A MORSE INDEX FORMULA FOR RADIAL SOLUTIONS OF LANE-EMDEN PROBLEMS

FRANCESCA DE MARCHIS, ISABELLA IANNI, FILOMENA PACELLA

Abstract. We consider the semilinear Lane-Emden problem:

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases} \qquad (\mathcal{E}_p)$$

where B is the unit ball of $\mathbb{R}^N$, $N \geq 3$, centered at the origin and $1 < p < p_S$, $p_S = \frac{N+2}{N-2}$.
We prove that for any radial solution u_p of ($\mathcal{E}_p$) with m nodal domains its Morse index $\mathsf{m}(u_p)$ is given by the formula

$$\mathsf{m}(u_p) = m + N(m-1)$$

if p is sufficiently close to p_S.

1. Introduction

We consider the classical Lane-Emden problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } B \\ u = 0 & \text{on } \partial B \subset \mathbb{R}^N \end{cases} \qquad (1.1)$$

where B is the unit ball of $\mathbb{R}^N$, $N \geq 3$, centered at the origin and $1 < p < p_S$, with $p_S = \frac{N+2}{N-2} = 2^* - 1$, where 2^* is the critical exponent for the Sobolev embedding $H^1_0(B) \hookrightarrow L^{2^*}(B)$.

In this paper we study the Morse index of the radial solutions of (1.1).

We recall that the *Morse index* $\mathsf{m}(u_p)$ of a solution u_p of (1.1) is the maximal dimension of a subspace $X \subset H^1_0(B)$ where the quadratic form associated to the linearized operator at u_p:

$$L_p = (-\Delta - p|u_p|^{p-1})$$

is negative definite. Equivalently, since B is a bounded domain, $\mathsf{m}(u_p)$ can be defined as the number of the negative Dirichlet eigenvalues of L_p counted with their multiplicity.

It is well known that (1.1) possess infinitely many radial solutions among which only one is positive (or negative) while all the others change sign and can be characterized by the number of their nodal regions. For a given radial solution u_p of (1.1) with m nodal domains, it has been proved in [19] that the *radial Morse index*, i.e. the number of the negative eigenvalues of L_p in the Sobolev space of radial functions $H^1_{0,rad}(B)$, is exactly m. Obviously the Morse index $\mathsf{m}(u_p)$, in $H^1_0(B)$, can be larger

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than m, because of the presence of negative non radial eigenvalues of L_p.
The knowledge of the Morse index is, in general, a very important qualitative property of a solution. In particular it helps to classify the solutions and study their stability or possible bifurcations.

A first estimate that we get for a radial solution u_p of (1.1) with m nodal domains is the following one (see Theorem 2.1):

$$\mathsf{m}(u_p) \geq m + N(m-1), \tag{1.2}$$

which improves a result in [1].

The main theorem of the present paper states that for p close to the critical exponent the extimate (1.2) is sharp. More precisely we prove:

Theorem 1.1. *Let $N \geq 3$ and u_p be a radial solution to (1.1) with $m \in \mathbb{N}^+$ nodal regions. Then*

$$\mathsf{m}(u_p) = m + N(m-1), \qquad \textit{for } p \textit{ sufficiently close to } p_S. \tag{1.3}$$

Let us make a few comments about this result pointing out some interesting features of the formula (1.3).

First, writing (1.3) as

$$\mathsf{m}(u_p) = m(N+1) - N,$$

we see that the Morse index $\mathsf{m}(u_p)$ grows linearly with respect to the number m of nodal domains, which corresponds also to the number of negative radial eigenvalues of the operator L_p (*cf.* [19]). This is somehow surprising since, in general, one would expect *many more* negative nonradial eigenvalues then the negative radial ones. Indeed if we look at the distribution of the radial and nonradial eigenvalues of the linear operator $(-\Delta)$ in $H_0^1(B)$ we observe that:

(i) on one side by a result of Brüning-Heintze and Donnelly [8, 9, 15] we get that

$$\lambda_{r,m} \sim Cm^2 \quad \text{ as } m \to +\infty$$

where $\lambda_{r,m}$ is the m-th radial eigenvalues of $(-\Delta)$, which implies that the number $n_r(m^2)$ of the radial eigenvalues of $(-\Delta)$ bounded by m^2 is m, more precisely

$$n_r(m^2) \sim m \quad \text{ as } m \to +\infty$$

(ii) on the other side by the classical Weil law (see e.g. [23]):

$$n(m^2) \sim Cm^N \quad \text{ as } m \to +\infty \quad (N \text{ is the dimension})$$

where $n(m^2)$ is the number of all the eigenvalues of $(-\Delta)$ in $H_0^1(B)$ less than or equal to m^2

In an equivalent way we can observe that if we consider a radial eigenfunction of $(-\Delta)$ in $H_0^1(B)$ with m nodal regions, i.e. corresponding to the eigenvalue $\lambda_{r,m}$, then its *Morse index* is just the number of the eigenvalues less than $\lambda_{r,m}$ which, by (i) and (ii), grows at a rate of order m^N and so faster then m (if $N \geq 2$) as $m \to +\infty$.
So L_p represents an example of a linear, Schrödinger type, operator determined by the potential $V_p(x) = p|u_p(x)|^{p-1}$, for p approaching p_S, for which (i) and (ii) do not hold, at least for the negative eigenvalues.

Another interesting consequence of all this could be derived studying (1.1) as $p \to 1$. In this case it is reasonable to conjecture the convergence of the Morse index $\mathsf{m}(u_p)$

to the Morse index of the Dirichlet radial eigenfunction of $(-\Delta)$ with m nodal regions (i.e. the eigenfunction corresponding to the radial eigenvalue $\lambda_{r,m}$) possibly augmented by the multiplicity of $\lambda_{r,m}$, which is 1. Indeed suitable normalizations of solutions of (1.1) converge to eigenfunctions of the Laplacian as $p \to 1$ (see [7, 17]). Therefore the previous considerations indicate that for large m the Morse index $\mathsf{m}(u_p)$ for p close to 1 is of order m^N, hence it is much bigger than $m + N(m-1)$, which is by (1.3) the Morse index of u_p for p close to p_S. So bifurcations from u_p should appear, as p ranges from 1 to p_S, showing that the structure of the solution set of (1.1) is richer than one could imagine.

Next we would like to point out another interesting fact: the formula (1.3) *does not hold in dimension* $N = 2$*, as* $p \to p_S = +\infty$. Indeed in the recent paper [13] we have proved the following:

Theorem 1.2 ([13])**.** *Let* u_p *be a radial sign-changing solution to* (1.1) *with* 2 *nodal regions, but with* $B \subset \mathbb{R}^2$ *and* $p_S = +\infty$*. Then*

$$\mathsf{m}(u_p) = 12 \qquad \text{for } p \text{ sufficiently large.}$$

Obviously $12 \neq m + N(m-1) = 4$ for $N = 2$ and $m = 2$. Note that in this case the value of $m(u_p)$ seems to be related to the Morse index of one of the radial solutions to the singular Liouville problem in $\mathbb{R}^2$ ([10]), see [13] for further details.

Let us describe the method for proving Theorem 1.1, which also clarifies the differences with the case $N = 2$.

Since the solutions u_p are radial, to study the spectrum of the linearized operator L_p we decompose it as a sum of the spectrum of a radial weighted operator and the spectrum of the Laplace-Beltrami operator on the unit sphere. To bypass the difficulty of dealing with a weighted eigenvalue problem with a singularity at the origin we approximate the ball B by annuli A_n with a small hole, showing that the number of negative eigenvalues of the linearized operator L_p is preserved (we refer to [13] for this). Then (see Section 4) it turns out that the Morse index $\mathsf{m}(u_p)$ is determined by the *size* of the first $(m-1)$ (radial) eigenvalues $\tilde{\beta}_i(p)$, $i = 1, \ldots, m-1$, of the weighted operator

$$\tilde{L}_p^n = |x|^2(-\Delta - V_p(x)) \tag{1.4}$$

in $H_0^1(A_n)$, where the potential $V_p(x)$ is $p|u_p(x)|^{p-1}$ and $n = n_p$ is properly chosen. In order to study these eigenvalues a good knowledge of the potential $V_p(x)$ is needed which, in turns, means to have accurate estimates on the solutions u_p. This is where the hypothesis on the exponent p enters.
If $N \geq 3$, in Section 3 we make a precise analysis of the asymptotic behavior of u_p as $p \to p_S$, which allows to get the needed estimates on the potential $V_p(x)$ for p close to the critical exponent. In particular we get that suitable rescalings of u_p in each nodal region converge to the same positive radial solution U of the critical equation in $\mathbb{R}^N$:

$$-\Delta U = U^{p_S} \ \text{ in } \mathbb{R}^N, \ N \geq 3. \tag{1.5}$$

This allows to detect precisely the asymptotic behavior, as $p \to p_S$, of the first eigenvalue $\tilde{\beta}_1(p)$ (and then, as a consequence, of all the other eigenvalues $\tilde{\beta}_i(p)$, $i = 2, \ldots, m-1$) by several nontrivial estimates (see Section 5).

In dimension 2 the procedure followed in [13] is similar but the striking difference with respect to the case $N \geq 3$ is that the limit problems, as $p \to +\infty$, for the positive and negative part of the nodal radial solutions u_p with 2 nodal domains are

different. Indeed it was proved in [18] that (assuming w.l.g. $u_p(0) > 0$) a suitable rescaling of u_p^+ converges to a regular solution of the Liouville problem in $\mathbb{R}^2$, while a suitable rescaling of u_p^- converges to a radial solution of a singular Liouville problem in $\mathbb{R}^2$ (see also [12]). So the estimates needed to compute the Morse index of u_p are completely different and the contribution from the annular nodal region is bigger and makes the Morse index of u_p higher with respect to the corresponding case in dimension $N \geq 3$. This difference reflects in the study of the asymptotic behavior of the first radial eigenvalue $\tilde{\beta}_1(p)$ (see Remark 5.11) which makes the proof in dimension $N \geq 3$ more delicate than that for $N = 2$.
We also point out that the assertion of Theorem 1.1 holds for radial solutions to (1.1) with *any number of nodal regions*, while in the case $N = 2$ the result of [13] has been obtained only for solutions with 2 nodal regions. This is because an asymptotic analysis of radial solutions with $m \geq 3$ is lacking in dimension $N = 2$. We believe that the strategy of the present paper could be pursued also in dimension $N = 2$ to get a result for general radial solutions. We plan to do this in a future paper.

A final comment is that the whole strategy for the Morse index computation (here as in [13]) relies on the peculiar behavior of the radial solutions which have all the nodal regions shrinking at the same point as $p \to p_S$ (as $p \to +\infty$ when $N = 2$). This property also induces an interesting blow-up (in time) phenomenon in the associated parabolic problem with initial data close to the radial stationary solutions (see [11, 14, 21]).

The paper is organized as follows. We start in Section 2 by proving a lower bound for the Morse index of radial solutions of semilinear elliptic Dirichlet problems with general autonomous nonlinearities. This part holds in any dimension $N \geq 2$ and extends previous results in [1] giving, as a special case, the estimate (1.2). In Section 3 we perform the asymptotic analysis of the radial solutions of (1.1) as $p \to p_S$. The results in this section are interesting in themselves and do not appear in previous papers. In Section 4 we approximate the eigenvalue problem in the ball by corresponding ones in approximating annuli and set the auxiliary weighted eigenvalue problems. In section 5 we study the radial eigenvalues of the weighted operator $\tilde{L}_p^n$ introduced in (1.4); in particular the analysis of the first one $\tilde{\beta}_1(p)$ is the central part of the section. The delicate estimates that we develop here are crucial for our proof; in order to obtain them we need to analyze accurately the contribution to the Morse index of each nodal region of u_p. Finally the proof of Theorem 1.1 is presented in Section 6.

Contents

2. A lower bound for the Morse index

We consider a semilinear elliptic problem with a general autonomous nonlinearity:

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} . \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is either a ball or an annulus centered at the origin and $f \in C^1(\mathbb{R})$.

For a solution u of (2.1) we denote by $\mathsf{m}(u)$ the *Morse index* of u, namely the number of the negative Dirichlet eigenvalues of L_u in Ω (counted with their multiplicity), where $L_u : H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$ is the linearized operator at u, namely

$$L_u(v) := -\Delta v - f'(u(x))v. \tag{2.2}$$

When the solution u is radial we also denote by $\mathsf{m}_{\mathsf{rad}}(u)$ the *radial Morse index* of u, i.e. the number of negative radial eigenvalues of the linearized operator L_u.

W prove here a result which improves the one in [1] and holds in any dimension $N \geq 2$.

Theorem 2.1. *Let u be a radial solution of (2.1) with $m \geq 2$ nodal domains. Then*

$$\mathsf{m}(u) \geq \mathsf{m}_{\mathsf{rad}}(u) + N(m-1), \tag{2.3}$$

Moreover, if f satisfies the condition

$$f(s) \leq f'(s)s,$$

then

$$\mathsf{m}_{\mathsf{rad}}(u) \geq m, \tag{2.4}$$

and hence

$$\mathsf{m}(u) \geq m + N(m-1).$$

Proof. Let us fix $m \in \mathbb{N}^+$ and let us denote by u_m a radial solution of (2.1) having m nodal regions. We use the partial derivatives of u_m to produce negative eigenvalues whose corresponding eigenfunctions are odd with respect to an hyperplane passing through the origin. Let us consider, for any $i = 1, \dots, N$, the hyperplane $T_i = \{x = (x_1, \dots, x_N) : x_i = 0\}$ and the domain $\Omega_i^- = \{x \in \Omega : x_i < 0\}$, i.e. Ω_i^- is the *half ball* or the *half annulus* determinated by T_i.
Then we denote by $A_1, \dots, A_m$ the nodal regions of u_m, counting them starting from the outer boundary in such a way that ∂A_1 contains $\partial\Omega$ if Ω is a ball or the outer boundary of Ω if Ω is an annulus. Since u_m is radial we have that A_j are annuli for $j \in \{1, \dots, m-1\}$ while A_m is a ball if Ω is a ball or another annulus if so is Ω. Let us first consider the case of the ball so that:

$$A_j = \{x \in \Omega : R_{j+1} < |x| < R_j\} \qquad j = 1, \dots, m-1$$

$$A_m = \{x \in \Omega : |x| < R_m\}$$

where R_j, $j = 2, \dots, m$, are the nodal radii and R_1 is the radius of the ball Ω.
We consider the derivatives $\frac{\partial u_m}{\partial x_i}$, $i = 1, \dots, N$, which satisfy the equation

$$L_{u_m}\left(\frac{\partial u_m}{\partial x_i}\right) = 0 \qquad \text{in } \Omega. \tag{2.5}$$

Using the symmetry of u_m we have:

$$\frac{\partial u_m}{\partial x_i} = 0 \qquad \text{on } \overline{\Omega} \cap T_i. \tag{2.6}$$

Then we consider the *half nodal regions*

$$A_{i,j}^- = A_j \cap \Omega_i^-, \qquad j = 1, \dots, m \text{ and } i = 1, \dots, N.$$

To simplify the notations let us fix $i = 1$ and focus on the function $\frac{\partial u_m}{\partial x_1}$ in the sets $A_{1,j}^-$, that we simply denote by A_j^-. Whatever we prove for $\frac{\partial u_m}{\partial x_1}$ will hold with obvious changes for the other derivatives $\frac{\partial u_m}{\partial x_i}$, $i = 2, \dots, N$.
Let us observe that for each nodal region A_j, writing $u_m(r) = u_m(|x|)$ there exists at least one value $r_j \in (R_j, R_{j+1})$, $j = 1, \dots, m-1$, such that

$$\frac{du_m}{dr}(r_j) = 0. \tag{2.7}$$

Notice that if the nonlinearity $f = f(s)$ satisfies the condition $s\,f(s) \geq 0$ then r_j is the unique radius in (R_j, R_{j+1}) such that (2.7) holds in A_j, $j = 1, \dots, m-1$.
Then, since u_m is radial we have that $\frac{\partial u_m}{\partial x_1} \equiv 0$ on the spheres

$$S_j = \{x \in \mathbb{R}^N \,;\, |x| = r_j\} \qquad j = 1, \dots, m-1. \tag{2.8}$$

Let us fix one $r_j \in (R_j, R_{j+1})$ for each $j = 1, \dots, m-1$ (i.e. just one value of the radius in the interval (R_j, R_{j+1}) such that (2.7) holds) and consider the sets

$$N_j^- = \{x \in \mathbb{R}^N \,:\, r_j > |x| > r_{j+1}\} \cap \Omega_1^- \qquad j = 1, \dots, m-2$$

and observe that for $j = 1, \dots, m-2$, by (2.5), (2.6) and (2.8)

$$\begin{cases} L_{u_m}\left(\frac{\partial u_m}{\partial x_1}\right) = 0 & \text{in } N_j^- \\ \frac{\partial u_m}{\partial x_1} = 0 & \text{on } \partial N_j^-. \end{cases} \tag{2.9}$$

Thus $\frac{\partial u_m}{\partial x_1}$ is an eigenfunction of the linearized operator L_{u_m} in N_j^- corresponding to the zero eigenvalue which is the first one or an higher one according to the fact that $\frac{\partial u_m}{\partial x_1}$ changes sign or not in N_j^-.
Moreover also in the set

$$N_{m-1}^- = \{x \in \mathbb{R}^N \,:\, r_{m-1} > |x| \geq 0\} \cap \Omega_1^-$$

the function $\frac{\partial u_m}{\partial x_1}$ satisfies (2.9) (for $j = m-1$). Hence also in N_{m-1}^- zero is an eigenvalue for L_{u_m} with corresponding eigenfunction $\frac{\partial u_m}{\partial x_1}$.
In conclusion we have obtained $(m-1)$ adjacent regions where an eigenvalue of L_{u_m} is zero. This implies that in the domain $N^- = \cup_{j=1}^{m-1} N_j^-$ the h-th eigenvalue λ_h of L_{u_m} is zero for some $h \geq m-1$.
Since N^- is strictly contained in Ω_1^-, by construction we have that the h-th eigenvalue λ_h of L_{u_m} in Ω_1^- is negative for some $h \geq m-1$, in particular $\lambda_{m-1} = \lambda_{m-1}(L_{u_m}) < 0$ in Ω_1^- and so are all $\lambda_n = \lambda_n(L_m)$ in Ω_1^- for $n \leq m-1$. By reflecting by oddness with respect to T_1 the corresponding eigenfunctions we get eigenfunctions of L_{u_m} in the whole Ω corresponding to the same $(m-1)$ negative eigenvalues λ_n, $n = 1, \dots, m-1$.
Repeating the same arguments for all $i = 1, \dots, N$ we get at least $(m-1)$ negative eigenvalues $\lambda_n(u_m)$ in the domains Ω_i^-, for each $i = 1, \dots, N$, which give eigenvalues of L_{u_m} in the whole Ω whose corresponding eigenfunctions are odd with respect to T_i, $i = 1, \dots, N$.
Note that, by symmetry,

$$\lambda_n(L_{u_m}, \Omega_i^-) = \lambda_n(L_{u_m}, \Omega_s^-) \qquad \text{for } i \neq s,\ \ i, s = 1, \dots, N \qquad n = 1, \dots, m-1$$

but the corresponding eigenfunctions are linearly independent, because they are odd with respect to orthogonal axes.
So the multiplicity of each eigenvalue λ_n of L_{u_m} in Ω is at least N so that we have

got at least $N(m-1)$ negative eigenvalues. Since the eigenfunctions we have found are not radial, adding $\mathsf{m}_{\mathsf{rad}}(u_m)$, we get the estimate (2.3).

If f satisfies the condition $f(u) \leq f'(u)u$ then it is easy to see that each (radial) nodal region gives the existence of one negative radial eigenvalue, so we get (2.4).

