# Robust observer design under measurement noise

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**Abstract:** We prove new results on robust observer design for systems with noisy measurement and bounded trajectories. A state observer is designed by dominating the incrementally homogeneous nonlinearities of the observation error system with its linear approximation, while gain adaptation and incremental observability guarantee an asymptotic upper bound for the estimation error depending on the limsup of the norm of the measuremen noise. The gain adaptation is implemented as the output of a stable filter using the squared norm of the measured output estimation error and the mismatch between each estimate and its saturated value.

*Keywords:* measurement noise, robust observers, gain adaptation, saturated estimates.

## 1. INTRODUCTION

Homogeneity and homogeneous approximations have been investigated have been exploited by many authors in the design of global state observers (Qian (2005), Qian & Lin (2006), Yang & Lin (2003), Andrieu et al. (2008)): the idea is to design a state observer for the homogeneous approximation of the system and convergence to zero of the estimation error is preserved under any perturbation which does not change the homogeneous approximation. The class of systems for which an observer can be designed by domination techniques has been enlarged by adding dynamic gain adaptation (Khalil & Saberi (1987), Bullinger & Allgower (1997), Lei et al. (2005), **?**, Andrieu et al. (2009)). The class of homogeneous systems has been enlarged by introducing (incremental) homogeneity in the upper bound in Battilotti (2014) and used together with gain adaptation and self-tuned saturations for designing global observers in Battilotti (2015a) for systems with bounded trajectories. Homogeneity in the upper bound gives enough a general framework for including triangular structures (feedback and feedforward systems), homogeneous and interlaced structures. Self-tuned saturations were previously used in Lei et al. (2005) in the observer design for feedback-linearizable systems with bounded trajectories. However, the gain adaptation is such that the dynamically adapted gain is non-decreasing along solutions. As known, this may lead to serious growth problems in the presence of measurement disturbance(Egardt (1979, Example 4.2), Peterson & Narendra (1982), Mareels (1984), Khalil & Saberi (1987)). This problem has been addressed by several authors (Egardt (1979), Mareels (1984), Peterson & Narendra (1982), Ioannou & Kokotovic (1984)), trying to reduce the adapted gain instead to let it grow with no bound, for example when the measured output estimation error is decreasing. In Vasilijevic & Khalil (2006) it is shown that measurement disturbance introduces an upper bound on the gain when good estimation performances are required. In this direction, we find the works of Ahrens & Khalil (2006), which relies on the knowledge of a bound for the nonlinearities of the system, and Boizot et al. (2010), which relies on the knowledge of a bound for the dynamic gain and the Lipschitz constant of the nonlinearities of the system. The effect of measurement disturbance on observer design has been studied, following Boizot et al. (2010), for a class of lower triangular systems with bounded trajectories and for a given class of observers in Sanfelice & Praly (2011), satisfying additional properties on the mismatch between the vector fields of the system and of the observer, by proving an upper bound (depending on the measurement noise) for the estimation error in the mean and an upper bound on the limsup of the estimation error in the mean. In the absence of measurement noise, this last bound can be made arbitrarily small by setting properly the parameters of the class of observers. This, however, do not discard a potential oscillatory behavior of the estimates (Mareels et al. (1999)).

In this paper, we prove new results on robust observer design in the presence of measurement disturbance for systems with bounded trajectories by using incremental homogeneity in the upper bound (Battilotti (2014)) and gain adaptation (Andrieu et al. (2008), Bullinger & Allgower (1997), Khalil & Saberi (1987), Lei et al. (2005)) with saturated estimates and dynamically tuned saturation levels (Lei et al. (2005))). A state observer is designed by dominating the incrementally homogeneous (in the upper bound) nonlinearities of the observation error system with its linear approximation. The gain adaptation and updating of the saturation levels is implemented through a stable filter which regulates its output by using a suitable function of the squared norm of the measured output estimation error. Our observer guarantees an upper bound on the limsup of the norm of the estimation error depending on the limsup of the norm of the measurement noise. As a particular case, if the measurement disturbance tends asymptotically to zero the estimation error itself tends to zero.

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The paper is organized as follows. In section 2 some notation is introduced. In section 3 the class of system is described and the problem is formulated. In section 4 an observer is presented together with the main result and the parameter observer design is discussed in section 4.1. In section 4.2 example and simulation are given and in section 4.3 the main result is proved. In the appendix the notion of incremental generalized homogeneity is shortly recalled together with some of its properties and related and auxiliary results.

## 2. NOTATION

(N1) $\mathbb{R}^n$ (resp. $\mathbb{R}^{n\times n}$) is the set of $n$-dimensional real column vectors (resp. $n \times n$ matrices). $\mathbb{R}_\geqslant$ (resp. $\mathbb{R}^n_\geqslant$, $\mathbb{R}^{n\times n}_\geqslant$) denotes the set of real non-negative numbers (resp. vectors in $\mathbb{R}^n$, matrices in $\mathbb{R}^{n\times n}$, with real non-negative entries). $\mathbb{R}_>$ (resp. $\mathbb{R}^n_>$) denotes the set of real positive numbers (resp. vectors in $\mathbb{R}^n$ with real positive entries).

