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Time-Changed Processes Governed by Space-Time Fractional Telegraph Equations

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In this work, we construct compositions of vector processes of the form $S_n^{2\beta}(c^2\mathcal{L}^\nu(t))$, $t > 0$, $\nu \in (0, \frac{1}{2}]$, $\beta \in (0, 1]$, $n \in \mathbb{N}$, whose distribution is related to space-time fractional n -dimensional telegraph equations. We present within a unifying framework the pde connections of n -dimensional isotropic stable processes $S_n^{2\beta}$ whose random time is represented by the inverse $\mathcal{L}^\nu(t)$, $t > 0$, of the superposition of independent positively skewed stable processes, $\mathcal{H}^\nu(t) = H_1^{2\nu}(t) + (2\lambda)^{\frac{1}{\nu}} H_2^\nu(t)$, $t > 0$, ($H_1^{2\nu}$, H_2^ν , independent stable subordinators). As special cases for $n = 1$, $\nu = \frac{1}{2}$ and $\beta = 1$, we examine the telegraph process T at Brownian time B ([14]) and establish the equality in distribution $B(c^2\mathcal{L}^{\frac{1}{2}}(t)) \stackrel{\text{law}}{=} T(|B(t)|)$, $t > 0$. Furthermore the iterated Brownian motion ([2]) and the two-dimensional motion at finite velocity with a random time are investigated. For all these processes, we present their counterparts as Brownian motion at delayed stable-distributed time.

Keywords Riemann-Liouville fractional calculus; Telegraph processes; Stable positively skewed r.v.’s; Subordinators; Fractional Laplacian; Mittag-Leffler functions; Time-changed processes; Airy functions.

Mathematics Subject Classification 60G51; 60G52; 35C05.

1. Introduction and Preliminaries

1.1. Introduction

The study of the interplay between fractional equations and stochastic processes has began in the middle of the 1980s with the analysis of simple time-fractional diffusion equations (see [10] for a rigorous work on this field, or more recently [1], where the compositions of Brownian sheets with Brownian motions are considered, and for a non-Euclidean counterpart see [7]). In some articles, the connection between fractional diffusion equations

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and stable processes is explored (see, e.g., [17, 21]). The iterated Brownian motion has distribution satisfying the following fractional equation

$$\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}u(x, t) = \frac{1}{2^{\frac{3}{2}}}\frac{\partial^2}{\partial x^2}u(x, t), \quad x \in \mathbb{R}, t > 0, \tag{1.1}$$

(see, e.g., [2]) and also the fourth-order equation

$$\frac{\partial}{\partial t}u(x, t) = \frac{1}{2^3}\frac{\partial^4}{\partial x^4}u(x, t) + \frac{1}{2\sqrt{2\pi t}}\frac{d^2}{dx^2}\delta(x), \quad x \in \mathbb{R}, t > 0, \tag{1.2}$$

see [6] (also for an interpretation of the iterated Brownian motion to model the motion of a gas in a crack).

When the fractional equation has a telegraph structure, with more than one time-fractional derivative involved, that is for $\nu \in (0, 1]$

$$\left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda\frac{\partial^\nu}{\partial t^\nu}\right)u(x, t) = c^2\frac{\partial^2}{\partial x^2}u(x, t), \quad x \in \mathbb{R}, t > 0, \lambda > 0, c > 0, \tag{1.3}$$

the relationship of its solution with the time-changed telegraph process is examined and established in Orsingher and Beghin [14]. The space-fractional telegraph equation (with M. Riesz space derivatives) has been considered in Orsingher and Zhao [15], while the connection between space-fractional equations and asymmetric stable processes has been established in Feller [8].

Fractional telegraph equations from the analytic point of view have been studied by many authors (see [19] for equations with n time derivatives). For their solutions have been worked out also numerical techniques (see, e.g., [13]). Telegraph equations have an extraordinary importance in electrodynamics (the scalar Maxwell equations are of this type), in the theory of damped vibrations and in probability because they are connected with finite velocity random motions.

In this article, we consider various types of processes obtained by composing symmetric stable processes $S_n^{2\beta}(t), t > 0, 0 < \beta \leq 1$, with the inverse of the sum of two independent stable subordinators, say $\mathcal{L}^\nu(t), t > 0, 0 < \nu \leq \frac{1}{2}$. These time-changed processes, $W_n(t) = S_n^{2\beta}(c^2\mathcal{L}^\nu(t)), t > 0$, have distributions, $w_\nu^\beta(\mathbf{x}, t), \mathbf{x} \in \mathbb{R}^n, t > 0$, which satisfy telegraph-type space-time fractional equations of the form

$$\left(\frac{c}{\partial t^{2\nu}}\partial^{2\nu} + 2\lambda\frac{c}{\partial t^\nu}\partial^\nu\right)w_\nu^\beta(\mathbf{x}, t) = -c^2(-\Delta)^\beta w_\nu^\beta(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^n, t > 0, \lambda > 0, \tag{1.4}$$

where $0 < \beta \leq 1, 0 < \nu \leq \frac{1}{2}$, subject to the initial condition

$$w_\nu^\beta(\mathbf{x}, 0) = \delta(\mathbf{x}). \tag{1.5}$$

The fractional Laplacian $(-\Delta)^\beta$, appearing in (1.4), is defined and analyzed in Section 3 below. The fractional derivatives appearing in (1.4) are meant in the Dzherbashyan-Caputo sense, that is,

$$\frac{c}{dt^\nu}d^\nu f(t) = \frac{1}{\Gamma(m-\nu)}\int_0^t\frac{d^m}{ds^m}f(s)ds, \quad m-1 < \nu < m, m \in \mathbb{N}, \tag{1.6}$$

for a function $f \in AC^m$ that is the space of functions with continuous derivatives up to the $(m - 1)$ -th order and having Lebesgue summable m -th order derivative (see

[12, p. 90]). Equation (1.4) includes, as particular cases, all fractional equations studied so far (including diffusion equations) and also the main equations of mathematical physics in the limit. Thus, the distribution of the composed process $S_n^{2\beta}(\mathcal{L}^\nu(t))$, $t > 0$, represents the fundamental solution of the most general n -dimensional time-space fractional telegraph equation. We give the general Fourier transform of the solution to (1.4) with initial condition (1.5) as

$$\begin{aligned} & \mathbb{E} e^{i\xi \cdot S_n^{2\beta}(c^2 \mathcal{L}^\nu(t))} \\ &= \frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - c^2 \|\xi\|^{2\beta}}} \right) E_{\nu,1}(r_1 t^\nu) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - c^2 \|\xi\|^{2\beta}}} \right) E_{\nu,1}(r_2 t^\nu) \right], \end{aligned} \tag{1.7}$$

where

$$r_1 = -\lambda + \sqrt{\lambda^2 - c^2 \|\xi\|^{2\beta}}, \quad r_2 = -\lambda - \sqrt{\lambda^2 - c^2 \|\xi\|^{2\beta}} \tag{1.8}$$

and,

$$E_{\nu,\psi}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\nu k + \psi)}, \quad \nu, \psi > 0, \tag{1.9}$$

is the two-parameters Mittag-Leffler function (see, e.g., [11] for a general overview on the Mittag-Leffler functions). Our analysis, therefore, includes all previous results in a unique framework and sheds an additional insight into the literature in this field.

An important role in our analysis is played by the time change based on the process $\mathcal{L}^\nu(t)$, $t > 0$. We consider first the sum of two independent positively skewed stable r.v.'s $H_1^{2\nu}(t)$ and $H_2^\nu(t)$, $t > 0$, $0 < \nu \leq \frac{1}{2}$,

$$\mathcal{H}^\nu(t) = H_1^{2\nu}(t) + (2\lambda)^{\frac{1}{\nu}} H_2^\nu(t), \quad t > 0, \tag{1.10}$$

whose distribution $h_\nu(x, t)$ is governed by the space fractional equation

$$\frac{\partial}{\partial t} h_\nu(x, t) = - \left(\frac{\partial^{2\nu}}{\partial x^{2\nu}} + 2\lambda \frac{\partial^\nu}{\partial x^\nu} \right) h_\nu(x, t), \quad x \geq 0, t > 0, 0 < \nu \leq \frac{1}{2}. \tag{1.11}$$

In (1.11), the fractional derivatives must be meant in the Riemann-Liouville sense which, for a function $f \in AC^m$, is defined as

$$\frac{d^\nu}{dx^\nu} f(x) = \frac{1}{\Gamma(m - \nu)} \frac{d^m}{dx^m} \int_0^x \frac{f(s)}{(x - s)^{\nu+1-m}} ds, \quad m - 1 < \nu < m, m \in \mathbb{N}. \tag{1.12}$$

We then take the inverse $\mathcal{L}^\nu(t)$, $t > 0$, to the process $\mathcal{H}^\nu(t)$, $t > 0$, defined as

$$\mathcal{L}^\nu(t) = \inf \left\{ s \geq 0 : H_1^{2\nu}(s) + (2\lambda)^{\frac{1}{\nu}} H_2^\nu(s) \geq t \right\}, \quad t > 0, \tag{1.13}$$

whose distribution is related to that of $\mathcal{H}^\nu(t)$, $t > 0$, by means of the formula

$$\Pr \{ \mathcal{L}^\nu(t) < x \} = \Pr \{ \mathcal{H}^\nu(x) > t \}. \tag{1.14}$$

The distribution $\ell_\nu(x, t)$ of $\mathcal{L}^\nu(t)$, $t > 0$, satisfies the time-fractional telegraph-type equation

$$\left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{\partial^\nu}{\partial t^\nu} \right) \ell_\nu(x, t) = -\frac{\partial}{\partial x} \ell_\nu(x, t), \quad x \geq 0, t > 0, 0 < \nu \leq \frac{1}{2}, \quad (1.15)$$

where the fractional derivatives appearing in (1.15) are again in the Riemann-Liouville sense. We are able to give explicit forms of the Laplace transforms of $\hat{\ell}_\nu(x, t)$ and $\ell_\nu(x, t)$ in terms of Mittag-Leffler functions for all values of $0 < \nu \leq \frac{1}{2}$. For example, for the distribution $\ell_\nu(x, t)$ of $\mathcal{L}^\nu(t)$ we have that, for $\gamma < \lambda^2$,

$$\begin{aligned} & \int_0^\infty e^{-\gamma x} \ell_\nu(x, t) dx \\ &= \frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) E_{\nu,1}(r_1 t^\nu) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) E_{\nu,1}(r_2 t^\nu) \right], \end{aligned} \quad (1.16)$$

where

$$r_1 = -\lambda + \sqrt{\lambda^2 - \gamma}, \quad r_2 = -\lambda - \sqrt{\lambda^2 - \gamma}. \quad (1.17)$$

The distribution $\ell_\nu(x, t)$ of $\mathcal{L}^\nu(t)$, $t > 0$, has the general form

$$\ell_\nu(x, t) = \int_0^t l_{2\nu}(x, s) h_\nu(t-s, 2\lambda x) ds + 2\lambda \int_0^t l_\nu(2\lambda x, s) h_{2\nu}(t-s, x) ds, \quad (1.18)$$

where the distributions of $H^{2\nu}$, H^ν , and that of their inverse processes $L^{2\nu}$ and L^ν appear. For our analysis it is relevant to obtain the distributions of $\mathcal{H}^{\frac{1}{3}}(t)$, $t > 0$, and $\mathcal{L}^{\frac{1}{3}}(t)$, $t > 0$. We also obtain explicitly the distributions of $H^{\frac{1}{3}}(t)$ and $H^{\frac{2}{3}}(t)$, $t > 0$, and also of their inverses $L^{\frac{1}{3}}(t)$ and $L^{\frac{2}{3}}(t)$, $t > 0$, in terms of Airy functions. By means of the convolutions of these distributions we arrive at the following cumbersome density of the random time $\mathcal{L}^{\frac{1}{3}}(t)$, $t > 0$,

$$\begin{aligned} \Pr \{ \mathcal{L}^{\frac{1}{3}}(t) \in dx \} &= \frac{2\lambda}{\sqrt{\pi}} \int_0^t ds \int_0^\infty dw e^{-w} w^{-\frac{1}{6}} \text{Ai} \left(-x \sqrt[3]{\frac{2^2 w}{3(t-s)^2}} \right) \text{Ai} \left(\frac{2\lambda x}{\sqrt[3]{3s}} \right) \\ &\quad \times \frac{3}{\sqrt[3]{3s}} \sqrt[3]{\frac{2^2}{3(t-s)^2}} \left[\frac{x}{2s} + \frac{s}{t-s} \right] dx. \end{aligned} \quad (1.19)$$

For $n = 1$, $\beta = 1$ and $\nu = 1$ in (1.4), we get the telegraph equation, which is satisfied by the distribution of the one-dimensional telegraph process

$$T(t) = V(0) \int_0^t (-1)^{N(s)} ds, \quad t > 0, \quad (1.20)$$

where $N(t)$, $t > 0$ is an homogeneous Poisson process, with parameter $\lambda > 0$, independent from the symmetric r.v. $V(0)$ (with values $\pm c$). Properties of this process (including first-passage time distributions) are studied in Foong and Kanno [9] and a telegraph process with random velocities has been recently considered by Stadje and Zacks [20].

For $n = 1, \beta = 1$ and $\nu = \frac{1}{2}$ the special equation

$$\begin{cases} \left(\frac{\partial}{\partial t} + 2\lambda \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} \right) w_{\frac{1}{2}}^1(x, t) = c^2 \frac{\partial^2}{\partial x^2} w_{\frac{1}{2}}^1(x, t), & x \in \mathbb{R}, t > 0, \\ w_{\frac{1}{2}}^1(x, 0) = \delta(x), \end{cases} \quad (1.21)$$

has solution coinciding with the distribution of $T(|B(t)|), t > 0$, where $|B(t)|, t > 0$, is a reflecting Brownian motion independent from T (see [14]). For $\lambda \rightarrow \infty, c \rightarrow \infty$, in such a way that $\frac{c^2}{\lambda} \rightarrow 1$ the fractional diffusion equation (1.1) is obtained from (1.21) and the composition $T(|B(t)|), t > 0$, converges in distribution to the iterated Brownian motion. Our result, specialized to this particular case gives the following unexpected equality in distribution

$$T(|B(t)|) \stackrel{\text{law}}{=} B\left(c^2 \mathcal{L}^{\frac{1}{2}}(t)\right), \quad t > 0, \quad (1.22)$$

where

$$\Pr\{B(c^2 \mathcal{L}^{\nu}(t)) \in dx\} = \frac{\lambda dx}{c\pi} \int_0^t \frac{1}{\sqrt{s(t-s)}} e^{-\frac{x^2}{4c^2s} - \frac{\lambda^2 s^2}{t-s}} \left(\frac{s}{2(t-s)} + 1 \right) ds, \quad (1.23)$$

and

$$\Pr\{T(|B(t)|) \in dx\} = \int_0^\infty \Pr\{T(s) \in dx\} \Pr\{|B(t)| \in ds\}. \quad (1.24)$$

The absolutely continuous component of the distribution of the telegraph process $T(t), t > 0$, reads

$$\Pr\{T(t) \in dx\} = \frac{dx e^{-\lambda t}}{2c} \left\{ \lambda I_0\left(\frac{\lambda}{c} \sqrt{c^2 t^2 - x^2}\right) + \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{c} \sqrt{c^2 t^2 - x^2}\right) \right\}, \quad (1.25)$$

where $|x| < ct, t > 0, c > 0$, and

$$I_0(x) = \sum_{k=0}^\infty \left(\frac{x}{2}\right)^{2k} \frac{1}{(k!)^2}. \quad (1.26)$$

For $n = 2, \beta = 1$ and $\nu = 1$, Equation (1.4) coincides with that of damped planar vibrations (we call it planar telegraph equation) and governs the vertical oscillations of thin deformable structures. The solution to

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} \right) r(x, y, t) = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) r(x, y, t), & x^2 + y^2 < c^2 t^2, t > 0, \\ r(x, y, 0) = \delta(x, y), \\ r_t(x, y, 0) = 0, \end{cases} \quad (1.27)$$

corresponds to the distribution $r(x, y, t)$ of the vector $\mathbf{T}(t) = (X(t), Y(t))$ related to a planar motion described in Orsingher and De Gregorio [16]. This random motion $\mathbf{T}(t), t > 0$, is performed at finite velocity c , possesses sample paths composed by segments whose orientation is uniform in $[0, 2\pi)$, and with changes of direction at Poisson times. The distribution $r(x, y, t)$ of $\mathbf{T}(t), t > 0$, is concentrated inside a circle C_{ct} of radius ct and

has an absolutely continuous component which reads

$$r(x, y, t) = \frac{\lambda}{2\pi c} \frac{e^{-\lambda t + \frac{\lambda}{c} \sqrt{c^2 t^2 - (x^2 + y^2)}}}{\sqrt{c^2 t^2 - (x^2 + y^2)}}, \quad (x, y) \in C_{ct}, t > 0. \tag{1.28}$$

If no Poisson event occurs, the moving particle reaches the boundary ∂C_{ct} of C_{ct} with probability $e^{-\lambda t}$. The vector process $\mathbf{T}(t)$, $t > 0$, taken at a random time represented by a reflecting Brownian motion, $|B(t)|$, has distribution

$$q(x, y, t) dx dy = \int_0^\infty \Pr \{X(s) \in dx, Y(s) \in dy\} \Pr \{|B(t)| \in ds\} \tag{1.29}$$

which satisfies the fractional equation

$$\left(\frac{\partial}{\partial t} + 2\lambda \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\right) q(x, y, t) = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) q(x, y, t), \quad (x, y) \in \mathbb{R}^2, t > 0. \tag{1.30}$$

However, the distribution of $\mathbf{B}_2(c^2 \mathcal{L}^{\frac{1}{2}}(t))$, $t > 0$, does not coincide with (1.29) (\mathbf{B}_2 is a two-dimensional Brownian motion). In this case the role of $T(t)$, $t > 0$, in (1.22) is here played by a process which is a slight modification of $\mathbf{T}(t)$, $t > 0$. We take the planar process with law

$$\tau(x, y, t) = \frac{\lambda e^{-\lambda t}}{2\pi c} \left[\frac{e^{\frac{\lambda}{c} \sqrt{c^2 t^2 - (x^2 + y^2)}} + e^{-\frac{\lambda}{c} \sqrt{c^2 t^2 - (x^2 + y^2)}}}{\sqrt{c^2 t^2 - (x^2 + y^2)}} \right], \quad x^2 + y^2 < c^2 t^2, t > 0, \tag{1.31}$$

which also solves Equation (1.27). The process with distribution

$$\begin{aligned} q(x, y, t) &= \int_0^\infty \tau(x, y, s) \left[\Pr \{|B(t)| \in ds\} + \frac{1}{2\lambda} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} \Pr \{|B(t)| \in ds\} \right] \\ &= \int_0^\infty \left(\tau(x, y, s) + \frac{\partial}{\partial s} \tau(x, y, s) \right) \Pr \{|B(t)| \in ds\}, \end{aligned} \tag{1.32}$$

has the same law of a planar Brownian motion at the time $\mathcal{L}^{\frac{1}{2}}(t)$, $t > 0$. The process $\mathfrak{T}(t)$, $t > 0$, possessing distribution (1.31) is obtained from $\mathbf{T}(t)$, $t > 0$, by disregarding displacements started off by even-order Poisson events.

