Lyapunov stabilization of discrete-time feedforward dynamics

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***Abstract*— The paper discusses stabilization of nonlinear discrete-time dynamics in feedforward form. First it is shown how to define a Lyapunov function for the uncontrolled dynamics via the construction of a suitable cross-term. Then, stabilization is achieved in terms of *u*-average passivity. Several constructive cases are analyzed.**

***Index Terms*— Lyapunov Methods; Stability of nonlinear systems; Algebraic/geometric methods**

I. INTRODUCTION

Nonlinear discrete-time control theory has been attracting a growing interest in the control community because of its impact into the sampled-data, or more generally hybrid context. Although important works bridge the gap between the continuous-time and discrete-time domains through different methodologies (e. g., [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]), hard difficulties still represent obstacles in extending results that are well-known and elegant in continuous time. These are essentially concerned with the generic nonlinearity in the control variable of the dynamics and the difficulty to settle the geometric structure underlying the evolutions.

As a first attempt to characterize accessibility properties of nonlinear discrete-time dynamics, an alternative differential-difference state-space representation (or (F_0,G)-form) was introduced in [11]. In this context, a discrete-time dynamics over $\mathbb{R}^n$ is described by two coupled differential-difference equations as

$$x^+ = F(x), \quad x^+ := x^+(0) \tag{1a}$$

$$\frac{\partial x^+(u)}{\partial u} = G(x^+(u),u). \tag{1b}$$

Denoting by $x^+(u)$ a curve in $\mathbb{R}^n$ parametrized by $u \in \mathbb{R}$, (1a) models the free evolution described by a smooth mapping $F(\cdot)$ while (1b) models the variational effect of the control by a vector field $G(\cdot,u)$, parameterized by u and assumed complete. Further exploiting this differential geometric framework, structural properties (e.g., invariance, decoupling [12]) have been characterized up to introducing the concept of u-average passivity [13]. This latter notion enables to relax the necessity of a direct throughput as usually required when defining passivity for discrete-time systems. Recently, u-average passivity based feedback design (or control Lyapunov design at large) has been introduced in [14] and is exploited in the present paper with reference to stabilization of cascade dynamics.

More precisely, asymptotic stabilization of cascade discrete-time dynamics exhibiting an upper-triangular (or feedforward) form is addressed. Discrete-time forwarding design was firstly addressed in [15] via the construction of a bounded solution to a suitable control-dependent inequality. Arguing so, the difficulty of solving the nonlinear algebraic equation which implicitly defines the feedback solution is overcome. In [16], a discrete-time forwarding design is proposed by exploiting the framework of Immersion and Invariance so relaxing the a-priori knowledge of a Lyapunov function for the first part of the cascade dynamics. In the present paper, we propose a two steps procedure based on control Lyapunov design and feedback average passivation so reminding of the continuous-time forwarding technique ([17], [18]). Preliminarily considering a two block cascade dynamics with nonlinear coupling mapping, a Lyapunov function is firstly constructed for the uncontrolled stable system via the computation of a suitable cross-term. Then, asymptotic stabilization is achieved in terms of u-average passivity. Constructive solutions are discussed based on specifications of the interconnection term. As a particular case, one recovers the case of dynamics in strict-feedforward form studied in [19] where the construction of a cross term reduces to the one of a coordinates transformation rendering the overall dynamics driftless. Finally, it is shown how similar cascade connected forms are recovered when representing input-delayed dynamics through dynamical extension. It follows that the proposed forwarding design procedure may represent an original control Lyapunov design for discrete-time input delayed dynamics.

The paper is organized as follows: in Section II, the existence of a cross-term is proven for the uncontrolled dynamics. It is employed in while in Section III for stabilizing feedforward dynamics through u-average passivity. In Section IV, case studies specifying the connection term structure are discussed. In Section V conclusions are set.

II. LYAPUNOV CROSS TERM FOR CASCADE DYNAMICS

Consider a two block cascade dynamics of the form

$$z_{k+1} = f(z_k) + \varphi(z_k,\xi_k), \quad \xi_{k+1} = a(\xi_k). \tag{2}$$

with $\xi \in \mathbb{R}^{n_\xi}$, $z \in \mathbb{R}^{n_z}$; f, φ and a are continuous functions in their arguments and $(z,\xi) = (0,0)$ is an equilibrium state. We note that the dynamics (2) is uncontrolled with nonlinear connecting map $\varphi(z,\xi)$. The following standing assumptions are introduced.

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A.1 $z_{k+1} = f(z_k)$ has a Globally Stable (GS) equilibrium at the origin with continuously differentiable, positive definite, radially unbounded Lyapunov function $W : \mathbb{R}^{n_z} \to \mathbb{R}_{\geq 0}$ such that $W(f(z)) - W(z) \leq 0$;

A.2 $\xi_{k+1} = a(\xi_k)$ has a Globally Asymptotically Stable (GAS) and Locally Exponentially Stable (LES) equilibrium at the origin with continuously differentiable, positive definite, radially unbounded Lyapunov function $U : \mathbb{R}^{n_\xi} \to \mathbb{R}_{\geq 0}$ such that $U(a(\xi)) - U(\xi) < 0$ for $\xi \neq 0$.

Assumptions **A.1** and **A.2** are not enough to deduce GS of the origin for the complete cascade. For this purpose, further assumptions are needed.

