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Local and nonlocal Venttsel' problems in fractal domains

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Contents

Introduction

The aim of this dissertation is to study local and nonlocal Venttsel' problems in fractal domains. Venttsel' problems appear in different field, e.g. engineering problems of hydraulic fracturing, water wave theory, phase-transition phenomena, fluid diffusion, as well as models of heat transfer, some climate models or non-isothermal phase separation in a confined container; among the others, we refer to [14], [77], [48], [28], [34], [75], [27], [35] and the references listed in.
From the physical point of view, such as e.g. in the framework of heat propagation, it turns out that the boundary or the interface acts as a preferential fast absorbing trail for the heat stream. It could be important for industrial applications (see [79], [80]) to enhance the surface effects with respect to the surrounded volume, by increasing the surface (or the length); hence fractal boundaries and interfaces turn out to be a good tool.
From the mathematical point of view, a Venttsel' problem is described by an evolution equation in the bulk coupled with an evolution equation on the boundary where the operators appearing in the bulk and the boundary equations are of the same order.
These boundary conditions, known in literature also as *dynamical boundary conditions*, were first introduced by Venttsel' in his pioneering work of 1959 (see [88]); they are the most general ones in literature, since they include Dirichlet, Neumann and Robin boundary conditions.
There is a huge literature on Venttsel' problems in regular, irregular, and fractal domains, in both local and nonlocal case; for the local case, among the others we refer to [2], [3], [31], [4], [86], [84], [90], [58, 59, 60, 61, 62] and the references listed in, while for the nonlocal case we refer to [56], [85], [91], [87] and the references listed in.
In view of concrete applications to real world problems, such as the ones mentioned above, the ambitious aim of this thesis is to study both theoretically and numerically Venttsel' problems in 2D and 3D fractal domains, such as Koch-type domains, which are prototypes of irregular domains.
More precisely, in addition to the study of the "continuous" problem (P) at hand, we are also interested in considering its numerical approximation as well as in carrying

out some numerical simulations to interpret the physical results.
This goal is achieved in 3 steps.

i) We prove existence and uniqueness of the "weak" solution of the "continuous" problem (P) in the 2D or 3D fractal domain.

ii) We construct a sequence of smoother approximating problems (P_n) in the corresponding natural pre-fractal (polygonal or polyhedral) domains which are non-convex.
After proving existence and uniqueness results for the weak solution, we study the asymptotic behavior of the approximating solutions as n goes to infinity. This is a crucial step because there is a jump of dimension when passing from the pre-fractal boundaries (which have dimension 1 or 2) to the fractal boundary (which has an Hausdorff dimension greater than 1 or 2 respectively).

iii) For any fixed n, we consider the numerical approximation of problem (P_n) by a FEM in space and a finite difference scheme in time. In order to obtain a priori error estimates and an optimal rate of convergence, it is crucial to have a priori regularity results for the weak solution as well as a suitable mesh which takes into account the geometrical singularities of the domain. It is still an open problem also in the local case to prove that the "approximated numerical solution" does converge to the fractal one.

We point out that this is a long term project and the research is still ongoing.

Steps *i*)-*iii*) for the case of linear nonlocal Venttsel' problems in 2D Koch-type domains are dealt by in Chapters 2 and 3. The M-convergence of the energy forms in the nonlocal case is still an open problem.
The quasi-linear case is more delicate. Step *i*) is completely analyzed both for the 2D and 3D case in Chapters 4 and 5. Step *ii*) is analyzed only in the local case; the nonlocal case is object of further investigations, as well the regularity issues and hence its numerical approximation.
We now give a detailed description of the first two chapters in which we consider a nonlocal Venttsel' problem in a piecewise smooth domain Ω. For the reader's convenience we describe the nonlocal boundary condition.
The (parabolic) nonlocal linear Venttsel' boundary condition on $\partial\Omega$ is defined as follows:

$$\frac{\mathrm{d}u}{\mathrm{d}t} - \Delta_\ell u + \frac{\partial u}{\partial \nu} + bu + \theta_s(u) = f,$$

where u is the unknown of the problem, b and f are given function in suitable Banach spaces, $\frac{\partial u}{\partial \nu}$ is the outward normal derivative, Δ_ℓ is the piecewise tangential Laplace operator on $\partial\Omega$, and the nonlocal term $\theta_s(u)$, for $s \in (0,1)$, is a linear and continuous operator on $H^s(\partial\Omega)$ defined as a double integral on the boundary (see (2.1.1) below). The presence of the nonlocal term in the boundary condition, in the framework of heat flow, accounts for a non-constant conductivity $K(x,y)$ on the boundary which scales according to a certain law:

$$\frac{k^{-1}}{|x-y|^{N+2s}} \leq K(x,y) \leq \frac{k}{|x-y|^{N+2s}}$$

where $x, y \in \mathbb{R}^N$, $s \in (0,1)$ and $k > 1$ (see [9] for more details and for a probabilistic interpretation of the associated process).

It has to be pointed out that a nonlocal term already appears (in a different form) in the original paper of Venttsel'. In any case a nonlocal term is important in all those diffusion models in which one wants to emphasize the interaction between the boundary and the bulk such as e.g. in the diffusion of sprays in the lungs.

More precisely, in Chapter 2 we consider an elliptic equation for the Laplace operator in a two-dimensional piecewise smooth domain Ω coupled with a nonlocal linear Venttsel' boundary condition. We prove existence, uniqueness and regularity results for the weak solution in weighted Sobolev spaces, which will turn out essential in the next Chapter.

The regularity in weighted Sobolev spaces of the weak solution of local Venttsel' problems in pre-fractal domains has been dealt in [63]. The presence of the nonlocal term requires to adopt new tools. The techniques used deeply rely on the fact that the nonlocal term can be regarded as a sort of "regional" fractional Laplacian of order s. To our knowledge, these are the first regularity results obtained for nonlocal Venttsel' problems in piecewise smooth domains. We first prove a priori estimates, by means of the so-called *Munchhausen trick* (see Section 2.2), and only after proving the existence and uniqueness of the weak solution, we prove that it has the desired regularity.

In Chapter 3, we consider the numerical approximation of a heat equation with nonlocal Venttsel' boundary conditions in a Koch-type pre-fractal (non-convex) domain Ω_n, for $n \in \mathbb{N}$ fixed.

We prove existence and uniqueness of the weak (strict) solution via a semigroup approach. We perform the numerical approximation by mixed methods: we first approximate the problem in the space variable by using a finite element method and then we use a finite difference scheme in time. Since the domain is not convex, the lack of the usual H^2-regularity of the weak solution deteriorates the rate of convergence.

In order to obtain a priori error estimates and an optimal rate of convergence, it is crucial to obtain regularity results of the weak solution as well as an ad-hoc mesh refined near the singular points.

By following the approach developed by Grisvard [37] for non-convex plane domains in the case of the Laplace operator and by adapting the results of Chapter 2, we prove regularity results of the strict solution in weighted Sobolev spaces.

We use an ad-hoc mesh, satisfying the so-called *Grisvard conditions* (see Section 3.6) [21]. We then approximate the problem also in the time variable by a θ-method for $\frac{1}{2} \leq \theta \leq 1$. Thanks to the regularity results of Sections 3.2 and 3.4, we achieve the optimal rate of convergence of our numerical scheme, i.e. we obtain the same rate of convergence as in the case of convex domains as well as in the linear local case (see [20]).

We present also some preliminary numerical results. We study the heat flow across a pre-fractal boundary where the nonlocal term is active only on a portion of its. As shown in Figures 3.5 and 3.6, the nonlocal term is responsible of a larger heat flux in the part of the boundary where it is active. From the point of view of the applications, this fact, as well as the irregular geometry, turn out to be important to drain or increase the heat in *a priori* fixed areas. These simulations show that the presence of the nonlocal term actually helps the process of heat flow through the boundary.

In Chapters 4 and 5, we consider local quasi-linear Venttsel' boundary conditions on fractal (and pre-fractal) sets in the 2D and 3D case respectively, involving a "fractal" p-Laplace Beltrami-type operator on the boundary for $p \geq 2$.

In the two-dimensional case, the existence and uniqueness in the fractal case has been dealt in [56]. In Chapter 4, in view of numerical approximation, we consider also the associated pre-fractal problems (P_n), for every $n \in \mathbb{N}$ fixed. We give existence and uniqueness results for the solution u_n of problem (P_n) via a nonlinear semigroup approach. We then prove the convergence (in a suitable sense) of the sequence $\{u_n\}_{n\in\mathbb{N}}$ to the unique solution of the fractal problem (P).

More precisely, we denote by $\Omega \subset \mathbb{R}^2$ the bounded domain with boundary K the Koch snowflake and by Ω_n, for $n \in \mathbb{N}$, the natural corresponding approximating domains with boundary K_n (the n-th approximation of K). We introduce two suitable nonlinear energy functionals in the fractal and pre-fractal case. These functionals are the sum of a nonlinear p-energy term in the bulk, a nonlinear fractal p-energy term on the boundary plus lower order terms. These functionals are proper, lower semicontinuous and convex on $L^2(\Omega, m)$ and $L^2(\Omega, m_n)$ respectively, where m and m_n are the measures defined in (1.4.1) and (1.4.2) respectively. By a nonlinear semigroup approach, we get existence and uniqueness of the "strong" solution of the corresponding abstract homogeneous Cauchy problems, involving the subdifferentials of the nonlinear energy

functionals.
In order to study the asymptotic behavior of the pre-fractal solutions $\{u_n\}$, the natural setting is that of varying Hilbert spaces. To this end, we use the notion of *M-convergence*, first introduced by Mosco [71, 72] in the case of a linear energy form defined on a fixed Hilbert space; this notion was later generalized to the case of varying Hilbert spaces by Kuwae and Shioya [51], and then adapted to the case of nonlinear energy functionals by Tölle [82].
The proof of the M-convergence of the energy functionals is divided in two parts and it is quite technical, since it is based on delicate tools like *decimation* and *harmonic extensions*.
The M-convergence of the functionals is equivalent to the G-convergence of their subdifferentials which in turn is equivalent to the convergence of the nonlinear semigroup generated by the subdifferentials (we refer to the pioneering papers [5] and [12] for the case of a fixed Banach space).
At last, we prove that the solutions of the abstract Cauchy problems solve, in a suitable sense, a quasi-linear PDE with quasi-linear local Venttsel' boundary conditions.
We stress the fact that the Koch snowflake domain is a prototype of a domain with irregular boundary for which it is possible to construct a p-energy form. The construction of a p-energy form as limit of suitable discrete energy forms (defined on the pre-fractal sets) is possible only for the Koch curve and its variants. In the other cases, such as the Sierpinski Gasket, it is not possible due to the lack of an explicit value of the renormalization factor in the energy, which for the Koch curve is $\frac{4^{(p-1)n}}{p}$.
In Chapter 5 we consider the quasi-linear local Venttsel' problem in a three-dimensional "fractal" cylindrical domain Q having as lateral surface $S = K \times I$, where $I = [0, 1]$ (see Figure 1.3). We introduce a new quasi-linear p-energy functional E_S on the fractal "manifold" S, with domain $\mathcal{D}(S)$, as well as the nonlinear energy functionals associated with the Venttsel' problem both in the fractal and pre-fractal case.
Following the standard patterns in the proof of the M-convergence, one has to approximate the functions in the domain of the energy functional in terms of smoother functions.
In the 2D case, a key tool is the Hölder regularity of the functions in the domain of the energy functional on the boundary. In the 3D case, we prove new density results in order to achieve the convergence. This is obtained by using delicate tools, such as trace and extension theorems for Besov spaces on arbitrary closed subsets of $\mathbb{R}^3$.
We then consider the quasi-linear evolution Cauchy problems with nonlocal Venttsel' boundary conditions both in the fractal and pre-fractal case. We prove existence and uniqueness results by nonlinear semigroup theory. As in the 2D case, the M-convergence

of the functionals allows us to prove (in a suitable sense) the convergence of the solutions of the pre-fractal problems to the limit fractal one.

The plan of the thesis is the following.

Chapter 1 is devoted to preliminaries. We introduce the notations, the geometry and the relevant functional spaces, both on fractal and pre-fractal sets. We state trace results in both cases. We also recall some results on the convergence in varying Hilbert spaces. We then introduce nonlinear fractal energy forms, which appear in the energy functionals in Chapters 4 and 5.

In Chapter 2 we focus on a nonlocal linear elliptic Venttsel' problem in an arbitrary two-dimensional piecewise smooth domain. The results of this chapter are contained in [24].

In Section 2.1 we state the problem. In Section 2.2 we give an a priori estimate for the solution of the nonlocal linear Venttsel' problem. Finally, in Section 2.3 we give an existence and uniqueness result for the weak solution of the problem and we prove that the second derivatives of the solution belong to a suitable weighted Lebesgue space.

In Chapter 3 we consider a nonlocal linear parabolic Venttsel' problem in a fixed pre-fractal domain having boundary K_n. The results of this chapter are contained in [19].

In Section 3.1 we introduce the energy form and we prove some properties. In Section 3.2 we adapt the results of Section 2.2 to the parabolic problem, in order to obtain a suitable a priori estimate. In Section 3.3 we prove via standard semigroup theory the existence and uniqueness of the weak solution of the nonlocal linear parabolic Venttsel' problem, and we prove that it satisfies the a priori estimate stated in Section 3.2. In Section 3.4 we give other regularity results for the weak solution of the problem; in particular, we prove that it belongs to a suitable weighted Sobolev space, and moreover we prove that it is in particular continuous up to the boundary. In Section 3.5 we prove some a priori estimates which ensure the stability of our numerical scheme. In Section 3.6, we introduce a suitable family of triangulations of the domain and we perform the numerical approximation of the problem as explained above. We stress the fact that we obtain an optimal rate of convergence of our numerical method. In Section 3.7 we show some numerical simulations. It turns out that the nonlocal term has a key role in the process of heat flow through the pre-fractal boundary K_n.

In Chapter 4 we consider quasi-linear local Venttsel' problems in fractal and pre-fractal two-dimensional domains and we investigate the convergence of the pre-fractal solutions to the fractal one. The results of this chapter are contained in [25].

In Section 4.1 we introduce the quasi-linear energy functionals in both the fractal and pre-fractal case and we prove some properties. In Section 4.2 we prove the

M-convergence of the pre-fractal energy functionals to the fractal one. In Section 4.3 we consider abstract Cauchy problems involving the subdifferentials of the fractal and pre-fractal energy functionals, we present existence and uniqueness results for these problems, and we prove that the solutions solve quasi-linear local Venttsel' problems for the p-Laplace operator, respectively. Finally, we prove that the pre-fractal solutions converge to the limit fractal one.

In Chapter 5 we extend the results of Chapter 4 to the three-dimensional case of a cylindrical Koch-type domain $Q = \Omega \times I$ having as lateral surface $S = K \times I$, where $I = [0, 1]$. The results of this chapter are contained in [26].

In Section 5.1 we generalize the results of Section 1.2 to the 3D case. In Section 5.2 we introduce the energy functionals considered in the three-dimensional case. In Section 5.3 we prove some key density results, which will play a crucial role in the proof of the main result of Section 5.4. In Section 5.4, we prove the M-convergence of the pre-fractal energy functionals to the fractal one and we prove analogous existence, uniqueness and convergence results.

Chapter 1

Preliminaries

In this chapter we introduce some notions and some results which we will use throughout this thesis. In the first section we give some notions on fractal sets, and in particular on the domains we will consider in dimension two and three. In the second section we recall generalities on Sobolev spaces and we give some results on traces of functions on both the fractal and pre-fractal sets. In the third section we introduce Besov spaces on fractal sets. In the fourth section we present the notion of convergence of variable Hilbert spaces. In the fifth section, we give some generalities on nonlinear fractal energy forms.
Throughout this thesis, C will denote possibly different constants.

1.1 Fractal sets

1.1.1 The Koch snowflake

We denote by $P = (x_1, x_2)$ points in $\mathbb{R}^2$, by $|P - P_0|$ the Euclidean distance and by $B(P_0, r) = \{P \in \mathbb{R}^2 : |P - P_0| < r\}$, $P_0 \in \mathbb{R}^2, r > 0$ the Euclidean ball. By the Koch snowflake K, we will denote the union of three com-planar Koch curves (see [30]) K_1, K_2 and K_3. We assume that the junction points A_1, A_3 and A_5 are the vertices of a regular triangle with unit side length, i.e. $|A_1 - A_3| = |A_1 - A_5| = |A_3 - A_5| = 1$. K_1 is the uniquely determined self-similar set with respect to a family Ψ^1 of four contractions $\psi_1^{(1)}, ..., \psi_4^{(1)}$, where $\psi_i^{(1)} \colon \mathbb{C} \to \mathbb{C}$, $i = 1, \dots, 4$, with respect to the same Lipschitz constant $L = \frac{1}{3}$ (see [32]):

$$\psi_1^{(1)}(z) = \frac{z}{3}, \qquad \psi_2^{(1)}(z) = \frac{z}{3} e^{i\pi/3} + \frac{1}{3},$$

$$\psi_3^{(1)}(z) = \frac{z}{3} e^{-i\pi/3} + \frac{1}{2} + i\frac{\sqrt{3}}{6}, \quad \psi_4^{(1)}(z) = \frac{z+2}{3}.$$

Let $V_0^{(1)} := \{A_1, A_3\}$, $\psi_{i_1\dots i_n} := \psi_{i_1} \circ \cdots \circ \psi_{i_n}$, $V_{i_1\dots i_n}^{(1)} := \psi_{i_1\dots i_n}^{(1)}(V_0^{(1)})$ and

$$V_n^{(1)} := \bigcup_{i_1\dots i_n=1}^{4} V_{i_1\dots i_n}^{(1)}.$$

We set $i|n = (i_1, i_2, \dots, i_n)$, $V_\star^{(1)} := \cup_{n\geq 0} V_n^{(1)}$. It holds that $K_1 = \overline{V_\star^{(1)}}$. Now let K_0 denote the unit segment whose endpoints are A_1 and A_3. We set $K_{i_1\dots i_n} = \psi_{i_1\dots i_n}(K_0)$ and $V(K_{i_1\dots i_n}) = V_{i_1\dots i_n}$.
In a similar way, it is possible to approximate K_2 and K_3 by the sequences $\{V_n^{(2)}\}_{n\geq 0}$ and $\{V_n^{(3)}\}_{n\geq 0}$ respectively, and denote their limits by $V_\star^{(2)}$ and $V_\star^{(3)}$.
In order to approximate K, we define the increasing sequence of finite sets of points $\mathcal{V}^n := \cup_{i=1}^3 V_n^{(i)}$, $n \geq 1$ and $\mathcal{V}_\star := \cup_{n\geq 1} \mathcal{V}^n$. It holds that $\mathcal{V}_\star = \cup_{i=1}^3 V_\star^{(i)}$ and $K = \overline{V_\star}$.
The Hausdorff dimension of the Koch snowflake is given by $d_f = \frac{\ln 4}{\ln 3}$. This fractal is no longer self-similar and, hence, not nested.
One can define, in a natural way, a finite Borel measure μ supported on K by

$$\mu := \mu_1 + \mu_2 + \mu_3, \tag{1.1.1}$$

where μ_i denotes the normalized d_f-dimensional Hausdorff measure, restricted to K_i, $i = 1, 2, 3$. Further, for any $n \geq 1$, we define a discrete measure on $V_n^{(i)}$ by:

$$\mu_n^i := \frac{1}{4^n} \sum_{p\in V_n^{(i)}} \delta_{\{p\}}, \tag{1.1.2}$$

where $\delta_{\{p\}}$ denotes the Dirac measure at the point p. In [64], the following result is proved.

Proposition 1.1.1. *The sequence of measures $\{\mu_n^i\}_{n\geq 1}$ is weakly convergent to the measure μ_i.*

In the following we denote by

$$K_{n+1} = \bigcup_{i=1}^{3} K_i^{(n+1)} \tag{1.1.3}$$

the closed polygonal curve approximating K at the $(n+1)$-th step, where $K_i^{(n+1)}$ denotes the so-called pre-fractal (polygonal) curve approximating K_i.
We now recall the definition of *d-set*.

Definition 1.1.2. *A closed nonempty set $\mathcal{M} \subset \mathbb{R}^D$ is a d-set (for $0 < d \leq D$) if there exist a Borel measure $\tilde{\mu}$ with $\operatorname{supp}\tilde{\mu} = \mathcal{M}$ and two positive constants c_1 and c_2 such that*

$$c_1 r^d \leq \tilde{\mu}(B(P, r) \cap \mathcal{M}) \leq c_2 r^d \quad \forall\, P \in \mathcal{M}. \tag{1.1.4}$$

The measure $\tilde{\mu}$ is called a d-measure.

We remark that, from Definition 1.1.2, it follows that K is a d-set with $d = d_f$ and the measure μ is a d_f-measure (see [32]). Moreover, since μ is supported on K, it is not ambiguous to write $\mu(B(P,r))$ in place of $\mu(B(P,r) \cap K)$ in (1.1.4).

We remark that the Koch snowflake can be also regarded as a fractal manifold (see [32]).

Let Ω denote the (open) two-dimensional domain with boundary K, and for every integer $n \geq 1$ let Ω_n be the bounded non-convex pre-fractal polygonal domain approximating Ω with boundary K_n (the corresponding closed polygonal curves approximating K). We denote the vertices by P_j for $j = 1, \ldots, 3\mathcal{N}$, where $\mathcal{N} = 4^n$ and, for each fixed P_j, we denote by η_j the interior angle of Ω_n at P_j. We denote by M every side of the polygonal curve, by $\overset{\circ}{M}$ the corresponding open segment (i.e. the segment without its endpoints) and by $V(M)$ its vertices. Let $R = \{j = 1, \ldots, 3\mathcal{N} : \eta_j > \pi\}$. The set $\mathcal{Q} = \{P_j\}_{j \in R}$ is the subset of vertices whose angles are "reentrant" (see Figure 1.1).

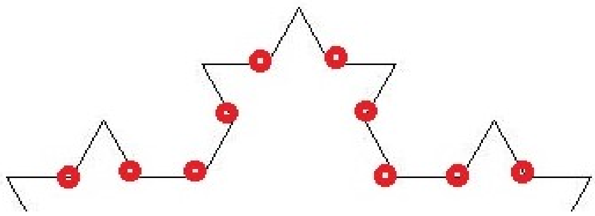

Figure 1.1: A zoom of the curve K_n with $n = 2$. The reentrant corners are circled in red.

We remark that the sequence $\{\Omega_n\}_{n \in \mathbb{N}}$ is an increasing sequence of sets exhausting Ω.

We denote by $\mathcal{T}$ the open equilateral triangle whose midpoints are the vertices A_1, A_3 and A_5 of K (see Figure 1.2).

1.1.2 The three-dimensional domain

In this section we introduce the three-dimensional domain which will be considered in Chapter 5.

Let Ω be the open bounded two-dimensional domain with boundary K introduced in the previous Section. By S we denote the cylindrical-type fractal surface

$$S = K \times I,$$

where $I = [0, 1]$.

We introduce on S the measure

$$\mathrm{d}g = \mathrm{d}\mu \times \mathrm{d}\mathcal{L}_1, \tag{1.1.5}$$

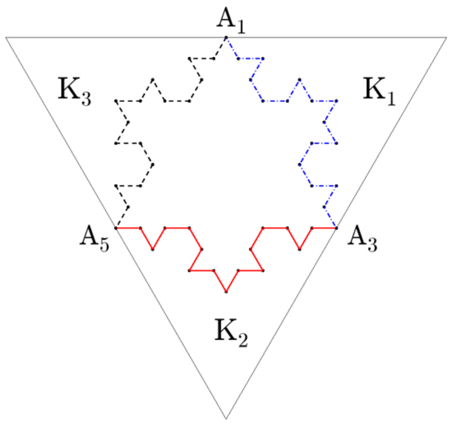

Figure 1.2: The Koch snowflake K.

where $\mathrm{d}\mathcal{L}_1$ is the one dimensional Lebesgue measure on I. The measure g is supported on S.
We point out that, in the sense of Definition 1.1.2, S is a (d_f+1)-set and the measure g is a (d_f+1)-measure.
By Q we denote the open cylindrical domain having S as "lateral surface" and the sets $\Omega \times \{0\}$ and $\Omega \times \{1\}$ as bases (see Figure 1.3).

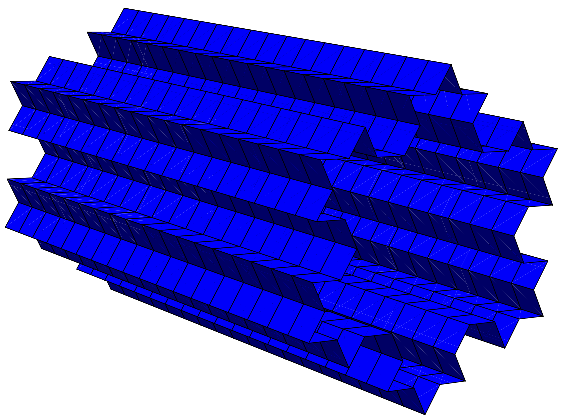

Figure 1.3: The fractal domain Q.

We denote by $P = (x, y) \in S$, where $x = (x_1, x_2)$ are the coordinates of the orthogonal projection of P on the plain containing K and y is the coordinate of the orthogonal projection of P on the interval $[0, 1]$, that is $(x_1, x_2) \in K$ and $y \in I$ (see Figure 1.4).
In a natural way, we define for every $n \in \mathbb{N}$ the approximating pre-fractal domains Q_n which are an increasing sequence exhausting Q. We also denote by $S_n = K_n \times I$ the

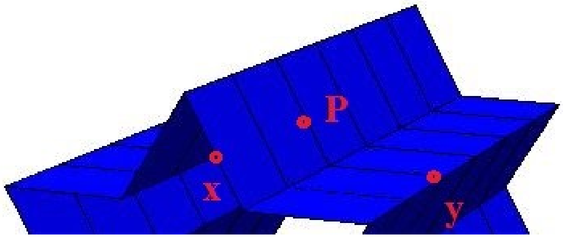

Figure 1.4: $P = (x, y)$: $x \in K$, $y \in I$.

lateral surface of Q_n, where K_n is the pre-fractal approximation of K introduced in Section 1.1.1. As in the fractal case, we denote points in S_n by the couple (x, y), where $x = (x_1, x_2) \in K_n$ and $y \in I$. In the notations of Section 1.1.1, we denote by $\mathcal{R}$ the open prism $\mathcal{T} \times [0, 1]$ with bases $\mathcal{T} \times \{0\}$ and $\mathcal{T} \times \{1\}$.

1.2 Sobolev spaces

By $L^p(\cdot)$ we denote the Lebesgue space with respect to the Lebesgue measure $d\mathcal{L}_2$ on subsets of $\mathbb{R}^2$, which will be left to the context whenever that does not create ambiguity. By $L^p(K)$ we denote the Banach space of p-summable functions on K with respect to the invariant measure μ. By ℓ we denote the natural arc length coordinate on each edge of K_n and we introduce the coordinates $x_1 = x_1(\ell)$, $x_2 = x_2(\ell)$, on every segment M of K_n.
Given $\mathcal{S}$ a closed set of $\mathbb{R}^2$, by $C(\mathcal{S})$ we denote the space of continuous functions on $\mathcal{S}$. Let $\mathcal{G}$ be an open set of $\mathbb{R}^2$. By $D(\mathcal{G})$ we denote the space of infinitely differentiable functions with compact support on $\mathcal{G}$. By $W^{s,p}(\mathcal{G})$, where $s \in \mathbb{R}^+$, we denote the usual (possibly fractional) Sobolev spaces (see [74]); $W^{s,p}_0(\mathcal{G})$ is the closure of $D(\mathcal{G})$ with respect to the $\|\cdot\|_{W^{s,p}}$-norm. $W^{-s,p'}(\mathcal{G})$ denotes the dual space of $W^{s,p}_0(\mathcal{G})$.
In the following, we will make use of trace spaces on boundaries of polygonal domains of $\mathbb{R}^2$. We give a characterization of Sobolev spaces on polygonal domains which is the most useful for our aim. By $W^{s,p}(K_n)$, for $s \geq 1$, we denote (see [13]) the set

$$\{u \in C(K_n) : u|_{\overset{\circ}{M}} \in W^{s,p}(\overset{\circ}{M})\}.$$

In the sequel, in the case $s = 1$, we consider $W^{1,p}(K_n)$ with the norm

$$\|u\|_{W^{1,p}(K_n)} = \left(\|u\|^p_{L^p(K_n)} + \|\mathrm{D}u\|^p_{L^p(K_n)}\right)^{\frac{1}{p}}.$$

By $W^{s,p}(K_n)$, for $0 < s \leq 1$, we denote the Sobolev space on K_n, defined by local Lipschitz charts as in [74]. We point out that, for $s = 1$, the two definitions coincide with equivalent norms.

In the following we will denote by $|A|$ the Lebesgue measure of a subset $A \subset \mathbb{R}^N$. For f in $W^{s,p}(\mathcal{G})$, we define the trace operator γ_0 as

$$\gamma_0 f(P) := \lim_{r \to 0} \frac{1}{|B(P,r) \cap \mathcal{G}|} \int\limits_{B(P,r) \cap \mathcal{G}} f(Q) \, \mathrm{d}\mathcal{L}_2 \tag{1.2.1}$$

at every point $P \in \overline{\mathcal{G}}$ where the limit exists. It is known that the limit (1.2.1) exists at *quasi every* $P \in \overline{\mathcal{G}}$ with respect to the (s,p)-capacity [1].
We now recall the results of Theorem 2.24 in [13], referring to [36] for a more general discussion.

Proposition 1.2.1. *Let Ω_n and K_n be as above and let $s > \frac{1}{p}$, with $s - \frac{1}{p}$ not integer. Then $W^{s-\frac{1}{p},p}(K_n)$ is the trace space to K_n of $W^{s,p}(\Omega_n)$ in the following sense:*

(i) *γ_0 is a continuous and linear operator from $W^{s,p}(\Omega_n)$ to $W^{s-\frac{1}{p},p}(K_n)$,*

(ii) *there is a continuous linear operator* Ext *from $W^{s-\frac{1}{p},p}(K_n)$ to $W^{s,p}(\Omega_n)$, such that $\gamma_0 \circ \mathrm{Ext}$ is the identity operator in $W^{s-\frac{1}{p},p}(K_n)$.*

In the sequel we denote by the symbol $f|_{K_n}$ the trace $\gamma_0 f$ to K_n. Sometimes we will omit the trace subscript and the interpretation will be left to the context.
The following theorem characterizes the trace on the polygonal K_n of a function belonging to the Sobolev space $W^{\beta,p}(\mathbb{R}^2)$ (for the definitions and the main properties of Sobolev spaces, see [1]).

Theorem 1.2.2. *Let K_n denote $\partial\Omega_n$. Let $u \in W^{\beta,p}(\mathbb{R}^2)$ for $\beta > 0$ and $\delta_n = (\frac{3}{4})^n = (3^{1-d_f})^n$. Then, for $\frac{1}{p} < \beta \leq \frac{2}{p}$,*

$$\|u\|^p_{L^p(K_n)} \leq \frac{C_\beta}{\delta_n} \|u\|^p_{W^{\beta,p}(\mathbb{R}^2)}, \tag{1.2.2}$$

where C_β is independent of n.

Proof. We point out that every $u \in W^{\beta,p}(\mathbb{R}^2)$ can be expressed in the following way:

$$u = G_\beta * g, \quad g \in L^p(\mathbb{R}^2), \quad \text{with } \|u\|_{W^{\beta,p}(\mathbb{R}^2)} = \|g\|_{L^p(\mathbb{R}^2)},$$

where G_β is the Bessel kernel of order β (see [43]). Then by Hölder inequality we have

$$\|u\|^p_{L^p(K_n)} = \int\limits_{K_n} |u|^p \, \mathrm{d}\ell = \int\limits_{K_n} \left| \int\limits_{\mathbb{R}^2} G_\beta(x-y) g(y) \, \mathrm{d}y \right|^p \mathrm{d}\ell \leq$$

$$\int\limits_{K_n} \left(\int\limits_{\mathbb{R}^2} |G_\beta(x-y)|^{ap} \, |g(y)|^p \, \mathrm{d}y \right) \left(\int\limits_{\mathbb{R}^2} |G_\beta(x-y)|^{(1-a)p'} \, \mathrm{d}y \right)^{\frac{p}{p'}} \mathrm{d}\ell,$$

where $0 < a < 1$ will be chosen later. Now, by using Lemma 1 on page 104 in [43], we get

$$\int_{\mathbb{R}^2} |G_\beta(x-y)|^{(1-a)p'} \, \mathrm{d}y \le C_1,$$

with C_1 independent of n, if

$$(2-\beta)(1-a)p' < 2. \tag{1.2.3}$$

Moreover, since K_n is a 1-set with constant $c_2 = C_3\,\delta_n^{-1}$ (see (1.1.4)), again from Lemma 1 on page 104 in [43] we get

$$\int_{K_n} |G_\beta(x-y)|^{ap} \, \mathrm{d}\ell \le C_4\,\delta_n^{-1},$$

with C_4 again independent of n, if

$$(2-\beta)ap < 1. \tag{1.2.4}$$

Hence, by choosing a in order to satisfy (1.2.3) and (1.2.4), it has to be

$$1 - \frac{2}{(2-\beta)p'} < a < \frac{1}{(2-\beta)p}$$

and, by imposing that $0 < a < 1$ and that $1 - \dfrac{2}{(2-\beta)p'} < \dfrac{1}{(2-\beta)p}$ we get the desired bounds on β.

Hence, by using Fubini's Theorem we get

$$\|u\|^p_{L^p(K_n)} \le C_1 \int_{K_n} \left(\int_{\mathbb{R}^2} |G_\beta(x-y)|^{ap}\, |g(y)|^p \, \mathrm{d}y \right) \mathrm{d}\ell =$$

$$C_1 \int_{\mathbb{R}^2} \left(\int_{K_n} |G(x-y)|^{ap} \, \mathrm{d}\ell \right) |g(y)|^p \, \mathrm{d}y \le C_1\, C_4\, \delta_n^{-1} \|g\|^p_{L^p(\mathbb{R}^2)} = C_\beta\, \delta_n^{-1} \|u\|^p_{W^{\beta,p}(\mathbb{R}^2)},$$

where C_β is a constant independent of n. $\square$

In order to introduce the notion of trace on suitable fractal sets, we give the definition of (ε,δ) *domain* (see [41]).

Definition 1.2.3. *Let $\mathcal{F} \subset \mathbb{R}^m$ be open and connected. For $x \in \mathcal{F}$, let $d(x) := \inf\limits_{y \in \mathcal{F}^c} |x-y|$. We say say that $\mathcal{F}$ is an (ε,δ) domain if, whenever $x, y \in \mathcal{F}$ with $|x-y| < \delta$, there exists a rectifiable arc $\gamma \in \mathcal{F}$ joining x to y such that*

$$\ell(\gamma) \le \frac{1}{\varepsilon}|x-y| \quad \text{and} \quad d(z) \ge \frac{\varepsilon |x-z||y-z|}{|x-y|} \text{ for every } z \in \gamma.$$

The following theorem is a consequence of Theorem 1 in Chapter V of [43] as the fractal K is a d-set.