1.2. Notations

For the reader convenience we list below the main notations used throughout the article.

- $S_n^{2\beta}(t) = (S_1^{2\beta}(t), S_2^{2\beta}(t), \dots, S_n^{2\beta}(t))$, $t > 0$, $0 < \beta \leq 1$, $n \in \mathbb{N}$ is a isotropic stable n -dimensional process with law $v_\beta(\mathbf{x}, t)$, $\mathbf{x} \in \mathbb{R}^n$, $t > 0$.
- $H^\nu(t)$, $t > 0$, $0 < \nu < 1$, is a totally positively-skewed stable process (stable subordinator), with law $h_\nu(x, t)$, $x \geq 0$, $t > 0$.
- $L^\nu(t)$, $t > 0$, is the inverse of $H^\nu(t)$, $t > 0$, and has law $l_\nu(x, t)$, $x \geq 0$, $t > 0$.
- $\mathcal{H}^\nu(t) = H_1^{2\nu}(t) + (2\lambda)^{\frac{1}{\nu}} H_2^\nu(t)$, $t > 0$, is the sum of two independent stable subordinators and has law $\hat{h}_\nu(x, t)$, $x \geq 0$, $t > 0$.
- $\mathcal{L}^\nu(t)$, $t > 0$, is the inverse of $\mathcal{H}^\nu(t)$, $t > 0$ and possesses distribution $\hat{l}_\nu(x, t)$, $x \geq 0$, $t > 0$.

- $T(t)$, $t > 0$, is a telegraph process with parameters $c > 0$ and $\lambda > 0$ and law $p_T(x, t)$, $-ct < x < ct$, $t > 0$.
- $W_n(t) = S_n^{2\beta}(c^2 \mathcal{L}^\nu(t))$, $t > 0$, has law $w_n^\beta(\mathbf{x}, t)$, $\mathbf{x} \in \mathbb{R}^n$, $t > 0$.
- $\mathcal{W}(t) = T(|B(t)|)$, $t > 0$, has distribution $w(x, t)$, $x \in \mathbb{R}$, $t > 0$.
- $T(t)$, $t > 0$, is the planar process with infinite directions, parameters $c, \lambda > 0$ and law $r(x, y, t)$, $(x, y) \in C_{ct} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < c^2 t^2\}$, $t > 0$.
- $\mathfrak{T}(t)$, $t > 0$, is the planar process with infinite directions, parameters $c, \lambda > 0$ and law $\mathfrak{r}(x, y, t)$, $(x, y) \in C_{ct} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < c^2 t^2\}$, $t > 0$, constructed by disregarding displacements started off only by even-labelled Poisson events.
- $\mathcal{Q}(t) = T(|B(t)|)$, $t > 0$, has law $q(x, y, t)$, $(x, y) \in \mathbb{R}^2$, $t > 0$.
- By \tilde{f} we denote the Laplace transform of the function f and by \hat{f} we denote its Fourier transform.

1.3. Preliminaries

Let us consider a stable process $S^\nu(t)$, $t > 0$, $0 < \nu \leq 2$, $\nu \neq 1$, with characteristic function

$$\mathbb{E}e^{i\xi S^\nu(t)} = e^{-\sigma|\xi|^\nu t(1-i\theta \operatorname{sign}(\xi) \tan \frac{\nu\pi}{2})} \tag{1.33}$$

where $\theta \in [-1, 1]$ is the skewness parameter and

$$\sigma = \cos \frac{\pi\nu}{2}. \tag{1.34}$$

For $\theta = 1$ the distribution corresponding to (1.33) is totally positively skewed and for $\theta = -1$ is totally negatively skewed. The stable process with stationary and independent increments, totally positively skewed will be denoted as $H^\nu(t)$, $t > 0$. We note that the density $h_\nu(x, t)$, of $H^\nu(t)$, is zero at $x = 0$ as the following calculation shows

$$\begin{aligned} h_\nu(0, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbb{E}e^{i\xi H^\nu(t)} d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sigma|\xi|^\nu t(1-i \tan \frac{\nu\pi}{2})} d\xi \\ &= \frac{1}{2\pi} \left[\int_0^{\infty} e^{-\sigma|\xi|^\nu t(1-i \tan \frac{\nu\pi}{2})} d\xi + \int_{-\infty}^0 e^{-\sigma|\xi|^\nu t(1+i \tan \frac{\nu\pi}{2})} d\xi \right] \\ &= \frac{1}{2\pi} \left[\int_0^{\infty} e^{-|\xi|^\nu t e^{-i\frac{\nu\pi}{2}}} d\xi + \int_0^{\infty} e^{-|\xi|^\nu t e^{i\frac{\nu\pi}{2}}} d\xi \right] \\ &= \frac{1}{2\pi} \left[\int_0^{\infty} e^{-z} \left(\frac{z}{t}\right)^{\frac{1}{\nu}-1} e^{i\frac{\pi}{2}} dz + \int_0^{\infty} e^{-z} \left(\frac{z}{t}\right)^{\frac{1}{\nu}-1} \frac{1}{t} e^{-i\frac{\pi}{2}} dz \right] \\ &= \frac{\cos \frac{\pi}{2}}{\pi} \int_0^{\infty} e^{-z} \left(\frac{z}{t}\right)^{\frac{1}{\nu}-1} \frac{1}{t} dz = 0. \end{aligned} \tag{1.35}$$

The positively skewed stable r.v. $H^\nu(t)$ has x -Laplace transform

$$\tilde{h}_\nu(\mu, t) = \mathbb{E}e^{-\mu H^\nu(t)} = e^{-t\mu^\nu}, \quad 0 < \nu < 1, \tag{1.36}$$

and, therefore, Fourier transform

$$\begin{aligned} \hat{h}_\nu(\xi, t) &= \mathbb{E}e^{i\xi H^\nu(t)} = \mathbb{E}(e^{-(i\xi)H^\nu(t)}) = e^{-t(|\xi|e^{-i\frac{\nu\pi}{2}} \operatorname{sign}(\xi))^\nu} \\ &= e^{-t|\xi|^\nu \cos \frac{\nu\pi}{2} (1-i \operatorname{sign}(\xi) \tan \frac{\nu\pi}{2})}. \end{aligned} \tag{1.37}$$

This shows once again that the skeweness parameter is $\theta = 1$.

The probability law $h_\nu(x, t)$, of $H^\nu(t)$, $t > 0$, solves the problem

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial^\nu}{\partial x^\nu}\right) h_\nu(x, t) = 0, & x > 0, t > 0, 0 < \nu < 1, \\ h_\nu(0, t) = 0, \\ h_\nu(x, 0) = \delta(x). \end{cases} \tag{1.38}$$

By taking the x -Laplace transform of the Riemann-Liouville fractional derivative appearing in (1.38) we have that

$$\begin{aligned} \mathcal{L}\left[\frac{\partial^\nu}{\partial x^\nu} h_\nu(x, t)\right](\mu) &= \int_0^\infty e^{-\mu x} \frac{\partial^\nu}{\partial x^\nu} h_\nu(x, t) dx \\ &= \int_0^\infty e^{-\mu x} \left[\frac{1}{\Gamma(1-\nu)} \frac{d}{dx} \int_0^x \frac{h_\nu(z, t)}{(x-z)^\nu} dz\right] dx \\ &= \int_0^\infty e^{-\mu x} \left[\frac{1}{\Gamma(1-\nu)} \int_0^x \frac{d}{dx} \frac{h_\nu(x-z, t)}{z^\nu} dz + \frac{h_\nu(0, t)}{\Gamma(1-\nu) x^\nu}\right] dx \\ &= \frac{h_\nu(0, t)}{\Gamma(1-\nu)} \int_0^\infty e^{-\mu x} x^{1-\nu-1} dx + \frac{1}{\Gamma(1-\nu)} \int_0^\infty \frac{dz}{z^\nu} \int_z^\infty dx e^{-\mu x} \frac{d}{dx} h_\nu(x-z, t) \\ &= h_\nu(0, t) \mu^{\nu-1} + \frac{1}{\Gamma(1-\nu)} \int_0^\infty e^{-\mu z} z^{-\nu} dz \int_0^\infty e^{-\mu x} \frac{d}{dx} h_\nu(x, t) dx \\ &= h_\nu(0, t) \mu^{\nu-1} + \left[\int_0^\infty e^{-\mu x} h_\nu(x, t) dx\right] \mu \frac{1}{\mu^{1-\nu}} - \mu^{\nu-1} h_\nu(0, t) = \mu^\nu \tilde{h}_\nu(\mu, t). \end{aligned} \tag{1.39}$$

Therefore,

$$\begin{cases} \frac{\partial}{\partial t} \tilde{h}_\nu(\mu, t) + \mu^\nu \tilde{h}_\nu(\mu, t) = 0, & \mu > 0, t > 0, \\ \tilde{h}_\nu(\mu, 0) = 1, \end{cases} \tag{1.40}$$

so that

$$\tilde{h}_\nu(\mu, t) = e^{-\mu^\nu t}. \tag{1.41}$$

In other words the density of a positively skewed stable r.v. solves the space-fractional problem (1.38).

We will also deal with the inverse process of $H^\nu(t)$, $t > 0$, say $L^\nu(t)$, $t > 0$, for which

$$\Pr\{H^\nu(x) > t\} = \Pr\{L^\nu(t) < x\}, \quad x > 0, t > 0. \tag{1.42}$$

Such a process has non-negative, nonstationary and nonindependent increments. Furthermore we recall that the law $l_\nu(x, t)$ of $L^\nu(t)$, can be written as

$$l_\nu(x, t) = \frac{1}{t^\nu} W_{-\nu, 1-\nu}\left(-\frac{x}{t^\nu}\right), \quad x \geq 0, t > 0, \tag{1.43}$$

where

$$W_{a,b}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k! \Gamma(ak + b)}, \quad x \in \mathbb{R}, a > -1, b \in \mathbb{C}, \tag{1.44}$$

is the Wright function, and has Laplace transform

$$\tilde{l}_\nu(x, \mu) = \int_0^\infty e^{-\mu t} l_\nu(x, t) dt = \int_0^\infty e^{-\mu t} \frac{1}{t^\nu} W_{-\nu, 1-\nu} \left(-\frac{x}{t^\nu} \right) dt = \mu^{\nu-1} e^{-x\mu^\nu}. \tag{1.45}$$

2. Sum of Stable Subordinators, $\mathcal{H}^\nu(t) = H_1^{2\nu}(t) + (2\lambda)^{\frac{1}{\nu}} H_2^\nu(t)$

For the construction of the vector process $\mathbf{W}_n(t) = \mathbf{S}_n^{2\beta}(c^2 \mathcal{L}^\nu(t))$, $t > 0$, whose distribution is driven by the general space-time fractional telegraph equation (1.4), we need the sum $\mathcal{H}^\nu(t)$, $t > 0$, of two independent positively skewed processes. The second step consists in constructing the process $\mathcal{L}^\nu(t)$, $t > 0$, inverse to $\mathcal{H}^\nu(t)$, $t > 0$. We now start by considering the following sum

$$\mathcal{H}^\nu(t) = H_1^{2\nu}(t) + (2\lambda)^{\frac{1}{\nu}} H_2^\nu(t), \quad t > 0, 0 < \nu \leq \frac{1}{2}, \tag{2.1}$$

with $H_1^{2\nu}$, H_2^ν , independent, positively skewed, stable random variables, $\lambda > 0$. The distribution of $\mathcal{H}^\nu(t)$ can be written as

$$h_\nu(x, t) = \int_0^x h_{2\nu}(y, t) h_\nu(x - y, 2\lambda t) dy. \tag{2.2}$$

Taking the double Laplace transform of (2.2), with respect to t and x , we get

$$\begin{aligned} \tilde{\tilde{h}}_\nu(\gamma, \mu) &= \int_0^\infty e^{-\mu t} \int_0^\infty e^{-\gamma x} h_\nu(x, t) dx dt = \int_0^\infty e^{-\mu t - t\gamma^{2\nu} - 2\lambda t\gamma^\nu} dt \\ &= \frac{1}{\gamma^{2\nu} + 2\lambda\gamma^\nu + \mu} = \left[\frac{1}{\gamma^\nu - r_2} - \frac{1}{\gamma^\nu - r_1} \right] \frac{1}{r_2 - r_1} \end{aligned} \tag{2.3}$$

where, for $0 < \mu < \lambda^2$,

$$\begin{cases} r_1 = -\lambda - \sqrt{\lambda^2 - \mu}, \\ r_2 = -\lambda + \sqrt{\lambda^2 - \mu}. \end{cases} \tag{2.4}$$

By means of formula

$$\int_0^\infty e^{-\gamma x} x^{\alpha-1} E_{\alpha,\alpha}(\eta x^\alpha) dx = \frac{1}{\gamma^\alpha - \eta}, \tag{2.5}$$

where $E_{\nu,\nu}(z)$ is the Mittag-Leffler function defined in (1.9), we can invert the x -Laplace transform in (2.3) obtaining, for $\mu < \lambda^2$,

$$\tilde{\tilde{h}}_\nu(x, \mu) = \frac{x^{\nu-1}}{2\sqrt{\lambda^2 - \mu}} \left[E_{\nu,\nu} \left(\left(-\lambda + \sqrt{\lambda^2 - \mu} \right) x^\nu \right) - E_{\nu,\nu} \left(\left(-\lambda - \sqrt{\lambda^2 - \mu} \right) x^\nu \right) \right]$$

$$\begin{aligned}
&= \frac{1}{2\sqrt{\lambda^2 - \mu}} \left[\frac{1}{-\lambda + \sqrt{\lambda^2 - \mu}} \frac{\partial}{\partial x} E_{\nu,1} \left(\left(-\lambda + \sqrt{\lambda^2 - \mu} \right) x^\nu \right) \right. \\
&\quad \left. - \frac{1}{-\lambda - \sqrt{\lambda^2 - \mu}} \frac{\partial}{\partial x} E_{\nu,1} \left(\left(-\lambda - \sqrt{\lambda^2 - \mu} \right) x^\nu \right) \right]. \tag{2.6}
\end{aligned}$$

Formula (2.6) gives the explicit form of the t -Laplace transform of $\tilde{h}_\nu(x, t)$ in terms of Mittag-Leffler functions. In view of formula

$$E_{\nu,1}(-\lambda t^\nu) = \frac{\sin \pi \nu}{\pi} \int_0^\infty \frac{e^{-\lambda^{\frac{1}{\nu}} t x} x^{\nu-1}}{x^{2\nu} + 1 + 2x^\nu \cos \pi \nu} dx, \quad 0 < \nu < 1, \tag{2.7}$$

we have that

$$\begin{aligned}
\tilde{h}_\nu(x, \mu) &= \frac{1}{2\sqrt{\lambda^2 - \mu}} \left[\frac{1}{-\lambda + \sqrt{\lambda^2 - \mu}} \frac{\partial}{\partial x} \int_0^\infty \frac{e^{-xy(\lambda - \sqrt{\lambda^2 - \mu})^{\frac{1}{\nu}}} y^{\nu-1} \sin \pi \nu dy}{\pi (y^{2\nu} + 1 + 2y^\nu \cos \pi \nu)} \right. \\
&\quad \left. + \frac{1}{\lambda + \sqrt{\lambda^2 - \mu}} \frac{\partial}{\partial x} \int_0^\infty \frac{e^{-xy(\lambda + \sqrt{\lambda^2 - \mu})^{\frac{1}{\nu}}} y^{\nu-1} \sin \pi \nu dy}{\pi (y^{2\nu} + 1 + 2y^\nu \cos \pi \nu)} \right] \\
&= \int_0^\infty \frac{dy y^\nu \sin \pi \nu}{\pi (y^{2\nu} + 1 + 2y^\nu \cos \pi \nu)} \frac{1}{2\sqrt{\lambda^2 - \mu}} \left[\left(\lambda - \sqrt{\lambda^2 - \mu} \right)^{\frac{1}{\nu}-1} \right. \\
&\quad \left. \times e^{-xy(\lambda - \sqrt{\lambda^2 - \mu})^{\frac{1}{\nu}}} - \left(\lambda + \sqrt{\lambda^2 - \mu} \right)^{\frac{1}{\nu}-1} e^{-xy(\lambda + \sqrt{\lambda^2 - \mu})^{\frac{1}{\nu}}} \right] \\
&= \mathbb{E} \left\{ \frac{\mathcal{U}^\nu}{2\sqrt{\lambda^2 - \mu}} \left[(-r_2)^{\frac{1}{\nu}-1} e^{-x\mathcal{U}^\nu(-r_2)^{\frac{1}{\nu}}} - (-r_1)^{\frac{1}{\nu}-1} e^{-x\mathcal{U}^\nu(-r_1)^{\frac{1}{\nu}}} \right] \right\} \\
&= \frac{1}{r_2 - r_1} \frac{\partial}{\partial x} \mathbb{E} \left[\frac{e^{-x\mathcal{U}^\nu(-r_2)^{\frac{1}{\nu}}}}{r_2} - \frac{e^{-x\mathcal{U}^\nu(-r_1)^{\frac{1}{\nu}}}}{r_1} \right], \tag{2.8}
\end{aligned}$$

where \mathcal{U}^ν is the Lamperti distribution with density

$$\frac{\Pr \{ \mathcal{U}^\nu \in du \}}{du} = \frac{\sin \pi \nu}{\pi} \frac{u^{\nu-1}}{1 + u^{2\nu} + 2u^\nu \cos \pi \nu}, \quad u > 0, \tag{2.9}$$

and represents the law of the ratio of two independent stable r.v.'s of the same order ν .