(N3) For any matrix $V \in \mathbb{R}^{p\times n}$ we denote by $V_{ij}$ the $(i,j)$-th entry of $V$ and for any vector $v \in \mathbb{R}^n$ we denote by $v_i$ the $i$-th element of $v$. We retain a similar notation for functions. For any $v \in \mathbb{R}^n$ we denote by diag$\{v\}$ the diagonal $n\times n$ matrix with diagonal elements $v_1,\dots,v_n$. Also, $|a|$ denotes the absolute value of $a \in \mathbb{R}$, $\|a\|$ denotes the euclidean norm of $a \in \mathbb{R}^n$, $\|A\|$ denotes the norm of $A \in \mathbb{R}^{n\times n}$ induced from the euclidean norm $\|\cdot\|$ and $\langle\langle a\rangle\rangle$ the column vector of the absolute values of the elements of $a \in \mathbb{R}^n$, i.e. $\langle\langle a\rangle\rangle := (|a_1|\cdots|a_n|)^T$. $\lambda_{min}(A)$ (resp. $\lambda_{max}(A)$) denotes the resp.minimum (maximum) eigenvalue of the matrix $A$.

(N3) We denote by $\mathbf{C}^j(\mathscr{X},\mathscr{Y})$, with $j \geqslant 0$, $\mathscr{X} \subset \mathbb{R}^n$ and $\mathscr{Y} \subset \mathbb{R}^p$, the set of $j$-times continuously differentiable functions $f : \mathscr{X} \to \mathscr{Y}$, $\mathbf{C}^0_0(\mathscr{X},\mathscr{Y})$ the set of uniformly continuous functions $f : \mathscr{X} \to \mathscr{Y}$, by $\mathbf{L}^\infty(\mathbb{R}_\geqslant,\mathscr{Y})$ the set of functions $f \in \mathbf{C}^0(\mathbb{R}_\geqslant,\mathscr{Y})$ such that $\sup_{\theta\geqslant 0}\|f(\theta)\| < +\infty$ and by $\mathbf{L}^j(\mathbb{R}_\geqslant,\mathscr{Y})$, with $j \geqslant 1$, the set of $f \in \mathbf{C}^0(\mathbb{R}_\geqslant,\mathscr{Y})$ such that $\int_0^\infty \|f(\theta)\|^j d\theta < +\infty$. For each $d \in \mathbf{L}^\infty(\mathbb{R}_\geqslant,\mathscr{Y})$, we have the sup norm of $d$ defined as $\|d\|_\infty := \sup_{t\geqslant 0}\|d(t)\|$. Moreover, $\mathcal{K}$ denotes the set of functions $f \in \mathbf{C}^0(\mathbb{R}_\geqslant,\mathbb{R}_\geqslant)$, strictly increasing with $f(0) = 0$.

(N4) A *saturation function* $\mathbf{sat}_h(\cdot)$ *with levels* $h \in \mathbb{R}^n_>$ is a function $\mathbf{sat}_h(x) := (\mathbf{sat}_{h_1}(x_1),\dots,\mathbf{sat}_{h_n}(x_n))^T$ such that for each $i = 1,\dots,n$ and $x_i \in \mathbb{R}$:

$$\mathbf{sat}_{h_i}(x_i)\begin{cases} x_i & |x_i| \leqslant h_i \\ \text{sign}(x_i)h_i & \text{otherwise.}\end{cases} \tag{1}$$

(N5) For any vectors $x \in \mathbb{R}^n, \mathfrak{r} \in \mathbb{R}^n_>$ and $\epsilon \in \mathbb{R}_>$, we define

$$\epsilon^{\mathfrak{r}} := (\epsilon^{\mathfrak{r}_1},\cdots,\epsilon^{\mathfrak{r}_n})^T,\ \epsilon^{\mathfrak{r}} \diamond x := (\epsilon^{\mathfrak{r}_1}x_1,\cdots,\epsilon^{\mathfrak{r}_n}x_n)^T \tag{2}$$

*viz.* $\epsilon^{\mathfrak{r}} \diamond x$ is the dilation of a vector $x$ with weights $\mathfrak{r}$.

(N6) for any vectors $x, y \in \mathbb{R}^n$ we write $x \preceq y$ if and only if $x_i \leqslant y_i$ for all $i = 1,\dots,n$. We retain the same notation for matrices $A, B \in \mathbb{R}^{n\times n}$: $A \preceq B$ if and only if $A_{ij} \leqslant B_{ij}$ for all $i,j = 1,\dots,n$. On the other hand $A \geqslant B$ (resp. $A > B$) for matrices $A, B \in \mathbb{R}^{n\times n}$ if and only if $A - B$ is positive semidefinite (resp. positive definite).

## 3. MAIN ASSUMPTIONS AND PROBLEM STATEMENT

Consider the system

$$\dot{x} = f(x) := [A + BF + HC]x + \phi(x), \tag{3}$$

$$y = h(x,d) := Cx + d \tag{4}$$

with state $x \in \mathbb{R}^n$, measurement $y \in \mathbb{R}$ and disturbance $d \in \mathbb{R}$. The triple $(A,B,C)$ is in prime form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{5}$$

$$C = [1\ 0\ \cdots\ 0\ 0] \tag{6}$$

with $F \in \mathbb{R}^{1\times n}$ and $H \in \mathbb{R}^{n\times 1}$. Moreover, $\phi$ is locally Lipschitz continuous with $\phi(0) = 0$ and $\frac{\partial\phi}{\partial x}(0) = 0$ so that $\dot{x} = [A + BF + HC]x$, $y = Cx + d$, represents the linear approximation of (3)-(4) around the origin.

