On partially minimum phase systems and nonlinear sampled-data control

Mattia Mattioni[1,2], Marwa Hassan[1], Salvatore Monaco[1] and Dorothée Normand-Cyrot[2]

***Abstract*— The concept of partially minimum phase systems is introduced and used with reference to the class of nonlinear systems exhibiting a linear output. It turns out that input-output feedback linearization with stability of the internal dynamics can be pursued via the use of a dummy output with respect to which the system is minimum-phase. The design strategy is extended to multirate sampled-data control and a working example illustrates the performances.**

***Index Terms*— Feedback linearization; Nonlinear output feedback; Sampled-data control**

I. INTRODUCTION

A huge number of control strategies is about assigning a target dynamic to a given system. Basically, the concerned design techniques require the inversion of some intrinsic dynamics of the plant that might filter the required behavior ([1], [2], [3], [4], [5], [6]). In the linear case, this corresponds to designing a feedback that assigns part of the eigenvalues coincident with the zeros of the system so making the corresponding dynamics unobservable. In the nonlinear case, similar considerations can be made via the inversion under feedback of the so-called zero-dynamics [7]. It results that the so-defined control will ensure stability in closed loop if and only if the zero-dynamics are asymptotically stable.

Though, the linear case suggests that stability in closed loop can be still pursued under state feedback via partial dynamic cancellation. As a matter of fact, one might design a feedback so to cancel only the stable zeros while leaving the remaining ones unchanged so performing a filtering action that should not compromise the required closed-loop behavior. Based on this idea, we consider non minimum phase nonlinear single-input single-output (SISO) systems that are controllable in first approximation and settle the problem in the context of Input-Output linearization. In that case, because the zero-dynamics are unstable, classical techniques cannot be implemented to solve the problem with stability. Based on the notion of *partially minimum phase* systems, the design we propose proceeds in two steps: considering the linear tangent model (LTM) of the original system, we first define a dummy output based on a suitable factorization of the numerator of its transfer function so that the corresponding linearized system is minimum-phase; then, we perform classical input-output linearization of the locally minimum-phase nonlinear system with the aforementioned dummy output. Finally, we show that when applying the resulting feedback to the original system, input-output linearization still holds with respect to the actual output while guaranteeing stability of the internal dynamics.

The proposed methodology is then applied to the sampled-data context; namely, measures of the output (say the state) are available only at some time instants and the control is piecewise constant over the sampling period. In this context, the problem under study is even more crucial because of the further zero-dynamics intrinsically induced by sampling that are generally unstable [8]. As a consequence, the minimum-phase property of a given nonlinear continuous-time system is not preserved by its sampled-data equivalent ([9], [10], [11], [12]). To overcome those issues, several solutions were proposed based on different sampling procedures ([10], [13], [14], [15], [16], [17]). Among these, the first one was based on multirate sampling in which the control signal is sampled-faster (say r times) than the measured variables. Accordingly, this sampling procedure introduces further degrees of freedom and prevents from the appearance of the unstable sampling zero dynamics while preserving the continuous-time relative degree ([9], [18]). As an alternative, in [13], [16] the authors exploited sampling via generalized hold function (GHF) in order to arbitrarily assign the zero-dynamics of the corresponding sampled-data equivalent system. Though, the relative degree is still not preserved in this case and the GHF method can be seen as a particular case of multirate sampling.

The paper is organized as follows: The problem is settled in Section II and motivated in Section III; the main result is in Section IV and extended to the sampled-data context in Section V. A simulated example is in Section VI. Section VII concludes the paper.

Notation and definitions: All the functions and vector fields defining the dynamics are assumed smooth and complete over the respective definition spaces. M_U (resp. M_U^I) denotes the space of measurable and locally bounded functions $u:\mathbb{R}\to U$ $(u:I\to U,\ I\subset\mathbb{R})$ with $U\subseteq\mathbb{R}$. $\mathscr{U}_\delta\subseteq M_U$ denotes the set of piecewise constant functions over time intervals of fixed length $\delta\in]0,T^*[$; i.e. $\mathscr{U}_\delta=\{u\in M_U\ \text{s.t.}\ u(t)=u_k,\forall t\in[k\delta,(k+1)\delta[;k\geq 0\}$. Given a vector field f, L_f denotes the Lie derivative operator, $\mathrm{L}_f=\sum_{i=1}^n f_i(\cdot)\frac{\partial}{\partial x_i}$. $e^{\mathrm{L}_f}x$ denotes the associated Lie series operator, $e^{\mathrm{L}_f}x:=x+\sum_{i\geq 1}\frac{\mathrm{L}_f^i x}{i!}$. A function $R(x,\delta)=O(\delta^p)$ is

Mattia Mattioni thanks the *Université Franco-Italienne/Università Italo-Francese* (UFI/UIF) for supporting his mobility from France to Italy within his PhD program.

[1] Dipartimento di Ingegneria Informatica, Automatica e Gestionale *A. Ruberti* (La Sapienza University of Rome); Via Ariosto 25, 00185 Rome, Italy {mattia.mattioni,marwa.hassan, salvatore.monaco}@uniroma1.it.

