# AN INTRINSIC APPROACH TO MANIFOLD CONSTRAINED VARIATIONAL PROBLEMS

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Abstract. Motivated by some questions in continuum mechanics and analysis in metric spaces, we give an intrinsic characterization of sequentially weak lower semicontinuous functionals defined on Sobolev maps with values into manifolds without embedding the target into Euclidean spaces.

## 0. Introduction

Equilibrium problems in several applied fields involve the minimization of energies defined on maps taking values into manifolds. We mention, for instance, Eriksen-Leslie theory (see [12], [39]) and de Gennes $Q$-tensor theory of liquid crystals (see [8], [3]), variational models for magnetostrictive materials, and more generally, variational theories for complex bodies, i.e. bodies whose macroscopic mechanical behavior is influenced prominently by the material substructures at different low scales (see [5], [28] and [29] for several examples of hyperelastic complex bodies and related energies).

A common approach to study the existence of ground states for these mechanical systems takes advantage of Nash isometric embeddings [33] to linearize the target constraint and recast the problem into the usual Euclidean setting. In this way the standard distributional calculus is restored, and classical tools of the direct methods in the Calculus of Variations can be applied (cp. with [4], [20], [7], [15], [16], [17], [32]).

In this note, instead, we give a characterization of sequentially weak lower semicontinuous functionals defined on spaces of Sobolev maps with values into a Riemannian manifold $\mathcal{M}$ based uniquely on the Riemannian properties of the target. To this aim, we follow the intrinsic definition of Sobolev maps into metric spaces introduced by Ambrosio [2] and Reshetnyak [35]. Building upon this, we provide first a natural notion of approximate differentiability for $\mathcal{M}$-valued maps, and then we give a definition of quasiconvexity for integrands defined on a suitable bundle over $\mathcal{M}$, which characterizes the sequential weak lower semicontinuity of the corresponding functional as in Morrey's celebrated results in the vectorial Calculus of Variations [30, 31].

The motivations for this work come from the attempt to develop a robust technique in order to study lower semicontinuity properties of energies defined on spaces of functions taking values into nonlinear singular spaces. Indeed, if on one hand semicontinuity is clearly independent from the chosen embedding for the target space, on the other hand many features enjoyed by classical vector valued Sobolev maps do not hold anymore for general metric space valued maps. Thus, several arguments exploited in the Euclidean framework cannot be modified to prove analogous results for non-flat singular codomains which do not admit an Euclidean embedding.

The aim of this note is then to show in a relatively simple but non-trivial case how semicontinuity can be qualified intrinsically avoiding as much as possible specific characteristics of the Euclidean structure. The techniques introduced here will be, indeed, further developed in [14] to tackle the analysis of more sophisticated and physically more plausible energies arising within multifields theories of complex bodies (see [28]), for which smooth embedding in Euclidean spaces are not available.

From a mechanical perspective the non-uniqueness of the isometric embedding in Nash theorems is relevant. For instance, in the general model building framework of complex bodies it leads to different representations of the microstructure. Therefore, the choice of one specific isometry has to be considered as a sort of additional constitutive assumption on the model (cp. with [28]). Hence, the intrinsic approach developed here frees the problem of establishing the lower semicontinuity property from this drawback.

To our knowledge, few investigations on the semicontinuity properties of energies defined on Sobolev maps taking values into non-standard settings are present in literature. We mention some previous results by Reshetnyak [37] for functionals on metric space valued Sobolev functions holding true under convexity assumptions of the integrands, a paper by Dacorogna et al. [7] developing an extrinsic approach to (embedded) manifold constrained variational problems, and a recent paper by C. De Lellis and the authors [9], where some of the ideas presented here are successfully employed to characterize lower semicontinuous energies for Almgren's multiple valued functions.

It is worth to point out that in our framework the hypotheses in Reshetnyak's paper turn out to be not necessary and, on the other hand, that several features enjoyed by Almgren's $Q$-valued functions, such as the Lipschitz approximation property, are no longer available for manifold constrained Sobolev maps. Hence, in this paper we establish a semicontinuity result which for some aspects is complementary to the ones known in literature and which, as said, will be used in conjunction with those in order to deal with more pertinent energies for what concerns some mechanical models for complex bodies.

A brief resume of the paper is as follows. The rest of the Introduction is dedicated to fix the basic notations and introduce the main relevant notions for the analysis we will develop: $W^{1,p}(\Omega,\mathcal{M})$ maps, for which we provide an intrinsic definition of approximate differential, and quasiconvexity. Section 1 is devoted to the proof of the approximate differentiability and the Calderón-Zygmund $L^p$-approximate differentiability of $\mathcal{M}$-valued Sobolev maps. In Section 2 we prove the main result of the paper: the characterization of weak lower semicontinuous functionals in terms of quasiconvexity of the corresponding energy densities (see Theorem 0.6). Eventually, in Appendix A we recall some technical results instrumental for our approach.

0.1. **Basic assumptions.** In order to illustrate the results, we introduce the following assumptions and notations. Throughout the whole paper, $\Omega$ will always be a bounded open subset of the Euclidean space $\mathbb{R}^m$ endowed with canonical base $\mathrm{e}_1,\dots,\mathrm{e}_m$.

In what follows we shall make quick recalls of some standard notions and results in Riemannian geometry mainly referring to the book [11] for precise references.

For what concerns the target space, $(\mathcal{M}^n, g)$ will always denote a connected Riemannian manifold of class at least $C^2$ and dimension $n$, often indicated simply by $\mathcal{M}$. It is understood that $\mathcal{M}$ satisfies the Hausdorff and countable basis axioms. Moreover, we shall always suppose that $\mathcal{M}$ is complete, i.e. the exponential map $\exp_u$ is defined on all of $T_u\mathcal{M}$ for every point $u \in \mathcal{M}$. Recall then that, by Hopf-Rinow's theorem, $\mathcal{M}$ endowed with the geodesic distance $d_{\mathcal{M}}$ is a complete metric space. In particular, for points $u \in \mathcal{M}$ we shall denote with $B_r(u) \subseteq \mathcal{M}$ the open ball with respect to the metric $d_{\mathcal{M}}$. Let us remark that with a slight abuse of notation the Euclidean ball in $\mathbb{R}^m$ centred in $x$ with radius $r > 0$ will be denoted by $B_r(x)$.

As usual, $T\mathcal{M}$ will be the tangent bundle: points of $T\mathcal{M}$ are couples $(u, \mathrm{v})$, where $u$ is in $\mathcal{M}$ and v is a tangent vector to $\mathcal{M}$ at $u$, in symbols $\mathrm{v} \in T_u\mathcal{M}$. In addition, we consider the vector bundle with base space $\mathcal{M}$ and total space the linear homomorphisms $\mathrm{Hom}(\mathbb{R}^m, T\mathcal{M})$, whose points are couples $(u, A)$ with $u$ in $\mathcal{M}$ and $A : \mathbb{R}^m \to T_u\mathcal{M}$ a linear map. For this bundle, $\pi : \mathrm{Hom}(\mathbb{R}^m, T\mathcal{M}) \to \mathcal{M}$ denotes the projection map on $\mathcal{M}$. Note that, with fixed $u$ in $\mathcal{M}$, $\mathrm{Hom}(\mathbb{R}^m, T_u\mathcal{M})$ can be identified with $(T_u\mathcal{M})^m$ through the identification

$$A \simeq (\mathrm{v}_1, \dots, \mathrm{v}_m) \quad \text{with} \quad \mathrm{v}_i = A\,\mathrm{e}_i \in T_u\mathcal{M}, \quad \text{for} \quad i = 1, \dots, m.$$

Since we are going to consider continuous functionals defined on such bundles, we specify that we endow $T\mathcal{M}$ with the induced Riemannian metric (see, for instance, [11, Chapter 3, exercise 2]) whose distance is given, for $(p, \mathrm{v}), (q, \mathrm{w}) \in T\mathcal{M}$, by

$$d_{T\mathcal{M}}\big((p, \mathrm{v}), (q, \mathrm{w})\big) := \inf_{\vartheta = (\gamma, X)} \int_0^1 \sqrt{|\dot\gamma(t)|^2_{g(\gamma(t))} + |\nabla_{\dot\gamma(t)} X(t)|^2_{g(\gamma(t))}}\, dt, \tag{0.1}$$

where the infimum is taken among all smooth curves

$$[0,1] \ni t \mapsto \vartheta(t) = (\gamma(t), X(t)) \in T\mathcal{M},$$

such that $\vartheta(0) = (p, \mathrm{v})$ and $\vartheta(1) = (q, \mathrm{w})$ – above $\nabla$ denotes always the Levi-Civita connection.