Theorem 1.2.4. *Let $u \in W^{\beta,p}(\mathbb{R}^2)$. Then, for $\frac{2-d_f}{p} < \beta$,*

$$\|u\|^p_{L^p(K)} \leq C^*_\beta \|u\|^p_{W^{\beta,p}(\mathbb{R}^2)}. \tag{1.2.5}$$

It is possible to prove that the domains Ω_n are (ε, δ) domains with parameters ε and δ independent of the (increasing) number of sides of K_n. Thus by the extension theorem for (ε, δ) domains due to Jones (Theorem 1 in [41]) we obtain the following Theorem 1.2.5, which provides an extension operator from $W^{1,p}(\Omega_n)$ to the space $W^{1,p}(\mathbb{R}^2)$ whose norm is independent of n.

Theorem 1.2.5. *There exists a bounded linear extension operator* $\mathrm{Ext}_J : W^{1,p}(\Omega_n) \to W^{1,p}(\mathbb{R}^2)$ *such that*

$$\|\mathrm{Ext}_J\, v\|^p_{W^{1,p}(\mathbb{R}^2)} \leq C_J \|v\|^p_{W^{1,p}(\Omega_n)} \tag{1.2.6}$$

with C_J independent of n.

We now present an extension theorem for fractional Sobolev spaces $W^{\beta,p}(\Omega)$. We observe that if $0 < \beta < 1$, estimate (1.2.7) can be deduced from Theorem 1 on page 103 in [43] (see also Theorem 3 on page 155 in [43]).

Theorem 1.2.6. *There exists a linear extension operator $\mathcal{E}\mathrm{xt}$ such that, for any $\beta > 0$ $\mathcal{E}\mathrm{xt} : W^{\beta,p}(\Omega) \to W^{\beta,p}(\mathbb{R}^2)$,*

$$\|\mathcal{E}\mathrm{xt}\, v\|^p_{W^{\beta,p}(\mathbb{R}^2)} \leq \bar{C}_\beta \|v\|^p_{W^{\beta,p}(\Omega)} \tag{1.2.7}$$

with $\bar{C}_\beta$ depending on β.

We now recall the Friedrichs inequality, see [69, page 24] for more details.

Proposition 1.2.7. *Let $u \in H^1(\Omega_n)$. There exists a positive constant C such that*

$$\|u\|^2_{L^2(\Omega_n)} \leq C \left(\|\mathrm{D}u\|^2_{L^2(\Omega_n)} + \|u\|^2_{L^2(K_n)} \right). \tag{1.2.8}$$

To conclude this section, we introduce weighted Sobolev spaces following [73].

Let $r = r(x)$ be the distance from the set of vertices. For $\gamma \in \mathbb{R}$ and $s = 0, 1, 2, \ldots$, we denote by $H^s_\gamma(\Omega_n)$ the weighted Sobolev space of functions such that the norm

$$\|u\|_{H^s_\gamma(\Omega_n)} = \left(\sum_{|\alpha| \leq s} \int_{\Omega_n} r^{2(\gamma - s + |\alpha|)} |\mathrm{D}^\alpha u(x)|^2 \, \mathrm{d}\mathcal{L}_2 \right)^{\frac{1}{2}}$$

is finite, and, for $s > 0$ integer, by $H^{s-\frac{1}{2}}_{\gamma}(K_n)$ the trace space of $H^s_\gamma(\Omega_n)$ equipped with the norm

$$\|u\|_{H^{s-\frac{1}{2}}_{\gamma}(K_n)} = \inf_{v=u \,\mathrm{on}\, K_n} \|v\|_{H^s_\gamma(\Omega_n)}.$$

For the details see (2.17), Chapter 2 in [73]. We introduce also the weighted Lebesgue space $L^2_\gamma(\Omega_n)$, for $\gamma \in \mathbb{R}$, as the space of functions for which the norm

$$\|u\|_{L^2_\gamma(\Omega_n)} = \left(\int_{\Omega_n} |u|^2 \, r^{2\gamma} \, \mathrm{d}\mathcal{L}_2 \right)^{\frac{1}{2}}$$

is finite. We point out that this space coincides with the space $H^0_\gamma(\Omega)$. We define also, for $\sigma \in \mathbb{R}$, the composite space

$$V^2_\sigma(\Omega_n, K_n) := \{u \in H^1(\Omega_n) \, : \, r^\sigma \mathrm{D}^2 u \in L^2(\Omega_n), \, \gamma_0 u \in H^2(K_n)\}. \tag{1.2.9}$$

1.3 Besov spaces

We now come to the definition of the Besov spaces $B^{p,p}_\alpha$ with α positive and non-integer (see [83] and [43]). Let $\mathcal{S}$ be a d-set in $\mathbb{R}^D$, $\alpha > 0$ non integer, $k = [\alpha]$ the integer part of α, j a D-dimensional multi-index of length $|j| \leq k$.
If f and $\{f^{(j)}\}$ are functions defined $\tilde{\mu}$–a.e. on $\mathcal{S}$, we set

$$R_j(P, P') = f^{(j)}(P) - \sum_{|j+i| \leq k} \frac{f^{(j+i)}(P')}{i!} (P - P')^i \ ,$$

where $f^{(0)} = f$ and i denotes a D-dimensional multi-index.

Definition 1.3.1. *We say that $f \in B^{p,p}_\alpha(\mathcal{S})$ if there exists a family $\{f^{(j)}\}$ with $|j| \leq k$, as above, such that $f^{(j)} \in L^p(\mathcal{S}, \tilde{\mu})$ and $\|\{a_n\}\|_{l_p} < \infty$ where a_n is the smallest number such that*

$$\left(3^{nd} \int \int_{|P-P'| < 3^{-n}} |R_j(P, P')|^p \mathrm{d}\tilde{\mu}(P) \, \mathrm{d}\tilde{\mu}(P') \right)^{1/p} \leq 3^{-n(\alpha - |j|)} a_n \ .$$

The norm of f in $B^{p,p}_\alpha(\mathcal{S})$ is

$$\|f\|_{B^{p,p}_\alpha(\mathcal{S})} = \|f\|_{L^p(\mathcal{S}, \tilde{\mu})} + \|\{a_n\}\|_{l_p} \ .$$

The family $\{f^{(j)}\}$ in the previous definition is uniquely determined by f, as shown in [43], for d-sets with $d > D - 1$.

Let us note that for $0 < \alpha < 1$ the norm $\|f\|_{B^{p,p}_\alpha(\mathcal{S})}$ can be written as

$$\|f\|_{L^p(\mathcal{S},\tilde{\mu})} + \left(\iint\limits_{|P-P'|<1} \frac{|f(P) - f(P')|^p}{|P-P'|^{d+p\alpha}} \, \mathrm{d}\tilde{\mu}(P) \, \mathrm{d}\tilde{\mu}(P') \right)^{1/p}.$$

We now state the trace theorem specialized to our case.

Proposition 1.3.2. *Let K be the Koch snowflake. Let $s > \frac{2-d}{p}$ and $s - \frac{2-d}{p} \notin \mathbb{N}$. Then $B^{p,p}_{s-\frac{2-d}{p}}(K)$ is the trace space to K of $W^{s,p}(\Omega)$ in the following sense:*

- (i) *γ_0 is a continuous linear operator from $W^{s,p}(\Omega)$ to $B^{p,p}_{s-\frac{2-d}{p}}(K)$,*
- (ii) *there is a continuous linear operator* Ext *from $B^{p,p}_{s-\frac{2-d}{p}}(K)$ to $W^{s,p}(\Omega)$ such that $\gamma_0 \circ \mathrm{Ext}$ is the identity operator in $B^{p,p}_{s-\frac{2-d}{p}}(K)$.*

For the proof we refer to Theorem 1 of Chapter VII in [43], see also [83].

From Proposition 1.3.2 it follows that when $s = 1$ the trace space of $W^{1,p}(\Omega)$ is $B^{p,p}_{1-\frac{2-d_f}{p}}(K)$.
In the sequel we denote by the symbol $f|_K$ the trace $\gamma_0 f$ to K. For the sake of simplicity we will omit the subscript.
In the following, we denote the dual of the Besov space on K with $(B^{p,p}_\alpha(K))'$. In [44] the authors proved, in the general framework of d-sets, that the space $(B^{p,p}_\alpha(K))'$ coincides with the space $B^{p',p'}_{-\alpha}(K)$, where p' is the conjugate exponent of p.

1.4 Convergence of Hilbert spaces

We introduce the notion of convergent Hilbert spaces that we will use in the next sections. For further details and proofs of the theorems see [51] and [46].
The Hilbert spaces we consider are real and separable.

Definition 1.4.1. *A sequence of Hilbert spaces $\{H_n\}_{n\in\mathbb{N}}$ converges to a Hilbert space H if there exists a dense subspace $C \subset H$ and a sequence $\{Z_n\}_{n\in\mathbb{N}}$ of linear operators $Z_n\colon C \subset H \to H_n$ such that*

$$\lim_{n\to\infty} \|Z_n u\|_{H_n} = \|u\|_H \textit{ for any } u \in C.$$

We define the space $\mathcal{H} = \{\cup_n H_n\} \cup H$ and define strong and weak convergence in $\mathcal{H}$.
From now on we assume $\{H_n\}_{n\in\mathbb{N}}$, H and $\{Z_n\}_{n\in\mathbb{N}}$ are as in Definition 1.4.1.

Definition 1.4.2 (Strong convergence in $\mathcal{H}$)**.** *A sequence of vectors* $\{u_n\}_{n\in\mathbb{N}}$ *strongly converges to* u *in* $\mathcal{H}$ *if* $u_n \in H_n$, $u \in H$ *and there exists a sequence* $\{\widetilde{u}_m\}_{m\in\mathbb{N}} \in C$ *tending to* u *in* H *such that*

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \|Z_n \widetilde{u}_m - u_n\|_{H_n} = 0$$

Definition 1.4.3 (Weak convergence in $\mathcal{H}$)**.** *A sequence of vectors* $\{u_n\}_{n\in\mathbb{N}}$ *weakly converges to* u *in* $\mathcal{H}$ *if* $u_n \in H_n$, $u \in H$ *and*

$$(u_n, v_n)_{H_n} \to (u, v)_H$$

for every sequence $\{v_n\}_{n\in\mathbb{N}}$ *strongly tending to* v *in* $\mathcal{H}$.

Remark 1.4.4. We note that the strong convergence implies the weak convergence (see [51]).

Lemma 1.4.5. *Let* $\{u_n\}_{n\in\mathbb{N}}$ *be a sequence weakly converging to* u *in* $\mathcal{H}$. *Then*

$$\sup_{n\to\infty} \|u_n\|_{H_n} < \infty, \quad \|u\|_H \leq \underline{\lim}_{n\to\infty} \|u_n\|_{H_n}.$$

Moreover, $u_n \to u$ *strongly if and only if* $\|u\|_H = \lim_{n\to\infty} \|u_n\|_{H_n}$.

Let us recall some characterizations of the strong convergence of a sequence of vectors $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$.

Lemma 1.4.6. *Let* $u \in H$ *and let* $\{u_n\}_{n\in\mathbb{N}}$ *be a sequence of vectors* $u_n \in H_n$. *Then* $\{u_n\}_{n\in\mathbb{N}}$ *strongly converges to* u *in* $\mathcal{H}$ *if and only if*

$$(u_n,\ v_n)_{H_n} \to (u, v)_H$$

for every sequence $\{v_n\}_{n\in\mathbb{N}}$ *with* $v_n \in H_n$ *weakly converging to a vector* v *in* $\mathcal{H}$.

Lemma 1.4.7. *A sequence of vectors* $\{u_n\}_{n\in\mathbb{N}}$ *with* $u_n \in H_n$ *strongly converges to a vector* u *in* $\mathcal{H}$ *if and only if*

$$\begin{aligned} \|u_n\|_{H_n} &\to \|u\|_H && \textit{and} \\ (u_n,\ Z_n(\varphi))_{H_n} &\to (u, \varphi)_H && \textit{for every } \varphi \in C. \end{aligned}$$

Lemma 1.4.8. *Let* $\{u_n\}_{n\in\mathbb{N}}$ *be a sequence with* $u_n \in H_n$. *If* $\|u_n\|_{H_n}$ *is uniformly bounded, then there exists a subsequence of* $\{u_n\}_{n\in\mathbb{N}}$ *which weakly converges in* $\mathcal{H}$.

Lemma 1.4.9. *For every $u \in H$ there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$, $u_n \in H_n$ strongly converging to u in $\mathcal{H}$.*

We now define the G-convergence of operators (see Definition 7.20 in [82]).

Definition 1.4.10. *Let $n \in \mathbb{N}$, $A_n \colon H_n \to 2^{H_n}$, $A \colon H \to 2^H$ be multivalued operators. We say that A_n G-converges to A, $A_n \xrightarrow{G} A$, if for every $[x, y] \in A$ (i.e. $x \in D(A)$ and $y \in A(x)$) there exists $[x_n, y_n] \in A_n$, $n \in \mathbb{N}$ such that $x_n \to x$ and $y_n \to y$ strongly in $\mathcal{H}$.*

In the following we denote by $L^2(\bar{\Omega}, m)$ the Lesbegue space with respect to the measure m with

$$dm = d\mathcal{L}_2 + d\mu, \tag{1.4.1}$$

and by the space $L^2(\Omega, m_n)$ the Lebesgue space with respect to the measure m_n with

$$dm_n = \chi_{\Omega_n} d\mathcal{L}_2 + \chi_{K_n} \delta_n d\ell, \tag{1.4.2}$$

where χ_{Ω_n} and χ_{K_n} denote the characteristic function of Ω_n and K_n respectively. Throughout this thesis we consider $H = L^2(\bar{\Omega}, m)$ where m is the measure in (1.4.1), and the sequence $\{H_n\}_{n\in\mathbb{N}}$ with $H_n = \{L^2(\Omega) \cap L^2(\Omega, m_n)\}$ where m_n is the measure in (1.4.2) with norms

$$\|u\|_H^2 = \|u\|^2_{L^2(\Omega)} + \|u|_K\|^2_{L^2(K,\mu)}, \quad \|u\|^2_{H_n} = \|u\|^2_{L^2(\Omega_n)} + \|u|_{K_n}\|^2_{L^2(K_n,\delta_n\ell)}.$$

Proposition 1.4.11. *Let $\delta_n = \left(\frac{3}{4}\right)^n$. Then the sequence $\{H_n\}_{n\in\mathbb{N}}$ converges in the sense of Definition 1.4.1 to H.*

For the proof, see Proposition 4.1 in [63], where C and Z_n in Definition 1.4.1 are respectively $C(\overline{\Omega})$ and the identity operator on $C(\overline{\Omega})$.

1.5 Nonlinear fractal energy forms

For $f : V_\star^{(i)} \to \mathbb{R}$, $i = 1, 2, 3$, we define for $1 < p < \infty$ and $n \in \mathbb{N}$,

$$\mathcal{E}^{(n)}_{p,i}[f] = \frac{1}{p} 4^{(p-1)n} \sum_{i_1,\dots,i_n=1}^{4} \sum_{\xi,\eta \in V_0^{(i)}} |f(\psi_{i_1\dots i_n}(\xi)) - f(\psi_{i_1\dots i_n}(\eta))|^p, \tag{1.5.1}$$

and

$$\mathcal{E}^{(n)}_p[f] = \sum_{i=1}^{3} \mathcal{E}^{(n)}_{p,i}[f]. \tag{1.5.2}$$

We note that the form $\mathcal{E}^{(n)}_p$ in (1.5.2) can be also written as

$$\mathcal{E}_p^{(n)}[f] = \frac{4^{(p-1)n}}{p} \sum_{M \in K_n} \sum_{r,s \in V(M)} |f(r) - f(s)|^p. \tag{1.5.3}$$

It has been shown in [16] that the sequence $\mathcal{E}_p^{(n)}[f]$ is non-decreasing; by defining for $f : V_\star^{(i)} \to \mathbb{R}$

$$\mathcal{E}_p^{(i)}[f] = \lim_{n\to\infty} \mathcal{E}_{p,i}^{(n)}[f], \tag{1.5.4}$$

the set

$$\mathcal{F}_{\star,i}^{(p)} = \{f : V_\star^{(i)} \to \mathbb{R} : \mathcal{E}_p[f] < \infty\} \tag{1.5.5}$$

does not degenerate to a space containing only constant functions.
Each $f \in \mathcal{F}_{\star,i}^{(p)}$ can be uniquely extended in $C(K_i)$. We denote this extension on K_i still by f and we define the space

$$D(\mathcal{E}_p^{(i)}) = \{f \in C(K_i) \,:\, \mathcal{E}_p^{(i)}[f] < \infty\}, \tag{1.5.6}$$

where $\mathcal{E}_p^{(i)}[f] := \mathcal{E}_p^{(i)}[f|_{V_\star^{(i)}}]$. Hence $D(\mathcal{E}_p^{(i)}) \subset C(K_i) \subset L^p(K_i, \mu)$. Moreover, $(\mathcal{E}_p^{(i)}, D(\mathcal{E}_p^{(i)}))$ is a non-negative energy functional in $L^p(K_i, \mu_i)$ and the following result holds (see [16]).

Theorem 1.5.1. *The following properties hold.*
i) $D(\mathcal{E}_p^{(i)})$ *is complete in the norm* $\|f\|_{D(\mathcal{E}_p^{(i)})} := \|f\|_{L^p(K^i,\mu_i)} + (\mathcal{E}_p^{(i)}[f])^{1/p}$.
ii) $D(\mathcal{E}_p^{(i)})$ *is dense in* $L^p(K_i, \mu_i)$.
iii) $D(\mathcal{E}_q^{(i)}) \subset D(\mathcal{E}_p^{(i)})$, *for* $1 < p \leq q < \infty$.

By proceeding as in Section 4.1 and 4.2 in [32] one can define on K a p-energy form $(\mathcal{E}_p, D(\mathcal{E}_p))$

$$\mathcal{E}_p[u] = \sum_{i=1}^{3} \mathcal{E}_p^{(i)}[u|_{K_i}] \tag{1.5.7}$$

for every $u \in D(\mathcal{E}_p)$, where $D(\mathcal{E}_p) = \{u \in C(K) : u|_{K_i} \in D(\mathcal{E}_p^{(i)})$ for $i = 1, 2, 3\}$.

Chapter 2

Regularity of the solution of nonlocal Venttsel' problems in piecewise smooth domains

In this chapter, we consider an elliptic nonlocal Venttsel' problem in a piecewise smooth domain $\Omega \subset \mathbb{R}^2$ formally stated as follows:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ -\Delta_\ell u = -\frac{\partial u}{\partial \nu} - bu - \theta_s(u) + g & \text{on } \partial\Omega, \end{cases} \tag{2.0.1}$$

where b, f and g are given functions, $\Delta_\ell = \dfrac{\partial^2}{\partial \ell^2}$ is the piecewise tangential Laplace operator, ν the unit vector of exterior normal, and $\theta_s \colon H^s(\partial\Omega) \to H^{-s}(\partial\Omega)$, for $s \in (0,1)$, is the so-called *nonlocal operator*.

We will first consider the case in which the boundary $\partial\Omega$ is polygonal, then we will consider the case of a piecewise smooth boundary. We are interested in the regularity of the solution of problem (2.0.1) in weighted Sobolev spaces. We preliminary prove an a priori estimate for the solution, then we prove the existence and uniqueness results. At last, we prove that the weak solution of problem (2.0.1) has the desired regularity.

2.1 Statement of the problem

We consider a domain $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\partial\Omega$. Namely, we suppose that $\partial\Omega$ is made by a finite number of segments, which form a finite number N of angles with opening α_j, for $j = 1, \dots, N$, and let us denote with α the opening of the largest angle in $\partial\Omega$ (see Figure 2.1).

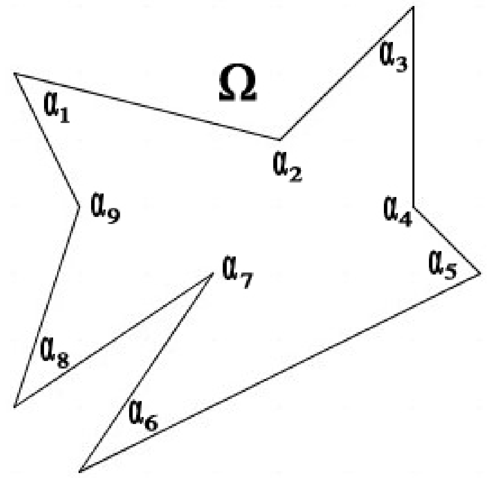

Figure 2.1: A possible example of domain Ω. In this case $N = 9$ and $\alpha = \alpha_7$.

We consider the problem formally stated in (2.0.1). Let $b \in C(\partial\Omega)$ be a strictly positive function. We set $\theta_s\colon H^s(\partial\Omega) \to H^{-s}(\partial\Omega)$ as follows: for every $u, v \in H^s(\partial\Omega)$

$$\langle \theta_s(u), v\rangle = \iint\limits_{\partial\Omega\times\partial\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+2s}}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y), \tag{2.1.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-s}(\partial\Omega)$ and $H^s(\partial\Omega)$. We remark that the nonlocal term $\theta_s(\cdot)$ can be regarded as an analogue of the fractional Laplace operator $(-\Delta)^s$ on the boundary.
We now define a bilinear form E as follows:

$$E(u,v) = \int\limits_{\Omega} \mathrm{D}u\,\mathrm{D}v\,\mathrm{d}\mathcal{L}_2 + \int\limits_{\partial\Omega} \mathrm{D}_\ell u\,\mathrm{D}_\ell v\,\mathrm{d}\ell + \int\limits_{\partial\Omega} b\,u\,v\,\mathrm{d}\ell + \langle\theta_s(u), v\rangle, \tag{2.1.2}$$

for every $u, v \in V^1(\Omega, \partial\Omega) := \{u \in H^1(\Omega) \,:\, \gamma_0 u \in H^1(\partial\Omega)\}$.
We consider the weak formulation of problem (2.0.1):

$$\text{Given } f \text{ and } g, \text{ find } u \in V^1(\Omega,\partial\Omega) \text{ such that } E(u,v) = \int\limits_{\Omega} f\,v\,\mathrm{d}\mathcal{L}_2 + \int\limits_{\partial\Omega} g\,v\,\mathrm{d}\ell \tag{2.1.3}$$

for every $v \in V^1(\Omega, \partial\Omega)$.

We define the space $V^2_\sigma(\Omega, \partial\Omega)$ in a natural way following (1.2.9):

$$V^2_\sigma(\Omega,\partial\Omega) := \{u \in H^1(\Omega) \,:\, r^\sigma \mathrm{D}^2 u \in L^2(\Omega),\ \gamma_0 u \in H^2(\partial\Omega)\}. \tag{2.1.4}$$

The hypothesis on f and g will be given in the theorems. In this chapter we do not indicate the dependence of C on the geometry of Ω.

2.2 A priori estimates

Theorem 2.2.1. *Let $f \in L^2_\sigma(\Omega)$ and $g \in L^2(\partial\Omega)$. Let $u \in V^2_\sigma(\Omega, \partial\Omega)$ be a solution of problem (2.0.1). Suppose that $s < 3/4$. Then there exists a positive constant $C = C(\sigma)$ such that*

$$\|u\|^2_{H^1(\Omega)} + \|r^\sigma \mathrm{D}^2 u\|^2_{L^2(\Omega)} + \|u\|^2_{H^2(\partial\Omega)} \leq C(\sigma)(\|u\|^2_{L^2(\partial\Omega)} + \|r^\sigma f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\partial\Omega)}), \tag{2.2.1}$$

provided

$$1 - \frac{\pi}{\alpha} < \sigma < \frac{1}{2}, \qquad \sigma \geq -\frac{1}{2} \tag{2.2.2}$$

(recall that α is the opening of the largest angle in $\partial\Omega$).

Proof. We use the so-called *Munchhausen trick*. We start by assuming that $\frac{\partial u}{\partial \nu}$ and $\theta_s(u)$ belong to $L^2(\partial\Omega)$, hence the right-hand side of the boundary equation in (2.0.1) belongs to $L^2(\partial\Omega)$. Hence $u \in H^2(\partial\Omega)$ and the following estimate holds:

$$\|u\|^2_{H^2(\partial\Omega)} \leq C\left(\left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(\partial\Omega)} + \|u\|^2_{L^2(\partial\Omega)} + \|\theta_s(u)\|^2_{L^2(\partial\Omega)} + \|g\|^2_{L^2(\partial\Omega)}\right). \tag{2.2.3}$$

In the following the norm of $\theta_s(u)$ in an Hilbert space H' has to be intended as the norm in H of the unique element F associated to the operator by Riesz's theorem.
First we estimate $\|\theta_s(u)\|^2_{L^2(\partial\Omega)}$. Since $u \in H^2(\partial\Omega)$, it is sufficient to consider the local behavior of u near the vertices. Without loss of generality, we can assume that the vertex is located at the origin. We introduce a smooth cutoff function η and rectify $\partial\Omega$ near the origin. Then $u\eta|_{\partial\Omega}$ becomes a function on $\mathbb{R}$ which is the sum of a smooth function and a term $c|t|\tilde{\eta}(t)$ (here $\tilde{\eta}$ is a one-dimensional cutoff function near the origin). The function $c|t|\tilde{\eta}(t)$ belongs to $H^\beta(\mathbb{R})$ for every $\beta < 3/2$ (this can be seen by using the definition of Sobolev space by the Fourier transform $\mathcal{F}$:

$$H^s(\mathbb{R}) = \{v \in \mathcal{S}' \mid (1 + |\xi|^2)^{s/2}\mathcal{F}[v] \in L^2(\mathbb{R})\},$$

where $\mathcal{S}'$ is the space of tempered distributions). Hence, since θ_s is a linear and continuous functional from $H^\beta(\partial\Omega)$ to $H^{\beta-2s}(\partial\Omega)$, this implies that $\theta_s(u) \in H^{\beta-2s}(\partial\Omega)$ and

$$\|\theta_s(u)\|^2_{H^{\beta-2s}(\partial\Omega)} \leq C\|u\|^2_{H^2(\partial\Omega)}, \tag{2.2.4}$$

where C depends on β and s.
We fix $\beta \in (2s, 3/2)$. From the compact embedding of $H^{\beta-2s}(\partial\Omega)$ in $L^2(\partial\Omega)$ we deduce that for every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that

$$\|\theta_s(u)\|^2_{L^2(\partial\Omega)} \leq \varepsilon\|\theta_s(u)\|^2_{H^{\beta-2s}(\partial\Omega)} + C(\varepsilon)\|\theta_s(u)\|^2_{H^{-s}(\partial\Omega)},$$

see Lemma 6.1, Chapter 2 in [74]. Similarly, we have

$$\|\theta_s(u)\|^2_{H^{-s}(\partial\Omega)} \le C\|u\|^2_{H^s(\partial\Omega)} \le \varepsilon\|u\|^2_{H^2(\partial\Omega)} + C(\varepsilon)\|u\|^2_{L^2(\partial\Omega)}.$$

Therefore we obtain the following estimate using (2.2.3) and (2.2.4):

$$\|u\|^2_{H^2(\partial\Omega)} \le C\left(\left\|\frac{\partial u}{\partial\nu}\right\|^2_{L^2(\partial\Omega)} + \|g\|^2_{L^2(\partial\Omega)} + \varepsilon\|u\|^2_{H^2(\partial\Omega)} + C(\varepsilon)\|u\|^2_{L^2(\partial\Omega)}\right).$$

By choosing ε sufficiently small we obtain

$$\|u\|^2_{H^2(\partial\Omega)} \le C\left(\left\|\frac{\partial u}{\partial\nu}\right\|^2_{L^2(\partial\Omega)} + \|u\|^2_{L^2(\partial\Omega)} + \|g\|^2_{L^2(\partial\Omega)}\right). \tag{2.2.5}$$

We now estimate $\left\|\frac{\partial u}{\partial\nu}\right\|^2_{L^2(\partial\Omega)}$. We consider a smooth function U on $\overline{\Omega}$ which is linear near the corners of $\partial\Omega$ and such that $(u-U)(P) = \mathrm{D}_\ell(u-U)(P) = 0$ in every vertex P of $\partial\Omega$. Since D^2U vanishes in neighborhoods of vertices, without loss of generality we can assume that for every $\gamma\in\mathbb{R}$

$$\|U\|^2_{H^1(\Omega)} + \|r^\gamma \mathrm{D}^2U\|^2_{L^2(\Omega)} + \|U\|^2_{H^2(\partial\Omega)} \le C(\gamma)\|u\|^2_{H^2(\partial\Omega)}. \tag{2.2.6}$$

If we consider the function $v = u - U$, from Hardy inequality applied on each segment of $\partial\Omega$ (see [38]) we obtain that $v\in H^2_{\gamma=0}(\partial\Omega)$:

$$\|v\|^2_{H^2_{\gamma=0}(\partial\Omega)} = \int_{\partial\Omega} |v|^2\, r^{-4}\,\mathrm{d}\ell + \int_{\partial\Omega} |\mathrm{D}v|^2\, r^{-2}\,\mathrm{d}\ell + \int_{\partial\Omega} |\mathrm{D}^2v|^2\,\mathrm{d}\ell \le$$

$$C\left(\int_{\partial\Omega} |\mathrm{D}v|^2\, r^{-2}\,\mathrm{d}\ell + \int_{\partial\Omega} |\mathrm{D}^2v|^2\,\mathrm{d}\ell\right) \le \tilde{C}\int_{\partial\Omega} |\mathrm{D}^2v|^2\,\mathrm{d}\ell \le \tilde{C}\|v\|^2_{H^2(\partial\Omega)} < \infty.$$

By rescaling we deduce $v\in H^{\frac{3}{2}}_{-\frac{1}{2}}(\partial\Omega)$, and

$$\|v\|_{H^{\frac{3}{2}}_{-\frac{1}{2}}(\partial\Omega)} \le C\|u\|_{H^2(\partial\Omega)}. \tag{2.2.7}$$

We point out that from (2.2.2) in particular $v\in H^{\frac{3}{2}}_{\sigma}(\partial\Omega)$. Now we consider v as the solution of the Dirichlet problem

$$\begin{cases} -\Delta v = f + \Delta U \in L^2_\sigma(\Omega), \\ v|_{\partial\Omega} \in H^{\frac{3}{2}}_\sigma(\partial\Omega). \end{cases} \tag{2.2.8}$$

From Theorem 3.1, Chapter 2 in [73] (with $l = 0$) it follows that $v\in H^2_\sigma(\Omega)$ if $|\sigma - 1| < \pi/\alpha$ (we recall that α is the opening of the largest angle in $\partial\Omega$). From (2.2.6) and (2.2.7), this implies

$$\|u\|^2_{H^1(\Omega)} + \|r^\sigma D^2u\|^2_{L^2(\Omega)} \le C(\sigma)(\|r^\sigma f\|^2_{L^2(\Omega)} + \|u\|^2_{H^2(\partial\Omega)}) \tag{2.2.9}$$

(to estimate the first term, we also used that $\sigma \leq 1$ in (2.2.2)).
By rescaling, we deduce that $\mathrm{D}u \in L^2_{\sigma-\frac{1}{2}}(\partial\Omega)$ and

$$\|\mathrm{D}u\|^2_{L^2_{\sigma-\frac{1}{2}}(\partial\Omega)} \leq \|u\|^2_{H^1(\Omega)} + \|r^\sigma D^2 u\|^2_{L^2(\Omega)}. \tag{2.2.10}$$

We define a cutoff function η_δ such that

$$\eta_\delta(r) = 1 \quad \text{for} \quad r > \delta, \qquad \eta_\delta(r) = 0 \quad \text{for} \quad r < \delta/2.$$

Now we introduce the following trace operator:

$$u \longrightarrow \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \eta_\delta \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} + (1-\eta_\delta)\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} =: \mathcal{K}_1(\delta)u + \mathcal{K}_2(\delta)u.$$

We remark that the operator $\mathcal{K}_1(\delta)\colon H^2_\sigma(\Omega) \to L^2(\partial\Omega)$ is compact. Using (2.2.9), we obtain for arbitrary $\varepsilon > 0$

$$\|\mathcal{K}_1(\delta)u\|^2_{L^2(\partial\Omega)} \leq \frac{\varepsilon}{2}(\|r^\sigma f\|^2_{L^2(\Omega)} + \|u\|^2_{H^2(\partial\Omega)}) + C(\varepsilon,\sigma,\delta)\|u\|^2_{L^2(\partial\Omega)}.$$

From (2.2.9) and (2.2.10) we deduce

$$\|\mathcal{K}_2(\delta)u\|^2_{L^2(\partial\Omega)} \leq C(\sigma)\delta^{\frac{1}{2}-\sigma}(\|r^\sigma f\|^2_{L^2(\Omega)} + \|u\|^2_{H^2(\partial\Omega)}).$$

This in turn implies that $\sigma < \frac{1}{2}$. Hence, by choosing $\delta(\sigma,\varepsilon)$ sufficiently small, we obtain

$$\left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(\partial\Omega)} \leq \varepsilon(\|r^\sigma f\|^2_{L^2(\Omega)} + \|u\|^2_{H^2(\partial\Omega)}) + C(\varepsilon,\sigma)\|u\|^2_{L^2(\partial\Omega)}.$$

Substituting the above inequality into (2.2.5) we obtain

$$\|u\|^2_{H^2(K_n)} \leq C\left(\varepsilon(\|r^\sigma f\|^2_{L^2(\Omega)} + \|u\|^2_{H^2(K_n)}) + C(\varepsilon,\sigma)\|u\|^2_{L^2(K_n)} + \|g\|^2_{L^2(K_n)}\right).$$

By choosing ε sufficiently small we obtain

$$\|u\|^2_{H^2(K_n)} \leq C\left(\|r^\sigma f\|^2_{L^2(\Omega)} + C(\sigma)\|u\|^2_{L^2(K_n)} + \|g\|^2_{L^2(K_n)}\right). \tag{2.2.11}$$

Taking into account (2.2.9), we get the thesis. $\square$

2.3 Solvability of the Venttsel' problem

We begin by proving the existence and uniqueness of the weak solution.
We point out that the we can equip the space $V^1(\Omega,\partial\Omega)$ with the natural norm

$$|||u|||_{V^1(\Omega,\partial\Omega)} := \left(\|u\|^2_{H^1(\Omega)} + \|u\|^2_{H^1(\partial\Omega)}\right)^{\frac{1}{2}}.$$

Instead, by Friedrichs inequality (see (1.2.8)), we equip $V^1(\Omega, \partial\Omega)$ with the equivalent Hilbertian norm

$$\|u\|_{V^1(\Omega,\partial\Omega)} = \left(\|\mathrm{D}u\|^2_{L^2(\Omega)} + \|\mathrm{D}_\ell u\|^2_{L^2(\partial\Omega)} + \|u\|^2_{L^2(\partial\Omega)}\right)^{\frac{1}{2}}.$$

Lemma 2.3.1. *The energy form $E[u] = E(u,u)$ generates an equivalent norm in $V^1(\Omega, \partial\Omega)$.*

Proof. Since $b \in C(\partial\Omega)$ and

$$\langle \theta_s(u), u\rangle \leq C\|u\|^2_{H^s(\partial\Omega)} \leq C\|u\|^2_{H^1(\partial\Omega)},$$

we obtain that $E[u] \leq C\|u\|^2_{V^1(\Omega,\partial\Omega)}$. Then, since $\langle \theta_s(u), u\rangle \geq 0$ and $\inf\limits_{\partial\Omega} b > 0$, we have

$$E[u] \geq C\|u\|^2_{V^1(\Omega,\partial\Omega)}.$$

□

The following existence and uniqueness result holds.