Theorem 2.1. *The law $\tilde{h}_\nu(x, t)$ of the process $\mathcal{H}^\nu(t) = H_1^{2\nu}(t) + (2\lambda)^{\frac{1}{\nu}} H_2^\nu(t)$ solves the fractional problem*

$$\begin{cases} \frac{\partial}{\partial t} \tilde{h}_\nu(x, t) = - \left(\frac{\partial^{2\nu}}{\partial x^{2\nu}} + 2\lambda \frac{\partial^\nu}{\partial x^\nu} \right) \tilde{h}_\nu(x, t), & x > 0, t > 0, 0 < \nu < \frac{1}{2}, \\ \tilde{h}_\nu(0, t) = 0, \\ \tilde{h}_\nu(x, 0) = \delta(x). \end{cases} \tag{2.10}$$

The fractional derivatives appearing in (2.10) are intended in the Riemann-Liouville sense.

Proof. By considering (1.37), we have that the Fourier transform of $h_\nu(x, t)$ is written as

$$\begin{aligned} \widehat{h}_\nu(\xi, t) &= \mathbb{E}e^{i\xi H^\nu(t)} = \mathbb{E}e^{i\xi [H^{2\nu}(t) + (2\lambda)^{\frac{1}{\nu}} H^\nu(t)]} = \mathbb{E}e^{i\xi H^{2\nu}(t)} e^{i\xi H^\nu(2\lambda t)} \\ &= e^{-t|\xi|^{2\nu} \cos \pi\nu (1 - i \operatorname{sign}(\xi) \tan \pi\nu) - 2\lambda t |\xi|^\nu \cos \frac{\pi\nu}{2} (1 - i \operatorname{sign}(\xi) \tan \frac{\pi\nu}{2})} \\ &= e^{-t(|\xi|e^{-\frac{i\pi}{2} \operatorname{sign}(\xi)})^{2\nu} - 2\lambda t (|\xi|e^{-\frac{i\pi}{2} \operatorname{sign}(\xi)})^\nu}, \end{aligned} \tag{2.11}$$

and, thus,

$$\begin{aligned} \frac{\partial}{\partial t} \widehat{h}_\nu(\xi, t) &= \left[- \left(|\xi| e^{-\frac{i\pi}{2} \operatorname{sign}(\xi)} \right)^{2\nu} - 2\lambda \left(|\xi| e^{-\frac{i\pi}{2} \operatorname{sign}(\xi)} \right)^\nu \right] \\ &\quad \times e^{-t(|\xi|e^{-\frac{i\pi}{2} \operatorname{sign}(\xi)})^{2\nu} - 2\lambda t (|\xi|e^{-\frac{i\pi}{2} \operatorname{sign}(\xi)})^\nu}. \end{aligned} \tag{2.12}$$

In view of the relationship

$$|\xi| e^{-\frac{i\pi}{2} \operatorname{sign}(\xi)} = -i\xi \tag{2.13}$$

we have that formula (2.12) can be rewritten as

$$\frac{\partial}{\partial t} \widehat{h}_\nu(\xi, t) = [-(-i\xi)^{2\nu} - 2\lambda (-i\xi)^\nu] e^{-t(-i\xi)^{2\nu} - 2\lambda t(-i\xi)^\nu}. \tag{2.14}$$

In (1.39), we have shown that

$$\mathcal{L} \left[\frac{\partial^\nu}{\partial x^\nu} h_\nu(x, t) \right] (\mu) = \int_0^\infty e^{-\mu x} \frac{\partial^\nu}{\partial x^\nu} h_\nu(x, t) dx = \mu^\nu \widetilde{h}_\nu(\mu, t) \tag{2.15}$$

and, thus, for a sufficiently good function f we have the following Fourier transform

$$\mathcal{F} \left[\frac{\partial^\nu}{\partial x^\nu} f(x) \right] (\xi) = \int_0^\infty e^{-(-i\xi)x} \frac{\partial^\nu}{\partial x^\nu} f(x) dx = (-i\xi)^\nu \widehat{f}(\xi). \tag{2.16}$$

In view of (2.16), we have that the Fourier transform of the right-hand side of the equation (2.10), equipped with the boundary conditions, is written as

$$\begin{aligned} & -\mathcal{F} \left[\frac{\partial^{2\nu}}{\partial x^{2\nu}} h_\nu(x, t) + 2\lambda \frac{\partial^\nu}{\partial x^\nu} h_\nu(x, t) \right] (\xi) = \\ &= - \int_0^\infty e^{-(-i\xi)x} \frac{\partial^{2\nu}}{\partial x^{2\nu}} h_\nu(x, t) dx - 2\lambda \int_0^\infty e^{-(-i\xi)x} \frac{\partial^\nu}{\partial x^\nu} h_\nu(x, t) dx \\ &= - \left((-i\xi)^{2\nu} + 2\lambda (-i\xi)^\nu \right) \widehat{h}_\nu(\xi, t) \\ &= - \left((-i\xi)^{2\nu} + 2\lambda (-i\xi)^\nu \right) e^{-t(|\xi|e^{-\frac{i\pi}{2} \operatorname{sign}(\xi)})^{2\nu} - 2\lambda t (|\xi|e^{-\frac{i\pi}{2} \operatorname{sign}(\xi)})^\nu} \\ &= - \left((-i\xi)^{2\nu} + 2\lambda (-i\xi)^\nu \right) e^{-t(-i\xi)^{2\nu} - 2\lambda t(-i\xi)^\nu}, \end{aligned} \tag{2.17}$$

which coincides with formula (2.14). This is tantamount to saying that the Fourier transform $\widehat{h}_\nu(\xi, t)$ is the solution to

$$\begin{cases} \frac{\partial}{\partial t} \widehat{h}_\nu(\xi, t) = -((-i\xi)^{2\nu} + 2\lambda(-i\xi)^\nu) \widehat{h}_\nu(\xi, t), & \xi \in \mathbb{R}, t > 0, \\ \widehat{h}_\nu(\xi, 0) = 1, \end{cases} \tag{2.18}$$

and this completes the proof. □

2.1. The Inverse Process $\mathcal{L}^\nu(t)$

Let $\mathcal{L}^\nu(t), t > 0$, be the inverse process of $\mathcal{H}^\nu(t), t > 0$, as defined in (1.13) for which

$$\Pr\{\mathcal{L}^\nu(t) < x\} = \Pr\{\mathcal{H}^\nu(x) > t\}, \quad x, t > 0, \tag{2.19}$$

and let $l_\nu(x, t)$ be the law of $\mathcal{L}^\nu(t), t > 0$. We have the following result.

Theorem 2.2. *The law $l_\nu(x, t)$ of the process $\mathcal{L}^\nu(t), t > 0$, solves the time-fractional boundary-initial problem*

$$\begin{cases} \left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{\partial^\nu}{\partial t^\nu} \right) l_\nu(x, t) = -\frac{\partial}{\partial x} l_\nu(x, t), & x > 0, t > 0, 0 < \nu < \frac{1}{2}, \\ l_\nu(x, 0) = \delta(x), \\ l_\nu(0, t) = \frac{t^{-2\nu}}{\Gamma(1-2\nu)} + 2\lambda \frac{t^{-\nu}}{\Gamma(1-\nu)}, \end{cases} \tag{2.20}$$

and has x-Laplace transform which reads, for $0 < \gamma < \lambda^2$,

$$\widetilde{l}_\nu(\gamma, t) = \frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) E_{\nu,1}(r_1 t^\nu) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) E_{\nu,1}(r_2 t^\nu) \right], \tag{2.21}$$

where

$$r_1 = -\lambda + \sqrt{\lambda^2 - \gamma}, \quad r_2 = -\lambda - \sqrt{\lambda^2 - \gamma}. \tag{2.22}$$

The fractional derivatives appearing in (2.20) are intended in the Riemann-Liouville sense.

Proof. We first show that the analytical solution to the problem (2.20) has double Laplace transform $\widetilde{\widetilde{l}}_\nu(\gamma, \mu)$ written as

$$\widetilde{\widetilde{l}}_\nu(\gamma, \mu) = \frac{\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}}{\mu^{2\nu} + 2\lambda\mu^\nu + \gamma}. \tag{2.23}$$

By taking the t -Laplace transform of the equation in (2.20) we have that

$$\mu^{2\nu} \widetilde{l}_\nu(x, \mu) + 2\lambda\mu^\nu \widetilde{l}_\nu(x, \mu) = -\frac{\partial}{\partial x} \widetilde{l}_\nu(x, \mu). \tag{2.24}$$

By taking into account the boundary condition and performing the x -Laplace transform of (2.24) we have that

$$(\mu^{2\nu} + 2\lambda\mu^\nu) \widetilde{\widetilde{l}}_\nu(\gamma, \mu) = \widetilde{l}_\nu(0, \mu) - \gamma \widetilde{\widetilde{l}}_\nu(\gamma, \mu). \tag{2.25}$$

Now, by considering the boundary condition, we get that

$$\begin{aligned} \tilde{l}_v(0, \mu) &= \int_0^\infty dt e^{-\mu t} l_v(0, t) = \int_0^\infty dt e^{-\mu t} \left[\frac{t^{-2\nu}}{\Gamma(1-2\nu)} + 2\lambda \frac{t^{-\nu}}{\Gamma(1-\nu)} \right] \\ &= \mu^{2\nu-1} + 2\lambda\mu^{\nu-1}, \end{aligned} \tag{2.26}$$

and, thus,

$$\tilde{l}_v(\gamma, \mu) = \frac{\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}}{\mu^{2\nu} + 2\lambda\mu^\nu + \gamma}. \tag{2.27}$$

Now we show that the double Laplace transform of the law $l_v(x, t)$ coincides with (2.23). We first recall that

$$\begin{aligned} \tilde{h}_v(\mu, x) &= \int_0^\infty dt e^{-\mu t} h_v(t, x) = \mathbb{E}e^{-\mu\mathcal{H}^\nu(x)} = \mathbb{E}e^{-\mu H^{2\nu}(x)} \mathbb{E}e^{-\mu H^\nu(2\lambda x)} \\ &= \tilde{h}_{2\nu}(\mu, x) \tilde{h}_\nu(\mu, 2\lambda x) = e^{-x\mu^{2\nu} - x2\lambda\mu^\nu}, \quad x > 0, \end{aligned} \tag{2.28}$$

where we used result (1.36). By considering the construction of the process $\mathcal{L}^\nu(t)$, $t > 0$, as the inverse process of $\mathcal{H}^\nu(t)$, $t > 0$, as stated in (2.19), we get

$$l_v(x, t) = \frac{\Pr\{\mathcal{L}^\nu(t) \in dx\}}{dx} = -\frac{\partial}{\partial x} \Pr\{\mathcal{H}^\nu(x) < t\} = -\frac{\partial}{\partial x} \int_0^t h_v(s, x) ds. \tag{2.29}$$

In view of (2.29), the double Laplace transform of $l_v(x, t)$ can be obtained observing that

$$\begin{aligned} \tilde{l}_v(\gamma, \mu) &= \int_0^\infty dx e^{-\gamma x} \int_0^\infty dt e^{-\mu t} \left[-\frac{\partial}{\partial x} \int_0^t h_v(s, x) ds \right] \\ &= -\int_0^\infty dx e^{-\gamma x} \frac{\partial}{\partial x} \int_0^\infty dt e^{-\mu t} \int_0^t h_v(s, x) ds \\ &= -\frac{1}{\mu} \int_0^\infty dx e^{-\gamma x} \frac{\partial}{\partial x} \tilde{h}_v(x, \mu) = -\frac{1}{\mu} \int_0^\infty dx e^{-\gamma x} \left[\frac{\partial}{\partial x} e^{-x\mu^{2\nu} - 2\lambda x\mu^\nu} \right] \\ &= (\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}) \int_0^\infty dx e^{-\gamma x - x\mu^{2\nu} - 2\lambda x\mu^\nu} = \frac{\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}}{\mu^{2\nu} + 2\lambda\mu^\nu + \gamma}, \end{aligned} \tag{2.30}$$

which coincides with (2.23). Now we pass to the derivation of the x -Laplace transform of $l_v(x, t)$. We can write

$$\begin{aligned} \tilde{l}_v(\gamma, \mu) &= \frac{\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}}{\mu^{2\nu} + 2\lambda\mu^\nu + \gamma} = \frac{\mu^{\nu-1}}{\mu^\nu - r_1} + \frac{\mu^{\nu-1}}{\mu^\nu - r_2} - \frac{\mu^{2\nu-1}}{(\mu^\nu - r_1)(\mu^\nu - r_2)} \\ &= \frac{\mu^{\nu-1}}{\mu^\nu - r_1} + \frac{\mu^{\nu-1}}{\mu^\nu - r_2} - \left[\frac{\mu^{\nu-(1-\nu)}}{\mu^\nu - r_1} - \frac{\mu^{\nu-(1-\nu)}}{\mu^\nu - r_2} \right] \frac{1}{2\sqrt{\lambda^2 - \gamma}}, \end{aligned} \tag{2.31}$$

where

$$r_1 = -\lambda + \sqrt{\lambda^2 - \gamma}, \quad r_2 = -\lambda - \sqrt{\lambda^2 - \gamma}. \tag{2.32}$$

Now we need the following results

$$\int_0^\infty e^{-\mu t} E_{\nu,1}(r_j t^\nu) dt = \frac{\mu^{\nu-1}}{\mu^\nu - r_j}, \quad j = 1, 2,$$

$$\int_0^\infty e^{-\mu t} t^{(1-\nu)-1} E_{\nu,1-\nu}(r_j t^\nu) dt = \frac{\mu^{2\nu-1}}{\mu^\nu - r_j}. \tag{2.33}$$

Therefore,

$$\tilde{l}_\nu(\gamma, t) = E_{\nu,1}(r_1 t^\nu) + E_{\nu,1}(r_2 t^\nu) - \frac{t^{-\nu}}{2\sqrt{\lambda^2 - \gamma}} [E_{\nu,1-\nu}(r_1 t^\nu) - E_{\nu,1-\nu}(r_2 t^\nu)]. \tag{2.34}$$

Since

$$E_{\nu,1-\nu}(z) = z E_{\nu,1}(z) + \frac{1}{\Gamma(1-\nu)} \tag{2.35}$$

we have that

$$\begin{aligned} \tilde{l}_\nu(\gamma, t) &= E_{\nu,1}(r_1 t^\nu) + E_{\nu,1}(r_2 t^\nu) - \frac{t^{-\nu}}{2\sqrt{\lambda^2 - \gamma}} [r_1 t^\nu E_{\nu,1}(r_1 t^\nu) - r_2 t^\nu E_{\nu,1}(r_2 t^\nu)] \\ &= \left(1 - \frac{-\lambda + \sqrt{\lambda^2 - \gamma}}{2\sqrt{\lambda^2 - \gamma}}\right) E_{\nu,1}(r_1 t^\nu) + \left(1 - \frac{\lambda + \sqrt{\lambda^2 - \gamma}}{2\sqrt{\lambda^2 - \gamma}}\right) E_{\nu,1}(r_2 t^\nu) \\ &= \frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \gamma}}\right) E_{\nu,1}(r_1 t^\nu) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - \gamma}}\right) E_{\nu,1}(r_2 t^\nu) \right], \end{aligned} \tag{2.36}$$

which coincides with (2.21).