A.3 the function $\varphi(z,\xi)$ satisfies the *linear growth assumption*; i.e. there exist $\mathscr{K}$-functions[1] $\gamma_1(\cdot), \gamma_2(\cdot)$ such that

$$\|\varphi(z,\xi)\| \leq \gamma_1(\|\xi\|)\|z\| + \gamma_2(\|\xi\|);$$

A.4 the function $W(z)$ verifies :

- given any $s(\cdot) : \mathbb{R}^{n_z} \to: \mathbb{R}^{n_z}$ and $d(\cdot,\cdot) : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_z}$

$$|W(s(z) + d(z,\xi)) - W(s(z))| \leq \Big|\frac{\partial W}{\partial z} d(z,\xi)\Big|;$$

- there exist $c, M \in \mathbb{R}_{>0}$ such that for $\|z\| > M$

$$\|\frac{\partial W}{\partial z}\|\|z\| \leq cW(z).$$

The above assumptions imply the possibility of constructing a Lyapunov function $V_0(\cdot)$ for the complete dynamics starting from the respective ones $W(\cdot)$ and $U(\cdot)$. Setting

$$V_0(z,\xi) = W(z) + U(\xi) + \Psi(z,\xi) \quad (3)$$

we aim at defining an additional continuous cross term $\Psi(z,\xi) : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}$ to dominate the part with not definite sign when computing the difference

$$\Delta_k V_0(z,\xi) = V_0(z_{k+1}, \xi_{k+1}) - V_0(z_k, \xi_k).$$

It is a matter of computations to verify that

$$\begin{aligned}\Delta_k V_0(z,\xi) =& \Delta_k U(\xi) + W(f(z_k)) - W(z_k)\\ &+ W(f(z_k) + \varphi(z_k,\xi_k)) - W(f(z_k)) + \Delta_k \Psi(z,\xi)\end{aligned}$$

with $\Delta_k U(\xi) < 0$ and $W(f(z_k)) - W(z_k) \leq 0$. It turns out that, for ensuring $\Delta_k V_0(z,\xi) \leq 0$, the cross term $\Psi(z,\xi)$ can be chosen to satisfy

$$\Delta_k \Psi(z,\xi) = -W(f(z_k) + \varphi(z_k,\xi_k)) + W(f(z_k)) \quad (4)$$

where the right hand side represents the part in $\Delta_k V_0$ whose sign is not definite. As a consequence, $\Psi(z,\xi)$ is defined as

$$\Psi(z,\xi) = \sum_{k=0}^{\infty} W(f(z_k) + \varphi(z_k,\xi_k)) - W(f(z_k)) \quad (5)$$

along the trajectories $(z_k, \xi_k) = (\tilde{z}(k,z,\xi), \tilde{\xi}(k,\xi))$ of (2) starting at $(z_0,\xi_0) = (z,\xi)$. The stability of the whole system follows from the existence of such function V_0.

Theorem 2.1: Under assumptions **A.1** to **A.4**

(i) $\Psi : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}$ exists and is continuous;

(ii) $V_0 : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}$ in (3) is positive-definite and radially unbounded.

As a consequence the origin is a GS equilibrium of (2).

[1] A function ρ is said of class $\mathscr{K}$ if its continuous, strictly increasing and $\rho(0) = 0$. It is said of class $\mathscr{K}_\infty$ if it is $\mathscr{K}$ and it is unbounded.

A. Some particular cases

Some constructive cases are discussed below in relation with the connection term $\varphi(z,\xi)$.

1) Strict-feedforward dynamics: Consider strict-feedforward dynamics described by

$$z_{k+1} = Fz_k + \varphi(\xi_k), \quad \xi_{k+1} = a(\xi_k) \quad (6)$$

with $\varphi(0) = 0$ and $F^\top F = I$. Assumption **A.1** is satisfied with $W(z) = z^\top z$ and **A.4** is obviated. Specifying (4) for (6), one gets that the cross term $\Psi(z,\xi)$ must satisfy

$$\Delta_k \Psi(z,\xi) = -2z_k^\top F^\top \varphi(\xi_k) - \varphi^\top(\xi_k)\varphi(\xi_k). \quad (7)$$

As a consequence $\Delta_k \Psi(z,\xi) = -\Delta_k W(z)$ and, according to (5), one sets

$$\begin{aligned}\Psi(z,\xi) =& \sum_{k=0}^{\infty} \left[z_{k+1}^\top(z,\xi) z_{k+1}(z,\xi) - z_k^\top(z,\xi) z_k(z,\xi)\right]\\ =& (z_k^\top(z,\xi) z_k(z,\xi))_\infty - z^\top z\end{aligned}$$

where $(z_k^\top(z,\xi) z_k(z,\xi))_\infty = \lim_{k\to\infty} z_k^\top(z,\xi) z_k(z,\xi)$ and $z_k(z,\xi)$ denotes the z-trajectory at time k starting at (z,ξ). According to (3), a Lyapunov function for (6) is thus

$$V_0(z,\xi) = U(\xi) + (z_k^\top(z,\xi) z_k(z,\xi))_\infty. \quad (8)$$

More in detail, the dynamics (6) possess two invariant sets: a stable set where the evolutions are described by $\xi_{k+1} = a(\xi_k)$; a center set where the evolutions are described by $z_{k+1} = Fz_k$. It is a matter of computations to verify that the projection of the trajectories of (6) onto the center set are described by the map

