamma/2)}\frac{r^{\gamma/2-1}\mathrm{e}^{-\frac{r}{2t}}}{(2t)^{\gamma/2}}, \quad r,t > 0.$$

On the other hand, we introduce the first-passage time process $T_t,\ t > 0$ of a Brownian motion $B(t), t > 0$ starting at 0: $T_t = \inf\{s \geq 0 : B(s) = t\}$. Set

$$f(s,t) = \Pr\{T_t \in \mathrm{d}s\}/\mathrm{d}s = \frac{t\,\mathrm{e}^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}}, \quad s,t > 0.$$

The pdf $f(s,t)$ is a solution of the heat equation

$$\frac{\partial^2 f}{\partial t^2}(s,t) = 2\frac{\partial f}{\partial s}(s,t), \quad s,t > 0 \tag{4.2}$$

and the Laplace transform of $f(s,t)$ with respect to s is

$$\int_0^{\infty} \mathrm{e}^{-\lambda s} f(s,t)\mathrm{d}s = \exp\{-t\sqrt{2\lambda}\}, \quad \lambda > 0. \tag{4.3}$$

We shall also consider the first-passage time process $T_t^{\mu},\ t > 0$ of a Brownian motion $B^{\mu}(t)$, $t > 0$ with drift μ starting at 0. Set

$$f_{\mu}(s,t) = \Pr\{T_t^{\mu} \in \mathrm{d}s\}/\mathrm{d}s = \frac{t\,\mathrm{e}^{-\frac{(t-\mu s)^2}{2s}}}{\sqrt{2\pi s^3}}, \quad s,t > 0.$$

The pdf $f_{\mu}(s,t)$ is a solution of the PDE

$$\frac{\partial^2 f_{\mu}}{\partial t^2}(s,t) - 2\mu\frac{\partial f_{\mu}}{\partial t}(s,t) = 2\frac{\partial f_{\mu}}{\partial s}(s,t), \quad s,t > 0. \tag{4.4}$$

4.2. *The iterated Bessel process*

We consider here the iterated Bessel process

$$I_R(t) = R_1^{\gamma}(R_2^{\gamma}(t)), \quad t > 0$$

where R_1^{γ} and R_2^{γ} are two independent Bessel processes of parameters γ_1 and γ_2 starting at 0. The pdf of $I_R(t)$ for a fixed time $t > 0$ is

$$q(r,t) = \Pr\{I_R(t) \in \mathrm{d}r\}/\mathrm{d}r = \frac{4}{\Gamma(\gamma_1/2)\Gamma(\gamma_2/2)}\int_0^{\infty} \frac{r^{\gamma_1-1}\mathrm{e}^{-\frac{r^2}{2s}}}{(2s)^{\gamma_1/2}}\frac{s^{\gamma_2-1}\mathrm{e}^{-\frac{s^2}{2t}}}{(2t)^{\gamma_2/3}}\mathrm{d}s.$$

The function q can be expressed by means of Fox functions. Recall that Fox functions are defined as

$$H^{m,n}_{p,q}\left[x\left|\begin{matrix}(a_i,\alpha_i)_{i=1,\dots,p}\\(b_j,\beta_j)_{j=1,\dots,q}\end{matrix}\right.\right]=\frac{1}{2\pi \mathrm{i}}\int_{\theta-\mathrm{i}\infty}^{\theta+\mathrm{i}\infty}\mathcal{M}^{m,n}_{p,q}\left[\eta\left|\begin{matrix}(a_i,\alpha_i)_{i=1,\dots,p}\\(b_j,\beta_j)_{j=1,\dots,q}\end{matrix}\right.\right]x^{-\eta}\mathrm{d}\eta$$

where $\theta\in\mathbb{R}$ and

$$\mathcal{M}^{m,n}_{p,q}\left[\eta\left|\begin{matrix}(a_i,\alpha_i)_{i=1,\dots,p}\\(b_j,\beta_j)_{j=1,\dots,q}\end{matrix}\right.\right]=\frac{\prod_{j=1}^{m}\Gamma(b_j+\eta\beta_j)\prod_{i=1}^{n}\Gamma(1-a_i-\eta\alpha_i)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\eta\beta_j)\prod_{i=n+1}^{p}\Gamma(a_i+\eta\alpha_i)}.$$

For a profound analysis of the Fox function and related topics, see [29]. This function is characterized by its Mellin transform

$$\int_0^\infty x^{\eta-1}H^{m,n}_{p,q}\left[x\left|\begin{matrix}(a_i,\alpha_i)_{i=1,\dots,p}\\(b_j,\beta_j)_{j=1,\dots,q}\end{matrix}\right.\right]\mathrm{d}x=\mathcal{M}^{m,n}_{p,q}\left[\eta\left|\begin{matrix}(a_i,\alpha_i)_{i=1,\dots,p}\\(b_j,\beta_j)_{j=1,\dots,q}\end{matrix}\right.\right].$$

Recall also the property valid for any $c\in\mathbb{R}$

$$H^{m,n}_{p,q}\left[x\left|\begin{matrix}(a_i,\alpha_i)_{i=1,\dots,p}\\(b_j,\beta_j)_{j=1,\dots,q}\end{matrix}\right.\right]=\frac{1}{x^c}H^{m,n}_{p,q}\left[x\left|\begin{matrix}(a_i+c\alpha_i,\alpha_i)_{i=1,\dots,p}\\(b_j+c\beta_j,\beta_j)_{j=1,\dots,q}\end{matrix}\right.\right],\quad c\in\mathbb{R}. \tag{4.5}$$

Theorem 4.1. *The pdf of $I_R(t)$ is given by*

$$q(r,t)=\frac{1}{(8t)^{1/4}}H^{2,0}_{2,2}\left[\frac{r}{(8t)^{1/4}}\left|\begin{matrix}\left(\frac{\gamma_1}{2},0\right); & \left(\frac{\gamma_2}{2},0\right)\\\left(\frac{\gamma_1}{2}-\frac{1}{2},\frac{1}{2}\right); & \left(\frac{\gamma_2}{2}-\frac{1}{4},\frac{1}{4}\right)\end{matrix}\right.\right],\quad r,t>0 \tag{4.6}$$

$$=\frac{1}{r(8t)^{1/4}}H^{2,0}_{2,2}\left[\frac{r}{(8t)^{1/4}}\left|\begin{matrix}\left(\frac{\gamma_1}{2},0\right); & \left(\frac{\gamma_2}{2},0\right)\\\left(\frac{\gamma_1}{2},\frac{1}{2}\right); & \left(\frac{\gamma_2}{2},\frac{1}{4}\right)\end{matrix}\right.\right],\quad r,t>0. \tag{4.7}$$

Proof. Let us compute the Mellin transform of the function $r\mapsto q(r,t)$. By (4.1),

$$\begin{aligned}\mathcal{M}[q(\cdot,t)](\eta)&=\int_0^\infty r^{\eta-1}q(r,t)\mathrm{d}r=\int_0^\infty p^{\gamma_2}(s,t)\left[\int_0^\infty r^{\eta-1}p^{\gamma_1}(r,s)\mathrm{d}r\right]\mathrm{d}s\\&=2^{\frac{\eta-1}{2}}\frac{\Gamma\left(\frac{\eta+\gamma_1-1}{2}\right)}{\Gamma\left(\frac{\gamma_1}{2}\right)}\int_0^\infty s^{\frac{\eta+1}{2}-1}p^{\gamma_2}(s,t)\mathrm{d}s\\&=\frac{\Gamma\left(\frac{\eta+\gamma_1-1}{2}\right)\Gamma\left(\frac{\eta+2\gamma_2-1}{4}\right)}{\Gamma\left(\frac{\gamma_1}{2}\right)\Gamma\left(\frac{\gamma_2}{2}\right)}(8t)^{\frac{\eta-1}{4}}\end{aligned}$$

where $\Re\{\eta\}>\max\{1-\gamma_1,1-2\gamma_2\}$. Observing that

$$\mathcal{M}^{2,0}_{2,2}\left[\eta\left|\begin{matrix}(a_1,\alpha_1); & (a_2,\alpha_2)\\(b_1,\beta_1); & (b_2,\beta_2)\end{matrix}\right.\right]=\frac{\Gamma(b_1+\beta_1\eta)\Gamma(b_2+\beta_2\eta)}{\Gamma(a_1+\alpha_1\eta)\Gamma(a_2+\alpha_2\eta)},$$

we deduce that

$$\mathcal{M}[q(\cdot,t)](\eta) = (8t)^{\frac{\eta-1}{4}} \mathcal{M}_{2,2}^{2,0}\left[\eta \,\middle|\, \begin{matrix} (\gamma_1/2, 0); & (\gamma_2/2, 0) \\ (\gamma_1/2 - 1/2, 1/2); & (\gamma_2/2 - 1/4, 1/2) \end{matrix}\right]$$

from which we immediately extract (4.6). Formula (4.7) comes from (4.6) and (4.5) with $c = 1$. □