Now we check that the Laplace transform (2.36) solves the fractional equation

$$\begin{aligned} \left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{\partial^\nu}{\partial t^\nu}\right) \tilde{l}_\nu(\gamma, t) &= -\gamma \tilde{l}_\nu(\gamma, t) + l_\nu(0, t) \\ &= -\gamma \tilde{l}_\nu(\gamma, t) + \frac{t^{-2\nu}}{\Gamma(1-2\nu)} + 2\lambda \frac{t^{-\nu}}{\Gamma(1-\nu)} \end{aligned} \tag{2.37}$$

which is the x -Laplace transform of the equation appearing in (2.20). Since

$$\frac{\partial^{2\nu}}{\partial t^{2\nu}} \tilde{l}_\nu(\gamma, t) - \frac{t^{-2\nu}}{\Gamma(1-2\nu)} = \frac{{}^C \partial^{2\nu}}{\partial t^{2\nu}} \tilde{l}_\nu(\gamma, t) \tag{2.38}$$

$$\frac{\partial^\nu}{\partial t^\nu} \tilde{l}_\nu(\gamma, t) - \frac{t^{-\nu}}{\Gamma(1-\nu)} = \frac{{}^C \partial^\nu}{\partial t^\nu} \tilde{l}_\nu(\gamma, t) \tag{2.39}$$

we, therefore, need to show that

$$\left(\frac{{}^C \partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{{}^C \partial^\nu}{\partial t^\nu}\right) \tilde{l}_\nu(\gamma, t) = -\gamma \tilde{l}_\nu(\gamma, t). \tag{2.40}$$

In light of

$$\frac{{}^C \partial^\nu}{\partial t^\nu} E_{\nu,1}(r_j t^\nu) = r_j E_{\nu,1}(r_j t^\nu), \quad j = 1, 2, \tag{2.41}$$

$$\frac{{}^C \partial^{2\nu}}{\partial t^{2\nu}} E_{\nu,1}(r_j t^\nu) = r_j^2 E_{\nu,1}(r_j t^\nu) + \frac{t^{-\nu} r_j}{\Gamma(1-\nu)}, \tag{2.42}$$

we are able to show that (2.21) solves (2.37). We first check result (2.42) as follows, for $0 < 2\nu < 1$

$$\begin{aligned} \frac{{}^C \partial^{2\nu}}{\partial t^{2\nu}} E_{\nu,1}(r_j t^\nu) &= \sum_{k=0}^{\infty} \frac{r_j^k}{\Gamma(\nu k + 1)} \frac{{}^C \partial^{2\nu}}{\partial t^{2\nu}} t^{\nu k} \\ &= \sum_{k=1}^{\infty} \frac{r_j^k}{\Gamma(\nu k + 1)} \frac{\nu k}{\Gamma(1-2\nu)} \int_0^t s^{\nu k-1} (t-s)^{-2\nu} ds \\ &= \sum_{k=1}^{\infty} \frac{r_j^k t^{\nu k-2\nu}}{\Gamma(\nu k)} \frac{1}{\Gamma(1-2\nu)} \int_0^1 s^{\nu k-1} (1-s)^{1-2\nu-1} ds \\ &= \sum_{k=1}^{\infty} \frac{r_j^k t^{\nu k-2\nu}}{\Gamma(\nu k - 2\nu + 1)} = \sum_{k=0}^{\infty} \frac{r_j^{k+1} t^{\nu k-\nu}}{\Gamma(\nu k - \nu + 1)} \\ &= r_j t^{-\nu} \left[\sum_{k=1}^{\infty} \frac{(r_j t^\nu)^k}{\Gamma(\nu k - \nu + 1)} + \frac{1}{\Gamma(1-\nu)} \right] \\ &= r_j^2 E_{\nu,1}(r_j t^\nu) + \frac{t^{-\nu} r_j}{\Gamma(1-\nu)}. \end{aligned} \tag{2.43}$$

Therefore,

$$\begin{aligned} &\left(\frac{{}^C \partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{{}^C \partial^\nu}{\partial t^\nu} \right) \tilde{l}_\nu(\gamma, t) \\ &= \frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) \frac{{}^C \partial^{2\nu}}{\partial t^{2\nu}} E_{\nu,1}(r_1 t^\nu) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) \frac{{}^C \partial^{2\nu}}{\partial t^{2\nu}} E_{\nu,1}(r_2 t^\nu) \right] \\ &\quad + 2\lambda \frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) \frac{{}^C \partial^\nu}{\partial t^\nu} E_{\nu,1}(r_1 t^\nu) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) \frac{{}^C \partial^\nu}{\partial t^\nu} E_{\nu,1}(r_2 t^\nu) \right] \\ &= \frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) \left(r_1^2 E_{\nu,1}(r_1 t^\nu) + \frac{t^{-\nu} r_1}{\Gamma(1-\nu)} \right) \right. \\ &\quad \left. + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) \left(r_2^2 E_{\nu,1}(r_2 t^\nu) + \frac{t^{-\nu} r_2}{\Gamma(1-\nu)} \right) \right] \\ &\quad + 2\lambda \frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) (r_1 E_{\nu,1}(r_1 t^\nu)) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) r_2 E_{\nu,1}(r_2 t^\nu) \right] \\ &= \frac{1}{2} \left[r_1 \left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) E_{\nu,1}(r_1 t^\nu) (r_1 + 2\lambda) + r_2 \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) \right. \\ &\quad \left. \times E_{\nu,1}(r_2 t^\nu) (r_2 + 2\lambda) \right] \end{aligned}$$

$$\begin{aligned}
 &= -\frac{\gamma}{2} \frac{\lambda + \sqrt{\lambda^2 - \gamma}}{\sqrt{\lambda^2 - \gamma}} E_{\nu,1}(r_1 t^\nu) - \frac{\gamma}{2} \frac{\sqrt{\lambda^2 - \gamma} - \lambda}{\sqrt{\lambda^2 - \gamma}} E_{\nu,1}(r_2 t^\nu) \\
 &= -\gamma \left[\frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) E_{\nu,1}(r_1 t^\nu) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) E_{\nu,1}(r_2 t^\nu) \right] \right] \\
 &= -\gamma \tilde{l}_\nu(\gamma, t).
 \end{aligned} \tag{2.44}$$

In the last steps we used the fact that

$$\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) \frac{r_1 t^{-\nu}}{\Gamma(1 - \nu)} + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - \gamma}} \right) \frac{r_2 t^{-\nu}}{\Gamma(1 - \nu)} = 0, \tag{2.45}$$

and

$$r_1 + 2\lambda = -r_2, \quad r_2 + 2\lambda = -r_1, \quad r_1 r_2 = \gamma. \tag{2.46}$$

Remark 2.1. The derivation of result (2.21) suggests an alternative proof for the Fourier transform (Theorem 2.2 in [14]) of the law of the time-fractional telegraph process.

Remark 2.2. From (2.31), we get the time Laplace transform of $\tilde{l}_\nu(x, t)$, for $x > 0$, $\mu > 0$, $0 < \nu < \frac{1}{2}$, as

$$\tilde{l}_\nu(x, \mu) = \mu^{2\nu-1} e^{-x\mu^{2\nu}} e^{-2\lambda x\mu^\nu} + 2\lambda \mu^{\nu-1} e^{-2\lambda x\mu^\nu} e^{-x\mu^{2\nu}}. \tag{2.47}$$

Since (see formulas (1.43) and (1.45))

$$\tilde{l}_\nu(x, \mu) = \int_0^\infty e^{-\mu t} \frac{1}{t^\nu} W_{-\nu, 1-\nu} \left(-\frac{x}{t^\nu} \right) dt = \mu^{\nu-1} e^{-x\mu^\nu} \tag{2.48}$$

and (see formula (1.41))

$$\tilde{h}_\nu(\mu, t) = \int_0^\infty e^{-\mu x} h_\nu(x, t) dx = e^{-t\mu^\nu}, \tag{2.49}$$

we are able to invert (2.47) and we obtain the explicit distribution of the process $\mathcal{L}^\nu(t)$, $t > 0$, which reads

$$\begin{aligned}
 \tilde{l}_\nu(x, t) &= \frac{\Pr\{\mathcal{L}^\nu(t) \in dx\}}{dx} \\
 &= \int_0^t l_{2\nu}(x, s) h_\nu(t - s, 2\lambda x) ds + 2\lambda \int_0^t l_\nu(2\lambda x, s) h_{2\nu}(t - s, x) ds \\
 &= \int_0^t \frac{1}{s^{2\nu}} W_{-2\nu, 1-2\nu} \left(-\frac{x}{s^{2\nu}} \right) h_\nu(t - s, 2\lambda x) ds \\
 &\quad + 2\lambda \int_0^t \frac{1}{s^\nu} W_{-\nu, 1-\nu} \left(-\frac{2\lambda x}{s^\nu} \right) h_{2\nu}(t - s, x) ds.
 \end{aligned} \tag{2.50}$$

The densities h_ν and $h_{2\nu}$ can be written down in terms of series expansion of stable laws (see, [17, p. 245]).

3. n -Dimensional Stable Laws and Fractional Laplacian

Let

$$S_n^{2\beta}(t) = \left(S_1^{2\beta}(t), S_2^{2\beta}(t), \dots, S_n^{2\beta}(t) \right), \quad t > 0, \beta \in (0, 1], \tag{3.1}$$

be the isotropic stable n -dimensional process with joint characteristic function

$$\begin{aligned} \widehat{v}_n^{2\beta}(\boldsymbol{\xi}, t) &= \widehat{v}_n^{2\beta}(\xi_1, \xi_2, \dots, \xi_n, t) = \mathbb{E} e^{i\boldsymbol{\xi} \cdot S_n^{2\beta}(t)} = e^{-t(\sqrt{\xi_1^2 + \xi_2^2 + \dots + \xi_n^2})^{2\beta}} \\ &= e^{-t\|\boldsymbol{\xi}\|^{2\beta}}. \end{aligned} \tag{3.2}$$

The density corresponding to the characteristic function $\widehat{v}_n^{2\beta}(\boldsymbol{\xi}, t)$ is given by

$$v_n^{2\beta}(\mathbf{x}, t) = v_n^{2\beta}(x_1, x_2, \dots, x_n, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi} \cdot \mathbf{x}} e^{-t\|\boldsymbol{\xi}\|^{2\beta}} d\boldsymbol{\xi}. \tag{3.3}$$

The equation governing the distribution $v_n^{2\beta}(\mathbf{x}, t)$ of the vector process $S_n^{2\beta}(t)$, $t > 0$, is

$$\left(\frac{\partial}{\partial t} + (-\Delta)^\beta \right) v_n^{2\beta}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \mathbb{R}^n, t > 0, \tag{3.4}$$

where the fractional negative Laplacian is related to the classical Laplacian by means of the following relationships (Bochner representation, see, e.g., [3, 5])

$$\begin{aligned} \frac{\sin \pi \beta}{\pi} \int_0^\infty d\lambda \lambda^{\beta-1} (\lambda - \Delta)^{-1} \Delta &= \frac{\sin \pi \beta}{\pi} \int_0^\infty \lambda^{\beta-1} \left(\int_0^\infty e^{-w(\lambda - \Delta)} dw \right) \Delta d\lambda \\ &= \frac{\sin \pi \beta}{\pi} \Gamma(\beta) \int_0^\infty w^{1-\beta-1} e^{-w(-\Delta)} \Delta dw = \frac{1}{\Gamma(1 - \beta)} \int_0^\infty w^{1-\beta-1} e^{-w(-\Delta)} \Delta dw \\ &= -(-\Delta)^\beta. \end{aligned} \tag{3.5}$$

A definition of the fractional negative Laplacian can be given in the space of the Fourier transforms as follows

$$-(-\Delta)^\beta u(\mathbf{x}) = -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\mathbf{x} \cdot \boldsymbol{\xi}} (\xi_1^2 + \xi_2^2 + \dots + \xi_n^2)^\beta \widehat{u}(\boldsymbol{\xi}) d\boldsymbol{\xi}, \tag{3.6}$$

where

$$\text{Dom}(-\Delta)^\beta = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\widehat{u}(\boldsymbol{\xi})|^2 (1 + \|\boldsymbol{\xi}\|^{2\beta}) d\boldsymbol{\xi} < \infty \right\}. \tag{3.7}$$

An equivalent alternative definition of the n -dimensional fractional Laplacian is

$$(-\Delta)^\beta u(\mathbf{x}) = c(\beta, n) \text{P.V.} \int_{\mathbb{R}^n} \frac{u(\mathbf{x}) - u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^{n+2\beta}} dy, \tag{3.8}$$

where the multiplicative constant $c(\beta, n)$ must be evaluated in such a way that

$$\int_{\mathbb{R}^n} e^{i\boldsymbol{\xi} \cdot \mathbf{x}} (-\Delta)^\beta u(\mathbf{x}) d\mathbf{x} = \|\boldsymbol{\xi}\|^{2\beta} \int_{\mathbb{R}^n} e^{i\boldsymbol{\xi} \cdot \mathbf{x}} u(\mathbf{x}) d\mathbf{x}. \tag{3.9}$$

Let us focus our attention on the one-dimensional case of (3.8). In this case, we have that, for $0 < 2\beta < 1$,

$$\begin{aligned} \left(-\frac{\partial^2}{\partial x^2}\right)^\beta u(x) &= c(\beta, 1) \text{P.V.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+2\beta}} dy \\ &= c(\beta, 1) \lim_{\epsilon \rightarrow 0} \left[\int_{-\infty}^{0-\epsilon} \frac{u(x) - u(x-z)}{|z|^{1+2\beta}} dz + \int_{0+\epsilon}^{\infty} \frac{u(x) - u(x-z)}{|z|^{1+2\beta}} dz \right] \\ &= c(\beta, 1) \lim_{\epsilon \rightarrow 0} \left[\int_{0+\epsilon}^{\infty} \frac{u(x) - u(x+z)}{z^{1+2\beta}} dz + \int_{0+\epsilon}^{\infty} \frac{u(x) - u(x-z)}{z^{1+2\beta}} dz \right] \\ &= \frac{\Gamma(1-2\beta)}{2\beta} c(\beta, 1) \left[\frac{1}{\Gamma(1-2\beta)} \frac{d}{dx} \left(\int_{-\infty}^x \frac{u(z) dz}{(x-z)^{2\beta}} - \int_x^{\infty} \frac{u(z) dz}{(z-x)^{2\beta}} \right) \right], \end{aligned} \tag{3.10}$$

where in the intermediate steps, we considered the relation between the Marchaud and the Weyl fractional derivatives. By setting

$$c(\beta, 1) = \frac{\beta}{\Gamma(1-2\beta) \cos \beta\pi}, \tag{3.11}$$

we have that, for $0 < 2\beta < 1$,

$$\begin{aligned} &-\left(-\frac{\partial^2}{\partial x^2}\right)^\beta u(x) \\ &= -\frac{1}{2 \cos \beta\pi} \left[\frac{1}{\Gamma(1-2\beta)} \frac{d}{dx} \int_{-\infty}^x \frac{u(z) dz}{(x-z)^{2\beta}} - \frac{1}{\Gamma(1-2\beta)} \frac{d}{dx} \int_x^{\infty} \frac{u(z) dz}{(z-x)^{2\beta}} \right] \\ &= -\frac{1}{2 \cos \beta\pi} \frac{1}{\Gamma(1-2\beta)} \frac{d}{dx} \int_{-\infty}^{\infty} \frac{u(z)}{|x-z|^{2\beta}} dz = \frac{\partial^{2\beta}}{\partial |x|^{2\beta}} u(x), \end{aligned} \tag{3.12}$$

where $\frac{\partial^{2\beta}}{\partial |x|^{2\beta}}$ represents the Riesz operator.

Remark 3.1. We notice that, for $0 < 2\beta < 1$,

$$\mathcal{F} \left[\frac{\partial^{2\beta}}{\partial |x|^{2\beta}} u(x) \right] (\xi) = -|\xi|^{2\beta} \widehat{u}(\xi). \tag{3.13}$$

This is due to the calculation

$$\begin{aligned} &\mathcal{F} \left[\frac{\partial^{2\beta}}{\partial |x|^{2\beta}} u(x) \right] (\xi) \\ &= -\frac{1}{2 \cos \beta\pi} \frac{1}{\Gamma(1-2\beta)} \left[\int_{-\infty}^{\infty} dx e^{i\xi x} \left(\frac{d}{dx} \int_{-\infty}^x \frac{u(z) dz}{(x-z)^{2\beta}} - \frac{d}{dx} \int_x^{\infty} \frac{u(z) dz}{(z-x)^{2\beta}} \right) \right] \\ &= \frac{i\xi}{2 \cos \beta\pi} \frac{1}{\Gamma(1-2\beta)} \left[\int_{-\infty}^{\infty} dx e^{i\xi x} \left(\int_{-\infty}^x \frac{u(z) dz}{(x-z)^{2\beta}} - \int_x^{\infty} \frac{u(z) dz}{(z-x)^{2\beta}} \right) \right] \\ &= \frac{i\xi}{2 \cos \beta\pi} \frac{1}{\Gamma(1-2\beta)} \left[\int_{-\infty}^{\infty} dz u(z) \left(\int_z^{\infty} \frac{e^{i\xi x} dx}{(x-z)^{2\beta}} - \int_{-\infty}^z \frac{e^{i\xi x} dx}{(z-x)^{2\beta}} \right) \right] \\ &= \frac{i\xi}{2 \cos \beta\pi} \frac{1}{\Gamma(1-2\beta)} \left[\int_{-\infty}^{\infty} e^{i\xi z} u(z) dz \left(\int_0^{\infty} \frac{e^{i\xi y}}{y^{2\beta}} dy - \int_0^{\infty} \frac{e^{-i\xi y}}{y^{2\beta}} dy \right) \right] \end{aligned}$$

$$\begin{aligned}
 &= -\frac{2\xi}{2 \cos \beta\pi} \frac{1}{\Gamma(1-2\beta)} \int_{-\infty}^{\infty} e^{i\xi z} u(z) dz \int_0^{\infty} \frac{\sin \xi y}{y^{2\beta}} dy \\
 &= -\frac{\xi}{\cos \beta\pi} \frac{1}{\Gamma(1-2\beta)} \frac{\widehat{u}(\xi)}{\Gamma(2\beta)} \int_0^{\infty} \int_0^{\infty} \sin \xi y e^{-wy} w^{2\beta-1} dw dy \\
 &= -\frac{\xi}{\cos \beta\pi} \frac{1}{\Gamma(1-2\beta)} \frac{\widehat{u}(\xi)}{\Gamma(2\beta)} \int_0^{\infty} dw w^{2\beta-1} \int_0^{\infty} dy e^{-wy} \left(\frac{e^{i\xi y} - e^{-i\xi y}}{2i} \right) \\
 &= -\frac{\xi^2}{\cos \beta\pi} \frac{1}{\Gamma(1-2\beta)} \frac{\widehat{u}(\xi)}{\Gamma(2\beta)} \int_0^{\infty} dw \frac{w^{2\beta-1}}{w^2 + \xi^2} \\
 &= -\frac{\xi^2}{\cos \beta\pi} \frac{1}{\Gamma(1-2\beta)} \frac{\widehat{u}(\xi)}{\Gamma(2\beta)} \int_0^{\infty} dw w^{2\beta-1} \int_0^{\infty} dy e^{-y(w^2+\xi^2)} \\
 &= -\frac{\xi^2}{2 \cos \beta\pi} \frac{1}{\Gamma(1-2\beta)} \frac{\widehat{u}(\xi)}{\Gamma(2\beta)} \frac{\Gamma(\beta) \Gamma(1-\beta)}{|\xi|^{2-2\beta}} = -|\xi|^{2\beta} \widehat{u}(\xi). \tag{3.14}
 \end{aligned}$$

This concludes the proof of (3.13).