We consider in (3)-(4) the class $\mathcal{D}(\Delta)$ of disturbances $d \in \mathbf{L}^\infty(\mathbb{R}_\geqslant,\mathbb{R})$ such that $\|d\|_\infty \leqslant \Delta$ and uniformly continuous on their domain. The problem is to give an estimate of the state of (3) using only the noisy measurement (4).

Our assumptions on the class of systems (3)-(4) are the following ones (see the appendix for few recalls on incremental homogeneity in the upper bound which we will abbreviate as i.h.u.b. throughout the paper):

(H0) (*incremental homogeneity*)
- (i) $A^T(\phi + HC)$ is incrementally homogeneous in the upper bound (i.h.u.b.) with quadruples $(\mathfrak{r}, \mathfrak{r} - \mathfrak{g}, \mathfrak{g}, A^T(\phi_U + H_U C))$, with $\phi_U(0,0) = 0$ and for some $H_U \in \mathbb{R}^{n\times 1}$,
- (ii) $(I - AA^T)(\phi + BF)$ is i.h.u.b. with quadruple $(\mathfrak{r}, (I - AA^T)(\mathfrak{r} + \mathfrak{g}), \mathfrak{g}, (I - AA^T)(\phi_U + BF_U)$ for some $F_U \in \mathbb{R}^{1\times n}$,
- (iii) the degrees $\mathfrak{g}$ and weights $\mathfrak{r}$ satisfy for each $j = 2,\dots,n$

$$2(\mathfrak{g}_j - \mathfrak{g}_{j-1}) + \mathfrak{g}_{j-1} + \mathfrak{r}_{j-1} \leqslant \mathfrak{r}_j - \mathfrak{g}_j \leqslant \mathfrak{g}_{j-1} + \mathfrak{r}_{j-1},$$

(H1) (*state boundedness*) $x(\cdot,x_0) \in \mathbf{L}^\infty(\mathbb{R}_\geqslant,\mathbb{R})$ for each $x_0 \in \mathbb{R}^n$, where $x(t,x_0)$ is the solution of (3) with initial condition $x_0$.

*Remark 1.* Assumption (H0) is a sufficient condition for the design of a local/semi-global state observer for systems (3)-(4) with $d \equiv 0$ (see Battilotti (2014)). Assumption (H0) covers a large class of nonlinear systems (3)-(4) (globally Lipschitz systems, triangular systems, homogeneous systems: see also Battilotti (2015b)).•

*Remark 2.* Assumption (H1) is somewhat restrictive. However, many physical systems have this property (Van Der Pol and Fitzhugh-Nagumo oscillators, Lorentz systems, ...). (H1) with (H0) are sufficient conditions for the design of a global state observer for systems (3)-(4) with $d \equiv 0$ (see Battilotti (2015a)). A very simple relaxation of (H1) is obtained for example by assuming additionally that $\phi$ and $\psi$ are globally Lipschitz. •

## 4. THE STRUCTURE OF THE OBSERVER AND MAIN RESULT

The observer we propose for (3)-(4) has the following interconnected structure. The first part of the filter is devoted to the estimation of $x$

$$\begin{aligned}\dot{\xi} &= A\xi + (BF + HC)\mathbf{sat}_{cz^{\mathfrak{r}}}(\xi) + \phi(\mathbf{sat}_{cz^{\mathfrak{r}}}(\xi)) \\ &\quad + L_z[y - C\xi],\ \xi(0) := \xi_0, \end{aligned} \tag{7}$$

where

$$L_z \coloneqq kz^{2\mathfrak{g}_1}(I - A^T G_z)^{-1}C^T,\ G_z \coloneqq \mathrm{diag}\{\Gamma z^{2A\mathfrak{g}}\} \tag{8}$$

with $c, k > 0$ and diagonal positive definite $\Gamma \in \mathbb{R}^{n\times n}$ (specified in section 4.1), while the second part of the filter is devoted to the gain adaptation and tuning of the saturation levels

$$\begin{aligned}\dot{z} &= z^{-2|\mathfrak{g}_n|}\sigma\Big(z^{2(\mathfrak{g}_1-\mathfrak{r}_1)}\max\Big\{q_z(\xi, y) \\ &\quad - h(\Delta)z^{2(\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)+1}\lambda\Big(\frac{q_z(\xi, y)}{z^{2(\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)+1}}\Big), 0\Big\}\Big), \\ &\quad z(0) := z_0 \geqslant 1, \end{aligned} \tag{9}$$

where

$$\begin{aligned}\sigma(s) &\coloneqq s/\sqrt{1+s^2}, \\ q_z(\xi, y) &\coloneqq |y - C\xi|^2 - 2\Delta^2 \\ &\quad + z^{2(\mathfrak{r}_1-\mathfrak{g}_1)}\|z^{\mathfrak{g}-\mathfrak{r}} \diamond (\xi - \mathbf{sat}_{cz^{\mathfrak{r}}}(\xi))\|^2. \end{aligned} \tag{10}$$

and $\lambda$ any smooth function such that

$$\lambda(s) = \begin{cases} 1 & s \leqslant h(\Delta) \\ \in (0,1) & s \in (h(\Delta), 2h(\Delta)) \\ 0 & s \geqslant 2h(\Delta). \end{cases} \tag{11}$$

with $h \in \mathcal{K}$ (specified in section 4.1).

