Immersion and Invariance stabilization of strict-feedback dynamics under sampling ⋆

Mattia Mattioni [a,b], Salvatore Monaco [a], Dorothée Normand-Cyrot [b]

[a] *Dipartimento di Ingegneria Informatica, Automatica e Gestionale 'Antonio Ruberti', Università di Roma "La Sapienza", via Ariosto 25, 00185 Roma, Italy.*

[b] *Laboratoire des Signaux et Systèmes, CNRS-CentraleSupelec, 3 rue Joliot Curie, 91190 Gif-sur-Yvette, France*

Abstract

The paper deals with sampled-data stabilization of continuous-time dynamics in strict-feedback form via Immersion and Invariance. Starting from the characterization of the sampled-data target dynamics and its invariant manifold, a multi-rate control law is designed to achieve attractiveness and invariance of such a manifold. Simulations on an academic example and a practical case illustrate the performances.

Key words: Asymptotic stabilization, digital implementation, sampled-data stabilization, nonlinear systems

1 Introduction

Stabilization of continuous-time (CT) strict-feedback dynamics has been widely investigated in the last decades. Several methodologies have been proposed exploiting the connected cascade structure. Among all, backstepping is certainly the most popular one and involves an iterative top-down Lyapunov-based procedure to compute the controller (Kokotović and Arcak [2001]). Strict-feedback structures can be assumed in a purely discrete-time (DT) context as well and similar top-down constructive procedures can be carried out for the design. However, several difficulties arise for the computation of the control solutions as they are only implicitly defined by nonlinear algebraic equations.
This last issue can be overcome in the sampled-data (SD) context where the discrete-time model represents the evolutions, at the sampling times, of the system under the action of piecewise constant control over the sampling intervals (Monaco and Normand-Cyrot [2001]). In that case, the strict-feedback structure is lost but, as it will be clarified in the sequel, the SD equivalent model inherits a nested structure which is useful for the design.
Several contributions discuss backstepping-like methods for SD dynamics (Nešić and Grüne [2005], Burlion et al. [2006], Postoyan et al. [2008]). A SD Lyapunov-based adaptive control strategy was proposed in Postoyan et al. [2008] by exploiting the triangular structure. In a recent work by Tanasa et al. [2016] Input-Lyapunov-Matching (ILM) was employed to design a multi-rate backstepping stabilizing controller.
Immersion and Invariance (I&I) has been introduced in continuous time as an alternative tool for nonlinear stabilization (Astolfi and Ortega [2003], Astolfi et al. [2008]). It relies on the idea of driving the trajectories of a nonlinear system towards the ones of an a-priori defined stable target dynamics while preserving their boundedness. Such an approach qualifies for its robustness with respect to higher order dynamics, applicability to real cases and simplicity, as illustrated in several practical domains (Rabai et al. [2013], Mannarino and Mantegazza [2014]). A first extension to nonlinear discrete-time systems in strict-feedback form was provided by Yalcin and Astolfi [2011].
How to preserve I&I stabilization under digital control remains a challenging problem. In Mattei et al. [2015], assuming part of the continuous-time dynamics stable, the sampled-data controller stretching the dynamics onto the associated continuous-time manifold guaran-

⋆ Supported by a public grant overseen by the French National Research Agency (ANR) as part of the *Investissement d'Avenir* program, through the "iCODE" project. M. Mattioni's mobility from France to Italy is supported by the *Université Franco-Italienne/Italo-Francese.*

Email addresses: mattia.mattioni@l2s.centralesupelec.fr (Mattia Mattioni), salvatore.monaco@uniroma1.it (Salvatore Monaco), cyrot@lss.supelec.fr (Dorothée Normand-Cyrot).

tees its attractiveness and thus I&I stabilization.
In the present paper, we discuss the same problem for systems in strict feedback form. In doing so, one has to keep in mind that the strict-feedback structure is lost under sampling and that the implementation of a piecewise constant controller via emulation of the continuous-time one is not satisfactory. Under the usual assumptions set on strict feedback dynamics, we show that I&I stabilizability under sampling can be preserved by adequately redefining a sampled-data target system and its associated invariant manifold (both parameterized by the sampling period). The stabilizing design is then carried out via multi-rate feedback strategies of order equal to the number of cascade connections. This is first detailed for strict-feedback systems with two-cascade connections while the extension to the case of m cascades is sketched as it follows the same lines. Preliminary results on the one-cascade case are in Mattioni et al. [2015b].
In conclusion, it is shown that the existence of a CT-I&I control for systems in strict-feedback form is sufficient to guarantee the existence of a m-rate SD-I&I feedback. The proof is constructive and the control solution admits an expansion in powers of the sampling period. In practice, only approximate solutions can be computed and implemented so affecting the overall performances.The stability properties of the closed-loop system under approximate controllers are discussed with respect to the length of the sampling period.

The paper is organized as follows. After some recalls and introductory concepts in Section 2, the main results are discussed in Sections 3 and 4. Constructive aspects and extensions are detailed in Section 5. In Section 6, examples and simulations are carried out.

2 Recalls and basic facts

2.1 Assumptions and Notations

Maps and vector fields are assumed smooth (i.e., infinitely differentiable of class C^∞) and forward complete to guarantee the existence of solutions and prevent from finite escape time. The sampling period $\delta \in]0, T^*[$ is assumed regular. $T^* > 0$ denotes the maximum allowable sampling period (MASP, Tanasa et al. [2016]). Given a vector field $f : \mathbb{R}^n \to T_x\mathbb{R}^n$, L_f denotes the associated Lie derivative operator, $L_f = \sum_{i=1}^{n} f_i(\cdot)\frac{\partial}{\partial x_i}$. e^{L_f} denotes the associated Lie series operator, $e^{L_f} = 1+\sum_{i\geq 1}\frac{L_f^i}{i!}$.When no ambiguity is possible, $L_f L_g$ stands for $L_f \circ L_g$. For any real valued function h on $\mathbb{R}^n$, one gets $e^{L_f}h(x) = e^{L_f}h\big|_x = h(e^{L_f}x)$, where $e^{L_f}x$ stands for $e^{L_f}Id\big|_x$ and I_d is the identity function over $\mathbb{R}^n$. The evaluation of a function at time $t = k\delta$ ($\big|_{t=k\delta}$) is omitted when it is clear from the context. The subscript $\big|_k$ is omitted as well when no confusion arises.
A function $R(x, \delta)$ is said to be of order δ^p, $p \geq 1$ ($R(x, \delta) = O(\delta^p)$) if whenever R is defined it can be written as $R(x, \delta) = \delta^{p-1}\tilde{R}(x, \delta)$ and $\exists\theta \in \mathcal{K}_\infty$ and $\delta^* \geq 0$, such that for each $\delta \leq \delta^*$, $\tilde{R}(x, \delta) \leq \theta(\delta)$.

2.2 Problem statement

In this paper we consider strict-feedback continuous-time dynamics (Khalil [2002]) in the general m-cascade form

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_{21}\\
\dot{x}_{2j} &= f_{2j}(x_1, x_{21}, \ldots, x_{2j}) + g_{2j}(x_1, x_{21}, \ldots, x_{2j})x_{2j+1}\\
\dot{x}_3 &= u
\end{aligned} \tag{1}$$

where $x_1 \in \mathbb{R}^p$, $x_2 = (x_{21}, \ldots, x_{2m-1})^\top \in \mathbb{R}^{m-1}$, $x_3 = x_{2m}$, $x_{2j}, u \in \mathbb{R}$ for $j = 1, \ldots, m-1$. We assume that $g_{2j}(\cdot) \neq 0$ (globally) and that the origin is the unique equilibrium of (1). From now on, the stabilizability of the x_1-dynamics via fictitious feedback $x_{21} = \gamma(x_1)$ is assumed.