4. Space-Time Fractional Telegraph Equation

We consider now the composition of an isotropic vector of stable processes $S_n^{2\beta}(t), t > 0$, defined in (3.1), with the positively-valued process, defined in (2.19),

$$\mathcal{L}^\nu(t) = \inf \left\{ s \geq 0 : \mathcal{H}^\nu(s) = H_1^{2\nu}(s) + (2\lambda)^{\frac{1}{\nu}} H_2^\nu(s) \geq t \right\}, \quad t > 0, \tag{4.1}$$

where $H_1^{2\nu}, H_2^\nu$ are independent positively skewed stable processes of order 2ν and ν , respectively. The distribution $w_\nu^\beta(\mathbf{x}, t)$ of the process $S_n^{2\beta}(c^2 \mathcal{L}^\nu(t)), t > 0, \beta \in (0, 1]$, is the fundamental solution to the space-time fractional telegraph equation

$$\left(\frac{c}{\partial t^{2\nu}} + 2\lambda \frac{c}{\partial t^\nu} \right) w_\nu^\beta(\mathbf{x}, t) = -c^2 (-\Delta)^\beta w_\nu^\beta(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^n, t > 0. \tag{4.2}$$

In our view, the next theorem generalizes some previous results because we here have fractionality in space and time and the equation (4.2) is defined in \mathbb{R}^n .

Theorem 4.1. For $\nu \in (0, \frac{1}{2}]$ and $\beta \in (0, 1]$ the solution to the Cauchy problem for the space-time fractional n-dimensional telegraph equation

$$\begin{cases} \left(\frac{c}{\partial t^{2\nu}} + 2\lambda \frac{c}{\partial t^\nu} \right) w_\nu^\beta(\mathbf{x}, t) = -c^2 (-\Delta)^\beta w_\nu^\beta(\mathbf{x}, t), & \mathbf{x} \in \mathbb{R}^n, t > 0 \\ w_\nu^\beta(\mathbf{x}, 0) = \delta(\mathbf{x}), \end{cases} \tag{4.3}$$

coincides with the probability law of the vector process

$$W_n(t) = S_n^{2\beta}(c^2 \mathcal{L}^\nu(t)), \quad t > 0, \tag{4.4}$$

and has Fourier transform which reads

$$\widehat{w}_\nu^\beta(\xi, t)$$

$$\begin{aligned}
 &= \frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - c^2 \|\xi\|^{2\beta}}} \right) E_{\nu,1}(r_1 t^\nu) \right. \\
 &\quad \left. + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - c^2 \|\xi\|^{2\beta}}} \right) E_{\nu,1}(r_2 t^\nu) \right], \tag{4.5}
 \end{aligned}$$

where

$$r_1 = -\lambda + \sqrt{\lambda^2 - c^2 \|\xi\|^{2\beta}}, \quad r_2 = -\lambda - \sqrt{\lambda^2 - c^2 \|\xi\|^{2\beta}}. \tag{4.6}$$

The time derivatives appearing in (4.3) must be meant in the Dzherbashyan-Caputo sense. The fractional Laplacian is defined in (3.6).

Proof. By taking the Laplace transform of (4.3) we have

$$\mu^{2\nu} \widetilde{w}_\nu^\beta(\mathbf{x}, \mu) - \mu^{2\nu-1} \delta(\mathbf{x}) + 2\lambda \left[\mu^\nu \widetilde{w}_\nu^\beta(\mathbf{x}, \mu) - \mu^{\nu-1} \delta(\mathbf{x}) \right] = -c^2 (-\Delta)^\beta \widetilde{w}_\nu^\beta(\mathbf{x}, \mu), \tag{4.7}$$

where we used the fact that (see [12] page 98, Lemma 2.24)

$$\mathcal{L} \left[\frac{C \partial^\nu}{\partial t^\nu} w_\nu^\beta(\mathbf{x}, t) \right] = \mu^\nu \widetilde{w}_\nu^\beta(\mathbf{x}, \mu) - \mu^{\nu-1} w_\nu^\beta(\mathbf{x}, 0). \tag{4.8}$$

Now the Fourier transform of (4.7) yields

$$(\mu^{2\nu} + 2\lambda\mu^\nu) \widetilde{w}_\nu^\beta(\xi, \mu) - (\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}) = -c^2 \|\xi\|^{2\beta} \widetilde{w}_\nu^\beta(\xi, \mu), \tag{4.9}$$

and, thus,

$$\widetilde{w}_\nu^\beta(\xi, \mu) = \frac{\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}}{\mu^{2\nu} + 2\lambda\mu^\nu + c^2 \|\xi\|^{2\beta}}, \quad \mu > 0, \xi \in \mathbb{R}^n. \tag{4.10}$$

The probability density of the process $W_n(t), t > 0$, defined in (4.4), can be written as

$$w_\nu^\beta(\mathbf{x}, t) = \int_0^\infty v_\beta(\mathbf{x}, c^2 s) l_\nu(s, t) ds, \tag{4.11}$$

and has Fourier transform equal to

$$\int_{\mathbb{R}^n} e^{i\xi \cdot \mathbf{x}} w_\nu^\beta(\mathbf{x}, t) d\mathbf{x} = \int_0^\infty e^{-c^2 s \|\xi\|^{2\beta}} l_\nu(s, t) ds. \tag{4.12}$$

In order to show that the Laplace transform of (4.12) coincides with (4.10), we have to derive the Laplace transform of $l_\nu(x, t)$, with respect to the time t . Since

$$\Pr \{ \mathcal{L}^\nu(t) < x \} = \Pr \{ \mathcal{H}^\nu(x) > t \} \tag{4.13}$$

we have that

$$\widetilde{l}_\nu(x, \mu) = \int_0^\infty e^{-\mu t} \frac{\partial}{\partial x} \int_t^\infty \Pr \{ \mathcal{H}^\nu(x) \in ds \} dt = \int_0^\infty e^{-\mu t} \left(-\frac{\partial}{\partial x} \int_0^t h_\nu(s, x) ds \right) dt$$

$$= -\frac{\partial}{\partial x} \frac{e^{-x\mu^{2\nu}-2\lambda x\mu^\nu}}{\mu} = (\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}) e^{-x\mu^{2\nu}-2\lambda x\mu^\nu}, \tag{4.14}$$

where we used result (2.28). Now we can complete the proof by taking the Laplace transform of (4.12) so that, in view of (4.14), we obtain

$$\begin{aligned} & \int_0^\infty e^{-\mu t} dt \int_0^\infty e^{-c^2 s \|\xi\|^{2\beta}} f_\nu(s, t) ds \\ &= (\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}) \int_0^\infty e^{-s c^2 \|\xi\|^{2\beta} - s \mu^{2\nu} - 2\lambda s \mu^\nu} ds = \frac{\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}}{\mu^{2\nu} + 2\lambda\mu^\nu + c^2 \|\xi\|^{2\beta}}, \end{aligned} \tag{4.15}$$

which coincides with (4.10). The unicity of Fourier-Laplace transform proves that the claimed result holds. The proof that the Fourier transform of $w_\nu^\beta(x, t)$ has the form (4.5) can be carried out by means of the calculation performed in Theorem 2.2. We have that

$$\begin{aligned} \widehat{w}_\nu^\beta(\xi, \mu) &= \frac{\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}}{\mu^{2\nu} + 2\lambda\mu^\nu + c^2 \|\xi\|^{2\beta}} = \frac{\mu^{\nu-1}}{\mu^\nu - r_1} + \frac{\mu^{\nu-1}}{\mu^\nu - r_2} - \frac{\mu^{2\nu-1}}{(\mu^\nu - r_1)(\mu^\nu - r_2)} \\ &= \frac{\mu^{\nu-1}}{\mu^\nu - r_1} + \frac{\mu^{\nu-1}}{\mu^\nu - r_2} - \left[\frac{\mu^{\nu-(1-\nu)}}{\mu^\nu - r_1} - \frac{\mu^{\nu-(1-\nu)}}{\mu^\nu - r_2} \right] \frac{1}{2\sqrt{\lambda^2 - c^2 \|\xi\|^{2\beta}}}, \end{aligned} \tag{4.16}$$

where

$$r_1 = -\lambda + \sqrt{\lambda^2 - c^2 \|\xi\|^{2\beta}}, \quad r_2 = -\lambda - \sqrt{\lambda^2 - c^2 \|\xi\|^{2\beta}}. \tag{4.17}$$

and, thus, by inverting (4.16) by means of (2.33), we obtain result (4.5). An alternative derivation of (4.5) can be carried out as follows

$$\begin{aligned} \widehat{w}_\nu^\beta(\xi, t) &= \int_{-\infty}^\infty e^{i\xi \cdot x} \int_0^\infty \Pr\{S_n^{2\beta}(c^2 s) \in dx\} \Pr\{\mathcal{L}^\nu(t) \in ds\} \\ &= \int_0^\infty e^{-c^2 s \|\xi\|^{2\beta}} \Pr\{\mathcal{L}^\nu(t) \in ds\} = (4.5) \end{aligned} \tag{4.18}$$

because of Theorem 2.2. □

4.1. The Case $\nu = \frac{1}{2}$, Subordinator with Drift

The fractional equation (4.2), for $n = 1, \nu = \frac{1}{2}$, reads

$$\left(\frac{\partial}{\partial t} + 2\lambda \frac{c \partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} \right) w_{\frac{1}{2}}^\beta(x, t) = c^2 \frac{\partial^{2\beta}}{\partial |x|^{2\beta}} w_{\frac{1}{2}}^\beta(x, t), \quad 0 < \beta < 1, \tag{4.19}$$

where $\frac{\partial^{2\beta}}{\partial |x|^{2\beta}}$ is the Riesz operator defined in (3.12). For $\beta = 1$ we have the special case

$$\left(\frac{\partial}{\partial t} + 2\lambda \frac{c \partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} \right) w_{\frac{1}{2}}^1(x, t) = c^2 \frac{\partial^2}{\partial x^2} w_{\frac{1}{2}}^1(x, t) \tag{4.20}$$

dealt with in Orsingher and Beghin [14]. The construction of the composition related to Equation (4.19) involves the subordinator

$$\mathcal{H}^{\frac{1}{2}}(t) = t + (2\lambda)^2 H^{\frac{1}{2}}(t), \quad t > 0, \tag{4.21}$$

where $H^{\frac{1}{2}}(t), t > 0$, is a positively skewed stable process and has the same law as the first-passage time of a Brownian motion through level $\frac{t}{\sqrt{2}}$. We note that $\mathcal{H}^{\frac{1}{2}}(t), t > 0$, has distribution with support $[t, \infty)$ and, thus, differs from $\mathcal{H}^\nu(t), t > 0, 0 < \nu < \frac{1}{2}$, which instead has support $[0, \infty)$. The distribution of (4.21) writes

$$\Pr \left\{ \mathcal{H}^{\frac{1}{2}}(t) < x \right\} = \int_0^{\frac{x-t}{(2\lambda)^2}} \frac{t}{\sqrt{2}} \frac{e^{-\frac{t^2}{4z}}}{\sqrt{2\pi z^3}} dz, \quad x > t > 0. \tag{4.22}$$

The inverse process

$$\mathcal{L}^{\frac{1}{2}}(t) = \inf \left\{ s : s + (2\lambda)^2 H^{\frac{1}{2}}(s) \geq t \right\} = \inf \left\{ s : \mathcal{H}^{\frac{1}{2}}(s) \geq t \right\} \tag{4.23}$$

is related to (4.21) by means of the relationship

$$\Pr \left\{ \mathcal{L}^{\frac{1}{2}}(t) < x \right\} = \Pr \left\{ \mathcal{H}^{\frac{1}{2}}(x) > t \right\} = \int_{\frac{t-x}{(2\lambda)^2}}^{\infty} \frac{x}{\sqrt{2}} \frac{e^{-\frac{x^2}{4z}}}{\sqrt{2\pi z^3}} dz. \tag{4.24}$$

From (4.24), we can extract the distributon of $\mathcal{L}^{\frac{1}{2}}(t), t > 0$, in the following manner

$$\begin{aligned} l_{\frac{1}{2}}(x, t) &= \frac{\Pr \left\{ \mathcal{L}^{\frac{1}{2}}(t) \in dx \right\}}{dx} = \frac{\partial}{\partial x} \int_{\frac{t-x}{(2\lambda)^2}}^{\infty} \frac{x e^{-\frac{x^2}{4z}}}{\sqrt{4\pi z^3}} dz \\ &= \frac{2\lambda x e^{-\frac{(2\lambda x)^2}{4(t-x)}}}{\sqrt{4\pi (t-x)^3}} + 2\lambda \frac{e^{-\frac{(2\lambda x)^2}{4(t-x)}}}{\sqrt{\pi(t-x)}}, \quad 0 < x < t. \end{aligned} \tag{4.25}$$

Remark 4.1. The distribution (4.25) can be also obtained from the general case (2.50), which for $\nu = \frac{1}{2}$ becomes, for $0 < x < t$,

$$\begin{aligned} l_{\frac{1}{2}}(x, t) &= \int_0^t \delta(s-x) h_{\frac{1}{2}}(t-s, 2\lambda x) ds + 2\lambda \int_0^t l_{\frac{1}{2}}(2\lambda x, s) \delta(x-(t-s)) ds \\ &= h_{\frac{1}{2}}(t-x, 2\lambda x) + 2\lambda l_{\frac{1}{2}}(2\lambda x, t-x) \\ &= \frac{2\lambda x e^{-\frac{(2\lambda x)^2}{4(t-x)}}}{\sqrt{4\pi (t-x)^3}} + 2\lambda \frac{e^{-\frac{(2\lambda x)^2}{4(t-x)}}}{\sqrt{\pi(t-x)}}. \end{aligned} \tag{4.26}$$

In the last step we used the fact that

$$L^{\frac{1}{2}}(t) \stackrel{law}{=} |B(t)|, \quad t > 0, \tag{4.27}$$

where $L^{\frac{1}{2}}(t), t > 0$, dealt with in Section 1.3, is the inverse of the totally positively skewed stable process $H^{\frac{1}{2}}(t), t > 0$.

The t -Laplace transform of (4.25) becomes

$$\begin{aligned} \tilde{l}_{\frac{1}{2}}(x, \mu) &= \int_x^\infty e^{-\mu t} l_{\frac{1}{2}}(x, t) dt \\ &= \frac{2\lambda x}{\sqrt{2}} \int_x^\infty e^{-\mu t} \frac{e^{-\frac{(2\lambda x)^2}{4(t-x)}}}{\sqrt{2\pi(t-x)^3}} dt + 2\lambda \int_x^\infty e^{-\mu t} \frac{e^{-\frac{(2\lambda x)^2}{4(t-x)}}}{\sqrt{\pi(t-x)}} dt \\ &= \frac{2\lambda x}{\sqrt{2}} e^{-\mu x} \int_0^\infty e^{-\mu t} \frac{e^{-\frac{(2\lambda x)^2}{4t}}}{\sqrt{2\pi t^3}} dt + 2\lambda e^{-\mu x} \int_0^\infty e^{-\mu t} \frac{e^{-\frac{(2\lambda x)^2}{4t}}}{\sqrt{\pi t}} dt \\ &= e^{-\mu x} e^{-2\lambda x \sqrt{\mu}} + 2\lambda \mu^{-\frac{1}{2}} e^{-\mu x} e^{-2\lambda x \sqrt{\mu}}. \end{aligned} \tag{4.28}$$

Finally, the x -Laplace transform of (4.28) becomes

$$\begin{aligned} \tilde{\tilde{l}}_{\frac{1}{2}}(\gamma, \mu) &= \int_0^\infty e^{-\gamma x} \left(\int_x^\infty e^{-\mu t} l_{\frac{1}{2}}(x, t) dt \right) dx \\ &= \frac{1}{\mu + \gamma + 2\lambda\sqrt{\mu}} + \frac{2\lambda}{\sqrt{\mu}} \frac{1}{\mu + \gamma + 2\lambda\sqrt{\mu}} = \frac{1 + 2\lambda\mu^{-\frac{1}{2}}}{\mu + \gamma + 2\lambda\sqrt{\mu}}, \end{aligned} \tag{4.29}$$

which coincides with (2.31), for $\nu = \frac{1}{2}$. Let us now consider the process $W_n(t) = S_n^{2\beta}(c^2 \mathcal{L}^\nu(t))$, $t > 0$, dealt with in Theorem 4.1. For $\beta = 1$, $n = 1$ and $\nu = \frac{1}{2}$ this process becomes

$$W_1(t) = S_1^2(c^2 \mathcal{L}^{\frac{1}{2}}(t)) = B(c^2 \mathcal{L}^{\frac{1}{2}}(t)), \quad t > 0 \tag{4.30}$$

where B represents a standard Brownian motion and $\mathcal{L}^{\frac{1}{2}}(t)$, $t > 0$, is the process defined in (4.23). With

$$p_{|B|}(x, t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{\pi t}}, \quad x > 0, t > 0, \tag{4.31}$$

we denote the law of the process $|B(t)|$, $t > 0$. In view of the previous results we are able to prove the following theorem.