Theorem 4.2. *Fix $t > 0$. The pdf $q(r,t)$ of the random variable $I_R(t)$ is a solution to the PDE*

$$\frac{\partial q}{\partial t}(r,t) = \frac{1}{8}\left(\frac{\partial^2}{\partial r^2} - \frac{\gamma_1 - 1}{r}\frac{\partial}{\partial r} + \frac{\gamma_1 - 1}{r^2}\right) \times \left(\frac{\partial^2}{\partial r^2} - \frac{(\gamma_1 + 2\gamma_2 - 3)}{r}\frac{\partial}{\partial r} + \frac{(\gamma_1 - 1)(2\gamma_2 - 1)}{r^2}\right) q(r,t) \tag{4.8}$$

which can also be written as

$$\frac{\partial q}{\partial t}(r,t) = \frac{1}{8}\left(\frac{\partial^2}{\partial r^2} - (\gamma_1 - 1)\frac{\partial}{\partial r}\frac{1}{r}\right) \times \left(\frac{\partial^2}{\partial r^2} - (\gamma_1 + 2\gamma_2 - 3)\frac{\partial}{\partial r}\frac{1}{r} + \frac{2(\gamma_1 - 2)(\gamma_2 - 1)}{r^2}\right) q(r,t).$$

Proof. According to the settings of Theorem 3.1, we have

$$\frac{\partial p^{\gamma_2}}{\partial t}(s,t) = \frac{\partial^2}{\partial s^2}\left(\frac{1}{2}p^{\gamma_2}(s,t)\right) - \frac{\partial}{\partial s}\left(\frac{\gamma_2 - 1}{2s}p^{\gamma_2}(s,t)\right),$$

we then see that (3.1) is fulfilled with $a_0(s) = 0, a_1(s) = \frac{\gamma_2 - 1}{2s}, a_2(s) = \frac{1}{2}$. Next, in view of (3.2), we have $\mathcal{D}_{0,r} = 0, \mathcal{D}_{1,r} = \frac{1}{2}\left(\mathcal{A}_r^{\gamma_2 *}\right)^2$ and

$$\begin{aligned} a_1(s)\frac{\partial p^{\gamma_1}}{\partial s}(r,s) &= \frac{\gamma_2 - 1}{2s} r^{\gamma_1 - 1}\frac{\partial k^{\gamma_1}}{\partial s}(r,s) = \frac{\gamma_2 - 1}{2s} r^{\gamma_1 - 1}\mathcal{A}_r^{\gamma_1} k^{\gamma_1}(r,s) \\ &= -\frac{\gamma_2 - 1}{2} r^{\gamma_1 - 1}\mathcal{A}_r^{\gamma_1}\left(\frac{1}{r}\frac{\partial k^{\gamma_1}}{\partial r}(r,s)\right) \\ &= -\frac{\gamma_2 - 1}{2} r^{\gamma_1 - 1}\mathcal{A}_r^{\gamma_1}\left(\frac{1}{r}\frac{\partial}{\partial r}\frac{p^{\gamma_1}(r,s)}{r^{\gamma_1 - 1}}\right). \end{aligned}$$

As a byproduct, we have obtained that

$$a_1(s)\frac{\partial p^{\gamma_1}}{\partial s}(r,s) = \mathcal{D}_{1,r}\, p^{\gamma_1}(r,s)$$

with $\mathcal{D}_{1,r} = -\frac{\gamma_2 - 1}{2} r^{\gamma - 1}\mathcal{A}_r^{\gamma_1}\left(\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r^{\gamma_1 - 1}}\right)$. By Theorem 3.1, we get $\frac{\partial q}{\partial t}(r,t) = \mathcal{D}_r q(r,t) + c(r,t)$ where clearly $c(r,t) = 0$ for $r, t > 0$ and

$$\begin{aligned} \mathcal{D}_r &= \mathcal{D}_{0,r} + \mathcal{D}_{1,r} + \mathcal{D}_{2,r} \\ &= \frac{1}{2}\left(\mathcal{A}_r^{\gamma_1 *}\right)^2 - \frac{\gamma_2 - 1}{2} r^{\gamma_1 - 1}\mathcal{A}_r^{\gamma_1}\left(\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r^{\gamma_1 - 1}}\right) \\ &= \frac{1}{8}\left(\frac{\partial^2}{\partial r^2} - (\gamma_1 - 1)\frac{\partial}{\partial r}\frac{1}{r}\right)^2 - \frac{\gamma_2 - 1}{4} r^{\gamma_1 - 1}\left(\frac{\partial^2}{\partial r^2} + \frac{\gamma_1 - 1}{r}\frac{\partial}{\partial r}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r^{\gamma_1 - 1}}\right). \end{aligned}$$

An elementary computation shows that

$$r^{\gamma_1-1}\left(\frac{\partial^2}{\partial r^2}+\frac{\gamma_1-1}{r}\frac{\partial}{\partial r}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r^{\gamma_1-1}}\right)=\left(\frac{\partial^2}{\partial r^2}-(\gamma_1-1)\frac{\partial}{\partial r}\frac{1}{r}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}-\frac{\gamma_1-1}{r^2}\right)$$

and then

$$\begin{aligned}\mathcal{D}_r &= \frac{1}{8}\left(\frac{\partial^2}{\partial r^2}-(\gamma_1-1)\frac{\partial}{\partial r}\frac{1}{r}\right)\\ &\quad\times\left(\frac{\partial^2}{\partial r^2}-(\gamma_1-1)\frac{\partial}{\partial r}\frac{1}{r}-\frac{2(\gamma_2-1)}{r}\frac{\partial}{\partial r}+\frac{2(\gamma_1-1)(\gamma_2-1)}{r^2}\right)\\ &= \frac{1}{8}\left(\frac{\partial^2}{\partial r^2}-(\gamma_1-1)\frac{\partial}{\partial r}\frac{1}{r}\right)\left(\frac{\partial^2}{\partial r^2}-\frac{\gamma_1+2\gamma_2-3}{r}\frac{\partial}{\partial r}+\frac{(\gamma_1-1)(2\gamma_2-1)}{r^2}\right)\end{aligned}$$

which proves Theorem 3.1. □

Remark 4.1. For $\gamma_1=\gamma_2=1$ Eq. (4.8) becomes

$$\frac{\partial q}{\partial t}=\frac{1}{8}\frac{\partial^4 q}{\partial r^4},\quad r,t>0. \tag{4.9}$$

Funaki (see [13]) has proved that the artificial process

$$Z^{(1)}(t)=\begin{cases}B_1(B_2(t)), & B_2(t)\ge 0\\ \mathrm{i}B_1(-B_2(t)), & B_2(t)\le 0\end{cases}\quad t>0$$

has a law satisfying Eq. (4.9). Hochberg and Orsingher [18] have shown that

$$Z^{(2)}(t)=\begin{cases}B_1(\mathrm{i}B_2(t)), & B_2(t)>0\\ \mathrm{i}B_1(-\mathrm{i}B_2(t)), & B_2(t)<0\end{cases}\quad t>0$$

instead has the law

$$q(r,t)=\int_{-\infty}^{\infty}\frac{e^{-\frac{r^2}{2\mathrm{i}s}}}{\sqrt{2\pi\mathrm{i}s}}\frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}}\mathrm{d}s \tag{4.10}$$

satisfying

$$\frac{\partial p}{\partial t}=-\frac{1}{8}\frac{\partial^4 p}{\partial r^4},\quad r\in\mathbb{R},\ t>0.$$

In [5], formula (4.10) is rewritten as

$$q(r,t)=\frac{1}{\sqrt{2\pi}}E\left(\frac{1}{|B(t)|^{1/2}}\cos\left(\frac{r^2}{2|B(t)|}-\frac{\pi}{4}\right)\right)$$

as can be ascertained by suitable manipulations. The pdf of the iterated Brownian motion $B_1(|B_2(t)|), t>0$ satisfies the fourth-order equation

$$\frac{\partial q}{\partial t}=\frac{1}{8}\frac{\partial^4 q}{\partial r^4}+\frac{1}{2\sqrt{2\pi t}}\frac{\partial^2\delta}{\partial r^2},\quad r\in\mathbb{R},\ t>0$$

(see [11,1]) and the fractional equation

$$\frac{\partial^{\frac{1}{2}} q}{\partial t^{\frac{1}{2}}}=\frac{1}{2^{\frac{3}{2}}}\frac{\partial^2 q}{\partial r^2},\quad r\in\mathbb{R},\ t>0$$

as shown in [32].