Assumption 2.1 *There exist functions $\gamma(\cdot) : \mathbb{R}^p \to \mathbb{R}$ with $\gamma(0) = 0$ and proper*[1] *$W(\cdot) : \mathbb{R}^p \to \mathbb{R}^+$, such that $(L_{f_1} + \gamma L_{g_1})W(x_1) < 0$ for all $x_1 \in \mathbb{R}^p/\{0\}$.*

Accordingly, *I&I stablizability* of (1) can be proven in the sense of Definition 1 in Astolfi and Ortega [2003]. With reference to standard arguments, one defines the target dynamics as $\dot{\xi}(t) = f_1(\xi) + g_1(\xi)\gamma(\xi)$ and the immersion mapping as $\pi(\xi) = col(\xi,\ \gamma_1(\xi), \cdots,\ \gamma_m(\xi))$ with, for $i = 1, ..., m-1$

$$\begin{aligned}
&\gamma_1(\xi) = \tilde{\gamma}_1(\xi) = \gamma(\xi) \qquad (2)\\
&\gamma_{i+1}(\xi) = \tilde{\gamma}_{i+1}(\xi, \gamma_1(\xi), \ldots, \gamma_i(\xi)) =\\
&g_{2i}^{-1}(\xi, \gamma_1(\xi), \ldots, \gamma_i(\xi))(\dot{\gamma}_i(\xi) - f_{2i}(\xi, \gamma_1(\xi), \ldots, \gamma_i(\xi))).
\end{aligned}$$

According to (2), $\dot{\gamma}_m(\xi) = c(\xi)$ defines the control constraining the state evolutions of (1) over the target. Setting

$$\begin{aligned}
z_1 &= \phi_1(x_1, x_{21}) = x_{21} - \tilde{\gamma}_1(x_1) \qquad (3)\\
z_j &= \phi_j(x_1, x_{21}, \ldots, x_{2j}) = x_{2j} - \tilde{\gamma}_j(x_1, x_{21}, \ldots, x_{2j})\\
z_m &= \phi_j(x_1, x_2, x_3) = x_3 - \tilde{\gamma}_m(x_1, x_2)
\end{aligned}$$

for $j = 2, \ldots m-1$, GAS of the equilibrium of (1) is achieved under the feedback

$$u_c = \psi(x, z) = -K(x)z + \dot{\tilde{\gamma}}_m(x), \quad K(x) > 0 \tag{4}$$

which guarantees manifold attractivity and trajectory boundedness of the extended dynamics over $\mathbb{R}^{p+2m}$

[1] $W : \mathbb{R}^n \to \mathbb{R}$ is *proper* if $\forall\ r > 0$, $W^{-1}([0, r]) = \{x \in \mathbb{R}^n\ \ W(x) \geq r\}$ is compact.

$$
\begin{aligned}
&\dot z_j = g_{2j}(x_1, z_1+\tilde\gamma_1(x_1),\dots,z_j+\tilde\gamma_j(x_1,x_{21},\dots,x_{2j}))z_{j+1}\\
&\dot z_m = u-\dot{\tilde\gamma}_m(x)\\
&\dot x_1 = f_1(x_1)+g_1(x_1)(z_1+\tilde\gamma_1(x_1))\\
&\dot x_{2j} = g_{2j}(x_1,x_{21},\dots,x_{2j})z_{j+1}+\dot{\tilde\gamma}_j(x_1,x_{21},\dots,x_{2j})\\
&\dot x_3 = u
\end{aligned}
\tag{5}
$$

for $j = 1,\dots,m-1$.

Remark 2.1 *Mappings $\tilde\gamma_i : \mathbb{R}^p \times \mathbb{R}^{i-1} \to \mathbb{R}$ are instrumental to define the off-manifold components z. In the sequel, with a slight abuse of notation, we will use γ_i instead of $\tilde\gamma_i$ when no confusion arises.*

In the following, we discuss the problem of preserving I&I stabilizability when the control variable $u(t)$ is piecewise constant; i.e. $u(t) = u_k$ for $t \in [k\delta,(k+1)\delta[,\ k \geq 0$. For this purpose, it is instrumental to redefine I&I stabilizability for nonlinear DT systems (Monaco and Normand-Cyrot [2015]) of the form

$$x_{k+1} = F(x_k,u_k) \tag{6}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}$ and x^* is the equilibrium.

Definition 2.1 *The DT dynamics (6) is said to be I&I stabilizable if there exist mappings*

$$
\begin{aligned}
&\alpha(\cdot):\mathbb{R}^p\to\mathbb{R}^p;\quad \pi(\cdot):\mathbb{R}^p\to\mathbb{R}^n;\quad c(\cdot):\mathbb{R}^p\to\mathbb{R}\\
&\phi(\cdot):\mathbb{R}^n\to\mathbb{R}^{n-p};\ \psi(\cdot,\cdot):\mathbb{R}^{n\times(n-p)}\to\mathbb{R}
\end{aligned}
$$

such that the following conditions hold:

H1. (Target dynamics) *The dynamics*

$$\xi_{k+1} = \alpha(\xi_k)$$

with state $\xi \in \mathbb{R}^p$ $(p < n)$ has a globally asymptotically stable equilibrium at $\xi^* \in \mathbb{R}^p$ and $x^* = \pi(\xi^*)$;

H2. (Immersion condition) *For all $\xi \in \mathbb{R}^p$*

$$F(\pi(\xi_k),c(\xi_k)) = \pi(\alpha(\xi_k));$$

H3. (Implicit manifold) *The following identity among sets holds*

$$\{x\in\mathbb{R}^n|\phi(x)=0\} = \{x\in\mathbb{R}^n|x=\pi(\xi)\ for\ \xi\in\mathbb{R}^p\};$$

H4. (Manifold attractivity and trajectory boundedness) *All trajectories of the extended dynamics*

$$
\begin{aligned}
z_{k+1} &= \phi(F(x_k,\psi(x_k,z_k)))\\
x_{k+1} &= F(x_k,\psi(x_k,z_k))
\end{aligned}
$$

with $z_0 = \phi(x_0)$, are bounded for all $k \geq 0$ while $\lim_{k\to\infty} z_k = 0$ *and $\psi(\pi(\xi),0) = c(\xi)$.*

I&I stabilizability implies that x^* is a globally asymptotically stable equilibrium of the closed-loop dynamics $x_{k+1} = F(x_k,\psi(x_k,\phi(x_k)))$.

3 SD I&I stabilization: the 2-cascade case

Consider the two-cascade connected dynamics

$$
\begin{aligned}
\dot x_1 &= f_1(x_1)+g_1(x_1)x_2\\
\dot x_2 &= f_2(x_1,x_2)+g_2(x_1,x_2)x_3\\
\dot x_3 &= u
\end{aligned}
\tag{7}
$$

and set, for the sake of brevity, $x = (x_1^\top, x_2, x_3)^\top$ and

$$
\begin{aligned}
\bar f(x) &= \begin{pmatrix} f_1(x_1)+g_1(x_1)x_2\\ f_2(x_1,x_2)\end{pmatrix},\ \bar g(x) = \begin{pmatrix}0\\ g_2(x_1,x_2)\end{pmatrix}\\
\tilde f(x) &= \begin{pmatrix}\bar f(x)+\bar g(x)x_3\\ 0\end{pmatrix},\ \tilde g(x) = \begin{pmatrix}\mathbf{0}_{2\times1}\\ 1\end{pmatrix}.
\end{aligned}
\tag{8}
$$

Assuming $u(t)$ constant over time intervals of length δ, the *equivalent SD single-rate model* (Monaco and Normand-Cyrot [1997]) is described by the map

$$x_{k+1} = F^\delta(x_k,u_k) = e^{\delta(L_{\tilde f}+u_kL_{\tilde g})}x\big|_{x_k}.$$

Through easy computations, one gets

$$
\begin{aligned}
x_{1k+1} &= F_{11}^\delta(x_1,x_2)+\frac{\delta^2}{2!}x_3F_{21}^\delta(x_1,x_2,x_3)\\
&\quad+\frac{\delta^3}{3!}uG_1^\delta(x_1,x_2,x_3,u)\\
x_{2k+1} &= F_2^\delta(x_1,x_2,x_3)+\frac{\delta^2}{2!}uG_2^\delta(x_1,x_2,x_3,u)\\
x_{3k+1} &= x_3+\delta u
\end{aligned}
\tag{9}
$$

with

$$
\begin{aligned}
G_1^\delta &= g_1(x_1)g_2(x_1,x_2)+3!\sum_{j\geq1}\delta^jG_{1j}(x_1,x_2,x_3,u)\\
G_2^\delta &= g_2(x_1,x_2)+2!\sum_{j\geq1}\delta^jG_{2j}(x_1,x_2,x_3,u)\\
F_{11}^\delta &+ \frac{\delta^2}{2!}x_3F_{21}^\delta = x_1+\delta(f_1+x_2g_1)\\
&+\sum_{j\geq1}\frac{\delta^{j+1}}{(j+1)!}(L_{\bar f}+x_3L_{\bar g})^j(f_1+x_2g_1)\\
F_2^\delta &= x_2+\delta(f_2+x_3g_2)+\sum_{j\geq1}\frac{\delta^{j+1}}{(j+1)!}(L_{\bar f}+x_3L_{\bar g})^j(f_2+x_3g_2)\\
G_{1j} &= (L_{\bar f}+x_3L_{\bar g})G_{1j-1}\\
&+u\frac{\partial}{\partial x_3}(G_{1j-1}+L_{\bar f}+x_3L_{\bar g})^{j+2}(f_1+x_2g_1))\\
G_{2j} &= (L_{\bar f}+x_3L_{\bar g})G_{2j-1}\\
&+u\frac{\partial}{\partial x_3}(G_{2j-1}+(L_{\bar f}+x_3L_{\bar g})^{j+1}(f_2+x_3g_2)).
\end{aligned}
$$