The estimator (7) is a copy of the system equations (3), except for saturating estimates inside the terms $BF + HC + \phi$ and $\psi$, plus an innovation term $L_z[y - C\xi]$. Note also that the gain matrix $L_z$ and the saturation levels are adapted according to the values of $z$. The dynamics of $z$ is implemented as a stable filter forced by the term $q_z$, which depends on the squared norm of the output estimation error $y - C\xi$ and the mismatch between $\xi$ and its saturated value $\mathbf{sat}_{cz^{\mathfrak{r}}}(\xi)$, dynamically weighted by adaptation of $z$.

The main result of this paper is the following. Let $\phi_0$ denote the vector of initial conditions $x_0$, $\xi_0$ and $z_0 \geqslant 1$. Let $x_t(x_0)$, resp. $\xi_t(\phi_0, d), z_t(\phi_0, d)$, denotes the solution of (3), resp. (7)-(9), ensuing from initial condition $x_0$, resp. $\phi_0$ with measurement disturbance $d \in \mathcal{D}(\Delta)$.

*Theorem 3.* Assume (H0) and (H1). There exist $c, k > 0$, $h \in \mathcal{K}$ and diagonal positive definite $\Gamma \in \mathbb{R}^{n\times n}$ such that the solution $x_t(x_0)$, $\xi_t(\phi_0, d), z_t(\phi_0, d)$ of (3)-(4)-(7)-(9) is defined and bounded for all $t \geqslant 0$, initial conditions $\phi_0$ and measurement disturbance $d \in \mathcal{D}(\Delta)$. In addition,

$$\lim_{t\to+\infty} z_t(\phi_0, d) = z_\infty, \tag{12}$$

$$\limsup_{t\to+\infty} \|x_t(x_0) - \xi_t(\phi_0, d)\|^2 \leqslant \frac{4\lambda^3_{max}(P)}{\lambda^2_{min}(Q)}\chi^2_{z_\infty}(\Delta) \tag{13}$$

with

$$\begin{aligned}\chi_{z_\infty}(\Delta) &:= \Big(\|BF + HC\| \\ &\quad + \sup_{\substack{\|w_1\|\leqslant 2nc\|z_\infty^{\mathfrak{r}}\| \\ \|w_2\|\leqslant 2n(\sqrt{\nu_{z_\infty}(\Delta)}+c\|z_\infty^{\mathfrak{r}}\|)}} \|\Phi_U(w_1, w_2)\|\Big)\sqrt{\nu_{z_\infty}(\Delta)} \\ &\quad +\sqrt{\mu_{z_\infty}(\Delta)}\|L_{z_\infty} - L\| + \|L\|\Delta \\ \mu_{z_\infty}(\Delta) &:= h(\Delta)z_\infty^{2(\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)+1} + 2\Delta^2, \\ \nu_{z_\infty}(\Delta) &:= \mu_{z_\infty}(\Delta)z_\infty^{2(\max_i \mathfrak{r}_i-\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)} \end{aligned} \tag{14}$$

and $P, Q$, resp. $L$, symmetric and positive definite $n \times n$ matrices, resp. a $1 \times n$ matrix, such that

$$\begin{aligned}&(x - \xi)^T P[(A + BF + HC)(x - \xi) + \phi(x) - \phi(\xi) \\ &-LC(x - \xi)] \leqslant -\|x - \xi\|_Q^2 \end{aligned} \tag{15}$$

for all $x, \xi \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is any compact set for which $x_t(x_0), \xi_t(\phi_0, d) \in \Omega$ for all $t \geqslant 0$.□

*Remark 4.* The inequality (15) is instrumental only to obtain the bound (13) on the estimation error and it is not needed in the observer design (see next section). Under assumption (H0) and according to Battilotti (2014), theorem V.1, there indeed exist symmetric and positive definite $n \times n$ matrices $P, Q$ and $1 \times n$ matrix $L$ (all depending on $\Omega$) such that (15) holds for all $x, \xi \in \Omega$. As it results from (13) the limsup of the norm of the estimation error is bounded by a $\mathcal{K}$-class function of $\Delta$, which is an upper bound for the supremum norm of $d$.

### *4.1 Choice of the observer parameters*

The observer (7)-(9) is characterized by the parameters $c, k > 0$, $h \in \mathcal{K}$ and diagonal positive definite $\Gamma$. These quantities are chosen as follows. Let $\phi_U, \psi_U, F_U, H_U$, $\mathfrak{r}$ and $\mathfrak{g}$ be as in assumption (H0) and let $\Delta$ be the upper bound for the sup norm of the measurement disturbance $d$. Towards the filter definition, the following calculations should be accomplished:

(i) find $k$ and $\Gamma$ such that for some $a > 0$

$$\begin{aligned}&2aI \leqslant \mathcal{X}(k, \Gamma) \coloneqq 2(kC^TC + A^T\Gamma A) \\ &-\Big[2(I + A^T\Gamma)(BF_U + H_UC) + A + A^T\Gamma^2 \\ &+2\max_{i\geqslant 2}|\mathfrak{g}_i|A^T\Gamma\Big](I - A^T\Gamma)^{-1} \\ &-(I - A^T\Gamma)^{-T}\Big[2(I + A^T\Gamma)(BF_U + H_UC) + A + A^T\Gamma^2 \\ &+2\max_{i\geqslant 2}|\mathfrak{g}_i|A^T\Gamma\Big]^T - 2\mathrm{diag}\{\mathfrak{r}_1, \ldots, \mathfrak{r}_n\}. \end{aligned} \tag{16}$$