Corollary 2.3.2. *Let $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$. Then there exists a unique weak solution in $V^1(\Omega, \partial\Omega)$ of problem (2.1.3). Moreover*

$$\|u\|_{V^1(\Omega,\partial\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}), \tag{2.3.1}$$

where C depends only on the coercivity constant of E.

Proof. Existence and uniqueness of the weak solution follow from Lax-Milgram Lemma, since the energy form E is coercive on $V^1(\Omega, \partial\Omega)$. As to (2.3.1), we take $v = u$ as test function in (2.1.3) and, from the coercivity of E, Cauchy-Schwarz inequality and Friedrichs inequality, we get

$$C\|u\|^2_{V^1(\Omega,\partial\Omega)} \leq E[u] = \int\limits_{\Omega} fu\,\mathrm{d}\mathcal{L}_2 + \int\limits_{\partial\Omega} gu\,\mathrm{d}\ell \leq$$
$$\|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\|u\|_{L^2(\partial\Omega)} \leq \|u\|_{V^1(\Omega,\partial\Omega)}(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}),$$

i.e. the thesis. □

We finally prove the desired regularity for the weak solution of the nonlocal Venttsel' problem.

Theorem 2.3.3. *Let σ satisfy condition (2.2.2). Let $f \in L^2_\sigma(\Omega)$ and $g \in L^2(\partial\Omega)$. Then problem (2.0.1) has a unique solution $u \in V^2_\sigma(\Omega, \partial\Omega)$, and the following inequality holds*

$$\|u\|^2_{H^1(\Omega)} + \|r^\sigma D^2 u\|^2_{L^2(\Omega)} + \|u\|^2_{H^2(\partial\Omega)} \leq C(\|r^\sigma f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\partial\Omega)}), \tag{2.3.2}$$

where C depends on σ and the coercivity constant of E.

Proof. We introduce the set of operators $\mathcal{L}_\mu\colon V^2_\sigma(\Omega,\partial\Omega)\to L^2_\sigma(\Omega)\times L^2(\partial\Omega)$ defined as

$$\mathcal{L}_\mu u := \left(-\Delta u, -\Delta_\ell u + bu + \mu\left(\frac{\partial u}{\partial\nu}+\theta_s(u)\right)\right).$$

We claim that the operator $\mathcal{L}_0$ is invertible. Indeed, it corresponds to the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega,\\ -\Delta_\ell u + bu = g & \text{on } \partial\Omega. \end{cases}$$

Here the equation in Ω and the boundary condition are decoupled. So we can first solve the boundary equation and then use its solution as the Dirichlet datum for the equation in the domain. Proceeding as in Theorem 2.2.1, we show that the solution belongs to $V^2_\sigma(\Omega,\partial\Omega)$ and inequality (2.3.2) holds. So the claim follows.
The estimates in Theorem 2.2.1 show that the operator

$$\mathcal{L}_\mu-\mathcal{L}_0\colon V^2_\sigma(\Omega,\partial\Omega)\to L^2_\sigma(\Omega)\times L^2(\partial\Omega);\qquad \mathcal{L}_\mu u-\mathcal{L}_0 u=\mu\left(0,\frac{\partial u}{\partial\nu}+\theta_s(u)\right)$$

is compact. Since, for $\mu=1$, $\operatorname{Ker}(\mathcal{L}_1)$ is trivial by Corollary 2.3.2, the operator $\mathcal{L}_1$ is also invertible, and the proof is complete. $\square$

We conclude this section with some remarks. If Ω is a convex polygon, then $\alpha<\pi$. So we can put $\sigma=0$ and obtain the following result.

Corollary 2.3.4. *Let Ω be a convex polygon. Let $f\in L^2(\Omega)$ and $g\in L^2(\partial\Omega)$. Then the problem (2.0.1) has a unique solution $u\in H^2(\Omega)\cap H^2(\partial\Omega)$, and the following inequality holds*

$$\|u\|^2_{H^2(\Omega)}+\|u\|^2_{H^2(\partial\Omega)}\le C(\|f\|^2_{L^2(\Omega)}+\|g\|^2_{L^2(\partial\Omega)}),$$

where C depends on the coercivity constant of E.

If Ω is not convex, then $\pi<\alpha<2\pi$. In this case we obtain the following result.

Theorem 2.3.5. *Let Ω be a non-convex polygon. Let $f\in L^2(\Omega)$ and $g\in L^2(\partial\Omega)$. Then the unique solution of the problem (2.0.1) admits the following decomposition:*

$$u(x)=\sum_{j\,:\,\alpha_j>\pi} c_j\chi(r_j)r^{\frac{\pi}{\alpha_j}}\sin(\pi\omega_j\alpha_j^{-1})+w(x). \tag{2.3.3}$$

Here (r_j,ω_j) are local polar coordinates in a neighborhood of the angle with opening α_j, χ is a cutoff function near the origin, and $w\in H^2(\Omega)\cap H^2(\partial\Omega)$. Moreover, the following inequality holds

$$\|w\|^2_{H^2(\Omega)}+\|w\|^2_{H^2(\partial\Omega)}+\sum_{j\,:\,\alpha_j>\pi}|c_j|^2\le C(\|f\|^2_{L^2(\Omega)}+\|g\|^2_{L^2(\partial\Omega)}),$$

where C depends on the coercivity constant of E.

Proof. Following the lines of the proof of Theorem 2.2.1, we obtain the Dirichlet problem for $v = u - U$

$$\begin{cases} -\Delta v \in L^2(\Omega) \\ v|_{\partial\Omega} \in H^{\frac{3}{2}}(\partial\Omega) \end{cases}$$

instead of (2.2.8). Theorem 3.4, Chapter 2 in [73] gives the representation (2.3.3) for v. Since U is smooth, the statement follows. □

Remark 2.3.6. Without any sign condition on the coefficient b, the problem (2.0.1) is not necessarily solvable, but it has the Fredholm property.

All our results easily hold for an arbitrary piecewise smooth domain $\Omega \subset \mathbb{R}^2$ without cusps.

Chapter 3

Numerical approximation of parabolic nonlocal Venttsel' problems in pre-fractal domains

In this chapter, we study a parabolic nonlocal Venttsel' problem for the Laplace operator in the domain Ω_n introduced in Section 1.1.1. The problem is formally stated as follows: for every $t \in [0, T]$

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = \Delta u + f & \text{in } \Omega_n, \\ -\Delta_{K_n} u = -\frac{\partial u}{\partial \nu} - bu - \theta_{\frac{1}{2}}(u) + f - \frac{\mathrm{d}u}{\mathrm{d}t} & \text{on } K_n, \\ u(0, x) = u_0(x) & \text{in } \overline{\Omega}_n. \end{cases} \tag{3.0.1}$$

where f, b and u_0 are given functions in suitable spaces, ν is the outward unit normal vector and Δ_{K_n} is the piecewise tangential Laplace operator on K_n.
The presence of the nonlocal term $\theta_{\frac{1}{2}}(u)$ seems to deteriorate the regularity of the weak solution up to the boundary. In order to prove the same regularity results obtained for the local version of the problem, we have to make use of Theorem 2.2.1 adapted to our case, to obtain the right regularity on K_n. After giving existence, uniqueness and regularity results, we perform a numerical approximation by mixed methods: FEM in space and finite differences in time. After proving a priori error estimates for the approximated problem, we show some numerical simulations which stress the importance of the nonlocal term in the process of heat flow.
Since in this chapter n will be fixed, we omit the subscript on Ω_n and we will simply write Ω.

3.1 The energy form

Let b be a positive continuous function on $\overline{\Omega}$. We set $\theta_{\frac{1}{2}}: H^{\frac{1}{2}}(K_n) \to H^{-\frac{1}{2}}(K_n)$ as follows: for every $u, v \in H^{\frac{1}{2}}(K_n)$

$$\langle \theta_{\frac{1}{2}}(u), v \rangle_{H^{-\frac{1}{2}}(K_n), H^{\frac{1}{2}}(K_n)} = \iint\limits_{K_n \times K_n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^2} \, \mathrm{d}\ell(x) \, \mathrm{d}\ell(y).$$

From now on $\langle \cdot, \cdot \rangle$ will denote the duality pairing between $H^{-\frac{1}{2}}(K_n)$ and $H^{\frac{1}{2}}(K_n)$.
We define now the energy form E as

$$E[u] = E_{\Omega}[u] + E_{K_n}[u] + \int\limits_{K_n} b\, |u|^2 \, \mathrm{d}\ell + \langle \theta_{\frac{1}{2}}(u), u \rangle \tag{3.1.1}$$

with domain

$$V(\Omega, K_n) = \{u \in H^1(\Omega) \,:\, \gamma_0 u \in H^1(K_n)\},$$

where

$$E_{\Omega}[u] = \int\limits_{\Omega} |\mathrm{D}u|^2 \, \mathrm{d}\mathcal{L}_2$$

and

$$E_{K_n}[u] = \int\limits_{K_n} |\mathrm{D}_{\ell} u|^2 \, \mathrm{d}\ell.$$

Here D_{ℓ} denotes the tangential derivative on K_n.
$V(\Omega, K_n)$ is a Hilbert space equipped with the norm

$$\|u\|_{V(\Omega,K_n)} = \left(\int\limits_{\Omega} |\mathrm{D}u|^2 \, \mathrm{d}\mathcal{L}_2 + \int\limits_{K_n} |\mathrm{D}_{\ell} u|^2 \, \mathrm{d}\ell + \|u\|^2_{L^2(\Omega,m)} \right)^{\frac{1}{2}},$$

where $\mathrm{d}m = \mathrm{d}\mathcal{L}_2 + \mathrm{d}\ell$. The space $V(\Omega, K_n)$ is non-trivial.
In order to prove the coercivity of the energy form E, we introduce an equivalent norm on $V(\Omega, K_n)$, which is defined as

$$|||u|||_{V(\Omega,K_n)} := \left(\|\mathrm{D}u\|^2_{L^2(\Omega)} + \Phi(u) \right)^{\frac{1}{2}}, \tag{3.1.2}$$

where $\Phi(u) := \|u\|^2_{H^1(K_n)} + \langle \theta_{\frac{1}{2}}(u), u \rangle$.

Proposition 3.1.1. *The norms $\|\cdot\|_{V(\Omega,K_n)}$ and $|||\cdot|||_{V(\Omega,K_n)}$ are equivalent.*

Proof. We note that $\|u\|^2_{V(\Omega,K_n)} \leq C_2 |||u|||^2_{V(\Omega,K_n)}$ thanks to (1.2.8). To prove that $\|u\|^2_{V(\Omega,K_n)} \geq C_1 |||u|||^2_{V(\Omega,K_n)}$ we note that

$$\langle\theta_{\frac{1}{2}}(u),u\rangle \leq \|u\|^2_{H^{\frac{1}{2}}(K_n)} \leq C\|u\|^2_{H^1(K_n)}.$$

□

We point out that, from our hypothesis on b, Proposition 3.1.1 in turn implies that the norm induced by $E[u]$ is equivalent to the norm $\|\cdot\|_{V(\Omega,K_n)}$.
We now prove some properties of the form E.

Proposition 3.1.2. *The form $E[u]$ defined in (3.1.1) is continuous and coercive on $V(\Omega,K_n)$.*

Proof. We start by proving the continuity of E on $V(\Omega,K_n)$. Since b is continuous on $\overline{\Omega}$, we have

$$E[u] \leq \|\mathrm{D}u\|^2_{L^2(\Omega)} + \|\mathrm{D}_\ell u\|^2_{L^2(K_n)} + \left(\max_{K_n} b\right) \|u\|^2_{L^2(K_n)} + \langle\theta_{\frac{1}{2}}(u),u\rangle \leq$$
$$\|u\|^2_{H^1(\Omega)} + c_1\,\|u\|^2_{H^1(K_n)} + \langle\theta_{\frac{1}{2}}(u),u\rangle \leq \|u\|^2_{H^1(\Omega)} + c_2\|u\|^2_{H^1(K_n)} \leq \max\{1,c_2\}\,\|u\|^2_{V(\Omega,K_n)}.$$

We prove the coercivity. By using again the continuity of b, we have

$$E[u] \geq \|\mathrm{D}u\|^2_{L^2(\Omega)} + \|\mathrm{D}_\ell u\|^2_{L^2(K_n)} + \left(\min_{K_n} b\right) \|u\|^2_{L^2(K_n)} + \langle\theta_{\frac{1}{2}}(u),u\rangle \geq$$
$$\|\mathrm{D}u\|^2_{L^2(\Omega)} + \min\left\{1,\min_{K_n} b\right\} \|u\|^2_{H^1(K_n)} + \iint_{K_n\times K_n} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y) \geq \bar{C}\|u\|^2_{V(\Omega,K_n)},$$

where $\bar{C}$ depends on b and C_1. □

Proposition 3.1.3. *The energy form $E[u]$ is closed in $L^2(\Omega,m)$, i.e. for every Cauchy sequence $\{u_k\} \subseteq V(\Omega,K_n)$ there exists $u \in V(\Omega,K_n)$ such that*

$$E[u_k-u] + \|u_k-u\|_{L^2(\Omega,m)} \to 0 \quad \textit{for } k\to+\infty.$$

Proof. Let $\{u_k\}$ be a Cauchy sequence in $V(\Omega,K_n)$, i.e. a sequence such that

$$E[u_k-u_j] + \|u_k-u_j\|_{L^2(\Omega,m)} \to 0 \quad \text{for } k,j\to+\infty.$$

We observe that in particular $\{u_k\}$ is a Cauchy sequence in $L^2(\Omega,m)$ and, since $L^2(\Omega,m)$ is a Banach space, there exists an element $u \in L^2(\Omega,m)$ such that

$$\|u_k-u\|_{L^2(\Omega,m)} \xrightarrow[k\to+\infty]{} 0.$$

We have to prove that

$$E[u_k-u] \xrightarrow[k\to+\infty]{} 0.$$

We note that from $E[u_k - u_j] \to 0$ when $k, j \to +\infty$, it follows that each term in (3.1.1) vanishes (because they are all non negative terms).

Since $\int_\Omega |\mathrm{D}(u_k - u_j)|^2 \, \mathrm{d}\mathcal{L}_2 \to 0$, it follows that $\{\mathrm{D}u_k\}$ is a Cauchy sequence in $L^2(\Omega)$, and the same holds for the terms in $L^2(K_n)$. Then $\{\mathrm{D}u_k\}$ is a Cauchy sequence in $L^2(\Omega, m)$, hence there exists an element $w \in L^2(\Omega, m)$ such that $\mathrm{D}u \to w$ in $L^2(\Omega, m)$. From Remark 4, Chapter 9 in [11], we know that $w = \mathrm{D}u$ a.e., so we have that $u \in V(\Omega, K_n)$.

It is trivial that $\int_{K_n} b|u_k - u|^2 \, \mathrm{d}\ell \to 0$ because b is a continuous function on $\overline{\Omega}$. It remains to study the term $\theta_{\frac{1}{2}}$:

$$\langle \theta_{\frac{1}{2}}(u_k - u), u_k - u \rangle_{H^{-\frac{1}{2}}(K_n), H^{\frac{1}{2}}(K_n)} =$$

$$\iint_{K_n \times K_n} \frac{|u_k(x) - u(x) - (u_k(y) - u(y))|^2}{|x - y|^2} \, \mathrm{d}\ell(x) \, \mathrm{d}\ell(y) \leq \|u_k - u\|^2_{H^{\frac{1}{2}}(K_n)} \leq$$

$$C\|u_k - u\|^2_{H^1(K_n)}$$

and the last term tends to 0 when $k \to +\infty$ because we know that $u_k \to u$ in $V(\Omega, K_n)$.

$\square$

Theorem 3.1.4. *The energy form $E[u]$ with its domain $V(\Omega, K_n)$ is a Dirichlet form on $L^2(\Omega, m)$.*

Proof. We have to prove that $E[u]$ is markovian. Since we know that $E[u]$ is closed, we can prove a sufficient condition for having markovianity, i.e. $\forall\, u \in V(\Omega, K_n)$

$$v := (u \vee 0) \wedge 1 \in V(\Omega, K_n) \text{ and } E[v] \leq E[u],$$

where $(u \vee v)(x) = \max\{u(x), v(x)\}$ and $(u \wedge v)(x) = \min\{u(x), v(x)\}$.

Let us consider the map $T\colon \mathbb{R} \to \mathbb{R}$ defined as $T(s) = ((s \vee 0) \wedge 1)$, then we set $v(x) := T(u(x))$. Now we approximate T with functions $T_\varepsilon \in C^1(\mathbb{R})$ such that

$$|T_\varepsilon(s) - T(s)| < \varepsilon \text{ and } \left|\frac{\mathrm{d}T_\varepsilon}{\mathrm{d}s}\right| \leq 1.$$

Since $T_\varepsilon \in C^1(\mathbb{R})$ and $u \in V(\Omega, K_n)$, it follows that $T_\varepsilon(u(x)) \in V(\Omega, K_n)$, then $T(u(x)) = v(x) \in V(\Omega, K_n)$. Now

$$\begin{aligned}\frac{\mathrm{d}T_\varepsilon(u(x_1(\ell), x_2(\ell)))}{\mathrm{d}\ell} &= \frac{\partial T_\varepsilon}{\partial u} \frac{\partial u}{\partial x_1} \frac{\mathrm{d}x_1(\ell)}{\mathrm{d}\ell} + \frac{\partial T_\varepsilon}{\partial u} \frac{\partial u}{\partial x_2} \frac{\mathrm{d}x_2(\ell)}{\mathrm{d}\ell} \\ &= \frac{\partial T_\varepsilon}{\partial u} (\mathrm{D}u, z(\ell)) = \frac{\partial T_\varepsilon}{\partial u} \mathrm{D}_\ell u,\end{aligned}$$

where $z(\ell) = (x_1'(\ell), x_2'(\ell))$. Using the properties of T_ε, it follows that

$$\left|\frac{\mathrm{d}T_\varepsilon(u(x_1(\ell),x_2(\ell)))}{\mathrm{d}\ell}\right|^2 \le \left|\frac{\partial T_\varepsilon}{\partial u}\right|^2 |\mathrm{D}_\ell u(x_1(\ell),x_2(\ell))|^2 \le |\mathrm{D}_\ell u(x_1(\ell),x_2(\ell))|^2.$$

Hence we have that

$$E_{K_n}[v] = E_{K_n}[T(u)] \le \int\limits_{K_n} |\mathrm{D}_\ell u(x(\ell))|^2 \,\mathrm{d}\ell = E_{K_n}[u].$$

We can repeat the same argument on $E_\Omega[u]$ to prove that $E_\Omega[v] \le E_\Omega[u]$. It is obvious that

$$\int\limits_{K_n} b|v|^2 \,\mathrm{d}\ell \le \int\limits_{K_n} b|u|^2 \,\mathrm{d}\ell.$$

We have only to prove that

$$\langle \theta_{\frac{1}{2}}(v), v\rangle \le \langle \theta_{\frac{1}{2}}(u), u\rangle.$$

We define the sets $A = \{x \in K_n \,:\, u(x) \le 0\}$, $B = \{x \in K_n \,:\, 0 < u(x) < 1\}$ and $C = \{x \in K_n \,:\, u(x) \ge 1\}$. Hence we can split the nonlocal term into a sum of integral terms in the following way:

$$\begin{aligned}\langle \theta_{\frac{1}{2}}(v), v\rangle = &\iint\limits_{A\times A} \frac{|v(x)-v(y)|^2}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y) + \iint\limits_{B\times A} \frac{|v(x)-v(y)|^2}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y) + \\ &\iint\limits_{C\times A} \frac{|v(x)-v(y)|^2}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y) + \iint\limits_{A\times B} \frac{|v(x)-v(y)|^2}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y) + \\ &\iint\limits_{B\times B} \frac{|v(x)-v(y)|^2}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y) + \iint\limits_{C\times B} \frac{|v(x)-v(y)|^2}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y) + \\ &\iint\limits_{A\times C} \frac{|v(x)-v(y)|^2}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y) + \iint\limits_{B\times C} \frac{|v(x)-v(y)|^2}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y) + \\ &\iint\limits_{C\times C} \frac{|v(x)-v(y)|^2}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y).\end{aligned}$$

We point out that, from the definition of v, we have that

$$\iint\limits_{A\times A} \frac{|v(x)-v(y)|^2}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y) = \iint\limits_{C\times C} \frac{|v(x)-v(y)|^2}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y) = 0$$

and $\displaystyle\iint\limits_{B\times B} \frac{|v(x)-v(y)|^2}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y) \le \langle \theta_{\frac{1}{2}}(u), u\rangle$. As to the other terms in the above sum, the following inequalities hold:

- on $B \times A = \{(x,y) \in K_n \,:\, u(x) \le 0, 0 < u(y) < 1\}$ we have that

$$|v(x)-v(y)| = u(y) \le u(y) - u(x) \le |u(x)-u(y)|,$$

hence $\displaystyle\iint_{B\times A} \frac{|v(x)-v(y)|^2}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y) \le \langle \theta_{\frac{1}{2}}(u), u\rangle$;

- on $C \times A = \{(x,y) \in K_n \,:\, u(x) \le 0, u(y) \ge 1\}$ we have that

$$|v(x)-v(y)| = 1 \le u(y)-u(x) \le |u(x)-u(y)|,$$

hence $\displaystyle\iint_{C\times A} \frac{|v(x)-v(y)|^2}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y) \le \langle \theta_{\frac{1}{2}}(u), u\rangle$;

- on $C \times B = \{(x,y) \in K_n \,:\, 0 < u(x) < 1, u(y) \ge 1\}$ we have that

$$|v(x)-v(y)| = |u(x)-1| = 1-u(x) \le u(y)-u(x) \le |u(x)-u(y)|,$$

hence $\displaystyle\iint_{C\times B} \frac{|v(x)-v(y)|^2}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y) \le \langle \theta_{\frac{1}{2}}(u), u\rangle$.

Similar arguments hold on $A\times B$, $A\times C$ and $B\times C$ respectively. Therefore we proved that $\langle \theta_{\frac{1}{2}}(v), v\rangle \le \langle \theta_{\frac{1}{2}}(u), u\rangle$. Hence $E[u]$ is markovian, then $E[u]$ with its domain $V(\Omega, K_n)$ is a Dirichlet form on $L^2(\Omega, m)$. □

For the main properties of Dirichlet forms, see [33].
Now we define the bilinear form associated to the energy form $E[u]$ as follows: for every $u, v \in V(\Omega, K_n)$

$$E(u,v) = \int_\Omega \mathrm{D}u\,\mathrm{D}v\,\mathrm{d}\mathcal{L}_2 + \int_{K_n} \mathrm{D}_\ell u\,\mathrm{D}_\ell v\,\mathrm{d}\ell + \int_{K_n} b\,u\,v\,\mathrm{d}\ell + \langle \theta_{\frac{1}{2}}(u), v\rangle. \tag{3.1.3}$$

Theorem 3.1.5. *For every $u, v \in V(\Omega, K_n)$, $E(u,v)$ is a closed symmetric bilinear form on $L^2(\Omega, m)$. Then there exists a unique self-adjoint non-positive operator A on $L^2(\Omega, m)$ such that*

$$E(u,v) = (-Au, v)_{L^2(\Omega,m)} \quad \forall\, u \in D(A), \forall\, v \in V(\Omega, K_n), \tag{3.1.4}$$

where $D(A) \subset V(\Omega, K_n)$ is the domain of A and it is dense in $L^2(\Omega, m)$.
For the proof see [45].
In Theorem 3.1.4 we proved that $(E_{K_n}, H^1_0(K_n))$ is a closed bilinear form on $L^2(K_n)$. Then, there exists a unique self-adjoint, non positive operator Δ_{K_n} on $L^2(K_n)$ (with domain $D(\Delta_{K_n})$ dense in $L^2(K_n)$) such that

$$E_{K_n}(u,v) = -\int_{K_n} (\Delta_{K_n} u)\, v\,\mathrm{d}\ell, \quad u \in D(\Delta_{K_n}), v \in H^1_0(K_n)$$

(see Chap. 6, Theorem 2.1 in [45]). Let now $H^{-1}(K_n)$ be the dual space of $H^1_0(K_n)$. We can also introduce the Laplace operator on K_n as a variational operator

$$\Delta_{K_n} \colon H^1_0(K_n) \to H^{-1}(K_n)$$

by

$$E_{K_n}(z,w) = -\langle \Delta_{K_n} z, w\rangle_{H^{-1}(K_n),H^1_0(K_n)} \tag{3.1.5}$$

for $z \in H^1_0(K_n)$ and for all $w \in H^1_0(K_n)$. We will use the same symbol Δ_{K_n} to define the Laplace operator both as a self-adjoint operator and as a variational operator. It will be clear from the context to which case we refer.

Remark 3.1.6. As it will be clear in (3.4.7), Δ_{K_n} will be the piecewise tangential Laplacian with domain $D(\Delta_{K_n}) = H^2(K_n)$

Theorem 3.1.7. *The self-adjoint non positive operator A associated to the Dirichlet form $E[u]$ is the generator of a strongly continuous analytic contraction semigroup $\{T_t, t \geq 0\}$ on $L^2(\Omega, m)$.*

Proof. The analyticity of $\{T_t\}$ follows from Proposition 3.1.2 (see Theorem 6.2, Chapter 4 in [81]). The contraction property follows from Lumer-Phillips Theorem (see Theorem 4.3, Chapter 1 in [76]). The strong continuity follows from Theorem 1.3.1 in [33]. □

3.2 A priori estimates in weighted Sobolev spaces

In this section we prove a priori estimates for the solution of problem (3.0.1). We stress the fact that the key issue is to prove that $u \in H^2(K_n)$, which does not follow as in the case of local Venttsel' problem (see [63]). In this section t is fixed.
We adapt the results of Chapter 2 for the elliptic problem to the parabolic problem (3.0.1). We recall the space $V^2_\sigma(\Omega, K_n)$ defined in (1.2.9). We state the following Theorem, i.e. Theorem 2.2.1 specialized to our case.

Theorem 3.2.1. *Let f and $\frac{\mathrm{d}u}{\mathrm{d}t}$ belong to $L^2(\Omega, m)$. For every $t \in [0, T]$ let $u \in V^2_\sigma(\Omega, K_n)$ be a solution of problem (3.0.1). Then there exists a positive constant $C = C(\sigma)$ such that*

$$\|u\|^2_{H^1(\Omega)} + \|r^\sigma \mathrm{D}^2 u\|^2_{L^2(\Omega)} + \|u\|^2_{H^2(K_n)} \leq C(\sigma)\left(\|u\|^2_{L^2(K_n)} + \|f\|^2_{L^2(\Omega,m)} + \left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|^2_{L^2(\Omega,m)}\right), \tag{3.2.1}$$

provided

$$\frac{1}{4} < \sigma < \frac{1}{2}. \tag{3.2.2}$$

We point out that Theorem 3.3.1 will state that $\frac{\mathrm{d}u}{\mathrm{d}t} \in L^2(\Omega, m)$, hence this hypothesis in the following will always be satisfied.

Proof. As in the proof of Theorem 2.2.1, we assume that $\frac{\partial u}{\partial \nu}$ and $\theta_{\frac{1}{2}}(u)$ belong to $L^2(K_n)$, hence the right-hand side of the second equation in (3.0.1) belongs to $L^2(K_n)$. Then the following estimate holds:

$$\|u\|^2_{H^2(K_n)} \leq C\left(\left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(K_n)} + \|u\|^2_{L^2(K_n)} + \|\theta_{\frac{1}{2}}(u)\|^2_{L^2(K_n)} + \|f\|^2_{L^2(K_n)} + \left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|^2_{L^2(K_n)}\right). \tag{3.2.3}$$

By proceeding exactly as in the proof of Theorem 2.2.1, we prove that $\theta_{\frac{1}{2}}(u) \in H^\beta(K_n)$ for $\beta < \frac{1}{2}$ and that the following estimate holds:

$$\|u\|^2_{H^2(K_n)} \leq C\left(\left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(K_n)} + \|u\|^2_{L^2(K_n)} + \|f\|^2_{L^2(K_n)} + \left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|^2_{L^2(K_n)}\right). \tag{3.2.4}$$

As to the estimate of $\|\frac{\partial u}{\partial \nu}\|^2_{L^2(K_n)}$, we consider a smooth function U on $\overline{\Omega}$ as in the proof of Theorem 2.2.1 such that (2.2.6) holds. Setting $v = u - U$, we obtain that $v \in H^2_{\gamma=0}(K_n)$, $v \in H^{\frac{3}{2}}_{-\frac{1}{2}}(K_n)$ and estimate (2.2.7) holds.

We point out that from (3.2.2) in particular $v \in H^{\frac{3}{2}}_\sigma(K_n)$. Now we consider v as the solution of the Dirichlet problem

$$\begin{cases} -\Delta v = f - \frac{\mathrm{d}u}{\mathrm{d}t} + \Delta U \in L^2(\Omega), \\ v|_{K_n} \in H^{\frac{3}{2}}_\sigma(K_n). \end{cases} \tag{3.2.5}$$

We note that, due to our hypothesis (3.2.2) on σ, in particular $f - \frac{\mathrm{d}u}{\mathrm{d}t} \in L^2_\sigma(\Omega)$. Hence, from Theorem 3.1, Chapter 2 in [73] (with $l = 0$) it follows that $v \in H^2_\sigma(\Omega)$ if $|\sigma - 1| < 3/4$, since the opening of the worst angle in K_n is $\frac{4\pi}{3}$. Moreover, from (2.2.6) and (2.2.7), the following estimate holds:

$$\begin{aligned} \|u\|^2_{H^1(\Omega)} + \|r^\sigma \mathrm{D}^2 u\|^2_{L^2(\Omega)} &\leq C(\sigma)\left(\left\|r^\sigma\left(f - \frac{\mathrm{d}u}{\mathrm{d}t}\right)\right\|^2_{L^2(\Omega)} + \|u\|^2_{H^2(K_n)}\right) \\ &\leq C(\sigma)\left(\left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} + \|u\|^2_{H^2(K_n)}\right). \end{aligned} \tag{3.2.6}$$

By rescaling, we deduce that $\mathrm{D}u \in L^2_{\sigma-\frac{1}{2}}(K_n)$ and

$$\|\mathrm{D}u\|^2_{L^2_{\sigma-\frac{1}{2}}(K_n)} \leq \|u\|^2_{H^1(\Omega)} + \|r^\sigma \mathrm{D}^2 u\|^2_{L^2(\Omega)}. \tag{3.2.7}$$

We introduce the following trace operators:

$$\mathcal{K}_1(\delta)u := \eta_\delta \frac{\partial u}{\partial \nu}\Big|_{K_n}, \quad \mathcal{K}_2(\delta)u := (1 - \eta_\delta)\frac{\partial u}{\partial \nu}\Big|_{K_n},$$

where η_δ is the cutoff function defined in the proof of Theorem 2.2.1. We remark that the operator $\mathcal{K}_1(\delta)\colon H^2_\sigma(\Omega) \to L^2(K_n)$ is compact and, by using (3.2.6), we obtain for arbitrary $\varepsilon > 0$

$$\|\mathcal{K}_1(\delta)u\|^2_{L^2(K_n)} \leq \frac{\varepsilon}{2}\left(\left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} + \|u\|^2_{H^2(K_n)}\right) + C(\varepsilon,\sigma,\delta)\|u\|^2_{L^2(K_n)}.$$

From (3.2.6) and (3.2.7) we deduce

$$\|\mathcal{K}_2(\delta)u\|^2_{L^2(K_n)} \leq C(\sigma)\delta^{\frac{1}{2}-\sigma}\left(\left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} + \|u\|^2_{H^2(K_n)}\right),$$

which as before implies $\sigma < \frac{1}{2}$. Hence, by choosing δ sufficiently small, it follows that

$$\left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(K_n)} \leq \varepsilon\left(\left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} + \|u\|^2_{H^2(K_n)}\right) + C(\varepsilon,\sigma)\|u\|^2_{L^2(K_n)}.$$

Substituting the above inequality into (3.2.4) we obtain

$$\begin{aligned}\|u\|^2_{H^2(K_n)} \leq C\varepsilon & \left(\left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} + \|u\|^2_{H^2(K_n)}\right) \\ & + C(\varepsilon,\sigma)\|u\|^2_{L^2(K_n)} + \|f\|^2_{L^2(K_n)} + \left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|^2_{L^2(K_n)}.\end{aligned}$$

By choosing ε sufficiently small we obtain

$$\|u\|^2_{H^2(K_n)} \leq C(\sigma)\left(\|f\|^2_{L^2(\Omega,m)} + \|u\|^2_{L^2(K_n)} + \left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|^2_{L^2(\Omega,m)}\right) \tag{3.2.8}$$

and, taking into account (3.2.6), we get the thesis. □

3.3 Existence and uniqueness results

We now consider the following abstract Cauchy problem, for $T > 0$ fixed:

$$(P)\begin{cases} u'(t) = Au(t) + f(t) & \text{for } t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{3.3.1}$$

where f and u_0 are given functions in suitable spaces and A is the operator associated to the energy form E. From semigroup theory we get the following existence and uniqueness result.

Theorem 3.3.1. *Let $\alpha \in (0,1)$, $f \in C^{0,\alpha}([0,T]; L^2(\Omega, m))$ and $u_0 \in D(A)$. We define*

$$u(t) = T_t\, u_0 + \int_0^t T_{t-\tau}\, f(\tau)\, \mathrm{d}\tau, \tag{3.3.2}$$

where T_t is the semigroup generated by the operator A. Then u defined in (3.3.2) is the unique strict *solution of problem (P), i.e. a function u such that $u'(t) = Au(t) + f(t)$ for all $t \in [0,T]$, $u(0) = u_0$ and*

$$u \in C^1([0,T]; L^2(\Omega, m)) \cap C([0,T]; D(A)).$$

Moreover the following estimate holds:

$$\|u\|_{C^1([0,T];L^2(\Omega,m))} + \|u\|_{C([0,T];D(A))} \leq C\left(\|u_0\|_{D(A)} + \|f\|_{C^{0,\alpha}([0,T];L^2(\Omega,m))}\right),$$

where C is a constant independent of n.