[2]Laboratoire des Signaux et Systèmes (L2S, CNRS); 3, Rue Joliot Curie, 91192, Gif-sur-Yvette, France {mattioni,cyrot}@l2s.centralesupelec.fr

said to be of order δ^p $(p \geq 1)$ if whenever it is defined it can be written as $R(x,\delta) = \delta^{p-1}\tilde{R}(x,\delta)$ and there exist function $\theta \in \mathscr{K}_\infty$ and $\delta^* > 0$ s. t. $\forall \delta \leq \delta^*$, $|\tilde{R}(x,\delta)| \leq \theta(\delta)$.

II. Problem settlement

We consider nonlinear feedback linearizable input-affine dynamics with linear output map of the form

$$\begin{aligned} \dot{x} =& f(x) + g(x)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}, y \in \mathbb{R} \\ y =& Cx \end{aligned} \tag{1}$$

verifying the following assumptions: (1) has relative degree $r \leq n$ and is partially minimum phase[1]; the Linear Tangent Model (LTM) at the origin

$$\begin{aligned} A &= \frac{\partial f}{\partial x}\Big|_0 = \begin{pmatrix} \mathbf{0} & I_{r-1} \\ & -\mathbf{a} \end{pmatrix}, \ B = g(0) = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \\ C &= \begin{pmatrix} b_0 & \dots & b_m & \mathbf{0} \end{pmatrix} \end{aligned} \tag{2}$$

is controllable. $\mathbf{a} = (a_0 \ \dots \ a_{n-1})$ is a row vector containing the coefficients of the associated characteristic polynomial. As a consequence, (2) rewrites

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{3}$$

and has relative degree $\hat{r}$ coinciding, at least locally, with r.

Remark 2.1: If (A,B,C) is not in the canonical controllable form (2), one preliminarily applies to (1) the linear transformation

$$\xi = Tx, \quad T = \begin{pmatrix} \gamma^\top & (\gamma A)^\top & \dots (\gamma A^{n-1})^\top \end{pmatrix}^\top$$

with $\gamma = \begin{pmatrix} \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} B & AB & \dots A^{n-1}B \end{pmatrix}^{-1}$ so transforming the system into the required form.

In this setting, one looks for a continuous-time feedback that ensures input-output linearization of (1) while guaranteeing stability of the internal dynamics. This will be achieved via partial dynamics cancellation. Then, the strategy will be extended to the sampled-data context through multirate sampled-data feedback.

III. Partial zero-dynamics cancellation

Let us start discussing how partial cancellation of the zero dynamics can be used to assign the dynamics under feedback. For, let (3) be the LTM at the origin of (1). Since (A,B) is controllable, the transfer function of the system is provided by

$$W(s) = C(sI - A)^{-1}B = \frac{N(s)}{D(s)}$$

with $N(s) = b_0 + b_1 s + \dots + b_m s^m$ and $D(s) = a_0 + a_1 s + \dots + a_{n-1}s^{n-1} + s^n$ and relative degree $\hat{r} = n - m$.

Given any factorization of the numerator $N(s) = N_1(s)N_2(s)$ and fixed $D(s)$, the dummy output $y_i = C_i x$ with $C_i = (b_0^i \dots b_{m_i}^i \ \mathbf{0})$ corresponds to the transfer function having

$$N_i(s) := b_0^i + b_1^i s + \dots + b_{m_i}^i s^{m_i}$$

[1]Consider a nonminimum phase nonlinear system (1) with LTM model at the origin (3) whose zeros are the roots of a not Hurwitz polynomial N(s); we say that it is *partially minimum phase* there exists a factorization of $N(s) = N_1(s)N_2(s)$ so that $N_2(s)$ is Hurwitz.

$(i = 1,2)$ as numerator and relative degree $r_i = n - m_i$ $(i = 1,2)$. Accordingly, the outputs y, y_1 and y_2 are related by

$$y(t) = N_1(\mathrm{d})y_2(t), \quad y(t) = N_2(\mathrm{d})y_1(t)$$

so getting for $j \neq i$ and $\mathrm{d} = \frac{\mathrm{d}}{\mathrm{d}t}$

$$y(t) = b_0^j y_i + b_1^j \frac{\mathrm{d}}{\mathrm{d}t} y_i + \dots + b_{m_j}^j \frac{\mathrm{d}^{m_j}}{\mathrm{d}t^{m_j}} y_i.$$

Remark 3.1: The feedback

$$u_i = F_i x + v, \quad F_i = -\frac{C_i A^{r_i}}{C_i A^{r_i - 1}B}, \quad i = 1,2$$

transforms (3) into a system with closed-loop transfer function given by

$$\begin{aligned} W^{F_i}(s) &= C(sI - A - BF_i)^{-1}B \\ &= \frac{N_j(s)}{s_i^r} = \frac{b_0 + b_1^j s + .. + b_{m_j}^j s^{m_j}}{s^{r_i}}, \quad j \neq i. \end{aligned}$$

Remark 3.2: It is a matter of computations to verify that the feedback $u = F_i x$ coincides with the one deduced from the Ackermann formula assigning the poles of the system to the roots of $p_i^*(s) = s^{r_i}N_i(s)$. As a consequence, it rewrites $u_i = F_i x$ with $F_i = -\gamma p_i^*(A)$ and $\gamma = \begin{pmatrix} \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}^{-1}$.