The case when Ω is an annulus follows in a similar, slightly easier, way, since the only difference is that the last nodal region A_m is an annulus, so that it does not need to be treated in a different way with respect to the other regions A_j, $j = 1, \dots, m-1$. □

We end this section recalling the following known result concerning the case when f is a power type nonlinearity and the domain Ω is a ball (see [4] for the case $m = 2$ and [19, Proposition 2.9] for any $m \in \mathbb{N}^+$)

Theorem 2.2 ([4, 19]). *Let Ω be a ball and $f(u) = |u|^{p-1}u$, $p \in (1, p_S)$, $p_S = \frac{N+2}{N-2}$ if $N \geq 3$, $p_S = +\infty$ if $N = 2$. Let u be a radial solution to (2.1) with $m \in \mathbb{N}^+$ nodal regions. Then*

$$\mathsf{m}_{\mathsf{rad}}(u) = m.$$

3. Asymptotic analysis of the nodal radial solutions

In this section we analyze the asymptotic behavior as $p \to p_S$ of any radial sign-changing solution of (1.1). It is well known that for any fixed $p \in (1, p_S)$ the radial solutions of problem (1.1) are infinitely many, precisely for each $m \in \mathbb{N}^+$ there is a unique (up to the sign, being the nonlinearity odd) radial solution to (1.1) with m nodal domains.

So for $m \in \mathbb{N}^+$ let us denote by u_p^{m} the unique nodal radial solution of (1.1) having m nodal regions and satisfying

$$u_p^{\mathrm{m}}(0) > 0. \tag{3.1}$$

The 1-dimensional profile of this solution is described in Figure 1. With abuse of notation we will write often $u_p^{\mathrm{m}}(r) = u_p^{\mathrm{m}}(|x|)$.

In the next proposition we state a few qualitative properties of the solutions u_p^{m}.

Proposition 3.1. *Let $p \in (1, p_S)$, then:*

(i) $u_p^{\mathrm{m}}(0) = \|u_p^{\mathrm{m}}\|_\infty$,

(ii) *in each nodal region the map $r \mapsto u_p^{\mathrm{m}}(r)$ has exactly one critical point (which is either a local maximum or a local minimum point, and they alternate),*

(iii) $\int_B |\nabla u_p^{\mathrm{m}}(y)|^2 dy = \int_B |u_p^{\mathrm{m}}(y)|^{p+1} dy \underset{p \to p_S}{\longrightarrow} m S_N^{\frac{N}{2}}$,

where S_N is the best constant for the Sobolev embedding $H_0^1(B) \hookrightarrow L^{2^*}(B)$:

$$\sqrt{S_N}\|v\|_{L^{2^*}(B)} \leq \|\nabla v\|_{L^2(B)}, \quad \forall v \in H_0^1(B). \tag{3.2}$$

The statement (i)–(iii) are known, in particular (i) and (ii) follow by o.d.e. arguments. Instead (iii) derives by the uniqueness of u_p^{m}. In fact on the one hand it is easy to see by the Sobolev embedding that for each nodal region B_p of u_p^{m} we have

$$\lim_{p \to p_S} \int_{B_p} |\nabla u_p^{\mathrm{m}}(y)|^2 dy \geq S_N^{\frac{N}{2}}. \tag{3.3}$$

On the other hand, for any fixed $m \in \mathbb{N}^+$, radial nodal solutions of (1.1) with m nodal regions and whose energy converges to $mS_N^{\frac{N}{2}}$ have been obtained in [22].

Now let us denote by $r^{\mathrm{m}}_{i,p}$, $i = 1, \dots m-1$, the nodal radii of u^{m}_p and for uniformity of notation, by $r^{\mathrm{m}}_{m,p}$ the radius of B. Then writing with abuse of notation $u^{\mathrm{m}}_p(r) = u^{\mathrm{m}}_p(|x|)$, we have

$$\begin{array}{l} 0 < r^{\mathrm{m}}_{1,p} < r^{\mathrm{m}}_{2,p} < \dots < r^{\mathrm{m}}_{m-1,p} < r^{\mathrm{m}}_{m,p} := 1 \\ u^{\mathrm{m}}_p(r^{\mathrm{m}}_{i,p}) = 0 \qquad \text{i=1,\dots,m.} \end{array} \tag{3.4}$$

Moreover we denote by $s^{\mathrm{m}}_{i,p}$, $i = 0, \dots m-1$, the unique maximum point of $|u^{\mathrm{m}}_p|$ in each nodal region, so

$$\begin{array}{l} s^{\mathrm{m}}_{0,p} = 0 \\ s^{\mathrm{m}}_{i,p} \in (r^{\mathrm{m}}_{i,p}, r^{\mathrm{m}}_{i+1,p}), \quad i = 1, \dots, m-1 \text{ (if } m \geq 2) \end{array} \tag{3.5}$$

and

$$(u^{\mathrm{m}}_p)'(s^{\mathrm{m}}_{i,p}) = 0, \quad i = 0, \dots, m-1.$$

Let us denote the m nodal regions of u^{m}_p by $B^{\mathrm{m}}_{i,p} \subset \mathbb{R}^N$, $i = 0, \dots, m-1$, namely:

$$\begin{array}{l} B^{\mathrm{m}}_{0,p} := \{x \in \mathbb{R}^N \ : \ |x| < r^{\mathrm{m}}_{1,p}\} \\ B^{\mathrm{m}}_{i,p} := \{x \in \mathbb{R}^N \ : \ r^{\mathrm{m}}_{i,p} < |x| < r^{\mathrm{m}}_{i+1,p}\}, \quad i = 1, \dots, m-1 \text{ (if } m \geq 2). \end{array} \tag{3.6}$$

Then we consider the restriction of $|u^{\mathrm{m}}_p|$ to the i-th nodal region

$$u^{\mathrm{m}}_{i,p} := |u^{\mathrm{m}}_p| \chi_{B^{\mathrm{m}}_{i,p}}, \quad i = 0, \dots, m-1. \tag{3.7}$$

and let us define

$$M^{\mathrm{m}}_{i,p} := \|u^{\mathrm{m}}_{i,p}\|_\infty = u^{\mathrm{m}}_{i,p}(s^{\mathrm{m}}_{i,p}) = |u^{\mathrm{m}}_p(s^{\mathrm{m}}_{i,p})|, \quad i = 0, \dots, m-1. \tag{3.8}$$

Observe that when $m = 2$ then $u^2_{0,p}$ and $u^2_{1,p}$ are respectively the positive and negative part of u^2_p.

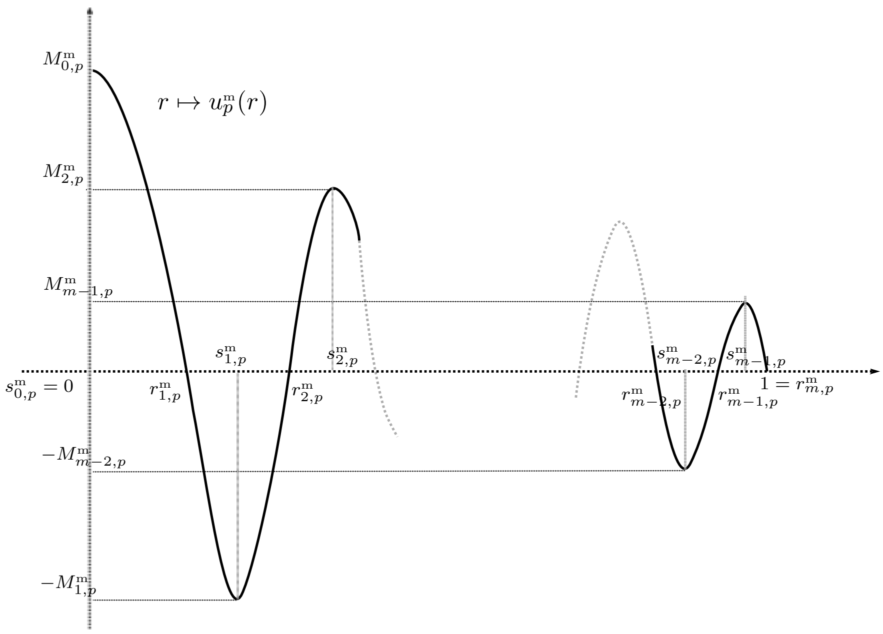

FIGURE 1. The radial solution $r \mapsto u^{\mathrm{m}}_p(r)$ of (1.1) having m nodal regions

Our next result establishes the relation among nodal radii in (3.4), maximum points in (3.5) and scaling parameters in (3.8) related to radial solutions of (1.1) with a different number of nodal regions, m and h respectively:

Lemma 3.2. *Let $m \in \mathbb{N}^+$, $m \geq 2$ and $h = 1, \dots, m-1$. Then for $j = 1, \dots, h$ we have:*

$$r^{\mathrm{h}}_{j,p} = \frac{r^{\mathrm{m}}_{j,p}}{r^{\mathrm{m}}_{h,p}}. \tag{3.9}$$

Moreover for $j = 0, \dots, h-1$ we have:

$$s^{\mathrm{h}}_{j,p} = \frac{s^{\mathrm{m}}_{j,p}}{r^{\mathrm{m}}_{h,p}} \tag{3.10}$$

$$s^{\mathrm{h}}_{j,p}(M^{\mathrm{h}}_{j,p})^{\frac{p-1}{2}} = s^{\mathrm{m}}_{j,p}(M^{\mathrm{m}}_{j,p})^{\frac{p-1}{2}} \tag{3.11}$$

$$(M^{\mathrm{h}}_{j,p})^{\frac{p-1}{2}} = r^{\mathrm{m}}_{h,p}(M^{\mathrm{m}}_{j,p})^{\frac{p-1}{2}}. \tag{3.12}$$

Proof. Let $h = 1, \dots, m-1$ and consider the restriction of the solution u^{m}_p to the first h nodal regions:

$$w^{\mathrm{m}}_{h,p} := u^{\mathrm{m}}_p \chi_{\bigcup_{n=0}^{h-1} B^{\mathrm{m}}_{n,p}}. \tag{3.13}$$

Then it is easy to check that the scaling $\widetilde{w}^{\mathrm{m}}_{h,p}(|x|)$ of $w^{\mathrm{m}}_{h,p}$ defined as

$$\widetilde{w}^{\mathrm{m}}_{h,p}(|x|) := (r^{\mathrm{m}}_{h,p})^{\frac{2}{p-1}} w^{\mathrm{m}}_{h,p}(r^{\mathrm{m}}_{h,p}|x|) \tag{3.14}$$

is a radial solution to (1.1) having h nodal regions and such that $\widetilde{w}^{\mathrm{m}}_{h,p}(0) > 0$. By uniqueness

$$\widetilde{w}^{\mathrm{m}}_{h,p} = u^{\mathrm{h}}_p. \tag{3.15}$$

As a consequence we immediately get (3.10) and (3.9). Moreover we also have:

$$M^{\mathrm{h}}_{0,p} = u^{\mathrm{h}}_p(0) \overset{(3.15)}{=} (r^{\mathrm{m}}_{h,p})^{\frac{2}{p-1}} u^{\mathrm{m}}_p(0) = (r^{\mathrm{m}}_{h,p})^{\frac{2}{p-1}} M^{\mathrm{m}}_{0,p},$$

which gives (3.12) in the case $j = 0$. Instead, when $j = 1, \dots, h-1$, we have:

$$M^{\mathrm{h}}_{j,p} = u^{\mathrm{h}}(s^{\mathrm{h}}_{j,p}) \overset{(3.15)}{=} (r^{\mathrm{m}}_{h,p})^{\frac{2}{p-1}} u^{\mathrm{m}}_p(r^{\mathrm{m}}_{h,p} s^{\mathrm{h}}_{j,p}) \overset{(3.10)}{=} (r^{\mathrm{m}}_{h,p})^{\frac{2}{p-1}} u^{\mathrm{m}}_p(s^{\mathrm{m}}_{j,p}) = (r^{\mathrm{m}}_{h,p})^{\frac{2}{p-1}} M^{\mathrm{m}}_{j,p},$$

which ends the proof of (3.12). Last by (3.12) and (3.10) we get (3.11). □

In the sequel, in order to make the reading more fluid, when there is no possibility of misunderstanding we may drop the dependence on m in our notations, writing, for instance, simply $u_{i,p}, r_{i,p}, M_{i,p}, \dots$ instead of $u^{\mathrm{m}}_{i,p}, r^{\mathrm{m}}_{i,p}, M^{\mathrm{m}}_{i,p} \dots$.

Similarly as in [5, Lemma 2.1] (where the case $m = 2$ is considered) we get

Proposition 3.3. *Let $m \in \mathbb{N}^+$. As $p \to p_S$ we have, for any $i = 0, \dots, m-1$:*

$$\int_B |\nabla u^{\mathrm{m}}_{i,p}(y)|^2 dy = \int_B |u^{\mathrm{m}}_{i,p}(y)|^{p+1} dy \longrightarrow S_N^{\frac{N}{2}} \tag{3.16}$$

$$\int_B |u^{\mathrm{m}}_{i,p}(y)|^{2^*} dy \longrightarrow S_N^{\frac{N}{2}} \tag{3.17}$$

$$\int_B |u^{\mathrm{m}}_{i,p}(y)|^{\frac{N}{2}(p-1)} \longrightarrow S_N^{\frac{N}{2}} \tag{3.18}$$

$$u^{\mathrm{m}}_p \rightharpoonup 0 \ \ in \ H^1_0(B) \tag{3.19}$$

$$M^{\mathrm{m}}_{i,p} \longrightarrow +\infty \tag{3.20}$$

Proof. (3.16) is a direct consequence of Proposition 3.1-(iii) and (3.3). The convergence results in (3.17) and (3.18) follow then from (3.16), indeed:

$$S_N^{\frac{N}{2}} \overset{(3.16)}{=} \lim_{p\to p_S} \frac{\left(\int_B |u^{\rm m}_{i,p}(y)|^{p+1}dy\right)^{\frac{2^*}{(p+1)}}}{|B|^{\frac{2^*}{p+1}-1}} \overset{\text{Hölder}}{\leq} \lim_{p\to p_S}\int_B |u^{\rm m}_{i,p}(y)|^{2^*}dy$$
$$\overset{(3.2)}{\leq} \lim_{p\to p_S} \frac{\|\nabla u^{\rm m}_{i,p}\|^{2^*}_{L^2(B)}}{S_N^{\frac{N}{N-2}}} \overset{(3.16)}{=} S_N^{\frac{N}{2}},$$

which proves (3.17) and similarly we get (3.18):

$$S_N^{\frac{N}{2}} \overset{(3.16)}{\underset{(3.17)}{=}} \lim_{p\to p_S} \frac{\left(\int_B |u^{\rm m}_{i,p}(y)|^{p+1}dy\right)^{\frac{N}{2}}}{\left(\int_B |u^{\rm m}_{i,p}(y)|^{2^*}dy\right)^{\frac{N-2}{2}}} \overset{\text{Hölder}}{\leq} \lim_{p\to p_S}\int_B |u^{\rm m}_{i,p}(y)|^{\frac{N}{2}(p-1)}dy$$
$$\overset{\text{Hölder}}{\leq} \lim_{p\to p_S}\int_B |u^{\rm m}_{i,p}(y)|^{p+1}dy \overset{(3.16)}{=} S_N^{\frac{N}{2}}.$$

The proof of (3.19) follows immediately by the fact that $(u^{\rm m}_{i,p})_p$ is (by (3.16) and (3.17)) a minimizing sequence for the Sobolev embedding $H^1_0(B) \hookrightarrow L^{2^*}(B)$, so that $u^{\rm m}_{i,p} \rightharpoonup 0$ in $H^1_0(B)$ as $p \to p_S$.
Finally the proof of (3.20) follows by (3.17) and (3.19), indeed fixing $\alpha \in (0, 2^*)$, then as $p \to p_S$:

$$S_N^{\frac{N}{2}} \overset{(3.17)}{\longleftarrow} \int_B |u^{\rm m}_{i,p}|^{2^*}dy \leq \|u^{\rm m}_{i,p}\|^{\alpha}_{\infty}\int_B |u^{\rm m}_{i,p}|^{2^*-\alpha}dy,$$

so, since by (3.19) and Rellich Theorem $\int_B |u^{\rm m}_{i,p}|^{2^*-\alpha} \to 0$ as $p \to p_S$, then necessarily (3.20) holds. □

We recall now the classical inequality due to Strauss ([24]), which holds for any $v \in H^1_{rad}(\mathbb{R}^N)$, $N \geq 3$:

$$|v(x)| \leq C_N \frac{\|\nabla v\|_{L^2(\mathbb{R}^N)}}{|x|^{\frac{N-1}{2}}} \qquad \text{for any } x \neq 0, \tag{3.21}$$

where $C_N > 0$ is a constant independent of v. From it we easily deduce:

Proposition 3.4. *Let $m \in \mathbb{N}^+$, $m \geq 2$. For $i = 1, \dots, m-1$ we have*

$$s^{\rm m}_{i,p} \to 0 \text{ (and so also } r^{\rm m}_{i,p} \to 0) \qquad \text{as } p \to p_S$$

Proof. Since $s^{\rm m}_{i-1,p} < r^{\rm m}_{i,p} < s^{\rm m}_{i,p}$, it is enough to show the result in the case $i = m-1$. So setting $s_p := s^{\rm m}_{m-1,p}$, we want to prove that $s_p \to 0$ as $p \to p_S$. If by contradiction $s_{p_n} \geq \alpha > 0$ for a sequence $p_n \to p_S$ as $n \to +\infty$, then by (3.21) and Proposition 3.1-(iii)

$$M^{\rm m}_{m-1,p_n} = |u^{\rm m}_p(s_{p_n})| \leq \frac{C_N}{|\alpha|^{\frac{N-1}{2}}}\|\nabla u^{\rm m}_{p_n}\|_{L^2(\mathbb{R}^N)} \longrightarrow \frac{C_N}{|\alpha|^{\frac{N-1}{2}}} m S_N^{\frac{N}{2}} \quad \text{as } n \to +\infty.$$

So the sequence $(M^{\rm m}_{m-1,p_n})_n$ would be bounded in contradiction with (3.20). □

The next propositions contain crucial estimates for $|u^{\rm m}_p|$ in each nodal region $B^{\rm m}_{i,p}$, $i = 0, \dots, m-1$.

Proposition 3.5. *Let $m \in \mathbb{N}^+$, then*

$$|u_p^{\rm m}(x)| \le \frac{M_{0,p}^{\rm m}}{\left[1 + \frac{(M_{0,p}^{\rm m})^{p-1}}{N(N-2)}|x|^2\right]^{\frac{N-2}{2}}} \qquad \forall\, x \in B_{0,p}^{\rm m}. \tag{3.22}$$

where $B_{0,p}^{\rm m} \subset \mathbb{R}^N$ is as in (3.6) and $M_{0,p}^{\rm m} > 0$ as in (3.8).

Proof. The ordinary differential equation satisfied by $u_p^{\rm m}$ can be turned by a suitable change of variable into an Emden-Fowler equation. Then the proof can be derived adapting the arguments contained in the papers [2, 3] of Atkinson and Peletier, who dealt with the Brezis-Nirenberg problem. Since the proof of the next Proposition 3.6 is similar but slightly more involved, we refer to it for the details. □

Next, if $u_p^{\rm m}$ changes sign (i.e. $m \ge 2$) we can estimate $|u_p^{\rm m}|$ in a similar way in suitable proper subsets $C_{i,p}^{\rm m} \subset B_{i,p}^{\rm m}$, $i = 1, \dots, m-1$. As one can see from the statement below, when $i = 1, \dots, m-2$ ($m \ge 3$), we make the assumption ($\mathcal{R}_i^{\rm m}$), which will be shown in Corollary 3.12 to be always satisfied.