Inequality (16) is alway solvable in the unknowns $c, k$ and $\Gamma$, on account of the fact that $\mathcal{X}(k, \Gamma)$ can be obtained recursively as follows (recall that $\Gamma_{i,i}$ denotes the $i$-th diagonal entry of $\Gamma$)

$$\begin{aligned}\mathcal{X}^{(n-1)} &\coloneqq 2\Gamma_{n-1,n-1}, \\ \mathcal{X}^{(n-j)} &\coloneqq \left[\begin{array}{c|c} 2\Gamma_{n-j,n-j} + \mathcal{Z}_1^{(n-j)} & (\mathcal{Z}_2^{(n-j)})^T \\ \hline \mathcal{Z}_2^{(n-j)} & \mathcal{X}^{(n-j+1)} \end{array}\right], = 2, \ldots, n, \\ \mathcal{X}^{(0)} &= \mathcal{X}(k, \Gamma) \end{aligned}$$

with $\Gamma_{00} \coloneqq k$ and $\mathcal{Z}_2^{(n-j)}, \mathcal{Z}_1^{(n-j)}$, $j = 2, \ldots, n$, are suitable functions of $\Gamma_{n-j+1,n-j+1}, \ldots, \Gamma_{n-1,n-1}$. Therefore, it is sufficient to pick any $\Gamma_{n-1,n-1} > 0$ and for each increasing $j = 2, \dots, n$ select $\Gamma_{n-j,n-j} > 0$ such that $\mathcal{X}^{(n-j)} > 0$. Finally, set $a := \frac{\lambda_{min}(\mathcal{X}^{(0)})}{2}$.

*(ii)* define $c > 0$ as follows: if $\Phi \in \mathbf{C}^0(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant^{n\times n})$ is a matrix for which $\Phi(0) = 0$ and for all $s \geqslant 0$

$$\phi_U(w, z) \preceq \Phi(s), \forall w, z \in \mathbb{R}^n : \|z\| \leqslant ns,\ \|w\| \leqslant ns \quad (17)$$

(we recall that $\preceq$ for matrices means $\leqslant$ for each entry), calculate $c > 0$ such that

$$\begin{aligned} aI \leqslant \Upsilon(c, k, \Gamma) &:= \mathcal{X}(k, \Gamma) \\ &- 2[(I + A^T\Gamma)\Phi(c) + kC^T\Psi(c)](I - A^T\Gamma)^{-1} \\ &- 2(I - A^T\Gamma)^{-T}[(I + A^T\Gamma)\Phi(c)]^T \end{aligned} \quad (18)$$

The number $c$ always exists on account of (18) and continuity of $\phi_U$ with $\phi_U(0, 0) = 0$.

*(iii)* define $h \in \mathcal{K}$ as follows:

$$\begin{aligned} \Theta &:= 9\|(I - A^T\Gamma)^{-1}\|^2 \\ &\quad + 2\left\|C^T(C + 2\Psi(c))(I - A^T\Gamma)^{-1}\right)\right\|^2 \quad (19) \\ h(\Delta) &:= 10k^2\Delta^2\Theta/a^2. \quad (20) \end{aligned}$$

### *4.2 Example and simulations*

The system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + (1 - x_1^2x_2^2)x_2,\ y = x_1 + d \end{aligned} \quad (21)$$

with measurement disturbance $d_t \in [-4, 4]$ satisfies assumptions (H0) and (H1) of theorem 3 with $\mathfrak{r}_1 = 1$, $\mathfrak{r}_2 = 2$, $\mathfrak{g}_1 = 8$ and $\mathfrak{g}_2 = 3$. Notice that $\phi(x) := (x_2, -x_1 + (1 - x_1^2x_2^2)x_2)^T$ is neither homogeneous nor homogeneous in the $\infty$-limit.

An observer has been designed according to our procedure and a simulation has been worked out with initial conditions $x(0) = (5, -5)^T$, $\xi(0) = (0, 0)^T$, $z(0) = 1$ and $\Delta = 4$. The saturation levels of the estimates are set with $c = 0.1$, the diagonal elements of $\Gamma$ are respectively 8 and 30 and $k = 100$. The states $x_{1,t}, x_{2,t}$ together with their estimates are shown versus time in Figs. 1,2 with disturbance $d_t = sin(10t)$.

### *4.3 Proof of the theorem 3*

Let $c, k$, $h(\Delta)$, $\Phi(c)$ and $\Gamma$ be selected as in section 4.1. Consider the following coordinate transformation

$$(x, \xi, z) \mapsto (x, \eta, z) : \eta := X_z^{-1}(x - \xi), \quad (22)$$

with $X_z := (I - A^TG_z)^{-1}$ (the identity matrix is $n \times n$).