Analogously, setting $\bar{\delta} = \frac{\delta}{2}$ and $u_k^i = u(t)$ for $t \in [k\delta + (i-1)\bar{\delta}, k\delta + i\bar{\delta}[$ $(i = 1, 2)$, the *equivalent SD two-rate model* is described by the composition

$$
\begin{aligned}
x_{k+1} =& F^{2\bar{\delta}}(x_k, u_k^1, u_k^2)\\
=& e^{\bar{\delta}(L_{\bar{f}} + u_k^1 L_{\bar{g}})} \circ e^{\bar{\delta}(L_{\bar{f}} + u_k^2 L_{\bar{g}})} x\big|_{x_k}
\end{aligned}
$$

and more explicitly as

$$
\begin{aligned}
x_{1k+1} =& F_{11}^{2\bar{\delta}}(x_1, x_2) + \frac{\bar{\delta}^2}{2!} x_3 F_{21}^{2\bar{\delta}}(x_1, x_2, x_3)\\
&+ \frac{\bar{\delta}^3}{3!}(10u^1 + u^2)\bar{G}_1^{\bar{\delta}}(x_1, x_2, x_3, u^1, u^2) \qquad (10)\\
x_{2k+1} =& F_2^{2\bar{\delta}}(x_1, x_2, x_3)\\
&+ \frac{\bar{\delta}^2}{2!}(3u^1 + u^2)\bar{G}_2^{\bar{\delta}}(x_1, x_2, x_3, u^1, u^2)\\
x_{3k+1} =& x_3 + \bar{\delta}(u^1 + u^2).
\end{aligned}
$$

Setting $u^1 = u^2 = u$ in (10), one recovers (9).
Although the strict-feedback structure is not preserved under sampling, (9) (resp. (10)) inherits from it some nice and useful properties. Indeed, x_{1k+1} depends on x_{2k} in $O(\delta^2)$ but on x_{3k} in $O(\delta^3)$ and on u_k in $O(\delta^4)$; x_{2k+1} depends on x_{3k} in $O(\delta^2)$ and on u_k in $O(\delta^3)$; x_{3k+1} depends on u_k in $O(\delta^2)$; hence, the strict-feedback structure turns into a scaling, with respect to the powers of δ, of the influence of each successive state and control variables.

Remark 3.1 *The nonlinear system (9) (resp. (10)) characterizes the exact sampled-data equivalent model of (7) when no approximation over δ is performed. It would be interesting to investigate on the preservation of the nested structure up to a certain order of approximation or making use of the δ-operator (Yuz and Goodwin [2005], Yuz and Goodwin [2014]) for characterizing the sampled equivalent models.*

3.1 On the choice of the SD target dynamics

The following result is instrumental to define the SD target dynamics and immersion mapping.

Proposition 3.1 *Consider the strict-feedback dynamics (7) under Assumption 2.1 and let (9) be its equivalent SD dynamics. Suppose $L_{g_1}W(x_1) \neq 0$ for all $x_1 \in \mathbb{R}^p - \{0\}$. Then, there exists $T^* > 0$ such that for each $\delta \in]0, T^*[$ and initial condition $\xi\big|_{t=k\delta} = \xi$, $k \geq 0$, the equalities*

$$
W(\xi_{k+1}) - W(\xi) = \int_{k\delta}^{(k+1)\delta} \frac{\partial W}{\partial \xi}(f_1 + \gamma_1 g_1)(\xi(\tau))d\tau \qquad (11)
$$

$$
v_1(\xi_{k+1}) = F_2^\delta(\xi, v_1, v_2) + \frac{\delta^2}{2!} v_3 G_2^\delta(\xi, v_1, v_2, v_3) \qquad (12)
$$

$$
v_2(\xi_{k+1}) = v_2 + \delta v_3 \qquad (13)
$$

with

$$
\xi_{k+1} = F_1^\delta(\xi, v_1, v_2) + \frac{\delta^3}{3!} v_3 G_1^\delta(\xi, v_1, v_2, v_3) \qquad (14)
$$

admit unique solutions $\gamma_1^\delta(\xi)$, $\gamma_2^\delta(\xi)$, $c^\delta(\xi) : \mathbb{R}^p \to \mathbb{R}$ in the form of asymptotic series expansions in δ; i.e. setting $v_i = \gamma_i^\delta(\xi)$ $(i = 1, 2)$ and $v_3 = c^\delta(\xi)$ with

$$
\begin{aligned}
\gamma_i^\delta(\xi) &= \gamma_{i0}(\xi) + \sum_{j\geq 1} \frac{\delta^j}{(j+1)!}\gamma_{ij}(\xi)\\
c^\delta(\xi) &= c_0(\xi) + \sum_{j\geq 1} \frac{\delta^j}{(j+1)!} c_j(\xi).
\end{aligned}
$$

Proof. The proof is reported in Appendix A. ◁

For the first terms, one gets

$$
\begin{aligned}
\gamma_i^\delta(\xi) =& \gamma_{i0}(\xi) + \frac{\delta}{2}\gamma_{i1}(\xi) + \frac{\delta^2}{3!}\gamma_{i2}(\xi) + O(\delta^3)\\
c^\delta(\xi) =& c_0(\xi) + \frac{\delta}{2}c_1(\xi) + \frac{\delta^2}{3!}c_2(\xi) + O(\delta^3)
\end{aligned}
$$

where $i = 1, 2$, $\gamma_{10}(\xi)$, $\gamma_{20}(\xi)$ and $c_0(\xi)$ coincide with the CT solutions (2) and

$$
\begin{aligned}
\gamma_{11}(\xi) =& 0, \qquad \gamma_{21}(\xi) = 0 && (15a)\\
c_1(\xi) =& (L_{f_1} + \gamma_{10}L_{g_1})^2\gamma_{10} + \dot{\gamma}_1 L_{g_1}\gamma_{20} && (15b)\\
\gamma_{12}(\xi) =& \ddot{\gamma}_1(\xi) = (L_{\bar{f}} + \gamma_{20}L_{\bar{g}})^2\gamma_{10} && (15c)\\
\gamma_{22}(\xi) =& \frac{1}{g_2}\Big(6\dddot{\gamma}_1 + 2\dot{\gamma}_1 L_{f_1}L_{g_1}\gamma_{10}\\
&+ 3\gamma_1\dot{\gamma}_1 L_{g_1}^2\gamma_{10} + \dot{\gamma}_1 L_{g_1}L_{f_1}\gamma_{20} - \frac{3}{2}g_2 c_1\\
&- (L_{\bar{f}} + \gamma_{20}L_{\bar{g}})^2(f_2 + \gamma_{20}g_2)\\
&- c_0(L_{\bar{f}} + \gamma_{20}L_{\bar{g}})g_2\Big) && (15d)\\
c_2(\xi) =& (L_{\bar{f}} + \gamma_{20}L_{\bar{g}})^3\gamma_{20} + 2\dot{\gamma}_1 L_{f_1}L_{g_1}\gamma_{20}\\
&+ 3\gamma_1\dot{\gamma}_1 L_{g_1}^2\gamma_{20} + \dot{\gamma}_1 L_{g_1}L_{f_1}\gamma_{20}. && (15e)
\end{aligned}
$$

According to the arguments reported in Appendix A, the existence of a CT solution $(\gamma_1(\cdot), \gamma_2(\cdot), c(\cdot))$ is sufficient to guarantee the existence of the sampled-data one.