4.3. Another iterated process

Let us consider now the iterated process

$$J_R(t) = R_1^{\gamma_1}(R_2^{\gamma_2}(t)^2)$$

where $R_1^{\gamma_1}$ and $R_2^{\gamma_2}$ are two independent Bessel processes with parameters γ_1 and γ_2 starting at 0. The pdf of $J_R(t)$ for a fixed $t > 0$ is

$$q(r,t) = \Pr\{J_R(t) \in \mathrm{d}r\}/\mathrm{d}r = \frac{2^{1-\frac{\gamma_1+\gamma_2}{2}}}{\Gamma(\gamma_1/2)\Gamma(\gamma_2/2)} \frac{r^{\gamma_1-1}}{t^{\gamma_2/2}} \int_0^\infty \mathrm{e}^{-\frac{r^2}{2s}-\frac{s}{2t}} \frac{\mathrm{d}s}{s}.$$

This function can be expressed in terms of the modified Bessel function K_0. Indeed, recalling that

$$\int_0^\infty \exp\left(-as - \frac{b}{s}\right) \frac{\mathrm{d}s}{s} = 2K_0(2\sqrt{ab}),$$

we have the following representation.

Theorem 4.3. *The pdf of* $J_R(t)$ *is given by*

$$q(r,t) = \frac{1}{2^{\frac{\gamma_1+\gamma_2}{2}-2}\Gamma(\gamma_1/2)\Gamma(\gamma_2/2)} \frac{r^{\gamma_1-1}}{t^{\gamma_2/2}} K_0\left(\frac{r}{\sqrt{t}}\right), \quad r,t > 0.$$

Concerning the PDE satisfying q we have the following result.

Theorem 4.4. *Fix* $t > 0$. *The pdf* $q(r,t)$ *of the random variable* $J_R(t)$ *is a solution to the PDE*

$$\begin{aligned} \frac{\partial q}{\partial t} &= -\frac{r}{2}\frac{\partial^3 q}{\partial r^3}(r,t) + \frac{(\gamma_1+\gamma_2-4)}{2}\frac{\partial^2 q}{\partial r^2}(r,t) \\ &\quad -\frac{(\gamma_1-1)(\gamma_2-1)}{2}\frac{\partial}{\partial r}\frac{q(r,t)}{r}, \quad r,t > 0. \end{aligned} \tag{4.11}$$

Proof. In order to apply Theorem 3.1, we first observe that

$$\frac{\partial \tilde{p}^{\gamma_2}}{\partial t}(s,t) = \frac{\partial^2}{\partial s^2}\left(2s\tilde{p}^{\gamma_2}(s,t)\right) - \frac{\partial}{\partial s}\left(\gamma_2\tilde{p}^{\gamma_2}(s,t)\right).$$

Therefore, with the settings of Theorem 3.1, condition (3.1) is fulfilled with $a_0(s) = 0$, $a_1(s) = \gamma_2$, $a_2(s) = 2s$. Furthermore, in view of (3.2), we have $\mathcal{D}_{0,r} = 0$, $\mathcal{D}_{1,r} = \gamma_2\left(\tilde{\mathcal{A}}_r^{\gamma_1}\right)^*$ and

$$\begin{aligned} a_2(s)\frac{\partial^2 p^{\gamma_1}}{\partial s^2}(r,s) &= 2sr^{\gamma_1-1}\frac{\partial^2 k^{\gamma_1}}{\partial s^2}(r,s) = 2sr^{\gamma_1-1}\left(\mathcal{A}_r^{\gamma_1}\right)^2 k^{\gamma_1}(r,s) \\ &= sr^{\gamma_1-1}\mathcal{A}_r^{\gamma_1}\left(\frac{\partial}{\partial r} + \frac{\gamma_1-1}{r}\right)\frac{\partial k^{\gamma_1}}{\partial r}(r,s) \\ &= -r^{\gamma_1-1}\mathcal{A}_r^{\gamma_1}\left(\frac{\partial}{\partial r} + \frac{\gamma_1-1}{r}\right)\left(rk^{\gamma_1}(r,s)\right) \\ &= -r^{\gamma_1-1}\mathcal{A}_r^{\gamma_1}\left(\frac{\partial}{\partial r} + \frac{\gamma_1-1}{r}\right)\frac{p^{\gamma_1}(r,s)}{r^{\gamma_1-2}}. \end{aligned}$$

As a byproduct, we have obtained

$$a_2(s)\frac{\partial^2 p^{\gamma_1}}{\partial s^2}(r,s) = \mathcal{D}_{2,r}\, p^{\gamma_1}(r,s)$$

with $\mathcal{D}_{2,r} = -r^{\gamma_1-1}\mathcal{A}_r^{\gamma_1}\left(\frac{\partial}{\partial r}+\frac{\gamma_1-1}{r}\right)\frac{1}{r^{\gamma_1-2}}$. By Theorem 3.1, we get $\frac{\partial q}{\partial t}(r,t) = \mathcal{D}_r q(r,t) + c(r,t)$ with $c(r,t)=0$ for $r,t>0$ and

$$\begin{aligned}\mathcal{D}_r &= \mathcal{D}_{0,r}+\mathcal{D}_{1,r}+\mathcal{D}_{2,r}\\ &= -r^{\gamma_1-1}\mathcal{A}_r^{\gamma_1}\left(\frac{\partial}{\partial r}+\frac{\gamma_1-1}{r}\right)\frac{1}{r^{\gamma_1-1}}+\gamma_2\left(\tilde{\mathcal{A}}_r^{\gamma_1}\right)^*\\ &= -\frac{1}{2}r^{\gamma_1-1}\left(\frac{\partial^2}{\partial r^2}+\frac{\gamma_1-1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial}{\partial r}+\frac{\gamma_1-1}{r}\right)\frac{1}{r^{\gamma_1-2}}\\ &\quad+\frac{\gamma_2}{2}\left(\frac{\partial^2}{\partial r^2}-\frac{\gamma_1-1}{r}\frac{\partial}{\partial r}+\frac{\gamma_1-1}{r^2}\right).\end{aligned}$$

Straightforward calculations lead to

$$\mathcal{D}_r = -\frac{r}{2}\frac{\partial^3}{\partial r^3}+\frac{\gamma_1+\gamma_2-4}{2}\frac{\partial^2}{\partial r^2}-\frac{(\gamma_1-1)(\gamma_2-1)}{2r}\frac{\partial}{\partial r}+\frac{(\gamma_1-1)(\gamma_2-1)}{2r^2}$$

and the proof of Theorem 4.4 is finished. □

4.4. The Bessel process at first-passage times of a Brownian motion

4.4.1. Brownian motion without drift

Let $T_t = \inf\{s\ge 0 : B(s)=t\}$ where B is a Brownian motion independent from the Bessel process $R^\gamma(t)$, $t>0$ starting from zero. In this section we study the new process $R^\gamma(T_t)$, $t>0$ concentrating our attention on its law and some related distributions. Stopping the Bessel process R^γ at the random time T_t can cause either a slowing down (with respect to the natural time) or a speed up of the time flow. The probability of slowing down is measured by the following integral

$$\Pr\{T_t\le t\} = \int_0^t \frac{t\,\mathrm{e}^{-\frac{t^2}{2x}}}{\sqrt{2\pi x^3}}\mathrm{d}x = \frac{1}{\sqrt{\pi}}\int_{t/2}^\infty \frac{\mathrm{e}^{-u}}{\sqrt{u}}\mathrm{d}u$$

which clearly decreases for all t. Furthermore, we observe that

$$\Pr\{T_t\le t\} = \sqrt{\frac{2}{\pi t}}\int_{t/2}^\infty \mathrm{e}^{-u}\mathrm{d}u \le \sqrt{\frac{2}{\pi t}}\mathrm{e}^{-\frac{t}{2}}$$

and this confirms the asymptotic speed up of the time flow implied by the subordinator T_t, $t>0$. We have now the explicit distribution of $R^\gamma(T_t)$ for a fixed $t>0$.

