Sampled-data stabilisation of a class of state-delayed nonlinear dynamics

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***Abstract*— The paper deals with the stabilisation of strict-feedback dynamics with a delay on the last component of the state. It is shown that the Immersion and Invariance approach provides a natural framework for solving the problem. An accademic simulated example is provided.**

***Index Terms*— Nonlinear stabilisation, Systems with delays, Nonlinear sampled-data systems.**

I. Introduction

Stabilisation under Immersion and Invariance - I&I -, proposed in [1] for continuous-time dynamics, has been the object of several investigations in the last decade. Several extensions and applicatisve results have been developed which identify a recognized control approach ([2], [3], [4]). It was extended to the discrete-time domain in [5] in relation with adaptive control in presence of parameter uncertainties.

More recently, in [6], it has been shown that the I&I approach provides a natural framework to deal with sampled-data stabilisation of input-delayed dynamics; while in [7] it has been fruitfully applied to design sampled-data controllers for dynamics which exhibit specific structures such as strict-feedback forms. Exploiting sampling to control systems with delayed inputs is a well known practice which has found renewed interest in the current literature ([8], [9], [10], [11], [12]). The present work follows these lines.

In this paper, the stabilization of a strict-feedback dynamics with delays on the last connecting state is addressed. More precisely and for simplicity we consider dynamics with one cascade of the form

$$\dot{x}_1(t) = f(x_1(t)) + g(x_1(t))x_2(t-\tau), \quad \dot{x}_2(t) = u(t) \qquad (1)$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}$, $u \in U \subseteq \mathbb{R}$, f and g are smooth vector fields on $\mathbb{R}^n$, i. e. C^∞, and τ denotes a delay acting on x_2, the connecting state.

The problem is set in the digital context assuming that the measures of the state are available at the sampling instants $t = k\delta$, $k \geq 0$ and the control is maintained constant over time intervals of length δ. The sampling period δ is chosen so that $\tau = N\delta$ for a positive $N \in \mathbb{N}^+$.

The idea developed in the sequel starts by noting that under a simple coordinates change, the delayed dynamics admits a higher but finite dimensional sampled-data equivalent model over which stabilization is reformulated in the I&I context with target given by the sampling of the delay-free dynamics. Then the design of the controller is achieved by driving the dynamics to the invariant manifold with boundedness of all the extended state trajectories.

The proposed solution combines two previous contributions of the authors:

- the sampled-data I&I stabilizer discussed in [6] which naturally identifies the target with the delay-free dynamics;
- the direct sampled-data I&I stabilizing in [7] to define the immersion mapping and feedback which render invariant the target manifold.

We note that the same type of state delays on connected dynamics was studied in [13] according to a continuous-time backstepping procedure.

This paper is organized as follows: in *Section II* the class of system under study is defined and some preliminary results are given; in *Section III* the main result is given and specified in the particular case of $\delta = \tau$ in *Section IV*; an academic example is discussed with some simulations in *Section V*.

II. Problem settlement and preliminary results

We summarize in the following the recurrent assumptions:

A) the sampling period δ, small enough, is a multiple of the delay τ, i.e. $\tau = N\delta$ for a suitable $N \in \mathbb{N}^+$;
B) the input $u(t)$ is set constant over time intervals of length δ; namely, $u(t) = u_k$ $t \in [k\delta, (k+1)\delta[$;
C) the delay free x_1-dynamics of (1) is smoothly stabilizable trough a *fictitious* continuous-time controller $x_2 = \gamma(x_1)$ and a control Lyapunov function, $W : \mathbb{R}^n \to \mathbb{R}$, is assumed known (see [14]);

$$[L_f + \gamma L_g]W(x_1) < 0 \quad \forall x_1 \in \mathbb{R}^n/\{0\}.$$

Accordingly, assuming the I&I framework [1], one defines for the delay-free dynamics:

- the target dynamics $\dot{\xi} = f(\xi) + g(\xi)\gamma(\xi)$;
- the immersion map $\pi(\xi) = \begin{pmatrix}\xi' & \gamma(\xi)\end{pmatrix}'$;
- the implicit manifold $z = \phi(x) = x_2 - \gamma(x_1)$, with $z(0) = x_2(0) - \gamma(x_1(0))$;
- the *on-the-manifold* control law $\varphi(\xi) = \dot{\gamma}(\xi)$ which renders invariant the manifold.

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Finally, the control law which makes the manifold attractive with boundedness of the trajectories of the full dynamics

$$\begin{aligned}\dot{x}_1 =& f(x_1) + g(x_1)[z + \gamma(x_1)]\\ \dot{x}_2 =& \psi(x,z)\\ \dot{z} =& \psi(x,z) - \dot{\gamma}(x_1)\end{aligned}$$

is set as $\psi(x,z) = \dot{\gamma}(x_1) - K(x)z$ with suitably chosen gain function $K(x)$.