With this metric at disposal, we define a metric structure on $\mathrm{Hom}(\mathbb{R}^m, T\mathcal{M})$ simply specifying the distance,

$$D\big((p, A), (q, B)\big) := \sqrt{\sum_{i=1}^m d_{T\mathcal{M}}\big((p, \mathrm{v}_i), (q, \mathrm{w}_i)\big)^2}, \tag{0.2}$$

where $A \simeq (\mathrm{v}_1, \dots, \mathrm{v}_m)$ and $B \simeq (\mathrm{w}_1, \dots, \mathrm{w}_m)$ with the above identification. We point out that, such choice being arbitrary, is however equivalent to any reasonable metric which is compatible with the one on $T\mathcal{M}$ in the case $m = 1$.

Throughout the paper the letter $C$ will denote a generic positive constant. We assume this convention since it is not essential to distinguish from one specific constant to another, leaving understood that the constant may change from line to line. The parameters on which each constant $C$ depends will be explicitely highlighted.

0.2. **Manifold constrained Sobolev maps.** Sobolev spaces of maps taking values into $\mathcal{M}$ are the functional analytic framework of the present paper. As explained in the introduction, we avoid any isometric embedding of the manifold into Euclidean spaces, hence we are led to consider an intrinsic notion suitable for our purposes. In addition, in view of the metric space analysis perspectives hinted to above, we shall follow the metric space approach developed in different contexts by Ambrosio [2] and Reshetnyak [35, 36, 37].

**Definition 0.1.** Let $p \in [1, +\infty]$. We say that a map $u$ belongs to $W^{1,p}(\Omega, \mathcal{M})$ if there exists $h \in L^p(\Omega)$ such that, for every $u_0 \in \mathcal{M}$,

(i) the map $x \mapsto d_{\mathcal{M}}(u(x), u_0)$ is $W^{1,p}(\Omega)$;
(ii) $|D\big(d_{\mathcal{M}}(u(x), u_0)\big)| \leq h(x)$ for $\mathcal{L}^m$-a.e. $x \in \Omega$.

**Remark 0.2.** It is very simple to see that maps $u \in W^{1,p}(\Omega, \mathcal{M})$ are stable under composition with Lipschitz functions, i.e. if $\varphi : \mathcal{M} \to \mathbb{R}^\nu$ is Lipschitz then $\varphi \circ u \in W^{1,p}(\Omega, \mathbb{R}^\nu)$ and $|D(\varphi \circ u)| \leq \mathrm{Lip}(\varphi)\, h$ (see, for example, [35]). Moreover, there exists an optimal $h$ fulfilling (ii) above, denoted by $|Du|$ and given by the following expression (see, for example, Reshetnyak [35]):

$$|Du|(x) = \sup_{\{u_i\}_{i\in\mathbb{N}}} |D\big(d_{\mathcal{M}}(u(x), u_i)\big)|,$$

where $\{u_i\}_{i\in\mathbb{N}} \subset \mathcal{M}$ is a countable dense set.

In the extrinsic theory, Sobolev spaces taking values into $\mathcal{M}$ are introduced by means of an isometric embedding or, more generally, via the Kuratowskij's isometric embedding into $\ell^\infty$ for separable metric space targets. Let us point out that the two approaches turn out to define the same function space in case the target domain can be embedded into the dual of a separable Banach space (cp. with [19, Theorem 1.7], see also [18, Section 3]).

Related notions have been introduced in the theory of harmonic maps with metric space targets in the works of Korevaar and Schoen [26] and Jost [22, 23], and in the theory of analysis on metric spaces by Heinonen et al. [21] and Ohta [34]. Equivalences for all these approaches and the one adopted here have been established partially in Reshetnyak [36] and Jost [24], and fully in Chiron [6].

Loosely speaking, in such general frameworks only the definition of the modulus of the gradient is given. On the contrary, exploiting the linear structure of the tangent spaces to $\mathcal{M}$, we shall show that an approximate differential, according to the following definition, can be introduced $\mathcal{L}^m$-a.e. on $\Omega$ intrinsically.

To begin with, we rephrase the classical notion of differentiability according to differential geometry into a metrical flavour. Let $u : \Omega \subset \mathbb{R}^m \to \mathcal{M}$ and $x \in \Omega$ be fixed. Then, the following assertions are equivalent:

(i) $u$ is differentiable at $x$ (according to differential geometry);
(ii) there exists a linear map $A : \mathbb{R}^m \to T_{u(x)}\mathcal{M}$ such that

$$d_{\mathcal{M}}\Big(u(y), \exp_{u(x)}(A(y-x))\Big) = o(|y-x|) \quad y \to x.$$

In addition, in the last case the map $A$ is unique and $du_x = A$.

Essentially, this follows from the biLipschitz property of the exponential map in a small neighborhood of the relevant point. We turn this pointwise characterization into a definition in an approximate sense.

**Definition 0.3.** Let $u : \Omega \to \mathcal{M}$ be a measurable function and $x$ be a point of approximate continuity of $u$. A linear map $A : \mathbb{R}^m \to T_{u(x)}\mathcal{M}$ is an approximate differential of $u$ at $x$ if for all $\varepsilon > 0$

$$\lim_{\rho \to 0^+} \rho^{-m} \mathcal{L}^m \left( \left\{ y \in B_\rho(x) : \, d_{\mathcal{M}} \left( u(y), \exp_{u(x)}(A(y-x)) \right) \geq \varepsilon \, |x-y| \right\} \right) = 0. \tag{0.3}$$

Clearly, the approximate differential, when it exists, is unique and we denote it by $du_x$ as in the smooth setting.

It is not hard to see that there exists a dimensional constant $C_m > 0$ such that $C_m^{-1} \|du_x\|_{g(u(x))} \leq |Du|(x) \leq C_m \|du_x\|_{g(u(x))}$ for $\mathcal{L}^m$-a.e. $x \in \Omega$, where $|Du|$ is the function introduced in Remark 0.2 and $\| \cdot \|_{g(u(x))}$ denotes the operatorial norm of $du_x$,

$$\|du_x\|_{g(u(x))} := \sup_{v \in \mathbb{R}^m, \, |v|=1} |du_x\,(v)|_{g(u(x))},$$

with $|\cdot|_{g(u)}$ the norm in $T_u\mathcal{M}$ induced by the metric $g$ (for more details, see Remark 1.7).

**Remark 0.4.** It is proved in [19, Theorem 2.17] that the notion of Sobolev maps in Definition 0.1 coincides with the classical one using an isometric embedding $i : \mathcal{M} \to \mathbb{R}^N$, namely $W^{1,p}(\Omega, \mathcal{M}) = W^{1,p}_i(\Omega, \mathcal{M})$, where

$$W^{1,p}_i(\Omega, \mathcal{M}) := \{ v \in W^{1,p}(\Omega, \mathbb{R}^N) : \, v(x) \in i(\mathcal{M}) \quad \mathcal{L}^m\text{-a.e. in } \Omega \}.$$

Moreover, for any $u \in W^{1,p}(\Omega, \mathcal{M})$, it is then easy to check that $d(i \circ u)_x = di_{u(x)} \circ du_x$ for $\mathcal{L}^m$-a.e. $x \in \Omega$, where $du_x$ is the map in Definition 0.3.