The feedback $u = F_i x + v$ places r_i eigenvalues of the system coincident with the zeros of $N_i(s)$ and the remaining ones to 0 so that stabilization in closed loop can be achieved via a further feedback v if and only if $N_i(s)$ is Hurwitz. The previous argument is the core idea of assigning the dynamics of the system via feedback through cancellation of the stable zeros only. Accordingly, if $N(s)$ is not Hurwitz (i.e. $N_j(s)$ has positive real part zeros) the closed-loop system will still have non stable zeros that will play an important role in filtering actions but that will not affect closed-loop stability. Concluding, given any controllable linear system one can pursue stabilization in closed loop via partial zeros cancelation: starting from a suitable factorization of the polynomial defining the zeros, this is achieved via the definition a dummy output with respect to which the system is minimum phase.

IV. Continuous-time feedback linearization of partially minimum phase systems

In what follows, we show how the idea developed in the linear context can be settled in the one of feedback linearization of nonlinear dynamics of the form (1) that are not minimum phase in first approximation.

Lemma 4.1: Consider the nonlinear system (1) and suppose that its LTM at the origin is controllable in the form (2) and non minimum phase with relative degree r. Denote by $N(s) = b_0 + b_1 s + \dots b_{n-r}s^{n-r}$ the not Hurwitz polynomial identifying the zeros of the LTM of (1) at the origin. Consider the maximal factorization of $N(s) = N_1(s)N_2(s)$

$$N_i(s) = b_0^i + b_1^i s + \dots b_{n-r_i}^i s^{n-r_i}, \quad i = 1,2 \tag{4}$$

such that $N_2(s)$ is a Hurwitz polynomial of degree $n - r_2$. Then, the system

$$\dot{x} = f(x) + g(x)u, \quad y_2 = C_2 x. \tag{5}$$

$C_2 = \begin{pmatrix} b_0^2 & b_1^2 & \dots & b_{n-r_2}^2 & \mathbf{0} \end{pmatrix}$ has relative degree r_2 and is locally minimum-phase.

Proof: By computing the linear approximation at the origin of (5), one gets that the matrices (A, B, C_2) are in the form (2) so that the entries of C_2 are the coefficients of $N_2(s)$ that is the numerator of the corresponding transfer function. By construction, $N_2(s)$ is a Hurwitz polynomial of degree $n - r_2$. It follows that, in a nearby of the origin, the relative degree of (5) is r_2. Furthermore, since the linear approximation of the zero-dynamics of (5) coincides with the zero-dynamics of its LTM model at the origin, one gets that (5) is minimum-phase. ■

Lemma 4.2: Consider the nonlinear system (5) and introduce the normal-form associated to $h_2(x) = C_2 x$

$$\begin{pmatrix} \zeta \\ \eta \end{pmatrix} = \phi(x) = \begin{pmatrix} h_2(x) & \dots & L_f^{r_2-1} h_2(x) & \phi_2^\top(x) \end{pmatrix}^\top \quad (6)$$

with $\phi_2(x)$ such that $L_g \phi_2(x) = 0$ so that

$$\dot{\zeta} = \hat{A}\zeta + \hat{B}(b(\zeta,\eta) + a(\zeta,\eta)u) \quad (7a)$$

$$\dot{\eta} = q(\zeta,\eta) \quad (7b)$$

$$y_2 = \begin{pmatrix} 1 & \mathbf{0} \end{pmatrix} \zeta. \quad (7c)$$

Then, the feedback

$$u = \frac{1}{a(\zeta,\eta)}(v - a(\zeta,\eta)) \quad (8)$$

solves the Input-Output Linearization problem with stable zero-dynamics.

Proof: The proof is straightforward from construction of y_2 in Lemma 4.1. ■

Remark 4.1: We recall that, in the original coordinates, the feedback (8) rewrites as

$$u = \gamma(x,v) := \frac{v - \mathrm{L}_f^{r_2} h_2(x)}{\mathrm{L}_g \mathrm{L}_f^{r_2-1} h_2(x)}. \quad (9)$$

Remark 4.2: By invoking the arguments in Section III, the original output $y = Cx$ rewrites as $y = N_1(\mathrm{d}) y_2$.

Theorem 4.1: Consider the nonlinear system (1) and suppose that its LTM at the origin is controllable in the form (2) and non minimum phase with relative degree r. Define the dummy output $y_i = h_i(x) = C_i x$ $(i = 1, 2)$ as in Lemma 4.1 and the state transformation (6) that puts the system into the form

$$\dot{\zeta} = \hat{A}\zeta + \hat{B}(b(\zeta,\eta) + a(\zeta,\eta)u) \quad (10a)$$

$$\dot{\eta} = q(\zeta,\eta) \quad (10b)$$

$$y = N_1(\mathrm{d}) y_2. \quad (10c)$$

Then, the feedback (8) solves the input-output linearization problem with stability of the internal dynamics.

Proof: From Lemmas 4.1 and 4.2, by expliciting $y = N_1(\mathrm{d}) y_2$ and exploiting (6) one gets

$$y = b_0^1 y_2 + b_1^1 \dot{y}_2 + \dots + b_{r_2-r}^1 y_2^{(r_2-r)} = \begin{pmatrix} C_1 & \mathbf{0} \end{pmatrix} \zeta$$

so that in closed loop (1) rewrites as

$$\dot{\zeta} = \hat{A}\zeta + \hat{B}v \quad (11a)$$

$$\dot{\eta} = q(\zeta,\eta) \quad (11b)$$

$$y = \begin{pmatrix} C_1 & \mathbf{0} \end{pmatrix} \zeta \quad (11c)$$

that exhibits a linear input-output behavior. Moreover, by construction, $y_2 \equiv 0$ implies $y \equiv 0$ so that the restriction of the trajectories of (11) onto the manifold identified by $y \equiv 0$ is described by the dynamics $\dot{\eta} = q(0,\eta)$ that has a locally asymptotically stable equilibrium by construction. Accordingly, when setting $v = F\zeta$ so that $\sigma(\hat{A} + \hat{B}F) \subset \mathbb{C}^-$, the closed-loop system has an asymptotically stable equilibrium at the origin. ■

The previous result shows that even if a nonlinear system is non-minimum phase, a suitable partition of the output can be performed on its LTM at the origin so that feedback linearization of the input-output behavior can be pursued while preserving stability of the internal dynamics.