Theorem 4.5. *Fix $t>0$. The pdf of $R^\gamma(T_t)$ reads*

$$q(r,t) = P\{R^\gamma(T_t)\in \mathrm{d}r\}/\mathrm{d}r = 2\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)}\frac{t r^{\gamma-1}}{\left(r^2+t^2\right)^{\frac{\gamma+1}{2}}},\quad r,t>0. \tag{4.12}$$

Proof. By the change of variable $s = (r^2 + t^2)/2w$ we have

$$\begin{aligned} q(r,t) &= \frac{2}{\Gamma\left(\frac{\gamma}{2}\right)} \int_0^\infty \frac{r^{\gamma-1} e^{-\frac{r^2}{2s}}}{(2s)^{\gamma/2}} \frac{t e^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds \\ &= \frac{2tr^{\gamma-1}}{2^{\frac{\gamma+1}{2}} \sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)} \int_0^\infty s^{-\frac{\gamma+3}{2}} e^{-\frac{1}{2s}(r^2+t^2)} ds \\ &= \frac{2tr^{\gamma-1}}{2^{\frac{\gamma+1}{2}} \sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)} \left(\frac{r^2+t^2}{2}\right)^{-\frac{\gamma+1}{2}} \int_0^\infty e^{-w} w^{(\gamma+1)/2-1} dw \\ &= \frac{2}{\sqrt{\pi}} \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)} \frac{tr^{\gamma-1}}{(r^2+t^2)^{\frac{\gamma+1}{2}}}, \quad r,t>0. \quad \square \end{aligned}$$

Remark 4.2. For $\gamma = n \in \mathbb{N}$ the process $R^n(T_t),\ t > 0$ can be represented as

$$R^n(T_t) = \sqrt{\sum_{j=1}^n B_j^2(T_t)}, \quad t > 0 \tag{4.13}$$

where $B_j(t), t > 0, j = 1, 2, \ldots, n$ are independent Brownian motions and the r.v.'s $B_j(T_t), t > 0$ possess a Cauchy distribution. Therefore (4.13) represents the Euclidean distance of an n-dimensional Cauchy random vector $(C_1(t), \ldots, C_n(t)), t > 0$.

Remark 4.3. We can obtain the μ-moments for $0 < \mu < 1$ of $R^\gamma(T_t)$. With the successive change of variable $r = t\sqrt{y}$ and $w = y/(1+y)$,

$$\begin{aligned} E\left[R^\gamma(T_t)^\mu\right] &= \int_0^\infty r^\mu \Pr\{R^\gamma(T_t) \in dr\} = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)} t^\mu \int_0^\infty \frac{y^{(\gamma+\mu)/2-1}}{(1+y)^{\frac{\gamma+1}{2}}} dy \\ &= \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)} t^\mu \int_0^1 w^{(\gamma+\mu)/2-1} (1-w)^{\frac{1-\mu}{2}-1} dw \\ &= \frac{\Gamma\left(\frac{\gamma+\mu}{2}\right)\Gamma\left(\frac{1-\mu}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)} t^\mu. \end{aligned}$$

We also note that for $\gamma = 1$, the pdf (4.12) coincides with a folded Cauchy law with scale parameter t and location parameter equal to zero.

Remark 4.4. For the distribution function of $R^\gamma(T_t), t > 0$ we have the following result

$$\begin{aligned} \Pr\{R^\gamma(T_t) > r\} &= \frac{2}{\sqrt{\pi}} \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)} t \int_r^\infty \frac{y^{\gamma-1}}{(t^2+y^2)^{\frac{\gamma+1}{2}}} dy \\ &= \frac{2}{\sqrt{\pi}} \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)} t \left[\frac{r^{\gamma-2}}{(\gamma-1)(t^2+r^2)^{\frac{\gamma-1}{2}}} + \frac{\gamma-2}{\gamma-1} \int_r^\infty \frac{y^{\gamma-3}}{(t^2+y^2)^{\frac{\gamma-1}{2}}} dy\right] \end{aligned} \tag{4.14}$$

for $\gamma > 1$. The recursive formula (4.14) yields some interesting particular cases

$$\Pr\{R^\gamma(T_t) > r\} = \begin{cases} \dfrac{t}{(t^2+r^2)^{1/2}}, & \gamma = 2, \\ \dfrac{4t}{\pi}\left[\dfrac{r}{t^2+r^2} + \dfrac{1}{2t}\left(\dfrac{\pi}{2} - \arctan\dfrac{r}{t}\right)\right], & \gamma = 3, \\ \dfrac{t}{(t^2+r^2)^{1/2}} + \dfrac{tr^2}{2(t^2+r^2)^{3/2}}, & \gamma = 4. \end{cases}$$

Remark 4.5. Another result related to distribution (4.12) states that

$$\Pr\left\{\frac{t^3}{t^2+R^\gamma(T_t)^2} \in \mathrm{d}r\right\}\Big/\mathrm{d}r = \frac{1}{B(\gamma/2, 1/2)}\frac{1}{t}\left(\frac{r}{t}\right)^{\frac{1}{2}-1}\left(1-\frac{r}{t}\right)^{\gamma/2-1} \tag{4.15}$$

for $0 < r < t, \gamma > 0$. To check this, it suffices to evaluate the following derivative

$$\Pr\left\{\frac{t^3}{t^2+R^\gamma(T_t)^2} \in \mathrm{d}r\right\}\Big/\mathrm{d}r = \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)} t \frac{\mathrm{d}}{\mathrm{d}r}\int_{\sqrt{\frac{t^3-rt^2}{r}}}^{\infty} \frac{y^{\gamma-1/2-1}}{(y^2+t^2)^{\frac{\gamma+1}{2}}}\mathrm{d}y.$$

For $\gamma = 1$ from (4.15) one obtains the law of sojourn time on $(0, \infty)$ of Brownian motion and the even-order pseudo-processes, while for odd values of γ the distribution of the sojourn time on the half-line for odd-order pseudo-processes emerges (for $\gamma = 3$ see [31], $\gamma = 2n+1, n > 2$ see [22]).

Remark 4.6. Set

$$S(t) = \frac{1}{R^\gamma(T_t)}, \quad \text{for } t > 0.$$

After some calculation we find that

$$P\{S(t) \in \mathrm{d}r\}/\mathrm{d}r = \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)}\frac{t}{(1+r^2t^2)^{\frac{\gamma+1}{2}}}, \quad r, t > 0. \tag{4.16}$$

We note that for $t = \frac{1}{\sqrt{n}}, \gamma = n$, density (4.16) coincides with a folded t-distribution with n degrees of freedom and its density takes the form

$$f(r; n) = \frac{2}{\sqrt{\pi n}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{\left(1+\frac{r^2}{n}\right)^{\frac{n+1}{2}}}, \quad r > 0. \tag{4.17}$$

For $n = 1$, density (4.17) coincides with a folded Cauchy and coincides with (4.12) for $\gamma = 1$ and at time $t = 1$.

Theorem 4.6. *Fix $\gamma > 0$ and $t > 0$. The pdf of $R^\gamma(T_t)$ is a solution to the following equation*

$$-\frac{\partial^2 q}{\partial t^2}(r,t) = \left(\frac{\partial^2}{\partial r^2} - (\gamma-1)\frac{\partial}{\partial r}\frac{1}{r}\right)q(r,t), \quad r, t > 0. \tag{4.18}$$

Proof. By differentiating the pdf $q(r,t) = \int_0^\infty p^\gamma(r,s) f(s,t)\mathrm{d}s$ twice with respect to t and using (4.2), we immediately get

$$\begin{aligned}\frac{\partial^2 q}{\partial t^2}(r,t) &= \int_0^\infty p^\gamma(r,s)\frac{\partial^2}{\partial t^2} f(s,t)\mathrm{d}s = 2\int_0^\infty p^\gamma(r,s)\frac{\partial f}{\partial s}(s,t)\mathrm{d}s \\ &= \left[2p^\gamma(r,s)f(s,t)\right]_{s=0}^{s=\infty} - 2\int_0^\infty \frac{\partial}{\partial s}p^\gamma(r,s)f(s,t)\mathrm{d}s \\ &= -\left(\frac{\partial^2}{\partial r^2} - (\gamma-1)\frac{\partial}{\partial r}\frac{1}{r}\right)q(r,t). \quad \square\end{aligned}$$

Remark 4.7. For $\gamma = 1$ the Bessel process coincides with the reflected Brownian motion so that $R^1(T_t),\ t > 0$ is a reflected Brownian motion stopped at the random time T_t and therefore becomes a folded Cauchy process. It is easy to prove that the Cauchy density

$$q(r,t) = \frac{t}{\pi(t^2+r^2)}, \quad r,t > 0$$

solves the Laplace equation and this agrees with (4.18).