A. The extended hybrid representation

Consider the continuous-time dynamics (1) and set $x_3(t) = x_2(t-\tau)$ so moving the delay into the input variable

$$\dot{x}_1(t) = f(x_1(t)) + g(x_1(t))x_3(t), \quad \dot{x}_3(t) = u(t-\tau) \tag{2}$$

so that the approach proposed in [6] can be used. Setting $\tau = N\delta$ and under Assumption A, the *hybrid extended dynamics* over $\mathbb{R}^{n+1+N}$ is defined for $t \in [k\delta, (k+1)\delta[$ as:

$$\begin{aligned}&\dot{x}_1(t) = f(x_1(t)) + g(x_1(t))x_3(t)\\ &\dot{x}_3(t) = v_k^1, \quad v_{k+1}^1 = v_k^2, \quad \dots \quad v_{k+1}^N = u_k.\end{aligned} \tag{3}$$

It results that the control design problem can be set on the sampled-data equivalent of (3), which is finite dimensional dynamics, with state extension of order N, strictly related to the delay length.

B. Sampled-data delay free I&I stabilization

Following [7], Assumption C provides sufficient conditions for the existence of an I&I sampled-data controller preserving GAS of the equilibrium when $\tau = 0$. Setting $\tau = 0$, one defines the equivalent sampled-data dynamics of (1) through integration over the time interval $[k\delta, (k+1)\delta[; k \geq 0$, as in [15]. It is provided in the form of a map parameterized by δ:

$$\begin{aligned}&x_{1k+1} = F^{\delta}(x_{1k}, x_{2k}) + \frac{\delta^2}{2!} u_k G^{\delta}(x_{1k}, x_{2k}, u_k)\\ &x_{2k+1} = x_{2k} + \delta u_k\end{aligned} \tag{4}$$

when $x_k = x(t)\big|_{t=k\delta}$. The following proposition summarises the results in [7], where a complete proof is given.

Proposition 2.1: Consider the nonlinear continuous-time dynamics in (1) under Assumptions A, B and C in the delay free case (i.e., $\tau = 0$). Then, its sampled-data equivalent dynamics (4) is I&I stabilizable with target dynamics

$$\xi_{k+1} = F^{\delta}(\xi_k, \gamma^{\delta}(\xi_k)) + \frac{\delta^2}{2!} \varphi^{\delta}(\xi_k) G^{\delta}(\xi_k, \gamma^{\delta}(\xi_k), \varphi^{\delta}(\xi_k)). \tag{5}$$

The mappings $\gamma^{\delta}(\cdot)$ and $\varphi^{\delta}(\cdot)$ are solutions of the two equalities:

$$W(\xi_{k+1}) = W(\xi_k) + \int_{k\delta}^{(k+1)\delta} L_{f+g\gamma} W(\xi(\tau)) d\tau \tag{6}$$

$$\gamma^{\delta}(\xi_{k+1}) = \gamma^{\delta}(\xi_k) + \delta \varphi^{\delta}(\xi_k). \tag{7}$$

We note that the mappings $\gamma^{\delta}(\cdot)$ and $\varphi^{\delta}(\cdot)$ are defined by their asymptotic series expansions in powers of δ as follows

$$\gamma^{\delta}(\xi_k) = \gamma_0(\xi_k) + \sum_{i\geq 0} \frac{\delta^i}{(i+1)!} \gamma_i(\xi_k)$$

$$\varphi^{\delta}(\xi_k) = \varphi_0(\xi_k) + \sum_{i\geq 0} \frac{\delta^i}{(i+1)!} \varphi_i(\xi_k).$$

Accordingly, both the immersion mapping $\pi^{\delta}(\xi) = \begin{pmatrix}\xi' & \gamma^{\delta}(\xi)\end{pmatrix}'$ and the implicit manifold characterisation $\phi^{\delta}(x) = x_2 - \gamma^{\delta}(x_1)$ are parameterized by the sampling period δ. Setting $\delta = 0$, one recovers the continuous-time solutions $(\pi(\cdot), \phi(\cdot), \varphi(\cdot))$.

We note that the equality (6) ensures Input Lyapunov Matching - ILM - at the sampling instants (see [15], [16]) of the closed loop behavior of the function $W(\cdot)$ on the target dynamics (5).This guarantees that the equilibrium of (5) is GAS. On the other hand, equality (7) guarantees the invariance of the manifold. Accordingly, it is implicitly defined as $\phi^{\delta}(x) = 0$. On these bases, the I&I stabilizing sampled-data feedback $u = \psi^{\delta}(x,z)$ is designed to drive z to zero while preserving boundedness of the complete state trajectories

$$\begin{aligned}&x_{1k+1} = F^{\delta}(x_{1k}, x_{2k}) + \frac{\delta^2}{2!} u_k G^{\delta}(x_{1k}, x_{2k}, u_k)\\ &x_{2k+1} = x_{2k} + \delta u_k, \quad z_{k+1} = z_k + \delta u_k - \gamma^{\delta}(x_{1k+1}) + \gamma^{\delta}(x_{1k}).\end{aligned}$$

It follows that the equilibrium of the closed-loop x dynamics is GAS in the delay free case.