It is an issue of few passages to prove that he solutions $\eta_t$ and $z_t$ have infinite escape time. Notice that for all $t \geqslant 0$

$$\begin{aligned} 1 \leqslant z_t &\leqslant \left((2|\mathfrak{g}_n| + 1)t + z_0^{2|\mathfrak{g}_n|+1}\right)^{\frac{1}{2|\mathfrak{g}_n|+1}} \quad (23) \\ 0 \leqslant \dot{z}_t &\leqslant z_t^{-2|\mathfrak{g}_n|} \leqslant 1. \quad (24) \end{aligned}$$

Now, let us consider a candidate Lyapunov function for the estimation error system. To this aim, define $V_z(\eta) :=$

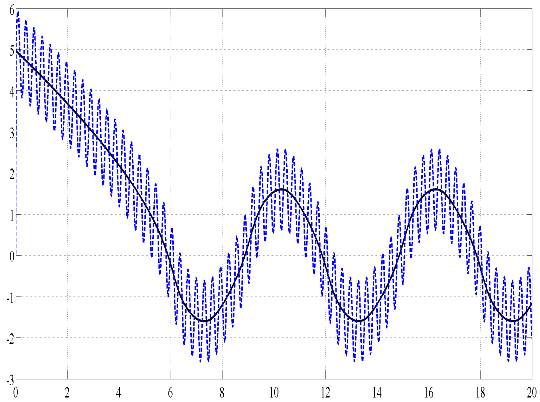

Fig. 1. State $x_1(t)$ (continuous line) and its estimate (dotted line) versus time with $d = sin(10t)$.

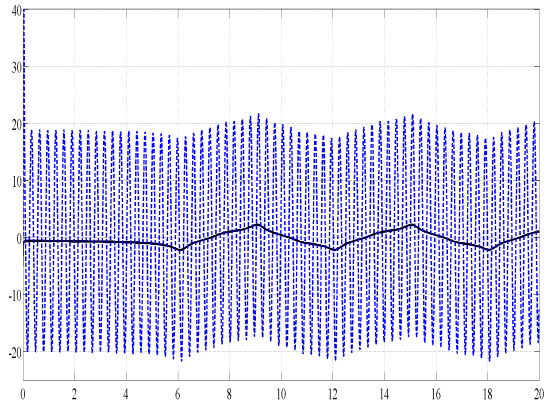

Fig. 2. State $x_2(t)$ (continuous line) and its estimate (dotted line) with $d = sin(10t)$.

$\|z^{-\mathfrak{r}} \diamond \eta\|^2$. We evaluate the time derivative of $V_z$ along the trajectories of (3)-(4)-(7)-(9) in the new coordinates. To do this, we will exploit assumption (H0) and the constructive condition (18), (23)-(24). We obtain for all $t \geqslant 0$

$$\begin{aligned} \dot{V}_{z_t} \leqslant &-\frac{a}{2}\|z_t^{\mathfrak{g}-\mathfrak{r}} \diamond \eta_t\|^2 + \frac{4}{a}\|z_t^{-\mathfrak{g}-\mathfrak{r}} \diamond \delta_{z_t}(x_t)\|^2 \\ &+ \frac{4k^2\Delta^2}{a} z_t^{2(\mathfrak{g}_1-\mathfrak{r}_1)} \end{aligned} \quad (25)$$

where

$$\begin{aligned} \delta_z(x) := [I - A^TG_z]\Big[&(BF+HC)(x - \mathbf{sat}(cz^{\mathfrak{r}}, x)) \\ &+\phi(x) - \phi\Big(\mathbf{sat}(cz^{\mathfrak{r}}, x)\Big)\Big]. \end{aligned}$$

Using once again the monotonicity of the degrees $\mathfrak{g}$,

$$\begin{aligned} \dot{V}_{z_t} \leqslant &-\frac{a}{2}z_t^{2\mathfrak{g}_n}V_{z_t} + \frac{4}{a}\|z_t^{-\mathfrak{g}-\mathfrak{r}} \diamond \delta_{z_t}(x_t)\|^2 \\ &+ \frac{4k^2\Delta^2}{a} z_t^{2(\mathfrak{g}_1-\mathfrak{r}_1)} \end{aligned} \quad (26)$$

*(C) We claim* $z \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$. Since $z_t$ is non-decreasing for $t \geqslant 0$, we have either $\lim_{t\to+\infty} z_t < +\infty$ or $\lim_{t\to+\infty} z_t = +\infty$. Assume by absurd that

$$\lim_{t\to+\infty} z_t = +\infty \tag{27}$$

There always exists $\bar{z} > 1$ and $\bar{T} > 0$ such that $z_t \geqslant \bar{z}$ for all $t \geqslant \bar{T}$ and

$$\mathbf{sat}(cz_t^{\mathfrak{r}}, x_t) = x_t,\ \forall t \geqslant \bar{T} \tag{28}$$

and, consequently,

$$\delta_{z_t}(x_t) = 0,\ \forall t \geqslant \bar{T}. \tag{29}$$

By a contradiction argument we can prove claim (C).

*(D) We claim* $\limsup_{t\to+\infty} q_{z_t}(\xi_t, y_t) \leqslant h(\Delta) z_\infty^{2(\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)+1}$.

From (26) and $z \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$ and $x \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}^n)$ (claim (C) and assumption (H1)), we have

$$\dot{V}_{z_t}|_{(??)} \leqslant -\frac{a\|z\|_\infty^{-2|\mathfrak{g}_n|}}{4} V_{z_t} + N_{\|x\|_\infty, \|z\|_\infty, \Delta}$$

for all $t \geqslant 0$ and for some $N_{\|x\|_\infty, \|z\|_\infty, \Delta} > 0$ which depends only on the sup norms $\|x\|_\infty$, $\|z\|_\infty$ and $\Delta$. This implies that

$$V_{z_t} \leqslant \max\{V_{z_0}, \frac{4\|z\|_\infty^{2|\mathfrak{g}_n|}}{a} N_{\|x\|_\infty, \|z\|_\infty, \Delta}\} \tag{30}$$

for all $t \geqslant 0$ and, therefore, $V_z(\eta) \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$. Since $z \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$ (claim (C)), we conclude that $\eta \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}^n)$ and, therefore, $\xi \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}^n)$ (see the change of coordinates (22)).