As a result of Proposition 3.1, the three lemmas below characterize respectively the sampled-data target dynamics, immersion mapping and invariant manifold.

Lemma 3.1 *Rewrite (14) by setting* $(v_1, v_2, v_3) = (\gamma_1^\delta(\xi), \gamma_2^\delta(\xi), c^\delta(\xi))$*, solutions of (11), (12), (13); i.e.*

$$\begin{aligned}\xi_{k+1} =&\alpha^\delta(\xi_k) = F_1^\delta(\xi, \gamma_1^\delta(\xi), \gamma_2^\delta(\xi)) + \\ &\frac{\delta^3}{3!} c^\delta(\xi) G_1^\delta(\xi, \gamma_1^\delta(\xi), \gamma_2^\delta(\xi), c^\delta(\xi)).\end{aligned} \tag{16}$$

*Then, (16) defines the target dynamics associated to (9). Accordingly, condition **H1** in Definition 2.1 is verified.*

Proof. Equality (11) guarantees ILM (Tanasa et al. [2016]) of the control Lyapunov function $W(\cdot)$ at $t = k\delta$ ($k \geq 0$). By assumption, $W(\cdot)$ verifies $(L_{f_1} + \gamma L_{g_1})W(\xi) < 0$, along the continuous-time trajectories. Because of matching, SD-GAS of the origin of (16) immediately follows and, thus, condition **H1**. ◃

Lemma 3.2 *The mapping* $\pi^\delta(\xi) = col(\xi,\ \gamma_1^\delta(\xi), \gamma_2^\delta(\xi))$ *satisfies the immersion condition **H2** in Definition 2.1 whenever* $\gamma_1^\delta(\cdot)$ *and* $\gamma_2^\delta(\cdot)$ *are the solutions of (11) and (12).*

Lemma 3.3 *Let Lemmas 3.1 and 3.2 hold, then* $z = \phi^\delta(x) = 0$ *with*[2]

$$\begin{aligned} z_1 = \phi_1^\delta(x_1, x_2) =& x_2 - \gamma_1^\delta(x_1) \\ z_2 = \phi_2^\delta(x_1, x_2, x_3) =& x_3 - \tilde{\gamma}_2^\delta(x_1, x_2)\end{aligned}$$

and $z_0 = \phi^\delta(x_0)$ *implicitly defines the* p*-dimensional manifold satisfying condition **H3** in Definition 2.1.*

4 Main result and I&I stabilizing feedback

It is now possible to set the main result.

Theorem 4.1 *Consider (1) with* $m = 2$ *under Assumption 2.1. Then, there exists* $T^* > 0$ *such that for each* $\delta \in]0, T^*[$*, any control law* $u = \psi^\delta(x, z)$ *designed to bring* z *to* 0 *with boundedness of the trajectories of the extended dynamics over* $\mathbb{R}^{p+4}$

$$\begin{aligned} x_{k+1} =& F^\delta(x, u) \\ z_{1k+1} =& F_2^\delta(x_1, z_1 + \gamma_1^\delta(x_1), z_2 + \gamma_2^\delta(x_1, z_1 + \gamma_1^\delta(x_1))) + \\ &\frac{\delta^2}{2!} u G_2^\delta(x_1, z_1 + \gamma_1^\delta(x_1), z_2 + \gamma_2^\delta(x_1, z_1 + \\ &\gamma_1^\delta(x_1)), u) - \gamma_1^\delta(x_{1k+1}) + \gamma_1^\delta(x_1) \\ z_{2k+1} =& z_2 + \delta u - \gamma_2^\delta(x_{1k+1}, z_{1k+1} + \gamma_1^\delta(x_{1k+1})) + \\ &\gamma_2^\delta(x_1, z_1 + \gamma_1^\delta(x_1))\end{aligned} \tag{17}$$

achieves GAS of the origin of the closed-loop dynamics $x_{k+1} = F^\delta(x_k, \psi^\delta(x_k, \phi^\delta(x_k)))$*. Equivalently, the SD dynamics (9) is* I&I stabilizable *with SD target (16).*

Proof. One has to prove that Definition 2.1 holds true for the dynamics (9). By construction, **H1**, **H2** are satisfied when defining the SD target dynamics and immersion mapping according to Lemmas 3.1-3.2. Furthermore, **H3** is verified by setting $z = \phi^\delta(x)$ as in Lemma 3.3. It follows that any SD feedback $u = \psi^\delta(x, z)$ designed to bring z to 0 with boundedness of trajectories of the extended dynamics (17) makes the equilibrium of (9) (and so the one of (1)) GAS. ◃

4.1 On the design of the I&I SD control law

Theorem 4.1 states the existence of a SD controller $u_k = \psi^\delta(x_k, z_k)$ stabilizing the equilibrium of (7). Some constructive aspects are discussed below.

We first note that, setting the output of (7) as $y = \phi_1(x)$, the corresponding system has relative degree $r = 2$ (the dimension of the cascade) with GAS p-dimensional zero-dynamics coinciding with the target. Following the idea in Monaco and Normand-Cyrot [2007] of preserving stability of the zero-dynamics via multi-rate feedback, a double-rate ($2R$) control will be employed in the present context.

Thus, in the sequel we will consider the double-rate equivalent model (10) with $u(t) = u_k^i$ for $t \in [k\delta + (i-1)\bar{\delta}, k\delta + i\bar{\delta}[$ ($i = 1, 2$). Since $\bar{\delta} < \delta < T^*$, Proposition 3.1 still holds and performing computations, the following extended dynamics over $\mathbb{R}^{p+4}$ is obtained

$$\begin{aligned} x_{k+1} =& F^{2\bar{\delta}}(x, u^1, u^2) \\ z_{1k+1} =& F_2^{\bar{\delta}}(x_1, z_1 + \gamma_1^{\bar{\delta}}(x_1), z_2 + \gamma_2^{\bar{\delta}}(x_1, z_1 + \gamma_1^{\bar{\delta}}(x_1))) + \\ &\frac{\bar{\delta}^2}{2!}(3u^1 + u^2)\bar{G}_2^{\bar{\delta}}(x_1, z_1 + \gamma_1^{\bar{\delta}}(x_1), z_2 + \\ &\gamma_2^{\bar{\delta}}(x_1, z_1 + \gamma_1^{\bar{\delta}}(x_1)), u^1, u^2) - \gamma_1^{\bar{\delta}}(x_{1k+1}) + \gamma_1^{\bar{\delta}}(x_1) \\ z_{2k+1} =& z_2 + \bar{\delta}(u^1 + u^2) - \gamma_2^{\bar{\delta}}(x_{1k+1}, z_{1k+1} + \gamma_1^{\bar{\delta}}(x_{1k+1})) \\ &+ \gamma_2^{\bar{\delta}}(x_1, z_1 + \gamma_1^{\bar{\delta}}(x_1)).\end{aligned} \tag{18}$$

The result is now obtained by reproducing the evolution of the continuous-time off-manifold component, at any sampling instant $t = k\delta$. It will be referred to as *Input-to-Partial State Matching* (I-PSM, Monaco and Normand-Cyrot [2007]). For this purpose, we denote by $z(t) = \phi(x(t))$ and $z_k = \phi^{\bar{\delta}}(x_k)$, respectively, the CT and SD off-manifold trajectories. The problem consists in defining a $2R$ controller $u_{sd} = col(u^1, u^2)$ verifying

$$\phi^{\bar{\delta}}(F^{2\bar{\delta}}(x_k, u^1, u^2))\big|_{\bar{\delta}=\frac{\delta}{2}} = e^{\delta(L_{\tilde{f}} + u_c L_{\tilde{g}})}\phi(x)\big|_{x_k} \tag{19}$$

for any $k \geq 0$ and with $\tilde{f}$ and $\tilde{g}$ in (8).

The left-hand side of (19) represents the SD evolution of the z-coordinate in (18). The right-hand side can be

[2] $\tilde{\gamma}_2^\delta$ in defined in a similar way as in Section 2.2

obtained by specifying (5) for $m = 2$ and $u = u_c$ as in (4). The control u_{sd} comes to be the implicit solution of (19) where each u^i is in the form of a series expansion in $\bar{\delta}$; i.e.

$$u^i = \sum_{j\geq 0} \frac{\bar{\delta}^j}{(j+1)!} u^i_j \qquad i = 1, 2. \tag{20}$$

The next result states the existence of the feedback $u_{sd} = \psi^{\bar{\delta}}(x_k)$ in the form (20) and solution to (19).