III. Main result

Consider the continuous-time dynamics (2) (or, equivalently, (1)) and its hybrid representation (3) over $\mathbb{R}^{n+1+N}$ when $\tau = N\delta$. Its sampled-data equivalent dynamics is described as

$$\begin{aligned}&x_{1k+1} = F^{\delta}(x_{1k}, x_{3k}) + \frac{\delta^2}{2!} v_k^1 G^{\delta}(x_{1k}, x_{3k}, v_k^1)\\ &x_{3k+1} = x_3 + \delta v_k^1, \; v_{k+1}^1 = v_k^2 \quad \dots \quad v_{k+1}^N = u_k\end{aligned} \tag{8}$$

or, in a more compact way, as $x_{k+1}^e = \bar{F}^{\delta}(x_k^e, u_k)$ with $x^e = col(x_1', \, x_3, \, v^1, \, \dots, \, v^N)' \in \mathbb{R}^{n+1+N}$. In [6], the authors define the GAS sampled-data I&I target dynamics as the closed-loop dynamics (4) under the delay-free feedback $\psi^{\delta}(\cdot,\cdot)$, as defined in Proposition 2.1. Hence, the attractive manifold is the one where the delay on the input is recovered. An alternative approach is stated by the following result.

Theorem 3.1: Consider the input-affine continuous-time dynamics in (1) with state delay $\tau = N\delta$ under Assumptions A, B and C. Let the extended sampled-data dynamics (8) with equilibrium $x_*^e = col(x_*', \; 0_{N\times 1}')$, then it is I&I stabilizable with target dynamics 5 and $\gamma^{\delta}(\cdot), \varphi^{\delta}(\cdot) : \mathbb{R}^n \to \mathbb{R}$ defined as in Proposition 2.1.

Proof: To prove the thesis, one has to show that the conditions in Theorem 2.2 in [6] are verified. For this purpose, suppose $\gamma^{\delta}(\cdot)$ and $\varphi^{\delta}(\cdot)$ defined according to Proposition 2.1 as solutions to the I-LM problem in (6-7) with control Lyapunov function $W : \mathbb{R}^n \to \mathbb{R}^+$. Consequently, one can define the extended immersion mapping $\bar{\pi}^{\delta} : \mathbb{R}^n \to \mathbb{R}^{n+1+N}$

$$\bar{\pi}^{\delta}(\xi_k) = \begin{pmatrix}\xi_k' & \gamma^{\delta}(\xi_k) & \varphi^{\delta}(\xi_k) & \dots & \varphi^{\delta}(\xi_{k+N-1})\end{pmatrix}' \tag{9}$$

and extended mapping $\bar{\phi}^\delta : \mathbb{R}^{n+N+1} \to \mathbb{R}^{N+1}$ as

$$\begin{aligned} z_{1k} &= \bar{\phi}_1^\delta(x_k, v_k) = x_{3k} - \gamma^\delta(x_{1k}) \\ z_{2k} &= \bar{\phi}_2^\delta(x_k, v_k) = v_k^1 - \varphi^\delta(x_{1k}) \\ &\dots \\ z_{N+1,K} &= \bar{\phi}_{N+1}^\delta(x_k, v_k) = v_k^N - \varphi^\delta(x_{1,k+N-1}) \end{aligned} \tag{10}$$

where $v = col(v^1, \dots, v^N)'$.

By construction, the three instrumental condition for I&I stabilization are satisfied (see Theorem 2.2 in [6]). More in detail, the target dynamics $\xi_{k+1} = \alpha^\delta(\xi)$ as in (5) has a GAS equilibrium $\xi_* \in \mathbb{R}^p$. Then, the *Immersion Condition* is satisfied by the choices (9)-(10) with $z_0 = \bar{\phi}(x_0, v_0)$. On these bases, it is straightforward that the sampled-data feedback $u_k = \bar{\psi}^\delta(x_k^e, z_k)$ designed in order to bring z to zero and make all the trajectories of the dynamics

$$\begin{aligned} z_{1k+1} =& z_{1k} + \delta[z_{2k} - \varphi^\delta(x_{1k})] + \gamma^\delta(x_{1k}) - \gamma^\delta(x_{1k+1}) \\ z_{2k+1} =& z_{3k} \quad \dots \quad z_{Nk+1} = z_{N+1k} \\ z_{N+1k+1} =& \bar{\psi}^\delta(x_k^e, z_k) - \varphi^\delta(x_{1k+N}) \\ x_{k+1}^e =& \bar{F}^\delta(x_k^e, \bar{\psi}^\delta(x_k^e, z_k)). \end{aligned} \tag{11}$$

bounded, globally asymptotically stabilizes the equilibrium of (1). ■

Remark 3.1: When the system dynamics reaches the invariant manifold, the feedback reduces to $c^\delta(\cdot)$ corresponding to the delay-free stabilizing feedback $\varphi^\delta(\cdot)$ in Proposition 2.1. When $N = 0$, the delay-free case, one recovers $c^\delta(\xi) = \varphi^\delta(\xi)$.

Remark 3.2: For a given τ, the pair $(N, \ \delta)$ has to be chosen as a trade off between computational effort and required performances on the closed-loop system.