We also notice that for Lipschitz maps $u$ the approximate differentials $du_x$ induce the family of seminorms introduced by Kirchheim [25] for the characterization of metric differentiability. More precisely, it is shown in [25] that, given a Lipschitz map $w : \Omega \to (X, \| \cdot \|)$, with $(X, \| \cdot \|)$ a Banach space, for $\mathcal{L}^m$-a.e. $x \in \Omega$ there exists a seminorm $MD(w, x)$ such that

$$\|w(y) - w(z)\| - MD(w, x)(y - z) = o(|y - x| + |z - x|) \quad \text{ as } y, z \to x.$$

Thus, once an isometric embedding $i : \mathcal{M} \to X$ of the metric space $(\mathcal{M}, d_{\mathcal{M}})$ into a Banach space $(X, \| \cdot \|)$ is chosen (in passing we note that this is always possible), it is easy to see that for a Lipschitz map $u$ it holds

$$MD(i \circ u, x)(y - z) = |du_x(y - z)|_{g(u(x))}.$$

Finally, we define weak (weak* if $p = +\infty$) convergence in $W^{1,p}(\Omega, \mathcal{M})$ through an equivalent characterization in the Euclidean case. To this aim, for any map $u \in W^{1,p}(\Omega, \mathcal{M})$, set

$$\|du\|_p^p := \int_\Omega \|du_x\|^p_{g(u(x))} \, dx \quad \text{for } \; p < +\infty;$$
$$\|du\|_\infty := \operatorname{ess} - \sup_{x \in \Omega} \|du_x\|_{g(u(x))} \quad \text{for } \; p = +\infty.$$

Given $u$, $u_k \in W^{1,p}(\Omega, \mathcal{M})$, we say that $(u_k)_{k \in \mathbb{N}}$ converges weakly (weakly*) to $u$ provided $u_k \to u$ in $L^p(\Omega, \mathcal{M})$ and $\sup_k \|du_k\|_p < +\infty$.

0.3. **Quasiconvexity and lower semicontinuity.** We consider continuous integrands $f : \Omega \times \mathrm{Hom}(\mathbb{R}^m, T\mathcal{M}) \to [0, +\infty)$. We say that $f$ is *admissible* in $W^{1,p}(\Omega, \mathcal{M})$ if, for $p \in [1, +\infty[$,

$$0 \leq f(x, u, A) \leq C \left(1 + d^p_{\mathcal{M}}(u, u_0) + \|A\|^p_{g(u)}\right),$$

where $u_0 \in \mathcal{M}$ is a fixed point and $C$ is a positive constant; or if $f$ extends continuously to $\bar{\Omega} \times \mathrm{Hom}(\mathbb{R}^m, T\mathcal{M})$ in case $p = +\infty$.

As a consequence of the existence of the approximate differential, if $p \in [1, +\infty]$ and $f$ is an admissible integrand, for any map $u \in W^{1,p}(\Omega, \mathcal{M})$, the following energy is well-defined:

$$F(u) = \int_\Omega f\big(x, u(x), du_x\big) dx. \tag{0.4}$$

We now introduce the notion of quasiconvexity for such functionals, which is a natural generalization of Morrey's definition. Set $C_r := [-r/2, r/2]^m$ for all $r > 0$. If $u \in \mathcal{M}$ and $\varphi \in C^\infty_c(C_1, T_u\mathcal{M})$, for all $x \in C_1$ we identify in the usual way the spaces $T_{\varphi(x)}(T_u\mathcal{M})$ and $T_u\mathcal{M}$. In particular, $d\varphi_x : \mathbb{R}^m \to T_{\varphi(x)}(T_u\mathcal{M}) \simeq T_u\mathcal{M}$ can be seen as an element of $\mathrm{Hom}(\mathbb{R}^m, T_u\mathcal{M})$ – thus giving sense to (0.5) below.

**Definition 0.5.** Let $f : \Omega \times \mathrm{Hom}(\mathbb{R}^m, T\mathcal{M}) \to \mathbb{R}$ be locally bounded. We say that $f$ is *quasiconvex* if, for every $(x, u, A) \in \Omega \times \mathrm{Hom}(\mathbb{R}^m, T\mathcal{M})$ and for every test function $\varphi \in C^\infty_c(C_1, T_u\mathcal{M})$,

$$f\big(x, u, A\big) \leq \int_{C_1} f\big(x, u, A + d\varphi_y\big) dy. \tag{0.5}$$

Finally, we are in the position to state the main result of the paper.

**Theorem 0.6.** *Let $p \in [1, +\infty]$ and $f : \Omega \times \mathrm{Hom}(\mathbb{R}^m, T\mathcal{M}) \to [0, +\infty)$ be a continuous admissible integrand. If $f$ is quasiconvex, then the functional $F$ in* (0.4) *is weakly (weakly*) lower semicontinuous in $W^{1,p}(\Omega, \mathcal{M})$. Conversely, if $F$ is weakly* lower semicontinuous in $W^{1,\infty}(\Omega, \mathcal{M})$, then $f$ is quasiconvex.*

We remark again that previous results in this setting usually regard the target manifold as isometrically embedded into a linear space (cp. with [4], [17], [7], [32]) in order to exploit extrinsic arguments such as the existence and regularity of a (local) closest point projection. Our proof, instead, is entirely relying within the metric theory of manifold valued Sobolev spaces according to Definition 0.1.

## 1. Manifold-valued Sobolev functions

In this section we shall establish some basic preliminary results concerning the theory of manifold constrained Sobolev maps which will be used in the proof of Theorem 0.6.

To begin with, we show that manifold valued Sobolev maps are Lipschitz continuous on big pieces of $\Omega$.

**Lemma 1.1.** *Let $u$ be in $W^{1,p}(\Omega, \mathcal{M})$, $p \in [1, +\infty]$. Then, there exists a family of Borel sets $\Omega_\lambda \subseteq \Omega$ such that $\mathcal{L}^m(\Omega \setminus \Omega_\lambda) \to 0$ as $\lambda \to +\infty$ and $u|_{\Omega_\lambda}$ is Lipschitz continuous.*

*Proof.* If $p = +\infty$, then it is easily recognized from Definition 0.1 that $u$ is Lipschitz continuous and, hence, there is nothing to prove. Therefore, we may assume $p < +\infty$. Let $h \in L^p(\Omega)$ be an admissible function in Definition 0.1 (ii) and set

$$\Omega_\lambda := \big\{x \in \Omega : M(h)(x) \leq \lambda\big\},$$

where $M$ is the maximal function operator (see [38] for the definition). Note that the set $\Omega_\lambda$ are increasing with respect to $\lambda$ and moreover, by the standard weak $L^p - L^p$ estimate for the maximal function,

$$\mathcal{L}^m(\Omega \setminus \Omega_\lambda) \leq C\,\lambda^{-p} \int_\Omega |h(x)|^p\,dx \to 0 \quad \text{as} \quad \lambda \to +\infty.$$

On the other hand, since by definition $|D(d(u(x), u_0))| \leq h(x)$ for every $u_0 \in \mathcal{M}$ and $\mathcal{L}^m$-a.e. $x \in \Omega$, it follows that

$$M\big(D(d(u(x), u_0))\big) \leq \lambda \quad \mathcal{L}^m\text{-a.e. in } \Omega_\lambda.$$

Then, a by now standard computation implies $\mathrm{Lip}(d(u, u_0)|_{\Omega_\lambda}) \leq C\,\lambda$, with $C$ a dimensional constant (see, for instance, [13, Section 6.6.3]). Since this holds for every $u_0 \in \mathcal{M}$, we infer easily that $u|_{\Omega_\lambda}$ is $C\lambda$-Lipschitz as well. □

**Remark 1.2.** The existence of a Lipschitz approximation to a Sobolev map with values into $\mathcal{M}$ is not implied by Lemma 1.1. The answer to the problem of density of regular mappings in $W^{1,p}(\Omega, \mathcal{M})$ is negative in general and depends on the topology of $\mathcal{M}$ (see, for example [18, Theorem 2.3]).

On the contrary, the Lipschitz approximation property holds true in the more singular case of Almgren's $Q$-valued functions (cp. with [10]).

Lemma 1.1 allows us to prove the almost everywhere approximate differentiability of Sobolev functions. In the proof below we keep using the notation introduced in Lemma 1.1 to which we refer.