Set

$$\alpha_z(\xi, y) := \max\Big\{q_z(\xi, y) \\ -h(\Delta) z_t^{2(\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)+1} \lambda\Big(\frac{q_z(\xi, y)}{z_t^{2(\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)+1}}\Big), 0\Big\}.$$

On account of the fact that $\xi, x \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}^n)$ and $z \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$, also $\dot{x}, \dot{\xi} \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}^n)$ with $\dot{z} \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R})$ so that $x, \xi \in \mathbf{C}^0_0(\mathbb{R}_\geqslant, \mathbb{R}^n)$ and $z \in \mathbf{C}^0_0(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$. Since $z \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$, by integration of the $\dot{z}_t$ equation we get $\alpha_z(\xi, y) \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant) \cap \mathbf{L}^1(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$. By use of lemmas **??** and **??** it is routine to prove $\alpha_z(\xi, y) \in \mathbf{C}^0_0(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$ so that it follows $\lim_{t\to+\infty} \alpha_{z_t}(\xi_t, y_t) = 0$ by virtue of Barbalat's lemma. From this and properties of lim sup we obtain

$$0 \geqslant \limsup_{t\to+\infty} q_{z_t}(\xi_t, y_t) - h(\Delta) z_\infty^{2(\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)+1} \tag{31}$$

which finally implies (D).

*(E).* As a consequence of claim (D) and definition of $q_z$, since $\lim_{t\to+\infty} z_t = z_\infty \in [1, +\infty)$ and using the monotonicity of the degrees $\{\mathfrak{g}_j\}_{j=1,\ldots,n}$

$$\limsup_{t\to+\infty} |y - C\xi|^2 \\ \leqslant \mu_{z_\infty}(\Delta) z_\infty^{2(\max_i \mathfrak{r}_i - \mathfrak{r}_1 + \mathfrak{g}_1 - \mathfrak{g}_n)} = \nu_{z_\infty}(\Delta) \tag{32}$$

Let be $\Omega \subset \mathbb{R}^n$ be a compact set including the origin such that $x_t, \xi_t \in \Omega$ for all $t \geqslant 0$. Under assumption (ii) and according to Battilotti (2014), theorem V.1, there exist symmetric and positive definite $n \times n$ matrices $P, Q$ and $1 \times n$ matrix $L$ (all depending on $\Omega$) such that (15) (with $\psi \equiv 0$) holds for all $x, \xi \in \Omega$. On the other hand, (3)-(4) can be rewritten as follows

$$\dot{x}_t = (A + BF + HC)x_t + \phi(x_t), \tag{33}$$
$$\dot{\xi}_t = (A + BF + HC)\xi_t + \phi(\xi_t) + LC(x_t - \xi_t) + \mathcal{W}_t,$$

where

$$\mathcal{W} := (BF + HC)(\mathbf{sat}_{cz^{\mathfrak{r}}}(\xi) - \xi) \\ +\phi(\mathbf{sat}_{cz^{\mathfrak{r}}}(\xi)) - \phi(\xi) + (L_z - L)(y - C\xi) + Ld.$$

Using (32) and the incremental properties of $\phi$ (assumption (i) of (H0)), recalling that $\limsup_{t\to+\infty} a_t b_t \leqslant \limsup_{t\to+\infty} a_t \limsup_{t\to+\infty} b_t$ and $\limsup_{t\to+\infty} f(b_t) \leqslant \sup_{\|b\|\leqslant 2n \limsup_{t\to+\infty} \|c_t\|} f(b)$ if $\|b_t\| \leqslant n\|c_t\|$ for all $t \geqslant 0$ with $b_t, c_t \in \mathbb{R}^n$),

$$\limsup_{t\to+\infty} \|\mathcal{W}_t\| \leqslant \Big(\|BF + HC\| \\ + \sup_{\substack{\|w_1\|\leqslant 2nc\|z_\infty^{\mathfrak{r}}\| \\ \|w_2\|\leqslant 2n(\sqrt{\nu_{z_\infty}(\Delta)} + c\|z_\infty^{\mathfrak{r}}\|)}} \|\Phi_U(w_1, w_2)\|\Big)\sqrt{\nu_{z_\infty}(\Delta)} \\ +\sqrt{\mu_{z_\infty}(\Delta)}\|L_{z_\infty} - L\| + \|L\|\Delta = \chi_{z_\infty}(\Delta) \tag{34}$$

Pick $\epsilon > 0$ and let $T_\epsilon > 0$ be such that

$$\|\mathcal{W}_{t+T_\epsilon}\| \leqslant \chi_{z_\infty}(\Delta) + \epsilon,\ \forall t \geqslant 0 \tag{35}$$

(which always exists by (34)). With $V_t = (1/2)\|x_t - \xi_t\|_P^2$ we have from (15) and (33) and for all $t \geqslant 0$

$$\dot{V}_{t+T_\epsilon}|_{(33)} \leqslant -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} V_{t+T_\epsilon} + \frac{\lambda_{max}^2(P)}{\lambda_{min}(Q)} \|\mathcal{W}_{t+T_\epsilon}\|^2,$$

so that, on account of (35),

$$\|x_{t+T_\epsilon} - \xi_{t+T_\epsilon}\|^2 \leqslant \frac{V_{T_\epsilon}}{\lambda_{max}(P)} e^{-\frac{\lambda_{min}(Q)t}{2\lambda_{max}(P)}} \\ +\frac{4\lambda_{max}^3(P)}{\lambda_{min}^2(Q)} (\chi_{z_\infty}(\Delta) + \epsilon)^2 [1 - e^{-\frac{\lambda_{min}(Q)t}{\lambda_{max}(P)}}]$$

Passing to the limit for $t \to +\infty$ on both sides and $\epsilon$ being arbitrary, we obtain the conclusions of theorem 3.