Proposition 3.6. *Let $\alpha \in (0, \frac{N-2}{2})$, $m \in \mathbb{N}^+$, $m \ge 2$ and $i \in \{1, \dots, m-1\}$. If $m \ge 3$ assume that*

$$\frac{s_{i,p}^{\rm m}}{r_{i+1,p}^{\rm m}} \longrightarrow 0 \qquad as\ p \to p_S, \quad \forall i \ne m-1. \tag{$\mathcal{R}_i^{\rm m}$}$$

Then there exists $\gamma = \gamma(\alpha, m) \in (0,1)$, $\gamma(\alpha, m) \to 1$ as $\alpha \to 0$ and $\delta_i = \delta_i(\alpha, m) \in (0, \frac{4}{N-2})$ such that for $p \ge p_S - \delta_i$ we have

$$|u_p^{\rm m}(x)| \le \frac{M_{i,p}^{\rm m}}{\left[1 + \frac{2\alpha}{N(N-2)^2}(M_{i,p}^{\rm m})^{p-1}|x|^2\right]^{\frac{N-2}{2}}} \qquad \forall\, x \in C_{i,p}^{\rm m}, \tag{3.23}$$

where

$$C_{i,p}^{\rm m} := \left\{x \in \mathbb{R}^N :\ \gamma^{-\frac{1}{N}} s_{i,p}^{\rm m} < |x| < r_{i+1,p}^{\rm m}\right\}\ (\subset B_{i,p}^{\rm m})$$

and $M_{i,p}^{\rm m} > 0$ is defined in (3.8).

Proof. We argue as in [20]. Since u_p is a radial solution to (1.1) and $s_{i,p}$ is a critical point for it then $u_{i,p} = |u_p|\chi_{B_{i,p}}$ satisfies in particular

$$\begin{cases} u''_{i,p}(r) + \frac{N-1}{r}u'_{i,p}(r) + (u_{i,p}(r))^p = 0 & r \in (s_{i,p}, r_{i+1,p}) \\ u'_{i,p}(s_{i,p}) = 0 \\ u_{i,p}(r_{i+1,p}) = 0 \\ u_{i,p}(s_{i,p}) = M_{i,p} \end{cases} \tag{3.24}$$

Let

$$t := \left(\frac{N-2}{r}\right)^{N-2}$$

and

$$y_p(t) := u_{i,p}\left(\frac{N-2}{t^{\frac{1}{N-2}}}\right)$$

then y_p satisfies an Emden-Fowler type ordinary differential equation:

$$\begin{cases} y''_p(t) + t^{-k}(y_p(t))^p = 0, & t \in (t_{1,p}, t_{2,p}) \\ y'_p(t_{2,p}) = 0 \\ y_p(t_{1,p}) = 0 \\ y_p(t_{2,p}) = M_{i,p} \end{cases} \tag{3.25}$$

where $k := 2\frac{N-1}{N-2}$, $t_{1,p} := \left(\frac{N-2}{r_{i+1,p}}\right)^{N-2}$, $t_{2,p} := \left(\frac{N-2}{s_{i,p}}\right)^{N-2}$ (notice that y_p, $t_{1,p}$ and $t_{2,p}$ depend also on i but we have omitted it in the notations for simplicity).

STEP 1. We show that

$$\left(y_p' t^{k-1} y_p^{1-k}\right)' + t^{k-2} y_p^{-k} t_{2,p}^{1-k} (y_p(t_{2,p}))^{p+1} \leq 0, \text{ for all } t \in (t_{1,p}, t_{2,p}) \tag{3.26}$$

Proof of STEP 1. We differentiate $y_p' t^{k-1} y_p^{1-k}$ and using $y_p'' + t^{-k} y_p^p = 0$ we get

$$\begin{aligned}\left(y_p' t^{k-1} y_p^{1-k}\right)' &= y_p'' t^{k-1} y_p^{1-k} + y_p'(k-1) t^{k-2} y_p^{1-k} - (k-1)(y_p')^2 t^{k-1} y_p^{-k} \\ &= -t^{-1} y_p^{p+1-k} + y_p'(k-1) t^{k-2} y_p^{1-k} - (k-1)(y_p')^2 t^{k-1} y_p^{-k} \\ &= -2(k-1) t^{k-2} y_p^{-k} \left(\frac{1}{2(k-1)} t^{1-k} y_p^{p+1} - \frac{1}{2} y_p' y_p + \frac{1}{2}(y_p')^2 t\right)\end{aligned}$$

Adding and subtracting $t^{k-2} y_p^{-k} t_{2,p}^{1-k} (y_p(t_{2,p}))^{p+1}$ we deduce

$$\left(y_p' t^{k-1} y_p^{1-k}\right)' + t^{k-2} y_p^{-k} t_{2,p}^{1-k} (y_p(t_{2,p}))^{p+1} = -2(k-1) t^{k-2} y_p^{-k} L_p(t)$$

where

$$L_p(t) := \frac{1}{2(k-1)} t^{1-k} y_p^{p+1} - \frac{1}{2} y_p' y_p + \frac{1}{2}(y_p')^2 t - \frac{1}{2(k-1)} t_{2,p}^{1-k} (y_p(t_{2,p}))^{p+1}$$

Hence (3.26) is proved if we show that

$$L_p(t) \geq 0 \quad \text{for all} \quad t \in (t_{1,p}, t_{2,p}), \tag{3.27}$$

which follows just observing that by definition $L_p(t_{2,p}) = 0$ and that $L_p'(t) \leq 0$ for $t \in (t_{1,p}, t_{2,p})$. Indeed by easy computations

$$L_p'(t) = \frac{p(N-2) - (N+2)}{2N} t^{1-k} y_p'(t) (y_p(t))^p$$

where $\frac{p(N-2)-(N+2)}{2N} < 0$ (since $p < p_S$), $y_p(t) > 0$ and $y_p'(t) \geq 0$ for $t \in (t_{1,p}, t_{2,p})$ (because $(u_{i,p})'(s) \leq 0$ for $s \in (s_{i,p}, r_{i+1,p})$).

STEP 2. We show that for any $\alpha \in (0, \frac{N-2}{2})$ there exist $\gamma = \gamma(\alpha) \in (0,1)$, $\delta_i = \delta_i(\alpha) > 0$ such that

$$y_p(t) \leq M_{i,p} \left[1 + \frac{2}{N}(M_{i,p})^{p-1} t^{-\frac{2}{N-2}} \alpha\right]^{-\frac{N-2}{2}}, \quad \text{for } t \in (t_{1,p}, \gamma^{\frac{N-2}{N}} t_{2,p}),\ p_S - p < \delta_i. \tag{3.28}$$

Proof of STEP 2. We integrate (3.26) between t and $t_{2,p}$ for all $t \in (t_{1,p}, t_{2,p})$. Since $y_p'(t_{2,p}) = 0$ and $y_p(t_{2,p}) = M_{i,p}$ we get

$$y_p'(t) t^{k-1} y_p(t)^{1-k} \geq t_{2,p}^{1-k} (M_{i,p})^{p+1} \int_t^{t_{2,p}} s^{k-2} y_p(s)^{-k} ds \quad \text{for all} \quad t \in (t_{1,p}, t_{2,p}).$$

Since $u_{i,p} \leq M_{i,p}$ by definition, it follows $y_p^{-k} \geq (M_{i,p})^{-k}$, so

$$y_p'(t) t^{k-1} y_p(t)^{1-k} \geq t_{2,p}^{1-k} (M_{i,p})^{p+1-k} \int_t^{t_{2,p}} s^{k-2} ds$$

$$= \frac{(M_{i,p})^{p+1-k}}{k-1}\left(1-\left(\frac{t}{t_{2,p}}\right)^{k-1}\right).$$

Multiplying both side by t^{1-k} we get

$$\frac{1}{2-k}(y_p(t)^{2-k})' = y_p'(t)y_p(t)^{1-k} \geq \frac{(M_{i,p})^{p+1-k}}{k-1}\left(t^{1-k}-\left(\frac{1}{t_{2,p}}\right)^{k-1}\right).$$

Integrating between t and $t_{2,p}$ and recalling that $y_p(t_{2,p}) = M_{i,p}$, we have

$$\begin{aligned}\frac{y_p(t)^{2-k}}{k-2} - \frac{(M_{i,p})^{2-k}}{k-2} &\geq \frac{(M_{i,p})^{p+1-k}}{k-1}\left(-\frac{t_{2,p}^{2-k}}{k-2}+\frac{t^{2-k}}{k-2}-\frac{1}{t_{2,p}^{k-2}}+\frac{t}{t_{2,p}^{k-1}}\right)\\ &= \frac{(M_{i,p})^{p+1-k}}{k-1}t^{2-k}g\left(\left(\frac{t}{t_{2,p}}\right)^{k-1}\right), \end{aligned} \tag{3.29}$$

where

$$g(s) := \frac{1}{k-2} + s - \frac{k-1}{k-2}s^{\frac{k-2}{k-1}}, \quad s \in [0,1].$$

Observe that

$$\begin{aligned} &g(0) = \frac{1}{k-2} = \frac{N-2}{2} > 0\\ &g(1) = 0\\ &g'(s) = 1 - s^{-\frac{1}{k-1}} < 0 \ \text{ in } (0,1). \end{aligned}$$

so $g(s) > 0$ for all $s \in (0,1)$. Moreover, if for any $\alpha \in (0, \frac{N-2}{2})$ there exists only one $\gamma = \gamma(\alpha) \in (0,1)$ such that $g(\gamma) = \alpha$, $g(s) > \alpha$ for all $s \in [0,\gamma)$ and $\gamma \to 1$ as $\alpha \to 0$.
Now remembering that in (3.29) $s := \left(\frac{t}{t_{2,p}}\right)^{k-1}$, it follows that $s < \gamma$ if and only if $t < \gamma^{\frac{1}{k-1}} t_{2,p}$. Let us observe that $t_{1,p} < \gamma^{\frac{1}{k-1}} t_{2,p}$ if and only if

$$s_{i,p}^{N-2} < \gamma^{\frac{1}{k-1}} r_{i+1,p},$$

which holds true, for any fixed $i \in \{1,\dots,m-1\}$, if $p_S - p < \delta_i$, for some number $\delta_i(\gamma) > 0$. In fact in the case $i = m-1$ we have, by definition, that $r_{i+1,p} \equiv 1$ so that the inequality follows directly from Proposition 3.4, while when $i \neq m-1$ it follows by the assumption ($\mathcal{R}_i^{\mathrm{m}}$).
Hence from (3.29) we have

$$y_p(t)^{2-k} - (M_{i,p})^{2-k} \geq \frac{(M_{i,p})^{p+1-k}(k-2)}{k-1}t^{2-k}\alpha, \quad \text{for } t \in (t_{1,p}, \gamma^{\frac{1}{k-1}} t_{2,p}),\ p_S - p < \delta_i$$

which gives (3.28).

STEP 3. Estimate for $u_{i,p}$.

Proof of STEP 3. By definition we have $y_p(t) = u_{i,p}\left(\frac{N-2}{t^{\frac{1}{N-2}}}\right)$, so by (3.28)

$$u_{i,p}\left(\frac{N-2}{t^{\frac{1}{N-2}}}\right) \leq M_{i,p}\left[1 + \frac{2}{N}(M_{i,p})^{p-1}t^{-\frac{2}{N-2}}\alpha\right]^{-\frac{N-2}{2}}$$

for $t \in (t_{1,p}, \gamma^{\frac{N-2}{N}} t_{2,p})$, $p_S - p < \delta_i$. The conclusion follows for $|x| = r := \frac{N-2}{t^{\frac{1}{N-2}}}$. $\square$

We consider now, for $m \in \mathbb{N}^+$, the m *tail sets*

$$T^{\mathrm{m}}_{i,p} := \bigcup_{j=i}^{m-1} B^{\mathrm{m}}_{j,p}, \qquad i = 0, \dots, m-1 \tag{3.30}$$

where $B^{\mathrm{m}}_{i,p}$ are the *nodal regions* of u^{m}_p defined in (3.6) (observe that $T^{\mathrm{m}}_{0,p} = B$, $T^{\mathrm{m}}_{1,p} = B \setminus B^{\mathrm{m}}_{0,p}, \dots, T^{\mathrm{m}}_{m-1,p} = B^{\mathrm{m}}_{m-1,p}$). We define the m rescaled functions

$$z^{\mathrm{m}}_{i,p}(x) := \frac{1}{M^{\mathrm{m}}_{i,p}}\, u^{\mathrm{m}}_p\Big(\frac{|x|}{(M^{\mathrm{m}}_{i,p})^{\frac{p-1}{2}}}\Big), \quad x \in \widetilde{T}^{\mathrm{m}}_{i,p} := (M^{\mathrm{m}}_{i,p})^{\frac{p-1}{2}} T^{\mathrm{m}}_{i,p}, \tag{3.31}$$
$$i = 0, \dots, m-1$$

which are radial, solve

$$\begin{cases} -\Delta z^{\mathrm{m}}_{i,p} = |z^{\mathrm{m}}_{i,p})|^{p-1} z^{\mathrm{m}}_{i,p} & \text{in } \widetilde{T}^{\mathrm{m}}_{i,p} \\ z^{\mathrm{m}}_{i,p} = 0 & \text{on } \partial(\widetilde{T}^{\mathrm{m}}_{i,p}) \\ z^{\mathrm{m}}_{i,p}(s^{\mathrm{m}}_{i,p}) = 1 \text{ and } (z^{\mathrm{m}}_{i,p})'(s^{\mathrm{m}}_{i,p}) = 0 & \end{cases} \tag{3.32}$$

and moreover, by the assumption (3.1), satisfy

$$(-1)^i z^{\mathrm{m}}_{i,p} > 0 \quad \text{in } \widetilde{B}^{\mathrm{m}}_{i,p} := (M^{\mathrm{m}}_{i,p})^{\frac{p-1}{2}} B^{\mathrm{m}}_{i,p}. \tag{3.33}$$

The main result of this section consists in proving that they all converge, up to the sign, to the same function

$$U(x) := \left(\frac{N(N-2)}{N(N-2)+|x|^2}\right)^{\frac{N-2}{2}}, \tag{3.34}$$

which is the unique positive bounded radial solution to the critical equation in $\mathbb{R}^N$:

$$\begin{cases} -\Delta U = U^{p_S} & \text{in } \mathbb{R}^N \\ U(0) = 1 \end{cases} \tag{3.35}$$

and satisfies

$$\int_{\mathbb{R}^N} |\nabla U|^2 dx = \int_{\mathbb{R}^N} U^{2^*} dx = S_N^{\frac{N}{2}}. \tag{3.36}$$

Precisely we show the following:

Theorem 3.7. *Let $m \in \mathbb{N}^+$. We have, as $p \to p_S$:*

$$z^{\mathrm{m}}_{0,p} \longrightarrow U \quad \text{in } C^2_{loc}(\mathbb{R}^N), \tag{3.37}$$
$$(-1)^i z^{\mathrm{m}}_{i,p} \longrightarrow U \quad \text{in } C^2_{loc}(\mathbb{R}^N \setminus \{0\}),\ \forall\, i = 1, \dots, m-1 \ \ (\text{if } m \geq 2) \tag{3.38}$$

As we will see, in order to prove Theorem 3.7 it is enough to scale each nodal region $B^{\mathrm{m}}_{i,p}$ as

$$\widetilde{B}^{\mathrm{m}}_{i,p} := (M^{\mathrm{m}}_{i,p})^{\frac{p-1}{2}} B^{\mathrm{m}}_{i,p}, \quad i = 0, \dots, m-1 \tag{3.39}$$

and show that the same result holds for the restriction of $z^{\mathrm{m}}_{i,p}$ to the set $\widetilde{B}^{\mathrm{m}}_{i,p}$, $i = 0, \dots, m-1$ (see Proposition 3.14 ahead). We point out that the study of the rescaled functions $z^{\mathrm{m}}_{i,p}\chi_{\widetilde{B}^{\mathrm{m}}_{i,p}}$, $i = 1, \dots, m-1$, is more delicate as compared to the study of the first rescaled function $z^{\mathrm{m}}_{0,p}\chi_{\widetilde{B}^{\mathrm{m}}_{0,p}}$. The main reason is that the radius $s^{\mathrm{m}}_{i,p}$, where the maximum of $|u^{\mathrm{m}}_p| = |u^{\mathrm{m}}_p(r)|$ is achieved in the nodal region $B^{\mathrm{m}}_{i,p}$, depends on p when $i \neq 0$, while $s^{\mathrm{m}}_{0,p} \equiv 0$, for any p.
Moreover let us observe that also the nodal radii $r^{\mathrm{m}}_{i,p}$ depend on p. When $i = 1, \dots, m-1$ we know by Proposition 3.4 that both $r^{\mathrm{m}}_{i,p}$ and $s^{\mathrm{m}}_{i,p}$ converge to zero as $p \to p_S$ and, before proving Theorem 3.7, we need to get precise information about

their rate of convergence. In particular in order to determine the limit problem we need to understand how $s^{\rm m}_{i,p}$ and $r^{\rm m}_{i,p}$ behave with respect to the rescaling parameters $(M^{\rm m}_{i,p})^{\frac{p-1}{2}}$.

To this aim for $m \in \mathbb{N}^+$, $m \geq 2$ and $i = 1, \dots, m-1$, let us define the following properties:

$$r^{\rm m}_{i,p}(M^{\rm m}_{i-1,p})^{\frac{p-1}{2}} \longrightarrow +\infty \qquad \text{as } p \to p_S \tag{$\mathcal{A}^{\rm m}_i$}$$

$$s^{\rm m}_{i,p}(M^{\rm m}_{i,p})^{\frac{p-1}{2}} \longrightarrow 0 \qquad \text{as } p \to p_S. \tag{$\mathcal{B}^{\rm m}_i$}$$

Clearly ($\mathcal{B}^{\rm m}_i$) implies

$$r^{\rm m}_{i,p}(M^{\rm m}_{i,p})^{\frac{p-1}{2}} \longrightarrow 0 \qquad \text{as } p \to p_S. \tag{$\mathcal{C}^{\rm m}_i$}$$

We can easily prove that the first property holds, indeed we have:

Proposition 3.8. *Let $m \in \mathbb{N}^+$, $m \geq 2$. Then*

$$(\mathcal{A}^{\rm m}_i) \text{ holds true} \qquad \text{for any } \ i = 1, \dots, m-1.$$

Proof. Let $i \in \{1, \dots, m-1\}$, we want to show that

$$r^{\rm m}_{i,p}(M^{\rm m}_{i-1,p})^{\frac{p-1}{2}} \longrightarrow +\infty \qquad \text{as } p \to p_S.$$

This follows directly from Lemma 3.2 and Proposition 3.3. Indeed, choosing $h := i$ and $j := i-1$ into (3.12) and using (3.20), we get:

$$r^{\rm m}_{i,p}(M^{\rm m}_{i-1,p})^{\frac{p-1}{2}} \overset{(3.12)}{=} (M^{\rm i}_{i-1,p})^{\frac{p-1}{2}} \overset{(3.20)}{\longrightarrow} +\infty \qquad \text{as } p \to p_S.$$

□

Property ($\mathcal{B}^{\rm m}_i$) is more difficult to be obtained. First we prove it for $i = m-1$ (Proposition 3.9 below) and then we extend it to the remaining cases (Proposition 3.11) by means of Lemma 3.2.

Proposition 3.9. *Let $m \in \mathbb{N}^+$, $m \geq 2$. Then*

($\mathcal{B}^{\rm m}_{m-1}$) (and hence also ($\mathcal{C}^{\rm m}_{m-1}$)) holds true .

We first get the following easy estimate.