**Corollary 1.3.** *Every map $u \in W^{1,p}(\Omega, \mathcal{M})$ is approximately differentiable $\mathcal{L}^m$-a.e. on $\Omega$, i.e. for $\mathcal{L}^m$-a.e. $x \in \Omega$, there exists a (unique) linear map $du_x : \mathbb{R}^m \to T_{u(x)}\mathcal{M}$ such that, for all $\varepsilon > 0$,*

$$\lim_{\rho \to 0^+} \rho^{-m} \mathcal{L}^m\left(\left\{y \in B_\rho(x) :\, d_{\mathcal{M}}\left(u(y), \exp_{u(x)}(du_x(y - x))\right) \geq \varepsilon\,|x - y|\right\}\right) = 0. \tag{1.1}$$

*Proof.* Fix $\lambda > 0$. Since $u|_{\Omega_\lambda}$ is Lipschitz continuous, there exists $\rho > 0$ such that we can cover $\Omega_\lambda$ with finitely many balls $B_r(x_i)$ such that $u(\Omega_\lambda \cap B_r(x_i)) \subset U_i$ with $(U_i, \varphi_i)$ a local chart, i.e. $U_i$ open in $\mathcal{M}$ and $\varphi_i : U_i \subset \mathcal{M} \to \mathbb{R}^n$ a coordinate map. Furthermore, being $\varphi_i \circ u$ Lipschitz continuous in $\Omega_\lambda \cap B_r(x_i)$, it is differentiable $\mathcal{L}^m$-a.e. there. More precisely, we can find a Lipschitz extension $w$ of $\varphi_i \circ u$ to the whole of $B_\rho(x_i)$ with the same Lipschitz constant. Finally, recall that Rademacher's theorem implies the classical differentiability of $w$ $\mathcal{L}^m$-a.e. on $B_r(x_i)$.

Consider points $x \in \Omega_\lambda$ such that:

(a) $\Omega_\lambda$ has density one in $x$;
(b) $x \in B_r(x_i)$ is such that $(\varphi_i \circ u)|_{\Omega_\lambda}$ is differentiable in $x$;
(c) $x$ is a Lebesgue point for $u$.

We shall show that for every $x$ enjoying (a)-(c), $u$ is approximately differentiable in $x$ with

$$du_x := d\varphi_i^{-1}|_{\varphi_i(u(x))} \circ d(\varphi_i \circ u)_x, \tag{1.2}$$

where the differentials appearing on the right hand side are the standard differential for Lipschitz maps. Set

$$E_\rho := \left\{ y \in B_\rho(x) : \, d_{\mathcal{M}} \left( u(y), \exp_{u(x)}(du_x(y-x)) \right) \geq \varepsilon \, |x-y| \right\},$$

and consider separately $E_\rho \cap \Omega_\lambda$ and $E_\rho \setminus \Omega_\lambda$. By item (a) it suffices to prove that $\mathcal{L}^m(E_\rho \cap \Omega_\lambda) = o(\rho^m)$ as $\rho \to 0^+$. To this aim, we note for all $y \in B_\rho(x) \cap \Omega_\lambda$ we have

$$d_{\mathcal{M}} \left( u(y), \exp_{u(x)} \left( du_x(y-x) \right) \right) \leq C |\varphi_i \circ u(y) - \varphi_i \circ \exp_{u(x)} \left( du_x(y-x) \right) |.$$

Note that the maps $\varphi_i \circ u$ and $\varphi_i \circ \exp_{u(x)} \left( du_x(\cdot - x) \right)$ share in $x$ the same differentials by (1.2) as well as the same common value. Hence, it follows that

$$d_{\mathcal{M}} \left( u(y), \exp_{u(x)} \left( du_x(y-x) \right) \right) \leq C \, f_1(y) + C \, f_2(y),$$

with

$$f_1(y) := |\varphi_i \circ u(y) - \varphi_i \circ u(x) - d(\varphi_i \circ u)_x(y-x)|,$$
$$f_2(y) := |\varphi_i \circ \exp_{u(x)} \left( du_x(y-x) \right) - \varphi_i \circ u(x) - d(\varphi_i \circ u)_x(y-x)|.$$

Thus, we infer

$$E_\rho \cap \Omega_\lambda \subseteq \left\{ y \in B_\rho(x) \cap \Omega_\lambda : \, f_1(y) \geq \frac{\varepsilon}{2C} |y-x| \right\} \bigcup \bigcup \left\{ y \in B_\rho(x) \cap \Omega_\lambda : \, f_2(y) \geq \frac{\varepsilon}{2C} |y-x| \right\}.$$

The Lipschitz continuity of $\varphi_i \circ u$ and $\varphi_i \circ \exp_{u(x)} \left( du_x(\cdot - x) \right)$ on $B_\rho(x) \cap \Omega_\lambda$ for small $\rho > 0$ implies that, for every point $x$ satisfying (a)–(c), $\mathcal{L}^m(E_\rho \cap \Omega_\lambda) = o(\rho^m)$.

Finally, since (a)–(c) hold for a subset of $\Omega_\lambda$ of full measure, and $\mathcal{L}^m(\Omega \setminus \Omega_\lambda) \to 0$ as $\lambda \to +\infty$, the proof is concluded. □

**Remark 1.4.** The representation formula given in (1.2) ensures measurability of the approximate differential $du$. In addition, uniqueness and locality follows straightforwardly from (1.1).

1.1. **$L^p$-approximate differentiability.** In this section we shall improve upon Corollary 1.3. More precisely, in Proposition 1.6 we shall establish the Calderón-Zygmund $L^p$-approximate differentiability property for maps in $W^{1,p}(\Omega, \mathcal{M})$.

We start off with a simple technical result.

**Lemma 1.5.** *For every $u \in \mathcal{M}$, let $r_u > 0$ be the injectivity radius of $\mathcal{M}$ in $u$. Then, for all $r \in (0, r_u/2)$, there exists a Lipschitz map $\theta_r : \mathcal{M} \to B_r(u)$ with $\theta_r|_{B_r(u)} = \mathrm{Id}$, $\theta_r|_{\mathcal{M} \setminus B_{2r}(u)} = u$ and $Lip(\theta_r) \leq C$, for some positive constant $C = C(u, \mathcal{M})$.*

*Proof.* With fixed $u \in \mathcal{M}$ and $r \in (0, r_u/2)$, let $\theta_r : \mathcal{M} \to \mathcal{M}$ be the map

$$\theta_r(z) := \gamma_z \big( (d_{\mathcal{M}}(z,u) \wedge (2\,r - d_{\mathcal{M}}(z,u))) \vee 0 \big),$$

where $\gamma_z : [0, +\infty) \to \mathcal{M}$ is any minimizing geodesic (hence parametrized by arclength) starting from $u$ and passing through $z$ (its existence is guaranteed by the Hopf-Rinow's theorem).

The choice of $r \in (0, r_u/2)$ makes $\theta_r$ well-defined. Indeed, minimizing geodesics are unique if $z \in B_{2r}(u)$, and for $z \in \mathcal{M} \setminus B_{2r}(u)$ the argument of $\gamma_z$ reduces to 0, so that $\gamma_z(0) = u$ for any geodesic. In particular, by the very definition $\theta_r|_{B_r(u)}$ is the identity, $\theta_r|_{\mathcal{M}\setminus B_{2r}(u)} = u$, and $\theta_r$ takes values into $B_r(u)$.

Eventually, to show that $\theta_r$ is Lipschitz continuous, we note that it suffices to prove that the restriction $\theta_r|_{B_{2r}(u)}$ enjoys such a property. The latter follows easily again from the choice of $r$ and the very definition of $\theta_r$. □

**Proposition 1.6.** *Let $u \in W^{1,p}(\Omega, \mathcal{M})$. Then, for $\mathcal{L}^m$-a.e. $x \in \Omega$ it holds*

$$\lim_{r\to 0^+} r^{-p-m} \int_{B_r(x)} d^p_{\mathcal{M}} \Big(u(y), \exp_{u(x)} (du_x(y-x))\Big)\, dy = 0. \tag{1.3}$$

*Proof.* We show (1.3) for all points $x \in \Omega$ of approximate differentiability of $u$, and of approximate continuity for $du$ and $h$ in Definition 0.1. Let $s < r_{u(x)}$, where $r_{u(x)}$ is as in Lemma 1.5 and let $\theta_s$ be the corresponding Lipschitz map. Then, we estimate the left hand side in (1.3) as follows:

$$\begin{aligned}\int_{B_r(x)} d^p_{\mathcal{M}} \Big(u(y), \exp_{u(x)} (du_x(y-x))\Big)\, dy &\le C \int_{B_r(x)} d^p_{\mathcal{M}} \left(u(y), \theta_s\circ u(y)\right) dy \\ &+ C\int_{B_r(x)} d^p_{\mathcal{M}} \Big(\theta_s\circ u(y), \exp_{u(x)} (du_x(y-x))\Big)\, dy =: I_1 + I_2. \end{aligned}\tag{1.4}$$