For the proof see Theorem 4.3.1 in [68].

Remark 3.3.2. If we suppose $u_0 \in \overline{D(A)}$ in Theorem 3.3.1, then the solution u of problem (3.3.1) is *classical*, i.e. it belongs to $C^1((0,T]; L^2(\Omega, m)) \cap C((0,T]; D(A)) \cap C([0,T]; L^2(\Omega, m))$ and it satisfies pointwise problem (3.3.1). In addition to that, we have that $u \in C^{1,\alpha}([\varepsilon, T]; L^2(\Omega, m)) \cap C^{0,\alpha}([\varepsilon, T]; D(A))$ for every $\varepsilon \in (0,T)$ (see Theorem 4.3.1 in [68]).

We now give the strong formulation of the abstract Cauchy problem (P).

Theorem 3.3.3. *Let u be the unique strict solution of (3.3.1) given by Theorem 3.3.1. Then, for every $t \in [0,T]$, it holds that*

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t}(t,x) = \Delta u(t,x) + f(t,x) & \text{for a.e. } x = (x_1, x_2) \in \Omega, \\ \frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{\partial u}{\partial \nu} + \Delta_{K_n} u - bu - \theta_{\frac{1}{2}}(u) + f & \text{in } H^{-\frac{1}{2}}(K_n), \\ u(0,x) = u_0(x) & \text{in } \overline{\Omega}. \end{cases} \tag{3.3.3}$$

Proof. For every fixed $t \in [0,T]$, we multiply the first equation in (3.3.1) by a test function $\varphi \in D(\Omega)$ and then we integrate on Ω. Then, by using (3.1.4), we obtain

$$\int_\Omega \frac{\mathrm{d}u}{\mathrm{d}t}\, \varphi\, \mathrm{d}\mathcal{L}_2 = \int_\Omega Au\, \varphi\, \mathrm{d}\mathcal{L}_2 + \int_\Omega f\, \varphi\, \mathrm{d}\mathcal{L}_2 = -E(u, \varphi) + \int_\Omega f\, \varphi\, \mathrm{d}\mathcal{L}_2.$$

Since φ has compact support in Ω, after integrating by parts, we get

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \Delta u + f \quad \text{in } (\mathcal{D}(\Omega))', \tag{3.3.4}$$

then, by density, equation (3.3.4) holds in $L^2(\Omega)$, so it holds for a.e. $x \in \Omega$. We remark that from this it follows that, for each fixed $t \in [0,T]$, $u \in V(\Omega) := \{u \in H^1(\Omega) \,|\, \Delta u \in L^2(\Omega)\}$, where Δu has to be intended in the distributional sense. Hence, we can apply Green formula for Lipschitz domains (see [7]) which yields in particular that $\frac{\partial u}{\partial \nu} \in H^{-\frac{1}{2}}(K_n)$:

$$\int_\Omega \mathrm{D}u\,\mathrm{D}v\,\mathrm{d}\mathcal{L}_2 = \left\langle \frac{\partial u}{\partial \nu}, v \right\rangle - \int_\Omega \Delta u\, v\,\mathrm{d}\mathcal{L}_2.$$

We now come to the dynamical boundary condition. Let $v \in V(\Omega, K_n)$. We take the scalar product in $L^2(\Omega, m)$ between the first equation in (3.3.1) and v, so we obtain

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}, v\right)_{L^2(\Omega,m)} = (Au, v)_{L^2(\Omega,m)} + (f, v)_{L^2(\Omega,m)}. \tag{3.3.5}$$

Then, by using again (3.1.4), we have that

$$\int_\Omega \frac{\mathrm{d}u}{\mathrm{d}t}\, v\,\mathrm{d}\mathcal{L}_2 + \int_{K_n} \frac{\mathrm{d}u}{\mathrm{d}t}\, v\,\mathrm{d}\ell =$$
$$-\int_\Omega \mathrm{D}u\,\mathrm{D}v\,\mathrm{d}\mathcal{L}_2 - \int_{K_n} \mathrm{D}_\ell u\,\mathrm{D}_\ell v\,\mathrm{d}\ell - \int_{K_n} b\,u\,v\,\mathrm{d}\ell - \langle \theta_{\frac{1}{2}}(u), v\rangle + \int_\Omega f\,v\,\mathrm{d}\mathcal{L}_2 + \int_{K_n} f\,v\,\mathrm{d}\ell.$$

Now, using Green formula for Lipschitz domains and the fact that equation (3.3.4) holds a.e. in Ω, we obtain $\forall\, v \in V(\Omega, K_n)$ and for each $t \in [0,T]$

$$\int_{K_n} \frac{\mathrm{d}u}{\mathrm{d}t}\, v\,\mathrm{d}\ell = -\left\langle \frac{\partial u}{\partial \nu}, v \right\rangle - \int_{K_n} \mathrm{D}_\ell u\,\mathrm{D}_\ell v\,\mathrm{d}\ell - \int_{K_n} b\,u\,v\,\mathrm{d}\ell - \langle \theta_{\frac{1}{2}}(u), v\rangle + \int_{K_n} f\,v\,\mathrm{d}\ell. \tag{3.3.6}$$

Since $H^1(K_n)$ is dense in $H^{\frac{1}{2}}(K_n)$ (see [7]), we deduce that the boundary condition

$$-\Delta_{K_n} u = -\frac{\partial u}{\partial \nu} - bu - \theta_{\frac{1}{2}}(u) + f - \frac{\mathrm{d}u}{\mathrm{d}t} \tag{3.3.7}$$

holds in $H^{-\frac{1}{2}}(K_n)$. $\square$

We now prove a better regularity in space of the solution of problem (3.3.3).

Theorem 3.3.4. *Let σ and f be as in Theorem 3.2.1. Then for every $t \in [0,T]$ the solution of problem (3.3.3) belongs to $V^2_\sigma(\Omega, K_n)$, and the following inequality holds:*

$$\|u\|^2_{H^1(\Omega)} + \|r^\sigma \mathrm{D}^2 u\|^2_{L^2(\Omega)} + \|u\|^2_{H^2(K_n)} \leq C\left(\|f\|^2_{L^2(\Omega,m)} + \left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|^2_{L^2(\Omega,m)}\right), \tag{3.3.8}$$

where C depends on σ.

Proof. We rewrite problem (3.3.3) as

$$\begin{cases} -\Delta u = f - \frac{\mathrm{d}u}{\mathrm{d}t} & \text{in } \Omega, \\ -\Delta_{K_n} u = -\frac{\partial u}{\partial \nu} - bu - \theta_{\frac{1}{2}}(u) + f - \frac{\mathrm{d}u}{\mathrm{d}t} & \text{on } K_n. \end{cases} \tag{3.3.9}$$

We note that, for every $t \in [0,T]$, $f - \frac{\mathrm{d}u}{\mathrm{d}t} \in L^2(\Omega, m)$. Hence, from elliptic regularity results of Theorem 2.2.1, we deduce that for every $t \in [0,T]$ $u \in V^2_\sigma(\Omega, K_n)$ and (3.3.8) holds. □

3.4 Regularity results in fractional Sobolev spaces

We now prove some regularity results for the strict solution u of (3.3.3).

Theorem 3.4.1. *Let u be the solution of problem (3.3.3). Then, for every fixed $t \in [0,T]$, $u \in H^s(\Omega)$ for $s < \frac{7}{4}$.*

Proof. Let us consider for every fixed $t \in [0,T]$ the weak solutions w and $\hat{w}$ in $H^1(\Omega)$ of the following auxiliary problems:

$$\begin{cases} \Delta \hat{w} = 0 & \text{in } \Omega \\ \hat{w} = u & \text{on } K_n, \end{cases} \tag{3.4.1}$$

$$\begin{cases} -\Delta w = -\frac{\mathrm{d}u}{\mathrm{d}t} + f & \text{in } \Omega \\ w = 0 & \text{on } K_n. \end{cases} \tag{3.4.2}$$

We point out that the regularity of the solution u of problem (3.3.3) follows from the regularity of $\hat{w}$ and w since

$$u = \hat{w} + w. \tag{3.4.3}$$

From Theorems 2 and 3 in [39], it follows that

$$\frac{\partial \hat{w}}{\partial \nu} \in L^2(K_n). \tag{3.4.4}$$

As to the solution w of problem (3.4.2), we remark that the right-hand side of the first equation belongs to $L^2(\Omega)$. From Kondrat'ev regularity results for the solutions of elliptic problems in corners (see [47]), since $f - \frac{\mathrm{d}u}{\mathrm{d}t} \in L^2_\mu(\Omega)$ for $\mu > 1/4$ (taking into account that the angles in Ω have opening equal to $\frac{\pi}{3}$ or $\frac{4\pi}{3}$), we get

$$\|\delta^\mu \mathrm{D}^\alpha w\|^2_{L^2(\Omega)} \le C(\mu, n) \left\| f - \frac{\mathrm{d}u}{\mathrm{d}t} \right\|_{L^2(\Omega)} \quad \text{for } |\alpha| = 2 \text{ and } \mu > \frac{1}{4}, \tag{3.4.5}$$

where $\delta = \delta(x)$ denotes the distance from the boundary. Now, by using Proposition 4.15 in [40], we have

$$\|w\|_{H^{2-\mu}(\Omega)} \leq c \left\{ \|\delta^\mu \sum_{|\alpha|=2} \mathrm{D}^\alpha w\|^2_{L^2(\Omega)} + \|w\|_{H^1(\Omega)} \right\}^{\frac{1}{2}}$$

and from (3.4.5) it follows that $w \in H^s(\Omega)$ for $s < 7/4$.
We now prove that $\hat{w}$ has the same regularity of w. Since $u \in H^2(K_n)$, in particular u belongs to $H^{\frac{3}{2}}(K_n)$. Then from the trace theorem (Proposition 1.2.1) there exists a function $\tilde{u}$ which belongs to $H^2(\Omega)$ and such that $\gamma_0 \tilde{u} = u$. If we consider then the function $\tilde{w} = \hat{w} - \tilde{u}$, this function belongs to $H^1(\Omega)$ and it is the weak solution of the auxiliary problem

$$\begin{cases} \Delta \tilde{w} = -\Delta \tilde{u} & \text{in } \Omega \\ \tilde{w} = 0 & \text{on } K_n. \end{cases} \tag{3.4.6}$$

Analogously, since $\Delta\tilde{u}$ belongs to $L^2(\Omega)$, we obtain that $\tilde{w}$ belongs to $H^2_\mu(\Omega)$ for $\mu > \frac{1}{4}$. This in particular implies that $\hat{w}$ belongs to $H^2_\mu(\Omega)$ for $\mu > \frac{1}{4}$, hence from Proposition 4.15 in [40] it follows that $u \in H^s(\Omega)$ for $s < \frac{7}{4}$. □

Remark 3.4.2. By proceeding as in Theorem 4.2 in [53], with the obvious changes, we can prove that $u \in H^2_\mu(\Omega)$ for $\mu > 1/4$, with weight given by the distance from the reentrant vertices (i.e. the vertices of the pre-fractal curve K_n).

Remark 3.4.3. From Theorem 3.4.1, we have that the solution of problem (P) is the solution of the following problem: for every $t \in [0, T]$,

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = \Delta u + f & \text{in } L^2(\Omega), \\ \frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{\partial u}{\partial \nu} + \Delta_{K_n} u - bu - \theta_{\frac{1}{2}}(u) + f & \text{in } L^2(K_n), \\ u(0,x) = u_0(x) & \text{a.e. } x \in \overline{\Omega}. \end{cases} \tag{3.4.7}$$

where Δ_{K_n} is the piecewise tangential Laplacian.

Since

$$u \in H^{\frac{7}{4}-\varepsilon}(\Omega) \quad \forall \varepsilon > 0, \tag{3.4.8}$$

from Sobolev embedding theorems (see Theorem 1.4.5.2 in [37]), we have that

$$u \in C^{0,\delta}(\overline{\Omega}) \text{ with } \delta = \tfrac{3}{4} - \varepsilon.$$

We remark that, just knowing that $u \in H^1(K_n)$, from Sobolev embedding theorems we deduce that $u \in C^{0,\frac{1}{2}}(K_n)$. From Theorem 3.4.1 we obtain a better regularity for the solution u of problem (3.4.7).
Moreover, since $u \in H^1(K_n)$ and $\mathrm{D}u \in H^{\frac{3}{4}-\varepsilon}(\Omega)$, from the trace theorem we have $\mathrm{D}u|_{K_n} \in H^{s_1}(K_n)$ for $0 < s_1 < \frac{1}{4}$.

Remark 3.4.4. For the reader's convenience, we now summarize the main regularity properties of the solution of problem (3.4.7) which will turn crucial in order to prove the a priori error estimates in Section 3.5: u belongs to $H^2_\mu(\Omega)$ for $\mu > 1/4$, it is Hölder continuous on $\overline{\Omega}$ with $\delta = \frac{3}{4} - \varepsilon$ and its trace belongs to $H^2(K_n)$.

3.5 A priori estimates

In this section we prove some a priori estimates for the solution u of problem (3.4.7).

Proposition 3.5.1. *Let u be the solution of problem (3.4.7). Then, for every fixed $t \in [0, T]$ it holds:*

$$\|u(t)\|^2_{L^2(\Omega,m)} + \int_0^t \|u(\tau)\|^2_{V(\Omega,K_n)} \,\mathrm{d}\tau \le C \left(\|u_0\|^2_{L^2(\Omega,m)} + \int_0^t \|f(\tau)\|^2_{L^2(\Omega,m)} \,\mathrm{d}\tau \right), \quad (3.5.1)$$

where C is a constant depending on t and on the coercivity constant of E.

Proof. We write the weak formulation of problem (3.4.7): for each $t \in [0, T]$,

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}(t), v \right)_{L^2(\Omega,m)} + E(u(t), v) = (f(t), v)_{L^2(\Omega,m)} \quad \forall\, v \in V(\Omega, K_n). \quad (3.5.2)$$

If we choose $v = u(t)$, thanks to the coercivity of E we have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\Omega,m)} + \|u(t)\|^2_{V(\Omega,K_n)} \le C_1 (f(t), v)_{L^2(\Omega,m)}. \quad (3.5.3)$$

We observe that, by using Cauchy-Schwartz and Young inequalities, we obtain

$$(f(t), u(t))_{L^2(\Omega,m)} \le \|f(t)\|_{L^2(\Omega,m)}\, \|u(t)\|_{L^2(\Omega,m)} \le \frac{1}{2}\|f(t)\|^2_{L^2(\Omega,m)} + \frac{1}{2}\, \|u(t)\|^2_{L^2(\Omega,m)}.$$

Hence estimate (3.5.3) can be written in the following way:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\Omega,m)} + \|u(t)\|^2_{V(\Omega,K_n)} \le C_2 (\|f(t)\|^2_{L^2(\Omega,m)} + \|u(t)\|^2_{L^2(\Omega,m)}). \quad (3.5.4)$$

From the differential version of Gronwall inequality (see [29, page 624]), (3.5.4) implies that

$$\|u(t)\|^2_{L^2(\Omega,m)} \le C_3 \left(\|u_0\|^2_{L^2(\Omega,m)} + \int_0^t \|f(\tau)\|^2_{L^2(\Omega,m)} \,\mathrm{d}\tau \right), \quad (3.5.5)$$

where C_3 depends on t. Now, by integrating (3.5.4) in $[0, t]$ and by using (3.5.5), we get

$$\int_0^t \|u(\tau)\|^2_{V(\Omega,K_n)} \,\mathrm{d}\tau \le C \left(\|u_0\|^2_{L^2(\Omega,m)} + \int_0^t \|f(\tau)\|^2_{L^2(\Omega,m)} \,\mathrm{d}\tau \right), \quad (3.5.6)$$

where C depends on the coercivity constant of E and on t. Putting together (3.5.5) and (3.5.6), we get the thesis. □

Theorem 3.5.2. *Let u be the solution of problem (3.4.7). Then it holds that*

$$\begin{aligned}\int_0^T \left\| \frac{\mathrm{d}u}{\mathrm{d}t}(\tau) \right\|^2_{L^2(\Omega,m)} \mathrm{d}\tau + \sup_{t\in[0,T]} \|u(t)\|^2_{V(\Omega,K_n)} \\ \leq \frac{1}{\min\{1,\bar{C}\}} \left(\tilde{C}\|u_0\|^2_{V(\Omega,K_n)} + \int_0^T \|f(\tau)\|^2_{L^2(\Omega,m)}\,\mathrm{d}\tau \right),\end{aligned} \tag{3.5.7}$$

where $\bar{C}$ is the coercivity constant of E, while the constant $\tilde{C}$ depends on n.

Proof. In order to prove this estimate, we use the *Faedo-Galerkin* method (see Section 7.1.2 in [29]). Let $\{\phi_j\}_{j=1}^{\infty}$ be a complete orthonormal basis of $V(\Omega,K_n)$, and $V^N = \mathrm{span}\{\phi_1,\dots,\phi_N\}$. We define $u_N\colon [0,T] \to V^N$ in the following way:

$$u_N(t) := \sum_{j=1}^{N} d_N^j(t)\,\phi_j, \tag{3.5.8}$$

where we select the coefficients $d_N^j(t)$, for $t \in [0,T]$ and $j = 1,\dots,N$ such that

$$d_N^j(0) = (u_0,\phi_j)_{L^2(\Omega,m)} \tag{3.5.9}$$

and

$$\left(\frac{\mathrm{d}u_N}{\mathrm{d}t}(t),\phi_j \right)_{L^2(\Omega,m)} + E(u_N(t),\phi_j) = (f(t),\phi_j)_{L^2(\Omega,m)} \quad \forall\, j = 1,\dots,N. \tag{3.5.10}$$

The existence and uniqueness of a function u_N of the form (3.5.8) follows from standard ODE theory.

Now we multiply equation (3.5.10) by $(d_N^j)'(t)$. Then, by taking the sum on $j = 1,\dots,N$, we obtain

$$\left(\frac{\mathrm{d}u_N}{\mathrm{d}t}(t), \frac{\mathrm{d}u_N}{\mathrm{d}t}(t) \right)_{L^2(\Omega,m)} + E\left(u_N(t), \frac{\mathrm{d}u_N}{\mathrm{d}t}(t) \right) = \left(f(t), \frac{\mathrm{d}u_N}{\mathrm{d}t}(t) \right)_{L^2(\Omega,m)}. \tag{3.5.11}$$

We point out that

$$\begin{aligned}E\left(u_N,\frac{\mathrm{d}u_N}{\mathrm{d}t}\right) = \int_\Omega \mathrm{D}u_N\,\mathrm{D}\left(\frac{\mathrm{d}u_N}{\mathrm{d}t}\right)\mathrm{d}\mathcal{L}_2 + \int_{K_n} \mathrm{D}_\ell u_N\,\mathrm{D}_\ell\left(\frac{\mathrm{d}u_N}{\mathrm{d}t}\right)\mathrm{d}\ell + \int_{K_n} b\,u_N\,\frac{\mathrm{d}u_N}{\mathrm{d}t}\,\mathrm{d}\ell\, + \\ \iint_{K_n\times K_n} \frac{(u_N(x)-u_N(y))(\frac{\mathrm{d}u_N}{\mathrm{d}t}(x) - \frac{\mathrm{d}u_N}{\mathrm{d}t}(y))}{|x-y|^2}\,\mathrm{d}\ell(x)\,\mathrm{d}\ell(y) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E[u_N].\end{aligned}$$

Then, from Cauchy-Schwartz and Young inequalities, (3.5.11) can be written in this way:

$$\frac{1}{2}\left\|\frac{\mathrm{d}u_N}{\mathrm{d}t}(t)\right\|^2_{L^2(\Omega,m)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E[u_N] \leq \frac{1}{2}\|f(t)\|^2_{L^2(\Omega,m)}. \tag{3.5.12}$$

Now, integrating (3.5.12) on $(0,t)$, using the coercivity of E and taking the supremum on $[0,T]$ we obtain

$$\begin{aligned}&\int_0^T \left\|\frac{\mathrm{d}u_N}{\mathrm{d}t}(\tau)\right\|^2_{L^2(\Omega,m)} \mathrm{d}\tau + \sup_{t\in[0,T]} \|u_N(t)\|^2_{V(\Omega,K_n)}\\ &\leq \frac{1}{\min\{1,\bar{C}\}}\left(E[u_N(0)] + \int_0^T \|f(\tau)\|^2_{L^2(\Omega,m)}\,\mathrm{d}\tau\right),\end{aligned} \tag{3.5.13}$$

where $\bar{C}$ is the coercivity constant of E. We now point out that, since $\langle\theta_{\frac{1}{2}}(u),u\rangle \leq \tilde{C}_1(n)\|u\|^2_{H^1(K_n)}$, it follows that

$$E[u_N(0)] \leq \tilde{C}_2(n)\|u_0\|^2_{V(\Omega,K_n)}.$$

Hence we get

$$\begin{aligned}&\int_0^T \left\|\frac{\mathrm{d}u_N}{\mathrm{d}t}(\tau)\right\|^2_{L^2(\Omega,m)} \mathrm{d}\tau + \sup_{t\in[0,T]} \|u_N(t)\|^2_{V(\Omega,K_n)}\\ &\leq \frac{1}{\min\{1,\bar{C}\}}\left(\tilde{C}_2(n)\|u_0\|^2_{V(\Omega,K_n)} + \int_0^T \|f(\tau)\|^2_{L^2(\Omega,m)}\,\mathrm{d}\tau\right),\end{aligned} \tag{3.5.14}$$

i.e. the thesis for u_N.
Now we want to prove the estimate for u. At first we observe that from (3.5.14) it follows that $u_N \in L^\infty([0,T];V(\Omega,K_n))$ and $\frac{\mathrm{d}u_N}{\mathrm{d}t} \in L^2([0,T];L^2(\Omega,m))$. Then, we point out that by multiplying (3.5.10) by d^j_N, we can obtain an estimate similar to (3.5.1) for u_N. Since

$$u_N \in L^2([0,T];V(\Omega,K_n)),$$

there exists a subsequence $\{u_{N_k}\}$ weakly converging to a function w in $L^2([0,T];V(\Omega,K_n))$, and also in $L^\infty([0,T];V(\Omega,K_n))$. Moreover, from (3.5.14) we deduce that $\frac{\mathrm{d}u_{N_k}}{\mathrm{d}t} \rightharpoonup \frac{\mathrm{d}w}{\mathrm{d}t}$ in $L^2([0,T];L^2(\Omega,m))$ when $k\to+\infty$.
We wish to prove that the limit w is the weak solution of (3.5.2). We define the following function for $m\leq N$:

$$v(t) := \sum_{j=1}^m d^j(t)\,\phi_j,$$

where $\{d^j\}_{j=1}^m$ are given smooth functions. After multiplying (3.5.10) by d^j, summing for $j=1,\dots,m$ and integrating on $[0,T]$, we obtain

$$\int_0^T \left[\left(\frac{\mathrm{d}u_N}{\mathrm{d}t}(\tau),v(\tau)\right)_{L^2(\Omega,m)} + E(u_N(\tau),v(\tau))\right]\mathrm{d}\tau = \int_0^T (f,v(\tau))\,\mathrm{d}\tau.$$

By setting $N = N_k$, from the weak convergence we get that w satisfies (3.5.2) for a.e. $t \in [0,T]$, hence u_N weakly converges to the solution of (3.4.7). Therefore, the thesis follows from the weak lower semicontinuity of the $L^\infty([0,T];V(\Omega,K_n))$ and $L^2([0,T];L^2(\Omega,m))$-norms and from (3.5.14) (we point out that the right-hand side of (3.5.14) does not depend on N). □

Corollary 3.5.3. *Let u be the solution of problem (3.4.7). Then it holds that*

$$\begin{aligned}\int_0^T \left(\left\| \frac{\mathrm{d}u}{\mathrm{d}t}(\tau) \right\|^2_{L^2(\Omega,m)} + \|u(\tau)\|^2_{H^2_\mu(\Omega)} \right) \mathrm{d}\tau + \sup_{t\in[0,T]} \|u(t)\|^2_{V(\Omega,K_n)} \\ \leq C \left(\|u_0\|^2_{V(\Omega,K_n)} + \int_0^T \|f(\tau)\|^2_{L^2(\Omega,m)}\, \mathrm{d}\tau \right),\end{aligned} \tag{3.5.15}$$

where C is a constant depending on μ, n, T and the coercivity constant of E.

Proof. Estimate (3.5.15) follows from (3.4.5), (3.5.1) and (3.5.7). □

3.6 Numerical approximation and a priori error estimates

We now focus our attention on the numerical approximation of problem (P). It will be carried out in two steps. In the former one we discretize by a Galerkin method the space variable only. We obtain an a priori error estimate for the semi-discrete solution. In the latter one we consider the fully discretized problem by a finite difference approach on the time variable.
In order to obtain optimal a priori error estimates, we use a suitable mesh (developed in [21] and [20], see Figure 3.1) which is compliant with the *Grisvard conditions*. More precisely, since the solution u is not in $H^2(\Omega)$, one has to use a suitable mesh refinement process which guarantees an optimal rate of convergence.
We consider a family of triangulations $\{T_{n,h}\}$, where n is the step of the approximation of the Koch snowflake and h is the mesh size of the triangulation, i.e.

$$h = \max_{S\in T_{n,h}} h_S,$$

where h_S is the diameter of the triangle $S \in T_{n,h}$. We require that this family of triangulations is *regular* and *conformal*.

Definition 3.6.1. *A family of triangulations $\{T_h\}$, $h > 0$, is regular if there exists a constant $\lambda \geq 1$ such that*

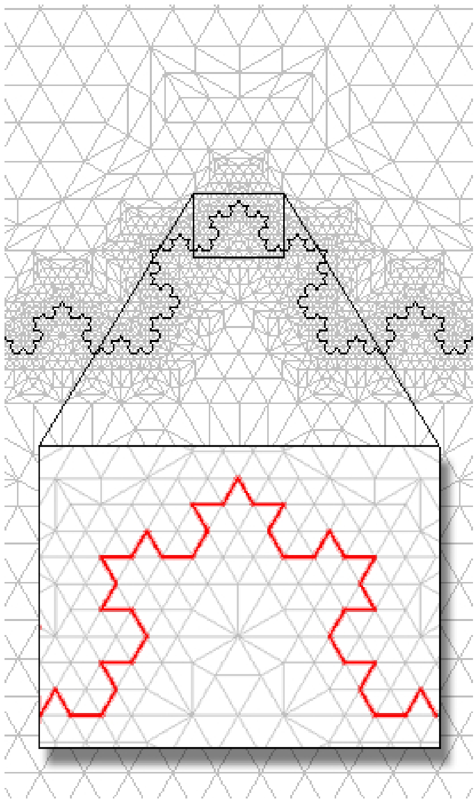

Figure 3.1: A zoom of the mesh considered.

$$\max_{S \in T_h} \frac{h_S}{\rho_S} \leq \lambda,$$

where ρ_S is the radius of the biggest circle inscribed in S. λ is the regularity constant of the mesh.
The family of triangulations $\{T_h\}$ is conformal if the intersection between two triangles of the family is either a vertex or an edge.

For more details on regular and conformal triangulations, we refer to [78].
We now have to make more assumptions on our family of triangulations $\{T_{n,h}\}$. Since the domain Ω we consider is not convex, the presence of the "reentrant" vertices deteriorates the regularity of both the strong and the numerical approximated solutions. Hence, we have to require that the mesh refinement process produces a family of triangulations compliant with some particular conditions, first formulated by Grisvard [37], which are adapted to our context in the following Definition.

Definition 3.6.2 (Grisvard conditions)**.** *A family of conformal and regular triangulations $\{T_{n,h}\}$ of Ω is compliant with the Grisvard conditions if each triangulation $T_{n,h}$*

of the family satisfies the following statements:

$$(a) \quad h_S \leq \lambda h^{\frac{1}{1-\mu}} \quad \textit{for every } S \textit{ having at least one reentrant vertex,} \tag{3.6.1}$$

$$(b) \quad h_S \leq \lambda h \inf_{x \in S} \tilde{r}(x)^\mu \quad \textit{for any other triangle } S, \tag{3.6.2}$$

where λ is the regularity constant of the mesh, μ is a suitable constant, $\tilde{r}$ is the so-called weighting distance defined as follows (where $\eta_n = \frac{1}{4}\frac{1}{3^n}$):

$$\tilde{r}(x) = \begin{cases} |x-P| & \textit{if } x \in B(P,\eta_n) \textit{ for some } P \in \mathcal{Q}, \\ 1 & \textit{if } x \notin \cup_{P\in\mathcal{Q}} B(P, 2\eta_n), \\ \frac{1-\eta_n}{\eta_n}(|x-P|-\eta_n)+\eta_n & \textit{otherwise,} \end{cases}$$

where $\mathcal{Q}$ is the set of reentrant vertices introduced in Section 1.1.1.

The number μ appearing in the above definition is the weight exponent of the Sobolev space to which u belongs (see Remark 3.4.4).
Finally we make two more assumptions on the mesh algorithm:

(c) $h \to 0$ when $j \to \infty$, where j is the iteration number of the mesh algorithm;

(d) the family $\{T_{n,h}\}$ is a sequence of *nested* refinements, i.e. all the nodes of $T_{n,h}$ are also nodes of $T_{n+1,h}$ and the mesh size $h \to 0$ when $n \to \infty$.

Hypothesis (c) guarantees the convergence of the finite element method, while hypothesis (d) allows a more accurate computation of the numerical solution and bounds the growth of the complexity of the numerical problems associated to the subsequent refinements.
We denote by $X_h := \{v \in C(\Omega) : v|_S \in \mathbb{P}_1 \, \forall \, S \in T_{n,h}\}$, where $\mathbb{P}_1$ denotes the space of polynomial functions of degree one. Let $I_h \colon H^2_\mu(\Omega) \to X_h$, with $\mu > 1/4$, be the X_h-interpolant operator, defined as:

$$I_h(u)|_S \in \mathbb{P}_1 \text{ for every } S \in T_{n,h} \text{ and } I_h(u) = u \text{ at any vertex of any } S \in T_{n,h}.$$

We note that I_h is well defined since u is in particular continuous on $\overline{\Omega}$ (see Remark 3.4.4). From Theorem 8.4.1.6 in [37], we deduce the following result.

Theorem 3.6.3. *Let $u \in H^2_\mu(\Omega)$. Let $\{T_{n,h}\}_{h>0}$ be a family of conformal and regular triangulations over Ω compliant with the Grisvard conditions (see Definition 3.6.2). Let $\{T_{n,h}\}_{h>0}$ be locally refined towards reentrant corners in the following sense: for $0 < \mu < 1$, we have as $h \to 0$*

1) $h_S \leq \sigma h^{\frac{1}{1-\mu}}$ for all $S \in T_{n,h}$ such that at least one vertex coincides with P_j for some $j \in \mathcal{Q}$;

2) $h_S \leq \sigma h \inf_S \tilde{r}^\mu$ for all $S \in T_{n,h}$ with no vertex coinciding with any P_j for every $j \in \mathfrak{Q}$.

Then there exists a constant C independent of h such that

$$\|\mathrm{D}(u - I_h(u))\|_{L^2(\Omega)} \leq C\, h \left\{ \sum_{|\beta|=2} \|\tilde{r}^\mu \mathrm{D}^\beta u\|^2_{L^2(\Omega)} \right\}^{\frac{1}{2}}.$$

We now discretize problem (3.4.7) in space first. We define the finite dimensional space of piecewise linear functions

$$X_{n,h} := \{v \in C(\overline{\Omega}) \,:\, v|_S \in \mathbb{P}_1 \,\forall\, S \in T_{n,h}\}.$$

We set $V_{n,h} := X_{n,h} \cap H^1(\Omega)$. We have that $V_{n,h} \subset V(\Omega, K_n)$, it is a finite dimensional space of dimension N_h, where N_h is the number of inner nodes of $T_{n,h}$. The semi-discrete approximation problem is the following:
for $f \in C^{0,\alpha}([0,T]; L^2(\Omega, m))$, $u_h^0 \in V_{n,h}$ such that $u_h^0 \to u_0$ in $L^2(\Omega, m)$ and for each $t \in [0,T]$, find $u_{n,h}(t) \in V_{n,h}$ such that

$$(\overline{P_{n,h}}) \begin{cases} \left(\dfrac{\mathrm{d}u_{n,h}}{\mathrm{d}t}(t), v_h \right)_{L^2(\Omega,m)} + E(u_{n,h}(t), v_h) = (f, v_h)_{L^2(\Omega,m)} & \forall\, v_h \in V_{n,h} \\ u_{n,h}(0) = u_h^0. \end{cases} \tag{3.6.3}$$

The existence and uniqueness of the semi-discrete solution $u_{n,h}(t) \in V_{n,h}$ of problem $(\overline{P_{n,h}})$ follows since problem $(\overline{P_{n,h}})$ is a Cauchy problem for a system of first order linear ordinary differential equations with constant coefficients (see e.g. [78]). By proceeding as in the proof of Proposition 3.5.1, we can prove an estimate similar to (3.5.1), hence we have the stability of the method.
We now recall some key estimates of the interpolation error (see Proposition 4, Lemma 1 and Theorem 5.1 in [20] and the references listed in).

Theorem 3.6.4. *Let $u(t)$ be the solution of problem (3.4.7) and let $I_h(u)$ be the interpolant polynomial of u. Then, for every $t \in [0,T]$ it holds that*

$$\|\mathrm{D}(u(t) - I_h(u(t)))\|^2_{L^2(\Omega)} + \|u(t) - I_h(u(t))\|^2_{H^1(K_n)} \leq c\, h^2 \left(\|u\|^2_{H^2_\mu(\Omega)} + \|u\|^2_{H^2(K_n)} \right), \tag{3.6.4}$$

where c is a positive constant.

Proposition 3.6.5. *Let $u(t)$ and $I_h(u(t))$ be as above. Then there exists a constant $C > 0$ independent of the triangle S such that*

$$\|u(t) - I_h(u(t))\|^2_{L^2(S)} \leq C\, h_S^4 \frac{1}{\rho_S^{2\mu}} \sum_{|\alpha|=2} \int_S r(x)^{2\mu}\, |\mathrm{D}^\alpha u(t)|^2 \,\mathrm{d}\mathcal{L}_2, \tag{3.6.5}$$

where h_S is the diameter of the triangle $S \in T_{n,h}$ and ρ_S is the radius of the biggest circle inscribed in S.

Proposition 3.6.6. *Let $u(t)$ and $I_h(u(t))$ be as above. Then for every $t \in [0,T]$ there exists a constant $C > 0$ such that*

$$\|u(t) - I_h(u(t))\|^2_{L^2(\Omega)} \leq C\, h^4 \|u(t)\|^2_{H^2_\mu(\Omega)}. \tag{3.6.6}$$

We now give an optimal error estimate with respect to the norm of $L^2([0,T]; V(\Omega, K_n))$ for piecewise linear polynomials only.