$$\phi(\xi) = -\sum_{\tau=k_0}^{\infty} F^{k_0-1-\tau} \varphi(\xi_\tau) \quad (9)$$

verifying the invariance equality

$$\phi(\xi_{k+1}) = F\phi(\xi_k) + \varphi(\xi_k). \quad (10)$$

Thus, under the coordinates change $\zeta = z - \phi(\xi)$, (6) is transformed into the decoupled dynamics

$$\zeta_{k+1} = F\zeta_k, \quad \xi_{k+1} = a(\xi_k). \quad (11)$$

Hence, a Lyapunov function for the cascade is given by $\tilde{V}_0(\zeta,\xi) = U(\xi) + \zeta^\top\zeta$. Exploiting the strict-feedforward form, one easily verifies that the two Lyapunov functions $\tilde{V}_0$ and $\tilde{V}$ coincide up to a coordinates change.

Proposition 2.1: Consider the strict-feedforward dynamics (6). Then, one has $V_0(z,\xi) = \tilde{V}_0(z + \phi(\xi), \xi)$ with $\phi(\xi) : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_z}$ described in (9). As a consequence, the cross-term takes the form

$$\Psi(z,\xi) = (\zeta - \phi(\xi))^\top(\zeta - \phi(\xi)) - z^\top z. \quad (12)$$

Proof: First, rewrite $\zeta^\top\zeta$ for $k_0 = 0$ as

$$(z+\sum_{\tau=0}^{\infty}F^{-1-\tau}\varphi(\xi_\tau))^\top(F^k)^\top F^k(z+\sum_{\tau=0}^{\infty}F^{-1-\tau}\varphi(\xi_\tau))$$
$$=\|z_k(z,\xi)+\sum_{\tau=0}^{\infty}F^{k-\tau-1}\varphi(\xi_\tau)-\sum_{\tau=0}^{k-1}F^{k-\tau-1}\varphi(\xi_\tau)\|^2.$$

Because $(F^k)^\top F^k = I$ and

$$z_k(z,\xi) = F^{k-k_0}z+\sum_{\tau=k_0}^{k-1}F^{k-\tau-1}\varphi(\xi_\tau)$$

then, letting $k\to\infty$, one gets

$$\zeta^\top\zeta = (z_k^\top(z,\xi))(z_k(z,\xi))_\infty.$$

Setting $\Psi(z,\xi) = (z-\phi(\xi))^\top(z-\phi(\xi))-z^\top z$, the cross term verifies (7) because of (10). ■

Remark 2.1: The cross-term in (8) depends on $\|z_k(z,\xi)\|^2$ that admits a limit for $k\to\infty$. This is not so in general for the solution $z_k(z,\xi)$, except in the particular case of $n_z = 1$. $V_0(z,\xi)$ can be thus computed even if a decoupling change of coordinates does not exist.

2) $W(f(z))\equiv W(z)$: Here, (4) specifies as

$$\Delta_k\Psi(z,\xi) = -W(f(z_k)+\varphi(z_k,\xi_k))+W(z_k) = -\Delta_k W(z)$$

so that the cross term takes the form

$$\Psi(z,\xi) = \sum_{k=0}^{\infty}\left[W(z_{k+1})-W(z_k)\right] = W_\infty(z,\xi)-W(z)$$

with $W_\infty(z,\xi) := \lim_{k\to\infty}W(z_k(z,\xi))$. Consequently, one gets

$$V_0(z,\xi) = U(\xi)+W_\infty(z,\xi).$$

3) $f(z) = z$: In such a case, one computes

$$z_\infty(z,\xi) = z+\lim_{N\to\infty}\sum_{k=0}^{N}\varphi(z_k,\xi_k)$$

and thus $W_\infty(z,\xi) = W(z_\infty(z,\xi))$. Accordingly, the map $(z,\xi)\mapsto(z_\infty,\xi)$ defines a *local* coordinates change since

$$\frac{\partial z_\infty}{\partial z} = I+\lim_{N\to\infty}\sum_{k=0}^{N}\frac{\partial\varphi}{\partial z}(z_k,\xi_k)$$

and the sum vanishes at $\xi = 0$. When the connection term $\varphi(\xi,z)$ does not depend on z, the above coordinates change is globally defined as one recovers a strict-feedforward form.

4) Particular structures of $\varphi(\xi)$: When the connection function $\varphi(\xi)$ is a finite polynomial of degree p, the cross term is quadratic of degree $2p$; the following example illustrates the case.

Example: Given

$$z_{k+1} = z_k+\frac{3}{4}\xi_k^2,\quad \xi_{k+1} = \frac{1}{2}\xi_k.$$

which verifies Assumptions **A.1** to **A.4** with $U(\xi) = \xi^2$ and $W(z) = z^2$. Assuming the connection term $\varphi(\cdot)$ to be a finite polynomial of degree 2, we set the cross term as a polynomial of degree 4, $\Psi(z,\xi) = a_1z\xi^2+a_2\xi^4$. Accordingly, one computes $a_1,a_2\in\mathbb{R}$ to solve (7) that specialises as

$$\frac{a_1}{2}(z+\frac{3}{4}\xi^2)\xi^2+\frac{a_2}{16}\xi^4-a_1z\xi^2-a_2\xi^4$$
$$=\frac{1}{16}\xi^4+\frac{1}{2}\xi^2(z+\frac{3}{4}\xi^2)-\xi^4-2z\xi^2.$$

III. Stabilization of extended cascade dynamics

The so built Lyapunov function $V_0(z,\xi)$ is now exploited to show u-average passivity of the extended controlled cascade and to compute the corresponding stabilizing feedback. Without loss of generality, the problem is set in the (F_0,G) formalism (1).