1) On the definition of the sampled-data control law: Theorem 3.1 states sufficient conditions for the existence of a I&I stabilizing controller $u_k = \bar{\psi}^\delta(x_k, v_k, z_k)$. In this section, we describe a multirate design strategy of order equal to the dimension of the off the manifold state component z. For, let us introduce the $(N+1)$-order multirate sampled-data dynamics associated with (11) when the I&I controller $\bar{\psi}^{i\bar{\delta}}(\cdot, \cdot)$ is denoted as u_k^i

$$\begin{aligned} z_{1k+1} =& z_{1k} + \bar{\delta} \sum_{i=2}^{N+1} [z_{ik} + \varphi^{\bar{\delta}}(x_{1k+\frac{i-2}{N+1}})] + \\ & \gamma^{\bar{\delta}}(x_{1k}) - \gamma^{\bar{\delta}}(x_{1k+1}) + \bar{\delta} u_k^1 \\ z_{2k+1} =& u_k^2 - \varphi^{\bar{\delta}}(x_{1k+1}) \quad \dots \quad z_{Nk+1} = u_k^N - \varphi^{\bar{\delta}}(x_{1k+\frac{2N-1}{N+1}}) \\ z_{N+1k+1} =& u_k^{N+1} - \varphi^{\bar{\delta}}(x_{1k+\frac{2N}{N+1}}) \\ x_{1k+1} =& F_1^{\bar{\delta}}(x_k^e, u_k^1) \quad x_{3k+1} = x_{3k} + \bar{\delta}[u_k^1 + \sum_{i=1}^{N} v_k^i] \\ v_{k+1}^1 =& u_k^2 \quad \dots \quad v_{k+1}^N = u_k^{N+1} \end{aligned} \tag{12}$$

in which the control $u(t)$ is maintained constant at values u_k^i over intervals of length $\bar{\delta} = \frac{\delta}{N+1}$ for all $t \in [k\delta + (i-1)\bar{\delta}, k\delta + i\bar{\delta}[$, $i = \frac{1}{N+1}$.

Remark 3.3: The prediction steps required with a single rate strategy is N; the multirate strategy requires, at most, $\frac{2N}{N+1}$ prediction steps.

The hypotheses of Theorem 2.2 in [6] are naturally preserved under the multirate controller. Though, an accurate rewriting of the immersion condition may be useful to point out that the so-defined sampled-data controller preserves manifold invariance under multirate-sampling. In particular, by defining $c^{i,\bar{\delta}}(\xi) = c^{\bar{\delta}}(\xi_{k+\frac{i}{N+1}})$ $(i = 1, \dots, N+1)$, one has that $\forall \xi \in \mathbb{R}^n$

$$\begin{aligned} \xi_{k+1} &= \alpha^{\bar{\delta}}(\xi_k) \\ \gamma^{\bar{\delta}}(\xi_{k+1}) &= \gamma^{\bar{\delta}}(\xi_k) + \bar{\delta} \sum_{i=2}^{N+1} \varphi^{\bar{\delta}}(\xi_{k+\frac{i-2}{N+1}}) + \bar{\delta} c^{1,\bar{\delta}}(\xi_k) \\ c^{2,\bar{\delta}}(\xi_k) &= \varphi^{\bar{\delta}}(x_{1k+1}) \quad \dots \quad c^{N,\bar{\delta}}(\xi_k) = \varphi^{\bar{\delta}}(x_{1k+\frac{2N-1}{N+1}}) \\ c^{N+1,\bar{\delta}}(\xi_k) &= \varphi^{\bar{\delta}}(x_{1k+\frac{2N}{N+1}}). \end{aligned} \tag{13}$$

Finally, one can see that the I&I stabilisation is achieved by the $(N+1)$-rate control u defined as

$$\begin{aligned} \bar{\delta} u_k^1 =& -\bar{\delta}\Gamma_1 z_{1k} - \gamma^{\bar{\delta}}(x_{1k}) + \gamma^{\bar{\delta}}(x_{1k+1}) - \\ & \bar{\delta} \sum_{i=2}^{N+1} [z_{ik} + \varphi^{\bar{\delta}}(x_{1k+\frac{i-2}{N+1}})] \\ u_k^2 =& -\Gamma_2 z_{2k} + \varphi^{\bar{\delta}}(x_{1k+1}) \\ & \dots \\ u_k^N =& -\Gamma_N z_{Nk} + \varphi^{\bar{\delta}}(x_{1k+\frac{2N-1}{N+1}}) \\ u_k^{N+1} =& -\Gamma_{N+1} z_{N+1k} + \varphi^{\bar{\delta}}(x_{1k+\frac{2N}{N+1}}) \end{aligned} \tag{14}$$

with suitably defined gains Γ_i $(i = 1, ..., N+1)$. More in detail, when such a controller is applied, one has that all trajectories of (12) are bounded for all $k \geq 0$ with

$$\lim_{k\to\infty} z_k = 0 \quad \bar{\psi}_k^{i\bar{\delta}}(\bar{\pi}^{\bar{\delta}}(\xi), \mathbf{0}) = c^{i\bar{\delta}}(\xi)$$

for $i = 1, \dots, N+1$.

Without loss of generality, the proof of the existence of such a solution is reported for the particular case of $\tau = \delta$.

IV. The case $\tau = \delta$

Let us discuss more in detail the design of the feedback $\bar{\psi}^\delta(x^e, z)$ in the single-rate case in which $\tau = \delta$. In such a case, Theorem 3.1 specifies as follows.