Remark 4.3: It is a matter of computations to verify that the LTM model of the closed-loop system (11) has transfer function $W(s) = \frac{N_1(s)}{s^{r_2}}$. Accordingly, one can interpret the nonlinear feedback (8) as the counterpart of the linear feedback presented in Section III; roughly speaking, when applying (8) to the original plant (1), one is inverting only the stable component of the zero-dynamics associated to y. As a consequence, as $y \to 0$, the trajectories of the closed-loop system are constrained onto the stable manifold associated to the dummy output $y_2 = C_2 x$ where they evolve according to $\dot{\eta} = q(0,\eta)$.

V. Feedback linearization of partially minimum phase systems under sampling

We now address the problem of preserving input-output linearization of (1) with stability under sampling by suitably exploiting the result in Theorem 4.1. As recalled in the introduction, the problem cannot be solved via standard (also known as single-rate) sampling procedures. In fact, considering $u(t) \in \mathscr{U}_\delta$ and $y(t) = y(k\delta)$ for $t \in [k\delta, (k+1)\delta[$ (δ the sampling period), the dynamics of (1) at the sampling instants is described by the single-rate sampled-data equivalent model

$$x_{k+1} = F^\delta(x_k, u_k), \quad y_k = h(x_k) \quad (12)$$

with $x_k := x(k\delta)$, $y_k := y(k\delta)$, $u_k := u(k\delta)$, $h(x) = Cx$ and $F^\delta(x_k, u_k) = e^{\delta(\mathrm{L}_f + u_k \mathrm{L}_g)} x\big|_{x_k}$. It is a matter of computations to verify that

$$y_{k+1} = h(x_k) + \sum_{i=1}^{r} \frac{\delta^i}{i!} \mathrm{L}_f^i h(x)\big|_{x_k} + \frac{\delta^r}{r!} u_k \mathrm{L}_g \mathrm{L}_f^r h(x)\big|_{x_k} + O(\delta^{r+1})$$

so that

$$\frac{\partial y_{k+1}}{\partial u_k} = \frac{\delta^r}{r!} \mathrm{L}_g \mathrm{L}_f^r h(x)\big|_{x_k} + O(\delta^{r+1}) \neq 0.$$

Thus, the relative degree of the sampled-data equivalent model of (1) is always falling to $r_d = 1$, despite the continuous-time one. As a consequence, whenever $r > 1$, the sampling process induces a further zero-dynamics of dimension $r-1$ (i.e., the so-called *sampling zero dynamics*, [9]) that is in general unstable for $r>1$. As a consequence, feedback linearization via single-rate sampling cannot be achieved while guaranteeing internal stability.

Multirate sampling enables us to preserve the relative degree and to avoid the appearance of the unstable sampling zero dynamics. Accordingly, one sets $u(t)=u_k^i$ for $t\in[(k+i-1)\delta,(k+i)\delta[$ for $i=1,\dots,r$ and $y(t)=y_k$ for $t\in[k\delta,(k+1)\delta[$ so that the multirate equivalent model of order r_2 of (1) gets the form

$$x_{k+1}=F_m^{\bar\delta}(x_k,u_k^1,\dots,u_k^{r_2}) \tag{13}$$

where $\bar\delta=\frac{\delta}{r_2}$ and

$$F_m^{\bar\delta}(x_k,u_k^1,\dots,u_k^{r_2})=e^{\bar\delta(\mathrm{L}_f+u_k^1\mathrm{L}_g)}\dots e^{\bar\delta(\mathrm{L}_f+u_k^{r_2}\mathrm{L}_g)}x\big|_{x_k}= F_m^{\bar\delta}(\cdot,u_k^{r_2})\circ\dots\circ F^{\bar\delta}(x_k,u_k^1).$$

In the sequel, we show how multirate feedback can be suitably employed with the arguments in Theorem 4.1 to achieve input-output linearization of (1) at the sampling instant $t=k\delta$ ($k\geq 0$) with stability regardless the minimum-phase property. Accordingly, we first design a multirate feedback $\mathbf{u}_k=\gamma(\bar\delta,x_k,\mathbf{v}_k)$ ($\mathbf{u}=\mathrm{col}(u^1,\dots,u^{r_2})$ and $\mathbf{v}=\mathrm{col}(v^1,\dots,v^{r_2})$) so to ensure input/output linearization of the $\mathbf{v}$-y_2 behavior of (5), at the sampling instants. This is achieved by considering the sampled-data dynamics (13) with augmented dummy output $Y_{2k}=H_2(x_k)$ composed of $y_2=C_2x$ and its first r_2-1 derivatives; namely, we consider

$$x_{k+1}=F_m^{\bar\delta}(x_k,u_k^1,\dots,u_k^{r_2}),\quad Y_{2k}=H_2(x_k) \tag{14}$$

with $\bar\delta=\frac{\delta}{r_2}$ and output vector

$$H_2(x)=\begin{pmatrix}h_2(x) & \mathrm{L}_f h_2(x) & \dots & \mathrm{L}_f^{r_2-1}h_2(x)\end{pmatrix}^\top$$

that has by construction a vector relative degree $r^\delta=(1,\dots,1)$.