A. Feedforward dynamics

Consider the two block controlled feedforward dynamics

$$z^+ = f(z)+\varphi(z,\xi),\quad z^+ := z^+(0) \tag{13a}$$
$$\frac{\partial z^+(u)}{\partial u} = G_z(z^+(u),\xi^+(u),u) \tag{13b}$$
$$\xi^+ = a(\xi),\quad \xi^+ := \xi^+(0) \tag{13c}$$
$$\frac{\partial\xi^+(u)}{\partial u} = B_\xi(\xi^+(u),u) \tag{13d}$$

with uncontrolled part defined in (2) and controlled vector fields $G_z(\cdot,\cdot,u)$ and $B_\xi(\cdot,u)$. In a more compact way, one writes over $\mathbb{R}^{n_z}\times\mathbb{R}^{n_\xi}$

$$x^+ = F(x),\quad ,\frac{\partial x^+(u)}{\partial u} = G(x^+(u),u),\quad x^+ := x^+(0)$$

with $x = col(z,\xi)$, $F(x) = col(f(z)+\varphi(z,\xi),a(\xi))$ and $G(x^+(u),u) = col(G_z(z^+(u),B_\xi(\xi^+(u)),u)$.

For any triplet (z_k,ξ_k,u_k), by integrating (13b)-(13d) over $[0,u_k[$ with initial condition (13a)-(13c), one recovers a feedforward dynamics in the form of a map

$$z_{k+1} = f(z_k)+\varphi(z_k,\xi_k)+g(z_k,\xi_k,u_k)$$
$$\xi_{k+1} = a(\xi_k)+b(\xi_k,u_k)$$

where $(z_{k+1},\xi_{k+1}) = (z^+(u_k),\xi^+(u_k))$ and

$$\frac{\partial g(z,\xi,u)}{\partial u} := G_z(z^+(u),\xi^+(u),u)$$
$$\frac{\partial b(\xi,u)}{\partial u} := B_\xi(\xi^+(u),u).$$

Property 3.1: Given any C^1-function $S:\mathbb{R}^{n_z}\times\mathbb{R}^{n_\xi}\to\mathbb{R}$, one can rewrite

$$S(x_{k+1}) = S(F(x_k))+\int_0^{u_k}\mathrm{L}_{G(\cdot,v)}S(x^+(v))\mathrm{d}v$$

where $\mathrm{L}_{G(\cdot,v)}S(x)$, denotes the usual Lie derivative of the function $S(\cdot)$ along $G(\cdot,v)$; i.e., $\mathrm{L}_{G(\cdot,v)}S(x) := \frac{\partial S}{\partial x}G(x,v)$. Furthermore, one has

$$\int_0^{u_k}\mathrm{L}_{G(\cdot,v)}S(x^+(v))\mathrm{d}v = u_k\int_0^1\mathrm{L}_{G(\cdot,\theta u_k)}S(x^+(\theta u_k))\mathrm{d}\theta.$$

B. u-average passivity and PBC design

GAS of the equilibrium can now be achieved through u-average passivity-based control as introduced in [14]. The following definitions are recalled.

Definition 3.1 (u-average passivity): The dynamics (13), with output $y = H(x,u)$ is u-average passive with positive definite storage function $S(\cdot)$ if the following inequality holds for any $u \in \mathbb{R}$

$$S(x^+(u)) - S(x) \leq \int_0^u H(x^+(v),v)\mathrm{d}v. \tag{14}$$

Definition 3.2 (ZSD): Given (13) with output $H(x,u)$, let $Z \subset \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi}$ be the largest positively invariant set contained in $\{x \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \mid H(x,0) = 0\}$. (13) is Zero-State-Detectable (ZSD) if $x = 0$ is asymptotically stable conditionally to Z.

Theorem 3.1: Consider (13) under **A.1** to **A.4**, then:

- (13) is u-average passive with respect to the output
$$H(z,\xi,u) = \mathrm{L}_{G(\cdot,u)}V_0(z,\xi) \tag{15}$$
and storage function $V_0(z,\xi)$;
- if, furthermore, (13) with output $H(z,\xi,0)$ is ZSD, the feedback u_d solving the equality
$$u_d = -\frac{1}{u_d}\int_0^{u_d} \mathrm{L}_{G(\cdot,v)}V_0(z^+(v),\xi^+(v),v)\mathrm{d}v \tag{16}$$
achieves GAS of the equilibrium $(z,\xi) = (0,0)$;
- if the linear approximation of (13) is stabilizable then (16) ensures LES of the closed-loop.