Proposition 4.1: Consider the continuous-time dynamics (1) satisfying Assumptions A, B and C with state delay $\tau = \delta$. Let the extended dynamics on $\mathbb{R}^{n+2}$ be

$$\begin{aligned} x_{1k+1} &= F^\delta(x_{1k}, x_{3k}) + \frac{\delta^2}{2!} v_k G^\delta(x_{1k}, x_{3k}, v_k) \\ x_{3k+1} &= x_3 + \delta v_k, \qquad v_{k+1} = u_k. \end{aligned} \tag{15}$$

Then it is I&I stabilizable with target dynamics (5) whose equilibrium is made GAS with suitable choice of $\gamma^\delta, \ \varphi^\delta : \mathbb{R}^n \to \mathbb{R}$.

Proof: The proof proceeds in the same way as in the one of Theorem 3.1, so it will be omitted. ■

A. On the design of the sampled-data stabilizer

In this section, a possible choice of the controller which satisfies the condition on *Manifold invariance and attractivity with trajectory boundedness* is proposed. When $\tau = \delta$, the double rate sampled-data equivalent model of the hybrid dynamics (3) over time intervals of length $\delta = 2\bar{\delta}$ is defined as in (12) with

$$\bar{\psi}^{1\bar{\delta}}(x_{1k}, v_k, z_k) = \bar{\psi}^{\bar{\delta}}(x_{1k}, z_k)$$

and

$$\bar{\psi}^{2\bar{\delta}}(x_{1k}, v_k, z_k) = \bar{\psi}^{\bar{\delta}}(x_{1k+\frac{1}{2}}, z_{k+\frac{1}{2}}).$$

Setting

$$\begin{aligned}\bar{\delta}\bar{\psi}^{1\bar{\delta}}(x_{1k}, v_k, z_k) =& \bar{\delta}\Gamma_1 z_{1k} + \gamma^{\bar{\delta}}(x_{1k+1}) \quad (16a)\\ &- \gamma^{\bar{\delta}}(x_{1k}) - \bar{\delta}[z_{2k} + \varphi^{\bar{\delta}}(x_{1k})]\\ \bar{\psi}^{2\bar{\delta}}(x_{1k}, v_k, z_k) =& \Gamma_2 z_{2k} + \varphi^{\bar{\delta}}(x_{1k+1}) \quad (16b)\end{aligned}$$

the reduced z-dynamics is

$$z_{1k+1} = [1 + \bar{\delta}\Gamma_1] z_{1k}, \quad z_{2k+1} = [1 + \Gamma_2] z_{2k}.$$

The existence of such a controller is proved in the following Proposition.

Proposition 4.2: Given the sampled-data dynamics in (15) verifying Theorem 3.1, then there exists a double-rate control ensuring, at each step, I&I stabilisation of the dynamics in (15).

Proof: Denoting by $\bar{\psi}^{1\bar{\delta}} = u_k^1$. The proof consists in verifying that there exist solutions in the form

$$\bar{\psi}^{j\bar{\delta}}(x_{1k}, v_k, z_k) = \bar{\psi}_0^j(x_{1k}, v_k, z_k) + \sum_{i\geq 1} \bar{\delta}^i \bar{\psi}_i^j(x_{1k}, v_k, z_k) \quad (17)$$

for $j = 1, 2$ to equalities (16a) and (16b).

The existence of a solution to (16b) is guaranteed since the right-hand side of the equality does not depend on $\bar{\psi}^{2\bar{\delta}}$ itself; hence, a series inversion is needed in order to compute the resulting controller. For, one rewrites $\gamma^{\bar{\delta}}(x_{1k+1})$ as the sum of two component:

$$\gamma^{\bar{\delta}}(x_{1k+1}) = \bar{\gamma}_1^{\bar{\delta}}(x_{1k}, z_k, 0) + \bar{\delta} u_k^1 \bar{\gamma}_2^{\bar{\delta}}(x_{1k}, z_k, u_k^1)$$

where

$$\bar{\gamma}_1^{\bar{\delta}}(x_{1k}, z_k, 0) = \gamma^{\bar{\delta}}(\tilde{F}^{\bar{\delta}}(x_{1k}, z_{1k} + \gamma^{\bar{\delta}}(x_{1k}), z_{2k} + \varphi^{\bar{\delta}}(x_{1k})))$$

does not depend on the control while

$$\begin{aligned}&\bar{\gamma}_2^{\bar{\delta}}(x_{1k}, z_k, u_k) =\\ &\frac{\delta}{2} \sum_{i\geq 1} \frac{\partial^i \gamma^{\bar{\delta}}}{\partial x_1^i}\Big|_{x_1 = \tilde{F}^{\bar{\delta}}} [\tilde{G}(x_{1k}, z_{1k} + \gamma^{\bar{\delta}}(x_{1k}), z_{2k} + \varphi^{\bar{\delta}}(x_{1k}), u_k^1)]^i\end{aligned}$$

is control dependent. One can now rewrite the equality among formal series in (16a) as

$$\begin{aligned}&\bar{\delta} S(\bar{\delta}, x_{1k}, z_k, u_k^1) = \bar{\delta} u_k^1 [1 - \bar{\gamma}_2^{\bar{\delta}}(x_{1k}, z_k, u_k)] - \bar{\delta}\Gamma_1(x_k^e) z_{1k} -\\ &\bar{\gamma}_1^{\bar{\delta}}(x_{1k}, z_k, 0) + \gamma^{\bar{\delta}}(x_{1k}) + \bar{\delta}[z_{2k} + \varphi^{\bar{\delta}}(x_{1k})] = 0.\end{aligned}$$