In this Section we refer to ([19], [18]) where these concepts are introduced and similar manipulations detailed with analog motivations.

At first, we compute the feedback $\mathbf{u}_k=\gamma(\bar\delta,x_k,\mathbf{v}_k)$ so that to reproduce, at the sampling instants $t=k\delta$, the trajectories of the dummy output of (5) and of its first r_2-1 derivatives in closed loop under the continuous-time linearizing feedback (9). The existence of the sampled-data control is stated in the following result.

Lemma 5.1: Consider the nonlinear system (5) under the hypotheses of Lemma 4.2 with multirate equivalent model of order r_2 provided by (14). Then, there exists a unique solution

$$\mathbf{u}^{\bar\delta}=\gamma(\bar\delta,x,\mathbf{v})=(\gamma^1(\bar\delta,x,\mathbf{v})\ \dots\ \gamma^{r_2}(\bar\delta,x,\mathbf{v}))^\top \tag{15}$$

to the input-output Matching (I-OM) equality

$$H_2(F_m^{\bar\delta}(x_k,\gamma^1(\bar\delta,x_k,\mathbf{v_k}),\dots,\gamma^{r_2}(\bar\delta,x_k,\mathbf{v_k}))= e^{r_2\bar\delta(\mathrm{L}_f+\gamma(\cdot,v)\mathrm{L}_g)}H_2(x)\big|_{x_k} \tag{16}$$

for any $x_k=x(k\delta)$ and $v(t)=v(k\delta):=v_k$, $\mathbf{v_k}=(v_k,\dots,v_k)$. Such a solution is in the form of a series expansion in powers of $\bar\delta$ around the continuous-time $\gamma(x,v)$ in (9); i.e., for $i=1,\dots,r_2$

$$\gamma^i(\bar\delta,x,\mathbf{v})=\gamma(x,\mathbf{v})+\sum_{j\geq 1}\frac{\bar\delta}{(j+1)!}\gamma_j^i(x,\mathbf{v}). \tag{17}$$

As a consequence, the feedback $\mathbf{u}_k^\delta=\gamma(\bar\delta,x_k,\mathbf{v}_k)$ ensures Input-Output linearization of (14) with stability of the internal dynamics.

Proof: First, we rewrite (16) as a formal series equality in the unknown $\mathbf{u}^{\bar\delta}$; i.e.,

$$\begin{pmatrix}\bar\delta^{r_2}S_1^{\bar\delta}(x,\mathbf{u}^{\bar\delta}) & \dots & \bar\delta S_1^{\bar\delta}(x,\mathbf{u}^{\bar\delta})\end{pmatrix}^\top \tag{18}$$

with, for $i=1,\dots,r_2$,

$$\bar\delta^i S_i^\delta(x,\mathbf{u}^{\bar\delta})=e^{\bar\delta(\mathrm{L}_f+u^1\mathrm{L}_g)}\dots e^{\bar\delta(\mathrm{L}_f+u^1\mathrm{L}_g)}\mathrm{L}_f^{i-1}h_2(x) - e^{r_2\bar\delta(\mathrm{L}_f+\gamma(\cdot,v)\mathrm{L}_g)}\mathrm{L}_f^{i-1}h_2(x).$$

Thus one looks for $\mathbf{u}=\gamma(\delta,x,v)$ satisfying

$$S^{\bar\delta}(x,\mathbf{u}^{\bar\delta})=\begin{pmatrix}S_1^{\bar\delta}(x,\mathbf{u}^{\bar\delta}) & \dots & S_1^{\bar\delta}(x,\mathbf{u}^{\bar\delta})\end{pmatrix}^\top=\mathbf{0} \tag{19}$$

where each term rewrites as $S_i^\delta(x,\mathbf{u}^{\bar\delta})=\sum_{s\geq 0}\delta^s S_{ij}(x,\mathbf{u}^{\bar\delta})$ with

$$S_{i0}(x,\mathbf{u}^{\bar\delta})=\Big(\Delta_j\mathbf{u}^{\bar\delta}-r_2^{r_2-i+1}\gamma(x,v)\Big)\mathrm{L}_g\mathrm{L}_f^{r_2-1}h_2(x) \tag{20}$$

and $\frac{\Delta_j}{j!}=\Big(\frac{j^{r_2-j+1}-(j-1)^{r_2-j+1}}{j!}\ \frac{(j-1)^{r_2-j+1}-(j-2)^{r_2-j+1}}{j!}\ \dots\ \frac{1}{j!}\Big)$. It results that $\mathbf{u}^\delta=\gamma(\delta,x,v)=(\gamma(x,v),\dots,\gamma(x,v))^\top$ solves (19) as $\delta\to 0$. More precisely, as $\delta\to 0$, one gets the equation