Theorem 3.6.7. *Let u be the solution of (3.4.7) and $u_{n,h}$ be the discrete solution of (3.6.3). Then, for each $t \in [0,T]$, it holds that*

$$\|u(t) - u_{n,h}(t)\|^2_{L^2(\Omega,m)} + \int_0^t \|u(\tau) - u_{n,h}(\tau)\|^2_{V(\Omega,K_n)}\, \mathrm{d}\tau \leq$$
$$\|u_0 - u_h^0\|^2_{L^2(\Omega,m)} + C\, h^2 \int_0^t \|f(\tau)\|^2_{L^2(\Omega,m)}\, \mathrm{d}\tau,$$

where C is a suitable constant independent of h.

For the proof we refer to Theorem 5.2 in [20] with small suitable changes.

We now consider the fully discretized problem, obtained by applying a finite difference scheme, the so-called θ-method, on the time variable. It is well known that the θ-method is unconditionally stable with respect both to the $L^2(\Omega)$ norm and to the $L^2(\Omega, m)$ norm provided $\frac{1}{2} \leq \theta \leq 1$. On the contrary, in the case of $0 \leq \theta < \frac{1}{2}$, one has to assume that $\{T_{n,h}\}$ is a quasi-uniform family of triangulations and that a restriction on the time step holds. Since the peculiarity of the family $\{T_{n,h}\}$ is not to be quasi-uniform, from now on we assume $\frac{1}{2} \leq \theta \leq 1$. An error estimate between the semi-discrete solution $u_{n,h}(t)$ and the fully discrete one $u^l_{n,h}$ can be obtained as in Theorem 6.1 in [20]. From this estimate and Theorem 3.6.7 we deduce the following convergence result.

Theorem 3.6.8. *We set $t_l = l\Delta t$ for $l = 0, 1, \ldots, \mathcal{M}$, $\Delta t > 0$ being the time step and $\mathcal{M}$ being the integer part of $T/\Delta t$. Assume that $f \in C^{0,\alpha}([0,T]; L^2(\Omega, m))$ and $\frac{\partial f}{\partial t} \in L^2([0,T] \times \Omega, \mathrm{d}t \times \mathrm{d}m)$. Let n be fixed and $u(t)$ be the solution of problem (3.4.7), and let $u^l_{n,h}$ be the fully discretized solution with the same initial datum u^0_h as given by the θ-method with $\frac{1}{2} \leq \theta \leq 1$. Then for every $l = 0, 1, \ldots, \mathcal{M}$*

$$\|u(t_l) - u^l_{n,h}\|^2_{L^2(\Omega,m)} \leq \|u_0 - u^0_h\|^2_{L^2(\Omega,m)} + C\, h^2 \left(\int_0^T \|f(\tau)\|^2_{L^2(\Omega,m)}\, \mathrm{d}\tau \right) +$$
$$C_\theta\, \Delta t^2 \left(\left\| \frac{\mathrm{d}u_{n,h}}{\mathrm{d}t}(0) \right\|^2_{L^2(\Omega,m)} + \int_0^T \left\| \frac{\partial f}{\partial t}(\tau) \right\|^2_{L^2(\Omega,m)} \mathrm{d}\tau \right),$$

where C_θ is a constant independent of $\mathcal{M}$, Δt and h and is a non-decreasing function of the continuity constant of $E(\cdot,\cdot)$ and T.

Remark 3.6.9. We point out that the norm $\left\|\frac{\mathrm{d}u_{n,h}}{\mathrm{d}t}(0)\right\|^2_{L^2(\Omega,m)}$ appearing in the above theorem can be estimated by $\|Au_0 + f(0)\|^2_{L^2(\Omega,m)}$. Indeed, proceeding as in Remark 11.3.1 in [78] with suitable changes, we take $u_h^0 = \Pi^k_{1,h}(u_0)$, where $\Pi^k_{1,h}$ is the "elliptic projection operator". Hence, taking into account (3.1.4), we get

$$\left\|\frac{\mathrm{d}u_{n,h}}{\mathrm{d}t}(0)\right\|^2_{L^2(\Omega,m)} = \left(Au_0 + f(0), \frac{\mathrm{d}u_{n,h}}{\mathrm{d}t}(0)\right)_{L^2(\Omega,m)},$$

and the thesis follows from Cauchy-Schwarz inequality.

3.7 Numerical results and conclusions

In this section we present some numerical results concerning the transmission problem defined at the end of Section 3.4. We consider the domain illustrated in Figure 3.2. A highly conductive pre-fractal interface $K_n = K_{n,a} \cup K_{n,b}$, delimiting a non-convex polygonal domain Ω_1, is placed at the center of a square domain Ω_2 to study the heat transmission across the pre-fractal. In order to appreciate its role in the transmission process, we consider the nonlocal term $\theta_{\frac{1}{2}}(u)$ active only on the portion $K_{n,a}$ of the pre-fractal interface (in red in the figure). Defining symmetric conditions with respect to the geometry of the problem (in terms of boundary conditions and heat sources), we will be able to compare the heat flux that crosses the interface where the nonlocal term is present with the heat flux that crosses the interface where the nonlocal term is not active, and evaluate, numerically, the influence of the nonlocal term in the heat exchange process.
The dimensional equations of the problem are:

$$\begin{cases} \rho\, C_p\, \frac{\mathrm{d}u}{\mathrm{d}t} = k\, \Delta u + f & \text{in } L^2(\Omega), \\ \rho_s\, C_{p,s}\, \frac{\mathrm{d}u}{\mathrm{d}t} = -(\nu_1\, k_1\, \mathrm{D}u_1 + \nu_2\, k_2\, \mathrm{D}u_2) + k_s\, (\Delta_{K_{n,a}} u - \theta_{\frac{1}{2}}(u)) + f & \text{in } L^2(K_{n,a}), \\ \rho_s\, C_{p,s}\, \frac{\mathrm{d}u}{\mathrm{d}t} = -(\nu_1\, k_1\, \mathrm{D}u_1 + \nu_2\, k_2\, \mathrm{D}u_2) + k_s\, \Delta_{K_{n,b}} u + f & \text{in } L^2(K_{n,b}), \\ u(0,x) = u_0(x) & \forall\, x \in \overline{\Omega}, \\ u(t,x) = 0 & \forall\, t, \forall\, x \in \partial\Omega \end{cases}$$

where

- $\Omega = \Omega_1 \cup \Omega_2$;

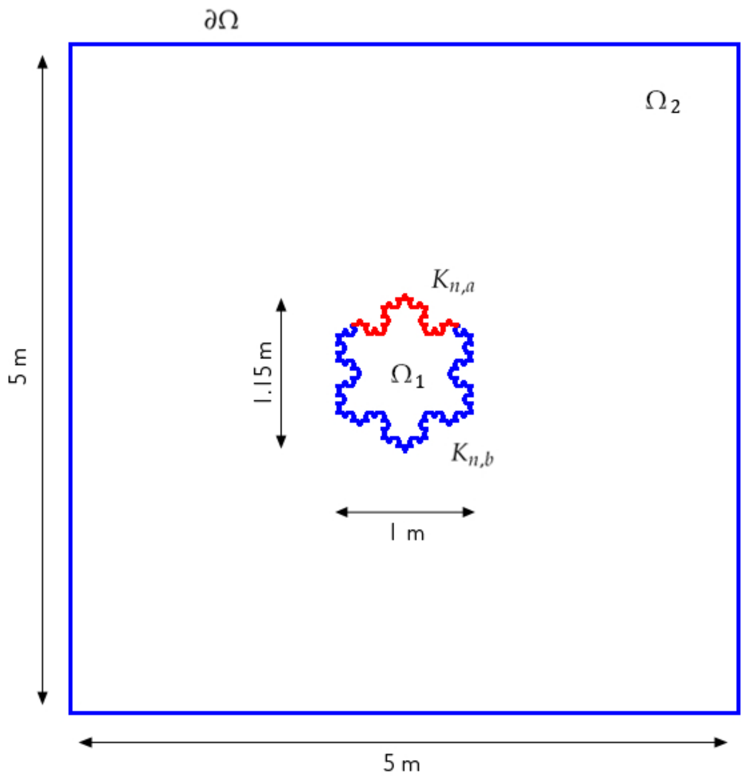

Figure 3.2: The domain of the problem.

- ρ is the material density in the bulk domain Ω (in Kg/m^3);
- ρ_s is the material density per meter in the boundary domain K_n (in Kg/m^2);
- C_p and $C_{p,s}$ are the heat capacity at constant pressure (in J/(Kg $\cdot$ K));
- k is the thermal conductivity in Ω (in W/(m $\cdot$ K)); $k_1 = k|_{\Omega_1}$ and $k_2 = k|_{\Omega_2}$;
- k_s is the thermal conductivity per meter in K_n (in W/K);
- ν_1 and ν_2 are the outwards normal vectors on K_n for Ω_1 and Ω_2 respectively;
- the term f represents a thermal source (in W/m^3);
- u is the unknown variable: the temperature in Kelvin degrees;
  $u_1 = u|_{\Omega_1}$ and $u_2 = u|_{\Omega_2}$.

The term bu which appears in the equations of the problem defined in Section 3.4 has been omitted here ($b = 0$) to emphasize the role of the nonlocal term $\theta_{\frac{1}{2}}(u)$ in the transmission problem. The operator $\theta_{\frac{1}{2}}(u)$ is defined by the duality pairing between $H^{-\frac{1}{2}}(K_n)$ and $H^{\frac{1}{2}}(K_n)$. For every $u, v \in H^{\frac{1}{2}}(K_n)$, we define

$$\langle \theta_{\frac{1}{2}}(u), v \rangle_{H^{-\frac{1}{2}}(K_n),H^{\frac{1}{2}}(K_n)} = \iint\limits_{K_n\times K_n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^2}\, \mathrm{d}\ell(x)\, \mathrm{d}\ell(y).$$

We observe that the term $\langle \theta_{\frac{1}{2}}(u), v \rangle$ can be rewritten in the following way:

$$\langle \theta_{\frac{1}{2}}(u), v \rangle =$$

$$\iint\limits_{K_n\times K_n} \frac{(u(x)-u(y))v(x)}{|x-y|^2}\, \mathrm{d}\ell(x)\, \mathrm{d}\ell(y) - \iint\limits_{K_n\times K_n} \frac{(u(x)-u(y))v(y)}{|x-y|^2}\, \mathrm{d}\ell(x)\, \mathrm{d}\ell(y) =$$
$$\iint\limits_{K_n\times K_n} \frac{(u(x)-u(y))v(x)}{|x-y|^2}\, \mathrm{d}\ell(x)\, \mathrm{d}\ell(y) + \iint\limits_{K_n\times K_n} \frac{(u(y)-u(x))v(y)}{|x-y|^2}\, \mathrm{d}\ell(x)\, \mathrm{d}\ell(y) =$$
$$2\iint\limits_{K_n\times K_n} \frac{(u(x)-u(y))v(x)}{|x-y|^2}\, \mathrm{d}\ell(x)\, \mathrm{d}\ell(y) = 2\int\limits_{K_n} (Iu)(x)v(x)\, \mathrm{d}\ell(x),$$

where $(Iu)(x) := \int\limits_{K_n} \frac{(u(x)-u(y))}{|x-y|^2}\, \mathrm{d}\ell(y)$. The last expression has been exploited for the implementation of the problem in the weak form. The simulations have been performed on Comsol V.3.5a, on a desktop computer with a quad-core Intel processor (i5-2320) running at 3.00 GHz and equipped with 8 GB RAM.
Table 3.1 shows the numerical values used for the parameters above defined.

ρ	ρ_s	C_p	$C_{p,s}$	k_s
8000	21000	450	150	10^6

Table 3.1: Numerical values used in the simulations for the physical coefficients.

The thermal conductivity k instead, has been defined variable in Ω as shown in Figure 3.3. The domain has been ideally divided into eight sectors. The thermal conductivity is constant within each sector and variable from one sector to the subsequent one, so as to have alternations of very low values ($k = 1$: isolating material) and high values ($k = 1000$: good conductive material) between adjacent sectors.
The alternation of high and low values of the thermal conductivity in adjacent sectors separated by the pre-fractal layer is used to force the heat flow *along* the pre-fractal on the east and west parts of the barrier, and *through* the barrier on the north and south parts.
The thermal source is defined as a 2D gaussian curve with a very low variance, in order to represent a flame concentrated at the center of Ω_1:

$$f(x) = 10^5\ e^{-0.5((x_1-\bar{x}_1)^2+(x_2-\bar{x}_2)^2)/0.001}$$

where $(\bar{x}_1, \bar{x}_2)$ are the coordinates of the center of Ω_1. Figure 3.4 shows a 3D representation of the source term.

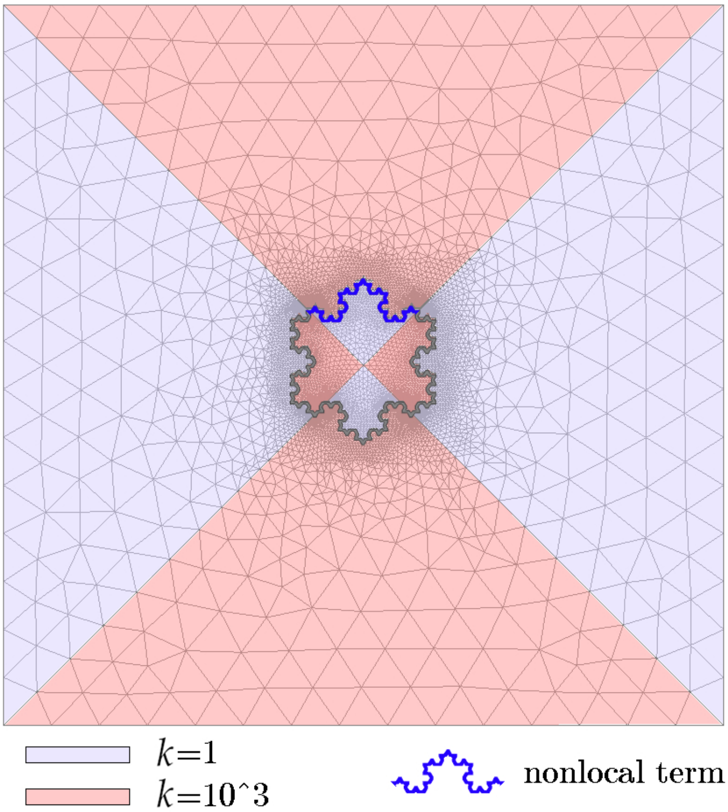

Figure 3.3: Definition of the thermal conductivity in the bulk domain Ω.

Taking into account our choices on the location of the source term and the boundary conditions, the heat flows from the center of Ω_1 (where the heat source has the maximum) towards the domain Ω_2 and reaches the boundary $\partial\Omega$ where the temperature is kept constant at 0 (Dirichlet conditions). In the east and west sectors in Ω_1, the heat produced by the source travels in the domain pushed by a high conductivity and reaches the pre-fractal barrier along the shortest possible path (ideally a straight line, which in the simulation takes the form of a slightly curved line because of numerical errors induced by the finite triangulation of the domain). As the barrier is reached, only a small part of the heat passes through it, because on the other side of the barrier there are two low conductivity areas that are holding the thermal flow. The heat mainly flows *along* the barrier (which is by assumption a highly conductive layer) until it reaches the north and south sectors, where, beyond the barrier there are again high conductivity areas.
Summarizing, the heat moves from the center of the domain Ω to the boundary $\partial\Omega$, and crosses the fractal layer mainly on the north and south sectors. The difference of the flow entity along these two main directions is due the role of the nonlocal term.

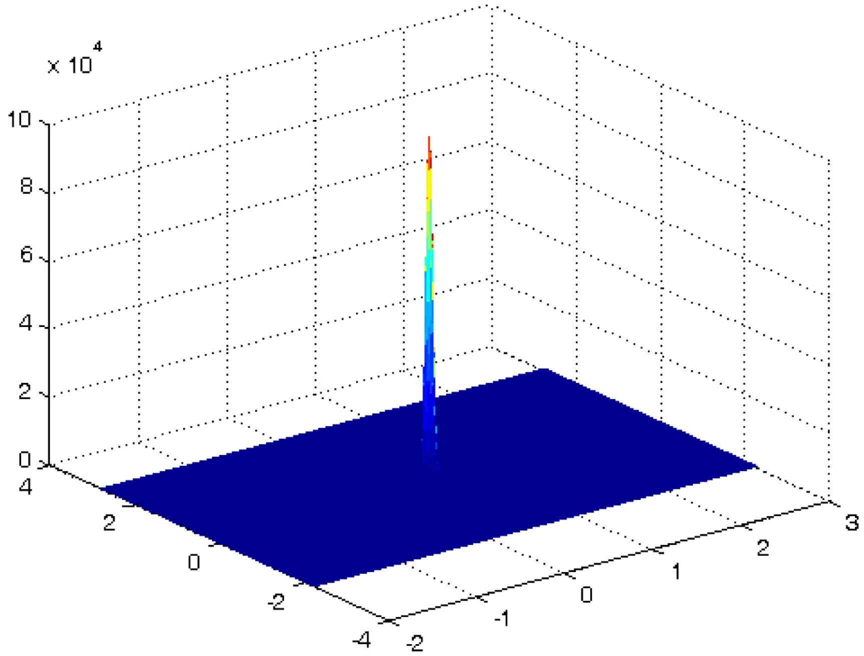

Figure 3.4: The source term $f(x)$.

The numerical simulations confirm that the nonlocal term is responsible for a larger flux across the barrier in the north sector.
Figure 3.5 shows the main streamlines of the heat flux for the stationary solution.
The streamlines have been drawn with a density in the domain proportional to the magnitude of the vector field to which they are tangent. Observe that the pre-fractal is crossed by much more lines in the north sector than everywhere else. This means that the amplitude of the heat flux that crosses the barrier in the north sector is much higher than in other sectors, and this is due to the presence of the nonlocal term.
Figure 3.6 shows a three-dimensional representation of the same streamlines. The height of the curves is proportional to the magnitude of the heat flow. This figure confirms that the heat flux across the barrier in the east and west sectors is negligible (the corresponding curves are almost completely flat). Most of the flux across the barrier takes place in the north and south sectors. But the former is populated by much more lines, in virtue of the fact that the nonlocal term acting in the north part of the pre-fractal is responsible of a larger heat flux across the barrier.
We conclude by noticing that the same results may be obtained also defining the problem on different domains. The Koch curve could be replaced by a more general symmetric pre-fractal of any order, or even by a pre-fractal of mixture type [22]. As

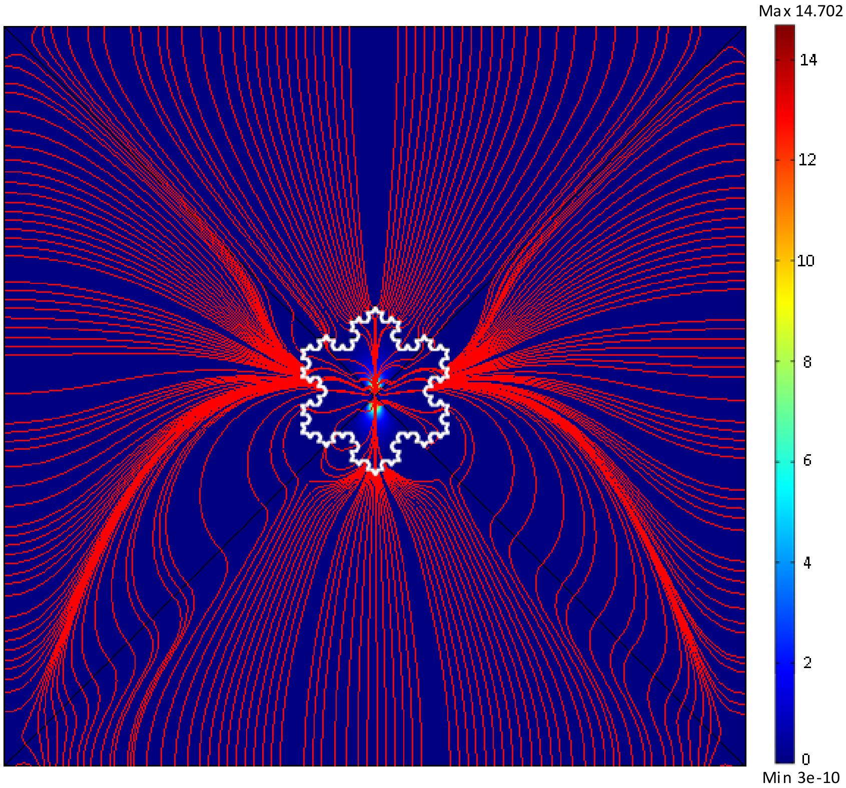

Figure 3.5: Heat flux streamlines at the stationary condition.

already pointed out in [20], the fractal geometry helps to achieve a larger heat flux across the barrier. Our experimental results suggest that by drawing a pre-fractal barrier of a proper material characterized by non-constant heat conductivity (which may be described by the nonlocal term $\theta_{\frac{1}{2}}(u)$) one could obtain a highly conductive layer with increased capability to drain the heat.

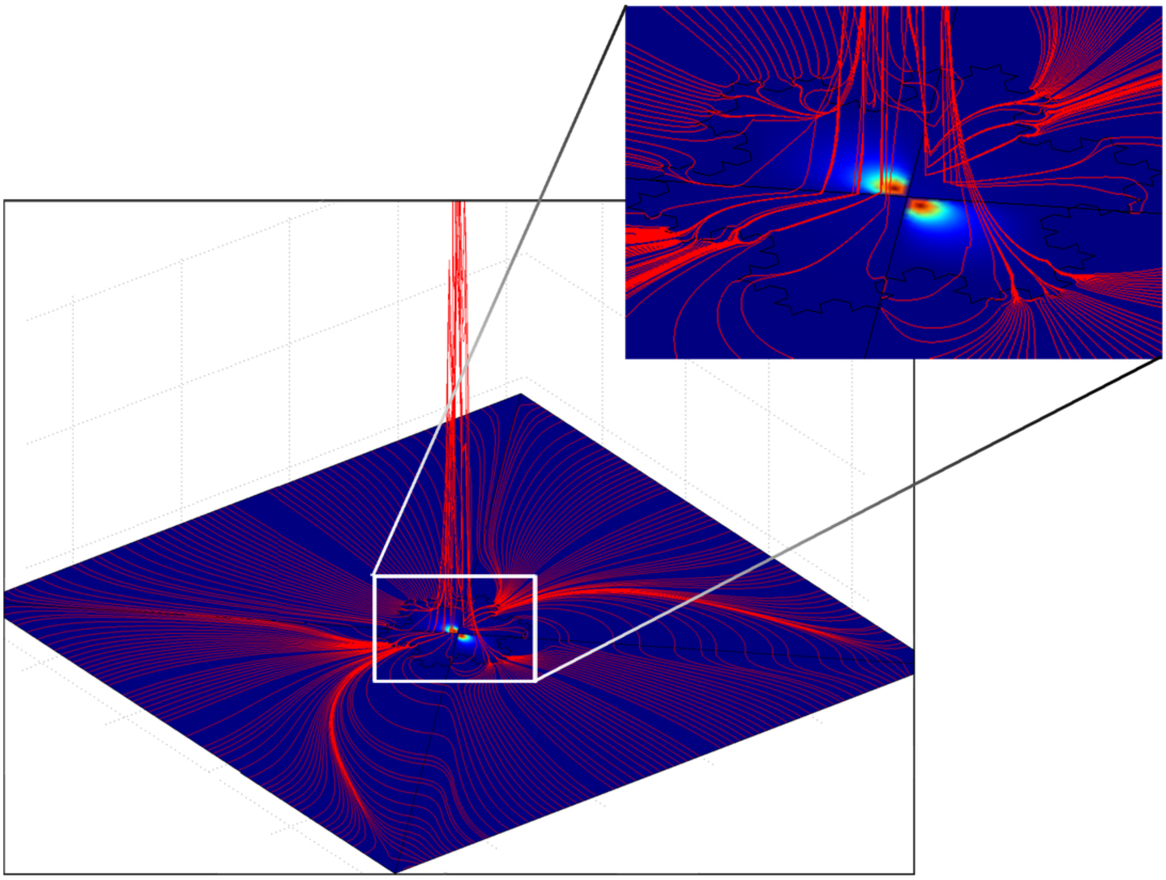

Figure 3.6: A 3D representation of the heat flux streamlines. The height of the curves is proportional to the amplitude of the heat flux.

Chapter 4

Quasilinear local Venttsel' problems in two-dimensional fractal domains

In this chapter, we investigate a quasilinear local Venttsel' problem for the p-Laplace operator, for $p \geq 2$. We study such problem in the two-dimensional domain Ω having fractal boundary K and in the corresponding approximating pre-fractal domains Ω_n with boundary K_n (see Section 1.1.1). We give existence and uniqueness results for both problems and we prove that the pre-fractal solutions converge to the limit fractal one. In order to do so, we prove the M-convergence of the pre-fractal functionals to the fractal one; this will require the use of *harmonic extensions*. Then, after proving the M-convergence, from the G-convergence of the subdifferentials we deduce the convergence of the solutions.

4.1 Energy functionals

We introduce the energy functionals for the fractal and pre-fractal problem respectively. Let H_n and H be the spaces defined in Section 1.4. Let be $p \geq 2$ and b a strictly positive continuous function in $\overline{\Omega}$. We set

$$\Phi_p[u] := \begin{cases} \frac{1}{p}\int\limits_{\Omega} |\mathrm{D}u|^p \,\mathrm{d}\mathcal{L}_2 + \frac{1}{p}\int\limits_{K} b|u|^p \,\mathrm{d}\mu + \mathcal{E}_p[u] & \text{if } u \in D(\Phi_p), \\ +\infty & \text{if } u \in H \setminus D(\Phi_p), \end{cases} \tag{4.1.1}$$

with domain

$$D(\Phi_p) := \left\{ u \in W^{1,p}(\Omega) \,:\, u|_K \in D(\mathcal{E}_p) \right\},$$

where $(\mathcal{E}_p, D(\mathcal{E}_p))$ is the nonlinear energy form defined in Section 1.5.

Proposition 4.1.1. *Φ_p is a weakly lower semicontinuous, proper and convex functional in H.*

For the proof see Proposition 2.3 in [56].

We define

$$E_p^{(n)}[u] = \frac{\delta_n^{1-p}}{p} \int\limits_{K_n} |\mathrm{D}u|^p \, \mathrm{d}\ell, \tag{4.1.2}$$

with domain

$$D(E_p^{(n)}) = W^{1,p}(K_n).$$

We now introduce the energy functional on the pre-fractal domain:

$$\Phi_p^{(n)}[u] := \begin{cases} \frac{1}{p}\int\limits_{\Omega} \chi_{\Omega_n} |\mathrm{D}u|^p \mathrm{d}\mathcal{L}_2 + \frac{\delta_n}{p} \int\limits_{K_n} b|u|^p \, \mathrm{d}\ell + E_p^{(n)}[u] & \text{if } u \in D(\Phi_p^{(n)}), \\ +\infty & \text{if } u \in H_n \setminus D(\Phi_p^{(n)}), \end{cases} \tag{4.1.3}$$

where

$$D(\Phi_p^{(n)}) := \left\{ u \in W^{1,p}(\Omega) \, : \, u|_{K_n} \in D(E_p^{(n)}) \right\}.$$

Proposition 4.1.2. $\Phi_p^{(n)}$ *is a weakly lower semicontinuous, proper and convex functional in* H_n*.*

Proof. From the definition of $\Phi_p^{(n)}$, it is clear that the functional is proper and convex. In order to prove weak lower semicontinuity, we prove it for each term in (4.1.3) and the thesis will follow from the properties of the liminf.
Let $\{v_h\} \in D(\Phi_p^{(n)})$ such that

$$v_h \rightharpoonup u \text{ in } W^{1,p}(\Omega_n) \text{ and } \gamma_0 v_h \rightharpoonup z \text{ in } W^{1,p}(K_n) \text{ for } h \to +\infty.$$

We have to show that $z = \gamma_0 u$ and that $\varliminf_{h\to\infty} \Phi_p^{(n)}[v_h] \geq \Phi_p^{(n)}[u]$. Since by hypothesis $v_h \in D(\Phi_p^{(n)})$, then $\mathrm{D}v_h \in (L^p(\Omega_n))^2$ and it is a bounded sequence. Hence there exists a subsequence, which we still denote by $\{\mathrm{D}v_h\}$, which weakly converges in $(L^p(\Omega_n))^2$ to a certain function w. Since $p \geq 2$, $L^p(\Omega_n)$ is embedded in $L^2(\Omega_n)$, and $\mathrm{D}v_h$ weakly converges to w also in $(L^2(\Omega_n))^2$, because D is a closed operator in $L^2(\Omega_n)$. Hence $w = \mathrm{D}u \in (L^2(\Omega_n))^2$ and in particular $v_h \rightharpoonup u$ in $H^1(\Omega_n)$. Hence, from the semicontinuity of the norm we have

$$\frac{1}{p} \varliminf_{h\to\infty} \int\limits_{\Omega_n} |\mathrm{D}v_h|^p \, \mathrm{d}\mathcal{L}_2 \geq \frac{1}{p} \int\limits_{\Omega_n} |\mathrm{D}u|^p \, \mathrm{d}\mathcal{L}_2.$$

Now, from the weak convergence in $H^1(\Omega_n)$, we deduce that $v_h \rightharpoonup u$ in $H^s(\Omega_n)$ for every $s \in (\frac{1}{2}, 1)$. Hence from trace theorem we have $\gamma_0 v_h \rightharpoonup \gamma_0 u$ in $H^{s-\frac{1}{2}}(K_n)$ and in particular in $L^2(K_n)$. From the uniqueness of the limit we get $z = \gamma_0 u$.
From the semicontinuity of the $W^{1,p}(K_n)$-norm, we get the thesis for the other two terms in (4.1.3), hence we get the thesis for $\Phi_p^{(n)}$. □

4.2 M-convergence of energy functionals and of their subdifferentials

We recall that the definition of M-convergence of quadratic energy forms was introduced by Mosco in [71] for a fixed Hilbert space and adapted to the case of varying Hilbert spaces by Kuwae and Shioya, see Definition 2.11 in [51]. This notion has been extended to the case of proper convex functionals in Banach spaces by Tölle (see Section 7.5, Definition 7.26 in [82]).
Let H_n be a sequence of Hilbert spaces converging to a Hilbert space H in the sense of Definition 1.4.1.

Definition 4.2.1. *A sequence of proper and convex functionals* $\left\{\Phi_p^{(n)}\right\}$ *defined in* H_n *M-converges to a functional* Φ_p *defined in* H *if the following hold:*

a) *for every* $\{v_n\} \in H_n$ *weakly converging to* $u \in H$ *in* $\mathcal{H}$

$$\varliminf_{n\to\infty} \Phi_p^{(n)}[v_n] \geq \Phi_p[u].$$

b) *for every* $u \in H$ *there exists a sequence* $\{w_n\}$*, with* $w_n \in H_n$ *strongly converging to* u *in* $\mathcal{H}$*, such that*

$$\varlimsup_{n\to\infty} \Phi_p^{(n)}[w_n] \leq \Phi_p[u].$$

We now state the main theorem of this section.

Theorem 4.2.2. *Let* $\delta_n = (3^{1-d_f})^n = \left(\frac{3}{4}\right)^n$*. Let* Φ_p *and* $\Phi_p^{(n)}$ *be defined as in* (4.1.1) *and* (4.1.3) *respectively. Then* $\Phi_p^{(n)}$ *M-converges to the functional* Φ_p*.*

We preliminary state the following propositions.

Proposition 4.2.3. *If* $\{v_n\}_{n\in\mathbb{N}}$ *weakly converges to a vector* u *in* $\mathcal{H}$*, then* $\{v_n\}_{n\in\mathbb{N}}$ *weakly converges to* u *in* $L^2(\Omega)$ *and* $\lim\limits_{n\to\infty} \delta_n \int\limits_{K_n} \varphi v_n \,\mathrm{d}\ell = \int\limits_{K} \varphi u \,\mathrm{d}\mu$ *for every* $\varphi \in C(\overline{\Omega})$*.*

For the proof see Proposition 4.4 in [63].

Proposition 4.2.4. *Let* $v_n \rightharpoonup u$ *in* $W^{1,p}(\Omega)$*,* $b \in C(\bar{\Omega})$*. Then*

$$\delta_n \int\limits_{K_n} b|v_n|^p \,\mathrm{d}\ell \to \int\limits_{K} b|u|^p \,\mathrm{d}\mu.$$

Proof. We first note that

$$\left|\delta_n \int_{K_n} b\,|v_n|^p \,\mathrm{d}\ell - \int_K b\,|u|^p \,\mathrm{d}\mu\right| \leq \left|\delta_n \int_{K_n} b\,|v_n|^p \,\mathrm{d}\ell - \delta_n \int_{K_n} b\,|u|^p \,\mathrm{d}\ell\right| + \left|\delta_n \int_{K_n} b\,|u|^p \,\mathrm{d}\ell - \int_K b\,|u|^p \,\mathrm{d}\mu\right|.$$

We set

$$A_n = \left|\delta_n \int_{K_n} b\,|v_n|^p \,\mathrm{d}\ell - \delta_n \int_{K_n} b\,|u|^p \,\mathrm{d}\ell\right|$$

and

$$B_n = \left|\delta_n \int_{K_n} b\,|u|^p \,\mathrm{d}\ell - \int_K b\,|u|^p \,\mathrm{d}\mu\right|$$

and we study these two terms separately.
For the first term it holds that

$$A_n \leq c_1\, \delta_n \|b\|_{C(\overline{\Omega})} \left(\|v_n - u\|_{L^p(K_n)}\right) \left(\|v_n\|_{L^p(K_n)} + \|u\|_{L^p(K_n)}\right)^{p-1}.$$

Since v_n weakly converges to u in $W^{1,p}(\Omega)$, it follows that v_n strongly converges to u in $W^{\alpha,p}(\Omega)$ for every $\alpha \in (0,1)$.
If we consider the extension of $v_n - u$ to $W^{\alpha,p}(\mathbb{R}^2)$ we have from Theorems 1.2.2 and 1.2.6

$$\delta_n \left(\|v_n - u\|_{L^p(K_n)}\right) \leq C_\alpha \,\|\mathrm{Ext}(v_n - u)\|_{W^{\alpha,p}(\mathbb{R}^2)} \leq c_2 \,\|v_n - u\|_{W^{\alpha,p}(\Omega)}.$$

Hence A_n goes to 0 when n tends to ∞.
We now prove that also B_n goes to 0. Since u belongs to $W^{1,p}(\Omega)$ there exists a sequence $\{g_m\} \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ such that $\|g_m - u\|_{W^{1,p}(\Omega)} \to 0$ as m goes to ∞ (see Proposition 4.4 in [41]). Then we have

$$B_n \leq \left|\delta_n \int_{K_n} b\,|u|^p \,\mathrm{d}\ell - \delta_n \int_{K_n} b\,|g_m|^p \,\mathrm{d}\ell\right| + \left|\delta_n \int_{K_n} b\,|g_m|^p \,\mathrm{d}\ell - \int_K b\,|g_m|^p \,\mathrm{d}\mu\right| + \left|\int_K b\,|g_m|^p \,\mathrm{d}\mu - \int_K b\,|u|^p \,\mathrm{d}\mu\right|.$$

Proceeding as in the case A_n we can estimate the first and the third term in the right-hand side with $\|u - g_m\|_{W^{1,p}(\Omega)}$ and hence we conclude that for every $\varepsilon > 0$ there exists

$m_\varepsilon \in \mathbb{N}$ such that these two terms are less than $c\,\varepsilon$. Then, if we choose $m > m_\varepsilon$ the second term in the right-hand side goes to 0 for n tending to ∞ for Proposition 1.4.11 (since $b\,g_m$ belongs to $C(\overline{\Omega})$). $\square$

We are now ready to prove Theorem 4.2.2.