The existence of a solution is proved by means of the Implicit Function Theorem. Indeed, for $\delta = 0$ one has

$$\begin{aligned}&S(0, x_{1k}, z_k, u_k^1) = \psi_0^1(x_{1k}, z_k) - \Gamma_1(x_k^e) z_{1k} + 2\frac{\partial \gamma_0}{\partial x_1}\Big|_{x_{1k}} \{f(x_1)\\ &+ g(x_1)[z_{1k} + \gamma_0(x_1)]\} + [z_{2k} + \varphi_0(x_1)] = 0\end{aligned}$$

where $\gamma_0(\cdot)$ and $\varphi_0(\cdot)$ are defined according to Proposition 2.1. Such an equality is solved if

$$\begin{aligned}\psi_0^1(x_{1k}, z_k) = &\Gamma_1(x_k^e) z_{1k} - 2\frac{\partial \gamma_0}{\partial x_1}\Big|_{x_{1k}} \{f(x_{1k}) +\\ &g(x_{1k})[z_{1k} + \gamma_0(x_{1k})]\} - [z_{2k} + \varphi_0(x_{1k})]\end{aligned}$$

i.e., the controller defined on the double-rate Euler sampled-data model of (3). Since the partial derivative

$$\frac{\partial S(\bar{\delta}, u)}{\partial u}\Big|_{\bar{\delta}=0} = 1$$

is non-zero for any (x_1, z), one can conclude that there exists, for $\bar{\delta}$ small enough, a control $u = \bar{\psi}^{1\bar{\delta}}(x, v, z)$ in a neighbourhood of $\bar{\psi}_0^1(x, v, z)$ such that

$$S(\bar{\delta}, \rho(\bar{\delta})) = 0 \iff u = \bar{\psi}^{1\bar{\delta}}(x, v, z) = \rho(\bar{\delta}),$$

where ρ is the formal inversion $\rho(\bar{\delta}) = S^{-1}(\bar{\delta}, \rho(\bar{\delta}))$. Such a solution can be defined as an asymptotic series of $\bar{\delta}$ in the form (17) with $\bar{\psi}_0^1(\cdot, \cdot) = \rho(0)$. The I&I stabilisation is guaranteed since the invariance of the multi-rate controller is verified at the inter-sampling and sampling instants as in (13). Hence, Theorem 3.1 is satisfied for a suitable choice of $\bar{\delta}\Gamma_1$ and Γ_2 (not necessarily static) in order to have boundedness of the whole state trajectories in 12, with $N = 1$. At this point, the choice of $\bar{\delta}\Gamma_1$ and Γ_2 can be performed by means of a control Lyapunov function defined as $V^{\delta}(x, v, z) = W(x_1) + \sum_{i=1}^{2} z_i^2$. ■

B. Some constructive aspects

In this part, some constructive aspects are sketched for the computation of the solution in an approximate context, [17]. More in detail, considering (12), with $N = 1$, one gets in $O(|z|^2)$ the approximation below

$$\begin{aligned}x_{1k+1} =& \tilde{F}^{\bar{\delta}}(x_{1k}, \gamma^{\bar{\delta}}(x_{1k}), \varphi^{\bar{\delta}}(x_{1k})) + P_1^{\bar{\delta}}(x_{1k}, z_{1k}, z_{2k}) z_{1k} +\\ &P_2^{\bar{\delta}}(x_{1k}, z_{1k}, z_{2k}) z_{2k} + \frac{\bar{\delta}^2}{2} \bar{\psi}^{1\bar{\delta}}(x_k, v_k, z_k)\\ &\tilde{G}(x_{1k}, z_{1k} + \gamma^{\bar{\delta}}(x_{1k}), z_{2k} + \varphi^{\bar{\delta}}(x_{1k}), \bar{\psi}^{1\bar{\delta}}(x_k, v_k, z_k))\end{aligned}$$

with, discarding the dependence on the state and the control,

$$P_1^{\bar{\delta}} = \frac{\partial \tilde{F}^{\bar{\delta}}}{\partial x_3}\Big|_{x_3 = \gamma^{\delta}(x_1)} \quad P_2^{\bar{\delta}} = \frac{\partial \tilde{F}^{\bar{\delta}}}{\partial v}\Big|_{v = \varphi^{\delta}(x_1)}.$$

Accordingly, one can write the Taylor expansion of $\gamma^{\bar{\delta}}(x_{1k+1})$ and $\varphi^{\bar{\delta}}(x_{1k+1})$ in $O(|z|^2)$ as

$$\begin{aligned}\gamma^{\bar{\delta}}(x_{1k+1}) =& \gamma^{\bar{\delta}}(\tilde{F}^{\bar{\delta}}) + \frac{\partial \gamma^{\bar{\delta}}}{\partial x_1}\big|_{\tilde{F}^{\bar{\delta}}}[P_1^{\bar{\delta}} z_{1k} + \\ & P_2^{\bar{\delta}} z_{2k} + \frac{\bar{\delta}^2}{2}\bar{\psi}^{1\bar{\delta}}\tilde{G}^{\bar{\delta}}] + O(|z|^2) + O(|\bar{\psi}^{1\bar{\delta}}|^2) \\ \varphi^{\bar{\delta}}(x_{1k+1}) =& \varphi^{\bar{\delta}}(\tilde{F}^{\bar{\delta}}) + \frac{\partial \varphi^{\bar{\delta}}}{\partial x_1}\big|_{\tilde{F}^{\bar{\delta}}}[P_1^{\bar{\delta}} z_{1k} + \\ & P_2^{\bar{\delta}} z_{2k} + \frac{\bar{\delta}^2}{2}\bar{\psi}^{1\bar{\delta}}\tilde{G}^{\bar{\delta}}] + O(|z|^2) + O(|\bar{\psi}^{1\bar{\delta}}|^2).\end{aligned} \tag{18}$$