$$S^{\bar\delta\to 0}(x,\mathbf{u}^{\bar\delta})=\Big(\Delta\mathbf{u}^{\bar\delta}-D\gamma(x,v)\Big)\mathrm{L}_g\mathrm{L}_f^{r_2-1}h_2(x)$$

with $\Delta=(\Delta_1^\top,\dots\Delta_{r_2}^\top)^\top$ and $D=\mathrm{diag}(r_2^{r_2},\dots,r_2)$. Furthemore, the Jacobian of $S^{\bar\delta}$ with respect to $\mathbf{u}^{\bar\delta}$ is

$$\nabla_{\mathbf{u}^{\bar\delta}}S^{\bar\delta}(x,(\gamma(x,v),\dots,\gamma(x,v))^\top)\big|_{\bar\delta\to 0}=\Delta\ \mathrm{L}_g\mathrm{L}_f^{r_2-1}h_2(x)$$

is full rank by definition of the continuous-time relative degree r_2 and because Δ is invertible (see [10] for details) so concluding, from the Implicit Function Theorem, the existence of $\delta\in]0,T^*[$ so that (16) admits a unique solution of the form (17) around the continuous-time solution $\gamma(x,v)$. Stability of the zero-dynamics is ensured by multirate sampling as proven in [10]. ■

The feedback control is in the form of a series expansion in powers of $\bar\delta$. Thus, iterative procedures can be carried out by substituting (17) into (16) and equating the terms with the same powers of $\bar\delta$ (see [19] where the explicit expression for the first terms are given). Unfortunately, only approximate solutions $\gamma^{[p]}(\bar\delta,x,v)$ can be implemented in practice through truncations of the series (17)) at finite order p in $\bar\delta$; namely, setting $\gamma^{[p]}(\bar\delta,x,v)=(\gamma^{1[p]}(\bar\delta,x,v),\dots,\gamma^{r_2[p]}(\bar\delta,x,v)$, one gets for $i=1,\dots,r_2$

$$\gamma^{i[p]}(\bar\delta,x,\mathbf{v})=\gamma(x,\mathbf{v})+\sum_{j=1}^{p}\frac{\bar\delta}{(j+1)!}\gamma_j^i(x,\mathbf{v}). \tag{21}$$

When $p=0$, one recovers the sample-and-hold (or emulated) solution $\gamma^{[0]}(\bar{\delta},x_k,\mathbf{v_k})=\gamma(x(k\delta),v(k\delta))$. Preservation of performances under approximate solutions has been discussed in [20] by showing that, although global asymptotic stability is lost, input-to-state stability (ISS) and practical global asymptotic stability can be deduced in closed loop even throughout the inter sampling instants.

Similarly to the continuous-time case, the next result shows that applying the feedback (15) to (1) ensures input-output linearization of the input-output behavior at any sampling instant $t=k\delta$ ($k\geq 0$) while preserving stability of the internal dynamics.

Theorem 5.1: Consider the nonlinear system (1) under the hypotheses of Theorem 4.1 with multirate equivalent model of order r_2 provided by

$$x_{k+1}=F_m^{\bar{\delta}}(x_k,u_k^1,\ldots,u_k^{r_2}),\quad y_k=\begin{pmatrix}C_1 & \mathbf{0}\end{pmatrix}H_2(x_k) \qquad (22)$$

and let the feedback (15) be the unique solution to the I-OM equality (16). Then the feedback $\mathbf{u}_k^{\delta}=\gamma(\bar{\delta},x_k,\mathbf{v}_k)$ ensures Input-Output linearization of (22) with stability of the internal dynamics.

Proof: We first note that y_k rewrites as a linear combination of Y_2. As a consequence, because the $\mathbf{v}$-Y_{2k} behavior is linear under (15), the $\mathbf{v}_k$-y_k is linear by construction. Moreover, we observe that $Y_2\equiv 0$ implies $y_k\equiv 0$ by definition. Thus, by construction of (15), as $y_k\to 0$, the closed-loop trajectories of (22) are forced onto the zero-manifold defined by $Y_2\equiv 0$ over which they are asymptotically stable. ■

Remark 5.1: Denote by z_i^c the zeros of the non Hurwitz polynomial $N_1(s)$ in Lemma 4.1. When considering the LTM model of (22) in closed loop under (15), one gets that, as $\bar{\delta}\to 0$, the closed-loop linearized system has exactly r_2-r zeros asymptotically approaching to the origin as $e^{\bar{\delta}z_i^c}$ (namely, as $\bar{\delta}\to 0$, $z_i^{\delta}\to e^{\bar{\delta}z_i^c}$, $i=1,\ldots,r_2$). Accordingly, by applying this result in the linear case, one gets that the feedback (15) is the one that assigns $n-r_2$ poles coincident with the stable zeros, without affecting the unstable ones.

Remark 5.2: Along the lines of the continuous-time case, when controlling (22) via the multirate feedback (15) one is constraining the trajectories of the closed-loop system onto the stable part of the zero-manifold identified by the non-minimum phase output.