For what concerns $I_2$, we note that, by assumption, $x$ turns out to be a point of approximate differentiability for the the vector-valued Sobolev map $\exp^{-1}_{u(x)} \circ \theta_s \circ u : \Omega \to T_{u(x)}\mathcal{M} \simeq \mathbb{R}^n$. In addition, $d(\exp^{-1}_{u(x)} \circ \theta_s \circ u)_x = du_x$ follows from $d(\theta_s)_{u(x)} = \mathrm{Id}$. Therefore, the local Lipschitz continuity of $\exp_{u(x)}$ implies

$$\begin{aligned} I_2 &\le C\int_{B_r(x)} \left|\exp^{-1}_{u(x)} \circ \theta_s \circ u(y) - d(\exp^{-1}_{u(x)} \circ \theta_s \circ u)_x(y-x)\right|^p_{g(u(x))} dy \\ &= o(r^{m+p}), \end{aligned}\tag{1.5}$$

by taking into account the classical Calderón-Zygmund $L^p$-differentiability of the (standard) Sobolev map $\exp^{-1}_{u(x)} \circ \theta_s \circ u$ (see Remark 0.2 and [13, Subsection 6.1.2]).

Regarding $I_1$, consider the set $H := \{y \in B_r(x) : d_{\mathcal{M}}(u(y), u(x)) > s\}$, and note that $d_{\mathcal{M}}(u(y), \theta_s \circ u(y)) = 0$ for $\mathcal{L}^m$-a.e. $y \in \Omega \setminus H$. Since $x$ is a point of approximate continuity of $u$, it follows that

$$\lim_{r\to 0^+} \frac{\mathcal{L}^m(B_r(x)\cap H)}{r^m} = 0. \tag{1.6}$$

Moreover, denoting by $\{u_i\}_{i\in\mathbb{N}}$ a dense subset of $\mathcal{M}$, the equality

$$d_{\mathcal{M}}(u, \theta_s\circ u) = \sup_{\{u_i\}_{i\in\mathbb{N}}} |d_{\mathcal{M}}(u, u_i) - d_{\mathcal{M}}(u_i, \theta_s\circ u)|,$$

yields that $d_{\mathcal{M}}(u, \theta_s \circ u) \in W^{1,p}(\Omega)$ and

$$|D\left(d_{\mathcal{M}}(u, \theta_s\circ u)\right)| \le \sup_{\{u_i\}_{i\in\mathbb{N}}} |D\left(d_{\mathcal{M}}(u, u_i)\right)| + |D\left(d_{\mathcal{M}}(u_i, \theta_s\circ u)\right)| \le C\, h. \tag{1.7}$$

In view of (1.6) and (1.7), we can apply Poincaré inequality and get

$$\begin{aligned} I_1 &\le C\, r^p \int_{B_r(x)} |D\left(d_{\mathcal{M}}(u(y), \theta_s\circ u(y))\right)|^p\, dy \\ &\le C\, r^p \int_{B_r(x)\cap H} |h(y)|^p\, dy = o(r^{m+p}). \end{aligned}$$

The last estimate, together with (1.5), finishes the proof of (1.3). □

**Remark 1.7.** From Proposition 1.6, it is not difficult to show that there exists a dimensional constant $C_m > 0$ such that

$$C_m^{-1} \|du_x\|_{g(u(x))} \le |Du|(x) \le C_m \|du_x\|_{g(u(x))}.$$

For, being $u|_{\Omega_\lambda}$ and the distance function Lipschitz continuous, the distributional gradient of $d_{\mathcal{M}}(u(\cdot), u_i)$, $i \in \mathbb{N}$, coincides with the pointwise approximate ones for $\mathcal{L}^m$-a.e. point in $\Omega_\lambda$. Hence, on one hand it is simple to verify that there exist dimensional constants $\gamma_m, C_m > 0$ (which can be computed explicitely) such that, for those points,

$$\begin{aligned}
\left|D\big(d_{\mathcal{M}}(u(\cdot), u_i)\big)|_{y=x}\right| &= \gamma_m \lim_{r\to 0} \fint_{B_r(x)\cap\Omega_\lambda} \frac{\left|d_{\mathcal{M}}(u(y), u_i) - d_{\mathcal{M}}(u(x), u_i)\right|}{r}\, dy \\
&\le \gamma_m \lim_{r\to 0} \fint_{B_r(x)} \frac{d_{\mathcal{M}}(u(y), u(x))}{r}\, dy \\
&\stackrel{(1.3)}{=} \gamma_m \lim_{r\to 0} \fint_{B_r(x)} \frac{d_{\mathcal{M}}\left(\exp_{u(x)}\left(du_x(y-x)\right), u(x)\right)}{r}\, dy \\
&= \gamma_m \lim_{r\to 0} \fint_{B_r(x)} \left| du_x\left(\frac{y-x}{r}\right)\right|_{g(u(x))} dy \le C_m \|du_x\|_{g(u(x))},
\end{aligned}$$

where we used that $d(\exp_{u(x)})_0 = \mathrm{Id}$, thus implying that $|Du|(x) \le C_m \|du_x\|_{g(u(x))}$. On the other hand, as shown by Ambrosio [2, Theorem 2.2 (ii)] for metric space valued BV functions (the proof remaining unchanged – even simplified – for the Sobolev class), $|Du|(x)$ is the approximate limit of the quotient $\frac{d_{\mathcal{M}}(u(y),u(x))}{|y-x|}$ for $\mathcal{L}^m$-a.e. $x \in \Omega$, thus we get

$$\begin{aligned}
|Du|(x) &\ge \limsup_{r\to 0} \frac{\gamma_m}{\omega_m\, r^m} \int_{B_r(x)\cap\Omega_\lambda} \frac{d_{\mathcal{M}}(u(y), u(x))}{r}\, dy \\
&\stackrel{(1.3)}{\ge} \limsup_{r\to 0} \frac{\gamma_m}{\omega_m\, r^m} \int_{B_r(x)\cap\Omega_\lambda} \frac{d_{\mathcal{M}}\left(\exp_{u(x)}\left(du_x(y-x)\right), u(x)\right)}{r}\, dy \\
&= \gamma_m \lim_{r\to 0} \fint_{B_r(x)} \frac{d_{\mathcal{M}}\left(\exp_{u(x)}\left(du_x(y-x)\right), u(x)\right)}{r}\, dy \\
&= \gamma_m \lim_{r\to 0} \fint_{B_r(x)} \left| du_x\left(\frac{y-x}{r}\right)\right|_{g(u(x))} dy \ge C_m^{-1} \|du_x\|_{g(u(x))},
\end{aligned}$$

since the left hand side of the last line is a norm for $du_x$.

Indipendently from the consideration above, note that, if $\varphi$ is any local coordinate chart, from the local representation $du_x = d\varphi^{-1}|_{\varphi(x)} \circ d(\varphi \circ u)_x$ (cp. with (1.1)) and Remark 0.2 it follows easily that $du \in L^p$ (it is indeed enough to choose local coordinates $\varphi$ with equi-bounded Lipschitz constants, for example defined on small normal neighborhoods).

## 2. Quasiconvexity and lower semicontinuity

In this section we prove Theorem 0.6.

2.1. **Necessity of quasiconvexity.** We shall start off by showing that if $F$ in (0.4) is weakly* lower semicontinuous in $W^{1,\infty}(\Omega,\mathcal{M})$ then $f$ is quasiconvex.