Lemma 3.10. *There exists $C_N := C_N(m) > 0$ and $\delta = \delta(m) > 0$ such that:*

$$\left|(u^{\rm m}_p)'(r)\right| \leq \frac{C_N}{r^{\frac{p+1}{p-1}}} \qquad \forall\, r \in (0,1),\ \forall\, (0<) p_S - p \leq \delta.$$

Proof. Writing (1.1) in polar coordinates it is easy to see that

$$\big((u^{\rm m}_p)'(r)\, r^{N-1}\big)' = -r^{N-1}|u^{\rm m}_p(r)|^{p-1}u^{\rm m}_p(r),$$

so integrating on $(0,r)$ (recall that $(u^{\rm m}_p)'(0) = 0$), by Hölder inequality, we have

$$\begin{aligned}
\left|(u^{\rm m}_p)'(r)\right| r^{N-1} &\leq \int_{\{|x|<r\}} |u^{\rm m}_p(x)|^p\, dx \\
&\leq \omega_N^{1-\frac{2p}{N(p-1)}}\, r^{N\left(1-\frac{2p}{N(p-1)}\right)} \left[\int_B |u^{\rm m}_p(x)|^{\frac{N}{2}(p-1)}\, dx\right]^{\frac{2p}{N(p-1)}}
\end{aligned}$$

and the conclusion follows from (3.18). □

Proof of Proposition 3.9. In order to shorten the notations let us set $s_p := s^{\mathrm{m}}_{m-1,p}$ and $M_p := M^{\mathrm{m}}_{m-1,p}$. Hence to prove $(\mathcal{B}^{\mathrm{m}}_{m-1})$ means to show that

$$s_p(M_p)^{\frac{p-1}{2}} \longrightarrow 0 \quad \text{ as } p \to p_S. \tag{3.40}$$

We also set $r_p := r^{\mathrm{m}}_{m-1,p}$ and we define

$$z_p := z^{\mathrm{m}}_{m-1,p}, \tag{3.41}$$

where $z^{\mathrm{m}}_{m-1,p}$ is the rescaled function defined in (3.31) for $i = m-1$, i.e. the one related to the last nodal region $T^{\mathrm{m}}_{m-1,p} = B^{\mathrm{m}}_{m-1,p}$. Recall (see (3.32) and (3.33) with $i = m-1$) that it satisfies

$$\begin{cases} -\Delta z_p = z_p^p & \text{in } \widetilde{B}^{\mathrm{m}}_{m-1,p} \\ z_p = 0 & \text{on } \partial(\widetilde{B}^{\mathrm{m}}_{m.1,p}) \\ z_p(s_p) = 1 \ \text{ and } \ (z_p)'(s_p) = 0 & \end{cases} \tag{3.42}$$

with $\widetilde{B}^{\mathrm{m}}_{m-1,p} = \left\{ r_p(M_p)^{\frac{p-2}{2}} < |x| < (M_p)^{\frac{p-2}{2}} \right\}$. Moreover z_p does not change sign in $\widetilde{B}^{\mathrm{m}}_{m-1,p}$ and w.l.g. let us assume that

$$z_p > 0 \ \text{ in } \widetilde{B}^{\mathrm{m}}_{m-1,p}.$$

We follow similar arguments as in the proofs of [20, Lemma 4-5] (which concern the study of the least-energy nodal radial solution for the Brezis-Nirenberg problem) and consider also (setting $s := |x|$) the one-dimensional rescaling of u^{m}_p:

$$w_p(s) := z_p\left(s + s_p(M_p)^{\frac{p-1}{2}}\right) = \frac{1}{M_p}\, u^{\mathrm{m}}_p\left(s_p + \frac{s}{(M_p)^{\frac{p-1}{2}}}\right), \qquad s \in (a_p, b_p),$$

where

$$\begin{aligned} a_p &:= (r_p - s_p)(M_p)^{\frac{p-1}{2}}, \\ b_p &:= (1 - s_p)(M_p)^{\frac{p-1}{2}}. \end{aligned}$$

Then w_p satisfies

$$\begin{cases} w_p''(s) + \dfrac{N-1}{s + s_p(M_p)^{\frac{p-1}{2}}} w_p'(s) + w_p(s)^p = 0 & s \in (a_p, b_p) \\ w_p'(0) = 0, \quad w_p(0) = 1 & \\ w_p \geq 0 & \end{cases}. \tag{3.43}$$

Also let us observe that by Proposition 3.4 and (3.20) one has that

$$b_p \to +\infty \quad \text{as } p \to p_S.$$

We divide the proof into two steps.

STEP 1. First we show that there exists $C > 0$ independent of p such that:

$$s_p(M_p)^{\frac{p-1}{2}} \leq C. \tag{3.44}$$

Proof of STEP 1. Assume by contradiction that up to a subsequence $s_p(M_p)^{\frac{p-1}{2}} \to +\infty$.

Up to a subsequence $a_p \to \bar{a}$, where $\bar{a} \in [-\infty, 0]$.

If $\bar{a} = -\infty$ or $\bar{a} < 0$, then passing to the limit into (3.43) we get that $w_p \to w$ in $C^1_{loc}(\bar{a}, +\infty)$ where w solves the limit problem

$$\begin{cases} w''(s) + w(s)^{p_S} = 0 & s \in (\bar{a}, +\infty) \\ w'(0) = 0, \quad w(0) = 1 & \end{cases} \tag{3.45}$$

and so in particular, by definition of w_p, $w>0$ in $(\bar a,+\infty)$. By a change of variable we have

$$\begin{aligned}\int_{\{r_p<|x|<1\}}|u_p^{\rm m}(x)|^{\frac N2(p-1)}dx &= \omega_N\int_{r_p}^1|u_p^{\rm m}(r)|^{\frac N2(p-1)}r^{N-1}dr\\ &\geq \omega_N s_p^{N-1}\int_{s_p}^1|u_p^{\rm m}(r)|^{\frac N2(p-1)}dr\\ &= \omega_N\left[s_p(M_p)^{\frac{p-1}{2}}\right]^{N-1}\int_0^{b_p}|w_p(s)|^{\frac N2(p-1)}ds \end{aligned}\tag{3.46}$$

and by Fatou's lemma

$$\liminf_{p\to p_S}\int_0^{b_p}|w_p(s)|^{\frac N2(p-1)}ds\geq\int_0^{+\infty}|w(s)|^{2^*}ds>0.$$

Hence passing to the limit into (3.46) we get

$$\lim_{p\to+\infty}\int_B|u^{\rm m}_{m-1,p}(x)|^{\frac N2(p-1)}dx=\lim_{p\to+\infty}\int_{\{r_p<|x|<1\}}|u_p^{\rm m}(x)|^{\frac N2(p-1)}dx=+\infty,$$

which is in contradiction with (3.18).

If $\bar a=0$ the previous argument fails because it could be $w\equiv 0$. So we consider the rescaled function z_p in (3.41) which is uniformly bounded and solves (3.42).
By definition $z_p\big(r_p(M_p)^{\frac{p-1}{2}}\big)=0$ and $z_p\big(s_p(M_p)^{\frac{p-1}{2}}\big)=1$ for any $p\in(1,p_S)$, so

$$\frac{\Big|z_p\big(s_p(M_p)^{\frac{p-1}{2}}\big)\ -\ z_p\big(r_p(M_p)^{\frac{p-1}{2}}\big)\Big|}{|a_p|}=\frac{1}{|a_p|}\to+\infty\qquad\text{as } p\to p_S.$$

where, since z_p is regular, one has

$$\frac{\Big|z_p\big(s_p(M_p)^{\frac{p-1}{2}}\big)\ -\ z_p\big(r_p(M_p)^{\frac{p-1}{2}}\big)\Big|}{|a_p|}=|(z_p)'(\xi_p)|$$

for some $\xi_p\in\Big(r_p(M_p)^{\frac{p-1}{2}},s_p(M_p)^{\frac{p-1}{2}}\Big)$. As a consequence

$$|(z_p)'(\xi_p)|\to+\infty\qquad\text{as } p\to p_S.\tag{3.47}$$

Since byProposition 3.1 we know that $(z_p)'>0$ in $\Big(r_p(M_p)^{\frac{p-1}{2}},s_p(M_p)^{\frac{p-1}{2}}\Big)$ and moreover by definition $z_p>0$, by writing the equation (3.42) in polar coordinates it is easy to see that

$$(z_p)''<0\quad\text{ in }\Big(r_p(M_p)^{\frac{p-1}{2}},s_p(M_p)^{\frac{p-1}{2}}\Big),$$

hence by (3.47)

$$(z_p)'\big(r_p(M_p)^{\frac{p-1}{2}}\big)\geq(z_p)'(\xi_p)\to+\infty\qquad\text{as } p\to p_S.\tag{3.48}$$

On the other side by Lemma 3.10 we also obtain

$$\big|(z_p)'\big(r_p(M_p)^{\frac{p-1}{2}}\big)\big|\leq\frac{C_N}{\big(r_p(M_p)^{\frac{p-1}{2}}\big)^{\frac{p+1}{p-1}}},\tag{3.49}$$

where, since $\bar a=0$, then $r_p(M_p)^{\frac{p-1}{2}}\to+\infty$, and so (3.49) gives a contradiction with (3.48).

STEP 2. We show (3.40).

Proof of STEP 2. We argue by contradiction assuming by the results of *STEP 1.* that, up to a subsequence, $s_p(M_p)^{\frac{p-1}{2}} \to s_0 > 0$ as $p \to p_S$. Then, since $0 < r_p < s_p$, we can have one of the following possibilities for a_p:

(i) $a_p \to 0$

(ii) $a_p \to \bar{a} < 0$.

Next we show that they both lead to a contradiction.

If we assume (i) we can repeat the same proof as in the case $\bar{a} = 0$ in *STEP 1.* The only difference is that now one has $r_p(M_p)^{\frac{p-1}{2}} \to s_0$, which still implies a uniform bound of $(z_p)'(r_p(M_p)^{\frac{p-1}{2}})$ by (3.49). This gives again a contradiction with (3.48).

Let us assume (ii) and define $r_0 := \bar{a} + s_0$. Clearly $r_0 \in [0, s_0)$ and $r_p(M_p)^{\frac{p-1}{2}} \to r_0$. If $r_0 > 0$, then we consider again the rescaled function z_p in (3.41) which is uniformly bounded and solves (3.42). So we get that $z_p \to z$ in $C^2_{loc}(\Pi_{r_0})$ as $p \to p_S$, where $\Pi_{r_0} := \{y \in \mathbb{R}^N : \ |y| > r_0\}$ and passing to the limit into (3.42) ($s_0 > r_0$), we have that z is a positive radial solution of

$$\begin{cases} -\Delta z = z^{p_S} \text{ in } \Pi_{r_0} \\ z'(s_0) = 0, \quad z(s_0) = 1 \end{cases} \tag{3.50}$$

In particular $z \not\equiv 0$. Next we show that z can be extended by continuity to zero on $\partial\Pi_{r_0}$, from which we get that $z \in H^1_0(\Pi_{r_0})$. In fact observe that $(z_p)'$ is uniformly bounded in $(r_p(M_p)^{\frac{p-1}{2}}, s_p(M_p)^{\frac{p-1}{2}})$ by a constant M. This is because we know that $(z_p)'$ is monotone decreasing in $(r_p(M_p)^{\frac{p-1}{2}}, s_p(M_p)^{\frac{p-1}{2}})$ and also, by (3.49) and $r_p(M_p)^{\frac{p-1}{2}} \to r_0 > 0$, that $(z_p)'(r_p(M_p)^{\frac{p-1}{2}})$ is uniformly bounded. As a consequence

$$z_p(s) \le M\left[s - r_p(M_p)^{\frac{p-1}{2}}\right], \quad s \in (r_p(M_p)^{\frac{p-1}{2}}, s_p(M_p)^{\frac{p-1}{2}})$$

and so, passing to the limit as $p \to p_S$ we get

$$z(s) \le M\left[s - r_0\right], \quad s \in (r_0, s_0),$$

from which the extension property follows.

Observe now that when $i = m - 1$ the uniform upper bound (3.23) for $u_p^{\rm m}$ in Proposition 3.6 holds (indeed let us recall that in the case $i = m-1$ the assumption $(\mathcal{R}_i^{\rm m})$ is not required). By scaling it gives the following upper bound for z_p:

$$|z_p(y)| \le \frac{1}{\left(1 + \frac{2\alpha}{N(N-2)^2}|y|^2\right)^{\frac{N-2}{2}}} \qquad \forall y \in \widetilde{C}^{\rm m}_{m-1,p},$$

where

$$\widetilde{C}^{\rm m}_{m-1,p} := \left\{y \in \mathbb{R}^N : \ \gamma^{-\frac{1}{N}} s_p(M_p)^{\frac{p-1}{2}} < |y| < (M_p)^{\frac{p-1}{2}}\right\} \subset \widetilde{B}^{\rm m}_{m-1,p}.$$

Moreover $|z_p| \le 1$ by definition, and so we get a uniform upper bound in the whole annulus $\widetilde{B}^{\rm m}_{m-1,p}$, precisely:

$$|z_p(y)| \le \begin{cases} 1, & y \in \widetilde{B}^{\rm m}_{m-1,p} \setminus \widetilde{C}^{\rm m}_{m-1,p} \\ \frac{1}{\left(1+\frac{2\alpha}{N(N-2)^2}|y|^2\right)^{\frac{N-2}{2}}}, & y \in \widetilde{C}^{\rm m}_{m-1,p}. \end{cases}$$

Hence we can use Lebesgue's theorem to prove

$$\int_{\Pi_{r_0}} |z|^{2^*}\, dx \overset{\text{Lebesgue}}{=} \lim_{p \to p_S} \int_{\widetilde{C}^{\rm m}_{m-1,p}} |z_p|^{\frac{N}{2}(p-1)}\, dx \tag{3.51}$$

$$= \lim_{p \to p_S} \int_B |u^{\rm m}_{m-1,p}|^{\frac{N}{2}(p-1)}\, dx \overset{(3.18)}{=} S_N^{\frac{N}{2}} \tag{3.52}$$

$(\frac{N}{2}(p-1) \to 2^*)$ and moreover, by Fatou's lemma

$$\int_{\Pi_{r_0}} |\nabla z|^2 \, dx \overset{\text{Fatou}}{\leq} \liminf_{p \to p_S} \int_{\widetilde{C}^{\mathrm{m}}_{m-1,p}} |\nabla z_p|^2 \, dx \tag{3.53}$$

$$= \liminf_{p \to p_S} \frac{(M_p)^{\frac{N}{2}(p-1)}}{(M_p)^{p+1}} \int_B |\nabla u^{\mathrm{m}}_{m-1,p}|^2 \, dy$$

$$\leq \lim_{p \to p_S} \int_B |\nabla u^{\mathrm{m}}_{m-1,p}|^2 \, dy \overset{(3.16)}{=} S_N^{\frac{N}{2}}, \tag{3.54}$$

where the last inequality follows from the fact that $\frac{N}{2}(p-1) \leq (p+1)$ for $p < p_S$ and $M_p > 1$ definitely (indeed $M_p \to +\infty$ by (3.20) with $i = m-1$). As a consequence of (3.51) and (3.53) the function z attains the best Sobolev constant S_N in Π_{r_0} and this is clearly impossible since it is known that S_N is not attained in domains strictly contained in $\mathbb{R}^N$. This concludes the proof in the case $r_0 > 0$.
Assume now $r_0 = 0$, then $z_p \to z$ in $C^2_{loc}(\mathbb{R}^N \setminus \{0\})$ as $p \to p_S$, where z is a radial, positive bounded solution to

$$\begin{cases} -\Delta z = z^{p_S} \text{ in } \mathbb{R}^N \setminus \{0\} \\ z'(s_0) = 0 \end{cases} . \tag{3.55}$$

Moreover by Fatou's lemma, as in (3.53), we have

$$\int_{\mathbb{R}^N} |\nabla z|^2 \, dx < \infty. \tag{3.56}$$

Integrating $-\left(z'(r)r^{N-1}\right)' \overset{(3.55)}{=} z^{p_S}(r)r^{N-1}$ we get

$$0 < \int_\delta^{s_0} z^{p_S}(r) r^{N-1} \, dr = z'(\delta)\delta^{N-1} \qquad \forall \delta \in (0, s_0),$$

where the left hand side is monotone decreasing in δ and so passing to the limit as $\delta \to 0^+$ we get

$$z'(\delta)\delta^{N-1} \to \alpha > 0,$$

namely $z'(r) \sim \frac{1}{r^{N-1}}$ around the origin and so

$$\int_{\mathbb{R}^N} |\nabla z(x)|^2 \, dx = \int_0^{+\infty} |z'(r)|^2 r^{N-1} \, dr = +\infty,$$

which contradicts (3.56). □

When $m \geq 3$ we need to prove property $(\mathcal{B}^{\mathrm{m}}_i)$ for the other indices $i \neq m-1$:

Proposition 3.11. *Let $m \in \mathbb{N}^+$, $m \geq 3$. Then*

$$(\mathcal{B}^{\mathrm{m}}_i) \text{ (and hence also } (\mathcal{C}^{\mathrm{m}}_i)) \text{ holds true} \quad \forall\, i = 1, \dots, m-2.$$

Proof. Let us fix $i \in \{1 \dots, m-2\}$, we want to show that $s^{\mathrm{m}}_{i,p}(M^{\mathrm{m}}_{i,p})^{\frac{p-1}{2}} \longrightarrow 0$ as $p \to p_S$.
The proof follows by Lemma 3.2 and Proposition 3.9. Indeed choosing $j := i$ and $h := i+1$ into (3.11) we get

$$s^{\mathrm{m}}_{i,p}(M^{\mathrm{m}}_{i,p})^{\frac{p-1}{2}} \overset{(3.11)}{=} s^{\mathrm{i+1}}_{i,p}(M^{\mathrm{i+1}}_{i,p})^{\frac{p-1}{2}} \overset{\text{(Proposition 3.9)}}{\longrightarrow} 0 \quad \text{ as } p \to p_S.$$

□

As a consequence of the properties $(\mathcal{A}^{\mathrm{m}}_i)$ and $(\mathcal{B}^{\mathrm{m}}_i)$ we may remove the assumption $(\mathcal{R}^{\mathrm{m}}_i)$ in the statement of Proposition 3.6, indeed:

Corollary 3.12. *Let $m \in \mathbb{N}^+$, $m \geq 3$. Then*

$$(\mathcal{R}_i^{\mathrm{m}}) \text{ holds } \forall\, i = 1, \dots, m-2 \tag{3.57}$$

As a consequence the results in Proposition 3.6 can be stated without the assumption $(\mathcal{R}_i^{\mathrm{m}})$.

Proof. By Proposition 3.8, Proposition 3.9 and Proposition 3.11 we have that the properties $(\mathcal{A}_i^{\mathrm{m}})$ and $(\mathcal{B}_i^{\mathrm{m}})$ are satisfied for any $i = 1, \dots, m-1$. Moreover observe that we haven't used $(\mathcal{R}_i^{\mathrm{m}})$ in order to obtain them. Indeed $(\mathcal{R}_i^{\mathrm{m}})$ appears only in the case $i \neq m-1$ of Proposition 3.6 and, up to now, we have used the estimate (3.23) of Proposition 3.6 only in the proof of Proposition 3.9, namely exactly in the case $i = m-1$when the assumption $(\mathcal{R}_i^{\mathrm{m}})$ is not needed to prove (3.23).
Last it is immediate to verify that

$$(\mathcal{A}_{i+1}^{\mathrm{m}}) \text{ and } (\mathcal{B}_i^{\mathrm{m}}) \quad \Longrightarrow \quad (\mathcal{R}_i^{\mathrm{m}}).$$

□

Remark 3.13. *Let us observe that the rate of divergence of the $M_{i,p}^{\mathrm{m}}$ for different indexes i cannot be the same, i.e. it immediately follows from $(\mathcal{A}_{i+1}^{\mathrm{m}})$ and $(\mathcal{C}_{i+1}^{\mathrm{m}})$ that:*

$$\frac{M_{i,p}^{\mathrm{m}}}{M_{i+1,p}^{\mathrm{m}}} \longrightarrow +\infty \quad \text{as } p \to p_S, \quad \forall\, i = 0, \dots, m-2. \tag{3.58}$$

For nodal low-energy *solutions ($m = 2$) of (1.1) with the points of maximum and minimum converging to the same point, this was already known by the results in* [5, Theorem 1.2].

Now, using the properties $(\mathcal{A}_i^{\mathrm{m}})$ and $(\mathcal{C}_i^{\mathrm{m}})$ (which follows by $(\mathcal{B}_i^{\mathrm{m}})$), we can prove the following result, from which Theorem 3.7 follows.