Remark 5.3: A purely digital single-rate feedback might be computed over (12) by settling Lemma 4.1 to this context. Assuming, for simplicity, that (1) is locally minimum-phase, one might define a partition of the original output $y_k=Cx_k$ based on the numerator $N^{\delta}(z)$ of transfer function of its LTM at the origin. Accordingly, one might deduce $y_2^{\delta}=C_2^{\delta}x_k$ with respect to which the original dynamics has no sampling zero dynamics and the $y=N(q)y_2^{\delta}$ where q denotes the shift operator and $N(q)$ is the polynomial defining the sampling zeros of the LTM. Though, an exact partition of the original output is hard to be found and only approximate solutions can be found based on the concept of limiting sampling zeros ([8], [16])

VI. The TORA example

An academic working example is proposed on the basis of the TORA system described in [21] (Section 4.4.1, model (4.4.2)). In this context, we consider the fictitious output

$$y=(\frac{2}{\varepsilon}(\varepsilon^2-1)\ \ 0\ \ 1-\varepsilon^2\ \ 1-\varepsilon^2)x$$

with respect to which the system is non-minimum phase and has relative degree $r=1$. By applying first the coordinates transformation in Remark 2.1 and following the lines of Section IV, we define the partition $N_1(s)=s-1$ and $N_2(s)=s^2+2s+1$ so that, in the original coordinates, we define the dummy

$$y_2=(0\ \ -\frac{2}{\varepsilon}(\varepsilon^2-1)\ \ 1-\varepsilon^2\ \ 0)x$$

with respect to which the system is minimum-phase in first approximation and has relative degree $r_2=2$. Accordingly, by applying Theorem 4.1, the feedback (8) with

$$\begin{aligned}\mathrm{L}_g\mathrm{L}_f h_2(x)=&\frac{\varepsilon^2-1}{\varepsilon^2\cos^2(x_3)-1}\\ \mathrm{L}_f^2 h_2(x)=&\frac{2x_2(\varepsilon^2-1)}{\varepsilon}-2x_4\cos(x_3)(\varepsilon^2-1)+\\ &+\frac{\varepsilon\cos(x3)(\varepsilon^2-1)(x_1-\varepsilon\sin(x3)(x_4^2+1))}{\varepsilon^2\cos(x_3)^2-1}\end{aligned}$$

and $v=-k_1h_2(x)-k_2\mathrm{L}_fh_2(x)$ achieves local asymptotic stabilization in closed loop for $k_1,k_2>0$.

To solve the problem under sampling, the multirate feedback $\gamma^{[1]}(\delta,x,\mathbf{v})$ in (21) is computed with first corrective terms

$$\gamma_1^1(x,v)=\frac{1}{3}\dot{\gamma}(x,\mathbf{v}),\quad \gamma_1^2(x,v)=\frac{5}{3}\dot{\gamma}(x,v)$$

and $\dot{\gamma}(x,v)=(\mathrm{L}_f+\gamma(x,v)\mathrm{L}_g)\gamma(x,v)$.

Figures 1 and 2 depict simulations of the aforementioned situations under the continuous-time feedback (8) and the approximate sampled-data one (21) with $p=1$ and for different values of the sampling period. The sample and hold solution is reported as well in a comparative sense. In particular, setting by $\eta=(\eta_1,\ \eta_2,\ \eta_3)^{\top}$, we denote the internal dynamics corresponding to the simulated situations. It is clear from Figure 1 that the continuous-time feedback computed via partial dynamic inversion yields feedback linearization while ensuring asymptotic stability in cosed-loop. Concerning sampled-data control, we note that, as δ increases, the emulated based solution fails in stabilizing (and linearizing the input-output behavior) in closed loop while the presented multirate strategy yields more than acceptable performances even in that case.

VII. Conclusions

The notion of partially minimum-phase systems is used to get feedback input-output linearization while preserving stability. The proposed approach is introduced in continuous time and extended to the sampled-data context through multirate to overcome the well-known pathologies induced by the sampling zero dynamics. The extension to systems

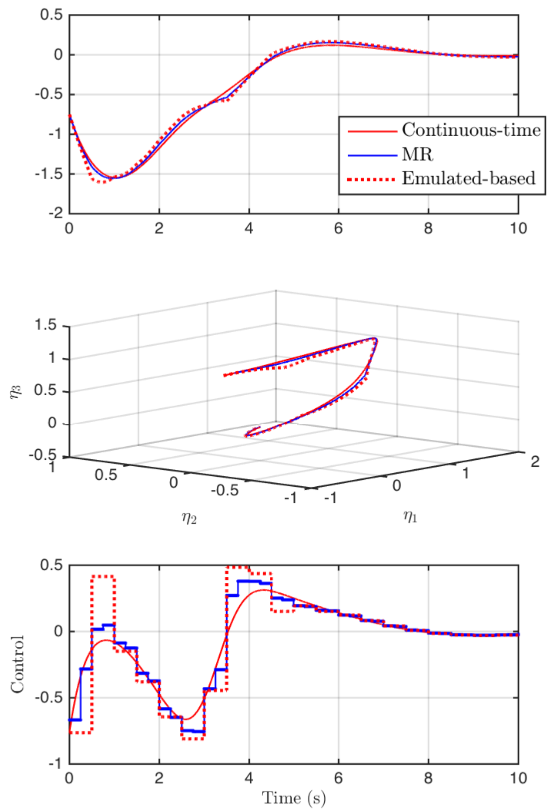

Fig. 1. $\delta = 0.5$ s.