Let $(x,u,A) \in \Omega \times \mathrm{Hom}(\mathbb{R}^m, TM)$ and $\varphi \in C_c^\infty(C_1, T_u\mathcal{M})$. Assume without loss of generality that $\varphi$ is extended to the whole of $\mathbb{R}^m$ by $C_1$-periodicity, and set $l_A(y) := A(y-x)$ for simplicity of notation. For $r > 0$ small enough to have $C_r(x) := x + C_r \subseteq \Omega$ and $k \in \mathbb{N}$, we define

$$\varphi_{r,k}(y) := \begin{cases} \exp_u\left(l_A(y) + \dfrac{r}{k}\,\varphi\left(\dfrac{k\,(y-x)}{r}\right)\right) & \text{if } y \in C_r(x), \\ \\ \exp_u(l_A(y)) & \text{if } y \in \Omega \setminus C_r(x). \end{cases}$$

Note that

$$\varphi_{r,k} \rightharpoonup^* \exp_u \circ\, l_A \quad \text{in} \quad W^{1,\infty}(\Omega,\mathcal{M}) \quad \text{as} \quad k \to +\infty.$$

By the semicontinuity assumption of $F$, and the very definition of $\varphi_{r,k}$, we infer that

$$F(\exp_u \circ\, l_A, C_r(x)) \le \liminf_{k\to+\infty} F(\varphi_{r,k}, C_r(x)). \tag{2.1}$$

Now we calculate explicitely the two sides of (2.1) by scaling back variables to the unit cube $C_1$. We begin with the left hand side, that gives

$$F(\exp_u \circ\, l_A, C_r(x)) = \int_{C_1} f\big(x + r\,z, \exp_u(r\,A\,z), d(\exp_u)_{rAz} \circ A\big)\, r^m\, dz.$$

Thus, by continuity of the integrand it follows that

$$\lim_{r\to 0} r^{-m} F(\exp_u \circ\, l_A, C_r(x)) = f(x,u,A). \tag{2.2}$$

On the other hand, the right hand side of (2.1) can be rewritten as

$$\begin{aligned} &F(\varphi_{r,k}, C_r(x)) = \\ &= r^m \int_{C_1} f\left(x + rz, \exp_u\left(rAz + \frac{r}{k}\varphi(k\,z)\right), d(\exp_u)_{rAz+\frac{r}{k}\varphi(rz)} \circ (A + d\varphi_{kz})\right) dz \\ &= r^m \int_{C_1} f(x,u,A + d\varphi_{kz})\, dz + r^m\,\omega(r), \end{aligned} \tag{2.3}$$

where $\omega(r)$ is defined by (2.3) and is clearly infinitesimal as $r$ goes to 0 because all the functions involved are continuous and $d(\exp_u)_0 = \mathrm{Id}$. Further, using the periodicity of $\varphi$ we get

$$F(\varphi_{r,k}, C_r(x)) = r^m \int_{C_1} f(x,u,A + d\varphi_z)\, dz + r^m\,\omega(r). \tag{2.4}$$

In conclusion, collecting (2.2) and (2.4), and taking the limit as $r$ goes to 0, (2.1) gives

$$f(x,u,A) \le \int_{C_1} f(x,u,A + d\varphi_z)\, dz,$$

thus proving the quasiconvexity of $f$.

2.2. **Sufficiency of quasiconvexity.** Let $f$ be as in the hypothesis of Theorem 0.6 and quasiconvex. We shall show that the corresponding functional $F$ in (0.4) is weakly (weakly$^*$ if $p = +\infty$) lower semicontinuous on $W^{1,p}(\Omega, \mathcal{M})$. We give the proof for $p < +\infty$ and leave to the reader the easy modification for the remaining case. We want to prove that, given $u_k \rightharpoonup u$, then

$$F(u) \leq \liminf_{k \to +\infty} F(u_k).$$

Let us reformulate conveniently the thesis. Note first that there is no loss of generality (up to extracting a subsequence which will never be renamed in the sequel) in assuming that the inferior limit above is in fact a limit. Moreover, in view of the growth hypothesis on $f$, we can assume as well that there exists a finite positive measure $\mu$ on $\Omega$ such that

$$f(x, u_k(x), (du_k)_x)\, \mathcal{L}^m \llcorner \Omega \rightharpoonup^* \mu.$$

Hence, it is clear that under these assumptions it suffices to show that

$$f(x, u(x), du_x) \leq \frac{d\mu}{d\mathcal{L}^m}(x) \text{ for } \mathcal{L}^m\text{-a.e. } x \in \Omega. \tag{2.5}$$

According to Lemma A.1, without relabeling the subsequence, we consider sets $\Omega_l$, $l \in \mathbb{N}$, such that properties in (i)–(iii) there are true for the sequence

$$\big(d_{\mathcal{M}}(u(x_0), u_k(x)))^p + \|d(u_k)_x\|^p_{g(u_k(x))}\big)_{k \in \mathbb{N}}.$$

In particular, there exists a superlinear function $\varphi$ such that, for all $l \in \mathbb{N}$,

$$\sup_{k \in \mathbb{N}} \int_{\Omega_l} \varphi\left((d_{\mathcal{M}}(u(x_0), u_k(x)))^p + \|d(u_k)_x\|^p_{g(u_k(x))}\right) dx < +\infty.$$

With fixed $l \in \mathbb{N}$, up to subsequences, we may assume the existence of a positive measure $\nu_l$ on $\Omega$ such that

$$\varphi\left((d_{\mathcal{M}}(u(x_0), u_k(x)))^p + \|d(u_k)_x\|^p_{g(u_k(x))}\right)\ \chi_{\Omega_l}(x)\, \mathcal{L}^m \llcorner \Omega \rightharpoonup^* \nu_l.$$

Finally, from the equi-boundness $\sup_k \|du_k\|_p < +\infty$, we assume as well that there exists a measure $\sigma$ such that

$$\|d(u_k)_x\|_{g(u_k(x))}\, \mathcal{L}^m \llcorner \Omega \rightharpoonup^* \sigma.$$

We are now in the position to specify the points $x$ for which we shall prove inequality (2.5). For, we consider the subset $\Omega'_l$ of points $x \in \Omega_l$ such that:

(a) the function $u$ is $L^p$-differentiable in $x$ according to (1.3);
(b) $\Omega_l$ has density one in $x$;
(c) $\dfrac{d\mu}{d\mathcal{L}^m}(x) + \dfrac{d\nu_l}{d\mathcal{L}^m}(x) + \dfrac{d\sigma}{d\mathcal{L}^m}(x) < +\infty$.

Clearly $\mathcal{L}^m(\Omega_l \setminus \Omega'_l) = 0$, so that $\Omega' := \cup_l \Omega'_l$ is a set of full measure in $\Omega$. We shall prove that inequality (2.5) is satisfied by all points belonging to $\Omega'$.

To this aim we modify the sequence $(u_k)_{k \in \mathbb{N}}$ in two steps.

2.2.1. *Truncation.* Fix $l \in \mathbb{N}$ and a point $x_0 \in \Omega_l'$. Then choose radii $\rho_k \to 0$ such that $\mu(\partial C_{\rho_k}(x_0)) = \nu(\partial C_{\rho_k}(x_0)) = \sigma(\partial C_{\rho_k}(x_0)) = 0$. Note that, by the choice of $\rho_k$ and item (c), we can extract a further subsequence (as usual not renamed) such that

$$\fint_{C_{\rho_k}(x_0)} d^p_{\mathcal{M}}(u_k(y), u(y))\, dy = o(\rho_k^p), \tag{2.6}$$

$$\lim_{k\to+\infty} \fint_{C_{\rho_k}(x_0)} f(y, u_k(y), d(u_k)_y)\, dy = \frac{d\mu}{d\mathcal{L}^m}(x_0) < +\infty. \tag{2.7}$$

$$\sup_k \fint_{C_{\rho_k}(x_0)\cap\Omega_l} \varphi\Big((d_{\mathcal{M}}(u(x_0), u_k(y)))^p + \|d(u_k)_y\|^p_{g(u_k(y))}\Big)\, dy < +\infty, \tag{2.8}$$

$$\sup_k \fint_{C_{\rho_k}(x_0)} \|d(u_k)_y\|^p_{g(u_k(y))}\, dy < +\infty. \tag{2.9}$$

Note that, in particular, from item (a) and (2.6) we get

$$\fint_{C_{\rho_k}(x_0)} d^p_{\mathcal{M}}\Big(u_k(y), \exp_{u(x_0)}(du_{x_0}(y-x_0))\Big)\, dy = o(\rho_k^p). \tag{2.10}$$