Proof: Computing $\Delta_k V_0(z,\xi) = V_0(z_{k+1},\xi_{k+1}) - V_0(z_k,\xi_k)$ along the dynamics (13) one gets (dropping the k-index in the right hand side)

$$\begin{aligned}\Delta_k V_0(z,\xi) &= U(a(\xi)) - U(\xi) + \int_0^u \mathrm{L}_{B_\xi(\cdot,v)}U(\xi^+(v))dv \\ &+ W(f(z)+\varphi(z,\xi)) - W(z) + \int_0^u \mathrm{L}_{G_z(\cdot,\xi^+(v),v)}W(z^+(v))dv \\ &+ \Psi(F(z,\xi)) - \Psi(z,\xi) + \int_0^u \mathrm{L}_{G(\cdot,v)}\Psi(z^+(v),\xi^+(v))dv.\end{aligned}$$

By construction of $\Psi(\cdot)$ for $u = 0$, one concludes u-average passivity with respect to the dummy output $H(\cdot,u) = \mathrm{L}_{G(\cdot,u)}V_0$ and storage function $V_0(\cdot)$; i.e.

$$\Delta_k V_0(z,\xi) \leq \int_0^u \mathrm{L}_{G(\cdot,v)}V_0(z^+(v),\xi^+(v))dv. \tag{17}$$

Accordingly, the control u solution to (16) achieves GAS of the equilibrium whenever (13) is ZSD with respect to $H(\cdot,0)$. LES follows from u-average passivity plus the stabilizability of the linear approximation of (13) at the origin. ■

Remark 3.1: The damping controller u_d solution of the equality (16) can equivalently be rewritten as the solution of

$$u_d = -\int_0^1 \mathrm{L}_{G(\cdot,\theta u_d)}V_0(x^+(\theta u_d))\mathrm{d}\theta. \tag{18}$$

To avoid the difficult problem of solving implicit equalities, approximate solutions can be computed. In [16], the authors provide an explicit and exactly computable expression of the feedback u which preserves u-average passivity and stability. The consequent solution is bounded by a positive constant $\mu \in \mathbb{R}$ and is defined as

$$u_{dap}(x) = -K(x)\mathrm{L}_{G(\cdot,0)}V_0(x^+(0))$$

for a suitable gain $K(\cdot) > 0$.

Example: Consider the discrete-time cascade dynamics

$$\begin{aligned}&z^+ = z+\xi, \quad \xi^+ = \xi \\ &\frac{\partial z^+(u)}{\partial u} = \frac{1}{2} - (\xi^+(u))^2, \quad \frac{\partial \xi^+(u)}{\partial u} = 1\end{aligned}$$

or, equivalently,

$$z_{k+1} = z + \xi + u(\frac{1}{2} - \xi^2) - u^2\xi - \frac{1}{3}u^3, \quad \xi_{k+1} = \xi + u$$

which verifies Assumption **A.1** with $W(z) = \frac{1}{2}z^2$ and Assumption **A.2** with preliminary feedback $u = -\frac{2}{3}\xi$ and $U(\xi) = \frac{1}{2}\xi^2$. The cross term $\Psi(z,\xi) = \frac{1}{2}(z+\xi+\frac{\xi^3}{3})^2 - \frac{1}{2}z^2$ verifies $\Delta_k V_0(z,\xi) = \Delta_k U(\xi) = -\frac{4}{9}\xi_k^2$. Finally, the u-average output and the consequent control are provided by

$$\begin{aligned}H(z,\xi,u) &= 4\xi + \frac{3}{2}z + \frac{13}{8}u + \frac{1}{2}\xi^3 \\ u &= -\frac{4}{7}z - \frac{32}{21}\xi - \frac{4}{21}\xi^3.\end{aligned}$$

IV. Some cases of study

A. The case of strict-feedforward dynamics

Consider the controlled strict-feedforward dynamics

$$z^+ = Fz + \varphi(\xi), \quad \frac{\partial z^+(u)}{\partial u} = G(\xi^+(u),u) \tag{19a}$$

$$\xi^+ = a(\xi), \quad \frac{\partial \xi^+(u)}{\partial u} = B(\xi^+(u),u) \tag{19b}$$

or equivalently

$$z_{k+1} = Fz_k + \varphi(\xi_k) + g(\xi_k,u_k), \quad \xi_{k+1} = a(\xi_k) + b(\xi_k,u_k)$$

with uncontrolled part (6) and by definition

$$\begin{aligned}g(\xi_k,u_k) &:= \int_0^{u_k} G(\xi^+(v),v)\mathrm{d}v \\ b(\xi_k,u_k) &:= \int_0^{u_k} B(\xi^+(v),v)\mathrm{d}v\end{aligned}$$

with $g(\cdot,0) = 0$ and $b(\cdot,0)$. As already detailed, when $u \equiv 0$, the coordinates change $\zeta = z - \phi(\xi)$ in (9) transforms the system into a decoupled dynamics of the form (11). Specyfying to (19), one gets

$$\zeta^+ = F\zeta_k, \quad \frac{\partial \zeta^+(u)}{\partial u} = G_\zeta(\xi^+(u),u) \tag{20a}$$

$$\xi^+ = a(\xi_k), \quad \frac{\partial \xi^+(u)}{\partial u} = B(\xi^+(v),v)\mathrm{d}v \tag{20b}$$

where

$$G_\zeta(\xi^+(u),u) = G(\xi^+(u),u) - \mathrm{L}_{B(\cdot,u)}\phi(\xi^+(u)).$$

As a consequence, Theorem 3.1 holds with output

$$Y_1(\zeta,\xi,u) = \mathrm{L}_{G_\zeta(\cdot,u)}\tilde{V}_0(\zeta,\xi). \tag{21}$$

Remark 4.1: When $F = I$ and $n_z = 1$, the coordinates change $\zeta = z - \phi(\xi)$ makes the ζ-dynamics driftless. Accordingly, one recovers the result in [19] proposed when assuming directly in (19), $\xi_{k+1} = u_k$ and $n_z = 1$.