One can now define the controls $\bar{\psi}^{1\bar{\delta}}$ and $\bar{\psi}^{2\bar{\delta}}$ by their asymptotic series expansions with respect to $\bar{\delta}$ truncated at the p-th order; namely,

$$\bar{\psi}^{j\bar{\delta},[p_j]}(x_1,v,z) = \bar{\psi}_0^j(x_1,v,z) + \sum_{i\geq 1}^{p} \bar{\delta}^i \bar{\psi}_i^j(x_1,v,z) \tag{19}$$

for $j=1,2$. Substituting (19) and (18) into (16a) and (16b), under suitable boundedness assumptions on $\Gamma_j(x_k,v_k)$, $j=1,2$, for the corresponding approximated dynamics in $O(\bar{\delta}^p)$ and $O(|z|^2)$, one has that $lim_{k\to\infty} z_k = 0$ with manifold invariance and boundedness of the approximated state trajectories. This implies that the computed feedback at least locally stabilizes the delayed continuous time dynamics in (1).

V. Example

Let us consider the system in strict-feedback form

$$\dot{x}_1(t) = x_1^2(t) + x_2(t-\tau), \quad \dot{x}_2(t) = u(t). \tag{20}$$

A. Continuous-time design - the delay free case

Let us consider $\tau = 0$. In the continuous time case, one has that the I&I control law which makes the origin globally asymptotically stable is

$$u_c(x) = -\Gamma_c z + \dot{\gamma}_c(x_1) \quad \varphi_c(x) = \dot{\gamma}_c(x_1) \quad K > a > 1$$

with $\Gamma_c = 2$ and $\gamma_c(x_1) = -x_1^2 - x_1$. The immersion mapping and invariant manifold are defined as in the proof of Proposition 2.1. The target dynamics is $\dot{\xi} = -\xi$.

B. Sampled-data design - the delay free case

Once again, suppose $\tau = 0$ and introduce the sampled-data equivalent model associated to (20) as below

$$\begin{aligned}x_{1k+1} &= x_{1k} + \delta(x_{1k}^2 + x_{2k}) + \delta^2 x_{1k}(x_{1k}^2 + x_{2k}) + \tfrac{\delta^2}{2!}u_k + O(\delta^3) \\ x_{2k+1} &= x_{2k} + \delta u_k.\end{aligned}$$

In this case, the resulting target dynamics is GAS by setting

$$\begin{aligned}\gamma_0(\xi_k) &= -\xi_k - \xi_k^2 \quad \gamma_2(\xi_k) = 2\xi_k^3 \\ \varphi_0(\xi_k) &= \xi_k + 2\xi_k^2 \quad \varphi_1(\xi_k) = -2\xi_k - 8\xi_k^2 - 4\xi_k^3\end{aligned}$$

where γ_0, γ_1, and c_0 are the terms of $\gamma^{\delta,[2]}$ and $\varphi^{\delta,[1]}$ which are defined according to Proposition 2.1. The final second-order approximated sampled-data I&I control law is provided by $\delta\psi_D(x_{1k},x_{2k},z_k) = -\Gamma_D z_k + \gamma^\delta(x_{1k+1}) - \gamma^\delta(x_{1k})$,where $\gamma^\delta(x_{1k+1})$ is computed as its Taylor extension around x_{1k} truncated at the second-order.

C. Sampled-data design - the case of $\tau = \delta$

According to Section II-A, one introduces $x_3(t) = x_2(t-\tau)$ and the sampled-equivalent extended dynamics associated to (20) as

$$\begin{aligned}x_{1k+1} &= x_{1k} + \delta(x_{1k}^2 + x_{3k}) + \delta^2 x_{1k}(x_{1k}^2 + x_{3k}) + \frac{\delta^2}{2!}v_k + O(\delta^3) \\ x_{3k+1} &= x_{3k} + \delta v_k \quad v_{k+1} = u_k.\end{aligned} \tag{21}$$

According to the sampled-data delay-free design, one introduces the target dynamics, immersion mapping and off-manifold component as in Proposition 4.1 and the problem results in finding $\bar{\psi}^\delta(x,v,z)$ such that $\lim_{k\to\infty} z_k = 0$ and $\bar{\psi}^\delta(\bar{\pi}^\delta(\xi),0) = c^\delta(\xi)$, with boundedness of the trajectories of the system with state $(z,\ x,\ v)$. Accordingly to Section V-A, one can define sampled-data controller by considering the double-rate sampled-equivalent model. Setting

$$\begin{aligned}\frac{\delta}{2}\bar{\psi}^{1\delta,[2]}(x_{1k},v_k,z_k) &= \frac{\delta}{2}\Gamma_1 z_{1k} + \gamma^\delta(x_{1k+1}) - \\ & \gamma^\delta(x_{1k}) - \frac{\delta}{2}[z_{2k} + \varphi^\delta(x_{1k})] \\ \bar{\psi}^{2\delta,[2]}(x_{1k},v_k,z_k) &= \Gamma_2 z_{2k} + \varphi^\delta(x_{1k+1})\end{aligned} \tag{22}$$

one ensures stability of the closed-loop sampled-data input-delayed dynamics.