Proposition 4.1 *Consider (1) and its equivalent single-rate dynamics (9) under the hypotheses of Theorem 4.1. For $\delta = 2\bar{\delta} \in]0, T^*[$, the formal series equality (19) admits a unique solution $u_{sd} = (u^1, u^2)$ in the form of (20) so ensuring I&I stabilization of the equilibrium.*

Proof. The proof is reported in Appendix B. $\triangleleft$

We note that the first term of the series expansion (20) is the CT controller $u_c(x)$ computed at $t = k\delta$ (emulated control). As already anticipated, the additional terms can be computed by an executable algorithm (see Tanasa et al. [2016] for details). As an example, one computes

$$\begin{aligned} u^1 =& u_c(x) + \frac{\bar{\delta}}{3}\dot{u}_c(x) + \frac{\bar{\delta}}{6g_2(x_1,x_2)} \times \\ & \Big(\gamma_{12} L_{g_1}\gamma_{10}(x_1) + 2z_2 \frac{\partial g_2}{\partial x_2}\Big)\big|_{x_2 = z_2 + \gamma_{10}} + O(\bar{\delta}^2) \\ u^2 =& u_c(x) + \frac{5\bar{\delta}}{3}\dot{u}_c(x) + O(\bar{\delta}^2). \end{aligned} \tag{21}$$

According to the above result, boundedness and convergence of the SD trajectories to the stable manifold are ensured via matching of the continuous-time ones. Nevertheless, matching is not required over the sampled-data manifold since, by construction, the evolutions over this surface are described by (16).

4.2 On approximate controllers

As shown in the previous part, the stabilizing sampled-data I&I controller is the implicit solution of a formal series equality. Accordingly, it is described by an asymptotic series expansion around u^i_0 ($i = 1, 2$) in the form of (20). GAS of the closed-loop equilibrium under this control implies the existence of a $\mathcal{KL}$ function β such that for each $k \geq 0$ and any initial condition x_0

$$|x_k| \leq \beta(|x_0|, k). \tag{22}$$

Nevertheless, implementation issues arise when considering that only approximations of the controller can be computed. To this end, define the *q-th order approximate controller* as the truncation of the series (20) at the q^{th} order of $\bar{\delta}$; namely,

$$u^{i[q]} = \sum_{j=0}^{q} \frac{\bar{\delta}^j}{(j+1)!} u^i_j \qquad i = 1, 2. \tag{23}$$

The stability property of the closed-loop system under such a controller is stated below.

Proposition 4.2 *Consider (10) with stabilizing feedback (20), then the approximated controller (23) of order q makes the equilibrium practically globally asymptotically stable in $\Theta(\bar{\delta}) = \{O(\bar{\delta}^{q+2}) \, : \, \bar{\delta} \in]0, T^*[\}$.*

Proof. Denote by x_{k+1} and $x^{[q]}_{k+1}$ the states of (10) under, respectively, the exact and approximate controllers from the same initial condition at $t = k\delta$. Then, at each instant $t = (k+1)\delta$, they coincide up to an error in $O(\bar{\delta}^{q+2})$. In virtue of (22) we can write, for all $k \geq 0$

$$\begin{aligned} |x^{[q]}_{k+1}| &\leq |x_{k+1}| + |x_{k+1} - x^{[q]}_{k+1}| \\ &\leq \beta(|x_{k+1}|, k) + \bar{\delta}^{q+1} R(\bar{\delta}, x_k) \end{aligned}$$

where R is a $\mathcal{K}_\infty$ function defined as the sum of the norms of the remaining terms of the dynamics (10). One concludes that the trajectories of the system converge to $B_{\bar{\delta}^{q+2}}(0)$, a neighborhood of the origin of radius $\bar{\delta}^{q+2}$. $\triangleleft$

5 Some extensions

5.1 The case of higher order cascades

In the case $m > 2$, all the properties we discussed still hold. The following result extends Theorem 4.1.

Theorem 5.1 *Consider the m-cascade connected dynamics (1) under Assumption 2.1, then I&I stabilizability is preserved under multi-rate feedback of order m.*

In order to prove the result, we have to guarantee the existence of functions $(\gamma^\delta_1(\cdot), \ldots, \gamma^\delta_m(\cdot), c^\delta(\cdot))$ satisfying **H1**, **H2** and **H3** of Definition 2.1. Then, the existence of a mR control $u_{sd} = (u^1, \ldots, u^m)$ satisfying **H4** has to be proven as well.

For this purpose, let (24) be the single-rate SD equivalent model associated to (1). Accordingly, one sets the target dynamics as

$$\xi_{k+1} = F^\delta_1(\xi, \gamma^\delta_1(\xi), \ldots, \gamma^\delta_m(\xi), c^\delta(\xi)) = \alpha^\delta(\xi)$$

where $(\gamma^\delta_1(\cdot), \cdots, \gamma^\delta_m(\cdot), c^\delta(\cdot))$ is the unique solution of the equalities

$$\begin{aligned} W(\alpha^\delta(\xi)) - W(\xi) =& \int_{k\delta}^{(k+1)\delta} \frac{\partial W}{\partial \xi}(f_1 + \gamma_1 g_1)(\xi(t))dt \\ \gamma^\delta_j(\alpha^\delta(\xi)) =& F^\delta_{2j}(\xi, \gamma^\delta_1(\xi), \ldots, \gamma^\delta_m(\xi), c^\delta(\xi)) \\ \gamma^\delta_m(\alpha^\delta(\xi)) =& \gamma^\delta_m(\xi) + \delta c^\delta(\xi) \end{aligned} \tag{25}$$

with $j = 1, \ldots, m-1$. Such a choice guarantees stability and invariance of the SD target. Analogously

$$
\begin{aligned}
x_{1k+1} =& F_1^\delta(x,u) = F_{11}^\delta(x_1,x_{21}) + \sum_{i>1}^{m-1} \frac{\delta^i}{i!} x_{2i} F_{1i}^\delta(x_1,x_{21},\dots,x_{2i}) + \frac{\delta^m}{m!} x_3 F_{1,m-1}^\delta(x) + \frac{\delta^{m+1}}{(m+1)!} u\, G_1^\delta(x,u) \\
x_{2,jk+1} =& F_{2j}^\delta(x,u) = F_{jj}^\delta(x_1,x_{21},\dots,x_{2j}) + \sum_{i>j}^{m-j-1} \frac{\delta^{i-j+1}}{(i-j+1)!} x_{2i} F_{ji}^\delta(x_1,x_{2,1},\dots,x_{2,i}) \\
&+ \frac{\delta^{m-j}}{(m-j)!} x_3 F_{j,m-1}^\delta(x) + \frac{\delta^{m-j+1}}{(m-j+1)!} u\, G_{2j}^\delta(x,u) \\
x_{3k+1} =& x_3 + \delta u.
\end{aligned}
\tag{24}
$$

to the case of $m = 2$, existence and uniqueness of $(\gamma_1^\delta(\cdot), \cdots, \gamma_m^\delta(\cdot), c^\delta(\cdot))$ are guaranteed by Assumption 2.1 plus the condition $L_{g_1} W \neq 0$ for all $\xi \neq 0$. Those solutions are in the form of series expansions around the CT ones.
Setting $z = \phi^\delta(x)$ as

$$
\begin{aligned}
z_1 = x_{21} - \tilde{\gamma}_1^\delta(x_1); \quad z_2 = x_{22} - \tilde{\gamma}_2^\delta(x_1,x_{21}); \\
\cdots = \dots, \quad z_m = x_3 - \tilde{\gamma}_m^\delta(x_1,x_2)
\end{aligned}
$$

the extended SD dynamics in (x,z) over R^{p+2m} can be written in a similar way as before. A multi-rate strategy of order m (mR) is computed to ensure I&I stabilization via an I-PSM control $u_{sd} = \psi^{\bar{\delta}}(x,z)$ ($\delta = m\bar{\delta}$). The existence of the control solution is ensured by the nonsingularity of the Jacobian of $S(x,z,u_{sd},\bar{\delta})$ (defined as in the proof of Theorem 4.1) with respect to u_{sd} computed for $\bar{\delta} \to 0$. As a matter of fat, it takes the form of a nonsingular matrix $K\bar{S}(x,z,0)$ with $K_{ji} = \sum_i^{m-j+1} \frac{1}{k!}$ and

$$
\bar{S}(x,z,0) \text{ s.t. } \begin{cases} \bar{S}_{11} = g_1(x_1) \prod_{n=1}^{m-1} g_{2n}(x_1,\dots,x_{2n}) & \\ \bar{S}_{ij} = \prod_{j=i-1}^{m-1} g_{2j}(x_1,\dots,x_{2j}), & \text{if } i = j \\ \bar{S}_{ij} = 0, & \text{if } j \neq i. \end{cases}
$$