We show now that we can reduce our computation to the case of a localized bounded sequence in $L^\infty$. To this aim, let $r_k > 0$ be such that $r_k \to 0$ and $\rho_k/r_k \to 0$ as $k \to +\infty$. Consider the maps $\theta_{r_k}$ provided by Lemma 1.5 with center $u(x_0)$. Set $v_k := \theta_{r_k} \circ u_k$ and

$$H_k := \big\{y \in C_{\rho_k}(x_0) :\, u_k(y) \neq v_k(y)\big\}.$$

Note that $H_k = \big\{y \in C_{\rho_k}(x_0) :\, d_{\mathcal{M}}(u_k(y), u(x_0)) > r_k\big\}$. From this we deduce that

$$\begin{aligned} r_k^p\, \mathcal{L}^m(H_k) &\le \int_{H_k} d^p_{\mathcal{M}}(u_k(y), u(x_0))\, dy \\ &\le C \int_{C_{\rho_k}(x_0)} d^p_{\mathcal{M}}\Big(u_k(y), \exp_{u(x_0)}(du_{x_0}(y-x_0))\Big)\, dy \\ &\quad + C \int_{H_k} d^p_{\mathcal{M}}\Big(\exp_{u(x_0)}(du_{x_0}(y-x_0)), u(x_0)\Big)\, dy \\ &\overset{(2.10)}{\le} o(\rho_k^{p+m}) + C\, \rho_k^p\, \mathcal{L}^m(H_k). \end{aligned} \tag{2.11}$$

The latter estimate implies that

$$\rho_k^{-m}\, \mathcal{L}^m(H_k) \le \big(1 - C\,\rho_k^p\, r_k^{-p}\big)^{-1}\, r_k^{-p} o(\rho_k^p); \tag{2.12}$$

hence, by recalling the choice of $r_k$, we infer that

$$\rho_k^{-m}\mathcal{L}^m(H_k) = o(1). \tag{2.13}$$

In turn, the previous inequality inserted in (2.11) implies also that

$$\int_{H_k} d^p_{\mathcal{M}}(u_k(y), u(x_0))\, dy = o(\rho_k^{p+m}). \tag{2.14}$$

Therefore, the Lipschitz continuity of $\theta_{r_k}$, the locality of the approximate differentials and the growth hypothesis on $f$, together with (2.8), (2.13) and (A.3) in

Lemma A.2, imply that

$$\rho_k^{-m}\int_{C_{\rho_k}(x_0)\cap\Omega_l}\big(f(y,u_k(y),d(u_k)_y)-f(y,v_k(y),d(v_k)_y)\big)dy\leq$$
$$\leq C\,\rho_k^{-m}\int_{H_k\cap\Omega_l}\Big((d_{\mathcal{M}}(u(x_0),u_k))^p+\|d(u_k)_y\|^p_{g(u_k(y))}\Big)\,dy=o(1).\quad(2.15)$$

Moreover, by definition of $v_k$ and $H_k$, we have that

$$\begin{aligned}
\fint_{C_{\rho_k}(x_0)} d^p_{\mathcal{M}}(v_k(y),u(y))\,dy &\overset{(2.6)}{\leq} C\fint_{C_{\rho_k}(x_0)} d^p_{\mathcal{M}}(v_k(y),u_k(y))\,dy+o(\rho_k^p)\\
&\leq C\,\rho_k^{-m}\int_{H_k} d^p_{\mathcal{M}}(v_k(y),u(x_0))\,dy+\\
&\quad + C\,\rho_k^{-m}\int_{H_k} d^p_{\mathcal{M}}(u_k(y),u(x_0))\,dy+o(\rho_k^p)\\
&\leq C\,r_k^p\,\rho_k^{-m}\,\mathcal{L}^m(H_k)+\\
&\quad + C\,\rho_k^{-m}\int_{H_k} d^p_{\mathcal{M}}(u_k(y),u(x_0))\,dy+o(\rho_k^p)\\
&\overset{(2.12),\,(2.14)}{\leq} o(\rho_k^p).
\end{aligned}\quad(2.16)$$

2.2.2. *Reduction to the flat case.* Since the $v_k$'s take values in $B_{r_k}(u(x_0))$, a set contained in a normal coordinate chart, we are able to reduce to the case of maps with values in a fixed tangent space,

$$w_k:=\exp^{-1}_{u(x_0)}\circ v_k : C_{\rho_k}(x_0)\to T_{u(x_0)}\mathcal{M}.$$

Let us first notice that (2.16) and item (a) in the definition of $\Omega_l'$ imply the estimate

$$\begin{aligned}
&\fint_{C_{\rho_k}(x_0)}|w_k(y)-du_{x_0}(y-x_0)|^p_{g(u(x_0))}\,dy\\
&\quad\leq C\fint_{C_{\rho_k}(x_0)} d^p_{\mathcal{M}}\Big(v_k(y),\exp_{u(x_0)}(du_{x_0}(y-x_0))\Big)\,dy\\
&\quad\leq C\fint_{C_{\rho_k}(x_0)}\Big(d^p_{\mathcal{M}}\left(v_k(y),u(y)\right)+d^p_{\mathcal{M}}\Big(u(y),\exp_{u(x_0)}(du_{x_0}(y-x_0))\Big)\Big)\,dy\\
&\quad=o(\rho_k^p).
\end{aligned}\quad(2.17)$$

Next, we show that the continuity of the integrand $f$ leads to

$$\lim_{k\to+\infty}\rho_k^{-m}\left|\int_{C_{\rho_k}(x_0)\cap\Omega_l}\big(f(y,v_k(y),d(v_k)_y)-f(x_0,u(x_0),d(w_k)_y)\big)\,dy\right|=0,\quad(2.18)$$

where, for every $y\in C_{\rho_k}(x_0)$ we identify, as usual, the tangent space to $T_{u(x_0)}\mathcal{M}$ at $w_k(y)$ with $T_{u(x_0)}\mathcal{M}$ itself.

To this aim, we notice that, for every $t > 0$, the integral on the left hand side of (2.18) is dominated by the sum of the two terms in the sequel:

$$I_t^k := C\,\rho_k^{-m} \int_{\{y\in C_{\rho_k}(x_0)\cap\Omega_l:\,\|d(v_k)_y\|_{g(v_k(y))}\geq t\}} \big(1 + \|d(v_k)_y\|^p_{g(v_k(y))}\big)\,dy,$$

and

$$J_t^k := \rho_k^{-m} \int_{\{y\in C_{\rho_k}(x_0)\cap\Omega_l:\,\|d(v_k)_y\|_{g(v_k(y))}< t\}} \big|f(y, v_k(y), d(v_k)_y) - f(x_0, u(x_0), d(w_k)_y)\big|\,dy.$$

Moreover, by (A.2) in Lemma A.2 and the equi-integrability of $dv_k$ in $\Omega_l$, which easily follows from (2.8) and the very definition of $v_k$ itself, we have that

$$\lim_{t\to+\infty} \sup_k I_t^k = 0.$$

Hence, to conclude (2.18) it is enough to show that for every $t > 0$ the term $J_t^k$ is infinitesimal as $k \to +\infty$.

For this, the uniform continuity of the integrand $f$ on compact sets provides us with a modulus of continuity $\omega_{f,t}$ such that

$$J_t^k \leq \omega_{f,t}\Big(\rho_k + \|D\big((u(x_0), dw_k), (v_k, dv_k)\big)\|_{L^\infty(C_{\rho_k}(x_0))}\Big),$$

where the distance $D$ appearing on the right hand side is the one introduced in (0.2) for $\mathrm{Hom}(\mathbb{R}^m, T\mathcal{M})$. Therefore, if we show that

$$\|D\big((u(x_0), dw_k), (v_k, dv_k)\big)\|_{L^\infty(C_{\rho_k}(x_0))} \leq C\,r_k, \tag{2.19}$$

we are done with (2.18).

The proof of (2.19) follows easily from the definition of the distance $D$. Indeed, consider any vector $\mathrm{e}_i$ of the standard basis of $\mathbb{R}^m$ and note that for $\mathcal{L}^m$-a.e. $y \in C_{r_k}(x_0)$, in the normal coordinates given by $\exp_{u(x_0)}$, the points $(v_k(y), d(v_k)_y(\mathrm{e}_i))$ and $(u(x_0), d(w_k)_y(\mathrm{e}_i))$ are represented respectively by $(p, W)$ and $(0, W)$, where $p$ are the coordinates of $v_k(y)$.