Proposition 3.14. *Let $m \in \mathbb{N}^+$ and let*

$$\widetilde{B}_{i,p}^{\mathrm{m}} := (M_{i,p}^{\mathrm{m}})^{\frac{p-1}{2}} B_{i,p}^{\mathrm{m}}, \;\; i = 0, \dots, m-1$$

where $B_{i,p}^{\mathrm{m}}$ are the nodal regions of u_p^{m} defined in (3.6) and the parameters $M_{i,p}^{\mathrm{m}} > 0$ are the ones introduced in (3.8). Then as $p \to p_S$ we have:

$$z_{0,p}^{\mathrm{m}} \chi_{\widetilde{B}_{0,p}^{\mathrm{m}}} \longrightarrow U \quad \text{in } C^2_{loc}(\mathbb{R}^N), \tag{3.59}$$

$$(-1)^i z_{i,p}^{\mathrm{m}} \chi_{\widetilde{B}_{i,p}^{\mathrm{m}}} \longrightarrow U \quad \text{in } C^2_{loc}(\mathbb{R}^N \setminus \{0\}), \; \forall\, i = 1, \dots, m-1 \;\; (\text{if } m \geq 2) \tag{3.60}$$

where the rescaled function $z_{i,p}^{\mathrm{m}}$ are defined in (3.31).

Proof. The proof of (3.59) is standard. Indeed, since the functions $z_{0,p}^{\mathrm{m}}$ are uniformly bounded, satisfy (3.32) in $\widetilde{B}_{0,p}^{\mathrm{m}}$ and property $(\mathcal{A}_1^{\mathrm{m}})$ holds, we have that the limit of the domain $\widetilde{B}_{0,p}^{\mathrm{m}}$ is the whole $\mathbb{R}^N$ and $z_{0,p}^{\mathrm{m}}$ converge in $C^2_{loc}(\mathbb{R}^N)$ to a solution z of (3.35). The limit function z has finite energy by Fatou's lemma, it is positive by (3.33) so it must necessarily be the function U in (3.34).

Similarly we prove (3.60). Indeed the rescaled functions $z_{i,p}^{\mathrm{m}}$, $i = 1, \dots, m-1$, are uniformly bounded and solve (3.32) in $\widetilde{B}_{i,p}^{\mathrm{m}}$. The limit of the domains $\widetilde{B}_{i,p}^{\mathrm{m}}$ is now $\mathbb{R}^N \setminus \{0\}$, this follows by the property $(\mathcal{C}_{m-1}^{\mathrm{m}})$ in the case $i = m-1$ and by the properties $(\mathcal{A}_{i+1}^{\mathrm{m}})$ and $(\mathcal{C}_i^{\mathrm{m}})$ in the other cases. By standard elliptic estimates, we have that $(-1)^i z^{\mathrm{m}}_{i,p} \to z$ in $C^2_{loc}(\mathbb{R}^N\setminus\{0\})$ where z is positive (by (3.33)) radial, solves

$$-\Delta z = z^{p_S} \qquad \text{in } \mathbb{R}^N\setminus\{0\}$$

and (as for the previous case) has finite energy.
Exactly as in Lemma 6 and Lemma 7 of [20] we get that z can be extended to a $C^1(\mathbb{R}^N)$ function such that $z(0)=1$, $\nabla z(0)=0$ and is a weak solution of (3.35) (in the whole $\mathbb{R}^N$). Hence z must be the function U of (3.34). □

Proof of Theorem 3.7. The proof is similar to the one of Proposition 3.14. Just observe that $z^{\mathrm{m}}_{i,p}$ is uniformly bounded in the whole rescaling of the tail set $\widetilde{T}^{\mathrm{m}}_{i,p}$ in (3.31), since it is uniformly bounded in $\widetilde{B}^{\mathrm{m}}_{i,p}$ (as already observed in the proof of Proposition 3.14) and moreover (3.58) holds true. Observe also that the limit of the domain $\widetilde{T}^{\mathrm{m}}_{0,p} = (M^{\mathrm{m}}_{0,p})^{\frac{p-1}{2}} B$ is clearly $\mathbb{R}^N$ (by (3.20)), while the limit of the domains $\widetilde{T}^{\mathrm{m}}_{i,p}$, when $i=1,\dots,m-1$, is the set $\mathbb{R}^N\setminus\{0\}$ (by (3.20) and property $(\mathcal{C}^{\mathrm{m}}_i)$). The result then follows similarly as in the proof of Proposition 3.14. □

We conclude the section with an estimate that will be important throughout the proof of Theorem 1.1:

Proposition 3.15. *Let $m\in\mathbb{N}^+$. There exist $\delta=\delta(m)>0$ and $C>0$ (independent of m) such that*

$$f^{\mathrm{m}}_p(|y|) := |y|^2 |u^{\mathrm{m}}_p(y)|^{p-1} \le C \qquad \textit{for any } y\in B \quad \textit{and} \quad p>p_S-\delta. \tag{3.61}$$

Proof. *Case I*: $\;r:=|y|\in[0,r^{\mathrm{m}}_{1,p}]$.
By Proposition 3.5 one has that $f^{\mathrm{m}}_p(r)\le \widetilde{g}_p\big(r(M^{\mathrm{m}}_{0,p})^{\frac{p-1}{2}}\big)$, where for $s\in[0,+\infty)$

$$\widetilde{g}_p(s) := \frac{s^2}{\Big(1+\frac{1}{N(N-2)}s^2\Big)^{\frac{(N-2)(p-1)}{2}}}.$$

Since $\frac{(N-2)(p-1)}{2}\ge\frac{3}{2}$ for p sufficiently close to p_S, it can be easily seen that there exist $\delta>0$ and $C>0$ such that

$$\widetilde{g}_p(s) \le \frac{s^2}{\Big(1+\frac{1}{N(N-2)}s^2\Big)^{\frac{3}{2}}} \le C \qquad \text{for any } s\in[0,+\infty) \text{ and } p>p_S-\delta.$$

This concludes the proof of *Case I*.

Case II: $\;r:=|y|\in(r^{\mathrm{m}}_{i,p}, r^{\mathrm{m}}_{i+1,p}]$, for some $i=1,\dots,m-1$.
Let us fix $\alpha\in(0,\frac{N-2}{2})$ and consider $\gamma=\gamma(\alpha,m)$ defined in Proposition 3.6. Then for any $r\in(r^{\mathrm{m}}_{i,p}, \gamma^{-\frac{1}{N}} s^{\mathrm{m}}_{i,p}]$ we use the property $(\mathcal{B}^{\mathrm{m}}_i)$ (which is satisfied by Propositions 3.9-3.11) to prove that:

$$f^{\mathrm{m}}_p(r) \le \gamma^{-\frac{2}{N}} (s^{\mathrm{m}}_{i,p})^2 |u^{\mathrm{m}}_p(r)|^{p-1} \overset{(3.8)}{\le} \gamma^{-\frac{2}{N}} (s^{\mathrm{m}}_{i,p})^2 (M^{\mathrm{m}}_{i,p})^{p-1} \underset{p\to p_S}{\overset{(\mathcal{B}^{\mathrm{m}}_i)}{\longrightarrow}} 0.$$

Then clearly there exists $C>0$ and there exists $\delta_i=\delta_i(m)>0$ such that $f^{\mathrm{m}}_p(r)\le C$, for any $r\in(r^{\mathrm{m}}_{i,p}, \gamma^{-\frac{1}{N}} s^{\mathrm{m}}_{i,p}]$ and for any $p\ge p_S-\delta_i$.
For $r\in(\gamma^{-\frac{1}{N}} s^{\mathrm{m}}_{i,p}, r^{\mathrm{m}}_{i+1,p}]$ by Proposition 3.6 and Corollary 3.12

$$f^{\mathrm{m}}_p(r) \le \widehat{g}_p\big(r(M^{\mathrm{m}}_{i,p})^{\frac{p-1}{2}}\big),$$

where for $s \in [0, +\infty)$

$$\widehat{g}_p(s) := \frac{s^2}{\left(1 + \frac{2\alpha}{N(N-2)^2} s^2\right)^{\frac{(N-2)(p-1)}{2}}}.$$

Exactly as in *Case I*, fixing $\delta > 0$ such that $\frac{(N-2)(p-1)}{2} \geq \frac{3}{2}$ it turns out that

$$\widehat{g}_p(s) \leq C \qquad \text{for any } s \in [0, +\infty) \text{ and } p > p_S - \delta,$$

and this ends the proof of *Case II*. □

4. Approximations of eigenvalues and auxiliary weighted problems

In the following we summarize the construction and the results obtained in Sections 3 and 4 of [13]. Along all the section $m \in N^+$ and $p \in (1, p_S)$ are fixed and u_p^{m} is the radial solution of (1.1) having m nodal regions, satisfying the sign condition (3.1) and already studied in the previous section.

Let $L_p^{\mathrm{m}} : H^2(B) \cap H_0^1(B) \to L^2(B)$ be the linearized operator at u_p^{m}, namely

$$L_p^{\mathrm{m}}(v) := -\Delta v - p|u_p^{\mathrm{m}}(x)|^{p-1} v. \tag{4.1}$$

The Dirichlet eigenvalues of L_p^{m} in B, counted with their multiplicity, are

$$\mu_1(m,p) < \mu_2(m,p) \leq \ldots \leq \mu_i(m,p) \leq \ldots,$$
$$\mu_i(m,p) \to +\infty \qquad \text{as } i \to +\infty.$$

Among these there are the *radial* Dirichlet eigenvalues, which also form a sequence, denoted by:

$$\beta_i(m,p), \quad i \in \mathbb{N}^+.$$

As in Section 2 the *Morse index* of u_p^{m} is denoted by $\mathsf{m}(u_p^{\mathrm{m}})$, while the *radial Morse index* of u_p^{m} (namely the number of negative radial eigenvalues of L_p^{m}) is denoted by $\mathsf{m}_{\mathsf{rad}}(u_p^{\mathrm{m}})$.

By Theorem 2.1 we know that

$$\mathsf{m}(u_p^{\mathrm{m}}) \geq m + N(m-1) \tag{4.2}$$

and by Theorem 2.2 that

$$\mathsf{m}_{\mathsf{rad}}(u_p^{\mathrm{m}}) = m. \tag{4.3}$$

As in [13], in order to compute the Morse index of u_p^{m}, we approximate the ball B with the annuli:

$$A_n := \{x \in \mathbb{R}^N \ : \ \frac{1}{n} < |x| < 1\}, \quad n \in \mathbb{N}^+, \tag{4.4}$$

and we denote by

$$\mu_i^n(m,p), \quad i \in \mathbb{N}^+$$

the Dirichlet eigenvalues of L_p^{m} in A_n counted according to their multiplicity and by

$$\beta_i^n(m,p), \quad i \in \mathbb{N}^+$$

the radial Dirichlet eigenvalues of L_p^{m} in A_n counted with their multiplicity. Finally we denote by

$$k_p^n(m) := \#\{\text{negative eigenvalues } \mu_i^n(m,p) \text{ of } L_p^{\mathrm{m}} \text{ in } A_n\}, \tag{4.5}$$

$$k_{p,rad}^n(m) := \#\{\text{negative radial eigenvalues } \beta_i^n(m,p) \text{ of } L_p^{\mathrm{m}} \text{ in } A_n\}. \tag{4.6}$$

As proved in [13] (Lemma 3.2 and Lemma 3.3 therein) the following holds:

Lemma 4.1. *For any fixed $m \in \mathbb{N}^+$ and any fixed $p \in (1, p_S)$ we have:*

$$\mu_i^n(m,p) \searrow \mu_i(m,p) \quad \text{and} \quad \beta_i^n(m,p) \searrow \beta_i(m,p) \quad \text{as} \quad n \to +\infty, \quad \forall\, i \in \mathbb{N}^+.$$

Hence there exists $n'_p = n'_p(m) \in \mathbb{N}^+$ such that

$$\mathsf{m}(u_p^{\mathrm{m}}) = k_p^n(m) \quad \text{and} \quad \mathsf{m}_{\mathsf{rad}}(u_p^{\mathrm{m}}) = k_{p,rad}^n(m), \quad \text{for } n \geq n'_p.$$

In order to make a decomposition of the spectrum of L_p^{m} we consider the auxiliary weighted linear operator $\widetilde{L_p^n}^{\mathrm{m}} : H^2(A_n) \cap H_0^1(A_n) \to L^2(A_n)$ defined by:

$$\widetilde{L_p^n}^{\mathrm{m}}(v) := |x|^2 \left(-\Delta v - p|u_p^{\mathrm{m}}(x)|^{p-1} v\right), \quad x \in A_n, \tag{4.7}$$

and denote by

$$\widetilde{\mu_i^n}(m,p), \quad i \in \mathbb{N}^+$$

its eigenvalues counted with their multiplicity. Observe that the corresponding eigenfunctions h satisfy

$$\begin{cases} -\Delta h(x) - p|u_p^{\mathrm{m}}(x)|^{p-1} h(x) = \widetilde{\mu_i^n}(m,p)\, \frac{h(x)}{|x|^2} & x \in A_n \\ \\ h = 0 \quad \text{on } \partial A_n. \end{cases}$$

Since u_p^{m} is radial we also consider the following linear operator $\widetilde{L_{p,rad}^n}^{\mathrm{m}} : H^2((\frac{1}{n},1)) \cap H_0^1((\frac{1}{n},1)) \to L^2((\frac{1}{n},1))$

$$\widetilde{L_{p,rad}^n}^{\mathrm{m}}(v) := r^2 \left(-v'' - \frac{(N-1)}{r} v' - p|u_p^{\mathrm{m}}(r)|^{p-1} v\right), \quad r \in (\frac{1}{n}, 1) \tag{4.8}$$

and denote by

$$\widetilde{\beta_i^n}(m,p), \quad i \in \mathbb{N}^+$$

its eigenvalues counted with their multiplicity. Obviously $\widetilde{\beta_i^n}(m,p)$ are nothing else than the radial eigenvalues of $\widetilde{L_p^n}^{\mathrm{m}}$. Let us also set

$$\widetilde{k_p^n}(m) := \#\{\text{negative eigenvalues } \widetilde{\mu_i^n}(m,p) \text{ of } \widetilde{L_p^n}^{\mathrm{m}}\}, \tag{4.9}$$

$$\widetilde{k_{p,rad}^n}(m) := \#\{\text{negative eigenvalues } \widetilde{\beta_i^n}(m,p) \text{ of the operator } \widetilde{L_{p,rad}^n}^{\mathrm{m}}\}. \tag{4.10}$$

Denoting by $\sigma(\cdot)$ the spectrum of a linear operator we recall that the following decomposition holds:

$$\sigma(\widetilde{L_p^n}^{\mathrm{m}}) = \sigma(\widetilde{L_{p,rad}^n}^{\mathrm{m}}) + \sigma(-\Delta_{S^{N-1}}), \quad \text{for any } n \in \mathbb{N}^+, \tag{4.11}$$

where $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on the unit sphere S^{N-1}, $N \geq 3$. The proof of (4.11) is not difficult, it can be found for example in [16]. So (4.11) means that, for any $n \in \mathbb{N}^+$:

$$\widetilde{\mu_j^n}(m,p) \;=\; \widetilde{\beta_i^n}(m,p) \;+\; \lambda_k, \quad \text{for } i,j \in \mathbb{N}^+, \quad k \in \mathbb{N}, \tag{4.12}$$

where λ_k are the eigenvalues of $-\Delta_{S^{N-1}}$, $N \geq 3$. Note that in (4.12) only $\widetilde{\beta_i^n}(m,p)$ depend on the exponent p, while the eigenvalues λ_k depend only on the dimension N and it is known ([6, Proposition 4.1]) that

$$\lambda_k = k(k+N-2), \quad k \in \mathbb{N}, \tag{4.13}$$

with multiplicity

$$N_k - N_{k-2}, \tag{4.14}$$

where

$$N_h := \binom{N-1+h}{N-1} = \frac{(N-1+h)!}{(N-1)!h!}, \quad \text{if } h \geq 0, \quad N_h = 0, \quad \text{if } h < 0. \tag{4.15}$$

Next result shows the equivalence between the number of the negative eigenvalues of the linearized operator L_p^{m} in A_n and that of the weighted operators:

Lemma 4.2. *We have:*

$$k_p^n(m) = \widetilde{k_p^n}(m) \qquad \text{and} \qquad k_{p,rad}^n(m) = \widetilde{k_{p,rad}^n}(m).$$

Proof. See [13, Lemma 4.2] □

Combining Lemma 4.1, Lemma 4.2, (4.2) and (4.3) we get:

Proposition 4.3. *Let $\in \mathbb{N}^+$ and $p \in (1, p_S)$. There exists $n'_p = n'_p(m) \in \mathbb{N}^+$ such that*

$$\mathsf{m}(u_p^{\mathrm{m}}) = \widetilde{k_p^n}(m) \qquad \text{and} \qquad \mathsf{m}_{\mathsf{rad}}(u_p^{\mathrm{m}}) = \widetilde{k_{p,rad}^n}(m), \quad \text{for } n \geq n'_p.$$

Hence

$$\widetilde{k_p^n}(m) \geq m + N(m-1) \qquad \text{and} \qquad \widetilde{k_{p,rad}^n}(m) = m, \quad \text{for } n \geq n'_p. \tag{4.16}$$

Because of the decomposition (4.12) and of Proposition 4.3 it is clear that in order to evaluate the Morse index $\mathsf{m}(u_p^{\mathrm{m}})$ (i.e. to prove Theorem 1.1) we have to estimate the negative eigenvalues $\widetilde{\beta_i^n}(m,p)$ of the weighted operator $\widetilde{L_{p,rad}^{n^{\mathrm{m}}}}$ which, by (4.16), are only the first m ones.

We conclude this section by an estimate of the last negative eigenvalue $\widetilde{\beta_m^n}(m,p)$. This result generalizes to any $m \in \mathbb{N}^+$ the analogous one already proved in [13, Proposition 4.5] in the case $m = 2$.

We emphasize that an estimate of the other negative eigenvalues $\widetilde{\beta_i^n}(m,p)$, $i = 1, \dots, m-1$, is much more difficult and it will be the object of the next section.

Proposition 4.4. *Let $m \in \mathbb{N}^+$ and $p \in (1, p_S)$. Let $n''_p = n''_p(m) := [\frac{1}{r_{1,p}^{\mathrm{m}}}] + 1$, where $r_{1,p}^{\mathrm{m}}$ is the first nodal radius of u_p^{m} as defined in (3.4). Then*

$$\widetilde{\beta_m^n}(m,p) > -(N-1) \quad \text{for any } n \geq n''_p.$$

Proof. Let $\eta(r) := \dfrac{\partial u_p^{\mathrm{m}}(r)}{\partial r}$, then by the choice of n''_p it follows that for any $n \geq n''_p$ one has $\frac{1}{n} < r_{1,p}^{\mathrm{m}}$ and so the function η satisfies

$$\begin{cases} \widetilde{L_{p,rad}^{n^{\mathrm{m}}}}\, \eta = -(N-1)\eta, & r \in (\frac{1}{n}, 1) \\ \eta(\frac{1}{n}) < 0 \\ \eta(1) \lesseqgtr 0 \text{ for } m \ \substack{\text{odd} \\ \text{even}} \end{cases}$$

(the inequalities on the boundary deriving from the assumption $u_p^{\mathrm{m}}(0) > 0$ in (3.1), moreover they are strict by the Hopf's Lemma). Moreover we know that, for $n \geq n''_p$, η has exactly $m-1$ zeros in the interval $(\frac{1}{n}, 1)$, given (if $m \geq 2$) by the points $s_{i,p}^{\mathrm{m}}$, $i = 1, \dots, m-1$, defined in (3.5).