By inverting the role of the Bessel process and that of the first-passage time we obtain a new process somehow related to $R^\gamma(T_t),\ t > 0$ which we denote by

$$T_{R^\gamma(t)} = \inf\{s \geq 0 : B(s) = R^\gamma(t)\}. \tag{4.19}$$

For $t > 0$, $T_{R^\gamma(t)}$ is the first instant where a Brownian motion B independent from R^γ attains the level $R^\gamma(t)$. The pdf of (4.19) is given by

$$\begin{aligned}\Pr\left\{T_{R^\gamma(t)} \in \mathrm{d}x\right\}/\mathrm{d}x &= \int_0^\infty \frac{s\mathrm{e}^{-\frac{s^2}{2x}}}{\sqrt{2\pi x^3}} \frac{2}{\Gamma\left(\frac{\gamma}{2}\right)} \frac{s^{\gamma-1}\mathrm{e}^{-\frac{s^2}{2t}}}{(2t)^\gamma/2}\mathrm{d}s \\ &= \frac{2}{2^{\frac{\gamma+1}{2}}\Gamma\left(\frac{\gamma}{2}\right)\sqrt{\pi}t^{\frac{\gamma}{2}}x^{\frac{3}{2}}} \int_0^\infty s^\gamma \mathrm{e}^{-\frac{s^2}{2}\left(\frac{1}{x}+\frac{1}{t}\right)}\mathrm{d}s \\ &= \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)} \frac{\sqrt{t}x^{\gamma/2-1}}{(x+t)^{\frac{\gamma+1}{2}}}, \quad x,t > 0,\ \gamma > 0.\end{aligned} \tag{4.20}$$

Remark 4.8. It can be easily checked that the following relationship holds

$$\sqrt{T_{R^\gamma(t^2)}} \overset{\text{i.d.}}{=} R^\gamma(T_t), \quad t > 0. \tag{4.21}$$

From (4.21) one can also infer that

$$T_{R^\gamma(t)} \overset{\text{i.d.}}{=} \left(R^\gamma(T_{\sqrt{t}})\right)^2, \quad t > 0. \tag{4.22}$$

In particular, for $\gamma = 1$, result (4.22) says that

$$T_{R^1(t)} \overset{\text{i.d.}}{=} \left(B(T_{\sqrt{t}})\right)^2 \overset{\text{i.d.}}{=} \left(C(\sqrt{t})\right)^2, \quad t > 0. \tag{4.23}$$

Remark 4.9. We note that the probability density (4.20) for $\gamma = 1, t = 1$ coincides with the ratio of two first-passage times through level 1 of two independent Brownian motions. In other words we have that

$$\Pr\left\{T_{R^1(1)} \in \mathrm{d}w\right\} = \Pr\left\{W_{1/2} \in \mathrm{d}w\right\} = \frac{1}{\pi}\frac{w^{-\frac{1}{2}}}{w+1}\mathrm{d}w, \quad w > 0 \tag{4.24}$$

where $W_{1/2} = T^1_{1/2}/T^2_{1/2}$ and $T^1_{1/2}, T^2_{1/2}$ are the first-passage times of B^1 and B^2 through level 1. Noticing that $T^1_{1/2}, T^2_{1/2}$ are stable r.v.'s of order 1/2, this statement is a special case of the following result. For stable positive, independent r.v.'s $T^1_\nu, T^2_\nu, 0 < \nu < 1$ with Laplace transform

$$E\mathrm{e}^{-\lambda T_\nu} = \mathrm{e}^{-\lambda^\nu}, \quad \lambda > 0 \tag{4.25}$$

the pdf of the ratio $W_\nu = T^1_\nu / T^2_\nu$ is given by the Lamperti law

$$\Pr\left\{W_\nu \in \mathrm{d}w\right\}/\mathrm{d}w = \frac{\sin \pi\nu}{\pi}\frac{w^{\nu-1}}{w^{2\nu} + 2w^\nu \cos\pi\nu + 1}, \quad w > 0 \tag{4.26}$$

(see e.g. [10,19,35]).

We now consider some subordinated processes involving the first-passage time of a Brownian motion with drift μ.

4.4.2. Brownian motion with drift

Let us consider the Bessel process $R^\gamma(T^\mu_t), t > 0$ subordinated by the first-passage time process $T^\mu_t, t > 0$ related to an independent Brownian motion $B^\mu(t), t > 0$ with drift μ, starting at 0. Let $q_\mu(r,t)$ denote the pdf of the random variable $R^\gamma(T^\mu_t)$: $q(r,t) = \Pr\{R^\gamma(T^\mu_t) \in \mathrm{d}r\}/\mathrm{d}r$.

Theorem 4.7. *Fix $t > 0$ and $\mu \in \mathbb{R}$. The pdf $q_\mu(r,t)$ admits the following form for $r > 0$*

$$q_\mu(r,t) = \frac{4t\,\mathrm{e}^{\mu t} r^{\gamma-1}}{2^{\frac{\gamma}{2}}\Gamma(\gamma/2)\sqrt{2\pi}}\left(\frac{\mu^2}{r^2+t^2}\right)^{\frac{\gamma+1}{4}} K_{\frac{\gamma+1}{2}}\left(|\mu|\sqrt{r^2+t^2}\right), \quad r, t > 0. \tag{4.27}$$

Proof. We have

$$\begin{aligned} q_\mu(r,t) &= \int_0^\infty \frac{2}{\Gamma(\gamma/2)}\frac{r^{\gamma-1}\mathrm{e}^{-\frac{r^2}{2s}}}{(2s)^{\frac{\gamma}{2}}}\frac{t\,\mathrm{e}^{-\frac{(t-\mu s)^2}{2s}}}{\sqrt{2\pi s^3}}\mathrm{d}s \\ &= \frac{2t\,\mathrm{e}^{\mu t} r^{\gamma-1}}{2^{\frac{\gamma}{2}}\Gamma(\gamma/2)\sqrt{2\pi}}\int_0^\infty s^{-\frac{\gamma+1}{2}-1}\mathrm{e}^{-\frac{r^2+t^2}{2s}-\frac{s\mu^2}{2}}\mathrm{d}s \\ &= \frac{4t\,\mathrm{e}^{\mu t} r^{\gamma-1}}{2^{\frac{\gamma}{2}}\Gamma(\gamma/2)\sqrt{2\pi}}\left(\frac{\mu^2}{r^2+t^2}\right)^{\frac{\gamma+1}{4}} K_{\frac{\gamma+1}{2}}\left(|\mu|\sqrt{r^2+t^2}\right). \end{aligned}$$

Result (4.27) emerges on applying the formula

$$\int_0^\infty x^{\nu-1}\exp\left\{-\beta x^p - \alpha x^{-p}\right\}\mathrm{d}x = \frac{2}{p}\left(\frac{\alpha}{\beta}\right)^{\frac{\nu}{2p}} K_{\frac{\nu}{p}}\left(2\sqrt{\alpha\beta}\right), \quad p, \alpha, \beta, \nu > 0 \tag{4.28}$$

where $\nu = (\gamma+1)/2, \alpha = \mu^2/2, \beta = (r^2+t^2)/2, p = 1$ and K_ν is the modified Bessel function (see [15, formula 3.478]). □

Remark 4.10. By applying the asymptotic formula for the modified Bessel function K_ν

$$K_\nu(x) \approx \frac{2^{\nu-1}\Gamma(\nu)}{x^\nu}, \quad \text{for } x \to 0^+$$

(see p. 929 of [15] and p. 136 of [26]) we have that

$$\begin{aligned} q_0(r,t) &= \frac{4}{2^{\frac{\gamma}{2}}\Gamma(\gamma/2)\sqrt{2\pi}} \frac{tr^{\gamma-1}}{(r^2+t^2)^{\frac{\gamma+1}{4}}} \frac{2^{\frac{\gamma+1}{2}}\Gamma((\gamma+1)/2)}{(r^2+t^2)^{\frac{\gamma+1}{4}}} \\ &= \frac{2\Gamma((\gamma+1)/2)}{\Gamma(\gamma/2)\sqrt{\pi}} \frac{tr^{\gamma-1}}{(r^2+t^2)^{\frac{\gamma+1}{2}}}, \quad r,t>0. \end{aligned}$$

The equation governing the distribution $q_\mu(r,t)$ is given in the next theorem.