Proof. **Proof of condition a)**. Let $v_n \in H_n$ be a weakly converging sequence in $\mathcal{H}$ to $u \in H$. We can suppose that $v_n \in D(\Phi_p^{(n)})$ and

$$\varliminf_{n\to\infty} \Phi_p^{(n)}[v_n] < \infty$$

(otherwise the thesis follows trivially). Then there exists a c independent of n such that

$$\frac{1}{p}\int_\Omega \chi_{\Omega_n} |\mathrm{D}v_n|^p \mathrm{d}\mathcal{L}_2 + \frac{\delta_n}{p}\int_{K_n} b|v_n|^p\,\mathrm{d}\ell + \frac{\delta_n^{1-p}}{p}\int_{K_n} |\mathrm{D}v_n|^p\,\mathrm{d}\ell \le c \tag{4.2.1}$$

In particular we have that $\|v_n\|_{W^{1,p}(\Omega_n)} < c$. For every $n \in \mathbb{N}$ from Theorem 1.2.5 there exists a bounded linear operator $\mathrm{Ext}\colon W^{1,p}(\Omega_n) \to W^{1,p}(\mathbb{R}^2)$ such that

$$\|\mathrm{Ext}\, v_n\|_{W^{1,p}(\mathbb{R}^2)} \le C\,\|v_n\|_{W^{1,p}(\Omega_n)} \le c\,C,$$

with C independent of n.
Now we denote by $\hat{v}_n = \mathrm{Ext}\, v_n|_\Omega$. Then $\hat{v}_n \in W^{1,p}(\Omega)$ and $\|\hat{v}_n\|_{W^{1,p}(\Omega)} \le c\,C$, hence there exists a subsequence, still denoted by $\hat{v}_n$, weakly converging to $\hat{v}$ in $W^{1,p}(\Omega)$. We point out that $\hat{v}_n$ strongly converges to $\hat{v}$ in $L^p(\Omega)$ and also in $L^2(\Omega)$ since $p \ge 2$. From Proposition 4.2.3, v_n weakly converges to u in $L^2(\Omega)$. We prove that $\hat{v} = u$ $\mathcal{L}_2$-a.e., that is

$$\int_\Omega (\hat{v} - u)\varphi\,\mathrm{d}\mathcal{L}_2 = 0$$

for each $\varphi \in L^2(\Omega)$. Indeed, we can write

$$\begin{aligned}\int_\Omega (\hat{v} - u)\varphi\,\mathrm{d}\mathcal{L}_2 &= \int_\Omega (\hat{v} - \hat{v}_n + \hat{v}_n - u)\varphi\,\mathrm{d}\mathcal{L}_2 \\ &= \int_\Omega (\hat{v} - \hat{v}_n)\varphi\,\mathrm{d}\mathcal{L}_2 + \int_{\Omega_n} (v_n - u)\varphi\,\mathrm{d}\mathcal{L}_2 + \int_{\Omega\setminus\Omega_n} (\hat{v}_n - u)\varphi\,\mathrm{d}\mathcal{L}_2.\end{aligned} \tag{4.2.2}$$

For every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that each term in the sum of the right-hand side of (4.2.2) is less than $\epsilon/3$. Since $\hat{v}_n \to \hat{v}$ in $L^2(\Omega)$ and $v_n \rightharpoonup u$ in $L^2(\Omega)$ we deduce our claim for the first two terms. As to $\int_{\Omega\setminus\Omega_n} (\hat{v}_n - u)\varphi\,\mathrm{d}\mathcal{L}_2$, from Hölder inequality we deduce that

$$\int_{\Omega\setminus\Omega_n} |(\hat{v}_n - u)\varphi|\,\mathrm{d}\mathcal{L}_2 \le \|\varphi\|_{L^2(\Omega\setminus\Omega_n)}(\|\hat{v}_n\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \le \epsilon/3,$$

since $|\Omega \setminus \Omega_n| \to 0$ as $n \to \infty$.
We now prove that

$$\varliminf_{n\to\infty} \int\limits_{\Omega} \chi_{\Omega_n} |\mathrm{D}v_n|^p \, \mathrm{d}\mathcal{L}_2 \geq \int\limits_{\Omega} |\mathrm{D}u|^p \, \mathrm{d}\mathcal{L}_2. \tag{4.2.3}$$

It is enough to prove that $\chi_{\Omega_n} \mathrm{D}v_n \rightharpoonup \mathrm{D}u$ in $L^p(\Omega)$, from here the claim will follow from the semicontinuity of the norm. Since $\chi_{\Omega_n} \mathrm{D}v_n = \chi_{\Omega_n} \mathrm{D}\hat{v}_n$, this amounts to prove that $\int_\Omega \chi_{\Omega_n} \mathrm{D}\hat{v}_n \varphi \, \mathrm{d}\mathcal{L}_2 \to \int_\Omega \mathrm{D}u\varphi \, \mathrm{d}\mathcal{L}_2$ for every $\varphi \in L^{p'}(\Omega)$.
It holds that

$$\int\limits_{\Omega} \mathrm{D}u\varphi \, \mathrm{d}\mathcal{L}_2 - \int\limits_{\Omega_n} \mathrm{D}\hat{v}_n\varphi \, \mathrm{d}\mathcal{L}_2 = \int\limits_{\Omega} (\mathrm{D}u - \mathrm{D}\hat{v}_n)\varphi \, \mathrm{d}\mathcal{L}_2 - \int\limits_{\Omega\setminus\Omega_n} \mathrm{D}\hat{v}_n\varphi \, \mathrm{d}\mathcal{L}_2.$$

The first term vanishes as $n \to \infty$ since $\mathrm{D}\hat{v}_n \rightharpoonup \mathrm{D}u$ in $L^p(\Omega)$. Now we estimate the second term $\int_{\Omega\setminus\Omega_n} |\mathrm{D}\hat{v}_n\varphi| \, \mathrm{d}\mathcal{L}_2$. We have

$$\int\limits_{\Omega\setminus\Omega_n} \mathrm{D}\hat{v}_n\varphi \, \mathrm{d}\mathcal{L}_2 \leq \|\varphi\|_{L^{p'}(\Omega\setminus\Omega_n)} \|\mathrm{D}\hat{v}_n\|_{L^p(\Omega)} \to 0.$$

Hence (4.2.3) holds. Now we prove that

$$\varliminf_{n\to\infty} \delta_n^{1-p} \int\limits_{K_n} |\mathrm{D}v_n|^p \, \mathrm{d}\ell \geq \mathcal{E}_p[u].$$

We begin by proving that

$$\mathcal{E}_p^{(n)}[u] \leq E_p^{(n)}[u]. \tag{4.2.4}$$

We fix a positive orientation on K_n, the anti-clockwise orientation, and we choose as origin A_1. This induces a natural orientation on the vertices P_j, for $j = 1, \dots, 3\mathcal{N}$, where $\mathcal{N} = 4^n$ and $P_1 = P_{3\mathcal{N}+1} = A_1$. Let now M_j be a segment of the n-th generation (i.e. $M_j \in K_n$). From the definition of $\mathcal{E}_p^{(n)}$ given in (1.5.3), we get

$$\mathcal{E}_p^{(n)}[u] = \frac{4^{(p-1)n}}{p} \sum_{j=1}^{3\mathcal{N}} (u(P_{j+1}) - u(P_j))^p = \frac{4^{(p-1)n}}{p} \sum_{j=1}^{3\mathcal{N}} \left| \int\limits_{M_j} |\mathrm{D}u| \, \mathrm{d}\ell \right|^p \leq$$
$$\frac{4^{(p-1)n}}{p} \sum_{j=1}^{3\mathcal{N}} |M_j|^{\frac{p}{p'}} \int\limits_{M_j} |\mathrm{D}u|^p \, \mathrm{d}\ell = \frac{1}{p} \left(\frac{4}{3}\right)^{(p-1)n} \sum_{j=1}^{3\mathcal{N}} \int\limits_{M_j} |\mathrm{D}u|^p \, \mathrm{d}\ell = E_p^{(n)}[u].$$

From Proposition 4.2.5, v_n is in particular continuous on K_n. Hence the function v_n is defined on the discrete set $\mathcal{V}^n$, so we extend it to a continuous function Hv_n on K. This extension is unique and it is obtained by constructing the discrete harmonic extension $Hv_n|_{\mathcal{V}_\star}$ of $v_n|_{\mathcal{V}^n}$ to the set $\mathcal{V}_\star$ and then taking the unique continuous extension of $Hv_n|_{\mathcal{V}_\star}$ to K. This iterative process is known as *decimation* in the physics literature (see [49], [50] and [15]). Then from (4.2.1) and (4.2.4) we have that

$$\mathcal{E}_p[Hv_n] = \sup_h \mathcal{E}_p^{(n)}[v_n] \leq c. \tag{4.2.5}$$

Moreover $Hv_n \in D(\mathcal{E}_p)$ and from (4.2.5) we have that $\{Hv_n\}$ is a bounded sequence in $D(\mathcal{E}_p)$. Then there exists a subsequence, still denoted by Hv_n, weakly converging to a function u^* in $D(\mathcal{E}_p)$ with

$$\mathcal{E}_p[u^*] \leq \varliminf_{n\to\infty} \mathcal{E}_p[Hv_n] = \varliminf_{n\to\infty} \mathcal{E}_p^{(n)}[v_n] \leq c \tag{4.2.6}$$

(this follows from the lower semi-continuity of the norm, from (4.2.5) and (4.2.4)). From Ascoli-Arzela Theorem it follows that

$$Hv_n \to u^* \text{ uniformly in } C(K) \text{ and } u^* \in C(K). \tag{4.2.7}$$

We have to prove now that $u^* = u|_K$ in $L^p(K)$. Since $\hat{v}_n$ weakly converges to u in $W^{1,p}(\Omega)$, it strongly converges to u in $W^{\alpha,p}(\Omega)$ for $0 < \alpha < 1$. Hence $\hat{v}_n|_K$ strongly converges to u in $B^{p,p}_\beta(K)$ with $\beta = \alpha - \frac{2-d_f}{p}$ and $\hat{v}_n|_K \to u|_K$ in $L^p(K)$ (in particular in $L^2(K)$).
Now let $\varphi \in C(\overline{\Omega})$. Then

$$\int_K (u^* - u|_K)\varphi \,\mathrm{d}\mu =$$
$$\int_K (u^* - Hv_n)\varphi \,\mathrm{d}\mu + \delta_n \int_{K_n} v_n \,\varphi \,\mathrm{d}\ell - \int_K u|_K \varphi \,\mathrm{d}\mu + \int_K Hv_n \,\varphi \,\mathrm{d}\mu - \delta_n \int_{K_n} v_n \,\varphi \,\mathrm{d}\ell.$$

We note that the first integral tends to 0 as n goes to infinity from the weak convergence, and that the difference between the second and the third integral also tends to 0 since $v_n \rightharpoonup u$ in $\mathcal{H}$ by assumption. We have to estimate

$$\int_K Hv_n \,\varphi \,\mathrm{d}\mu - \delta_n \int_{K_n} v_n \,\varphi \,\mathrm{d}\ell.$$

We note that, for every $P \in K_n$ and $P_i^* \in \mathcal{V}^n$, from the uniform boundedness of v_n and the uniform Hölder continuity of v_n on K_n (see Proposition 4.2.5), for every $\varepsilon > 0$ there exists $\bar{n} > 0$ such that for every $n \geq \bar{n}$

$$|\varphi(P)v_n(P) - \varphi(P_i^*)v_n(P_i^*)| \leq \varepsilon \, c + c_\varphi c_H \, 3^{-\beta n}. \tag{4.2.8}$$

Now we consider $P_i^* \in \mathcal{V}^n \cap M_i^n$, where $M_i^n := \psi_{i|n}(K_n)$. Then from (4.2.8) we have that

$$\delta_n \int\limits_{K_n} v_n \, \varphi \, \mathrm{d}\ell \le \delta_n \sum_{i=1}^{3\mathcal{N}} \int\limits_{M_i^n} |(v_n(P)\varphi(P) - v_n(P_i^*)\varphi(P_i^*))| \, \mathrm{d}\ell +$$

$$\delta_n \sum_{i=1}^{3\mathcal{N}} \int\limits_{K_n} |v_n(P_i^*)\varphi(P_i^*)| \, \mathrm{d}\ell \le \delta_n(\varepsilon \, c + c_\varphi c_H \, 3^{-\beta n})\delta_n^{-1} + \delta_n \sum_{i=1}^{3\mathcal{N}} \int\limits_{K_n} |v_n(P_i^*)\varphi(P_i^*)| \, \mathrm{d}\ell.$$

The first term vanishes as $n \to \infty$ for the arbitrariness of ε. Now we point out that

$$\delta_n \sum_{i=1}^{3\mathcal{N}} \int\limits_{K_n} v_n(P_i^*)\varphi(P_i^*) \, \mathrm{d}\ell = 4^{-n} \sum_{i=1}^{3\mathcal{N}} v_n(P_i^*)\varphi(P_i^*) = 4^{-n} \sum_{i=1}^{3\mathcal{N}} Hv_n(P_i^*)\varphi(P_i^*) =$$

$$\mu(\psi_{i|n}(K)) \sum_{i=1}^{3\mathcal{N}} Hv_n(P_i^*)\varphi(P_i^*) = \int\limits_K Hv_n(P_i^*)\varphi(P_i^*) \, \mathrm{d}\mu.$$

Hence we get, for $n \to \infty$,

$$\int\limits_K (Hv_n(P) \, \varphi(P) - Hv_n(P_i^*)\varphi(P_i^*)) \, \mathrm{d}\mu \to 0$$

since Hv_n is equi-Hölder continuous on K. We conclude the proof taking into account the liminf properties of the sum and Proposition 4.2.4.

Proof of condition b). We have to prove that for every $u \in H$ there exists $\{w_n\}_{n\in\mathbb{N}}$ strongly converging to u in $\mathcal{H}$ such that

$$\Phi_p[u] \ge \varlimsup_{n\to\infty} \Phi_p^{(n)}[w_n].$$

We can suppose that $u \in D(\Phi_p)$. Indeed, if $u \notin D(\Phi_p)$ then $\Phi_p[u] = +\infty$ and from Lemma 1.4.9 it follows that there exists a sequence $\{v_n\}_{n\in\mathbb{N}}$ converging to u in $\mathcal{H}$ and hence $\varlimsup_{n\to\infty} \Phi_p^{(n)}[v_n] \le \Phi_p[u] = +\infty$.

Let then $u \in D(\Phi_p)$, i.e. $u \in W^{1,p}(\Omega)$ and $u|_K \in D(\mathcal{E}_p)$. For the case $p = 2$, we refer to [63]. Here we consider the case $p > 2$. Since $p > 2$, then u belongs to $C(\overline{\Omega})$ (see [66]).

We extend by continuity u to $\overline{\mathcal{T}}$, where $\mathcal{T}$ is the triangle defined in Section 1.1.1, and we denote its extension by $\hat{u}$. Following the same approach as in [65] and [57] with some suitable modifications, we introduce a quasi-uniform triangulation τ_n of $\mathcal{T}$ made by equilateral triangles T_n^j such that the vertices of the pre-fractal curve K_n are nodes of the triangulation at the n-th level. Let $\mathcal{S}_n$ be the space of all the functions being continuous on $\overline{\mathcal{T}}$ and affine on the triangles of τ_n. We denote by $\mathcal{M}_n$ the nodes of τ_n, i.e. the set of the vertices of all T_n^j. For a given continuous function u, we denote by $I_n u$ the function which is affine on every $T_n^j \in \tau_n$ and which interpolates u in the nodes $P_{j,i} \in \mathcal{M}_n \cap \overline{\Omega}_n$. We set $w_n = I_n \hat{u}$ and we prove that $\{w_n\}$ strongly converges to u in $\mathcal{H}$, which is equivalent to prove that (see Lemma 1.4.6) $(w_n, v_n)_{H_n} \to (u, v)_H$ for every sequence $\{v_n\}$ weakly converging to a vector v in $\mathcal{H}$.

We know that
$$\|w_n - u\|_{W^{1,p}(\mathcal{T})} \to 0 \tag{4.2.9}$$
as n goes to ∞ (see [23]) and hence $\|w_n - u\|_{W^{1,p}(\Omega)} \to 0$.
From Theorem 1.2.2, there exists a constant c independent of n such that $\|w_n - u\|_{L^2(K_n)} \leq c\,\delta_n^{-\frac{1}{2}}\,\|w_n - u\|_{W^{1,p}(\Omega)}$.
Then we have

$$0 \leq |(w_n, v_n)_{H_n} - (u,v)_H| = \left| \int_{\Omega_n} w_n v_n \,\mathrm{d}\mathcal{L}_2 + \delta_n \int_{K_n} w_n v_n \,\mathrm{d}\ell - \int_{\Omega} uv \,\mathrm{d}\mathcal{L}_2 - \int_K uv\,\mathrm{d}\mu \right| =$$
$$\left| (w_n - u, v_n)_{L^2(\Omega_n)} + \delta_n \int_{K_n} (w_n - u) v_n \,\mathrm{d}\ell + (u, v_n)_{H_n} - (u,v)_H \right| \leq$$
$$\leq \left|(w_n - u, v_n)_{L^2(\Omega_n)}\right| + \left|(\sqrt{\delta_n}(w_n - u), \sqrt{\delta_n} v_n)_{L^2(K_n)}\right| + |(u, v_n)_{H_n} - (u,v)_H| \leq$$
$$\leq \|w_n - u\|_{L^2(\Omega)}\,\|v_n\|_{L^2(\Omega)} + \sqrt{\delta_n}\,\|w_n - u\|_{L^2(K_n)}\,\sqrt{\delta_n}\,\|v_n\|_{L^2(K_n)} + |(u, v_n)_{H_n} - (u,v)_H|$$
The claim follows since $v_n \rightharpoonup v$ in $\mathcal{H}$, therefore $\sup_n \|v_n\|_{H_n} < \infty$, and $\sqrt{\delta_n}\,\|w_n - u\|_{L^2(K_n)} \leq c\,\|w_n - u\|_{W^{1,p}(\Omega)}$.
We now prove condition b) of Definition 4.2.1 for the sequence w_n. We note that from Proposition 4.2.4
$$\lim_{n\to\infty} \delta_n \int_{K_n} b|w_n|^p \,\mathrm{d}\ell = \int_K b|u|^p \,\mathrm{d}\mu.$$
We have that
$$\int_{\Omega_n} |\mathrm{D}w_n|^p \,\mathrm{d}\mathcal{L}_2 \leq \int_{\Omega} |\mathrm{D}w_n|^p \,\mathrm{d}\mathcal{L}_2,$$
then, by taking the limit for $n \to \infty$, we have the thesis (since $\|\mathrm{D}(w_n - u)\|_{L^p(\Omega)} \to 0$ for $n \to \infty$).
We have only to prove that
$$\overline{\lim_{n\to\infty}} \frac{\delta_n^{1-p}}{p} \int_{K_n} |\mathrm{D}w_n|^p \,\mathrm{d}\ell \leq \mathcal{E}_p[u].$$
We now show that
$$\mathcal{E}_p^{(n)}[u] = \frac{\delta_n^{1-p}}{p} \int_{K_n} |\mathrm{D}w_n|^p \,\mathrm{d}\ell, \tag{4.2.10}$$
by using the parametrization of K_n by means of the arc length with origin $P_1 = A_1$.
Since $w_n = I_n \hat{u}$, we have that
$$w_n = m_j\, l + q_j \quad , l \in [l_j, l_{j+1}],$$

where $l_j = (j-1)\,3^{-n}$ for $j = 1, \ldots, 3\mathcal{N}$. Hence we get

$$\frac{\delta_n^{1-p}}{p} \int\limits_{K_n} |\mathrm{D}w_n|^p \, \mathrm{d}\ell = \frac{\delta_n^{1-p}}{p} \sum_{j=1}^{3\mathcal{N}} m_j^p (l_{j+1} - l_j) =$$
$$\frac{4^{(p-1)n}}{p} \sum_{j=1}^{3\mathcal{N}} (w_n(P_{j+1}) - w_n(P_j))^p = \mathcal{E}_p^{(n)}[w_n] = \mathcal{E}_p^{(n)}[u].$$

The claim then follows from the monotonicity of $\mathcal{E}_p^{(n)}[u]$.
Taking into account the limsup property of the sum the conclusion of the theorem follows. $\square$

Proposition 4.2.5. *Let $v_n \in D(\Phi_p^{(n)})$ be weakly converging in $\mathcal{H}$ to u with $\varliminf\limits_{n\to\infty} \Phi_p^{(n)}[v_n] < \infty$. Then v_n is uniformly bounded and Hölder continuous uniformly in n on K_n.*

Proof. We begin by proving the Hölder continuity. We start by assuming $P, Q \in M_j$. Then

$$|v_n(P) - v_n(Q)| \leq \int\limits_{\overline{PQ}} |\mathrm{D}v_n| \, \mathrm{d}\ell \leq \left(\int\limits_{\overline{PQ}} |\mathrm{D}v_n|^p \, \mathrm{d}\ell \right)^{\frac{1}{p}} |P-Q|^{\frac{1}{p'}}.$$

We point out that from (4.2.1) we have that

$$\int\limits_{K_n} |\mathrm{D}v_n|^p \, \mathrm{d}\ell \leq \, c\, \delta_n^{p-1},$$

where c is independent of n. Then, since $|P-Q| \leq 3^{-n}$, by setting $\beta := \frac{d_f}{p'}$ we get

$$|v_n(P) - v_n(Q)| \leq c\,|P-Q|^\beta\, \delta_n^{\frac{1}{p'}}\, 3^{-n(\frac{1}{p'} - \beta)} = c\,|P-Q|^\beta \frac{3^{\frac{n}{p'} - \frac{n(1-d_f)}{p'}}}{4^{\frac{n}{p'}}} =$$
$$c\,|P-Q|^\beta \left(\frac{3^{d_f}}{4}\right)^{\frac{n}{p'}} = c\,|P-Q|^\beta.$$

This proves the uniform Hölder continuity in n of v_n. If P and Q do not belong to the same segment, the proof can be carried out by a chain argument.
We now prove the uniform boundedness. From the uniform convergence of Hv_n to u^* in $C(K)$ (see (4.2.7)), it follows that for every $\varepsilon > 0$ there exists $\bar{n} > 0$ such that, for every $n > \bar{n}$,

$$|Hv_n(P) - u^*(P)| < \varepsilon \quad \forall\, P \in K.$$

Then it holds that, since $u^* \in C(K)$,

$$\|Hv_n\|_{L^\infty(K_n)} \leq \|Hv_n - u^*\|_{L^\infty(K_n)} + \|u^*\|_{C(K)} \leq \varepsilon + \|u^*\|_{C(K)}.$$

Now take $\bar{P} \in \mathcal{V}^n$. Since $Hv_n(\bar{P}) = v_n(\bar{P})$, then

$$|v_n(\bar{P})| = |Hv_n(\bar{P})| \leq \bar{c}.$$

Let now $P \in K_n$. Then

$$|v_n(P)| \leq |v_n(P) - v_n(\bar{P})| + |v_n(\bar{P})|,$$

hence the thesis follows. □

In the following Theorem we deduce the G-convergence of the associated subdifferentials.

Theorem 4.2.6. $\Phi_p^{(n)}$ *M-converges to* Φ_p *in* $\mathcal{H}$ *if and only if* $\partial\Phi_p^{(n)}$ *G-converges to* $\partial\Phi_p$.

For the proof see Theorem 7.46 in [82]. This result will be crucial for the convergence of the solutions of the nonlinear abstract Cauchy problems.

4.3 Convergence of the solutions of the abstract Cauchy problems

We now consider the abstract homogeneous Cauchy problem

$$(P)\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + \mathcal{A}u \ni 0, \quad t \in [0,T] \\ u(0) = u_0, \end{cases}$$

where $\mathcal{A}$ is the subdifferential of Φ_p, T is a fixed positive number, and u_0 is a given function. We now recall some results on the properties of nonlinear semigroups generated by the (opposite of) subdifferential of a proper convex lower semicontinuous functional on a real Hilbert space (see Theorem 1 and Remark 2 in [10], see also [8]). According to [8, Section 2.1, chapter II], we say that a function $u : [0,T] \to H$ is a strong solution of (P) if $u \in C([0,T];H)$, $u(t)$ is differentiable a.e. in $(0,T)$, $u(t) \in D(-\mathcal{A})$ a.e and $\frac{\mathrm{d}u}{\mathrm{d}t} + \mathcal{A}u \ni 0$ for a.e. $t \in [0,T]$.

Theorem 4.3.1. *Let* $\varphi : H \to (-\infty, +\infty]$ *be a proper, convex, lower semicontinuous functional on a real Hilbert space* H*, with effective domain* $D(\varphi)$*. The subdifferential* $\partial\varphi$ *is a maximal monotone m-accretive operator. Moreover,* $\overline{D(\varphi)} = \overline{D(\partial\varphi)}$. $-\partial\varphi$ *generates a (nonlinear)* C_0*-semigroup* $\{T(t)\}_{t\geq 0}$ *on* $\overline{D(\varphi)}$ *in the following sense: for each* $u_0 \in \overline{D(\varphi)}$*, the function* $u := T(\cdot)u_0$ *is the unique strong solution of the problem*

$$\begin{cases} u \in C(\mathbb{R}_+; H) \cap W^{1,\infty}_{loc}((0,\infty); H) \ \textit{and} \ u(t) \in D(\varphi) \ \textit{a.e.}, \\ \dfrac{\mathrm{d}u}{\mathrm{d}t} + \partial\varphi(u) \ni 0 \ \textit{a.e. on} \ \mathbb{R}_+, \\ u(0,x) = u_0(x). \end{cases}$$

In addition, $-\partial\varphi$ generates a (nonlinear) semigroup $\{\tilde{T}(t)\}_{t\geq 0}$ on H, where for every $t \geq 0$, $\tilde{T}(t)$ is the composition of the semigroup $T(t)$ on $\overline{D(\varphi)}$ with the projection on the convex set $\overline{D(\varphi)}$.

In our case it turns out that, from Theorem 4.3.1, the subdifferentials $\partial\Phi_p$ and $\partial\Phi_p^{(n)}$ are maximal, monotone and m-accretive operators on H and H_n respectively. Then, if we denote with $T_p(t)$ and $T_p^{(n)}(t)$ the nonlinear semigroups generated by $-\partial\Phi_p$ and $-\partial\Phi_p^{(n)}$ respectively, these semigroups are strongly continuous and contractive on H and H_n (see Proposition 2.5 in [56] for the fractal case).

Theorem 2.7 in [56] states the following result.

Theorem 4.3.2. *If $u_0 \in \overline{D(-\mathcal{A})}$, then (P) has a unique strong solution $u \in C([0,T];H)$ defined as $u = T_p(\cdot)u_0$ such that $u \in W^{1,2}((\delta,T);H)$ for every $\delta \in (0,T)$. Moreover $u \in D(-\mathcal{A})$ a.e. for $t \in (0,T)$, $\sqrt{t}\frac{\mathrm{d}u}{\mathrm{d}t} \in L^2(0,T;H)$ and $\Phi_p[u] \in L^1(0,T)$.*

Moreover, from Theorem 2.6 in [56], if we set $\alpha = 1 - \frac{2-d_f}{p}$, it can be proved that the solution u of problem (P) solves the following problem $(\tilde{P})$ on Ω for $t \in (0,T]$ in the following weak sense:

$$(\tilde{P})\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} - \Delta_p u = 0, \qquad \text{in } L^{p'}(\Omega) \\ \left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, \psi \right\rangle_{L^2(K,\mathrm{d}\mu),L^2(K,\mathrm{d}\mu)} + \left\langle \frac{\partial u}{\partial \nu}|\mathrm{D}u|^{p-2}, \psi \right\rangle_{(B^{p,p}_{\alpha}(K))',B^{p,p}_{\alpha}(K)} + \\ \langle b|u|^{p-2}u, \psi\rangle_{L^{p'}(K,\mathrm{d}\mu),L^p(K,\mathrm{d}\mu)} + \mathcal{E}_p(u,\psi) = 0 \quad \text{for every } \psi \in D(\mathcal{E}_p), \\ u(0,x) = u_0(x) \qquad \text{in } L^2(\overline{\Omega}, m). \end{cases}$$

We now come to the pre-fractal case. For each $n \in \mathbb{N}$ fixed, we consider the abstract homogeneous Cauchy problem

$$(P_n)\begin{cases} \frac{\mathrm{d}u_n}{\mathrm{d}t} + \mathcal{A}_n u_n \ni 0, \quad t \in [0,T] \\ u_n(0) = u_0^{(n)}, \end{cases}$$

where $\mathcal{A}_n$ is the subdifferential of $\Phi_p^{(n)}$, T is a fixed positive number, and $u_0^{(n)}$ is a given function.

Before stating existence and uniqueness results we give a characterization of $\mathcal{A}_n$.

Theorem 4.3.3. *Let $u_n(t)$ belong to $D(\Phi_p^{(n)})$ for a.e. $t \in (0,T]$, and f be in H_n. Then $f \in \partial\Phi_p^{(n)}[u_n]$ if and only if*

$$(\bar{P}_n)\begin{cases} -\Delta_p u_n = f \quad \textit{in } L^{p'}(\Omega_n), \\ \left\langle \frac{\partial u_n}{\partial \nu_n}|\mathrm{D}u_n|^{p-2}, \psi\right\rangle_{W^{-\frac{1}{p'},p'}(K_n), W^{\frac{1}{p'},p}(K_n)} + \delta_n \left\langle b|u_n|^{p-2}u_n, \psi\right\rangle_{L^{p'}(K_n), L^p(K_n)} \\ -\delta_n^{1-p}\left\langle \Delta_p u_n, \psi\right\rangle_{W^{-1,p'}(K_n), W^{1,p}(K_n)} = \delta_n \left\langle f, \psi\right\rangle_{L^2(K_n), L^2(K_n)} \textit{ for every } \psi \in W^{1,p}(K_n), \end{cases}$$

where $\frac{\partial u_n}{\partial \nu_n}$ *denotes the outward normal derivative across* K_n.

