HAMILTON PATHS WITH LASTING SEPARATION

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June 24, 2018

Abstract

We determine the asymptotics of the largest cardinality of a set of Hamilton paths in the complete graph with vertex set $[n]$ under the condition that for any two of the paths in the family there is a subpath of length k entirely contained in only one of them and edge–disjoint from the other one.

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1 Introduction

In recent years, in a series of papers we have studied the size of the largest family of Hamilton paths in a fixed complete graph with vertex set $[n]$ such that for any pair of the Hamilton paths from the family their union contains a fixed small subgraph [5], [7], [1]. The central question in the last mentioned paper is concerned with the case when the union of the paths must contain a cycle of prescribed length. Subsequently, answering a question left open in [1], I. Kovács and D. Soltész [3] have proved that the size of the largest family of Hamiltonian paths the pairwise union of which contains an odd cycle does not decrease if instead of an arbitrary odd cycle the union contains a triangle. (The analogous statement for even cycles vs. cycles of length 4 is false, [1].) These problems are rooted in zero-error information theory, especially in the graph capacity problem of Claude Shannon [9]. In [2] we introduced far-reaching generalisations of Shannon's problem with the aim of answering well-known questions in extremal combinatorics, including Rényi's problem of the maximum size of a family of pairwise qualitatively independent partitions of a finite set. The concept of capacity of permutations is a further natural extension of graph capacity [4] and has led to similar questions about Hamilton paths.

The questions studied in the present paper are intimately connected to those raised in [1], even though the conditions studied here are not in terms of subgraphs in the pairwise union of Hamilton paths. Our main concern is to understand the rough order of magnitude of the cardinality of the largest set of paths satisfying various conditions in order to develop an intuition for their hierarchy.

Somewhat surprisingly, our present conditions allow for large sets of superexponential size.

2 A simple problem

Let us start with the case $k = 2$ since already this exhibits the main features of what we see in general.

Theorem 1 *Let $M(n, 2)$ be the largest cardinality of a family of Hamilton paths in the complete graph K_n with vertex set $[n]$ such that for any two of them there is a path of 3 vertices with both edges contained in the same path and missing in the other one. We claim*

$$\lfloor \frac{n}{2} \rfloor! \leq M(n, 2) \leq 2^{n/2} n \lfloor \frac{n}{2} \rfloor!$$

Proof.

The lower bound on $M(n, 2)$ is given by the construction in Theorem 1 from [1]. For n even, let us consider the complete bipartite graph $K_{n/2,n/2}$. Let the two colour classes be A and B. Let us fix an arbitrary order of the vertices in A. Then each order of the vertices of B defines a Hamilton path starting in the first element of the order of A and alternating between the two classes in the prescribed order. In other words, the path goes

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from the first vertex of A into the first vertex of B, then returns to the second vertex of A, etc. In particular, for any two of these paths the order of the visit of the elements of B is different. The first vertex in which the two orders of B differ, is clearly the middle vertex of a path of length two both of whose edges belong only to one of the two paths. In the case of an odd n let A have cardinality $\lceil n/2 \rceil$ and have the fixed order. The rest is as before.

For the upper bound, consider first the case of n even. Fix a perfect matching in K_n and consider all the Hamilton paths containing all its edges. These various paths are defined by specifying an arbitrary order in which the edges of the matching appear in the Hamilton path alongside with the order of appearance of the two vertices for any of these edges. Then, clearly, we obtain $2^{n/2-1}\frac{n}{2}!$ Hamilton path in this manner and these are exactly those that contain the given matching. Every Hamilton path in K_n contains exactly one perfect matching. Clearly, two paths containing the same perfect matching cannot have a "private" path on 3 vertices, since every second edge of a Hamilton path is an edge from the fixed matching, and therefore is a common edge of both of the paths. This implies that an optimal construction of $M(n,2)$ Hamilton paths cannot have more paths than there are perfect matchings in K_n and this gives our upper bound in case of even n. The case of odd n can be treated similarly. For the upper bound, we have to note that even though every Hamilton path defines two edge–disjoint near–perfect matchings, the number of these near–perfect matchings is

$$2^{\lfloor n/2 \rfloor} n \lfloor \frac{n}{2} \rfloor!$$

and it is still true that two paths containing a same near–perfect matching cannot satisfy our condition, which gives the upper bound.