5.2 A nonlinear dynamics on the last component

Up to now we considered a strict-feedback dynamics in which the last component of the cascade is an integrator. Now, we assume a more general strict-feedback dynamics (with $m = 2$) over $\mathbb{R}^{p+2}$

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_2(x_1,x_2) + g_2(x_1,x_2)x_3 \\
\dot{x}_3 &= a(x_1,x_2,x_3) + b(x_1,x_2,x_3)v
\end{aligned}
$$

with $b \neq 0$ $\forall x \in \mathbb{R}^{p+2}$. Suppose as usual that v is a piecewise constant signal over time intervals of length δ. The problem is still solvable following the same lines as before with some extra calculus. In order to recover the situation we dealt with, it is possible to first solve the problem for $\dot{x}_3 = u_k$ (with piecewise constant u_k) and then look for a preliminary piecewise-constant feedback v_k such that, for each $k > 0$ and u_k, the following holds

$$
\delta u_k = \int_{k\delta}^{(k+1)\delta} \big[a(x(\tau)) + b(x(\tau))v_k\big] d\tau.
$$

More in particular, this gives

$$
\delta v_k = \Big[\int_{k\delta}^{(k+1)\delta} b(x(\tau)) d\tau \Big]^{-1} \int_{k\delta}^{(k+1)\delta} \big[u_k - a(x(\tau))\big] d\tau.
$$

The existence of $\big[\int_{k\delta}^{(k+1)\delta} b(x(\tau)) d\tau \big]^{-1}$ over the integration interval is ensured by the invertibility of $b(\cdot)$. It is easy to show that the resulting control v_k yields I&I stabilization of the equilibrium of the above system in the sense of Definition 2.1.

6 Examples and simulations

6.1 An academic example

Consider the system over $\mathbb{R}^3$

$$
\dot{x}_1 = x_1^2 + x_2; \quad \dot{x}_2 = x_3; \quad \dot{x}_3 = u. \tag{26}
$$

Setting $x_2 = \gamma(x_1) = -x_1 - x_1^2$ with $W(x_1) = \frac{1}{2}x_1^2$, one easily verifies that the x_1-system has a GAS equilibrium. Thus, one defines the scalar target dynamics as $\dot{\xi} = -\xi$. By construction, the immersion mapping and on-manifold control are $\pi(\xi) = col(\xi\ \gamma(\xi)\ \dot{\gamma}(\xi))$ and $c(\xi) = \ddot{\gamma}(\xi)$. Setting $z_1 = x_2 + x_1 + x_1^2 = 0$ and $z_2 = x_3 + (1 + 2x_1)(x_1^2 + x_2)$, suitably initialized, one gets the extended dynamics over $\mathbb{R}^5$

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + z_1, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u \\
\dot{z}_1 &= z_2, \quad \dot{z}_2 = u - \ddot{\gamma}(x_1).
\end{aligned}
$$

The control $u_c = -z_1 - z_2 + \ddot{\gamma}(x_1)$ makes the off-manifold component go to zero while preserving boundedness of the state trajectories. Hence the equilibrium of the system is I&I stabilized. Consider now the single-rate SD equivalent model described by the polynomial maps

$$
\begin{aligned}
x_{1k+1} =& x_1 + \delta(x_1^2 + x_2) + \delta^2(x_1^3 + x_1 x_2 + \frac{x_3}{2}) + \\
& \frac{\delta^3}{3}(3x_1^2 + x_2)(x_1^2 + x_2) + \frac{\delta^3}{3} x_1 x_3 + \frac{\delta^3}{3!} u + O(\delta^4) \\
x_{2k+1} =& x_2 + \delta x_3 + \frac{\delta^2}{2!} u, \quad x_{3k+1} = x_3 + \delta u.
\end{aligned}
$$

The SD target dynamics is then defined according to Proposition 3.1 by computing

$$
\begin{aligned}
&\gamma_1^\delta(\xi) = -\xi - \xi^2 + O(\delta^2), \quad \gamma_2^\delta(\xi) = \xi + 2\xi^2 + O(\delta^2) \\
&c^\delta(\xi) = -\xi - 4\xi^2 + O(\delta).
\end{aligned}
$$

Now, one defines the z component according to Lemma 3.3 and describes the two-rate SD z-dynamics as the polynomial maps

$$
\begin{aligned}
z_{1k+1} =& z_1 + 2\bar{\delta} z_2 + \frac{3\bar{\delta}^2}{2}(u^1 + u^2 + x_1 - z_1 + z_2 - 6x_1 z_1 + \\
& 2x_1 z_2 + 4x_1^2 + 2z_1^2) + \frac{\bar{\delta}^3}{3!}(7u^1 + u^2 + 14u^1 x_1 + \\
& 2u^2 x_1 + 48x_1 z_1 - 32x_1 z_2 + 48z_1 z_2 + 32x_1 z_1^2 - \\
& 96x_1^2 z_1 + 32x_1^2 z_2 - 16x_1^2 + 64x_1^3 - 32z_1^2) + O(\bar{\delta}^4) \\
z_{2k+1} =& z_2 + \bar{\delta}(u^1 + u^2 + 2x_1 - 2z_1 + 2z_2 - 12x_1 z_1 + \\
& 4x_1 z_2 + 8x_1^2 + 4z_1^2) + \bar{\delta}^2(\frac{3u^1}{2} + \frac{u^2}{2} + 3u^1 x_1 + \\
& u^2 x_1 + 12x_1 z_1 - 8x_1 z_2 + 12z_1 z_2 + 8x_1 z_1^2 - \\
& 24x_1^2 z_1 + 8x_1^2 z_2 - 4x_1^2 + 16x_1^3 - 8z_1^2) + O(\bar{\delta}^3)
\end{aligned}
$$

with $\delta = 2\bar{\delta}$. Now one computes $u_{sd}^{[1]\bar{\delta}} = (u_0^i + \frac{\bar{\delta}}{2} u_1^i)_{i=1,2}$ according to (21) with $u_0^1 = u_0^2 = u_c$ and

$$
\begin{aligned}
u_1^1 =& \frac{u_0^2}{3} - \frac{5u_0^1}{3} - \frac{4z_2}{3} - \frac{10u_0^1 x_1}{3} + \frac{2u_0^2 x_1}{3} - 8x_1 z_1 + \\
& \frac{16x_1 z_2}{3} - 8z_1 z_2 + \frac{4z_2}{3} - \frac{16x_1 z_1^2}{3} + 16x_1^2 z_1 - \\
& \frac{16x_1^2 z_2}{3} + \frac{8x_1^2}{3} - \frac{32x_1^3}{3} + \frac{16z_1^2}{3} + \frac{4z_1}{3} \\
u_1^2 =& \frac{80x_1 z_2}{3} - \frac{7u_0^2}{3} - \frac{20z_2}{3} - \frac{26u_0^1 x_1}{3} - \frac{14u_0^2 x_1}{3} - \\
& 40x_1 z_1 - \frac{13u_0^1}{3} - 40z_1 z_2 + \frac{20z_2}{3} - \frac{80x_1 z_1^2}{3} + \\
& 80x_1^2 z_1 - \frac{80x_1^2 z_2}{3} + \frac{40x_1^2}{3} - \frac{160x_1^3}{3} + \frac{80z_1^2}{3} + \frac{20z_1}{3}.
\end{aligned}
$$

Figures 1 and 2 depict simulation results, when the approximate two-rate SD control $u_{sd}^{[1]\bar{\delta}}$ (SD I&I) is compared with the emulated-based one (EB I&I). The evolutions of the closed-loop system under CT control (CT I&I) are reported, too.