Let w be an eigenfunction of $\widetilde{L_{p,rad}^{n^{\mathrm{m}}}}$ associated with the eigenvalue $\widetilde{\beta_m^n}(m,p)$, namely

$$\begin{cases} \widetilde{L_{p,rad}^{n^{\mathrm{m}}}}\, w = \widetilde{\beta_m^n}(m,p)\, w, & r \in (\frac{1}{n}, 1) \\ w(\frac{1}{n}) = 0 \\ w(1) = 0. \end{cases}$$

It is well known that w has exactly m nodal regions.

Assume by contradiction that $\widetilde{\beta_m^n}(m,p) \leq -(N-1)$.

If $\widetilde{\beta_m^n}(m,p) = -(N-1)$, then η and w are two solutions of the same Sturm-Liouville equation

$$(r^{N-1}v')' + \left[p|u_p(r)|^{p-1} r^{N-1} + \frac{\widetilde{\beta_m^n}(m,p)}{r^{3-N}} \right] v = 0, \qquad r \in (\frac{1}{n}, 1)$$

and they are linearly independent because $\eta(1) \neq 0 = w(1)$. As a consequence (Sturm Separation Theorem) the zeros of η and w must alternate. Since η has $m-1$ zeros, w must then have $m-1$ nodal regions and this gives a contradiction. If $-(N-1) > \widetilde{\beta^n_m}(m,p)$, then by the Sturm Comparison Theorem, η must have a zero between any two consecutive zeros of w. As a consequence, since we know that w has $m-1$ zeros in $(\frac{1}{n},1)$ and that also the boundary points $\frac{1}{n}$ and 1 are zeros, then η must have m zeros in $(\frac{1}{n},1)$, which gives again a contradiction. □

5. Asymptotic analysis of the eigenvalues $\widetilde{\beta^n_i}(m,p)$, $i=1,\dots,m-1$

This section is devoted to study the asymptotic behavior, as $p \to p_S$, of the first $(m-1)$ eigenvalues $\widetilde{\beta^n_i}(m,p)$, $i=1,\dots,m-1$, of the auxiliary weighted radial operator $\widetilde{L^{n}}^{\mathrm{m}}_{p,rad}$ defined in (4.8), when u^{m}_p is the radial solution to (1.1) having m nodal regions, for $m \in N^+$, which satisfies $u^{\mathrm{m}}_p(0) > 0$.

Recall that, for each $n \in \mathbb{N}^+$, the operator $\widetilde{L^{n}}^{\mathrm{m}}_{p,rad}$ in (4.8) is defined in the annulus

$$A_n = \{x \in \mathbb{R}^N \ : \ \frac{1}{n} < |x| < 1\}.$$

For our purposes it is convenient to *chose the number* n in dependence of p (and m) as follows:

$$n^{\mathrm{m}}_p := \max\{n'_p, n''_p, [(M^{\mathrm{m}}_{0,p})^{(p-1)}] + 1\}, \tag{5.1}$$

where $n'_p = n'_p(m)$ is defined in Proposition 4.3, while $n''_p = n''_p(m)$ is as in Proposition 4.4.
Then for any $i \in \mathbb{N}^+$ we consider the family of eigenvalues defined as

$$\widetilde{\beta}_i(m,p) := \widetilde{\beta^n_i}(m,p) \qquad \text{when} \ \ n = n^{\mathrm{m}}_p. \tag{5.2}$$

Notice that the definition of n^{m}_p in (5.1) and (4.16) imply that $\widetilde{\beta}_i(m,p) < 0$, for $i=1,\dots,m-1$, for every $p \in (1,p_S)$.
In order to shorten the notation for the operator, we set:

$$\widetilde{L}^{\mathrm{m}}_{p,rad} := \widetilde{L^{n}}^{\mathrm{m}}_{p,rad} \qquad \text{when } n = n^{\mathrm{m}}_p. \tag{5.3}$$

The main result of this section is about the asymptotic behavior of the first eigenvalue $\widetilde{\beta}_1(m,p)$ as $p \to p_S$:

Proposition 5.1. *Let* $m \in \mathbb{N}^+$.

$$\liminf_{p \to p_S} \ \widetilde{\beta}_1(m,p) \geq -(N-1). \tag{5.4}$$

An immediate consequence of the previous proposition is the following:

Corollary 5.2. *Let* $m \in \mathbb{N}^+$.

$$\liminf_{p \to p_S} \ \widetilde{\beta}_i(m,p) \geq -(N-1), \qquad \textit{for all} \ \ i=1,\dots,m-1.$$

Remark 5.3. *In the next section, while proving Theorem 1.1, we will show the reverse inequality:*

$$\widetilde{\beta}_i(m,p) < -(N-1), \qquad \forall\, i=1,\dots,m-1, \qquad \textit{for } p \textit{ close to } p_S$$

(see (6.11)). Combining this with Corollary 5.2 we will obtain the precise value of the limit:

$$\widetilde{\beta}_i(m,p) \to -(N-1) \quad \text{as} \ \ p \to p_S, \qquad \forall\, i = 1,\dots,m-1 \tag{5.5}$$

(see (6.13)).

The result in Proposition 5.1 is the core of the proof of Theorem 1.1. Since its proof is very long and needs various nontrivial estimates, let us first explain the strategy.

In order to get (5.4) we consider, for any fixed $p \in (1, p_S)$, the (radial and positive) eigenfunction ϕ_p^{m} of $\widetilde{L}^{\mathrm{m}}_{p,rad}$ (defined as in (5.3)) associated with the first eigenvalue $\widetilde{\beta}_1(m,p)$, namely

$$\begin{cases} -\phi_p^{\mathrm{m}}{}'' - \frac{(N-1)}{r}\phi_p^{\mathrm{m}}{}' - p|u_p^{\mathrm{m}}|^{p-1}\phi_p^{\mathrm{m}} = \widetilde{\beta}_1(m,p)\,\dfrac{\phi_p^{\mathrm{m}}}{r^2}, & r \in (\dfrac{1}{n_p^{\mathrm{m}}}, 1) \\ \phi_p^{\mathrm{m}}(\frac{1}{n_p^{\mathrm{m}}}) = \phi_p^{\mathrm{m}}(1) = 0. \end{cases} \tag{5.6}$$

To obtain the result one would like to pass to the limit as $p \to p_S$ into (5.6) and deduce the value of $\lim_{p\to p_S} \widetilde{\beta}_1(m,p)$ by studying the limit eigenvalue problem.

Since the term $p|u_p^{\mathrm{m}}|^{p-1}$ in the equation (5.6) is not bounded, it is more convenient to scale properly the eigenfunctions ϕ_p^{m} and pass to the limit into the equation satisfied by the scalings. The right possible scalings are the $\widehat{\phi_p^{\mathrm{m}}}^i$, $i = 0,\dots,m-1$, defined in (5.18) below, which satisfy the equations in (5.19) where the eigenvalue $\widetilde{\beta}_1(m,p)$ again appears. Note that the scaling parameter in the definition of $\widehat{\phi_p^{\mathrm{m}}}^i$ is given by the value $M^{\mathrm{m}}_{i,p}$ of the L^∞-norm of u_p^{m} in the corresponding i-th nodal region.

Of course this procedure is efficient if at least one among the $\widehat{\phi_p^{\mathrm{m}}}^i$ does not vanish in the limit. Since we cannot guarantee that this is always the case (see *CASE 2.* in the proof of Proposition 5.1) we combine it with a different strategy which consists in considering a suitable *limit eigenvalue problem* (with the operator $\widetilde{L}^*$ in Section 5.1) and exploiting the variational characterization of its first eigenvalue. This reduces the proof to analyzing the *difference* between a *limit potential* V and the actual potential $V^{\mathrm{m}}_{0,p}$ defined in (5.7) and (5.20) below, exploiting the asymptotic behavior of u_p^{m} studied in Section 3. In particular we need to evaluate the contribution to the limit of $\widetilde{\beta}_1(m,p)$ given by the first nodal region $B^{\mathrm{m}}_{0,p}$ of u_p^{m}, which is contained in Lemma 5.8 below, and the contribution given by the other nodal regions of u_p^{m} and this is done in Lemma 5.9 and Lemma 5.10 where the behavior of the function $f_p^{\mathrm{m}}(r) := |r|^2|u_p^{\mathrm{m}}(r)|^{p-1}$ in the nodal regions $B^{\mathrm{m}}_{i,p}$, $i = 1,\dots,m-1$, of u_p^{m} is studied.

To make easier the understanding of the proof of Proposition 5.1 we have divided this section as follows:

- in Section 5.1 we introduce the limit weighted eigenvalue problem;
- in Section 5.2 we collect all the preliminary results about ϕ_p^{m} as well as the properties of its scalings $\widehat{\phi_p^{\mathrm{m}}}^i$, $i = 0,\dots,m-1$;
- in Section 5.3 we estimate u_p^{m} in $B^{\mathrm{m}}_{0,p}$;
- in Section 5.4 we estimate u_p^{m} in $B \setminus B^{\mathrm{m}}_{0,p}$;
- in Section 5.5 we complete the proof of Proposition 5.1.

5.1. **A limit weighted eigenvalue problem.**

Let $N \geq 3$ and consider the weighted linear operator

$$\widetilde{L}^* v := |x|^2 \left[-\Delta v - V(x) v \right], \qquad x \in \mathbb{R}^N$$

where

$$V(x) := p_S\, U(x)^{p_S - 1} = \frac{N+2}{N-2} \left(\frac{N(N-2)}{N(N-2) + |x|^2} \right)^2 \tag{5.7}$$

with U as in (3.34), i.e. U is the unique positive bounded solution to the critical equation (3.35) in $\mathbb{R}^N$.

We want to define the first eigenvalue of $\widetilde{L}^*$. Let $D^{1,2}(\mathbb{R}^N)$ be the Hilbert space defined as the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the Dirichlet norm $\|v\|_{D^{1,2}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla v(x)|^2 dx \right)^{\frac{1}{2}}$ and let us denote by $D^{1,2}_{rad}(\mathbb{R}^N)$ its subspace made of radial functions.

Let us set

$$\widetilde{\beta}^* := \inf_{\substack{v \in D^{1,2}_{rad}(\mathbb{R}^N) \\ v \neq 0}} \frac{\int_{\mathbb{R}^N} \left(|\nabla v(x)|^2 - V(x) v(x)^2 \right) dx}{\| \frac{v}{|x|} \|^2_{L^2(\mathbb{R}^N)}}. \tag{5.8}$$

Observe that this definition is well posed since the Hardy inequality ([**?**, **?**, **?**]) holds:

$$\| \frac{v}{|x|} \|_{L^2(\mathbb{R}^N)} \leq \frac{2}{(N-2)} \|v\|_{D^{1,2}(\mathbb{R}^N)}, \qquad \text{for any } v \in D^{1,2}(\mathbb{R}^N), \quad N \geq 3 \tag{5.9}$$

and so

$$\int_{\mathbb{R}^N} V(x) v(x)^2 dx \leq \sup_{\mathbb{R}^N} (V(x)|x|^2) \int_{\mathbb{R}^N} \frac{v(x)^2}{|x|^2} dx \overset{(5.9)}{\leq} C \|v\|^2_{D^{1,2}(\mathbb{R}^N)},$$

where we have used that $\sup_{\mathbb{R}^N}(V(x)|x|^2) < +\infty$.
It is useful for the sequel to introduce also the weighted Hilbert space

$$L^2_{\frac{1}{|x|}}(\mathbb{R}^N) := \left\{ v : \mathbb{R}^N \to \mathbb{R} \ : \ \frac{v}{|x|} \in L^2(\mathbb{R}^N) \right\}, \tag{5.10}$$

endowed with the scalar product $(u, v) := \int_{\mathbb{R}^N} \frac{u(x)v(x)}{|x|^2} dx$. Note that $D^{1,2}_{rad}(\mathbb{R}^N) \hookrightarrow L^2_{\frac{1}{|x|}}(\mathbb{R}^N)$ continuously by Hardy inequality.
In [13] the precise value of $\widetilde{\beta}^*$ has been computed in any dimension and this will be a crucial step towards the proof of Theorem 1.1. We summarize the results for $\widetilde{\beta}^*$ obtained in [13] in the next theorem.

Theorem 5.4. *For any* $N \geq 3$

$$\widetilde{\beta}^* = -(N-1)$$

and it is achieved at the function

$$\eta^*(x) = \frac{|x|}{\left(1 + \frac{|x|^2}{N(N-2)}\right)^{\frac{N}{2}}},$$

which solves the eigenvalue problem

$$-\Delta \eta(x) - V(x) \eta(x) = \lambda \frac{\eta(x)}{|x|^2} \qquad x \in \mathbb{R}^N \setminus \{0\} \tag{5.11}$$

with eigenvalue

$$\lambda = \widetilde{\beta}^*.$$

Moreover if there exists $\eta \in C^2(\mathbb{R}^N \setminus \{0\}) \cap D^{1,2}_{rad}(\mathbb{R}^N)$, $\eta \geq 0$, $\eta \neq 0$ *radial solution to* (5.11) *with* $\lambda \leq 0$, *then*

$$\lambda = -(N-1), \tag{5.12}$$

namely $\widetilde{\beta}^*$ *is the unique nonpositive radial eigenvalue for problem* (5.11).

Proof. See Section 5 of [13]. □

5.2. **Properties of the eigenfuntion and its scalings.**

For any $m \in \mathbb{N}^+$ and $p \in (1, p_S)$ let us set

$$A_p^{\mathrm{m}} := A_{n_p^{\mathrm{m}}} = \left\{ y \in \mathbb{R}^N \ : \ \frac{1}{n_p^{\mathrm{m}}} < |y| < 1 \right\} \tag{5.13}$$

with n_p^{m} defined in (5.1) and let ϕ_p^{m} be the (radial and positive) solution to (5.6) normalized in such a way that

$$\left\| \frac{\phi_p^{\mathrm{m}}}{|y|} \right\|_{L^2(A_p^{\mathrm{m}})} = 1. \tag{5.14}$$

Lemma 5.5. *For any* $m \in \mathbb{N}^+$, *there exist* $\delta = \delta(m) > 0$ *and* $C > 0$ *(independent of* m*) such that*

$$\sup\{ \|\nabla \phi_p^{\mathrm{m}}\|^2_{L^2(A_p^m)} \ : \ p \in (p_S - \delta, p_S)\} \leq C.$$

Proof. From (5.6) and recalling that, by (3.61), there exists $\delta = \delta(m) > 0$ such that $p|u_p^{\mathrm{m}}(y)|^{p-1}|y|^2 \leq C$, for any $y \in B$ and $p > p_S - \delta$, we have:

$$\begin{aligned}
\int_{A_p^{\mathrm{m}}} |\nabla \phi_p^{\mathrm{m}}(y)|^2 dy &= \int_{A_p^{\mathrm{m}}} p|u_p^{\mathrm{m}}(y)|^{p-1}|y|^2 \frac{\phi_p^{\mathrm{m}}(y)^2}{|y|^2} dy + \widetilde{\beta}_1(m,p) \int_{A_p^{\mathrm{m}}} \frac{\phi_p^{\mathrm{m}}(y)^2}{|y|^2} dy \\
&\leq C \int_{A_p^{\mathrm{m}}} \frac{\phi_p^{\mathrm{m}}(y)^2}{|y|^2} dy + \widetilde{\beta}_1(m,p) \int_{A_p^{\mathrm{m}}} \frac{\phi_p^{\mathrm{m}}(y)^2}{|y|^2} dy \\
&\overset{(5.14)}{=} C + \widetilde{\beta}_1(m,p) \qquad (5.15) \\
&\leq C,
\end{aligned}$$

since $\widetilde{\beta}_1(m,p) < 0$. □

Next result gives a first, still inaccurate, bound from below of $\widetilde{\beta}_1(m,p)$ that will be useful in the sequel.

Lemma 5.6. *For any* $m \in \mathbb{N}^+$, *there exist* $\delta = \delta(m) > 0$ *and* $C > 0$ *(independent of* m*) such that*

$$-C \leq \widetilde{\beta}_1(m,p) \ (< 0), \qquad \textit{for any} \ \ p \in (p_S - \delta, p_S). \tag{5.16}$$

Proof. The proof follows directly from (5.15). □

Let

$$\widehat{A_p^{\mathrm{m}}}^{\,i} := (M_{i,p}^{\mathrm{m}})^{\frac{p-1}{2}} A_p^{\mathrm{m}} = \left\{ y \in \mathbb{R}^N \ : \ \frac{(M_{i,p}^{\mathrm{m}})^{\frac{p-1}{2}}}{n_p^{\mathrm{m}}} < |y| < (M_{i,p}^{\mathrm{m}})^{\frac{p-1}{2}} \right\}, \tag{5.17}$$

for $i=0,\dots,m-1$, where $A_p^{\rm m}$ is as in (5.13) and consider the m scalings of $\phi_p^{\rm m}$, defined by

$$\widehat{\phi_p^{\rm m}}^i(x) := \frac{1}{(M_{i,p}^{\rm m})^{\frac{(p-1)(N-2)}{4}}}\phi_p^{\rm m}\Big(\frac{|x|}{(M_{i,p}^{\rm m})^{\frac{p-1}{2}}}\Big), \quad \text{for } x\in \widehat{A_p^{\rm m}}^i, \quad i=0,\dots,m-1, \tag{5.18}$$

which, by (5.6), satisfy the equations

$$\begin{cases} -\Delta \widehat{\phi_p^{\rm m}}^i - V_{i,p}^{\rm m}(x)\widehat{\phi_p^{\rm m}}^i = \widetilde{\beta}_1(m,p)\dfrac{\widehat{\phi_p^{\rm m}}^i}{|x|^2}, & x\in \widehat{A_p^{\rm m}}^i \\ \widehat{\phi_p^{\rm m}}^i = 0 \quad \text{on } \partial \widehat{A_p^{\rm m}}^i \end{cases} \tag{5.19}$$

where

$$V_{i,p}^{\rm m}(x) := p\frac{1}{(M_{i,p}^{\rm m})^{p-1}}\left|u_p^{\rm m}\Big(\frac{|x|}{(M_{i,p}^{\rm m})^{\frac{p-1}{2}}}\Big)\right|^{p-1}. \tag{5.20}$$

Note that by (3.20), (5.1) and (3.58) we have that

$$\widehat{A_p^{\rm m}}^i \to \mathbb{R}^N\setminus\{0\} \qquad \text{as } p\to p_S, \ \forall\, i=0,\dots,m-1. \tag{5.21}$$

Moreover observe that when $x\in \widetilde{T}_{i,p}^{\rm m}\cap \widehat{A_p^{\rm m}}^i$

$$V_{i,p}^{\rm m}(x) = p|z_{i,p}^{\rm m}(x)|^{p-1}, \ i=0,\dots,m-1 \tag{5.22}$$

where $\widetilde{T}_{i,p}^{\rm m}$ and $z_{i,p}^{\rm m}$ are the rescaled sets and functions defined in (3.31), hence by Theorem 3.7, we have that, as $p\to p_S$:

$$\widetilde{T}_{i,p}^{\rm m}\cap \widehat{A_p^{\rm m}}^i = \left\{\begin{array}{ll} \widehat{A_p^{\rm m}}^0 & \text{if } i=0 \\ \widetilde{T}_{i,p}^{\rm m} & \text{if } i=1,\dots,m-1 \end{array}\right\} \longrightarrow \mathbb{R}^N\setminus\{0\}, \qquad \forall\, i=0,\dots,m-1 \tag{5.23}$$

and also that

$$V_{0,p}^{\rm m} \longrightarrow V \ \text{ in } \ C^0_{loc}(\mathbb{R}^N) \tag{5.24}$$

$$V_{i,p}^{\rm m}\chi_{\widetilde{T}_{i,p}^{\rm m}} \longrightarrow V \ \text{ in } \ C^0_{loc}(\mathbb{R}^N\setminus\{0\}), \qquad \forall\, i=1,\dots,m-1, \tag{5.25}$$

where V is defined in (5.7).