$\square$

We can adapt our previous reasoning to the case of $k > 2$. We start by $k = 4$ and will see that the rest is immediate.

Theorem 2 *Let $M(n,4)$ be the largest cardinality of a set of Hamilton paths in K_n with vertex set $[n]$ such that for any two of them there is a path of 5 vertices all the edges of which belong to just one and the same of the two paths. Then, if n is a multiple of 4,*

$$(n/4)! \leq M(n,4) \leq 2^{7n/4}(n/4)!.$$

Proof.

We just have to adapt the previous proof. In order to prove the lower bound, we consider the complete bipartite graph $K_{n/2.n/2}$ with its independent sets A and B of equal size. We fix an order of the vertices in A while in B we first partition the vertices into disjoint ordered couples. We will vary the order of the couples leaving the order fixed within each of the $n/4$ couples. For each of the $(n/4)!$ of the orders of the elements of B so obtained, we define a Hamilton path in the whole bipartite graph as follows. Let all our Hamilton paths start with the first vertex from A. From here the path goes to the first

vertex of the first couple from B. From the latter the path continues to the second vertex of A, and then goes back to the second vertex of the first couple of B. In other words, after having chosen our permutation of the vertices of B, the construction is defined precisely as in the previous proof of Theorem 1. The number of the paths so obtained equals the number of the considered permutations of B which is $(n/4)!$. Since the order of the couples is different, therefore, wherever the two permutations differ, both of them will generate a path of 4 edges, and all of which being adjacent to one of the vertices of the corresponding couple from B. Furthermore, since the two couples are disjoint, the two paths of length 4 do not have common edges, and none of these edges appear elsewhere in the union of the two paths.

In order to establish our upper bound, consider the set of edges $\{4i + 1, 4i + 2\}$ for $i = 0, 1, 2, \ldots, n/4 - 1$. To any set of edges such as this we associate all the Hamilton paths that visit these edges in any order but with the order of the vertices fixed within each edge and with the further restriction that between any pair of successively visited edges the path traverses exactly two arbitrary new vertices not belonging to any of the edges. It is then clear that no pair of Hamilton paths associated to a fixed edge set in this manner satisfies our pairwise condition since each of them has every fourth of its edges in common with all of the other ones. The number of such paths is clearly

$$\frac{(n/4)!(n/2)!}{2^{n/4}}$$

and of all these at most one can belong to our set of Hamilton paths satisfying the pairwise condition. Hence the total number of paths in our family is at most

$$\frac{2^{n/4}n!}{(n/4)!(n/2)!} = \frac{2^{n/4}(n/4)!n!}{(n/4)!(n/4)!(n/2)!} \leq 2^{7n/4}(n/4)!$$

$\square$

As a matter of fact, a similar reasoning gives the correct order of magnitude for $M(n, k)$ if k is even. We just state this and leave the details of the proof to the reader.

Theorem 3 *Let $M(n, k)$ be the largest cardinality of a set of Hamilton paths in K_n with vertex set $[n]$ such that for any two of them there is a path of $k + 1$ vertices all the edges of which belong to just one and the same of the two paths. Then, if n is a multiple of k,*

$$(n/k)! \leq M(n, k) \leq 2^{n[h(2/k)+3/k]}(n/k)!,$$

where $h(t) = -t \log t - (1 - t) \log(1 - t)$ is the binary entropy function with logarithm to the base 2.

If $k = 3$, things change a little bit in the construction.