Hence, we can estimate the distance between the two points by the length of the curve which in normal coordinates reads as $\vartheta := (\gamma, X) : [0, 1] \ni t \to (t\,p, W)$. Now, since $\gamma$ is a geodesic radius, we have $\dot\gamma(t) = |p|_{g(u(x_0))}\,\hat r$, where $\hat r$ is the radial versor. Moreover, by adopting Einstein convention of summing over the repeated indices and thus setting $\hat r = r^j\,\partial_j$ and $W = W^i\,\partial_i$, the definition of the Christoffel symbols yields that

$$\nabla_{\dot\gamma(t)} X(t) = |p|_{g(u(x_0))}\, r^j\, W^i\, \Gamma^m_{ij}(\gamma(t))\,\partial_m,$$

from which it follows

$$|\nabla_{\dot\gamma(t)} X(t)|_{g(\gamma(t))} \leq C\,|p|_{g(u(x_0))}\, t \sup_{i,j,m} |\Gamma^m_{ij}(\gamma(t))|.$$

Since in normal coordinates the Christoffel symbols are zero at the origin, it follows that, for $\mathcal{L}^m$-a.e. $y \in C_{r_k}(x_0)$,

$$d_{T\mathcal{M}}\big((u(x_0), d(w_k)_y(\mathrm{e}_i)), (v_k(y), d(v_k)_y(\mathrm{e}_i))\big) \leq$$
$$\leq \int_0^1 \sqrt{|\dot\gamma(t)|^2_{g(\gamma(t))} + |\nabla_{\dot\gamma(t)} X(t)|^2_{g(\gamma(t))}}\, dt$$
$$\leq C|p|_{g(u(x_0))}\left(1 + \sup_{\substack{u\in B_{r_k}(u(x_0)),\\ i,j,m}} |\Gamma^m_{ij}(u)|\right) \leq C\, r_k,$$

thus leading clearly to (2.19).

Now we can conclude the argument as in the standard vectorial case. Scaling back to the unit cube, we define a map $z_k : C_1 \to T_{u(x_0)}\mathcal{M}$ by

$$z_k(y) := \rho_k^{-1} w_k(\rho_k\, y + x_0).$$

Clearly, we can regard the map $z_k$ as taking values in $\mathbb{R}^n$ endowed with the metric $g(u(x_0))$. In addition, its differential can be represented by the corresponding Jacobian matrix; so that, with a slight abuse of notations, from now on we shall think of the integrands appearing below as standard variational ones.

By the definition of $z_k$, (2.9) and (2.17), we have that

$$\int_{C_1} |z_k(y) - du_{x_0}(y)|^p_{g(u(x_0))}\, dy \overset{(2.17)}{=} o(1),$$
$$\sup_k \int_{C_1} \|d(z_k)_y\|^p_{g(u(x_0))}\, dy \leq C \sup_k -\!\!\!\!\!\!\int_{C_{\rho_k}(x_0)} \|d(u_k)_y\|^p_{g(u(y))}\, dy \overset{(2.9)}{<} +\infty, \tag{2.20}$$

while (2.15) and (2.18) lead to

$$\lim_{k\to+\infty} \rho_k^{-m} \int_{C_{\rho_k}(x_0)\cap\Omega_l} f(y, u_k(y), d(u_k)_y)\, dy =$$
$$= \lim_{k\to+\infty} \int_{C_1\cap\rho_k^{-1}(\Omega_l - x_0)} f(x_0, u(x_0), d(z_k)_y)\, dy. \tag{2.21}$$

Eventually, up to passing to a subsequence, since $x_0$ is a point of density of $\Omega_l$, we may assume that the increasing family of sets $P_j := \cap_{k\geq j}\, \rho_k^{-1}(\Omega_l - x_0)$ satisfy $\mathcal{L}^m(C_1 \setminus P_j) \to 0$ as $j \to +\infty$. Then, classical sequential weak lower semicontinuity results for Carathéodory quasiconvex functionals defined on (standard) Sobolev spaces (see for instance [1, Theorem II.4] and [27, Theorem 1.1]) yield, for every $j \in \mathbb{N}$,

$$\liminf_{k\to+\infty} \int_{C_1\cap\,\rho_k^{-1}(\Omega_l - x_0)} f(x_0, u(x_0), d(z_k)_y)\, dy \geq$$
$$\geq \liminf_{k\to+\infty} \int_{C_1\cap P_j} f(x_0, u(x_0), d(z_k)_y)\, dy \geq \mathcal{L}^m(C_1\cap P_j) f(x_0, u(x_0), du_{x_0}).$$

This inequality together with (2.7) and (2.21) concludes the proof of (2.5) by taking the limit as $j \to +\infty$.

## Appendix A. Equi-integrability

As usual, in the following $\Omega \subset \mathbb{R}^m$ denotes a bounded open set. We say that a Borel function $\varphi : [0, +\infty) \to [0, +\infty)$ is superlinear at infinity if

$$\lim_{t \to +\infty} \frac{\varphi(t)}{t} = +\infty.$$

A sequence of functions $(z_k)_{k\in\mathbb{N}} \subset L^1(\Omega)$ is said to be equi-integrable if there exists a function $\varphi$ superlinear at infinity such that

$$\sup_k \int_\Omega \varphi(|z_k|)\, dx < +\infty.$$

We state the following two lemmas which can be easily deduced from [9, Lemma A.1 and Lemma 1.3], respectively. We provide some details of the proof of the first one for the sake of convenience.

**Lemma A.1.** *Let $(z_k)_{k\in\mathbb{N}}$ be a bounded sequence in $L^1(\Omega)$. Then, there exists a subsequence $(k_j)_{j\in\mathbb{N}}$ and a sequence of subsets $\Omega_l \subset \Omega$ such that:*

(i) $\Omega_l \subseteq \Omega_{l+1}$ *for every* $l \in \mathbb{N}$,
(ii) $\mathcal{L}^m(\Omega \setminus \Omega_l) = o(1)$ *as* $l \to +\infty$,
(iii) $(z_{k_j}\chi_{\Omega_l})_{j\in\mathbb{N}}$ *is equi-integrable uniformly in* $l \in \mathbb{N}$, *i.e. the same superlinear function* $\varphi$ *can be taken for every* $l$.

*Proof.* [9, Lemma A.1] provides us with a subsequence $(k_j)_{j\in\mathbb{N}}$ such that $(z_{k_j} \vee (-2^j) \wedge 2^j)_{j\in\mathbb{N}}$ is equi-integrable (actually, the truncation levels in the proof of [9, Lemma A.1] are selected as $\pm j$ but this choice is clearly not essential).

Set $\Omega_l := \cap_{j\geq l}\{x \in \Omega : |z_{k_j}(x)| \leq 2^j\}$, then items (i) and (iii) are satisfied by construction. Furthermore,

$$\mathcal{L}^m(\Omega \setminus \Omega_l) \leq 2^{-l+1} \sup_k \|z_k\|_{L^1}, \tag{A.1}$$

and the conclusion then follows. □

The next result can be obtained exactly as [9, Lemma 1.3].

**Lemma A.2.** *Let $(\Omega_l)_{l\in\mathbb{N}}$ be an increasing family of sets in $\Omega$ with $\mathcal{L}^m(\Omega \setminus \Omega_l) = o(1)$ as $l \to +\infty$. Let $z_k \in L^1(\Omega)$ with $z_k \geq 0$, and assume that, for every $l \in \mathbb{N}$,*

$$\sup_k \fint_{C_{\rho_k}\cap\Omega_l} \varphi(z_k) < +\infty,$$

*where $\rho_k \to 0$, and $\varphi$ is superlinear at infinity. Then, for every $l \in \mathbb{N}$ it holds*

$$\lim_{t\to+\infty} \left( \sup_k \rho_k^{-m} \int_{\{z_k \geq t\}\cap\Omega_l} z_k \right) = 0, \tag{A.2}$$

*and, for sets $A_k \subseteq C_{\rho_k}$ such that $\mathcal{L}^m(A_k) = o(\rho_k^m)$,*

$$\lim_{k\to+\infty} \rho_k^{-m} \int_{A_k\cap\Omega_l} z_k = 0. \tag{A.3}$$

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