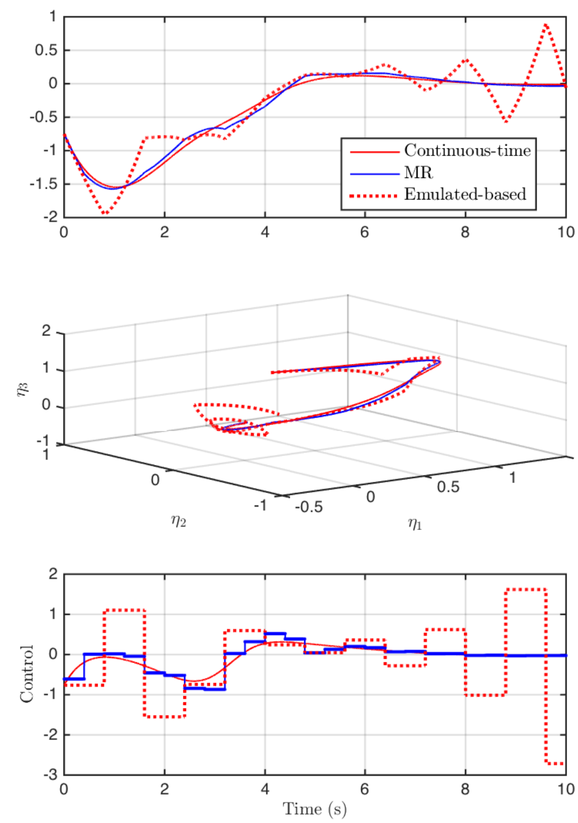

Fig. 2. $\delta = 0.7$ s.

exhibiting a nonlinear output mapping is the objective of further investigations.

References

[1] A. Isidori and C. I. Byrnes, "Output regulation of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 35, no. 2, pp. 131–140, Feb 1990.

[2] A. De Luca, *Zero Dynamics in Robotic Systems*. Boston, MA: Birkhäuser Boston, 1991, pp. 68–87.

[3] C. D. Persis and A. Isidori, "A geometric approach to nonlinear fault detection and isolation," *IEEE Transactions on Automatic Control*, vol. 46, no. 6, pp. 853–865, Jun 2001.

[4] R. Ortega, A. Van Der Schaft, F. Castanos, and A. Astolfi, "Control by interconnection and standard passivity-based control of port-hamiltonian systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 11, pp. 2527–2542, 2008.

[5] F. D. Priscoli, A. Isidori, L. Marconi, and A. Pietrabissa, "Leader-following coordination of nonlinear agents under time-varying communication topologies," *IEEE Transactions on Control of Network Systems*, vol. 2, no. 4, pp. 393–405, 2015.

[6] A. Astolfi, D. Karagiannis, and R. Ortega, *Nonlinear and adaptive control with applications*. Springer Publishing Company, 2008.

[7] A. Isidori, *Nonlinear Control Systems*. Springer-Verlag, 1995.

[8] K. Åström, P. Hagander, and J. Sternby, "Zeros of sampled systems," *Automatica*, vol. 20, no. 1, pp. 31 – 38, 1984.

[9] S. Monaco and D. Normand-Cyrot, "Zero dynamics of sampled nonlinear systems," *Systems & Control Letters*, vol. 11, no. 3, pp. 229 – 234, 1988.

[10] ——, "Multirate sampling and zero dynamics: from linear to nonlinear," in *Nonlinear Synthesis*. C. Byrnes, A. Isidori Eds., Birkauser, 1991, pp. 200–213.

[11] J. Yuz and G. C. Goodwin, *Sampled-data models for linear and nonlinear systems*. Springer-Verlag London, 2014.

[12] G. C. Goodwin, J. C. Aguero, M. E. C. Garridos, M. E. Salgado, and J. I. Yuz, "Sampling and sampled-data models: The interface between the continuous world and digital algorithms," *IEEE Control Systems*, vol. 33, no. 5, pp. 34–53, Oct 2013.

[13] P. Kabamba, "Control of linear systems using generalized sampled-data hold functions," *IEEE transactions on Automatic Control*, vol. 32, no. 9, pp. 772–783, 1987.

[14] T. Hagiwara, T. Yuasa, and M. Araki, "Stability of the limiting zeros of sampled-data systems with zero-and first-order holds," *International Journal of Control*, vol. 58, no. 6, pp. 1325–1346, 1993.

[15] M. Ishitobi, "Stability of zeros of sampled system with fractional order hold," *IEE Proceedings-Control Theory and Applications*, vol. 143, no. 3, pp. 296–300, 1996.

[16] D. S. Carrasco, G. C. Goodwin, and J. I. Yuz, "Modified euler-frobenius polynomials with application to sampled data modelling," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3972–3985, 2017.

[17] Y. Yamamoto, K. Yamamoto, and M. Nagahara, "Tracking of signals beyond the nyquist frequency," in *2016 IEEE 55th Conference on Decision and Control (CDC)*, Dec 2016, pp. 4003–4008.

[18] S. Monaco and D. Normand-Cyrot, "Issues on nonlinear digital systems," *Semi-Plenary Conference, European Journal of Control*, vol. 7, no. 2-3, pp. 160–178, 2001.

[19] ——, *On Nonlinear Digital Control*. Chapman & Hall, 1997, vol. 3, pp. 127–153.

[20] M. Mattioni, S. Monaco, and D. Normand-Cyrot, "Immersion and invariance stabilization of strict-feedback dynamics under sampling," *Automatica*, vol. 76, pp. 78 – 86, 2017.

[21] R. Sepulchre, M. Janković, and P. Kokotović, *Constructive nonlinear control*. Springer New York, 1997.