Theorem 4 *Let $M(n, 3)$ be the largest cardinality of a set of Hamilton paths in K_n with vertex set $[n]$ such that for any two of them there is a path of 4 vertices all the edges of which belong to just one and the same of the two paths. Then, if n is a multiple of 3,*

$$(n/3 - 1)! \leq M(n, 3) \leq [3 \cdot 2^{-1/3}]^n (n/3)!$$

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Proof.

We give a construction for the lower bound. Let n be a multiple of 3. We consider the complete bipartite graph $K_{n/3,2n/3}$. Let the bipartition into independent sets of this graph have the classes A and B with $|A| = n/3$. Let us partition the vertices of B into disjoint couples. We will consider all the permutations of these couples among themselves, with the order of the two vertices within each couple being fixed. For an arbitrary permutation of the couples in B let us connect, for every i the i'th vertex in the fixed permutation of the vertices of A with the two vertices of the i'th couple of the permutation of B. More precisely, let us fix the first couple and consider all the permutations of the remaining ones. Any of these permutations can be completed into a Hamilton path if we draw an edge between the second vertex of each couple and the first vertex of the consecutive couple of a permutation of the couples in B. We claim that for any pair of permutations of the couples of B, the corresponding Hamilton paths satisfy our condition. To see this, let us consider an arbitrary pair of these permutations. These will differ somewhere, and let us look at the first couple for which this happens. Suppose that in one of these permutations the couple in this position is formed by (a, b) while in the other one the corresponding couple is (c, d). We will refer to the partition having the couple (a, b) in the given position as the first of these two permutations. Then, by assumption, a, b, c and d are 4 different vertices. In the preceding position we have the same couple in the two permutations and we denote by x its last element. Also, let us denote by y the vertex adjacent to both a and b. We claim that the path with vertices x, a, y, b defined by the first partition is edge–disjoint from the Hamilton path defined by the other permutation. As a matter of fact, in the path defined by the partition the vertex x has an incident edge appearing in both paths, and this is the edge leading to x. By definition, the path continues in case of the first permutation by the edge (x, a) while the outgoing edge from x is different for the other permutation, namely, (x, c). But then the path defined by the second permutation does not contain any other edge incident to x. Further, in the path defined by the first permutation y is adjacent to both of a and b while in the path corresponding to the second permutation it is adjacent to c and d. This means that the edges (a, y) and (y, b) belong tho the first path but not to the second one, thereby establishing our claim that the three consecutive edges $(x, a), (a, y)$ and (y, b) all belong exclusively to the first of our two Hamilton paths. On the other hand, the total number of our Hamilton paths is $(n/3 - 1)!$ which proves the lower bound.

In order to establish the upper bound, let us consider the set of edges $\{3i + 1, 3i + 2\}$, for $i = 0, 1, 2, \ldots, n/3 - 1$. Let us also consider all the Hamilton paths in K_n that visit these edges in any order in such a manner that the path visits the adjacent vertices of the edge set in the fixed order while between any subsequent pair of these edges in the path there is exactly one extra vertex, not contained in any of them. It is clear that no pair of these Hamilton paths satisfies our condition since for among any three consecutive edges in any of these graphs there is one edge contained in all of our paths. For any fixed set of edges and intermediate points the number of Hamilton paths visiting them in the

described manner is

$$2^{n/3}\frac{n}{3}!\frac{n}{3}!$$

This gives the upper bound

$$M(n,3) \leq \frac{n!}{2^{n/3}[\frac{n}{3}!]^2} \leq \frac{3^n}{2^{n/3}} \cdot (n/3)!$$

□

Just as for the case of k even, we can use the ideas in the previous demonstration to determine the asymptotics of $M(n,k)$ for any fixed odd value of k. We will illustrate this in the case of $k = 5$. The rest is similar.