Still denoting by $\widehat{\phi_p^{\rm m}}^i$ the extension to 0 of $\widehat{\phi_p^{\rm m}}^i$ outside of $\widehat{A_p^{\rm m}}^i$, we also have that $\widehat{\phi_p^{\rm m}}^i$ is bounded in $D^{1,2}_{rad}(\mathbb{R}^N)$, indeed:

Lemma 5.7. *For any $m\in\mathbb{N}^+$, there exist $\delta=\delta(m)>0$ and $C>0$ (independent of m) such that*

$$\sup\{\|\nabla \widehat{\phi_p^{\rm m}}^i\|_{L^2(\mathbb{R}^N)} \, : \, p\in(p_S-\delta,p_S)\}\le C. \tag{5.26}$$

Moreover

$$\left\|\frac{\widehat{\phi_p^{\rm m}}^i}{|x|}\right\|_{L^2(\mathbb{R}^N)} = 1. \tag{5.27}$$

Proof. The proof of (5.26) and (5.27) follows directly from the definitions of $\widehat{\phi_p^{\rm m}}^i$. Indeed we have

$$\int_{\mathbb{R}^N}\frac{\widehat{\phi_p^{\rm m}}^i(x)^2}{|x|^2}dx = \int_{A_p^{\rm m}}\frac{\phi_p^{\rm m}(y)^2}{|y|^2}dy \overset{(5.14)}{=} 1$$

and, observing that $\nabla\widehat{\phi_p^{\mathrm{m}}}^i(x) = (M_{i,p}^{\mathrm{m}})^{-\frac{N(p-1)}{4}}\nabla\phi_p^{\mathrm{m}}\Big(\frac{|x|}{(M_{i,p}^{\mathrm{m}})^{\frac{p-1}{2}}}\Big)$, we also get

$$\int_{\mathbb{R}^N}|\nabla\widehat{\phi_p^{\mathrm{m}}}^i(y)|^2dy = \int_{A_p^{\mathrm{m}}}|\nabla\phi_p^{\mathrm{m}}(x)|^2dx \le C \tag{5.28}$$

by Lemma 5.5. □

5.3. An estimate in the first nodal region $B_{0,p}^{\mathrm{m}}$ of u_p^{m}.

In this section, investigating accurately the contribution given by the restriction of u_p^{m} to the *first* nodal region $B_{0,p}^{\mathrm{m}}$ intersected with the annulus A_p^{m} introduced in (5.13), we derive an estimate that will be used later in the proof of Proposition 5.1. More precisely we consider the set

$$F_p^{\mathrm{m}} := A_p^{\mathrm{m}} \cap B_{0,p}^{\mathrm{m}} = \big\{y\in\mathbb{R}^N \;:\; \frac{1}{n_p^{\mathrm{m}}} < |y| < r_{1,p}^{\mathrm{m}}\big\} \overset{(5.1)}{\neq} \emptyset, \tag{5.29}$$

where n_p^{m} is defined in (5.1) and $r_{1,p}^{\mathrm{m}}$ is the first nodal radius of u_p^{m} (see (3.4)) and prove the following:

Lemma 5.8. *Let $m\in\mathbb{N}^+$. For any $\varepsilon>0$ there exists $R_\varepsilon>0$ (independent of m) such that*

$$\lim_{p\to p_S}\int_{\widehat{F_p^{\mathrm{m}}}^0\cap\{|x|>R\}} V_{0,p}^{\mathrm{m}}(x)\,\widehat{\phi_p^{\mathrm{m}}}^0(x)^2\,dx \;\le\; \varepsilon, \qquad \text{for all } R\ge R_\varepsilon,$$

where

$$\widehat{F_p^{\mathrm{m}}}^0 := (M_{0,p}^{\mathrm{m}})^{\frac{p-1}{2}}F_p^{\mathrm{m}}, \tag{5.30}$$

$\widehat{\phi_p^{\mathrm{m}}}^0$ *is as in* (5.18) *and* $V_{0,p}^{\mathrm{m}}$ *satisfies* (5.22).

Proof. We divide the proof into two steps.

STEP 1. We show that for any $R>0$

$$\lim_{p\to p_S}\int_{\widehat{F_p^{m}}^0\cap\{|x|>R\}}|z_{0,p}^m(x)|^{\frac{N}{2}(p-1)}dx \;=\; \int_{\{|x|>R\}}U(x)^{\frac{2N}{N-2}}dx, \tag{5.31}$$

where U is the function in (3.34).

Proof of STEP 1. On one side by the choice of n_p^{m} in (5.1) we have that

$$\int_{\{|y|<\frac{1}{n_p^{\mathrm{m}}}\}}|u_p^{\mathrm{m}}(y)|^{\frac{N}{2}(p-1)}dy \le \omega_N\frac{(M_{0,p}^{\mathrm{m}})^{\frac{N}{2}(p-1)}}{(n_p^{\mathrm{m}})^N} \overset{(5.1)}{\le} \frac{1}{(n_p^{\mathrm{m}})^{\frac{N}{2}}} \underset{p\to p_S}{\longrightarrow} 0, \tag{5.32}$$

so, by the definition of $z_{0,p}^{\mathrm{m}}$ (see (3.31)), by (3.18) and (5.32) we have

$$\begin{aligned}\int_{\widehat{F_p^{\mathrm{m}}}^0}|z_{0,p}^{\mathrm{m}}(x)|^{\frac{N}{2}(p-1)}dx &= \int_{F_p^{\mathrm{m}}}|u_p^{\mathrm{m}}(y)|^{\frac{N}{2}(p-1)}dy\\ &= \int_{B_{0,p}^{\mathrm{m}}}|u_p^{\mathrm{m}}(y)|^{\frac{N}{2}(p-1)}dy - \int_{\{|y|<\frac{1}{n_p^{\mathrm{m}}}\}}|u_p^{\mathrm{m}}(y)|^{\frac{N}{2}(p-1)}dy\\ &\underset{p\to p_S}{\overset{(3.18)+(5.32)}{\longrightarrow}} S_N^{\frac{N}{2}} \overset{(3.36)}{=} \int_{\mathbb{R}^N}U(x)^{\frac{2N}{N-2}}dx.\end{aligned} \tag{5.33}$$

On the other side as $p\to p_S$, since $z_{0,p}^{\mathrm{m}}\to U$ in $C^2_{loc}(\mathbb{R}^N)$, $r_{1,p}^{\mathrm{m}}(M_{0,p}^{\mathrm{m}})^{\frac{p-1}{2}}\to+\infty$ by ($\mathcal{A}_1^{\mathrm{m}}$) (which holds by Proposition 3.8) and (5.32) holds, we deduce

$$\int_{\widehat{F_p^{\mathrm{m}}}^0\cap\{|x|\le R\}}|z_{0,p}^{\mathrm{m}}(x)|^{\frac{N}{2}(p-1)}dx \underset{p\to p_S}{\longrightarrow} \int_{\{|x|\le R\}}U(x)^{\frac{2N}{N-2}}dx, \tag{5.34}$$

for any $R > 0$. Combining (5.33) and (5.34) we get

$$\int_{\widehat{F_p^{\mathrm{m}}}^0 \cap \{|x|>R\}} |z_{0,p}^{\mathrm{m}}(x)|^{\frac{N}{2}(p-1)} dx \underset{p\to p_S}{\longrightarrow} \int_{\{|x|>R\}} U(x)^{\frac{2N}{N-2}} dx$$

STEP 2. End of the proof.

Proof of STEP 2. By using Hölder inequality with exponents $\frac{N}{2}$, $\frac{N}{N-2}$, the Sobolev embedding theorem and Lemma 5.7 we get, for any $R > 0$ and for any $p > p_S - \delta$ (where $\delta = \delta(m)$ as in Lemma 5.7):

$$\begin{aligned}
&\int_{\widehat{F_p^{\mathrm{m}}}^0 \cap \{|x|>R\}} V_{0,p}^{\mathrm{m}}(x)\, \widehat{\phi_p^{\mathrm{m}}}^0(x)^2\, dx \ = \\
&\overset{(5.22)}{=} \int_{\widehat{F_p^{\mathrm{m}}}^0 \cap \{|x|>R\}} p|z_{0,p}^{\mathrm{m}}(x)|^{p-1}\, \widehat{\phi_p^{\mathrm{m}}}^0(x)^2\, dx \\
&\overset{\text{Hölder}}{\leq} p_S \left[\int_{\widehat{F_p^{\mathrm{m}}}^0 \cap \{|x|>R\}} |z_{0,p}^{\mathrm{m}}(x)|^{\frac{N}{2}(p-1)} dx \right]^{\frac{2}{N}} \left\| \widehat{\phi_p^{\mathrm{m}}}^0 \right\|^2_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \\
&\overset{\text{Sobolev}}{\leq} \frac{p_S}{\sqrt{S_N}} \left[\int_{\widehat{F_p^{\mathrm{m}}}^0 \cap \{|x|>R\}} |z_{0,p}^{\mathrm{m}}(x)|^{\frac{N}{2}(p-1)} dx \right]^{\frac{2}{N}} \left\| \nabla \widehat{\phi_p^{\mathrm{m}}}^0 \right\|^2_{L^2(\mathbb{R}^N)} \\
&\overset{\text{Lemma 5.7}}{\leq} C \left[\int_{\widehat{F_p^{\mathrm{m}}}^0 \cap \{|x|>R\}} |z_{0,p}^{\mathrm{m}}(x)|^{\frac{N}{2}(p-1)} dx \right]^{\frac{2}{N}}. \qquad (5.35)
\end{aligned}$$

Let $\varepsilon > 0$ and $R_\varepsilon > 0$ such that

$$\int_{\{|x|>R\}} U(x)^{\frac{2N}{N-2}} dx \leq \frac{\varepsilon}{C} \quad \text{for} \quad R \geq R_\varepsilon. \qquad (5.36)$$

Passing to the limit into (5.35), by *STEP 1* and (5.36) we then have

$$\lim_{p\to p_S} \int_{\widehat{F_p^{\mathrm{m}}}^0 \cap \{|x|>R\}} V_{0,p}^{\mathrm{m}}(x)\, \widehat{\phi_p^{\mathrm{m}}}^0(x)^2 dx \quad \leq \quad \varepsilon \quad \text{for} \quad R \geq R_\varepsilon.$$

□

5.4. Estimates in the remaining nodal regions $\cup_{i=1}^{m-1} B_{i,p}^{\mathrm{m}}$ of u_p^{m}.

Let us consider the radial function f_p^{m} defined in (3.61):

$$f_p^{\mathrm{m}}(y) = |y|^2 |u_p^{\mathrm{m}}(y)|^{p-1}, \qquad y \in B. \qquad (5.37)$$

The next two lemmas provide estimates of f_p^{m} when $|y|$ belongs to suitable subsets of $[r_{1,p}^{\mathrm{m}}, 1]$, where $r_{1,p}^{\mathrm{m}}$ is the *first nodal radius* of u_p^{m} as defined in (3.4).

Lemma 5.9. *Let $m \in \mathbb{N}^+$. For any $\varepsilon > 0$ there exists $\widehat{K_\varepsilon}(= \widehat{K_\varepsilon}(m)) > 1$ such that for any $K \geq \widehat{K_\varepsilon}$, there exists $\delta_{K,\varepsilon}(= \delta_{K,\varepsilon}(m)) > 0$ such that, for any $i = 1, \ldots, m-1$, the set*

$$\emptyset \neq G_{i,p,K}^{\mathrm{m}} := \left\{ y \in \mathbb{R}^N \ : \ r_{i,p}^{\mathrm{m}} < |y| < \frac{1}{K} (M_{i,p}^{\mathrm{m}})^{-\frac{p-1}{2}} \right\} \subset B_{i,p}^{\mathrm{m}}, \quad \text{for } p \geq p_S - \delta_{K,\varepsilon} \qquad (5.38)$$

and

$$\max_{y \in \overline{\cup_{i=1}^{m-1} G_{i,p,K}^{\mathrm{m}}}} f_p^{\mathrm{m}}(y) \leq \varepsilon, \quad \text{for } p \geq p_S - \delta_{K,\varepsilon}. \qquad (5.39)$$

Proof. Let us fix $i \in \{1\dots,m-1\}$. Observe that by the limit properties ($\mathcal{C}_i^{\mathrm{m}}$) and either ($\mathcal{A}_{i+1}^{\mathrm{m}}$) when $i \neq m-1$ or (3.20) when $i=m-1$ (see Proposition 3.8, 3.9 and 3.11 in Section 3), we get

$$r_{i,p}^{\mathrm{m}}(M_{i,p}^{\mathrm{m}})^{\frac{p-1}{2}} \to 0 \quad \text{and} \quad r_{i+1,p}^{\mathrm{m}}(M_{i,p}^{\mathrm{m}})^{\frac{p-1}{2}} \to +\infty, \qquad \text{as } p \to p_S.$$

So for any fixed $K>1$ there exists $\delta_{K,i}(=\delta_{K,i}(m))>0$ such that

$$r_{i,p}^{\mathrm{m}} < \frac{1}{K}(M_{i,p}^{\mathrm{m}})^{-\frac{p-1}{2}} < r_{i+1,p}^{\mathrm{m}}, \qquad \text{for } p \geq p_S - \delta_{K,i}. \tag{5.40}$$

So for $K>1$ and $p \geq p_S-\delta_{K,i}$ it is well defined

$$c_{K,p,i}(=c_{K,p,i}(m)) := \max_{y\in \overline{G_{i,p,K}^{m}}} f_p^{\mathrm{m}}(y).$$

Next we show that for any $\varepsilon>0$ there exists $\widehat{K_{\varepsilon,i}}(=\widehat{K_{\varepsilon,i}}(m))>1$ such that for any $K\geq \widehat{K_{\varepsilon,i}}$, there exists $\delta_{K,i,\varepsilon}(=\delta_{K,i,\varepsilon}(m)) \in (0,\delta_{K,i}]$ such that

$$c_{K,p,i} \leq \varepsilon, \qquad \text{for } \ p \geq p_S - \delta_{K,i,\varepsilon}. \tag{5.41}$$

Arguing by contradiction, we can assume that there exists $\alpha>0$ such that for all $n\in\mathbb{N}$, there exist $K_n(=K_n(m))\geq n$ and $p_n(=p_n(m)) \geq p_S - \delta_{K_n,i}$ such that

$$c_{n,i} := c_{K_n,p_n,i} \geq \alpha^2. \tag{5.42}$$

Since $p_n \geq p_S - \delta_{K_n,i}$, by (5.40) we have that $r_{i,p_n}^{\mathrm{m}} < \frac{1}{K_n}(M_{i,p_n}^{\mathrm{m}})^{-\frac{p_n-1}{2}} < r_{i+1,p_n}^{\mathrm{m}}$. For any $n\in\mathbb{N}$ let $r_n(=r_n(i,m))\in\mathbb{R}$ be the radius such that

$$\begin{cases} r_{i,p_n}^{\mathrm{m}} \leq r_n \leq \frac{1}{K_n}(M_{i,p_n}^{\mathrm{m}})^{-\frac{p_n-1}{2}} \\[2ex] f_{p_n}^{\mathrm{m}}(r_n) = (r_n)^2|u_{p_n}^{\mathrm{m}}(r_n)|^{p_n-1} = c_{n,i}. \end{cases}$$

Then

$$(r_n)^2(M_{i,p_n}^{\mathrm{m}})^{p_n-1} = (r_n)^2|u_{p_n}^{\mathrm{m}}(s_{i,p_n}^{\mathrm{m}})|^{p_n-1} \geq (r_n)^2|u_{p_n}^{\mathrm{m}}(r_n)|^{p_n-1} = c_{n,i} \overset{(5.42)}{\geq} \alpha^2 > 0.$$

On the other side by construction

$$(r_n)^2(M_{i,p_n}^{\mathrm{m}})^{p_n-1} \leq \frac{1}{(K_n)^2} \leq \frac{1}{n^2}, \quad \text{for all } \ n\in\mathbb{N}$$

which gives a contradiction and so proves (5.41).
The conclusion of the proof follows setting

$$\widehat{K_\varepsilon}(m) := \max\{\widehat{K_{\varepsilon,i}}(m),\ i=1,\dots,m-1\}$$
$$\delta_{K,\varepsilon}(m) := \min\{\delta_{K,i,\varepsilon}(m),\ i=1,\dots,m-1\}$$

so by (5.40) we get (5.38), while (5.41) proves (5.39). □

Lemma 5.10. *Let $m\in\mathbb{N}^+$. For any $\varepsilon>0$ there exist $\delta_\varepsilon(=\delta_\varepsilon(m))>0$ and $K_\varepsilon(=K_\varepsilon(m)) \geq \widehat{K_\varepsilon}$ (where $\widehat{K_\varepsilon}>1$ is defined in Lemma 5.9) such that for any $i=1,\dots,m-1$ the set*

$$\emptyset \neq H_{i,p,\varepsilon}^{\mathrm{m}} := \left\{y\in\mathbb{R}^N \ : \ K_\varepsilon(M_{i,p}^{\mathrm{m}})^{-\frac{p-1}{2}} < |y| < r_{i+1,p}^{\mathrm{m}}\right\} \subset B_{i,p}^{\mathrm{m}}, \quad \textit{for } p\geq p_S-\delta_\varepsilon \tag{5.43}$$

and

$$\max_{y\in\bigcup_{i=1}^{m-1}\overline{H_{i,p,\varepsilon}^{\mathrm{m}}}} f_p^{\mathrm{m}}(y) \leq \varepsilon, \qquad \textit{for } p\geq p_S-\delta_\varepsilon. \tag{5.44}$$

Proof. We divide the proof into three steps.

STEP 1. Let $m \in \mathbb{N}^+$, $i \in \{1,\dots,m-1\}$ and define

$$g^m_{p,i}(r) := \frac{(M^{\mathrm{m}}_{i,p})^{p-1} r^2}{\left[1 + \frac{2\alpha}{N(N-2)^2}(M^{\mathrm{m}}_{i,p})^{p-1} r^2\right]^{\frac{N-2}{2}(p-1)}},$$

where $\alpha \in (0, \frac{N-2}{2})$ is fixed. We show that there exists $\widehat{K} > 0$ (independent of i and m) and $\widehat{\delta}_i (= \widehat{\delta}_i(m)) > 0$ such that:

$$\widehat{K}(M^{\mathrm{m}}_{i,p})^{-\frac{p-1}{2}} < r^{\mathrm{m}}_{i+1,p}, \quad \text{if } p \geq p_S - \widehat{\delta}_i$$

and the function $g^m_{p,i}$ is monotone decreasing in $[\widehat{K}(M^{\mathrm{m}}_{i,p})^{-\frac{p-1}{2}}, r^{\mathrm{m}}_{i+1,p}]$, for any $p \geq p_S - \widehat{\delta}_i$.