Theorem 4.2. *The law of (4.30) coincides with the law of the composition*

$$\mathcal{W}(t) = T(|B(t)|), \quad t > 0, \tag{4.32}$$

where T is the telegraph process (1.20) with parameters $c > 0$, $\lambda > 0$ and law $p_T(x, t)$ which has characteristic function

$$\begin{aligned} \widehat{p}_T(\xi, t) &= \frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - c^2 \xi^2}} \right) e^{-\lambda t + t \sqrt{\lambda^2 - c^2 \xi^2}} + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - c^2 \xi^2}} \right) e^{-\lambda t - t \sqrt{\lambda^2 - c^2 \xi^2}} \right]. \end{aligned} \tag{4.33}$$

In other words, we have the following equality in distribution

$$B\left(c^2\mathcal{L}^{\frac{1}{2}}(t)\right) \stackrel{\text{law}}{=} T(|B(t)|), \quad t > 0. \tag{4.34}$$

Proof. First, we show that the Fourier-Laplace transform of the law $w^{\frac{1}{2}}(x, t)$ of the process $W_1(t) = S_1^2(c^2\mathcal{L}^{\frac{1}{2}}(t)) = B(c^2\mathcal{L}^{\frac{1}{2}}(t)), t > 0$, is written as in (4.15) for $\nu = \frac{1}{2}, \beta = 1, n = 1$, and reads

$$\widehat{w^{\frac{1}{2}}}(\xi, \mu) = \frac{1 + 2\lambda\mu^{-\frac{1}{2}}}{\mu + 2\lambda\sqrt{\mu} + c^2\xi^2}. \tag{4.35}$$

We have that

$$\begin{aligned} \widetilde{w^{\frac{1}{2}}}(x, \mu) &= \int_0^\infty e^{-\mu t} \left(\int_0^t p_B(x, c^2s) \ell_{\frac{1}{2}}(s, t) ds \right) dt \\ &= \int_0^\infty p_B(x, c^2s) ds \int_s^\infty e^{-\mu t} \ell_{\frac{1}{2}}(s, t) dt \\ &= \int_0^\infty p_B(x, c^2s) ds \left[\int_s^\infty e^{-\mu t} \left(\frac{2\lambda s e^{-\frac{(2\lambda s)^2}{4(t-s)}}}{\sqrt{4\pi(t-s)^3}} + 2\lambda \frac{e^{-\frac{(2\lambda s)^2}{4(t-s)}}}{\sqrt{\pi(t-s)}} \right) dt \right] \\ &= \int_0^\infty p_B(x, c^2s) \left(e^{-s(\mu+2\lambda\sqrt{\mu})} + 2\lambda\sqrt{\mu} e^{-s(\mu+2\lambda\sqrt{\mu})} \right) ds \\ &= \int_0^\infty \frac{e^{-\frac{x^2}{4c^2s}}}{\sqrt{4\pi c^2s}} e^{-s(\mu+2\lambda\sqrt{\mu})} ds + 2\lambda\mu^{-\frac{1}{2}} \int_0^\infty \frac{e^{-\frac{x^2}{4c^2s}}}{\sqrt{4\pi c^2s}} e^{-s(\mu+2\lambda\sqrt{\mu})} ds, \tag{4.36} \end{aligned}$$

and, thus, by taking the Fourier transform we get

$$\begin{aligned} \widehat{w^{\frac{1}{2}}}(\xi, \mu) &= \int_0^\infty e^{-sc^2\xi^2} e^{-s(\mu+2\lambda\sqrt{\mu})} ds + 2\lambda\mu^{-\frac{1}{2}} \int_0^\infty e^{-sc^2\xi^2} e^{-s(\mu+2\lambda\sqrt{\mu})} ds \\ &= \frac{1 + 2\lambda\mu^{-\frac{1}{2}}}{\mu + 2\lambda\sqrt{\mu} + c^2\xi^2}. \tag{4.37} \end{aligned}$$

Now we are going to prove that the law $w(x, t)$ of the process $\mathcal{W}(t), t > 0$, has Fourier-Laplace transform that coincides with (4.35). We have that

$$w(x, t) = \int_0^\infty p_T(x, s) p_{|B|}(s, t) ds, \tag{4.38}$$

and thus the Fourier transform of $w(x, t)$ reads

$$\begin{aligned} \widehat{w}(\xi, t) &= \int_{-\infty}^\infty e^{i\xi x} dx \int_0^\infty p_T(x, s) p_{|B|}(s, t) ds \\ &= \frac{1}{2} \int_0^\infty \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - c^2\xi^2}} \right) e^{-\lambda s + s\sqrt{\lambda^2 - c^2\xi^2}} \right] \end{aligned}$$

$$+ \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - c^2\xi^2}} \right) e^{-\lambda s - s\sqrt{\lambda^2 - c^2\xi^2}} \Big] p_{|B|}(s, t) ds. \tag{4.39}$$

Passing now to the Laplace transform, we have

$$\begin{aligned} \tilde{w}(\xi, \mu) &= \frac{1}{2} \int_0^\infty e^{-\mu t} dt \int_0^\infty \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - c^2\xi^2}} \right) e^{-\lambda s + s\sqrt{\lambda^2 - c^2\xi^2}} \right. \\ &\quad \left. + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - c^2\xi^2}} \right) e^{-\lambda s - s\sqrt{\lambda^2 - c^2\xi^2}} \right] \frac{e^{-\frac{s^2}{4t}}}{\sqrt{\pi t}} ds \\ &= \frac{1}{2} \int_0^\infty \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - c^2\xi^2}} \right) e^{-\lambda s + s\sqrt{\lambda^2 - c^2\xi^2}} \right. \\ &\quad \left. + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - c^2\xi^2}} \right) e^{-\lambda s - s\sqrt{\lambda^2 - c^2\xi^2}} \right] \frac{e^{-s\sqrt{\mu}}}{\sqrt{\mu}} ds \\ &= \frac{1}{2\sqrt{\mu}} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - c^2\xi^2}} \right) \left(\frac{1}{\lambda + \sqrt{\mu} - \sqrt{\lambda^2 - c^2\xi^2}} \right) \right. \\ &\quad \left. + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - c^2\xi^2}} \right) \left(\frac{1}{\lambda + \sqrt{\mu} + \sqrt{\lambda^2 - c^2\xi^2}} \right) \right] \\ &= \frac{(\lambda + \sqrt{\lambda^2 - c^2\xi^2})(\lambda + \sqrt{\mu} + \sqrt{\lambda^2 - c^2\xi^2})}{(2\sqrt{\mu}\sqrt{\lambda^2 - c^2\xi^2})(\mu + 2\lambda\sqrt{\mu} + c^2\xi^2)} \\ &\quad + \frac{(\sqrt{\lambda^2 - c^2\xi^2} - \lambda)(\lambda + \sqrt{\mu} - \sqrt{\lambda^2 - c^2\xi^2})}{(2\sqrt{\mu}\sqrt{\lambda^2 - c^2\xi^2})(\mu + 2\lambda\sqrt{\mu} + c^2\xi^2)} \\ &= \frac{1 + 2\lambda\mu^{-\frac{1}{2}}}{\mu + 2\lambda\sqrt{\mu} + c^2\xi^2}, \tag{4.40} \end{aligned}$$

which coincides with (4.35). □

This shows that for each t we have the following equality in distribution

$$T(|B(t)|) \stackrel{\text{law}}{=} B\left(c^2\mathcal{L}^{\frac{1}{2}}(t)\right), \quad t > 0, \tag{4.41}$$

where the role of the Brownian motion is interchanged in the two members of (4.41). Thus, by suitably slowing down the time in (4.41), we obtain the same distributional effect of a telegraph process taken at a Brownian time.

Remark 4.2. The probability distribution of the process

$$W_1(t) = B(c^2\mathcal{L}^{\frac{1}{2}}(t)), \quad t > 0, \tag{4.42}$$

can be written as

$$\begin{aligned}
 w_{\frac{1}{2}}^1(x, t) &= \frac{\lambda}{c\pi} \int_0^t \frac{1}{\sqrt{s(t-s)}} e^{-\frac{x^2}{4c^2s} - \frac{\lambda^2s^2}{t-s}} \left[\frac{s}{2(t-s)} + 1 \right] ds \\
 &= \frac{\lambda}{c\pi} \int_0^t \frac{1}{\sqrt{s(t-s)}} e^{-\frac{x^2}{4c^2s} - \frac{\lambda^2s^2}{t-s}} \left[\frac{1}{2} \left(1 + \frac{t}{t-s} \right) \right] ds \\
 &\stackrel{y=\lambda s}{=} \frac{\sqrt{\lambda}}{c\pi} \int_0^{\lambda t} e^{-\frac{\lambda x^2}{4c^2y}} e^{-\frac{y^2}{t-\frac{y}{\lambda}}} \frac{1}{\sqrt{y}\sqrt{t-\frac{y}{\lambda}}} \left[\frac{1}{2} \left(1 + \frac{t}{t-\frac{y}{\lambda}} \right) \right] dy. \tag{4.43}
 \end{aligned}$$

Taking the limit for $c \rightarrow \infty, \lambda \rightarrow \infty, \frac{c^2}{\lambda} \rightarrow 1$, formula (4.43) becomes

$$\lim_{\substack{\lambda, c \rightarrow \infty \\ \frac{c^2}{\lambda} \rightarrow 1}} y_{\frac{1}{2}}^1(x, t) = 2 \int_0^\infty \frac{e^{-\frac{x^2}{4y}} e^{-\frac{y^2}{t}}}{\sqrt{4\pi y} \sqrt{\pi t}} dy \tag{4.44}$$

which coincides with the distribution of an iterated Brownian motion $B_1(|B_2(t)|)$, $t > 0$, with $B_j, j = 1, 2$, independent Brownian motions. From (4.43), we can see that the distribution of $W_1(t), t > 0$, has a bell-shaped structure. Finally, we show that the density $w_{\frac{1}{2}}^1(x, t)$ integrates to unity in force of the calculation

$$\begin{aligned}
 \int_{-\infty}^\infty w_{\frac{1}{2}}^1(x, t) dx &= \int_{-\infty}^\infty dx \int_0^t ds \frac{e^{-\frac{x^2}{4s}}}{\sqrt{4\pi s}} l_{\frac{1}{2}}(s, t) = \int_0^t ds \left(\frac{\partial}{\partial s} \int_{\frac{t-s}{(2\lambda)^2}}^\infty \frac{s e^{-\frac{s^2}{4z}}}{\sqrt{4\pi z^3}} dz \right) \\
 &= \left[\int_{\frac{t-s}{(2\lambda)^2}}^\infty \frac{s e^{-\frac{s^2}{4z}}}{\sqrt{4\pi z^3}} dz \right]_{s=0}^{s=t} = \int_0^\infty \frac{t e^{-\frac{t^2}{4z}}}{\sqrt{4\pi z^3}} dz = 1. \tag{4.45}
 \end{aligned}$$

In the intermediate step, formula (4.25) has been applied.

Remark 4.3. The characteristic function of the process $T^{2\beta}(t), t > 0$, whose distribution satisfies

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} \right) p_T^{2\beta}(x, t) = c^2 \frac{\partial^{2\beta}}{\partial |x|^{2\beta}} p_T^{2\beta}(x, t), & 0 < \beta < 1, \beta \neq \frac{1}{2} \\ p_T^{2\beta}(x, 0) = \delta(x), \\ \left. \frac{\partial}{\partial t} p_T^{2\beta}(x, t) \right|_{t=0} = 0, \end{cases} \tag{4.46}$$

reads

$$\begin{aligned}
 \mathbb{E} e^{i\xi T^{2\beta}(t)} &= \frac{e^{-\lambda t}}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - c^2|\xi|^{2\beta}}} \right) e^{t\sqrt{\lambda^2 - c^2|\xi|^{2\beta}}} \right. \\
 &\quad \left. + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - c^2|\xi|^{2\beta}}} \right) e^{-t\sqrt{\lambda^2 - c^2|\xi|^{2\beta}}} \right] \tag{4.47}
 \end{aligned}$$

see [15]. Therefore, by performing the same steps as in theorem (4.2), we prove that

$$S_1^{2\beta}(\mathcal{L}^{\frac{1}{2}}(t)) \stackrel{\text{law}}{=} T^{2\beta}(|B(t)|), \quad t > 0. \tag{4.48}$$

4.2. The Case $\nu = \frac{1}{3}$, Convolutions of Airy Functions

We first recall that the totally positively skewed stable process $H^{\frac{1}{3}}(t)$, $t > 0$ has law

$$\Pr \left\{ H^{\frac{1}{3}}(t) \in dx \right\} = \frac{t}{x^{\frac{2}{3}}\sqrt{3x}} \text{Ai} \left(\frac{t}{\sqrt[3]{3x}} \right) dx, \quad x > 0, t > 0, \tag{4.49}$$

where $\text{Ai}(\cdot)$ is the Airy function. Result (4.49) can be obtained from the general series expansion of the stable law of order $\frac{1}{3}$ (see [17], p. 245) which reads

$$\begin{aligned} h_{\frac{1}{3}}(x, 1) &= \frac{1}{3\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma\left(\frac{k+1}{3}\right)}{k!} x^{-\frac{1}{3}(k+1)-1} \sin\left(\frac{\pi}{3}(k+1)\right) \\ &= \frac{1}{3\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma\left(\frac{k+1}{3}\right)}{k!} x^{-\frac{k+1}{3}-1} (-1)^k \sin\left(\frac{2\pi(k+1)}{3}\right) \\ &= \frac{1}{3} \frac{3^{\frac{2}{3}}}{x^{\frac{2}{3}}\sqrt{3x}} \text{Ai} \left(\frac{1}{\sqrt[3]{3x}} \right) = \frac{1}{x^{\frac{2}{3}}\sqrt{3x}} \text{Ai} \left(\frac{1}{\sqrt[3]{3x}} \right), \end{aligned} \tag{4.50}$$

where we used formula (4.10) of [17], which reads

$$\text{Ai}(w) = \frac{3^{-\frac{2}{3}}}{\pi} \sum_{k=0}^{\infty} \left(3^{\frac{1}{3}}w\right)^k \frac{\sin\left(\frac{2\pi(k+1)}{3}\right)}{k!} \Gamma\left(\frac{k+1}{3}\right). \tag{4.51}$$

Since

$$H^{\frac{1}{3}}(t) \stackrel{\text{law}}{=} t^3 H^{\frac{1}{3}}(1), \tag{4.52}$$

we have result (4.49). From the relationship between $H^{\frac{1}{3}}(t)$, $t > 0$, and the inverse process $L^{\frac{1}{3}}(t)$, $t > 0$,

$$\Pr \left\{ H^{\frac{1}{3}}(t) < x \right\} = \Pr \left\{ L^{\frac{1}{3}}(x) > t \right\} \tag{4.53}$$

we extract the density of $L^{\frac{1}{3}}(x)$, $x > 0$,

$$\begin{aligned} \frac{\Pr \left\{ L^{\frac{1}{3}}(x) \in dt \right\}}{dt} &= -\frac{\partial}{\partial t} \int_0^x \frac{t}{s^{\frac{2}{3}}\sqrt{3s}} \text{Ai} \left(\frac{t}{\sqrt[3]{3s}} \right) ds \\ &= -\int_0^x \frac{1}{s^{\frac{2}{3}}\sqrt{3s}} \text{Ai} \left(\frac{t}{\sqrt[3]{3s}} \right) ds - \int_0^x \frac{t}{s^{\frac{2}{3}}\sqrt{3s}} \text{Ai}' \left(\frac{t}{\sqrt[3]{3s}} \right) \frac{ds}{\sqrt[3]{3s}} \end{aligned} \tag{4.54}$$

Since

$$\frac{\partial}{\partial s} \text{Ai} \left(\frac{t}{\sqrt[3]{3s}} \right) = -\frac{t}{3s^{\frac{2}{3}}\sqrt{3s}} \text{Ai}' \left(\frac{t}{\sqrt[3]{3s}} \right) \tag{4.55}$$

we conclude that, for $x > 0, t > 0$,

$$l_{\frac{1}{3}}(t, x) = \frac{\Pr \left\{ L^{\frac{1}{3}}(x) \in dt \right\}}{dt}$$

$$\begin{aligned}
 &= \int_0^x \frac{-1}{s\sqrt[3]{3s}} \text{Ai}\left(\frac{t}{\sqrt[3]{3s}}\right) ds + \int_0^x \frac{3}{\sqrt[3]{3s}} \frac{\partial}{\partial s} \text{Ai}\left(\frac{t}{\sqrt[3]{3s}}\right) ds \\
 &= \int_0^x \frac{-1}{s\sqrt[3]{3s}} \text{Ai}\left(\frac{t}{\sqrt[3]{3s}}\right) ds + \left[\frac{3}{\sqrt[3]{3s}} \text{Ai}\left(\frac{t}{\sqrt[3]{3s}}\right) \right]_{s=0}^{s=x} + \int_0^x \frac{ds}{s\sqrt[3]{3s}} \text{Ai}\left(\frac{t}{\sqrt[3]{3s}}\right) \\
 &= \frac{3}{\sqrt[3]{3x}} \text{Ai}\left(\frac{t}{\sqrt[3]{3x}}\right). \tag{4.56}
 \end{aligned}$$

In the last step we took into account the asymptotic expansion 7.2.19 of Bleistein and Handelsman [4].