Theorem 4.8. *Fix $\mu \ge 0$ and $t > 0$. The pdf of $R^\gamma(T_t^\mu)$ solves the following PDE*

$$\left(2\mu\frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2}\right) q_\mu(r,t) = \left(\frac{\partial^2}{\partial r^2} - (\gamma-1)\frac{\partial}{\partial r}\frac{1}{r}\right) q_\mu(r,t), \quad r,t>0. \tag{4.29}$$

Proof. We apply the differential operator $\frac{\partial^2}{\partial t^2} - 2\mu\frac{\partial}{\partial t}$ to the function

$$q_\mu(r,t) = \int_0^\infty p^\gamma(r,s) f_\mu(s,t)\mathrm{d}s, \quad r\ge 0,\ t>0.$$

We readily have in light of (4.4)

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - 2\mu\frac{\partial}{\partial t}\right) q_\mu(r,t) &= \int_0^\infty p^\gamma(r,s) 2\frac{\partial f_\mu}{\partial s}(s,t)\mathrm{d}s \\ &= -2\int_0^\infty \frac{\partial p^\gamma}{\partial s}(r,s) f_\mu(s,t)\mathrm{d}s \\ &= -\left(\frac{\partial^2}{\partial r^2} - (\gamma-1)\frac{\partial}{\partial r}\frac{1}{r}\right) q_\mu(r,t). \quad \square \end{aligned}$$

For $\mu = 0$ in (4.29) one retrieves Eq. (4.18).

5. Some generalized compositions

We somehow generalize the previous results. First consider the twice iterated Brownian first-passage time. By

$$I_T^2(t) = T^1_{T_t^2} = \inf\{s_1 \ge 0 : B^1(s_1) = \inf\{s_2 \ge 0 : B^2(s_2) = t\}\}, \quad t>0 \tag{5.1}$$

we mean a process which represents the first instant T^1 where a Brownian motion B^1 hits the level T_t^2 and T_t^2 represents the first instant where a Brownian motion B^2 (independent of B^1) hits level t. Clearly the pdf of the r.v. (5.1) is given by

$$\Pr\{I_T^2(t) \in \mathrm{d}x\}/\mathrm{d}x = \int_0^\infty \frac{s e^{-\frac{s^2}{2x}}}{\sqrt{2\pi x^3}} \frac{t e^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}}\mathrm{d}s. \tag{5.2}$$

Consider now the n-stage iterated Brownian first-passage time recursively defined by

$$I_T^n(t) = T^1_{I_T^{n-1}(t)}, \quad t > 0, \ n \geq 2$$

or explicitly by

$$I_T^n(t) = \inf\{s_1 \geq 0 : B^1(s_1) = \inf\{s_2 \geq 0 : B^2(s_2) = \cdots = \inf\{s_n \geq 0 : B(s_n) = t\}\ldots\}\}.$$

The corresponding pdf is expressed by

$$\begin{aligned} f^n(x,t) &= \Pr\{I^n_{T_t} \in \mathrm{d}x\}/\mathrm{d}x \\ &= \int_0^\infty \ldots \int_0^\infty \frac{s_1 \mathrm{e}^{-\frac{s_1^2}{2x}}}{\sqrt{2\pi x^3}} \frac{s_2 \mathrm{e}^{-\frac{s_2^2}{2s_1}}}{\sqrt{2\pi s_1^3}} \ldots \frac{t\mathrm{e}^{-\frac{t^2}{2s_{n-1}}}}{\sqrt{2\pi s_{n-1}^3}} \mathrm{d}s_1 \ldots \mathrm{d}s_{n-1}. \end{aligned} \tag{5.3}$$

We have the following theorem.

Theorem 5.1. *Fix $t > 0$. The pdf of $I_T^n(t)$ satisfies the following PDE*

$$\frac{\partial^{2^n} f^n}{\partial t^{2^n}}(x,t) = 2^{2^n-1} \frac{\partial f^n}{\partial x}(x,t) \tag{5.4}$$

and possesses the simple Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda x} f^n(x,t)\mathrm{d}x = \exp\left\{-2^{1-\frac{1}{2^n}} t \lambda^{\frac{1}{2^n}}\right\}. \tag{5.5}$$

Proof. In view of (4.2), by successive integration by parts we have that

$$\begin{aligned} \frac{\partial f^n}{\partial x}(x,t) &= \int_0^\infty \ldots \int_0^\infty \frac{\partial f}{\partial x}(x,s_1) f(s_1,s_2)\ldots f(s_{n-1},t)\mathrm{d}s_1\ldots\mathrm{d}s_{n-1} \\ &= \frac{1}{2}\int_0^\infty \ldots \int_0^\infty \frac{\partial^2 f}{\partial s_1^2}(x,s_1) f(s_1,s_2)\ldots f(s_{n-1},t)\mathrm{d}s_1\ldots\mathrm{d}s_{n-1} \\ &= \frac{1}{2}\int_0^\infty \ldots \int_0^\infty f(x,s_1)\frac{\partial^2 f}{\partial s_1^2}(s_1,s_2)\ldots f(s_{n-1},t)\mathrm{d}s_1\ldots\mathrm{d}s_{n-1} \\ &= \frac{1}{2^{1+2}}\int_0^\infty \ldots \int_0^\infty f(x,s_1)\frac{\partial^4 f}{\partial s_2^4}(s_1,s_2)\ldots f(s_{n-1},t)\mathrm{d}s_1\ldots\mathrm{d}s_{n-1}. \end{aligned}$$

By iterating this procedure, we finally get

$$\begin{aligned} \frac{\partial f^n}{\partial x}(x,t) &= \frac{1}{2^{1+2+\cdots+2^{n-2}}}\int_0^\infty \ldots \int_0^\infty f(x,s_1)f(s_1,s_2)\ldots \frac{\partial^{2^{n-1}} f}{\partial s_{n-1}^{2^{n-1}}}(s_{n-1},t)\mathrm{d}s_1\ldots ds_{n-1} \\ &= \frac{1}{2^{1+2+\cdots+2^{n-1}}}\int_0^\infty \ldots \int_0^\infty f(x,s_1)f(s_1,s_2)\ldots \frac{\partial^{2^n} f}{\partial t^{2^n}}(s_{n-1},t)\mathrm{d}s_1\ldots ds_{n-1} \\ &= \frac{1}{2^{2^n-1}}\frac{\partial^{2^n} f^n}{\partial t^{2^n}}(x,t). \end{aligned}$$

Concerning the Laplace transform of f^n, we successively have by (4.3)

$$\begin{aligned}
&\int_0^\infty e^{-\lambda x} f^n(x,t)\mathrm{d}x \\
&= \int_0^\infty \dots \int_0^\infty \left[\int_0^\infty e^{-\lambda x} f(x,s_1)\mathrm{d}x\right] f(s_1,s_2)\dots f(s_{n-1},t)\mathrm{d}s_1\dots\mathrm{d}s_{n-1} \\
&= \int_0^\infty \dots \int_0^\infty \left[\int_0^\infty e^{-2^{\frac{1}{2}}\lambda^{\frac{1}{2}} s_1} f(s_1,s_2)\mathrm{d}s_1\right] f(s_2,s_3)\dots f(s_{n-1},t)\mathrm{d}s_2\dots\mathrm{d}s_{n-1} \\
&= \int_0^\infty \dots \int_0^\infty \left[\int_0^\infty e^{-2^{\frac{1}{2}+\frac{1}{4}}\lambda^{\frac{1}{4}} s_2} f(s_2,s_3)\mathrm{d}s_2\right] f(s_3,s_4)\dots f(s_{n-1},t)\mathrm{d}s_3\dots\mathrm{d}s_{n-1}.
\end{aligned}$$

By iterating this procedure we get

$$\begin{aligned}
\int_0^\infty e^{-\lambda x} f^n(x,t)\mathrm{d}x &= \int_0^\infty e^{-2^{\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^{n-1}}}\lambda^{\frac{1}{2^{n-1}}} s_{n-1}} f(s_{n-1},t)\mathrm{d}s_{n-1} \\
&= \exp\left\{-2^{\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n}}\lambda^{\frac{1}{2^n}} t\right\} = \exp\left\{-2^{1-\frac{1}{2^n}}\lambda^{\frac{1}{2^n}} t\right\}. \quad \square
\end{aligned}$$

Remark 5.1. If we take the Laplace transform of Eq. (5.4) we get that for $\mathcal{L}(\lambda,t) = \int_0^\infty e^{-\lambda x} f^n(x,t)\mathrm{d}x = E\left[e^{-\lambda I_T^n(t)}\right]$

$$\frac{\partial^{2^n}\mathcal{L}}{\partial t^{2^n}}(\lambda,t) = 2^{2^n-1}\lambda\mathcal{L}(t,\lambda). \tag{5.6}$$

It is straightforward to realize that (5.5) satisfies Eq. (5.6).