Remark 4.2: In [16], the strict-feedforward stabilization is set in the Immersion and Invariance (I&I) framework, [20] when $n_z = 1$. Assuming **A.2**, a stable set over which the closed loop ξ-dynamics evolves is exhibited. The design aims at driving the off-stable set state components ζ to zero while ensuring boundedness of the full state trajectories. I&I is less demanding since the knowledge of a Lyapunov function $U(\xi)$ for the ξ-system is not necessary. On the other hand, the cross term approach covers a wider range of cases.

B. Stabilization of input-delayed dynamics

The result is now applied to design u-average passivity-based controllers for discrete-time dynamics affected by input delay of the form

$$z_{k+1} = f(z_k) + \varphi(z_k, u_{k-1}). \tag{22}$$

Setting the usual extension $\xi_k = u_{k-1}$, (22) rewrites as

$$z_{k+1} = f(z_k) + \varphi(z_k, \xi_k), \quad \xi_{k+1} = u_k \tag{23}$$

that clearly takes the form of (13) with $g(z, \xi, u) = 0$ and $a(\xi) = 0$. Assuming GS the origin of the dynamics $z_{k+1} = f(z_k)$ with C^1 and radially unbounded Lyapunov function $W(z)$ and setting $U(\xi) = \xi^2$, the Lyapunov function $V_0(z, \xi)$ for (23) takes the form $V_0(z, \xi) = \xi^2 + W(z) + \Psi(z, \xi)$ with cross term solution of

$$\Delta_k \Psi(z, \xi)\big|_{u \equiv 0} = -W(f(z) + \varphi(z, \xi)) + W(f(z)).$$

Under the assumptions in Theorem 3.1, one specifies the output map $H_{del}(z, \xi) = \frac{\partial V_0}{\partial \xi}(z, \xi)$ with respect to which (23) is u-average passive so satisfying the inequality

$$V_0(f(z) + \varphi(z, \xi), u) - V_0(z, \xi) \leq \int_0^u \frac{\partial V_0}{\partial \xi}(f(z) + \varphi(z, \xi), v)\mathrm{d}v.$$

Accordingly, the control u_{del} solution of the equality

$$u_{del} = -\frac{1}{u_{del}} \int_0^{u_{del}} \frac{\partial V_0}{\partial \xi}(f(z) + \varphi(z, \xi), v)\mathrm{d}v$$

stabilizes the equilibrium provided the ZSD property holds.

This comment can be generalized to multiple input delays and to a z-dynamics explicitly depending on u as well. This is of peculiar interest when the problem of stabilizing a continuous-time time-delay system is set in the sampled-data context and reformulated as a discrete-time stabilizing one over an extended state space [21].

V. Conclusions

Stabilization of discrete-time dynamics in feedforward form via Lyapunov-based and passivity-based methodologies has been addressed. The study is detailed for the case of two interconnected dynamics by constructing a Lyapunov function through the definition of a suitable cross-term. When considering dynamics issued from sampling, a similar approach has been developed in [22], taking advantage of the primitive continuous-time properties. Work is progressing regarding multi-block cascade dynamics and analyzing the variety of control problems involving these structures.

References

[1] S. Monaco and D. Normand-Cyrot, "Nonlinear systems in discrete time," in *Algebraic and geometric methods in nonlinear control theory*. Springer, 1986, pp. 411–430.

[2] C. Mohtadi, "Bode's integral theorem for discrete-time systems," in *IEE Proceedings D (Control Theory and Applications)*, vol. 137, no. 2. IET, 1990, pp. 57–66.

[3] L. Wei and C. Byrnes, "Design of discrete-time nonlinear control systems via smooth feedback," *IEEE Transactions on Automatic Control*, vol. 39, no. 11, pp. 2340–2346, 1994.

[4] D. Nešić, A. R. Teel, and E. D. Sontag, "Formulas relating kl stability estimates of discrete-time and sampled-data nonlinear systems," *Systems & Control Letters*, vol. 38, no. 1, pp. 49–60, 1999.

[5] D. Kazakos and J. Tsinias, "The input-to-state stability condition and global stabilization of discrete-time systems," *IEEE Transactions on Automatic Control*, vol. 39, no. 10, pp. 2111–2113, 1994.

[6] Z.-P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, no. 6, pp. 857–869, 2001.

[7] E. M. Navarro-LóPez and E. Fossas-Colet, "Feedback passivity of nonlinear discrete-time systems with direct input–output link," *Automatica*, vol. 40, no. 8, pp. 1423–1428, 2004.

[8] N. Kazantzis, "A new approach to the zero-dynamics assignment problem for nonlinear discrete-time systems using functional equations," *Systems & control letters*, vol. 51, no. 3, pp. 311–324, 2004.

[9] B. Jakubczyk, "Feedback linearization of discrete-time systems," *Systems & Control Letters*, vol. 9, no. 5, pp. 411 – 416, 1987.