With a similar calculation we obtain the law $h_{\frac{2}{3}}(x, t)$ of the process $H_{\frac{2}{3}}(t)$, $t > 0$, which is expressed in terms of the Airy function. From the general series expression of the stable law (see [17]), we have that

$$\begin{aligned}
 &h_{\frac{2}{3}}(x, 1) \\
 &= \frac{2}{3\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma\left(\frac{2}{3}(k+1)\right)}{k!} x^{-\frac{2}{3}(k+1)-1} \sin\left(\frac{2\pi}{3}(k+1)\right) \\
 &= \frac{2}{3\pi\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{-\frac{2}{3}(k+1)-1}}{k! 2^{1-\frac{2}{3}(k+1)}} \Gamma\left(\frac{k+1}{3}\right) \sin\left(\frac{2\pi}{3}(k+1)\right) \int_0^{\infty} dw e^{-w} w^{\frac{k+1}{3}+\frac{1}{2}-1} \\
 &= \frac{1}{x} \sqrt[3]{\frac{2^2}{3x^2}} \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-w} w^{-\frac{1}{6}} \text{Ai}\left(-\sqrt[3]{\frac{2^2 w}{3x^2}}\right) dw, \tag{4.57}
 \end{aligned}$$

and, thus, in force of the fact that $H_{\frac{2}{3}}(t) \stackrel{\text{law}}{=} t^{\frac{3}{2}} H_{\frac{2}{3}}(1)$,

$$h_{\frac{2}{3}}(x, t) = \frac{t}{\sqrt{\pi}} \frac{1}{x} \int_0^{\infty} dw e^{-w} w^{-\frac{1}{6}} \sqrt[3]{\frac{2^2}{3x^2}} \text{Ai}\left(-t\sqrt[3]{\frac{2^2 w}{3x^2}}\right). \tag{4.58}$$

Remark 4.4. We check that the distribution (4.58) integrates to unity. We have that

$$\begin{aligned}
 &\int_0^{\infty} h_{\frac{2}{3}}(x, t) dx \\
 &= \frac{t}{\sqrt{\pi}} \int_0^{\infty} dw e^{-w} w^{-\frac{1}{6}} \sqrt[3]{\frac{2^2}{3}} \int_0^{\infty} dx x^{-\frac{2}{3}-1} \text{Ai}\left(-t\sqrt[3]{\frac{2^2 w}{3x^2}}\right) \\
 &\stackrel{y=x^{-\frac{2}{3}}t\sqrt[3]{\frac{2^2 w}{3}}}{=} \frac{t}{\sqrt{\pi}} \int_0^{\infty} dw e^{-w} w^{-\frac{1}{6}} \sqrt[3]{\frac{2^2}{3}} \frac{3}{2} \left(t\sqrt[3]{\frac{2^2 w}{3}}\right)^{-1} \int_0^{\infty} dy \text{Ai}(-y) \\
 &= \frac{1}{\sqrt{\pi}} \int_0^{\infty} dw e^{-w} w^{-\frac{1}{6}} \sqrt[3]{\frac{2^2}{3}} \left(\sqrt[3]{\frac{2^2 w}{3}}\right)^{-1} \\
 &= \frac{1}{\sqrt{\pi}} \int_0^{\infty} dw e^{-w} w^{-\frac{1}{6}-\frac{1}{3}} = \frac{1}{\sqrt{\pi}} \int_0^{\infty} dw e^{-w} w^{\frac{1}{2}-1} = 1, \tag{4.59}
 \end{aligned}$$

where we used the fact that

$$\int_0^\infty dy \operatorname{Ai}(-y) = \frac{2}{3}. \tag{4.60}$$

For the law of the process $L^{\frac{2}{3}}(x)$, $x > 0$, we, therefore, have that

$$\begin{aligned} \Pr \left\{ L^{\frac{2}{3}}(x) < t \right\} &= \Pr \left\{ H^{\frac{2}{3}}(t) > x \right\} \\ &= \int_0^\infty \int_x^\infty \frac{t}{\sqrt{\pi}} \frac{1}{z} \sqrt{\frac{2^2}{3z^2}} \operatorname{Ai} \left(-t \sqrt[3]{\frac{2^2 w}{3z^2}} \right) e^{-w} w^{-\frac{1}{6}} dw dz \end{aligned} \tag{4.61}$$

and, thus,

$$\begin{aligned} l_{\frac{2}{3}}(t, x) &= \int_0^\infty \int_x^\infty \frac{dw dz}{z\sqrt{\pi}} \sqrt{\frac{2^2}{3z^2}} \operatorname{Ai} \left(-t \sqrt[3]{\frac{2^2 w}{3z^2}} \right) e^{-w} w^{-\frac{1}{6}} \\ &\quad - \int_0^\infty \int_x^\infty \frac{t}{z\sqrt{\pi}} \sqrt{\frac{2^2}{3z^2}} \sqrt{\frac{2^2 w}{3z^2}} \operatorname{Ai}' \left(-t \sqrt[3]{\frac{2^2 w}{3z^2}} \right) dz dw \\ &= \int_0^\infty \int_x^\infty \frac{dw dz}{z\sqrt{\pi}} \sqrt{\frac{2^2}{3z^2}} \operatorname{Ai} \left(-t \sqrt[3]{\frac{2^2 w}{3z^2}} \right) e^{-w} w^{-\frac{1}{6}} \\ &\quad - \frac{3}{2} \int_x^\infty \int_0^\infty \frac{1}{\sqrt{\pi}} \sqrt{\frac{2^2}{3z^2}} e^{-w} w^{-\frac{1}{6}} \frac{\partial}{\partial z} \operatorname{Ai} \left(-t \sqrt[3]{\frac{2^2 w}{3z^2}} \right) dw dz \\ &= \int_0^\infty \int_x^\infty \frac{dw dz}{z\sqrt{\pi}} \sqrt{\frac{2^2}{3z^2}} \operatorname{Ai} \left(-t \sqrt[3]{\frac{2^2 w}{3z^2}} \right) e^{-w} w^{-\frac{1}{6}} \\ &\quad - \left[\frac{3}{2\sqrt{\pi}} \int_0^\infty dw \sqrt{\frac{2^2}{3z^2}} e^{-w} w^{-\frac{1}{6}} \operatorname{Ai} \left(-t \sqrt[3]{\frac{2^2 w}{3z^2}} \right) \right]_{z=x}^{z=\infty} \\ &\quad - \int_0^\infty \int_x^\infty \frac{dw dz}{z\sqrt{\pi}} \sqrt{\frac{2^2}{3z^2}} \operatorname{Ai} \left(-t \sqrt[3]{\frac{2^2 w}{3z^2}} \right) e^{-w} w^{-\frac{1}{6}} \\ &= \frac{3}{2\sqrt{\pi}} \int_0^\infty \sqrt{\frac{2^2}{3x^2}} e^{-w} w^{-\frac{1}{6}} \operatorname{Ai} \left(-t \sqrt[3]{\frac{2^2 w}{3x^2}} \right) dw. \end{aligned} \tag{4.62}$$

For checking that (4.62) integrates to unity one can perform calculation similar to that of Remark 4.4.

Now we have all the information to get the distribution of the process $\mathcal{L}^{\frac{1}{3}}(t), t > 0$, by means of formula (2.50). We have that

$$\begin{aligned}
 \ell_{\frac{1}{3}}(x, t) &= \frac{\Pr \left\{ \mathcal{L}^{\frac{1}{3}}(t) \in dx \right\}}{dx} \\
 &= \int_0^t \ell_{\frac{2}{3}}(x, t-s) h_{\frac{1}{3}}(s, 2\lambda x) ds + 2\lambda \int_0^t \ell_{\frac{1}{3}}(2\lambda x, s) h_{\frac{2}{3}}(t-s, x) ds \\
 &= \int_0^t ds \left[\frac{3}{2\sqrt{\pi}} \int_0^\infty dw \sqrt[3]{\frac{2^2}{3(t-s)^2}} e^{-w} w^{-\frac{1}{6}} \text{Ai} \left(-x \sqrt[3]{\frac{2^2 w}{3(t-s)^2}} \right) dw \right] \\
 &\quad \times \frac{2\lambda x}{s\sqrt[3]{3s}} \text{Ai} \left(\frac{2\lambda x}{\sqrt[3]{3s}} \right) + 2\lambda \int_0^t ds \frac{3}{\sqrt[3]{3s}} \text{Ai} \left(\frac{2\lambda x}{\sqrt[3]{3s}} \right) \\
 &\quad \times \frac{s}{\sqrt{\pi}(t-s)} \int_0^\infty dw e^{-w} w^{-\frac{1}{6}} \sqrt[3]{\frac{2^2}{3(t-s)^2}} \text{Ai} \left(-x \sqrt[3]{\frac{2^2 w}{3(t-s)^2}} \right) \\
 &= \frac{2\lambda}{\sqrt{\pi}} \int_0^t ds \int_0^\infty dw e^{-w} w^{-\frac{1}{6}} \text{Ai} \left(-x \sqrt[3]{\frac{2^2 w}{3(t-s)^2}} \right) \text{Ai} \left(\frac{2\lambda x}{\sqrt[3]{3s}} \right) \\
 &\quad \times \frac{3}{\sqrt[3]{3s}} \sqrt[3]{\frac{2^2}{3(t-s)^2}} \left[\frac{x}{2s} + \frac{s}{t-s} \right]. \tag{4.63}
 \end{aligned}$$

Result (4.63) permits us to write explicitly the solution of the fractional telegraph equation (1.4) for $\nu = \frac{1}{3}, \beta = 1$ and $n = 1$, as

$$w_{\frac{1}{3}}(x, t) = \int_0^\infty \frac{e^{-\frac{x^2}{4c^2s}}}{\sqrt{4\pi c^2s}} \ell_{\frac{1}{3}}(s, t) ds, \quad x \in \mathbb{R}, t > 0. \tag{4.64}$$

4.3. The Planar Case

Let us consider the planar process

$$\mathbf{T}(t) = (X(t), Y(t)), \quad t > 0, \tag{4.65}$$

with infinite directions and finite velocity c , investigated in Orsingher and De Gregorio [16], which has probability law (see formula 1.2 therein)

$$r(x, y, t) = \frac{\lambda}{2\pi c} \frac{e^{-\lambda t + \frac{\lambda}{c} \sqrt{c^2 t^2 - (x^2 + y^2)}}}{\sqrt{c^2 t^2 - (x^2 + y^2)}}, \quad x^2 + y^2 < c^2 t^2, t > 0, \tag{4.66}$$

which satisfies the telegraph equation

$$\left(\frac{\partial^2}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} \right) r(x, y, t) = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) r(x, y, t). \tag{4.67}$$

The distribution of $T(t), t > 0$, has a singular component uniformly distributed on the circle $\partial C_{ct} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = c^2 t^2\}$ with probability mass equal to $e^{-\lambda t}$. The process $T(t), t > 0$, describes a random motion where directions change at Poisson paced times and the orientation of each segment of the sample paths is uniform in $[0, 2\pi)$.

Let $q(x, y, t)$ be the distribution obtained by means of the composition of the process $T(t)$ with a reflecting Brownian motion with law

$$p_{|B|}(s, t) = \frac{e^{-\frac{s^2}{4t}}}{\sqrt{\pi t}}, \quad t > 0, s > 0, \tag{4.68}$$

which satisfies the equation

$$\frac{c}{\partial t^{\frac{1}{2}}} p_{|B|}(s, t) = -\frac{\partial}{\partial s} p_{|B|}(s, t) \tag{4.69}$$

and also

$$\frac{\partial}{\partial t} p_{|B|}(s, t) = \frac{\partial^2}{\partial s^2} p_{|B|}(s, t) \tag{4.70}$$

We have the following theorem.

Theorem 4.3. *The law of the composition*

$$Q(t) = T(|B(t)|), \quad t > 0 \tag{4.71}$$

written as

$$q(x, y, t) = \int_0^\infty r(x, y, s) p_{|B|}(s, t) ds, \tag{4.72}$$

satisfies the two-dimensional time-fractional equation

$$\left(\frac{\partial}{\partial t} + 2\lambda \frac{c}{\partial t^{\frac{1}{2}}} \right) q(x, y, t) = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) q(x, y, t), \quad x, y \in \mathbb{R}, t > 0, \tag{4.73}$$

subject to the initial condition $q(x, y, 0) = \delta(x, y)$.

Proof. By considering (4.72) and (4.69), we can write

$$\begin{aligned} \frac{c}{\partial t^{\frac{1}{2}}} q(x, y, t) &= \int_0^\infty r(x, y, s) \frac{c}{\partial t^{\frac{1}{2}}} p_{|B|}(s, t) ds \\ &= \int_0^\infty r(x, y, s) \left(-\frac{\partial}{\partial s} p_{|B|}(s, t) \right) ds \\ &= \left[-p_{|B|}(s, t) r(x, y, s) \right]_{s=0}^{s=\infty} + \int_0^\infty p_{|B|}(s, t) \frac{\partial}{\partial s} r(x, y, s) ds. \end{aligned} \tag{4.74}$$

In the previous step, it must be taken into account that the boundary ∂C_{cs} is excluded. From (4.72) and (4.70), we have that

$$\frac{\partial}{\partial t} q(x, y, t) = \int_0^\infty r(x, y, s) \frac{\partial}{\partial t} p_{|B|}(s, t) ds = \int_0^\infty r(x, y, s) \frac{\partial^2}{\partial s^2} p_{|B|}(s, t) ds$$

$$\begin{aligned}
 &= \left[r(x, y, s) \frac{\partial}{\partial s} p_{|B|}(s, t) \right]_{s=0}^{s=\infty} - \int_0^\infty \frac{\partial}{\partial s} r(x, y, s) \frac{\partial}{\partial s} p_{|B|}(s, t) ds \\
 &= - \left[p_{|B|}(s, t) \frac{\partial}{\partial s} r(x, y, s) \right]_{s=0}^{s=\infty} + \int_0^\infty p_{|B|}(s, t) \frac{\partial^2}{\partial s^2} r(x, y, s) ds \tag{4.75}
 \end{aligned}$$

Thus, by looking at (4.67), (4.74), and (4.75) we obtain

$$\begin{aligned}
 &\frac{\partial}{\partial t} q(x, y, t) + 2\lambda \frac{c \partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} q(x, y, t) \\
 &= \int_0^\infty p_{|B|}(s, t) \left[\frac{\partial^2}{\partial s^2} r(x, y, s) + 2\lambda \frac{\partial}{\partial s} r(x, y, s) \right] ds \\
 &= \int_0^\infty p_{|B|}(s, t) c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) r(x, y, s) ds = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) q(x, y, t). \tag{4.76}
 \end{aligned}$$

which means that $q(x, y, t)$ satisfies equation (4.73). □

It is easy to show that the process $\mathbf{Q}(t) = \mathbf{T}(|B(t)|)$, $t > 0$, has not the same law of the process $\mathbf{W}_2(t) = \mathbf{B}_2(c^2 \mathcal{L}^{\frac{1}{2}}(t))$, $t > 0$. However, it is possible to construct a planar process, say $\mathfrak{T}(t)$, $t > 0$ (which is a slightly different version of $\mathbf{T}(t)$, $t > 0$) composed with a suitable “time process” which has the same distribution as $\mathbf{W}_2(t)$, $t > 0$. The planar random motion $\mathfrak{T}(t)$, $t > 0$, with distribution

$$\tau(x, y, t) = \frac{\lambda e^{-\lambda t}}{2\pi c} \left[\frac{e^{\frac{\lambda}{c} \sqrt{c^2 t^2 - (x^2 + y^2)}} + e^{-\frac{\lambda}{c} \sqrt{c^2 t^2 - (x^2 + y^2)}}}{\sqrt{c^2 t^2 - (x^2 + y^2)}} \right], \tag{4.77}$$

where $(x, y) \in C_{ct} = \{(x, y) : x^2 + y^2 < c^2 t^2\}$, can be constructed starting from the model dealt with in Orsingher and De Gregorio [16]. The distribution is based on the solution to the planar telegraph equation

$$\left(\frac{\partial^2}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} \right) \tau(x, y, t) = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \tau(x, y, t), \tag{4.78}$$

namely,

$$\tau(x, y, t) = \frac{e^{-\lambda t}}{\sqrt{c^2 t^2 - (x^2 + y^2)}} \left[A e^{\frac{\lambda}{c} \sqrt{c^2 t^2 - (x^2 + y^2)}} + B e^{-\frac{\lambda}{c} \sqrt{c^2 t^2 - (x^2 + y^2)}} \right], \tag{4.79}$$

with $A = B = \frac{\lambda}{2\pi c}$ and, thus, we can easily check that

$$\int_{C_{ct}} dx dy \tau(x, y, t) = 1 - e^{-2\lambda t}. \tag{4.80}$$

We take a particle starting from the origin, moving at finite velocity c , and changing direction (chosen with uniform distribution) at Poisson times and neglect displacements started off by even-labeled times. The sample paths of this motion are constructed by piecing together only odd-order displacements of the planar motion $\mathbf{T}(t)$, $t > 0$. The process just described

has distribution (4.77) as shown below

$$\begin{aligned}
 & \tau(x, y, t) \\
 &= \frac{\Pr\{\mathfrak{T}(t) \in d\mathbf{x}\}}{d\mathbf{x}} = \frac{\lambda e^{-\lambda t}}{2\pi c} \left[\frac{e^{\frac{\lambda}{c}\sqrt{c^2t^2-(x^2+y^2)}} + e^{-\frac{\lambda}{c}\sqrt{c^2t^2-(x^2+y^2)}}}{\sqrt{c^2t^2-(x^2+y^2)}} \right] \\
 &= \frac{\lambda^2}{c^2} \frac{1}{\pi} e^{-\lambda t} \left[\sum_{k=0}^{\infty} \left(\frac{\lambda}{c} \sqrt{c^2t^2-(x^2+y^2)} \right)^{2k-1} \frac{1}{(2k)!} \right] \\
 &= \frac{\lambda^2}{c^2} \frac{1}{\pi} \sum_{k=0}^{\infty} \left(\frac{\lambda}{c} \right)^{2k-1} (2k+1) (c^2t^2-(x^2+y^2))^{k-\frac{1}{2}} \frac{e^{-\lambda t}}{(2k)!(2k+1)} \frac{(\lambda t)^{2k+1}}{(\lambda t)^{2k+1}} \\
 &= 2 \sum_{k=0}^{\infty} \Pr\{X(t) \in dx, Y(t) \in dy | N(t) = 2k+1\} e^{-\lambda t} \frac{(\lambda t)^{2k+1}}{(2k+1)!} \\
 &= 2 \sum_{k=0}^{\infty} \Pr\{\mathbf{T}(t) \in d\mathbf{x} | N(t) = 2k+1\} e^{-\lambda t} \frac{(\lambda t)^{2k+1}}{(2k+1)!}, \tag{4.81}
 \end{aligned}$$

where, for $x^2 + y^2 < c^2t^2$ (see [16]),

$$\frac{\Pr\{X(t) \in dx, Y(t) \in dy | N(t) = n\}}{dx dy} = \frac{n}{2n(ct)^n} (c^2t^2 - (x^2 + y^2))^{\frac{n}{2}-1}, \tag{4.82}$$

and

$$2e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} 2 \Pr\{N(t) = 2k+1\} = 1 - e^{-2\lambda t}. \tag{4.83}$$

The factor 2 appearing in (4.81) and (4.83) can be interpreted as follows. The displacements generated by an even number of Poisson events are disregarded and replaced by displacements produced by an odd number of deviations. Therefore, odd-order Poisson events ignite twice the displacements considered in (4.81).