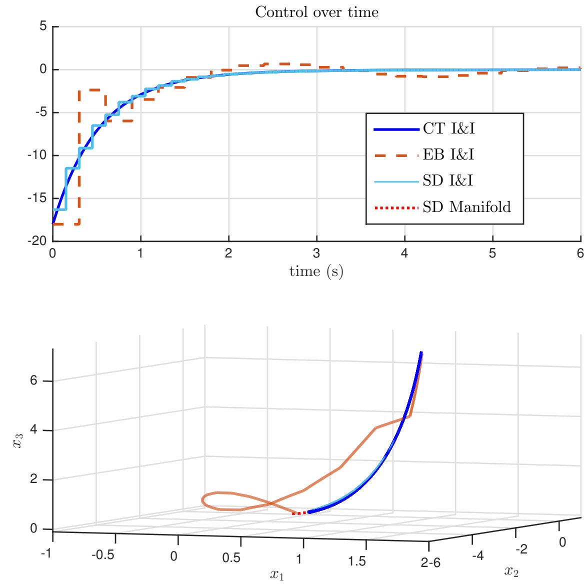

Fig. 1. Invariance with $\delta = 0.3s$

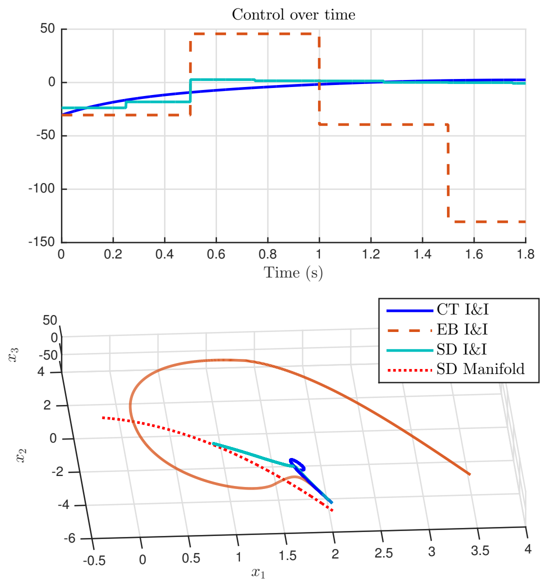

Fig. 2. Attractivity with $\delta = 0.5s$

Simulations are carried out for different sampling periods. We focus the attention on two main aspects to show the tolerance of the SD controllers in preserving the I&I conditions with respect to increasing values of δ: *invariance* of the manifold for initial conditions $x_0 = col(2, -5.5, 10)$ and, in particular, $z_0 = col(0, 0)$ (Fig. 1); *attractivity* of the manifold for initial conditions $x_0 = col(2, -5.5, 10.5)$ and $z_0 = col(0.5, 0.5)$ (Fig. 2). Though figures are omitted for the sake of space, we verified that for $\delta = 0.01s$ both controllers preserve invariance. How-

ever, for $\delta = 0.3$ s, the emulated-based controller does not succeed in doing it (this occurs already for $\delta = 0.1$s) though stability of the equilibrium is achieved.
As far as attractivity is concerned, both controllers still preserve it for lower values of δ. Though, for $\delta = 0.5$ s, the emulated-based control yields instability while the SD I&I one still stabilizes the closed-loop. Even in this case, the control effort of the here-presented control is more than acceptable.

6.2 The spacecraft example

Attitude control provides a nice case study for multi-input systems. Following Krstic and Tsiotras [1999], the attitude motion of a rigid spacecraft is described by equations which exhibit a strict-feedback form

$$\dot{\rho} = H(\rho)\omega, \quad \dot{\omega} = J^{-1}S(\omega)J\omega + J^{-1}u$$

where $\rho \in \mathbb{R}^3$ denotes the Caley-Rodrigues parameters describing the body orientation, $\omega \in \mathbb{R}^3$ the angular velocities in a body-fixed frame and $u \in \mathbb{R}^3$ the active control torques. Along the lines of Section 4., a first-order approximate double-rate control $u^{i[1]} = u_0^i + \frac{\delta}{2}u_1^i$ $(i = 1, 2)$ is computed, similarly to (21), with

$$\begin{aligned} u_0^i =& u_c = -(S(z)J - k_1S(\rho)J + k_1JH(\rho))(z - k_1\rho) - Jk_2z \\ z =& \omega + k_1\rho, \quad k_2 > \frac{1}{k_1} > 0. \end{aligned}$$

and u_1^i, following (21), with $\gamma_{12}(\rho) = 0$ and

$$\begin{aligned} \dot{u}_c = & - \big[-k_2S(z)J - k_1S(H(\rho)\rho)J - k_1S(H(\rho)z)J + \\ & k_1K\widehat{H(\rho)}\big] \left[k_1^2H(\rho)\rho - (k_2 + k_1H(\rho))z\right] + Jk_2^2z. \end{aligned}$$

Simulations are carried for different values of the sampling period. We refer to Krstic and Tsiotras [1999] for more details on the choice of the parameters and time scaling of the simulated model. State space evolutions and the amplitude of the controls are depicted in Fig. 3 showing that good performances are achieved even for quite a large sampling interval, $\delta = 1$ s. Though further simulations are omitted, we just note that EB solutions yield degradated performances for lower values of the sampling period.

7 Conclusion

It is shown that a SD-I&I feedback law can be designed for nonlinear systems in strict-feedback form whenever the I&I problem admits a solution in continuous time. This is achieved via a multi-rate feedback redesign involving the redefinition of the target system and of the consequent manifold which is hence made invariant. The new manifold and the stabilizing control are both parameterized by the sampling period δ, as usual in a sampled-data context. In general, only approximate solutions can be computed so yielding practical stability of the closed-loop. Simulations are reported for testing the proposed strategy.
The presented result assumes nonlinear systems in strict-feedback form which embeds quite a large class of dynamics. This result has been applied for stabilization of strict-feedback systems with state delay (Mattioni et al. [2015a]). Work is progressing to relax this assumption and extend the result to general nonlinear systems in input-affine form.

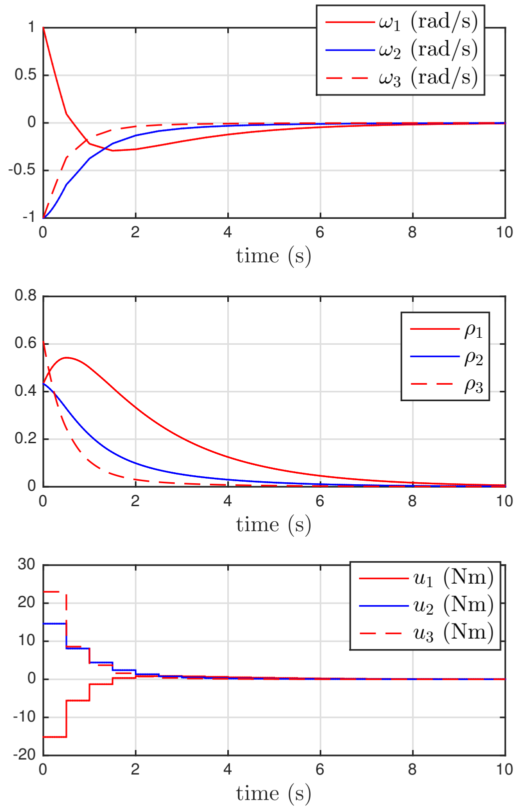

Fig. 3. $\delta = 1$ s and initial displacement $\theta = 240°$.