Proof. Let $f \in \partial\Phi_p^{(n)}[u_n]$, i.e. $\Phi_p^{(n)}[v] - \Phi_p^{(n)}[u_n] \geq (f, v - u_n)_{H_n}$ for every $v \in D(\Phi_p^{(n)})$:

$$\begin{aligned} \int_{\Omega_n} f(v-u_n)\,\mathrm{d}\mathcal{L}_2 + \delta_n \int_{K_n} f(v-u_n)\,\mathrm{d}\ell \quad \leq \\ \frac{1}{p}\int_{\Omega} \chi_{\Omega_n}(|\mathrm{D}v|^p - |\mathrm{D}u_n|^p)\,\mathrm{d}\mathcal{L}_2 + \frac{\delta_n}{p}\int_{K_n} b(|v|^p - |u_n|^p)\,\mathrm{d}\ell \quad + \\ \frac{\delta_n^{1-p}}{p}\int_{K_n} (|\mathrm{D}v|^p - |\mathrm{D}u_n|^p)\,\mathrm{d}\ell. \end{aligned} \tag{4.3.1}$$

By choosing $v = u_n + t\psi$, with $\psi \in D(\Phi_p^{(n)})$ and $0 < t \leq 1$ in (4.3.1), we obtain

$$\begin{aligned} t\int_{\Omega_n} f\,\psi\,\mathrm{d}\mathcal{L}_2 + t\delta_n \int_{K_n} f\,\psi\,\mathrm{d}\ell \quad \leq \\ \frac{1}{p}\int_{\Omega} \chi_{\Omega_n}(|\mathrm{D}(u_n+t\psi)|^p - |\mathrm{D}u_n|^p)\,\mathrm{d}\mathcal{L}_2 + \frac{\delta_n}{p}\int_{K_n} b(|u_n+t\psi|^p - |u_n|^p)\,\mathrm{d}\ell + \\ \frac{\delta_n^{1-p}}{p}\int_{K_n} (|\mathrm{D}(u_n+t\psi)|^p - |\mathrm{D}u_n|^p)\,\mathrm{d}\ell. \end{aligned} \tag{4.3.2}$$

Now, if $\psi \in \mathcal{D}(\Omega_n)$, from (4.3.2) we have that

$$\int_{\Omega_n} f\,\psi\,\mathrm{d}\mathcal{L}_2 \leq \frac{1}{p}\int_{\Omega_n} \frac{(|\mathrm{D}(u_n+t\psi)|^p - |\mathrm{D}u_n|^p)}{t}\,\mathrm{d}\mathcal{L}_2.$$

Then, by passing to the limit for $t \to 0^+$, we get

$$\int_{\Omega_n} f\,\psi\,\mathrm{d}\mathcal{L}_2 \leq \int_{\Omega_n} |\mathrm{D}u_n|^{p-2}\mathrm{D}u_n\,\mathrm{D}\psi\,\mathrm{d}\mathcal{L}_2.$$

By taking $-\psi$ in (4.3.2) we obtain the opposite inequality, and hence we get

$$\int_{\Omega_n} f\psi\,\mathrm{d}\mathcal{L}_2 = \int_{\Omega_n} |\mathrm{D}u_n|^{p-2}\mathrm{D}u_n\mathrm{D}\psi\,\mathrm{d}\mathcal{L}_2.$$

In order to apply Green formula for Lipschitz domains (see [13] and [7])

$$\int_{\Omega_n} |\mathrm{D}u|^{p-2}\mathrm{D}u\mathrm{D}\psi \,\mathrm{d}\mathcal{L}_2 = \left\langle \frac{\partial u}{\partial \nu_n}|\mathrm{D}u|^{p-2}, \psi \right\rangle_{W^{-\frac{1}{p'},p'}(K_n), W^{\frac{1}{p'},p}(K_n)} - \int_{\Omega_n} \Delta_p u\psi \,\mathrm{d}\mathcal{L}_2$$

we ask that $w := |\mathrm{D}u_n|^{p-2}\mathrm{D}u_n \in (L^{p'}_{\mathrm{div}}(\Omega_n))^2 := \{w \in (L^{p'}(\Omega_n))^2 \,:\, \mathrm{div}\, w \in L^{p'}(\Omega_n)\}$. Since $p \geq 2$, then $p' \leq 2$, therefore if we choose $f \in L^2(\Omega_n)$ in particular $f \in L^{p'}(\Omega_n)$. Hence, taking into account that $\psi \in \mathcal{D}(\Omega_n)$, it holds that $-\Delta_p u_n = f$ in $L^{p'}(\Omega_n)$ (in particular $-\Delta_p u_n = f$ in $L^2(\Omega_n)$) then it holds a.e. in Ω_n.
We go back to (4.3.2). Dividing by $t > 0$ and passing to the limit for $t \to 0^+$, we get

$$\int_{\Omega_n} f\psi \,\mathrm{d}\mathcal{L}_2 + \delta_n \int_{K_n} f\psi \,\mathrm{d}\ell \leq \int_{\Omega_n} |\mathrm{D}u_n|^{p-2}\mathrm{D}u_n\mathrm{D}\psi \,\mathrm{d}\mathcal{L}_2 + \delta_n \int_{K_n} b|u_n|^{p-2}u_n \,\psi \,\mathrm{d}\ell$$
$$+\delta_n^{1-p} \int_{K_n} |\mathrm{D}u_n|^{p-2}\mathrm{D}u_n\mathrm{D}\psi \,\mathrm{d}\ell.$$

As above, by taking $-\psi$ we obtain the opposite inequality, hence we get the equality. Then, by using Green formula for Lipschitz domains and since $-\Delta_p u_n = f$ in $L^{p'}(\Omega_n)$, we have

$$\begin{aligned}\delta_n \int_{K_n} f\psi \,\mathrm{d}\ell = &\left\langle \frac{\partial u_n}{\partial \nu_n}|\mathrm{D}u_n|^{p-2}, \psi \right\rangle_{W^{-\frac{1}{p'},p'}(K_n), W^{\frac{1}{p'},p}(K_n)} + \delta_n \int_{K_n} b|u_n|^{p-2}u_n \,\psi \,\mathrm{d}\ell \\ &+ \delta_n^{1-p} \int_{K_n} |\mathrm{D}u_n|^{p-2}\mathrm{D}u_n\mathrm{D}\psi \,\mathrm{d}\ell.\end{aligned} \tag{4.3.3}$$

We can define Δ_p as a variational operator $\Delta_p\colon W^{1,p}_0(K_n) \to W^{-1,p'}(K_n)$ in the following way:

$$\int_{K_n} |\mathrm{D}z|^{p-2}\mathrm{D}z\mathrm{D}w = - < \Delta_p \, z, w >_{W^{-1,p'}(K_n), W^{1,p}(K_n)} \tag{4.3.4}$$

for $z, w \in W^{1,p}_0(K_n)$. Then from (4.3.3) we have that

$$\delta_n f = \delta_n b|u_n|^{p-2}u_n - \delta_n^{1-p}\Delta_p u_n + \frac{\partial u_n}{\partial \nu_n}|\mathrm{D}u_n|^{p-2} \tag{4.3.5}$$

holds in $W^{-\frac{1}{p'},p'}(K_n)$.
We want now to prove the converse. Let then $u_n \in D(\Phi_p^{(n)})$ be the weak solution of problem $(\bar{P}_n)$. We have then to prove that $\Phi_p^{(n)}[v] - \Phi_p^{(n)}[u_n] \geq (f, v - u_n)_{H_n}$ for every $v \in D(\Phi_p^{(n)})$. By using the inequality

$$\frac{1}{p}(|a|^p - |b|^p) \geq |b|^{p-2}b(a-b) \tag{4.3.6}$$

one gets

$$\Phi_p^{(n)}[v]-\Phi_p^{(n)}[u_n]\geq\int\limits_{\Omega_n}|\mathrm{D}u_n|^{p-2}\mathrm{D}u_n\mathrm{D}v\,\mathrm{d}\mathcal{L}_2-\int\limits_{\Omega_n}|\mathrm{D}u_n|^p\,\mathrm{d}\mathcal{L}_2\ +$$
$$\delta_n^{1-p}\int\limits_{K_n}|\mathrm{D}u_n|^{p-2}\mathrm{D}u_n\mathrm{D}v\,\mathrm{d}\ell-\delta_n^{1-p}\int\limits_{K_n}|\mathrm{D}u_n|^p\,\mathrm{d}\ell+$$
$$\delta_n\int\limits_{K_n}b|u_n|^{p-2}u_nv\,\mathrm{d}\ell-\delta_n\int\limits_{K_n}b|u_n|^p\,\mathrm{d}\ell. \tag{4.3.7}$$

Since u_n is the weak solution of $(\bar{P}_n)$, by using as test functions v and u_n we have

$$\Phi_p^{(n)}[v]-\Phi_p^{(n)}[u_n]\geq(f,v)_{H_n}-(f,u_n)_{H_n},$$

i.e. the thesis. □

By proceeding as in Theorem 2.6 and Theorem 2.7 in [56] one can prove the following result.

Theorem 4.3.4. *If $u_0^{(n)}\in\overline{D(-\mathcal{A}_n)}$, then (P_n) has a unique strong solution $u_n\in C([0,T];H_n)$ defined as $u_n=T_p^{(n)}(\cdot)u_0^{(n)}$ such that $u_n\in W^{1,2}((\delta,T);H_n)$ for every $\delta\in(0,T)$. Moreover $u_n\in D(-\mathcal{A}_n)$ a.e. for $t\in(0,T)$, $\sqrt{t}\frac{\mathrm{d}u_n}{\mathrm{d}t}\in L^2(0,T;H_n)$ and $\Phi_p^{(n)}[u_n]\in L^1(0,T)$.*

Moreover from Theorem 4.3.3 it follows that the solution u_n of problem (P_n) solves for each $n\in\mathbb{N}$ the following problem $(\tilde{P}_n)$ on Ω_n for $t\in(0,T]$ in the following weak sense:

$$(\tilde{P}_n)\begin{cases}\frac{\mathrm{d}u_n}{\mathrm{d}t}-\Delta_p u_n=0, \qquad \text{in } L^{p'}(\Omega_n)\\ \delta_n\left\langle\frac{\mathrm{d}u_n}{\mathrm{d}t},\psi_n\right\rangle_{L^2(K_n),L^2(K_n)}+\left\langle\frac{\partial u_n}{\partial\nu_n}|\mathrm{D}u_n|^{p-2},\psi_n\right\rangle_{W^{-\frac{1}{p'},p'}(K_n),W^{\frac{1}{p'},p}(K_n)}+\\ \delta_n\left\langle b|u_n|^{p-2}u_n,\psi_n\right\rangle_{L^{p'}(K_n),L^p(K_n)}-\delta_n^{1-p}\left\langle\Delta_p u_n,\psi_n\right\rangle_{W^{-1,p'}(K_n),W^{1,p}(K_n)}=0\\ \qquad\forall\ \psi_n\in W^{1,p}(K_n),\\ u_n(0,x)=u_0^{(n)}(x) \qquad \text{in } L^2(\Omega)\cap L^2(\Omega,m_n).\end{cases}$$

Theorem 4.2.2, Theorem 4.2.6 and Theorem 7.24 in [82] allow us to deduce that the pre-fractal solutions converge in a suitable sense to the limit fractal one.

Theorem 4.3.5. *Let H_n, H, $\Phi_p^{(n)}$, Φ_p and δ_n be as in Theorem 4.2.2. Let $T_p^{(n)}(t)$, $T_p(t)$, $u_0^{(n)}$ and u_0 be as in Theorems 4.3.2 and 4.3.4. If $u_0^{(n)}\to u_0$ strongly in $\mathcal{H}$, then $T_p^{(n)}(t)\,u_0^{(n)}\xrightarrow[n\to\infty]{}T_p(t)\,u_0$ strongly in $\mathcal{H}$ for every $t\geq0$.*

Remark 4.3.6. We point out that the existence and uniqueness of the strong solution for problems $(\tilde{P})$ and $(\tilde{P}_n)$ can be proved also for the nonhomogeneous problems (see Theorem 2.7 in [56] for the fractal case). But in this case the asymptotic behavior of the solutions is still an open problem.

Chapter 5

Quasilinear local Venttsel' problems in three-dimensional fractal domains

In this chapter we extend the results of Chapter 4 to the three-dimensional case. Let Q, Q_n, S and S_n be as defined in Section 1.1.2. In the following we denote by $L^2(\overline{Q}, m)$ the Lesbegue space with respect to the measure m with

$$\mathrm{d}m = \mathrm{d}\mathcal{L}_3 + \mathrm{d}g, \tag{5.0.1}$$

where $\mathcal{L}_3$ is the three-dimensional Lebesgue measure and g is the measure defined in (1.1.5). By the space $L^2(Q, m_n)$ we denote the Lebesgue space with respect to the measure m_n with

$$\mathrm{d}m_n = \chi_{Q_n}\mathrm{d}\mathcal{L}_3 + \chi_{S_n}\delta_n\mathrm{d}\sigma, \tag{5.0.2}$$

where χ_{Q_n} and χ_{S_n} denote the characteristic function of Q_n and S_n respectively and $\mathrm{d}\sigma = \mathrm{d}\ell \times \mathrm{d}\mathcal{L}_1$ is the measure on every affine face $S_n^{(j)}$ of S_n.

In the setting of varying Hilbert spaces introduced in Section 1.4, we consider $H = L^2(\overline{Q}, m)$ where m is the measure in (5.0.1), and the sequence $\{H_n\}_{n\in\mathbb{N}}$ with $H_n = \{L^2(Q) \cap L^2(Q, m_n)\}$ where m_n is the measure in (5.0.2) with norms

$$\|u\|^2_H = \|u\|^2_{L^2(Q)} + \|u|_S\|^2_{L^2(S,g)}, \ \ \|u\|^2_{H_n} = \|u\|^2_{L^2(Q_n)} + \|u|_{S_n}\|^2_{L^2(S_n,\delta_n\sigma)}.$$

One can easily prove that, if we set $\delta_n = (\frac{3}{4})^n$, Theorem 1.4.11 holds and the spaces H_n converge in the sense of Kuwae and Shioya to the space H.

5.1 Preliminaries

Most of the results presented in Chapter 1 hold true also for the three-dimensional case. For the sake of clarity, we recall the main results of Chapter 1 specialized to the 3D case.

We define the Sobolev space $W^{s,p}(Q)$ and the trace operator γ_0 exactly as in Section 1.2, by replacing $\mathbb{R}^2$ with $\mathbb{R}^3$ (see e.g. [40]). We denote by $W^{1,p}(S_n)$ the Sobolev space (on the polyhedral domain S_n) of functions for which the norm

$$\|u\|^p_{W^{1,p}(S_n)} = \int\limits_I \left(\|u\|^p_{L^p(K_n)} + \|\mathrm{D}u\|^p_{L^p(K_n)} + \|\mathrm{D}_y u\|^p_{L^p(K_n)}\right) \mathrm{d}\mathcal{L}_1$$

is finite [74].

Proposition 5.1.1. *Let Q_n and S_n be as in Section 1.1.2. Let $\frac{1}{p} < s < 1 + \frac{1}{p}$. Then $W^{s-\frac{1}{p},p}(S_n)$ is the trace space to S_n of $W^{s,p}(Q_n)$ in the following sense:*

1. *γ_0 is a continuous and linear operator from $W^{s,p}(Q_n)$ to $W^{s-\frac{1}{p},p}(S_n)$;*
2. *there exists a continuous linear operator* Ext *from $W^{s-\frac{1}{p},p}(S_n)$ to $W^{s,p}(Q_n)$ such that $\gamma_0 \circ \mathrm{Ext}$ is the identity operator in $W^{s-\frac{1}{p},p}(S_n)$.*

We now fix

$$\beta = 1 - \frac{2 - d_f}{p}. \tag{5.1.1}$$

We recall the definition of Besov space on S only for this particular β, which is the case of interest.

Definition 5.1.2. *We say that $f \in B^{p,p}_\beta(S)$ if $f \in L^p(S, g)$ and it holds*

$$\|f\|_{B^{p,p}_\beta(S)} < +\infty,$$

where

$$\|f\|_{B^{p,p}_\beta(S)} = \|f\|_{L^p(S,g)} + \left(\iint\limits_{|P-P'|<1} \frac{|f(P) - f(P')|^p}{|P - P'|^{2d_f+p-1}} \mathrm{d}g(P)\mathrm{d}g(P') \right)^{\frac{1}{p}} \tag{5.1.2}$$

We recall the trace theorem specialized to our case.

Theorem 5.1.3. *Let Γ denote S, $\Omega \times \{0\}$ and $\Omega \times \{1\}$. $B^{p,p}_\alpha(\Gamma)$ is the trace space of $W^{1,p}(Q)$ that is:*

1. *There exists a linear and continuous operator $\gamma_0 : W^{1,p}(Q) \to B^{p,p}_\alpha(\Gamma)$.*

2. *There exists a linear and continuous operator* $\mathrm{Ext} : B^{p,p}_{\alpha}(\Gamma) \to W^{1,p}(Q)$*, such that* $\gamma_0 \circ \mathrm{Ext}$ *is the identity operator on* $B^{p,p}_{\alpha}(\Gamma)$*.*

For the proof we refer to Theorem 1 of Chapter VII in [43], see also [83]. In the case $\Gamma = S$, then the smoothness index α is equal to $1 - \frac{2-d_f}{p}$. If $\Gamma = \Omega \times \{0\}$ or $\Gamma = \Omega \times \{1\}$, then $\alpha = 1 - \frac{1}{p}$; we point out that in this case the Besov space $B^{p,p}_{1-\frac{1}{p}}(\Gamma)$ coincides with the fractional Sobolev space $W^{1-\frac{1}{p},p}(\Gamma)$.

We recall also the definition of Besov spaces on an arbitrary closed subset $F \subset \mathbb{R}^3$ given in [42, page 356]. Let μ_F be a measure supported on F. The Besov space $\tilde{B}^{p,p}_{\gamma}(F)$ with respect to μ_F is the space of functions such that the following norm is finite:

$$|||u|||^p_{\tilde{B}^{p,p}_{\gamma}(F)} = \|u\|^p_{L^p(F)} + \sum_{j=0}^{+\infty} 3^{j(\gamma p-3)} \iint\limits_{|x-y|<3^{-j}} \frac{|u(x)-u(y)|^p}{m_j(x) m_j(y)} \,\mathrm{d}\mu_F(x)\,\mathrm{d}\mu_F(y), \quad (5.1.3)$$

where $m_j(x) := \mu_F(B(x, 3^{-j}))$. From Proposition 2 in [42], it follows that this norm is equivalent to the following norm:

$$\|u\|^p_{\tilde{B}^{p,p}_{\gamma}(F)} = \|u\|^p_{L^p(F)} + \iint\limits_{|x-y|<1} \frac{|u(x)-u(y)|^p}{|x-y|^{\gamma p-3}(\mu_F(B(x,|x-y|)))^2} \,\mathrm{d}\mu_F(x)\,\mathrm{d}\mu_F(y). \quad (5.1.4)$$

We now give the three-dimensional version of Theorem 1.2.2, which can be proved in the same way with small suitable changes.

Theorem 5.1.4. *Let* $u \in W^{\tilde{\beta},p}(\mathbb{R}^3)$ *and* $\delta_n = (\frac{3}{4})^n = (3^{1-d_f})^n$*. Then, for* $\frac{1}{p} < \tilde{\beta} \leq \frac{3}{p}$*,*

$$\|u\|^p_{L^p(S_n)} \leq \frac{C_{\tilde{\beta}}}{\delta_n} \|u\|^p_{W^{\tilde{\beta},p}(\mathbb{R}^3)}, \quad (5.1.5)$$

where $C_{\tilde{\beta}}$ *is independent of* n*.*

We conclude this section by recalling the 3D versions of the extension theorems presented in Section 1.2.

Theorem 5.1.5. *Let* $u \in W^{\tilde{\beta},p}(\mathbb{R}^3)$*. Then, for* $\frac{2-d_f}{p} < \tilde{\beta}$*,*

$$\|u\|^p_{L^p(S)} \leq C^*_{\tilde{\beta}} \|u\|^p_{W^{\tilde{\beta},p}(\mathbb{R}^3)}. \quad (5.1.6)$$

Theorem 5.1.6. *There exists a bounded linear extension operator* $\mathrm{Ext}_J : W^{1,p}(Q_n) \to W^{1,p}(\mathbb{R}^3)$*, such that*

$$\|\mathrm{Ext}_J\, v\|^p_{W^{1,p}(\mathbb{R}^3)} \leq C_J \|v\|^p_{W^{1,p}(Q_n)} \quad (5.1.7)$$

with C_J *independent of* n*.*

Theorem 5.1.7. *There exists a linear extension operator* $\mathcal{E}\mathrm{xt}$ *such that, for any* $\tilde{\beta} > 0$ $\mathcal{E}\mathrm{xt} : W^{\tilde{\beta},p}(Q) \to W^{\tilde{\beta},p}(\mathbb{R}^3)$,

$$\|\mathcal{E}\mathrm{xt}\, v\|^p_{W^{\tilde{\beta},p}(\mathbb{R}^3)} \leq \bar{C}_{\tilde{\beta}} \|v\|^p_{W^{\tilde{\beta},p}(Q)} \tag{5.1.8}$$

with $\bar{C}_{\tilde{\beta}}$ *depending on* $\tilde{\beta}$.

5.2 Energy functionals

From now on, let $p > 2$ (for the case $p = 2$, we refer to [54] and [55]). By proceeding as in [17], we construct a p-energy form on K (which has the role of Euclidean p-Lagrangian $\mathrm{d}\mathcal{L}(u, v) = |\mathrm{D}u|^{p-2}\mathrm{D}u\mathrm{D}v\, \mathrm{d}\mathcal{L}_3$) by defining a p-Lagrangian measure $\mathcal{L}^p_K$ on K. The corresponding p-energy form on K is given by

$$\mathcal{E}_K(u, v) = \int_K \mathrm{d}\mathcal{L}^p_K(u, v)$$

with domain $\mathcal{D}(K) = \{u \in L^p(K, \mu) : \mathcal{E}_K[u] < +\infty\}$ dense in $L^p(K, \mu)$.

Proposition 5.2.1. $\mathcal{D}(K)$ *is a Banach space equipped with the following norm*

$$\|u\|_{\mathcal{D}(K)} = (\|u\|^p_{L^p(K)} + \mathcal{E}_K[u])^{\frac{1}{p}}. \tag{5.2.1}$$

As in [18] the following result can be proved.

Proposition 5.2.2. *For* $p > 1$, $\mathcal{D}(K)$ *is embedded in* $C^{0,\eta}(K)$, *with*

$$\eta = \left(1 - \frac{1}{p}\right) \frac{\ln 4}{\ln 3}.$$

Remark 5.2.3. We point out that, for $p > \frac{\ln 4}{\ln 4 - \ln 3}$, the Hölder exponent η in Proposition 5.2.2 is greater than one. In this case, for the Koch snowflake K, from Corollary 4.2 in [18], the space $C^{0,\eta}(K)$ does not degenerate to the space of constant functions.

We now define the energy form on S:

$$E_S[u] = \frac{1}{p} \int_I \mathcal{E}_K[u] \mathrm{d}\mathcal{L}_1 + \frac{1}{p} \int_K \int_I |\mathrm{D}_y u|^p \mathrm{d}\mathcal{L}_1 \mathrm{d}\mu \tag{5.2.2}$$

with domain $\mathcal{D}(S)$ defined as

$$\mathcal{D}(S) = \overline{C(S) \cap L^p([0, 1]; \mathcal{D}(K)) \cap W^{1,p}([0, 1]; L^p(K))}^{\|\cdot\|_{\mathcal{D}(S)}}, \tag{5.2.3}$$

where $\|\cdot\|_{\mathcal{D}(S)}$ is the intrinsic norm

$$\|u\|_{\mathcal{D}(S)} = (E_S[u] + \|u\|^p_{L^p(S,g)})^{\frac{1}{p}}. \tag{5.2.4}$$

We now give an embedding result for the domain $\mathcal{D}(S)$. Unlike the two-dimensional case where there is a characterization of the functions in $\mathcal{D}(K)$ in terms of the so-called Lipschitz spaces (see Theorem 4.1 in [18]), for $\mathcal{D}(S)$ we do not have such characterization, but the following result holds.

Proposition 5.2.4. *$\mathcal{D}(S)$ is continuously embedded in $B^{p,p}_{\bar\beta}(S)$, for any $0 < \bar\beta < 1$.*

Proof. We follow the proof in [52], adapted to our case.
We recall that

$$\mathcal{D}(S) := \overline{C(S) \bigcap L^p([0,1]; \mathcal{D}(K)) \bigcap W^{1,p}([0,1]; L^p(K))}^{\|\cdot\|_{\mathcal{D}(S)}}.$$

Following [67], we define $B^{p,p}_{d_f-\varepsilon,1}(S) := L^p([0,1]; B^{p,p}_{d_f-\varepsilon}(K)) \bigcap W^{1,p}([0,1]; L^p(K))$ for $\varepsilon > 0$.

From Theorem 4.1 in [18] and Proposition 3, Chapter V in [43], it holds that $\mathcal{D}(K) = B^{p,\infty}_{d_f}(K)$. Moreover, this last space is continuously embedded in $B^{p,p}_{d_f-\varepsilon}(K)$ for $\varepsilon > 0$ (see Proposition 5, Chapter VIII in [43]). Hence, from the definition of $\mathcal{D}(S)$, we deduce that $\mathcal{D}(S) \subset B^{p,p}_{d_f-\varepsilon,1}(S)$. Moreover, the embedding is continuous, i.e. there exists a positive constant C such that

$$\|u\|_{B^{p,p}_{d_f-\varepsilon,1}(S)} \le C\|u\|_{\mathcal{D}(S)}. \tag{5.2.5}$$

From the definition of $B^{p,p}_{d_f-\varepsilon,1}(S)$-norm we get

$$\begin{aligned}
\|u\|^p_{B^{p,p}_{d_f-\varepsilon,1}(S)} = \int_0^1 \left(\|u\|^p_{B^{p,p}_{d_f-\varepsilon}(K)} + \|u\|^p_{L^p(K)} + \|\mathrm{D}_y u\|^p_{L^p(K)} \right) \mathrm{d}\mathcal{L}_1 \le \\
C \int_0^1 \left(\|u\|^p_{B^{p,\infty}_{d_f}(K)} + \|u\|^p_{L^p(K)} + \|\mathrm{D}_y u\|^p_{L^p(K)} \right) \mathrm{d}\mathcal{L}_1 \le \\
C \int_0^1 \left(\|u\|^p_{\mathcal{D}(K)} + \|u\|^p_{L^p(K)} + \|\mathrm{D}_y u\|^p_{L^p(K)} \right) \mathrm{d}\mathcal{L}_1.
\end{aligned}$$

From the definition of E_S and of the norm in $\mathcal{D}(K)$, we get

$$\|u\|_{B^{p,p}_{d_f-\varepsilon,1}(S)} \le C(E_S[u] + \|u\|^p_{L^p(S)}) = C\|u\|^p_{\mathcal{D}(S)},$$

i.e. the thesis.
For any Banach space X and for any $0 < \bar{\beta} < 1$

$$W^{1,p}([0,1];X) \subset W^{\bar{\beta},p}([0,1];X).$$

Moreover if $\bar{\beta}$ is not integer, it holds

$$W^{\bar{\beta},p}([0,1];X) \equiv B^{p,p}_{\bar{\beta}}([0,1];X).$$

Hence if $0 < \bar{\beta} < 1$

$$B^{p,p}_{d_f-\varepsilon,1}(S) \subset L^p([0,1]; B^{p,p}_{d_f-\varepsilon}(K)) \bigcap B^{p,p}_{\bar{\beta}}(0,1;L^p(K)) \subset$$

$$L^p([0,1]; B^{p,p}_{\bar{\beta}}(K)) \bigcap B^{p,p}_{\bar{\beta}}([0,1];L^p(K)) = B^{p,p}_{\bar{\beta}}(S),$$

where the last equivalence can be proved following [67]. We now prove that there exists a positive constant C such for every $0 < \bar{\beta} < 1$

$$\|u\|_{B^{p,p}_{\bar{\beta}}(S)} \leq C\|u\|_{\mathcal{D}(S)}. \tag{5.2.6}$$

Indeed, from the above remarks, we get

$$\|u\|^p_{B^{p,p}_{\bar{\beta}}(S)} \leq C\left(\int_0^1 \|u\|^p_{B^{p,p}_{d_f-\varepsilon}(K)}\,\mathrm{d}\mathcal{L}_1 + \|u\|^p_{B^{p,p}_{\bar{\beta}}([0,1];L^p(K))}\right) = C(\|u\|^p_{L^p([0,1];B^{p,p}_{d_f-\varepsilon}(K))} + \|u\|^p_{W^{\bar{\beta},p}([0,1];L^p(K))}) \leq C(\|u\|^p_{L^p([0,1];B^{p,p}_{d_f-\varepsilon}(K))} + \|u\|^p_{W^{1,p}([0,1];L^p(K))}) = C\|u\|^p_{B^{p,p}_{d_f-\varepsilon,1}(S)}.$$

From (5.2.5) we get (5.2.6). Hence the theorem is proved. □

Now we introduce the energy functional on Q. Let us consider the space

$$V(Q,S) = \left\{u \in W^{1,p}(Q) \,:\, u|_S \in \mathcal{D}(S),\, u|_{\tilde{\Omega}} = 0\right\}, \tag{5.2.7}$$

where $\tilde{\Omega} := (\Omega \times \{0\}) \cup (\Omega \times \{1\})$.
Let b be a continuous and strictly positive function on $\overline{Q}$. We consider the energy functional Φ_p defined as follows:

$$\Phi_p[u] := \begin{cases} \frac{1}{p}\int\limits_Q |\mathrm{D}u|^p\,\mathrm{d}\mathcal{L}_3 + E_S[u|_S] + \frac{1}{p}\int\limits_S b|u|^p\,\mathrm{d}g & \text{if } u \in V(Q,S), \\ +\infty & \text{if } u \in H \setminus V(Q,S). \end{cases} \tag{5.2.8}$$

We denote by $L^p(\overline{Q}, m)$ the Lebesgue space with respect to the measure m defined in (5.0.1). The following Proposition easily follows from Proposition 4.1.1.

Proposition 5.2.5. *Φ_p is a weakly lower semicontinuous, proper and convex functional in H.*

We now set

$$E_p^{(n)}[u] = \frac{\delta_n^{1-p}}{p} \int\limits_I \left(\int\limits_{K_n} |Du|^p d\ell \right) d\mathcal{L}_1 + \frac{\delta_n}{p} \int\limits_{K_n} \left(\int\limits_I |D_y u|^p d\mathcal{L}_1 \right) d\ell, \tag{5.2.9}$$

with domain

$$D(E_p^{(n)}) = W^{1,p}(S_n).$$

We now introduce the energy functional on the pre-fractal domain:

$$\Phi_p^{(n)}[u] := \begin{cases} \frac{1}{p} \int\limits_Q \chi_{Q_n} |Du|^p d\mathcal{L}_3 + \frac{\delta_n}{p} \int\limits_{S_n} b|u|^p \, d\sigma + E_p^{(n)}[u] & \text{if } u \in V(Q, S_n), \\ +\infty & \text{if } u \in H_n \setminus V(Q, S_n), \end{cases} \tag{5.2.10}$$

where

$$V(Q, S_n) := \left\{ u \in W^{1,p}(Q) \, : \, u|_{S_n} \in D(E_p^{(n)}), \, u|_{\tilde{\Omega}_n} = 0 \right\},$$

where we define $\tilde{\Omega}_n := (\Omega_n \times \{0\}) \cup (\Omega_n \times \{1\})$.
By proceeding as in Proposition 4.1.2, we can prove the following result.

Proposition 5.2.6. $\Phi_p^{(n)}$ *is a weakly lower semicontinuous, proper and convex functional in* H_n.

5.3 Density results

In the notations of [67, page 8], we introduce the following space:

$$W(0,1) := L^p([0,1]; \mathcal{D}(K)) \bigcap W^{1,p}([0,1]; L^p(K)). \tag{5.3.1}$$

This is a Banach space equipped with the norm

$$\|u\|_{W(0,1)} = (\|u\|^p_{L^p([0,1];\mathcal{D}(K))} + \|D_y u\|^p_{L^p([0,1];L^p(K))})^{\frac{1}{p}}. \tag{5.3.2}$$

The following results hold.

Proposition 5.3.1. *The space* $D([0,1]; \mathcal{D}(K))$ *is densely embedded in* $W(0,1)$*, that is*

$$\overline{D([0,1]; \mathcal{D}(K))}^{\|\cdot\|_{W(0,1)}} = W(0,1). \tag{5.3.3}$$

Proof. One can easily adapt the proof of Theorem 2.1 page 11 in [67] to the case of Banach spaces, by replacing all the L^2 spaces with the corresponding L^p spaces. $\square$

Proposition 5.3.2. $D([0,1];\mathcal{D}(K)) \subset C(S)$.

Proof. See Proposition 5.2 in [54]. $\square$

Theorem 5.3.3. *The space $D([0,1];\mathcal{D}(K))$ is dense in $\mathcal{D}(S)$ with respect to the intrinsic norm $\|\cdot\|_{\mathcal{D}(S)}$.*

Proof. One can adapt the proof of Theorem 5.3 in [54] with small suitable changes. $\square$

We now state the main Theorem of the section.

Theorem 5.3.4. *Let Q, S and $V(Q,S)$ be defined as in Section 1.1.2 and (5.2.7) respectively. For every $u \in V(Q,S)$ there exists $\psi_h \in V(Q,S)\bigcap C(\overline{Q})$ such that:*

(1) $\|\psi_h - u\|_{W^{1,p}(Q)} \to 0$ *for* $h \to \infty$;

(2) $\|\psi_h - u\|_{L^p(\overline{Q},m)} \to 0$ *for* $h \to \infty$;

(3) $E_S[\psi_h - u] \to 0$ *for* $h \to \infty$.

In order to prove this Theorem, we need a preliminary proposition on trace and extension operators.

Proposition 5.3.5. *Let β be as defined in (5.1.1). Let γ_0 and* Ext *be the trace and the extension operator defined in Theorem 5.1.3 respectively. Then*

(1) If $u \in C(\mathbb{R}^3)\bigcap W^{1,p}(\mathbb{R}^3)$ then $\gamma_0 u \in C(S)\bigcap B^{p,p}_\beta(S)$.

(2) If $u \in C(S)\bigcap B^{p,p}_\beta(S)$ then $\operatorname{Ext} u \in C(\mathbb{R}^3)\bigcap W^{1,p}(\mathbb{R}^3)$.

Proof. One can adapt the proof of Proposition 5.5 in [54] with the obvious changes when considering the case $p \geq 2$ instead of $p = 2$. $\square$

We are now ready to prove Theorem 5.3.4.

Proof. We follow the spirit of the proof of Theorem 5.4 in [54].
Let us consider $u \in V(Q,S)$, then $u|_S \in \mathcal{D}(S)$. We set

$$\tilde{u} = \begin{cases} u|_S & \text{on } S, \\ 0 & \text{on } \partial Q \setminus S, \end{cases}$$

where $\partial Q \setminus S = \tilde{\Omega} = (\Omega \times \{0\}) \cup (\Omega \times \{1\})$. We point out that $u|_S \in B^{p,p}_\alpha(S)$ for every $\alpha \in (0,1)$ from Proposition 5.2.4.

We now consider the Besov space $\tilde{B}^{p,p}_\gamma(\partial Q)$ on the closed set $\partial Q = S \cup \tilde{\Omega}$. We remark that ∂Q is neither a 2-set nor a (d_f+1)-set because S and $\tilde{\Omega}$ have different Hausdorff dimensions. Hence we make use of the more general theory of Besov spaces on arbitrary closed sets introduced by Jonsson in [42].

We prove that there exists $\varepsilon > 0$ such that $\tilde{u} \in \tilde{B}^{p,p}_{1+\varepsilon}(\partial Q)$. We denote by $\tilde{\mu}$ the measure supported on ∂Q defined as

$$\mathrm{d}\tilde{\mu} = \chi_S \mathrm{d}g + \chi_{\tilde{\Omega}} \mathrm{d}\mathcal{L}_2.$$

The measure $\tilde{\mu}$ satisfies the following property: there exist two positive constants $\tilde{c}_1$ and $\tilde{c}_2$ such that

$$\tilde{c}_1 r^2 \leq \tilde{\mu}(B(x,r)) \leq \tilde{c}_2 r^{d_f+1} \quad \text{for every } x \in \partial Q. \tag{5.3.4}$$

Taking into account (5.3.4), from (5.1.4) it follows that

$$\|\tilde{u}\|^p_{\tilde{B}^{p,p}_\gamma(\partial Q)} \leq \tilde{c} \left(\|\tilde{u}\|^p_{L^p(\partial Q)} + \iint\limits_{|x-y|<1} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x-y|^{\gamma p+1}} \,\mathrm{d}\tilde{\mu}(x)\,\mathrm{d}\tilde{\mu}(y) \right).$$

The last integral is finite if and only if $\gamma < \frac{1+2d_f}{p}$. Moreover, since $\tilde{u} = 0$ on $\tilde{\Omega}$, then $\|\tilde{u}\|_{L^p(\partial Q)} = \|u|_S\|_{L^p(S)}$.