Remark 5.2. If we consider the generalization of (4.12), that is

$$R^\gamma(I_T^n(t)), \quad t>0$$

the corresponding probability law satisfies the 2^n-th order equation

$$-\frac{\partial^{2^n} q}{\partial t^{2^n}} = 2^{2^n-2}\left(\frac{\partial^2}{\partial r^2} - (\gamma-1)\frac{\partial}{\partial r}\frac{1}{r}\right)q, \quad r>0,\ t>0.$$

Remark 5.3. The following shows that there is a strict connection between the iterated first-passage time I_T^n and the iterated Brownian motion. Indeed distribution (5.2) can be written, for $n \geq 2$, as

$$\Pr\{I_T^n(t)\in \mathrm{d}x\}/\mathrm{d}x = \frac{t}{2^{n-1}x}\Pr\{B^n(|\dots|B^1(x)|\dots|)\in \mathrm{d}t\}/\mathrm{d}t, \quad x>0,\ t>0.$$

6. Compositions of hyperbolic Brownian motions on the Poincaré half-space

6.1. Case of dimension 2

We consider the classical model of hyperbolic space represented by the Poincaré half-space $H_2^+ = \{(x,y)\in\mathbb{R}^2 : y>0\}$ with the metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}.$$

The hyperbolic Brownian motion $B_2^{hp}(t), t > 0$ on H_2^+ is the diffusion process with generator $\mathcal{H}_2$ defined as

$$\mathcal{H}_2 = \frac{y^2}{2}\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}$$

and its transition function is the solution to the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{H}_2 u, \quad x \in \mathbb{R},\ y > 0 \\ u(x, y, 0) = \delta(y-1)\delta(x). \end{cases} \tag{6.1}$$

It is convenient to study the hyperbolic Brownian motion in terms of hyperbolic coordinates (η, α) where η is the hyperbolic distance of (x, y) from the origin $(0, 1)$ of H_2^+. In explicit terms (η, α) and (x, y) are related by

$$\cosh\eta = \frac{x^2 + y^2 + 1}{2y}, \qquad \tan\alpha = \frac{x^2 + y^2 - 1}{2x}.$$

Conversely the formulas transforming (x, y) into (η, α) are

$$\begin{cases} x = \frac{\sinh\eta\cos\alpha}{\cos\eta - \sinh\eta\sin\alpha}, & \eta > 0 \\ y = \frac{1}{\cosh\eta - \sinh\eta\sin\alpha}, & -\frac{\pi}{2} < \alpha < \frac{\pi}{2}. \end{cases}$$

Some details on these formulas can be found in [9,17,25,33]. The Cauchy problem (6.1) can be converted into hyperbolic coordinates as follows

$$\frac{\partial u}{\partial t} = \frac{1}{2}\left[\frac{1}{\sinh\eta}\frac{\partial}{\partial\eta}\left(\sinh\eta\frac{\partial}{\partial\eta}\right)u + \frac{1}{\sinh^2\eta}\frac{\partial^2 u}{\partial\alpha^2}\right], \quad \eta > 0,\ t > 0 \tag{6.2}$$

subject to the initial condition

$$u(\eta, \alpha, 0) = \delta(\eta) \quad \text{for all } \alpha \in [0, 2\pi).$$

If we concentrate on the distribution of the hyperbolic distance of the Brownian motion particle from the origin we disregard the dependence in (6.2) from α and study

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{1}{\sinh\eta}\frac{\partial}{\partial\eta}\left(\sinh\eta\frac{\partial}{\partial\eta}\right)u \\ u(\eta, 0) = \delta(\eta). \end{cases} \tag{6.3}$$

It is well known that the solution to (6.3) has the following form

$$k_2(\eta, t) = \frac{e^{-\frac{t}{8}}}{\sqrt{\pi t^3}}\int_\eta^\infty \frac{\varphi\, e^{-\frac{\varphi^2}{2t}}}{\sqrt{\cosh\varphi - \cosh\eta}}d\varphi, \quad \eta > 0,\ t > 0. \tag{6.4}$$

Remark 6.1. If we pass from problem (6.3) to

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{\sinh\eta}\frac{\partial}{\partial\eta}\left(\sinh\eta\frac{\partial}{\partial\eta}\right)u \\ u(\eta, 0) = \delta(\eta) \end{cases}$$

(by means of the change of variable $t' = t/2$) we obtain a somewhat different distribution which reads

$$k_2(\eta, t') = \frac{e^{-\frac{t'}{4}}}{\sqrt{\pi (2t')^3}} \int_\eta^\infty \frac{\varphi\, e^{-\frac{\varphi^2}{4t'}}}{\sqrt{\cosh\varphi - \cosh\eta}} d\varphi, \quad \eta > 0,\ t > 0. \tag{6.5}$$

In the first paper Gertsenshtein and Vasiliev [14] (and also in the subsequent literature) the factor 1/2 does not appear and the heat kernel is (6.5) (up to some constant). A detailed derivation of (6.4) and (6.5) is given in [25].

It is well known that the pdf of the distance from 0 of the hyperbolic Brownian motion at a fixed time $t > 0$ is given by

$$p_2(\eta, t) = \sinh\eta\, k_2(\eta, t) = \frac{\sinh\eta\, e^{-\frac{t}{8}}}{\sqrt{\pi t^3}} \int_\eta^\infty \frac{\varphi\, e^{-\frac{\varphi^2}{2t}}}{\sqrt{\cosh\varphi - \cosh\eta}} d\varphi, \quad \eta > 0,\ t > 0. \tag{6.6}$$

This function solves the adjoint equation

$$\frac{\partial p_2}{\partial t} = \frac{1}{2}\left\{ \frac{\partial^2 p_2}{\partial \eta^2} - \frac{\partial}{\partial \eta}(\coth\eta\, p_2) \right\}, \quad \eta > 0,\ t > 0. \tag{6.7}$$

For distribution (6.6) further characterizations are possible. Indeed, we can rewrite $p_2(\eta, t)$ as follows

$$p_2(\eta, t) = \sqrt{2} e^{-\frac{t}{8}} \int_\eta^\infty \frac{\sinh\eta}{\sqrt{\cosh\varphi - \cosh\eta}} g(\varphi, t) d\varphi$$

where $g(\varphi, t) = \Pr\{T_\varphi \in dt\}/dt$ is the pdf of $T_\varphi = \inf\{t > 0 : B(t) = \varphi\}$. Moreover,

$$p_2(\eta, t) = \frac{e^{-\frac{t}{8}}}{\sqrt{2\pi t^3}} E\left\{ \mathbb{I}_{[R^2(t) > \eta]} \frac{\sinh\eta}{\sqrt{\cosh R^2(t) - \cosh\eta}} \right\}, \quad \eta > 0,\ t > 0$$

where $R^2(t), t > 0$ is the two-dimensional Bessel process described above and the mean-value is taken with respect to the distribution of Bessel process $R^2(t), t > 0$.

We give now an alternative form of $p_2(\eta, t)$ in terms of the Euclidean distance. Indeed, the distribution of $B_2^{hp}(t), t > 0$ in H_2^+ can be written as

$$\begin{aligned} p_2(\eta, t) &= -2 \frac{e^{-\frac{t}{8}}}{\sqrt{\pi t}} \frac{d}{d\eta} \int_\eta^\infty \frac{\varphi\, e^{-\frac{\varphi^2}{2t}}}{t} \sqrt{\cosh\varphi - \cosh\eta}\, d\varphi \\ &= -2 \frac{e^{-\frac{t}{8}}}{\sqrt{\pi t}} \frac{d}{d\eta} E\left\{ \mathbb{I}_{[R^2(t) > \eta]} \sqrt{\cosh R^2(t) - \cosh\eta} \right\}, \quad \eta > 0,\ t > 0. \end{aligned} \tag{6.8}$$

If we take a Euclidean right triangle with one cathetus of length $\sqrt{\cosh\eta}$ and hypotenuse $\sqrt{\cosh\varphi}$, then $\sqrt{\cosh\varphi - \cosh\eta}$ represents the length of the second cathetus. Therefore the integrals above represent the length of the second cathetus weighted by means of the probability distribution of the Bessel process in the plane. Thus, formula (6.8) highlights the relation between the distribution of the hyperbolic distance in H_2^+ and the corresponding Euclidean distance in $\mathbb{R}^2$. We can recognize the additional factor of (6.8) as a gamma distribution with parameters 1/2, 1/8 (up to some normalizing constant).