## 5. CONCLUSIONS

We have presented a class of nonlinear observers for systems with noisy measurements and bounded trajectories. Our observer guarantees an upper bound for the limsup of the norm of the estimation error depending on the limsup of the norm of the measurement noise. In future research we will consider disturbances affecting also the state equations and unbounded state trajectories.

## Appendix A. INCREMENTAL HOMOGENEITY IN THE GENERALIZED SENSE: A REVIEW

The notion of (incremental) homogeneity has been introduced in Battilotti (2014) in the context of semi-global stabilization and observer design problems. Here we recall this notion in a slightly more general form.

### *A.1 Definitions*

*Definition 5.* A parametrized function $\phi_z \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}^l)$, $z \in \mathbb{R}_>$, is said to be incrementally homogeneous (i.h.) with quadruple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \phi)$ if there exist $\mathfrak{d} \in \mathbb{R}^l, \mathfrak{h} \in \mathbb{R}^n$,

$\mathfrak{r} \in \mathbb{R}^n_>$ and $\phi \in \mathbf{C}^0(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^{l\times n})$ such that for all $\epsilon > 0$ and $w', w'' \in \mathbb{R}^n$

$$\begin{aligned}&\phi_\epsilon(\epsilon^{\mathfrak{r}} \diamond w') - \phi_\epsilon(\epsilon^{\mathfrak{r}} \diamond w'')\\ &= \epsilon^{\mathfrak{d}} \diamond \Big(\phi(w', w'')\Big(\epsilon^{\mathfrak{h}} \diamond (w' - w'')\Big)\Big)\end{aligned}$$

The notion of incremental homogeneity incapsulates as a particular case the notion of homogeneity (see for example **?**). When $w''$ is set to 0 in definition 5 we say that $\phi_z$ is homogeneous with quadruple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \phi)$.

Note that the function $\phi_z$ may be parametrized by the dilating parameter itself. The function $\phi_z(x) \coloneqq x_1 + x_2^3$ (in this case $\phi_z$ does not depend on the dilating parameter) is i.h. with quadruple $(\mathfrak{r}, 0, \mathfrak{h}, \phi)$, where $\mathfrak{r} \coloneqq (1,2)^T$, $\mathfrak{h} \coloneqq (1,6)^T$ and $\phi(w', w'') \coloneqq (1, (w_2')^2 + (w_2'')^2 + w_2''w_2')$.

There are functions, like $\sin x$, which are not i.h. but behaves in the upper bound as i.h. function. This motivates the following definition ($\langle\!\langle a \rangle\!\rangle$ denotes the column vector of the absolute values of the elements of $a \in \mathbb{R}^n$).

*Definition 6.* A parametrized function $\phi \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}^l)$, $z \in \mathbb{R}_>$, is said to be incrementally homogeneous in the upper bound (i.h.u.b.) with quadruple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \phi_U)$ if there exist $\mathfrak{d} \in \mathbb{R}^l, \mathfrak{h} \in \mathbb{R}^n$, $\mathfrak{r} \in \mathbb{R}^n_>$, $\phi_U \in \mathbf{C}^0(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^{l\times n}_\geqslant)$ such that for all $\epsilon \geqslant 1$ and $w', w'' \in \mathbb{R}^n$

$$\begin{aligned}&\langle\!\langle \phi_\epsilon(\epsilon^{\mathfrak{r}} \diamond w') - \phi_\epsilon(\epsilon^{\mathfrak{r}} \diamond w'') \rangle\!\rangle\\ &\leq \epsilon^{\mathfrak{d}} \diamond \Big(\phi_U(w', w'')\Big\langle\!\Big\langle \epsilon^{\mathfrak{h}} \diamond (w' - w'') \Big\rangle\!\Big\rangle\Big)\end{aligned}$$

When $w''$ is set to 0 in definition 6 we will simply say that $\phi_z$ is homogeneous in the upper bound with quadruple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \phi_U)$.

The function $\phi_z(x) \coloneqq z\big[x_2 \; x_2^3 g(x_1)\big]^T$, $g \in \mathbf{C}^0(\mathbb{R}, \mathbb{R})$ any bounded and globally Lipschitz function, is i.h.u.b. with triple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \phi_U)$, where $\mathfrak{r} \coloneqq (1,2)^T$, $\mathfrak{d} \coloneqq (3,7)^T$, $\mathfrak{h} \coloneqq (1,0)^T$ and the matrix $\phi_U(w', w'')$ defined as

$$\begin{aligned}&[\phi_U(w', w'')]_{11} \coloneqq 0, [\phi_U(w', w'')]_{12} \coloneqq 1,\\ &[\phi_U(w', w'')]_{21} \coloneqq (w_2'')^3 \frac{|g(w_1') - g(w_1'')|}{|w_1' - w_1''|},\\ &[\phi_U(w', w'')]_{22} \coloneqq |(w_2')^2 + (w_2'')^2 + w_2'w_2''||g(w_1')|.\end{aligned}$$

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