We now estimate the $\tilde{B}^{p,p}_\gamma(\partial Q)$-norm of $\tilde{u}$ with the $B^{p,p}_\alpha(S)$-norm of $u|_S$, for $\alpha \in (0,1)$, hence $\gamma p + 1 < d_f + 1 + \alpha p$, that is

$$\gamma < \frac{d_f}{p} + \alpha. \tag{5.3.5}$$

We remark that, if γ satisfies (5.3.5), it also satisfies the condition $\gamma < \frac{1+2d_f}{p}$, and we have that

$$\|\tilde{u}\|^p_{\tilde{B}^{p,p}_\gamma(\partial Q)} \leq \tilde{c} \left(\|u|_S\|^p_{L^p(S)} + \|u|_S\|^p_{B^{p,p}_\alpha(S)} \right) < \infty.$$

For the arbitrariness of $0 < \alpha < 1$, there exists $\tilde{\alpha} > 1 - \frac{d_f}{p}$ such that

$$1 < \gamma < \tfrac{d_f}{p} + \tilde{\alpha}.$$

Therefore, there exists $\varepsilon > 0$ such that $\tilde{u} \in \tilde{B}^{p,p}_{1+\varepsilon}(\partial Q)$. Since ∂Q is a closed set in $\mathbb{R}^3$, from Theorem 1 in [42] there exists an extension operator $\mathrm{Ext}_{\partial Q}$ from $\tilde{B}^{p,p}_{1+\varepsilon}(\partial Q)$ to $B^{p,p}_{1+\varepsilon}(\mathbb{R}^3)$. This space coincides with $W^{1+\varepsilon,p}(\mathbb{R}^3)$. Hence, if we set

$$\widehat{u} := (\mathrm{Ext}_{\partial Q} \tilde{u})|_Q,$$

this function in particular belongs to $W^{1,p}(Q)$.
We now prove (1). From Theorem 5.3.3 there exists $\{\varphi_h\} \subset D(0,1;\mathcal{D}(K))$ such that

$$\|\varphi_h - u|_S\|_{\mathcal{D}(S)} \to 0 \text{ when } h \to \infty.$$

Let now $\widehat{\varphi_h} := \operatorname{Ext} \varphi_h$. Then from Proposition 5.3.5 (see [43])

$$\widehat{\varphi_h} \in W^{1,p}(Q) \bigcap C(\overline{Q}).$$

We now prove that $\|\widehat{\varphi_h} - \widehat{u}\|_{W^{1,p}(Q)} \to 0$. Indeed, from Theorem 5.1.3 and the inclusion of $\mathcal{D}(S)$ in $B^{p,p}_{\beta}(S)$ (see Proposition 5.2.4),

$$\|\widehat{\varphi_h} - \widehat{u}\|_{W^{1,p}(Q)} \leq C_1 \|\varphi_h - u|_S\|_{B^{p,p}_{\beta}(S)} \leq \|\varphi_h - u|_S\|_{\mathcal{D}(S)} \to 0$$

from the density Theorem 5.3.3.
Now let us consider the function $u - \widehat{u}$. This function belongs to $W^{1,p}(Q)$ and it is such that $(u-\widehat{u})|_{\partial Q} = 0$, then $u - \widehat{u} \in W^{1,p}_0(Q)$ (see Theorem 3 in [89]). There exists $\{\eta_m\}_{m\in\mathbb{N}} \subset C^1_0(\overline{Q})$ such that

$$\|\eta_m - (u - \widehat{u})\|_{W^{1,p}(Q)} \to 0. \tag{5.3.6}$$

Let $\{\psi_{h,m}\}$ denote the doubly indexed sequence of function $\{\widehat{\varphi_h} - \eta_m\}$. The sequence $\{\psi_{h,m}\}$ belongs to $W^{1,p}(Q) \bigcap C(\overline{Q})$. From Corollary 1.16 in [6] we deduce that $\{\psi_{m,h}\}$ converges to u in $W^{1,p}(Q)$ as $h \to \infty$. In fact there exists an increasing mapping $h \to m(h)$, tending to ∞ as $h \to \infty$, such that

$$\begin{aligned}\overline{\lim_{h\to\infty}} \|u - \psi_{h,m(h)}\|_{W^{1,p}(Q)} = \overline{\lim_{h\to\infty}} \|u - \widehat{\varphi_h} - \eta_{m(h)}\|_{W^{1,p}(Q)} \leq \\ \overline{\lim_{h\to\infty}} (\|u - \widehat{u} - \eta_{m(h)}\|_{W^{1,p}(Q)} + \|\widehat{\varphi_h} - \widehat{u}\|_{W^{1,p}(Q)}).\end{aligned}$$

Hence by applying Corollary 1.16 in [6] to the right hand side of the above inequality it follows that

$$\overline{\lim_{h\to\infty}} \|u - \psi_{h,m(h)}\|_{W^{1,p}(Q)} \leq \lim_{m\to\infty} \lim_{h\to\infty} \{ \|u - \widehat{u} - \eta_m\|_{W^{1,p}(Q)} + \|\widehat{\varphi_h} - \widehat{u}\|_{W^{1,p}(Q)} \}.$$

The two terms in the sum tend to zero when $m, h \to \infty$, then

$$\overline{\lim_{h\to\infty}} \|\psi_{h,m(h)} - u\|_{W^{1,p}(Q)} = 0, \tag{5.3.7}$$

and also $\underline{\lim}_{h\to\infty} \|\psi_{h,m(h)} - u\|_{W^{1,p}(Q)} = 0$. Hence we conclude that

$$\|\psi_{h,m(h)} - u\|_{W^{1,p}(Q)} \to 0 \text{ when } h \to \infty.$$

From now on we denote by $\psi_h = \psi_{h,m(h)}$. We now prove (2), that is

$$\|\psi_h - u\|_{L^p(\overline{Q},m)} = \|\psi_h - u\|_{L^p(Q)} + \|\psi_h - u\|_{L^p(S)} \to 0. \tag{5.3.8}$$

The first term in (5.3.8) tends to zero when $h \to \infty$ since

$$\|\psi_h - u\|_{L^p(Q)} \leq \|\psi_h - u\|_{W^{1,p}(Q)}.$$

We now prove that also the second term in (5.3.8) tends to zero:

$$\begin{aligned} \|\psi_h - u\|_{L^p(S)} &= \|\widehat{\varphi_h}|_S - \eta_h|_S - u|_S\|_{L^p(S)} \\ &\equiv \|\varphi_h - u|_S\|_{L^p(S)} \leq \|\varphi_h - u|_S\|_{\mathcal{D}(S)}, \end{aligned}$$

and the last quantity tends to zero from the density of $D(0,1;\mathcal{D}(K))$ in $\mathcal{D}(S)$. This proves that $\psi_h \to u$ in $L^p(\overline{Q}, m)$.
We now prove (3):

$$E_S[(u - \psi_h)|_S] = E_S[u|_S - \psi_h|_S] \equiv E_S[u|_S - \varphi_h] \leq \|u|_S - \varphi_h\|_{\mathcal{D}(S)} \to 0.$$

Hence the theorem is proved. $\square$

We remark that we can prove a result similar to Theorem 5.3.4 also for the pre-fractal case. We define the space

$$W^{(n)}(0,1) = L^p([0,1]; W^{1,p}(K_n)) \cap W^{1,p}([0,1]; L^p(K_n)).$$

Similarly to Proposition 5.3.1, we can prove that $D(0,1;W^{1,p}(K_n))$ is dense in $W^{(n)}(0,1)$. But it turns out that

$$W^{(n)}(0,1) \equiv W^{1,p}(S_n).$$

We also point out that we can prove as in Theorem 5.3.2 that $D(0,1;W^{1,p}(K_n)) \subset C(S_n)$. Hence the following result holds.

Theorem 5.3.6. *For every $u \in V(Q, S_n)$ there exists $\psi_h \in V(Q, S_n) \cap C(\overline{Q})$ such that:*

(1) $\|\psi_h - u\|_{W^{1,p}(Q)} \to 0$ for $h \to \infty$;

(2) $\|\psi_h - u\|_{L^p(Q,m_n)} \to 0$ for $h \to \infty$;

(3) $E_p^{(n)}[\psi_h - u] \to 0$ for $h \to \infty$.

Proof. Let $u \in V(Q, S_n)$, hence $u|_{S_n} \in D(E_p^{(n)}) = W^{1,p}(S_n)$. From the density of $D(0,1;W^{1,p}(K_n))$ in $W^{1,p}(S_n)$, there exists a sequence $\{\varphi_h\} \subset D(0,1;W^{1,p}(K_n))$ such that

$$\|\varphi_h - u\|_{W^{1,p}(S_n)} \to 0 \text{ for } h \to \infty.$$

Since $\{\varphi_h\} \subset D(0,1;W^{1,p}(K_n))$, in particular it belongs to $W^{1-\frac{1}{p},p}(S_n)$. From the trace Theorem 5.1.1 there exists an extension $\hat{\varphi}_h$ belonging to $W^{1,p}(Q_n)$; then, from Theorem 5.1.6, there exists an extension $\tilde{\varphi}_h \in W^{1,p}(\mathbb{R}^3)$. We point out that, since $\varphi_h \in C(S_n)$, as in Proposition 5.3.5 we can prove that the extension of φ_h is continuous on $\overline{Q}$. We set $\psi_h := \tilde{\varphi}_h|_Q$, hence $\psi_h \in W^{1,p}(Q)$. From Theorem 5.1.6 and Theorem 5.1.1 we get

$$\|\psi_h - u\|_{W^{1,p}(Q)} \leq C_1\|\tilde{\varphi}_h - u\|_{W^{1,p}(\mathbb{R}^3)} \leq C_2\|\hat{\varphi}_h - u\|_{W^{1,p}(Q_n)} \leq C_3\|\varphi_h - u\|_{W^{1-\frac{1}{p},p}(S_n)} \leq C_4\|\varphi_h - u\|_{W^{1,p}(S_n)},$$

and the last quantity tends to 0 for $h \to \infty$ from the density of $D(0,1;W^{1,p}(K_n))$ in $W^{1,p}(S_n)$.

As to (2), the following holds from (1) and the density of $D(0,1;W^{1,p}(K_n))$ in $W^{1,p}(S_n)$:

$$\|\psi_h - u\|^p_{L^p(Q,m_n)} = \|\psi_h - u\|^p_{L^p(Q_n)} + \delta_n\|\varphi_h - u\|^p_{L^p(S_n)} \leq C_1\|\psi_h - u\|^p_{W^{1,p}(Q)} + C_2\|\varphi_h - u\|^p_{W^{1,p}(S_n)} \to 0.$$

We now come to (3):

$$E^{(n)}_p[\psi_h - u] \leq C\|\varphi_h - u\|^p_{W^{1,p}(S_n)} \to 0.$$

Hence the theorem is proved. □

5.4 M-Convergence of the energy functionals and convergence results

The main theorem of this section is the following.

Theorem 5.4.1. *Let $\delta_n = (3^{1-d_f})^n = \left(\frac{3}{4}\right)^n$. Let Φ_p and $\Phi^{(n)}_p$ be defined as in (5.2.8) and (5.2.10) respectively. Then $\Phi^{(n)}_p$ M-converges to the functional Φ_p.*

We preliminary state the following propositions.

Proposition 5.4.2. *If $\{v_n\}_{n\in\mathbb{N}}$ weakly converges to a vector u in $\mathcal{H}$, then $\{v_n\}_{n\in\mathbb{N}}$ weakly converges to u in $L^2(Q)$ and $\lim_{n\to\infty} \delta_n \int_{S_n} \varphi v_n \,\mathrm{d}\sigma = \int_S \varphi u \,\mathrm{d}g$ for every $\varphi \in C(\overline{Q})$.*

For the proof see Proposition 6.6 in [55].

Proposition 5.4.3. *Let $v_n \rightharpoonup u$ in $W^{1,p}(Q)$, $b \in C(\overline{Q})$. Then*

$$\delta_n \int_{S_n} b|v_n|^p \,\mathrm{d}\sigma \to \int_S b|u|^p \,\mathrm{d}g.$$

Proof. The proof follows from Proposition 4.2.4 simply by integrating on I. □

Proof. (Theorem 5.4.1) The proof is similar to the proof of Theorem 4.2.2 with suitable changes. We have to prove conditions a) and b) in Definition 4.2.1.

Proof of condition a). Let $v_n \in H_n$ be a weakly converging sequence in $\mathcal{H}$ to $u \in H$. We can suppose that $v_n \in V(Q, S_n)$ and

$$\varliminf_{n\to\infty} \Phi_p^{(n)}[v_n] < \infty$$

(otherwise the thesis follows trivially). Then there exists a c independent of n such that

$$\frac{1}{p}\int_Q \chi_{Q_n}|\mathrm{D}v_n|^p \mathrm{d}\mathcal{L}_3 + \frac{\delta_n}{p}\int_{S_n} b|v_n|^p\,\mathrm{d}\sigma + \frac{\delta_n^{1-p}}{p}\int_{S_n} |\mathrm{D}v_n|^p\,\mathrm{d}\sigma + \frac{\delta_n}{p}\int_{S_n} |\mathrm{D}_y v_n|^p\,\mathrm{d}\sigma \le c. \quad (5.4.1)$$

Let us suppose that v_n is continuous on $\overline{Q}$. From (5.4.1), in particular we have that $\|v_n\|_{W^{1,p}(Q_n)} < c$. For every $n \in \mathbb{N}$ from Theorem 5.1.6 there exists a bounded linear operator $\mathrm{Ext}\colon W^{1,p}(Q_n) \to W^{1,p}(\mathbb{R}^3)$ such that

$$\|\mathrm{Ext}\, v_n\|_{W^{1,p}(\mathbb{R}^3)} \le C\,\|v_n\|_{W^{1,p}(Q_n)} \le c\,C,$$

with C independent of n.

We now set $\hat{v}_n = \mathrm{Ext}\, v_n|_Q$. We then proceed as in the proof of condition a) in Theorem 4.2.2, by using Proposition 5.4.2 this time, and we conclude that

$$\varliminf_{n\to\infty} \int_Q \chi_{Q_n}|\mathrm{D}v_n|^p\,\mathrm{d}\mathcal{L}_3 \ge \int_Q |\mathrm{D}u|^p\,\mathrm{d}\mathcal{L}_3.$$

Moreover, the following

$$\varliminf_{n\to\infty} \frac{\delta_n^{1-p}}{p}\int_{S_n} |\mathrm{D}v_n|^p\,\mathrm{d}\sigma \ge \frac{1}{p}\int_I \mathcal{E}_K[u]\,\mathrm{d}\mathcal{L}_1$$

holds as a consequence of Theorem 4.2.2 and Fatou Lemma. We are left to prove that

$$\varliminf_{n\to\infty} \frac{\delta_n}{p}\int_{S_n} |\mathrm{D}_y v_n|^p\,\mathrm{d}\sigma \ge \frac{1}{p}\int_S |\mathrm{D}_y u|^p\,\mathrm{d}g. \quad (5.4.2)$$

First we point out that, since v_n weakly converges to u in $W^{1,p}(Q)$, it follows that v_n strongly converges to u in $W^{s,p}(Q)$ for every $s \in (0,1)$. Hence, from Theorem 5.1.3, $v_n|_S$ strongly converges to $u|_S$ in $B^{p,p}_{s-\frac{2-d_f}{p}}(S)$, so in particular $v_n|_S$ strongly converges to $u|_S$ in $L^p(S)$.

We now set $w_n := \mathrm{D}_y v_n \in L^p(Q)$. In order to prove (5.4.2), we preliminary prove that

$$\|w_n\|_{L^p(S)} \le c.$$

From the density of $C^\infty(\overline{Q})$ in $W^{1,p}(Q)$ (see [70, Theorem 2, page 28]), there exists a sequence $\{w_n^h\}_h \in C^\infty(\overline{Q})$ such that $w_n^h \xrightarrow[h\to\infty]{} w_n$ in $L^p(S_n)$. We want to prove that $\|w_n^h\|_{L^p(S)} \leq c$.
By proceeding as in the proof of Theorem 4.5 in [57], since w_n^h is continuous on S, we can estimate the above norm in terms of the corresponding *Darboux sums*, and we get

$$\int\limits_S |w_n^h|^p \,\mathrm{d}g \leq \delta_n \int\limits_{S_n} |w_n^h| \,\mathrm{d}\sigma. \tag{5.4.3}$$

Passing to the upper limit as $h \to \infty$, since w_n^h strongly converges to w_n in $L^p(S_n)$, from (5.4.1) we get

$$\overline{\lim_{h\to\infty}} \|w_n^h\|_{L^p(S)} \leq c.$$

Since w_n^h is bounded in $L^p(S)$, there exists a subsequence (still denoted by w_n^h) weakly converging to a function w_n^* in $L^p(S)$ for $h \to \infty$. Moreover, from the lower semicontinuity of the norm, we have

$$\|w_n^*\|_{L^p(S)} \leq c.$$

The above inequality implies that there exists a subsequence of w_n^*, again denoted by w_n^*, weakly converging to a function w^* in $L^p(S)$. By using again the lower semicontinuity of the norm, we get

$$\|w^*\|_{L^p(S)} \leq \underline{\lim}_{n\to\infty} \int\limits_S |w_n^*|^p \,\mathrm{d}g \leq \underline{\lim}_{n\to\infty} \underline{\lim}_{h\to\infty} \int\limits_S |w_n^h|^p \,\mathrm{d}g \leq \underline{\lim}_{n\to\infty} \underline{\lim}_{h\to\infty} \delta_n \int\limits_{S_n} |w_n^h|^p \,\mathrm{d}\sigma =$$
$$\underline{\lim}_{n\to\infty} \delta_n \int\limits_{S_n} |w_n|^p \,\mathrm{d}\sigma = \underline{\lim}_{n\to\infty} \delta_n \int\limits_{S_n} |\mathrm{D}_y v_n|^p \,\mathrm{d}\sigma,$$

where in the last inequality we used (5.4.3). Hence (5.4.2) follows if we prove that $w^* = \mathrm{D}_y u$ a.e. in $L^p(S)$.
By using the definition of weak convergence and distributional derivative, we get $\forall\, \varphi \in L^{p'}(S)$

$$\int\limits_S w^*\varphi \,\mathrm{d}g = \lim_{n\to\infty} \int\limits_S w_n^*\varphi \,\mathrm{d}g = \lim_{n\to\infty} \lim_{h\to\infty} \int\limits_S w_n^h\varphi \,\mathrm{d}g = \lim_{n\to\infty} \int\limits_S w_n\varphi \,\mathrm{d}g =$$
$$\lim_{n\to\infty} \int\limits_S \mathrm{D}_y v_n \varphi \,\mathrm{d}g = -\lim_{n\to\infty} \int\limits_S v_n \mathrm{D}_y\varphi \,\mathrm{d}g = -\int\limits_S u\mathrm{D}_y\varphi \,\mathrm{d}g = \int\limits_S \mathrm{D}_y u\varphi \,\mathrm{d}g,$$

i.e. the thesis. We conclude the proof taking into account the liminf properties of the sum and Proposition 5.4.3.
If v_n is not continuous on $\overline{Q}$, from Theorem 5.3.6 there exists $w_n \in V(Q, S_n) \cap C(\overline{Q})$ such that $\|v_n - w_n\|_{W^{1,p}(Q)} \leq \frac{1}{n}$, $\|v_n - w_n\|_{L^p(Q,m_n)} \leq \frac{1}{n}$ and $\Phi_p^{(n)}[w_n] \leq \Phi_p^{(n)}[v_n] + \frac{1}{n}$.

By triangle inequality we easily have that w_n tends to u weakly in $\mathcal{H}$. Hence from the previous step we have

$$\Phi_p^{(n)}[u] \leq \varliminf_{n\to\infty} \Phi_p^{(n)}[w_n] \leq \varliminf_{n\to\infty} \left(\Phi_p^{(n)}[v_n] + \frac{1}{n}\right) = \varliminf_{n\to\infty} \Phi_p^{(n)}[v_n],$$

i.e. the thesis.

Proof of condition b). As before, we can suppose that $u \in V(Q,S)$, i.e. $u \in W^{1,p}(Q)$ and $u|_K \in \mathcal{D}(S)$. The difference from the proof of condition b) in Theorem 4.2.2 is that now u is no longer continuous (since we are in dimension three). We have then to consider two cases.

Step 1. We suppose that $u \in C(\overline{Q})$, hence $u \in H$. We extend by continuity u to $\overline{\mathcal{R}}$ and we put $\widehat{u}$ this extension. Following the same approach of Theorem 4.2.2, we introduce a quasi uniform triangulation τ_n of $\mathcal{R}$ made by equilateral tetrahedron R_n^j such that the vertices of the pre-fractal surface S_n are nodes of the triangulation at the n-th level. We denote by $I_n u$ the interpolant polynomial of u and we set $w_n = I_n\hat{u}$. As in the proof of condition b) in Theorem 4.2.2, we can prove that $\{w_n\}$ strongly converges to u in $\mathcal{H}$.

We now prove condition b) for the sequence w_n. We note that from Proposition 5.4.3

$$\lim_{n\to\infty} \delta_n \int\limits_{S_n} b|w_n|^p \,\mathrm{d}\sigma = \int\limits_{S} b|u|^p \,\mathrm{d}g.$$

We have that

$$\int\limits_{Q_n} |\mathrm{D}w_n|^p \,\mathrm{d}\mathcal{L}_3 \leq \int\limits_{Q} |\mathrm{D}w_n|^p \,\mathrm{d}\mathcal{L}_3,$$

then, by taking the limit for $n \to \infty$, we have the thesis (since $\|\mathrm{D}(w_n - u)\|_{L^p(Q)} \to 0$ for $n \to \infty$).

We have only to prove that

$$\varlimsup_{n\to\infty} E_p^{(n)}[w_n] \leq E_S[u|_S].$$

Since $w_n = I_n\hat{u}$, we have that

$$w_n = m_j\, l + h_i y + q_j \quad , \, l \in [l_j, l_{j+1}],\, y \in [y_i, y_{i+1}],$$

where $l_j = (j-1)\,3^{-n}$ and $y_i = (i-1)\,3^{-n}$ for $j = 1, \ldots, 3\mathcal{N}$, $i = 1, \ldots, M$. Hence we get

$$\frac{\delta_n^{1-p}}{p} \int\limits_I \mathrm{d}y \int\limits_{K_n} |\mathrm{D}w_n|^p \,\mathrm{d}\ell = \frac{\delta_n^{1-p}}{p} \sum_{i=1}^{M} \sum_{j=1}^{3\mathcal{N}} m_j^p (l_{j+1} - l_j)(y_{i+1} - y_i) \leq$$
$$\frac{4^{(p-1)n}}{p} \sum_{i=1}^{M} \sum_{j=1}^{3\mathcal{N}} (w_n(P_{j+1,i+1}) - w_n(P_{j,i}))^p =$$

$$\frac{4^{(p-1)n}}{p}\sum_{i=1}^{M}\sum_{j=1}^{3\mathcal{N}}(u(P_{j+1,i+1})-u(P_{j,i}))^p \le \int_I \mathcal{E}_K[u]\mathrm{d}\mathcal{L}_1.$$

Passing to the upper limit, we get

$$\overline{\lim_{n\to\infty}}\frac{\delta_n^{1-p}}{p}\int_I \mathrm{d}y\int_{K_n}|\mathrm{D}w_n|^p\,\mathrm{d}\ell \le \int_I \mathcal{E}_K[u]\mathrm{d}\mathcal{L}_1.$$

In the same way one can prove that

$$\overline{\lim_{n\to\infty}}\frac{\delta_n}{p}\int_I \mathrm{d}y\int_{K_n}|\mathrm{D}_y w_n|^p\,\mathrm{d}\ell \le \int_K\int_I |\mathrm{D}_y u|^p\mathrm{d}\mathcal{L}_1\mathrm{d}\mu.$$

Taking into account the limsup property of the sum the conclusion of the theorem follows.

Step 2. If $u\in V(Q,S)$, but u is not continuous, from Theorem 5.3.4 there exists $\psi_h\in V(Q,S)\bigcap C(\overline{Q})$ such that $\psi_h\to u$ in H and $\|\psi_h-u\|_{V(Q,S)}\to 0$. Let $h\in\mathbb{N}$ fixed such that $\|\psi_h-u\|_{V(Q,S)}\le\frac{1}{h}$ and $\|\psi_h-u\|_H\le\frac{1}{h}$. By $\tilde{\psi}_h$ we denote a continuous extension in $\overline{\mathcal{R}}$.

From Step 1 we have that for every fixed $h\in\mathbb{N}$ $I_n\tilde{\psi}_h$ strongly converges to $\tilde{\psi}_h$ in $\mathcal{H}$, $I_n\tilde{\psi}_h$ converges to $\tilde{\psi}_h$ in $W^{1,p}(\mathcal{R})$ when $n\to\infty$ and

$$\overline{\lim_{n\to\infty}}\,\Phi_p^{(n)}[I_n\tilde{\psi}_h]\le\Phi_p[\tilde{\psi}_h].$$

Passing to the upper limit for $h\to\infty$ to both sides of the above inequality we obtain

$$\overline{\lim_{h\to\infty}}\left(\overline{\lim_{n\to\infty}}\,\Phi_p^{(n)}[I_n\tilde{\psi}_h]\right)\le\overline{\lim_{h\to\infty}}\,\Phi_p[\tilde{\psi}_h]=\Phi_p[u].$$

We now want to apply Corollary 1.16 in [6] for proving that there exists an increasing mapping $n\to h(n)$ such that, denoting by $w_n=I_n\tilde{\psi}_{h(n)}$, we have that w_n converges to u in $\mathcal{H}$ and $\overline{\lim_{n\to\infty}}\,\Phi_p^{(n)}[w_n]\le\Phi_p[u]$. To this aim we have to prove that

$$\overline{\lim_{h\to\infty}}\,\overline{\lim_{n\to\infty}}|(w_{n,h},v_n)_{H_n}-(u,v)_H|\le 0$$

for every $\{v_n\}$ weakly converging to v in $\mathcal{H}$. Indeed we have

$$|(w_{n,h},v_n)_{H_n}-(u,v)_H|\le|(w_{n,h},v_n)_{H_n}-(\tilde{\psi}_n,v)_H+(\tilde{\psi}_h-u,v)_H|\le$$
$$|(w_{n,h},v_n)_{H_n}-(\tilde{\psi}_h,v)_H|+\|\tilde{\psi}_h-u\|_H\|v\|_H\le|(w_{n,h},v_n)_{H_n}-(\tilde{\psi}_h,v)_H|+\tfrac{c}{h}$$

Passing to the upper limit for $n\to\infty$, we obtain

$$\overline{\lim_{n\to\infty}}|(w_{n,h},v_n)_{H_n}-(u,v)_H|\to 0.$$

Then Corollary 1.16 in [6] provides the thesis. □

We point out that, from Theorem 4.2.6, the M-convergence of the functional implies the G-convergence of their subdifferentials, as in the two-dimensional case.

We now consider the abstract homogeneous Cauchy problem

$$(P)\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t}+\mathcal{A}u\ni 0, & t\in[0,T]\\ u(0)=u_0, \end{cases}$$

where $\mathcal{A}$ is the subdifferential of Φ_p, T is a fixed positive number, and u_0 is a given function. As in the two-dimensional case, from Theorem 4.3.1 the subdifferentials $\partial\Phi_p$ and $\partial\Phi_p^{(n)}$ are maximal, monotone and m-accretive operators on H and H_n respectively. Then, if we denote with $T_p(t)$ and $T_p^{(n)}(t)$ the nonlinear semigroups generated by $-\partial\Phi_p$ and $-\partial\Phi_p^{(n)}$ respectively, these semigroups are strongly continuous and contractive on H and H_n. Hence, we get an existence and uniqueness result for the solution $u=T_p(\cdot)u_0$ of problem (P) analogous to Theorem 4.3.2. Moreover, the solution u of problem (P) solves the following problem $(\tilde{P})$ on Q for $t\in(0,T]$ in the following weak sense (where β is given by (5.1.1)):

$$(\tilde{P})\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t}-\Delta_p u=0, \qquad \text{in } L^{p'}(Q)\\ \left\langle \frac{\mathrm{d}u}{\mathrm{d}t},\psi\right\rangle_{L^2(S,\mathrm{d}G),L^2(S,\mathrm{d}g)}+\left\langle \frac{\partial u}{\partial\nu}|\mathrm{D}u|^{p-2},\psi\right\rangle_{(B^{p,p}_{\beta}(S))',B^{p,p}_{\beta}(S)}+\\ \langle b|u|^{p-2}u,\psi\rangle_{L^{p'}(S,\mathrm{d}g),L^p(S,\mathrm{d}g)}+E_S(u,\psi)=0 \quad \text{for every } \psi\in\mathcal{D}(S),\\ u=0 \qquad \text{in } W^{-\frac{1}{p'},p'}(\tilde{\Omega}),\\ u(0,P)=u_0(P) \qquad \text{in } L^2(\overline{Q},m), \end{cases}$$

where we recall that $\tilde{\Omega}=(\Omega\times\{0\})\cup(\Omega\times\{1\})$.
As to the pre-fractal case, for each $n\in\mathbb{N}$ fixed we consider the abstract homogeneous Cauchy problem

$$(P_n)\begin{cases} \frac{\mathrm{d}u_n}{\mathrm{d}t}+\mathcal{A}_n u_n\ni 0, & t\in[0,T]\\ u_n(0)=u_0^{(n)}, \end{cases}$$

where $\mathcal{A}_n$ is the subdifferential of $\Phi_p^{(n)}$, T is a fixed positive number, and $u_0^{(n)}$ is a given function.
We can give a characterization result for $\mathcal{A}_n$, which is analogous to Theorem 4.3.3. We recall that $\tilde{\Omega}_n:=(\Omega_n\times\{0\})\cup(\Omega_n\times\{1\})$.

Theorem 5.4.4. *Let $u_n(t)$ belong to $V(Q,S_n)$ for a.e. $t\in(0,T]$, and f be in H_n. Then $f\in\partial\Phi_p^{(n)}[u_n]$ if and only if*

$$(\bar{P}_n)\begin{cases} -\Delta_p u_n = f \qquad\qquad \textit{in } L^{p'}(Q_n), \\ \left\langle \frac{\partial u_n}{\partial \nu_n}|\mathrm{D}u_n|^{p-2}, \psi \right\rangle_{W^{-\frac{1}{p'},p'}(S_n), W^{\frac{1}{p'},p}(S_n)} \qquad\qquad + \delta_n \left\langle b|u_n|^{p-2}u_n, \psi\right\rangle_{L^{p'}(S_n), L^p(S_n)} \\ -\delta_n^{1-p}\left\langle \Delta_p u_n, \psi\right\rangle_{W^{-1,p'}(S_n), W^{1,p}(S_n)} - \delta_n \left\langle \Delta_{p,y} u_n, \psi\right\rangle_{W^{-1,p'}(S_n), W^{1,p}(S_n)} \\ = \delta_n \left\langle f, \psi \right\rangle_{L^2(S_n), L^2(S_n)} \qquad \textit{for every } \psi \in W^{1,p}(S_n), \\ u_n = 0 \qquad \textit{in } W^{-\frac{1}{p'},p'}(\tilde{\Omega}_n), \end{cases}$$

where $\frac{\partial u_n}{\partial \nu_n}$ *denotes the normal derivative across* S_n *and* $\Delta_{p,y} := \mathrm{div}(|\mathrm{D}_y|^{p-2}\mathrm{D}_y)$.

Proof. The proof follows the one of Theorem 4.3.3, but we have to suitably define $\Delta_{p,y}$ as a variational operator from $W_0^{1,p}(S_n)$ to $W^{-1,p'}(S_n)$ in the following way:

$$\int_{S_n} |\mathrm{D}_y z|^{p-2}\mathrm{D}_y z \mathrm{D}_y w \,\mathrm{d}\sigma = - < \Delta_{p,y}\, z, w >_{W^{-1,p'}(S_n), W^{1,p}(S_n)} \tag{5.4.4}$$

for $z, w \in W_0^{1,p}(S_n)$.

In addition to that, we have to use a different version of the Green formula, since in this case $\partial Q_n = S_n \cup \tilde{\Omega}_n$:

$$\int_{Q_n} |\mathrm{D}u|^{p-2}\mathrm{D}u\mathrm{D}\psi \,\mathrm{d}\mathcal{L}_3 = \left\langle \frac{\partial u}{\partial \nu_n}|\mathrm{D}u|^{p-2}, \psi|_{S_n}\right\rangle_{W^{-\frac{1}{p'},p'}(S_n), W^{\frac{1}{p'},p}(S_n)} + \left\langle \frac{\partial u}{\partial \nu_n}|\mathrm{D}u|^{p-2}, \psi|_{\tilde{\Omega}_n}\right\rangle_{W^{-\frac{1}{p'},p'}(\tilde{\Omega}_n), W^{\frac{1}{p'},p}(\tilde{\Omega}_n)} - \int_{Q_n} \Delta_p u \psi \,\mathrm{d}\mathcal{L}_3.$$

Hence we have that the analogous of condition (4.3.5) is

$$\delta_n f = \delta_n b|u_n|^{p-2}u_n - \delta_n^{1-p}\Delta_p u_n + \frac{\partial u_n}{\partial \nu_n}|\mathrm{D}u_n|^{p-2} - \delta_n \Delta_{p,y} u_n \tag{5.4.5}$$

and it holds in $W^{-\frac{1}{p'},p'}(S_n)$. Moreover, we have $u_n = 0$ in $W^{-\frac{1}{p'},p'}(\tilde{\Omega}_n)$. $\square$

From the properties of $-\mathcal{A}_n$ and $T_p^{(n)}(t)$, we get an existence and uniqueness result for the solution $u_n = T_p^{(n)}(\cdot)u_0^{(n)}$ of problem (P_n) analogous to Theorem 4.3.4. Moreover, from Theorem 5.4.4 it follows that the solution u_n of problem (P_n) solves for each

$n \in \mathbb{N}$ the following problem $(\tilde{P}_n)$ on Q_n for $t \in (0, T]$ in the following weak sense:

$$
(\tilde{P}_n)
\begin{cases}
\frac{\mathrm{d}u_n}{\mathrm{d}t} - \Delta_p u_n = 0, \qquad \text{in } L^{p'}(Q_n) \\
\delta_n \left\langle \frac{\mathrm{d}u_n}{\mathrm{d}t}, \psi_n \right\rangle_{L^2(S_n), L^2(S_n)} + \left\langle \frac{\partial u_n}{\partial \nu_n} |\mathrm{D}u_n|^{p-2}, \psi_n \right\rangle_{W^{-\frac{1}{p'},p'}(S_n), W^{\frac{1}{p'},p}(S_n)} \\
+\delta_n \left\langle b|u_n|^{p-2} u_n, \psi_n \right\rangle_{L^{p'}(S_n), L^p(S_n)} - \delta_n^{1-p} \left\langle \Delta_p u_n, \psi_n \right\rangle_{W^{-1,p'}(S_n), W^{1,p}(S_n)} \\
-\delta_n \left\langle \Delta_{p,y} u_n, \psi_n \right\rangle_{W^{-1,p'}(S_n), W^{1,p}(S_n)} = 0 \quad \forall\, \psi_n \in W^{1,p}(S_n). \\
u_n = 0 \qquad \text{in } W^{-\frac{1}{p'},p'}(\tilde{\Omega}_n), \\
u_n(0, P) = u_0^{(n)}(P) \qquad \text{in } L^2(Q) \cap L^2(Q, m_n)
\end{cases}
$$

Theorem 5.4.1, Theorem 4.2.6 and Theorem 7.24 in [82] allow us to deduce that the pre-fractal solutions converge in a suitable sense to the limit fractal one as in the two-dimensional case.

Theorem 5.4.5. *Let H_n, H, $\Phi_p^{(n)}$, Φ_p and δ_n be as in Theorem 5.4.1. Let $u_0^{(n)} \in \overline{D(-\mathcal{A}_n)}$ and $u_0 \in \overline{D(-\mathcal{A})}$. If $u_0^{(n)} \to u_0$ strongly in $\mathcal{H}$, then $T_p^{(n)}(t)\, u_0^{(n)} \xrightarrow[n \to \infty]{} T_p(t)\, u_0$ strongly in $\mathcal{H}$ for every $t \geq 0$.*

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