We now examine the hyperbolic Brownian motion $B_2^{hp}(t), t > 0$ stopped at the first-passage time $T_t, t > 0$ of an independent standard Brownian motion B defined as $T_t = \inf\{s \geq 0 : B(s) = t\}$. In other words we study the process

$$J_2(t) = B_2^{hp}(T_t), \quad t > 0. \tag{6.9}$$

The pdf of $J_2(t)$ for a fixed $t >$ is given for $\eta > 0$ by

$$\begin{aligned} p_{J_2}(\eta, t) &= \int_0^\infty p_2(\eta, s) \frac{t\, e^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} \mathrm{d}s \\ &= \frac{t \sinh \eta}{\sqrt{2\pi}} \int_0^\infty \int_\eta^\infty \frac{\varphi\, e^{-\frac{\varphi^2+t^2}{2s}}}{\sqrt{\cosh \varphi - \cosh \eta}} \frac{e^{-\frac{s}{8}}}{s^3} \mathrm{d}s \mathrm{d}\varphi \\ &= \frac{t \sinh \eta}{2\sqrt{2\pi}} \int_\eta^\infty \frac{\varphi\, \mathrm{d}\varphi}{\sqrt{\cosh \varphi - \cosh \eta}} \frac{1}{(\varphi^2 + t^2)} K_2\left(\frac{1}{2}\sqrt{\varphi^2 + t^2}\right) \end{aligned} \tag{6.10}$$

where we have used formula (4.28). In analogy with representation (6.8) we can give the following expression for the distribution of hyperbolic Brownian motion stopped at T_t, for $\eta > 0, t > 0$

$$p_{J_2}(\eta, t) = -\frac{1}{2\sqrt{2}} \frac{\mathrm{d}}{\mathrm{d}\eta} E\left\{\mathbb{I}_{[C(t)>\eta]} C(t) \sqrt{\cosh C(t) - \cosh \eta}\, K_2\left(\frac{1}{2}\sqrt{(C(t))^2 + t^2}\right)\right\}$$

where $C(t), t > 0$ is a Cauchy process. For distribution (6.10) we can state the following result.

Theorem 6.1. *The pdf of $J_2(t)$ solves the following Cauchy problem*

$$-\frac{\partial^2 p_{J_2}}{\partial t^2} = \left(\frac{\partial^2}{\partial \eta^2} - \frac{\partial}{\partial \eta} \coth \eta\right) p_{J_2}, \quad p_{J_2}(\eta, 0) = \delta(\eta), \ \eta, t > 0.$$

Proof. The distribution of (6.9) can be written as

$$p_{J_2}(\eta, t) = \int_0^\infty p_2(\eta, s) f(s, t) \mathrm{d}s.$$

In view of (6.7) we have therefore that

$$\begin{aligned} -\frac{\partial^2 p_{J_2}}{\partial t^2}(\eta, t) &= -\int_0^\infty p_2(\eta, s) 2 \frac{\partial f}{\partial s}(s, t) \mathrm{d}s \\ &= 2\int_0^\infty f(s, t) \frac{\partial p_2}{\partial s}(\eta, s) \mathrm{d}s = \frac{\partial^2 p_{J_2}}{\partial \eta^2}(\eta, t) - \frac{\partial}{\partial \eta}\left(\coth \eta\, p_{J_2}(\eta, t)\right). \quad \square \end{aligned}$$

6.2. *Case of dimension 3*

The hyperbolic Brownian motion on $H_3^+ = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ is the diffusion with the generator

$$\mathcal{H}_3 = z^2 \left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right\} - z\frac{\partial}{\partial z}.$$

The distribution of the hyperbolic distance of Brownian motion in H_3^+ possesses the form

$$p_3(\eta,t)=\sinh^2\eta k_3(\eta,t)=\frac{\sinh\eta\,\mathrm{e}^{-t}}{2\sqrt{\pi t^3}}\eta\,\mathrm{e}^{-\frac{\eta^2}{4t}},\quad \eta>0,\ t>0 \tag{6.11}$$

where $k_3(\eta,t)$ is the kernel

$$k_3(\eta,t)=\frac{\mathrm{e}^{-t}}{2\sqrt{\pi t^3}}\frac{\eta\,\mathrm{e}^{-\frac{\eta^2}{4t}}}{\sinh\eta},\quad \eta>0,\ t>0.$$

The function k_3 is the solution to

$$\begin{cases}\dfrac{\partial u}{\partial t}=\dfrac{1}{\sinh^2\eta}\dfrac{\partial}{\partial\eta}\left(\sinh^2\eta\dfrac{\partial}{\partial\eta}\right)u\\ u(\eta,0)=\delta(\eta).\end{cases} \tag{6.12}$$

Remark 6.2. By means of the transformation $t=t'/2$, Eq. (6.12) is converted into

$$\frac{\partial u}{\partial t}(\eta,t)=\frac{1}{2}\frac{1}{\sinh^2\eta}\frac{\partial}{\partial\eta}\left(\sinh^2\eta\frac{\partial}{\partial\eta}\right)u(\eta,t)$$

and formula (6.11) leads to the different pdf

$$p_3(\eta,t)=2\frac{\sinh\eta\mathrm{e}^{-\frac{t}{2}}}{\sqrt{2\pi t^3}}\eta\mathrm{e}^{-\frac{\eta^2}{2t}},\quad \eta>0,\ t>0. \tag{6.13}$$

Distribution (6.11) solves the PDE

$$\frac{\partial p_3}{\partial t}(\eta,t)=\frac{\partial^2 p_3}{\partial\eta^2}-2\frac{\partial}{\partial\eta}\left(\coth\eta p_3(\eta,t)\right)$$

which involves the adjoint of the operator appearing in (6.12).

We now consider the process $J_3(t),t>0$ obtained by composing the three-dimensional hyperbolic Brownian motion $B_3^{hp}(t),t>0$ with $T_t=\inf\{s\geq 0: B(s)=t\}$ where B is a Brownian motion independent of B_3^{hp}. The pdf of $J_3(t)$ is equal to

$$\begin{aligned}p_{J_3}(\eta,t)&=\int_0^\infty p_3(\eta,s)f(s,t)\mathrm{d}s=\eta\sinh\eta\int_0^\infty \mathrm{e}^{-s}\frac{\mathrm{e}^{-\frac{\eta^2}{4s}}}{2\sqrt{\pi}s^{\frac{3}{2}}}\frac{t\,\mathrm{e}^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}}\mathrm{d}s\\ &=\frac{2\sqrt{2}}{\pi}\frac{\eta t\sinh\eta}{\left(\eta^2+2t^2\right)}K_2\left(\sqrt{\eta^2+2t^2}\right),\quad \eta>0,\ t>0.\end{aligned} \tag{6.14}$$

For the governing equation of (6.14) we present the following result.

Theorem 6.2. *The pdf of* $J_3(t),t>0$ *solves*

$$-\frac{\partial^2 p_{J_3}}{\partial t^2}=\left(\frac{\partial^2}{\partial\eta^2}-2\frac{\partial}{\partial\eta}\coth\eta\right)p_{J_3},\quad \eta,t>0$$

subject to the initial condition $p_{J_3}(\eta,0)=\delta(\eta)$.

Proof. We have

$$-\frac{\partial^2 p_{J_3}}{\partial t^2}(\eta, t) = -\int_0^\infty p_3(\eta, s) 2 \frac{\partial f}{\partial s}(s, t) \mathrm{d}s$$
$$= 2\int_0^\infty f(s, t) \frac{\partial p_3}{\partial s}(\eta, s) \mathrm{d}s = \left(\frac{\partial^2}{\partial \eta^2} - 2\frac{\partial}{\partial \eta} \coth \eta\right) p_{J_3}(\eta, t). \quad \square$$

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