D. Simulations

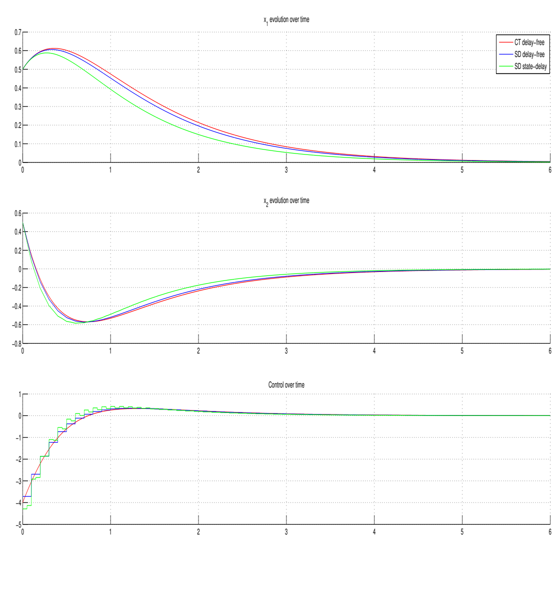

Fig. 1. $\delta = 0.1$ s and $x_0 = (0.5, 0.5)^T$

Simulations are carried out on the example in Section V for different sampling periods δ. The control law is defined according to an I&I double-rate design when $\tau = \delta$ with gains $\Gamma_1,\ \Gamma_2 = 1$. All the simulations are performed for the initial condition $x = \begin{pmatrix}0.5 & 0.5\end{pmatrix}^T$. The control approach presented in this paper is compared with the continuous-time and sampled-data ones (respectively in [1] and in [7]), when the former ones are applied to the delay-free dynamics. In general it can be pointed out that the so-defined feedback leads to good performances even with respect to the delay-free case. This is achieved since the control law is not explicitly designed in order to predict the delayed-state, but to stabilize the dynamics with no information on the delay-free controller. As a matter of fact, the proposed controller directly stabilizes the delayed-dynamics by leading it to the invariant manifold where the implicit prediction aim is fulfilled. Promising performances are obtained when δ increases with still limited control efforts and reasonable smoothness of the trajectories.

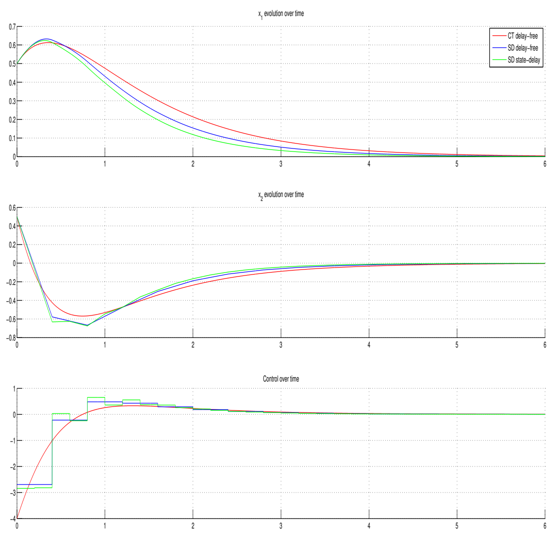

Fig. 2. $\delta = 0.4$ s and $x_0 = (0.5, 0.5)^T$

VI. Conclusions

In this paper a multi-rate sampled-data I&I controller is proposed for a special class of dynamics in which one state is affected by delays. The performances are shown through simulations on an academic example. The proposed approach can be extended to the case of dynamics with delayed interconnection (e.g. through the state component x_2) taking advantage of possible intrinsic properties of the sampled-data equivalent models [18].

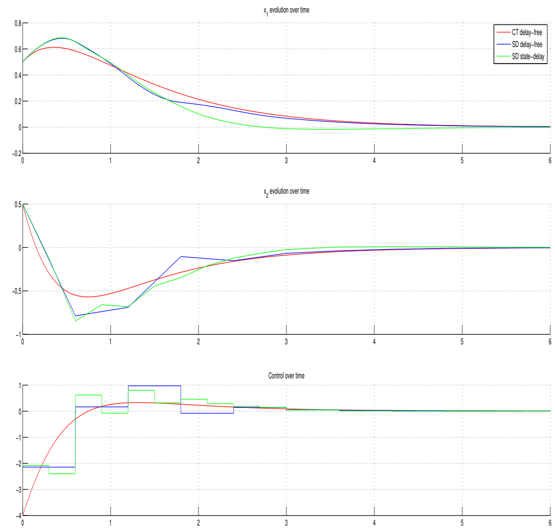

Fig. 3. $\delta = 0.6$ s and $x_0 = (0.5, 0.5)^T$

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