Theorem 4.4. *The composition with distribution*

$$\mathfrak{q}(x, y, t) = \int_0^\infty ds \tau(x, y, s) \left[p_{|B|}(s, t) + \frac{1}{2\lambda} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} p_{|B|}(s, t) \right], \tag{4.84}$$

which satisfies the time-fractional equation

$$\left(\frac{\partial}{\partial t} + 2\lambda \frac{c \partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} \right) \mathfrak{q}(x, y, t) = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \mathfrak{q}(x, y, t), \tag{4.85}$$

has the same law of the process $\mathbf{W}_2(t) = \mathbf{B}_2(c^2 \mathcal{L}^{\frac{1}{2}}(t))$.

Proof. We begin by evaluating the Fourier-Laplace transform of (4.84).

$$\widehat{\mathfrak{q}}(\xi, \alpha, \mu)$$

$$\begin{aligned}
 &= \int_0^\infty ds \int_0^\infty dt e^{-\mu t} \int_{C_{ct}} dx dy e^{i\xi x + i\alpha y} \tau(x, y, s) \left[p_{|B|}(s, t) + \frac{1}{2\lambda} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} p_{|B|}(s, t) \right] \\
 &= \frac{2\lambda + \sqrt{\mu}}{2\lambda\sqrt{\mu}} \int_0^\infty ds \int_{C_{ct}} dx dy e^{i\xi x + i\alpha y} \tau(x, y, s) e^{-s\sqrt{\mu}}. \tag{4.86}
 \end{aligned}$$

Now we need the Fourier transform of the law $\tau(x, y, t)$ of the process $\mathfrak{T}(t)$, $t > 0$, which reads

$$\begin{aligned}
 &\widehat{\tau}(\xi, \alpha, t) \\
 &= \frac{\lambda e^{-\lambda t}}{2\pi c} \iint_{C_{ct}} e^{i\xi x + i\alpha y} \left[\frac{e^{\frac{\lambda}{c}\sqrt{c^2 t^2 - (x^2 + y^2)}} + e^{-\frac{\lambda}{c}\sqrt{c^2 t^2 - (x^2 + y^2)}}}{\sqrt{c^2 t^2 - (x^2 + y^2)}} \right] dx dy \\
 &= \frac{\lambda e^{-\lambda t}}{2\pi c} \int_0^{2\pi} d\theta \int_0^{ct} d\rho \rho e^{i\rho(\xi \cos\theta + \alpha \sin\theta)} \frac{\lambda}{c} \frac{e^{\frac{\lambda}{c}\sqrt{c^2 t^2 - \rho^2}} + e^{-\frac{\lambda}{c}\sqrt{c^2 t^2 - \rho^2}}}{\sqrt{c^2 t^2 - \rho^2}} \\
 &= \frac{2\lambda^2 e^{-\lambda t}}{c^2} \int_0^{ct} \rho \sum_{m=0}^\infty \left(\frac{\lambda}{c} \sqrt{c^2 t^2 - \rho^2} \right)^{2m-1} \frac{1}{(2m)!} J_0(\rho\sqrt{\xi^2 + \alpha^2}) d\rho \\
 &= \frac{2\lambda e^{-\lambda t}}{c} \sum_{m=0}^\infty \frac{(\frac{\lambda}{c})^{2m}}{(2m)!} \sum_{k=0}^\infty \frac{(-1)^k \left(\frac{\sqrt{\xi^2 + \alpha^2}}{2} \right)^{2k}}{(k!)^2} \int_0^{ct} (c^2 t^2 - \rho^2)^{m-\frac{1}{2}} \rho^{2k+1} d\rho \\
 &= \frac{2\lambda e^{-\lambda t}}{c} \sum_{m=0}^\infty \frac{(\frac{\lambda}{c})^{2m}}{(2m)!} \sum_{k=0}^\infty \frac{(-1)^k \left(\frac{\sqrt{\xi^2 + \alpha^2}}{2} \right)^{2k}}{2(k!)^2 (ct)^{-(2m+2k+1)}} \int_0^1 y^k (1-y)^{m-\frac{1}{2}} dy \\
 &= \frac{\lambda}{c} e^{-\lambda t} \sum_{m=0}^\infty \left(\frac{\lambda}{c} \right)^{2m} \frac{1}{(2m)!} \sum_{k=0}^\infty (-1)^k \left(\frac{\sqrt{\xi^2 + \alpha^2}}{2} \right)^{2k} \frac{(ct)^{2m+2k+1} \Gamma(m + \frac{1}{2})}{k! \Gamma(k + m + 1 + \frac{1}{2})}. \tag{4.87}
 \end{aligned}$$

Thus, from (4.86), we have that

$$\widetilde{\mathfrak{q}}(\xi, \alpha, \mu) = \frac{2\lambda + \sqrt{\mu}}{2\lambda\sqrt{\mu}} \int_0^\infty ds \widehat{\tau}(\xi, \alpha, s) e^{-s\sqrt{\mu}} = \frac{1 + 2\lambda\mu^{-\frac{1}{2}}}{\mu + 2\lambda\sqrt{\mu} + c^2 (\xi^2 + \alpha^2)} \tag{4.88}$$

in force of the calculation

$$\begin{aligned}
 &\int_0^\infty ds \widehat{\tau}(\xi, \alpha, s) e^{-s\sqrt{\mu}} \\
 &= \frac{\lambda}{c} \int_0^\infty ds e^{-\lambda s} \sum_{m=0}^\infty \frac{\lambda^{2m}}{c^{2m}(2m)!} \sum_{k=0}^\infty \frac{(-1)^k \left(\frac{\sqrt{\xi^2 + \alpha^2}}{2} \right)^{2k} \Gamma(m + \frac{1}{2})}{k! (cs)^{-(2m+2k+1)} \Gamma(k + m + 1 + \frac{1}{2})} e^{-s\sqrt{\mu}}
 \end{aligned}$$

$$\begin{aligned}
 &= \lambda \sum_{m=0}^{\infty} \frac{\sqrt{\pi} 2^{1-2m} \Gamma(2m)}{\lambda^{-2m} (2m)! \Gamma(m)} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\sqrt{\xi^2 + \alpha^2}}{2}\right)^{2k} c^{2k}}{k! \Gamma(k + m + 1 + \frac{1}{2})} \int_0^{\infty} e^{-s(\lambda + \sqrt{\mu})} s^{2m+2k+1} ds \\
 &= \frac{\lambda}{2(\lambda + \sqrt{\mu})^2} \sum_{m=0}^{\infty} \frac{\lambda^{2m} \sqrt{\pi} 2^{1-2m}}{m! (\lambda + \sqrt{\mu})^{2m}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\sqrt{\xi^2 + \alpha^2}}{2}\right)^{2k}}{k! (\lambda + \sqrt{\mu})^{2k} c^{-2k}} \frac{\Gamma(2k + 2m + 2)}{\Gamma(k + m + 1 + \frac{1}{2})} \\
 &= \frac{\sqrt{\pi} \lambda}{2(\lambda + \sqrt{\mu})^2} \sum_{m=0}^{\infty} \frac{\lambda^{2m} 2^{1-2m}}{m! (\lambda + \sqrt{\mu})^{2m}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\sqrt{\xi^2 + \alpha^2}}{2}\right)^{2k}}{k! (\lambda + \sqrt{\mu})^{2k} c^{-2k}} \frac{\Gamma(k + m + 1)}{2^{1-2(k+m+1)} \sqrt{\pi}} \\
 &= \frac{2\lambda}{(\lambda + \sqrt{\mu})^2} \sum_{m=0}^{\infty} \frac{\lambda^{2m}}{m! (\lambda + \sqrt{\mu})^{2m}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\sqrt{\xi^2 + \alpha^2}\right)^{2k}}{k! (\lambda + \sqrt{\mu})^{2k} c^{-2k}} \int_0^{\infty} e^{-u} u^{k+m} du \\
 &= \frac{2\lambda}{(\lambda + \sqrt{\mu})^2} \int_0^{\infty} du e^{u \frac{\lambda^2}{(\lambda + \sqrt{\mu})^2} - u \frac{c^2(\xi^2 + \alpha^2)}{(\lambda + \sqrt{\mu})^2} - u} = \frac{\frac{2\lambda}{(\lambda + \sqrt{\mu})^2}}{1 - \frac{\lambda^2}{(\lambda + \sqrt{\mu})^2} + \frac{c^2(\xi^2 + \alpha^2)}{(\lambda + \sqrt{\mu})^2}} \\
 &= \frac{2\lambda}{(\lambda + \sqrt{\mu})^2 - \lambda^2 + c^2(\xi^2 + \alpha^2)} = \frac{2\lambda}{\mu + 2\lambda\sqrt{\mu} + c^2(\xi^2 + \alpha^2)}. \tag{4.89}
 \end{aligned}$$

The Fourier-Laplace transform of the law of the process $B_2(c^2 \mathcal{L}^{\frac{1}{2}}(t))$ is written as in (4.15) for $n = 2, \beta = 1$, and $\nu = \frac{1}{2}$ as the following calculation shows

$$\begin{aligned}
 \widehat{w}_{\frac{1}{2}}^1(\xi, \alpha, t) &= \int_0^{\infty} \widehat{p}_B(\xi, \alpha, c^2 s) \widetilde{l}_{\frac{1}{2}}(s, \mu) ds \\
 &= \left(1 + 2\lambda\mu^{-\frac{1}{2}}\right) \int_0^{\infty} e^{-\mu s - (\xi^2 + \alpha^2)c^2 s} \left[e^{-2\lambda s \sqrt{\mu}} + 2\lambda \frac{e^{-2\lambda s \sqrt{\mu}}}{\sqrt{\mu}} \right] ds \\
 &= \frac{1 + 2\lambda\mu^{-\frac{1}{2}}}{2\lambda\sqrt{\mu} + \mu + c^2(\xi^2 + \alpha^2)}. \tag{4.90}
 \end{aligned}$$

In the previous calculation, we use the Laplace transform of $l_{\frac{1}{2}}(x, t)$ obtained in (4.28). The proof is complete since (4.90), coincides with (4.88) and with the Fourier-Laplace transform of (4.85). \square

Remark 4.5. Since for the first passage time $\tau_{\frac{s}{\sqrt{2}}} = \inf \left\{ z : B(z) = \frac{s}{\sqrt{2}} \right\}$ of a Brownian motion through level $\frac{s}{\sqrt{2}}$ we have that

$$\int_0^{\infty} e^{-\mu t} \Pr \left\{ \tau_{\frac{s}{\sqrt{2}}} \in dt \right\} = e^{-s\sqrt{\mu}}, \tag{4.91}$$

and

$$\int_0^{\infty} e^{-\mu t} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} p_{|B|}(s, t) dt = e^{-s\sqrt{\mu}} \tag{4.92}$$

we can write

$$\begin{aligned} \int_0^\infty \mathfrak{r}(x, y, s) \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} p_{|B|}(s, t) ds &= \int_0^\infty \mathfrak{r}(x, y, s) \frac{s}{\sqrt{2}} \frac{e^{-\frac{s^2}{4t}}}{\sqrt{2\pi t^3}} ds \\ &= \int_0^\infty \frac{\partial}{\partial s} \mathfrak{r}(x, y, s) \frac{e^{-\frac{s^2}{4t}}}{\sqrt{\pi t}} ds = \int_0^\infty \frac{\partial}{\partial s} \mathfrak{r}(x, y, s) p_{|B|}(s, t) ds. \end{aligned} \tag{4.93}$$

This representation of the second term of (4.84) is extremely interesting because by integrating (4.93) in C_{ct} we get

$$\int_0^\infty \frac{\partial}{\partial s} (1 - e^{-2\lambda s}) p_{|B|}(s, t) ds = 2\lambda \int_0^\infty e^{-2\lambda s} p_{|B|}(s, t) ds \tag{4.94}$$

and yields the missing probability of the first term of (4.84).

Remark 4.6. We check that the law

$$q(x, y, t) = \int_0^\infty \mathfrak{r}(x, y, s) \left[p_{|B|}(s, t) + \frac{1}{2\lambda} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} p_{|B|}(s, t) \right] ds \tag{4.95}$$

integrates to unity. By taking the t -Laplace transform, the integral with respect to (x, y) becomes

$$\begin{aligned} &\iint_{C_{ct}} dx dy \int_0^\infty dt e^{-\mu t} q(x, y, t) \\ &= \int_0^\infty (1 - e^{-2\lambda s}) \left[\int_0^\infty e^{-\mu t} \left(p_{|B|}(s, t) + \frac{1}{2\lambda} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} p_{|B|}(s, t) \right) dt \right] ds \\ &= \int_0^\infty (1 - e^{-2\lambda s}) \left[\frac{e^{-s\sqrt{\mu}}}{\sqrt{\mu}} + \frac{e^{-s\sqrt{\mu}}}{2\lambda} \right] ds \\ &= \left(\frac{1}{\sqrt{\mu}} + \frac{1}{2\lambda} \right) \left[\int_0^\infty e^{-s\sqrt{\mu}} ds - \int_0^\infty e^{-s(2\lambda + \sqrt{\mu})} ds \right] \\ &= \frac{2\lambda + \sqrt{\mu}}{2\lambda\sqrt{\mu}} \left(\frac{1}{\sqrt{\mu}} - \frac{1}{2\lambda + \sqrt{\mu}} \right) = \frac{1}{\mu} = \int_0^\infty e^{-\mu t} dt. \end{aligned} \tag{4.96}$$

The same check can be done directly by taking into account formulas (4.93) and (4.94).

Relationships similar to $B(c^2 \mathcal{L}^{\frac{1}{2}}(t)) \stackrel{\text{law}}{=} T(|B(t)|), t > 0$, and the analogous one in the plane, cannot be established in spaces of dimension $n \geq 3$, because random motions governed by telegraph equations in such spaces have not been constructed. Random flights in \mathbb{R}^n have been studied [16] but their distributions are not related to higher-dimensional telegraph equations.

References

1. Allouba, H., and Nane, E. 2013. Interacting time-fractional and Δ^v PDE's systems via Brownian-time and inverse-stable-Lévy-time Brownian sheets. *Stochastics and Dynamics* 13(1). Article ID 1250012.
2. Allouba, H., and Zheng, W. 2001. Brownian-time processes: The PDE connection and the half-derivative generator. *Ann. Probab.* 29:1780–1795.
3. Balakrishnan, A.V. 1960. Fractional powers of closed operators and semigroups generated by them. *Pacific J. Math.* 10:419–437.
4. Bleistein, N., and Handelsman, R.A. 1986. *Asymptotic Expansions of Integrals*. Dover, New York.
5. Bochner, S. 1949. Diffusion equation and stochastic processes. *Proc. Nat. Acad. Sciences USA* 35:368–370.
6. DeBlassie, R.D. 2004. Iterated Brownian motion in an open set. *Ann. Appl. Probab.* 14:1529–1558.
7. D'Ovidio, M., Orsingher, E., and Toaldo, B. 2014. Fractional telegraph-type equations and hyperbolic Brownian motion. *Statistics and Probability Letters* 89(1):131–137.
8. Feller, W. 1952. On a generalization of Marcel Riesz' potentials and the semi-groups generated by them. *Comm. Sém. Math. Univ. Lund Medd. Lunds Univ. Mat. Sem.* 72–81.
9. Foong, S.K., and Kanno, S. 1994. Properties of the telegrapher's random process with or without a trap. *Stoch. Proc. Appl.* 53(1):147–173.
10. Fujita, Y. 1990. Integrodifferential equation which interpolates the heat equation and the wave equation. I-II. *Osaka J. Math I* 27(2):797–804. [Part II: 27(4):309–321, 1990]
11. Haubold, H.J., Mathai, A.M., and Saxena, R.K. 2011. Mittag-Leffler functions and their applications. *J. Appl. Math.* Article ID 298628.
12. Kilbas, A.A., Srivastava, H.M., and Trujillo, J.J. 2006. *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Scienc, Amsterdam.
13. Momani, S. 2005. Analytic and approximate solutions of the space- and time-fractional telegraph equations. *Appl. Math. Comput.* 170(2):1126–1134.
14. Orsingher, E., and Beghin, L. 2004. Time-fractional telegraph equations and telegraph processes with Brownian time. *Probab. Theory Related Fields* 128(1):141–160.
15. Orsingher, E., and Zhao, X. 2003. The space-fractional telegraph equation and the related fractional telegraph process. *Chin. Ann. Math. B* 24(1):1–12.
16. Orsingher, E., and De Gregorio, A. 2007. Random flights in higher spaces. *J. Theoret. Probab.* 20(4): 769–806.
17. Orsingher, E., and Beghin, L. 2009. Fractional diffusion equations and processes with randomly varying time. *Ann. Probab.* 37:206–249.
18. Saichev, A.I., and Zaslavsky, G.M. 1997. Fractional kinetic equations: Solutions and applications. *Chaos* 7: 753–764.
19. Saxena, R.K., Mathai, A.M., and Haubold, H.J. 2006. Reaction-diffusion systems and nonlinear waves. *Astrophys. Space Sci.* 305:297–303.
20. Stadge, W., and Zacks, S. 2004. Telegraph processes with random velocities. *J. Appl. Prob.* 41:665–678.
21. Zolotarev, V.M. 1986. *One-Dimensional Stable Distributions*. Translations of Mathematical Monographs, vol. 65. American Mathematical Society. [trans. from the Russian by H. H. McFaden; translation ed. Ben Silver]