[10] P. Pepe, G. Pola, and M. Di Benedetto, "On Lyapunov-Krasovskii characterizations of stability notions for discrete-time systems with unknown time-varying time-delays," in *IEEE 55th Conference on Decision and Control (CDC)*, 2016, pp. 447–452.

[11] S. Monaco and D. Normand-Cyrot, "A unified representation for nonlinear discrete-time and sampled dynamics," *Journal of Mathematical Systems Estimation and Control*, vol. 7, pp. 477–503, 1997.

[12] C. Califano, S. Monaco, and D. Normand-Cyrot, "Non-linear noninteracting control with stability in discrete-time: a geometric framework," *International Journal of Control*, vol. 75, no. 1, pp. 11–22, 2002.

[13] S. Monaco, D. Normand-Cyrot, and F. Tiefensee, "From passivity under sampling to a new discrete-time passivity concept," in *47th IEEE Conference on Decision and Control (CDC)*, 2008, pp. 3157–3162.

[14] S. Monaco and D. Normand-Cyrot, "Nonlinear average passivity and stabilizing controllers in discrete-time," *Systems & Control Letters*, vol. 60, pp. 431–439, 2011.

[15] F. Mazenc and H. Nijmeijer, "Forwarding in discrete-time nonlinear systems," *International Journal of Control*, vol. 71, no. 5, pp. 823–835, 1998.

[16] S. Monaco, D. Normand-Cyrot, and M. Mattioni, "Stabilization of feedforward discrete-time dynamics through immersion and invariance," in *2016 American Control Conference (ACC)*, July 2016, pp. 264–269.

[17] R. Sepulchre, M. Janković, and P. Kokotović, *Constructive nonlinear control*. Springer New York, 1997.

[18] L. Praly, R. Ortega, and G. Kaliora, "Stabilization of nonlinear systems via forwarding mod {L g V}," *IEEE Transactions on Automatic Control*, vol. 46, no. 9, pp. 1461–1466, 2001.

[19] S. Monaco and D. Normand-Cyrot, "Stabilization of nonlinear discrete-time dynamics in strict-feedforward form," in *European Control Conference 13*, 2013, pp. 2186–2191.

[20] A. Astolfi and R. Ortega, "Immersion and invariance: a new tool for stabilization and adaptive control of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 4, pp. 590–606, 2003.

[21] S. Monaco, D. Normand-Cyrot, and M. Mattioni, "Sampled-data stabilization of nonlinear dynamics with input delays through immersion and invariance," *IEEE Transactions on Automatic Control*, vol. PP, no. 99, pp. 1–1, 2016.

[22] M. Mattioni, S. Monaco, and D. Normand-Cyrot, "Sampled-data stabilization of feedforward dynamics with Lyapunov cross-term," in *2016 IEEE 55th Conference on Decision and Control (CDC)*, Dec 2016, pp. 1322–1327.

APPENDIX

Let us first prove **(i)**. Being the equilibrium of the dynamics $\xi_{k+1} = a(\xi_k)$ LES, there exist a real constant $|\alpha| < 1$ and a function $\gamma(\cdot) \in \mathscr{K}$ so that $\tilde{\xi}(s,\xi) \le \gamma(\|\xi\|)|\alpha|^s$ for any $s \ge 0$. Then, because of Assumption **A.4**, the following inequality holds

$$W(f(z_k)+\varphi(z_k,\xi_k)) - W(z_k) \le \tag{24}$$
$$W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k)) \le \Big|\frac{\partial W}{\partial z}\varphi(z_k,\xi_k)\Big| \le$$
$$\|\frac{\partial W}{\partial z}\|(\gamma(\|\xi\|))|\alpha|^k + \gamma(\|\xi\|))|\alpha|^k\|z_k\|) \le c\gamma(\|\xi\|)|\alpha|^k W(z_k).$$

Accordingly, $W(z)$ is not decreasing along the trajectories of (2) and $\|z_k\|$ and $\|\frac{\partial W}{\partial z}(z_k)\|$ are bounded on $[0,\infty)$ (because $W(z)$ is radially unbounded). Consequently, one can write

$$W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k)) \le \gamma_1(\|(z,\xi)\|)\alpha^k \tag{25}$$

so getting that $W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k))$ is summable over $[0,\infty)$ and (5) exists and is bounded for all (z,ξ).

Continuity of $\Psi(\cdot)$ in (5) comes from the fact that it is the composition and the sum of continuous functions on $[0,\infty)$.

As far as **(ii)** is concerned, positive definiteness of $V_0(\cdot)$ is obtained by exploiting the radial unboundedness of $W(z)$.

$$\begin{aligned} W(z_k) &= W(z) + \sum_{t=0}^{k-1}\big[W(f(z_t)+\varphi(z_t,\xi_t)) - W(z_t)\big] \\ &= W(z) + \sum_{t=0}^{k-1}\big[W(f(z_t)+\varphi(z_t,\xi_t)) - W(f(z_t))\big] \\ &+ \sum_{t=0}^{k-1}\big[W(f(z_t)) - W(z_t)\big] \end{aligned}$$

where the term $W(f(z_t)) - W(z_t)$ is non-increasing for any $t \ge 0$. By substracting both sides of the last equality by $W(f(z_t)) - W(z_t)$ and taking the limit for $k \to \infty$ one gets

$$\begin{aligned} &W_\infty(z) - \sum_{t=0}^{\infty}\big[W(f(z_t)) - W(z_t)\big] = \\ &W(z) + \sum_{t=0}^{\infty}\big[W(f(z_t)+\varphi(z_t,\xi_t)) - W(f(z_t))\big] \end{aligned}$$

where $W_\infty(z) = \lim_{k\to\infty} W(z_k)$ and $\Psi(z,\xi) = \sum_{t=0}^{\infty}\big[W(f(z_t)+\varphi(z_t,\xi_t)) - W(f(z_t))\big]$. Hence, one gets that $V_0(z,\xi)$ rewrites as

$$V_0(z,\xi) = W_\infty(z) - \sum_{t=0}^{\infty}\big[W(f(z_t)) - W(z_t)\big] + U(\xi) \ge 0. \tag{26}$$