A Proof of Proposition 3.1

The proof is constructive by solving (11),(12) and (13) via a bottom-up approach. We assume (v_1, v_2, v_3) in the form of asymptotic series expansions in the parameter δ as in Proposition 3.1. Hence, we substitute (v_1, v_2, v_3) with $(\gamma_1^\delta(\cdot), \gamma_2^\delta(\cdot), c^\delta(\cdot))$ into (11),(12) and (13) and compare the terms with the same power in δ. For each order, the existence of a unique solution is deduced from the strict-feedback form which ensures non singularity of the terms to be inverted. The first steps are detailed below. Rewriting (13) as $\gamma_2^\delta \circ \alpha^\delta(\xi) = \gamma_2^\delta(\xi) + \delta c^\delta(\xi)$ and replacing $\gamma_2^\delta \circ \alpha^\delta(\xi)$ with its Taylor expansion around $\gamma_2^\delta(\xi)$

$$\gamma_2^\delta \circ \alpha^\delta(\xi) = \gamma_2^\delta(\xi) + \sum_{i\geq 1} \frac{1}{i!}\frac{\partial^i \gamma_2^\delta}{\partial \xi^i}(\alpha^\delta(\xi) - \xi)^i$$

one gets the equality

$$\delta c^\delta(\xi) = \sum_{i\geq 1} \frac{1}{i!}\frac{\partial^i \gamma_2^\delta}{\partial \xi^i}(\alpha^\delta(\xi) - \xi)^i. \qquad \text{(A.1)}$$

Replacing $\gamma_j^\delta(\cdot)$ $(j = 1, 2)$ and $c^\delta(\cdot)$ with their corresponding series expansions into (A.1), one compares the terms of the same power of δ. For the first term, one gets $c_0(\xi) = (L_{f_1} + \gamma_{10} L_{g_1})\gamma_{20}(\xi)$, recovering the CT solution. In this way, one recursively solves (A.1) by dealing at each-step with a linear equation in the unknown $c_j(\cdot)$ in terms of $c_i(\cdot)$ $(i < j)$, $\gamma_{1s}(\cdot)$ and $\gamma_{2s}(\cdot)$ $(s \leq j)$ (see (15) for the terms (c_1, c_2)). Rewriting $F_2^\delta(\xi, v_1, v_2)$ as $F_2(\xi, v_1, v_2)\big|_{\delta=0} + \delta\tilde{F}_2^\delta(\xi, v_1, v_2)$ in (12) and applying the same procedure, one gets $\gamma_2^\delta(\cdot)$

$$\sum_{i\geq 1} \frac{1}{i!}\frac{\partial^i \gamma_1^\delta}{\partial \xi^i}(\alpha^\delta(\xi) - \xi)^i = \qquad \text{(A.2)}$$
$$\delta\tilde{F}_2^\delta(\xi, \gamma_1^\delta(\xi), \gamma_2^\delta(\xi)) + \frac{\delta^2}{2!}c^\delta(\xi)\ G_2^\delta(\xi, \gamma_1^\delta(\xi), \gamma_2^\delta(\xi), c^\delta(\xi))$$

since, by construction, $F_2^\delta(\cdot)\big|_{\delta=0} = v_1$. Again, such an equation is recursively solved, at each step, through a linear equation in γ_{2j} in terms of c_i, γ_{2i} $(i < j)$ and γ_{1s} $(s \leq j)$. For the first-term, one has $f_2(\xi, \gamma_{10}(\xi)) + \gamma_{20}(\xi) g_2(\xi, \gamma_{10}(\xi)) = (L_{f_1} + \gamma_{10} L_{g_1})\gamma_{10}(\xi)$, whose solution still coincides with the CT one.
Finally, rewriting (11) as the formal series equality in $\gamma_1^\delta(\xi)$

$$\Delta_k W(\xi) - e^{\delta(L_{f_1} + \gamma L_{g_1})} W\big|_\xi = \delta Q^\delta(\xi, \gamma_1^\delta(\xi)) = 0 \qquad \text{(A.3)}$$

with $\Delta_k W(\xi) = W(\alpha^\delta(\xi)) - W(\xi)$ and,by definition, $Q^\delta = Q_0 + \sum_{j\geq 1} \delta^j Q_i$, the first equality to verify is

$$Q_0(\xi, \gamma_{10}) = (L_{f_1} + \gamma_{10} L_{g_1})W\big|_\xi - (L_{f_1} + \gamma L_{g_1})W\big|_\xi = 0$$

which gives $\gamma_{10}(\xi) = \gamma(\xi)$, so recovering again the CT I&I solution described in (2). From the Implicit Function Theorem, a solution to (A.3) exists in a neighborhood of $\gamma_{10}(\xi)$ because for all $\xi \in \mathbb{R}^n$

$$\Big|\frac{\partial Q^\delta(\xi, v, \delta)}{\partial v}\Big|_{\delta\to 0, v=\gamma_{10}} = L_g W\big|_\xi \neq 0. \qquad \text{(A.4)}$$

B Proof of Proposition 4.1

The proof is constructive and works out by rewriting (19) as a formal series equality in the unknown (u^1, u^2)

$$\begin{pmatrix} \delta^2 S_1^\delta(x, z, u^1, u^2) \\ \delta S_2^\delta(x, z, u^1, u^2) \end{pmatrix} = \phi^{\frac{\delta}{2}}(x_{k+1}) - e^{\delta(L_{\tilde{f}} + u_c L_{\tilde{g}})}\phi(x)\big|_{x_k}$$

with $x_{k+1} = F^{2\bar{\delta}}(x, u_{ds})$ as in (10). Thus, one looks for a pair (u^1, u^2) satisfying

$$\Big(S_1^\delta(x, z, u^1, u^2),\ S_2^\delta(x, z, u^1, u^2)\Big)^\top = \Big(0,\ 0\Big)^\top \qquad \text{(B.1)}$$

in which $S_i^\delta = S_{i0} + \sum_{j\geq 1} \delta^j S_{ij}$ for $i = 1, 2$. Recalling that $\gamma_{i0}(\cdot) = \gamma_i(\cdot)$ for $i = 1, 2$, it results that $u^1 = u_c$ and $u^2 = u_c$ solves (B.1) for $\delta \to 0$. More precisely, one gets

$$\begin{aligned} 2S_{10}(x, z, u^1, u^2) =& \frac{3u^1 + u^2}{4} g_2(x_1, z_1 + \gamma_{10}(x_1)) - \\ & u_c g_2(x_1, z_1 + \gamma_{10}(x_1)) = 0 \\ S_{20}(x, z, u^1, u^2) =& \frac{u^1 + u^2}{2} - u_c = 0. \end{aligned}$$

Furthermore, provided the Jacobian matrix

$$\frac{\partial}{\partial u}\begin{pmatrix} S_1^\delta(x, z, u^1, u^2) \\ S_2^\delta(x, z, u^1, u^2) \end{pmatrix}\Bigg|_{\delta\to 0,\ u^1=u_c,\ u^2=u_c} = g_2(x_1, x_2) \qquad \text{(B.2)}$$

is full-rank, one concludes from the Implicit Function Theorem the existence of a small enough T^* such that for any $\delta \in]0, T^*[$, (B.1) admits a unique solution in the form of an asymptotic series expansion around the CT u_c. The condition $g_2(x_1, x_2) \neq 0$ is guaranteed by the CT strict-feedback structure. Since u_c ensures convergence of the CT z-components to 0 with boundedness of the extended state trajectories, The feedback (u^1, u^2) verifies **H4** of Definition 2.1. Thus, the thesis follows.

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Mattia Mattioni received his Bachelor Degree and Master of Science in Control Engineering (both Magna cum Laude) from *La Sapienza Università di Roma* (Italy). In 2015, he participated to the double-degree program *STIC&A* with *Université Paris Sud XI*. He is currently a PhD Student at *L2S* (Univ. Paris Sud 11, France) in a joint program with *DIAG A. Ruberti* (La Sapienza Università di Roma). His research is mainly concerned with the stabilization of nonlinear systems via sampled-data feedback.

Salvatore Monaco (SM 96, F 03) Laurea in 1974; research fellow from 1976, and full Professor in Systems Theory from 1986 at the University of Rome " Sapienza". His research activity is in the field of nonlinear control systems, discrete-time and digital systems, applied research in spacecraft control. Member of Scientific Councils and Associations: Founding member of EUCA (European Union Control Association), 1991-97; ASI (Italian Space Agency) Scientific Committee (1989-95); ASI Technological Committee (1996-99); ASI Evaluation Committee (2000-01); UFI/UIF (Università Italo-Francese), Chairman of the Scientific Committee (2001-05, 2015-).

Dorothée Normand-Cyrot PhD78 in Mathematics, Univ. Paris VII and Doctor83 es-Sciences in Physics, Univ. Paris11, she has been with the CNRS-L2S, France from 1981 and is currently Research Director. Visiting Professor at the DIAG, Univ. Rome "Sapienza", Italy, she promoted bi-lateral research activities and PhD doctorate training from 1987. IEEE-CSS Fellow (class 2005), her research activity is in the field of nonlinear discrete-time systems and digital control with applications; author and co-author of about 185 scientific publications, supervisor of nearly 30 PhD students. Among others, she was Funding Member of the European Union Control Association EUCA91 and Editor at Large of the European Journal of Control from 2003.