Theorem 5 *Let $M(n,5)$ be the largest cardinality of a set of Hamilton paths in K_n such that for any two of them one contains a subpath of 5 edges belonging eclusively to one and the same Hamilton path. Then, if n is a multiple of 5, we have*

$$(n/5 - 1)! \leq M(n,5) \leq 2^{n[h(2/5)+1/5]} \cdot (n/5)!$$

Proof. In order to prove the lower bound, consider the bipartite complete graph $K_{2n/5,3n/5}$ with vertex set $[n]$. Let the two independent sets be A and B with $|A| = 2n/5$. Let us fix an order of the elements of A. Further, let us partition these elements into disjoint couples of consecutive elements. In the set B let us partition the elements in an arbitrary manner into disjoint triples and let us fix an arbitrary order of the three elements within each of these groups. Let us permute the triples of B arbitrarily except for one that we will keep fixed as the first triple, say. Each of these permutations will give rise to a different Hamilton path in our bipartite complete graph as follows. Each of these paths will start in the first element of the first triple of B and will alternate between the elements of the first triple of B and the first couple of A. Once the path arrives to the last element of the first triple of B it interrupts alternation to continue with the first vertex of the second triple of B. From here the path goes to the first element of the second couple of A to resume alternation. We keep repeating this procedure until all the vertices are covered. The number of the corresponding Hamilton paths is clearly $(n/5 - 1)!$ which give the lower bound, once we realise that any pair of the Hamilton paths so obtained satisfies our pairwise condition. To see that any pair of these paths satisfy our pairwise condition, let us look at the corresponding permutations of the elements of B. We will refer to these as the first and the second permutation. Let (a, b, c) be the first triple of elements in the first permutation that differs, and in fact, is disjoint from the triple in the same positions in the second permutation. Let further x be the last vertex of the triple preceding (a, b, c) in the first permutation and let (y, z) be the corresponding couple in A. Then, as in the proof for $k = 3$, we see that the consecutive edges $(x, a), (a, y), (y, b), (b, z)$ and (z, c) define a subpath in the Hamilton path corresponding to the first permutation and none of these edges belongs to the Hamilton path corresponding to the second permutation.

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The proof of the upper bound is more straightforward. We consider the set of edges $E = \{\{5i + 1, 5i + 2\}, \text{for } i = 0, 1, 2, \ldots, n/5 - 1\}$. We consider the set of all the Hamilton paths that visit these edges in any order under the restriction that between any two edges the path passes exactly 3 different vertices not contained in any of them. Each of these vertices is visited just once. Since every fifth edge in each such path is from our set E and thus belongs to all of the paths, it follows that no pair of paths from this set satisfies our condition. On the other hand, the number of paths corresponding to a fixed set E in this manner is

$$2^{n/5} \cdot (n/5)!(3n/5)!.$$

This leads to the upper bound

$$M(n, 5) \leq \frac{n!}{2^{n/5} \cdot (n/5)!(3n/5)!} \leq \frac{n!(n/5)|}{2^{n/5} \cdot (n/5)!(n/5)|(3n/5)!} \leq 2^{n[h(2/5)+1/5]} \cdot (n/5)!.$$

$\square$

In a similar manner, we can prove that $M(n, k)$ has the order of magnitude $(n/k)!$ for all odd values of k.

3 Kernel and product structure

Interestingly enough, our near–optimal constructions in this paper are in terms of kernel structures. One could even think, especially in the case of $M(n, 2)$, that the construction is optimal. The family of Hamilton paths we thus suspect to be optimal has a so–called *kernel structure.* This means that all the paths in the construction have a fixed projection while the rest varies arbitrarily. As a matter of fact, we fix a linearly ordered set of $n/2$ vertices and consider all the Hamilton paths in K_n which have these vertices as odd–indexed vertices in the order of transition through the vertices of K_n. All the other constructions have a similar structure. In a sense, a kernel structure can be considered as a Cartesian product. More generally, it is interesting to analyse when it is that in a problem in extremal combinatorics, the extremal solutions have a Cartesian product structure. In Shannon's graph capacity problem, in all the solved basic cases, the optimal construction has a product structure. (This is so even for the famous pentagon graph, as it was proved by Lovász [8].)

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