From the radially unboundedness of $W(\cdot)$ and $U(\cdot)$ one has that if $V_0(z,\xi) = 0$ then $\xi = 0$. By construction, $V_0(z,0) = W(z)$ so concluding that $V_0(z,\xi) = 0$ implies $(z,\xi) = (0,0)$. This last inequality proves that $V_0(\cdot)$ is positive-definite.

To prove its radial unboundedness we first point out that $V_0(z,\xi) \to \infty$ as $\|\xi\| \to \infty$ for any z because of (26). Hence, one has to show that

$$\lim_{\|z\|\to+\infty}\Big[W_\infty(z) - \sum_{t=0}^{\infty}\big(W(f(z_t)) - W(z_t)\big)\Big] = +\infty. \tag{27}$$

This is achieved by lower bounding (27) by means of a radially unbounded function deduced from $W(z)$. For, fix ξ so that $\gamma(\|\xi\|)$ in (24) is a constant C. Accordingly, for any $k \ge 0$, we write

$$\begin{aligned} &|W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k))| \le \\ &\|\frac{\partial W}{\partial z}\|(C|\alpha|^k + C|\alpha|^k\|z_k\|). \end{aligned}$$

It follows that

$$\begin{aligned} &W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k)) \ge \\ &-|W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k))| \ge \\ &-2\|\frac{\partial W}{\partial z}\|C|\alpha|^k\|z_k\| - C(1-\|z_k\|)\|\frac{\partial W}{\partial z}\||\alpha|^k. \end{aligned}$$

When $1 - \|z_k\| > 0$, the term $-C(1-\|z_k\|)\frac{\partial W}{\partial z}\||\alpha|^k$ can be discarded without affecting the inequality. On the other hand, when $1 - \|z_k\| \le 0$, it is bounded by $K_2|\alpha|^k$ so that

$$\begin{aligned} &W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k)) \ge \\ &-2\|\frac{\partial W}{\partial z}\|C|\alpha|^k\|z_k\| - K_2|\alpha|^k. \end{aligned}$$

Using **A.4** we obtain

$$W(f(z_k)+\varphi(z_k,\xi_k)) - W(z_k) \ge \tag{28}$$
$$\begin{cases} -K|\alpha|^k W(z_k) - K_2|\alpha|^k + W(f(z_k)) - W(z_k), & \|z\| > r \\ -K_1|\alpha|^k W(z_k) - K_2|\alpha|^k + W(f(z_k)) - W(z_k), & \|z\| \le r \end{cases}$$

with $r \ge 1$ and real K, K_1, K_2.From (28) one gets the following lower bounds on $W(z_k)$.

When $\|z\| > r$ and $k \in [0,t)$

$$\begin{aligned} W(z_k) \ge \phi(k,0)W(z) + \sum_{t=0}^{k-1}\phi(k-1,t)\big[-K_2|\alpha_1|^t + \\ W(f(z_t)) - W(z_t)\big] \end{aligned}$$

When $\|z\| \le r$ and $k \in [0,t)$

$$W(z_k) \ge W(z) + \sum_{t=0}^{k-1}\big[-K_1|\alpha|^t - K_2|\alpha|^t + W(f(z_t)) - W(z_t)\big]$$

with $\phi(k,t) = \prod_{j=t}^{k}(1 - K|\alpha|^j)$. Accordingly, by mixing both the bounds, one gets

$$W(z_k) \ge \phi(k,0)W(z) + \sum_{t=0}^{k-1}(-K_1|\alpha|^t - K_2|\alpha|^t + W(f(z_t)) - W(z_t))$$

so that for all $k \ge 0$, $\phi(k,0)$ admits a lower bound K_3 and

$$W(z_k) \ge K_3 W(z) + \sum_{t=0}^{k-1}\big[W(f(z_t)) - W(z_t)\big] + r_k$$

with $r_k := \sum_{t=0}^{k-1}\big[-K_1|\alpha|^t - K_2|\alpha|^t\big]$ converging to a bounded solution r^* over $[0,\infty)$. Taking the limit as $k \to \infty$, one obtains

$$W_\infty(z,\xi) - \sum_{t=0}^{k-1}\big[W(f(z_t)) - W(z_t)\big] \ge K_3 W(z) + r^*.$$

We note that r^* and K_3 may depend on ξ but are independent of z so that (27) holds. Finally, by construction $V_0(z_{k+1},\xi_{k+1}) - V_0(z_k,\xi_k)) = W(f(z_k)) - W(z_k) + U(a(\xi_k)) - U(\xi_k) \le 0$ so concluding the proof.