Proof of STEP 1. Let $\widehat{K} := 2\left[\frac{N(N-2)^2}{2\alpha}\right]^{\frac{1}{2}} (> 0)$. Since, by (3.20) for $i = m-1$ and property $(\mathcal{A}^{\mathrm{m}}_{i+1})$ (which holds true by Proposition 3.8) for $i \neq m-1$, we have that

$$r^{\mathrm{m}}_{i+1,p}(M^{\mathrm{m}}_{i,p})^{\frac{p-1}{2}} \to +\infty \quad \text{as} \quad p \to p_S,$$

then there exists $\delta_{\widehat{K},i} (= \delta_{\widehat{K},i}(m)) > 0$ such that

$$\widehat{K}(M^{\mathrm{m}}_{i,p})^{-\frac{p-1}{2}} < r^{\mathrm{m}}_{i+1,p}, \quad \text{if } p \geq p_S - \delta_{\widehat{K},i}.$$

Moreover by easy computations

$$(g^{\mathrm{m}}_{p,i})'(r) = \frac{2(M^{\mathrm{m}}_{i,p})^{p-1} r}{\left[1 + \frac{2\alpha}{N(N-2)^2}(M^{\mathrm{m}}_{i,p})^{p-1} r^2\right]^{\frac{(N-2)}{2}(p-1)+1}} \left[1 - \frac{[(p-1)(N-2)-2]\,\alpha (M^{\mathrm{m}}_{i,p})^{p-1}}{N(N-2)^2} r^2\right]$$

hence $(g^{\mathrm{m}}_{p,i})'(r) \leq 0$ if and only if

$$r \geq \left[\frac{N(N-2)^2}{[(p-1)(N-2)-2]\,\alpha}\right]^{\frac{1}{2}} (M^{\mathrm{m}}_{i,p})^{-\frac{p-1}{2}}.$$

Since by our choice of $\widehat{K}$ we have

$$\left[\frac{N(N-2)^2}{[(p-1)(N-2)-2]\,\alpha}\right]^{\frac{1}{2}} \longrightarrow \frac{\widehat{K}}{2} \quad \text{as} \quad p \to p_S,$$

there exists $\widetilde{\delta} > 0$ such that if $p > p_S - \widetilde{\delta}$ then $(g^{\mathrm{m}}_{p,i})'(r) \leq 0$ for $r \geq \widehat{K}(M^{\mathrm{m}}_{i,p})^{-\frac{p-1}{2}}$. To conclude the proof of *STEP 1* it is enough to take $\widehat{\delta}_i (= \widehat{\delta}_i(m)) := \min\{\delta_{\widehat{K},i}(m), \widetilde{\delta}\}$.

STEP 2. Let $m \in \mathbb{N}^+$. Let us fix $i \in \{1,\dots,m-1\}$ and $\varepsilon > 0$. We show that there exist $\delta_{\varepsilon,i} (= \delta_{\varepsilon,i}(m)) > 0$ and $K_\varepsilon (= K_\varepsilon(m)) \geq \widehat{K_\varepsilon}$ (where $\widehat{K_\varepsilon} > 1$ is defined in Lemma 5.9) such that

$$\emptyset \neq H^{\mathrm{m}}_{i,p,\varepsilon} \subset B^{\mathrm{m}}_{i,p}, \qquad \textit{for } p \geq p_S - \delta_{\varepsilon,i} \tag{5.45}$$

and

$$\max_{y \in \overline{H^{\mathrm{m}}_{i,p,\varepsilon}}} f^{\mathrm{m}}_p(y) \leq \varepsilon, \qquad \textit{for } p \geq p_S - \delta_{\varepsilon,i}. \tag{5.46}$$

Proof of STEP 2. By Proposition 3.6 and Corollary 3.12 we know that there exist $\gamma = \gamma(\alpha, m) \in (0,1)$, $\gamma(\alpha, m) \to 1$ as $\alpha \to 0$ and $\delta_i = \delta_i(\alpha, m) > 0$ such that

$$f_p^{\mathrm{m}}(r) \leq g_{p,i}^{\mathrm{m}}(r), \qquad \text{for } \ r \in (\gamma^{-\frac{1}{N}} s_{i,p}^{\mathrm{m}}, r_{i+1,p}^{\mathrm{m}}], \ p \geq p_S - \delta_i \tag{5.47}$$

Observe that by property ($\mathcal{B}_i^{\mathrm{m}}$) (which holds true by Propositions 3.9 and 3.11)

$$s_{i,p}^{\mathrm{m}}(M_{i,p}^{\mathrm{m}})^{\frac{p-1}{2}} \to 0, \quad \text{ as } \ p \to p_S,$$

so there exists $\widetilde{\delta}_{\widehat{K},i}(m) > 0$ such that

$$\gamma^{-\frac{1}{N}} s_{i,p}^{\mathrm{m}} < \widehat{K}(M_{i,p}^{\mathrm{m}})^{-\frac{p-1}{2}}, \quad \text{for } p \geq p_S - \widetilde{\delta}_{\widehat{K},i}(m) \tag{5.48}$$

where $\widehat{K}$ is the number obtained in *STEP 1*.
Observe also that since, by (3.20) for $i = m-1$ and property ($\mathcal{A}_{i+1}^{\mathrm{m}}$) (which holds true by Proposition 3.8) for $i \neq m-1$, we have that

$$r_{i+1,p}^{\mathrm{m}}(M_{i,p}^{\mathrm{m}})^{\frac{p-1}{2}} \to +\infty \ \text{ as } \ p \to p_S,$$

then for any $K \geq \widehat{K}$ there exists $\delta_{K,i}(m) > 0$ such that

$$\widehat{K}(M_{i,p}^{\mathrm{m}})^{-\frac{p-1}{2}} \leq K(M_{i,p}^{\mathrm{m}})^{-\frac{p-1}{2}} < r_{i+1,p}^{\mathrm{m}}, \quad \text{for } p \geq p_S - \delta_{K,i}(m). \tag{5.49}$$

By *STEP 1*, (5.47), (5.48) and (5.49) we have that for any $K \geq \widehat{K}$ and for $p \geq p_S - \min\{\delta_i(\alpha, m), \widetilde{\delta}_{\widehat{K},i}(m), \delta_{K,i}(m), \widehat{\delta}_i\}$ (where $\widehat{\delta}_i (= \widehat{\delta}_i(m))$ is the one in *STEP 1*)

$$f_p^{\mathrm{m}}(r) \leq g_{p,i}^{\mathrm{m}}(r) \leq g_{p,i}^{\mathrm{m}}(K(M_{i,p}^{\mathrm{m}})^{-\frac{p-1}{2}}) \qquad \text{for } r \in (K(M_{i,p}^{\mathrm{m}})^{-\frac{p-1}{2}}, r_{i+1,p}^{\mathrm{m}}]. \tag{5.50}$$

Moreover if $p > p_S - \frac{2}{N-2}$ then $\frac{N-2}{2}(p-1) > 1$, so

$$g_{p,i}^{\mathrm{m}}(K(M_{i,p}^{\mathrm{m}})^{-\frac{p-1}{2}}) = \frac{K^2}{\left[1 + \frac{2\alpha}{N(N-2)^2}K^2\right]^{\frac{N-2}{2}(p-1)}} \longrightarrow 0 \qquad \text{as } \ K \to +\infty \tag{5.51}$$

The conclusion follows combining (5.51) with (5.50).

STEP 3. Conclusion.

Proof of STEP 3. The proof follows by *STEP 2.* taking

$$\delta_\varepsilon (= \delta_\varepsilon(m)) := \min\{\delta_{\varepsilon,i}(m), \ i = 1, \dots, m-1\}.$$

□

5.5. **Proof of Proposition 5.1.**

Proof. Arguing by contradiction let us assume that (5.4) does not hold. Then there exist $\varepsilon > 0$ and a sequence $p_j \to p_S$, as $j \to +\infty$, such that

$$\widetilde{\beta}_1(m, p_j) \to -(N-1) - 10\,\varepsilon, \quad \text{as } j \to +\infty. \tag{5.52}$$

Corresponding to this number $\varepsilon > 0$ we can take $K_\varepsilon > 1$ as in Lemma 5.10. Then we consider the $m-1$ scalings $\widehat{\phi_{p_j}^{\mathrm{m}}}^{i}$, $i = 1, \dots, m-1$, defined in (5.18) and observe that, by (5.27)

$$\liminf_{j \to +\infty} \int_{\{|x| \in [\frac{1}{K_\varepsilon}, K_\varepsilon]\}} \frac{\widehat{\phi_{p_j}^{\mathrm{m}}}^{i}(x)^2}{|x|^2} dx \quad \in [0,1], \qquad \forall\, i = 1, \dots, m-1.$$

Hence there exists a subsequence, that we still denote by p_j, for which one of the following two statements holds:

CASE 1. There exists $\alpha_\varepsilon \in (0,1]$ and $\kappa \in \{1,\dots,m-1\}$ such that:

$$\int_{\{|x|\in[\frac{1}{K_\varepsilon},K_\varepsilon]\}} \frac{\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}(x)^2}{|x|^2}dx \geq \alpha_\varepsilon, \quad \forall\, j \in \mathbb{N}. \tag{5.53}$$

CASE 2.

$$\int_{\{|x|\in[\frac{1}{K_\varepsilon},K_\varepsilon]\}} \frac{\widehat{\phi^{\mathrm{m}}_{p_j}}^{i}(x)^2}{|x|^2}dx \longrightarrow 0 \quad \text{as } \; j \to +\infty, \qquad \forall\, i = 1,\dots,m-1. \tag{5.54}$$

In *CASE 1* we will prove that

$$\widetilde{\beta}_1(m,p_j) \to -(N-1), \qquad \text{as } \; j \to +\infty, \tag{5.55}$$

which contradicts (5.52).
In *CASE 2* we will show that there exists $j_\varepsilon \in \mathbb{N}$ such that

$$\widetilde{\beta}_1(m,p_j) \geq -(N-1) - 9\,\varepsilon, \quad \text{for any } j \geq j_\varepsilon, \tag{5.56}$$

which also contradicts (5.52). So the assertion (5.4) will be proved.

Proof in CASE 1.

We will pass to the limit as $j \to +\infty$ into the equation (5.19) satisfied by the scaling $\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}$. Since (5.23) implies that, for any fixed $\rho \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, $\mathrm{supp}(\rho) \subset (\widetilde{T}^{\mathrm{m}}_{\kappa,p_j} \cap \widehat{A^{\mathrm{m}}_{p_j}}^{\kappa})$ for j sufficiently large, by (5.19) we have

$$\int_{\mathbb{R}^N\setminus\{0\}} \nabla\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}\nabla\rho\, dx - \int_{\mathbb{R}^N\setminus\{0\}} V^{\mathrm{m}}_{\kappa,p_j}(x)\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}\rho\, dx - \widetilde{\beta_1}(m,p_j)\int_{\mathbb{R}^N\setminus\{0\}} \frac{\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}\rho}{|x|^2}\, dx = 0, \tag{5.57}$$

where in particular $V^{\mathrm{m}}_{\kappa,p_j}$ satisfies (5.22).
By Lemma 5.7 we know that $\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}$ is bounded in the reflexive space $D^{1,2}_{rad}(\mathbb{R}^N)$, hence there exists $\widehat{\phi} = \widehat{\phi}^{\mathrm{m}}_{\kappa} \in D^{1,2}_{rad}(\mathbb{R}^N)$ such that up to a subsequence

$$\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa} \rightharpoonup \widehat{\phi} \qquad \text{in } D^{1,2}_{rad}(\mathbb{R}^N) \quad \text{ as } j \to +\infty \tag{5.58}$$

and so, by the continuous embedding $D^{1,2}_{rad}(\mathbb{R}^N) \hookrightarrow L^2_{\frac{1}{|x|}}(\mathbb{R}^N)$ (defined in (5.10)), we also have

$$\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa} \rightharpoonup \widehat{\phi} \qquad \text{in } L^2_{\frac{1}{|x|}}(\mathbb{R}^N) \quad \text{ as } j \to +\infty. \tag{5.59}$$

Moreover, for any bounded set $M \subset \mathbb{R}^N$, by the compact embedding $H^1(M) \hookrightarrow L^2(M)$ we have

$$\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa} \to \widehat{\phi} \qquad \text{in } L^2(M) \quad \text{ as } j \to +\infty \tag{5.60}$$

and so also

$$\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa} \to \widehat{\phi} \qquad a.e. \text{ in } \mathbb{R}^N \quad \text{ as } j \to +\infty. \tag{5.61}$$

Observe that by (5.61) $\widehat{\phi} \geq 0$. Next we show that

$$\widehat{\phi} \not\equiv 0. \tag{5.62}$$

Indeed by assumption (5.53)

$$\int_{\{|x|\in[\frac{1}{K_\varepsilon},K_\varepsilon]\}} \frac{\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}(x)^2}{|x|^2}dx \geq \alpha_\varepsilon > 0, \quad \text{ for any } j \in \mathbb{N}. \tag{5.63}$$

Hence taking $M=\{x\in\mathbb{R}^N\,:\,|x|\in[\frac{1}{K_\varepsilon},K_\varepsilon]\}$, by (5.60) we have, as $j\to+\infty$, that

$$\int_{\{|x|\in[\frac{1}{K_\varepsilon},K_\varepsilon]\}}\frac{\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}(x)^2}{|x|^2}dx\leq K_\varepsilon^2\int_{\{|x|\in[\frac{1}{K_\varepsilon},K_\varepsilon]\}}\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}(x)^2dx\longrightarrow K_\varepsilon^2\int_{\{|x|\in[\frac{1}{K_\varepsilon},K_\varepsilon]\}}\widehat{\phi}(x)^2dx.$$

Combining this with (5.63) we get

$$\int_{\{|x|\in[\frac{1}{K_\varepsilon},K_\varepsilon]\}}\widehat{\phi}(x)^2dx\geq\frac{\alpha_\varepsilon}{K_\varepsilon^2}>0,$$

thus proving (5.62).
We pass to the limit as $j\to+\infty$ into (5.57) as follows. By Lemma 5.6 there exists $\widetilde{\beta}_1^{\mathrm{m}}\leq 0$ such that up to a subsequence

$$\widetilde{\beta}_1(m,p_j)\to\widetilde{\beta}_1^{\mathrm{m}}\qquad\text{as } j\to+\infty,\tag{5.64}$$

by (5.58)

$$\int_{\mathbb{R}^N\setminus\{0\}}\nabla\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}\nabla\rho\;dx\;\to\int_{\mathbb{R}^N\setminus\{0\}}\nabla\widehat{\phi}\,\nabla\rho\;dx\quad\text{as }\;j\to+\infty,$$

by (5.59)

$$\int_{\mathbb{R}^N\setminus\{0\}}\frac{\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}\rho}{|x|^2}dx\to\int_{\mathbb{R}^N\setminus\{0\}}\frac{\widehat{\phi}\,\rho}{|x|^2}dx\quad\text{as }\;j\to+\infty,\tag{5.65}$$

for any test function ρ as in (5.57). Finally we show that

$$\int_{\mathbb{R}^N\setminus\{0\}}V^{\mathrm{m}}_{\kappa,p_j}(x)\,\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}\rho\;dx\;\to\int_{\mathbb{R}^N\setminus\{0\}}V(x)\,\widehat{\phi}\,\rho\;dx\quad\text{as }\;j\to+\infty,$$

where $V(x)$ is the potential defined in (5.7). Indeed:

$$\begin{aligned}&\left|\int_{\mathbb{R}^N\setminus\{0\}}V^{\mathrm{m}}_{\kappa,p_j}(x)\,\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}\rho\;dx\;-\int_{\mathbb{R}^N\setminus\{0\}}V(x)\,\widehat{\phi}\,\rho\;dx\right|\leq\\&\leq\sup_{\mathrm{supp}(\rho)}\left(|x|^2|V^{\mathrm{m}}_{\kappa,p_j}(x)-V(x)|\right)\int_{\mathbb{R}^N\setminus\{0\}}\frac{\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}|\rho|}{|x|^2}\;dx\;+\\&\qquad\qquad+\left|\int_{\mathbb{R}^N\setminus\{0\}}\frac{(\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}-\widehat{\phi})\overbrace{|x|^2V(x)\rho(x)}^{:=\widetilde{\rho}(x)}}{|x|^2}dx\right|\\&\leq\sup_{\mathrm{supp}(\rho)}\left(|x|^2|V^{\mathrm{m}}_{\kappa,p_j}(x)-V(x)|\right)\;C_\rho\;\left\|\frac{\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}}{|x|}\right\|_{L^2(\mathbb{R}^N)}+\left|\int_{\mathbb{R}^N\setminus\{0\}}\frac{(\widehat{\phi^{\mathrm{m}}_{p_j}}^{\kappa}-\widehat{\phi})\widetilde{\rho}}{|x|^2}dx\right|\\&\longrightarrow 0\qquad\text{as } j\to+\infty,\end{aligned}$$

where for the first term we have used (5.27) and the convergence result in (5.25) (observe that $supp(\rho)\subset(\widetilde{T}^{\mathrm{m}}_{\kappa,p_j}\cap\widehat{A^{\mathrm{m}}_{p_j}}^{\kappa})$ and so $V^{\mathrm{m}}_{\kappa,p_j}$ satisfies (5.22)) while for the second term the convergence follows from (5.65) since $\widetilde{\rho}:=\rho|x|^2V(x)\in C_0^\infty(\mathbb{R}^N\setminus\{0\})$.
As a consequence by passing to the limit into (5.57) we get

$$\int_{\mathbb{R}^N\setminus\{0\}}\nabla\widehat{\phi}\,\nabla\rho\;dx-\int_{\mathbb{R}^N\setminus\{0\}}V(x)\,\widehat{\phi}\,\rho\;dx-\widetilde{\beta}_1^{\mathrm{m}}\int_{\mathbb{R}^N\setminus\{0\}}\frac{\widehat{\phi}\,\rho}{|x|^2}\;dx=0,\tag{5.66}$$

for any $\rho \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, namely $\widehat{\phi}$ is (a weak and so classical) nontrivial non-negative solution to the limit equation

$$-\widehat{\phi}'' - \frac{N-1}{s}\widehat{\phi}' - V(s)\widehat{\phi} = \widetilde{\beta}_1^{\mathrm{m}} \frac{\widehat{\phi}}{s^2} \qquad s \in (0, +\infty). \tag{5.67}$$

where $\widetilde{\beta}_1^{\mathrm{m}}$ satisfies (5.64).
By Theorem 5.4 (see (5.12)) it follows that $\widetilde{\beta}_1^{\mathrm{m}} = -(N-1)$ namely, up to a subsequence

$$\widetilde{\beta}_1(m, p_j) \to -(N-1) \quad \text{as} \quad j \to +\infty,$$

thus obtaining (5.55).

Proof in CASE 2.

Let $\widetilde{\beta}^*$ be as in (5.8), then by Theorem 5.4 we know that $\widetilde{\beta}^* = -(N-1)$ and so, taking $\widehat{\phi_{p_j}^{\mathrm{m}}}^0$ as in (5.18), we have

$$\begin{aligned} -(N-1) \overset{\text{Theorem 5.4}}{=} \widetilde{\beta}^* &\overset{(5.8)+(5.27)}{\le} \int_{\mathbb{R}^N} \left(|\nabla \widehat{\phi_{p_j}^{\mathrm{m}}}^0(x)|^2 - V(x) \widehat{\phi_{p_j}^{\mathrm{m}}}^0(x)^2 \right) dx \\ &\overset{(5.19)}{=} \widetilde{\beta}_1(m, p_j) + \int_{\widehat{A_{p_j}^{\mathrm{m}}}^0} \left[V_{0,p_j}^{\mathrm{m}}(x) - V(x) \right] \widehat{\phi_{p_j}^{\mathrm{m}}}^0(x)^2 dx, \end{aligned} \tag{5.68}$$

where the set $\widehat{A_{p_j}^{\mathrm{m}}}^0$ is defined in (5.17), V_{0,p_j}^{m} satisfies (5.22) in $\widehat{A_{p_j}^{\mathrm{m}}}^0$ and V is as in (5.7).

Next we estimate the term $\int_{\widehat{A_{p_j}^{\mathrm{m}}}^0} \left[V_{0,p_j}^{\mathrm{m}}(x) - V(x) \right] \widehat{\phi_{p_j}^{\mathrm{m}}}^0(x)^2 dx$. As before $\varepsilon > 0$ is fixed as in (5.52). Let R_ε be as in Lemma 5.8 and fix $R > 0$ such that

$$R \ge \max\{1, R_\varepsilon, N(N-2), \frac{N\sqrt{(N+2)(N-2)}}{\sqrt{\varepsilon}}\}. \tag{5.69}$$

We have

$$\begin{aligned} \int_{\widehat{A_{p_j}^{\mathrm{m}}}^0} \left[V_{0,p_j}^{\mathrm{m}}(x) - V(x) \right] \widehat{\phi_{p_j}^{\mathrm{m}}}^0(x)^2 dx \quad \le \quad & \int_{\widehat{A_{p_j}^{\mathrm{m}}}^0 \cap \{|x| \le R\}} \left| V_{0,p_j}^{\mathrm{m}}(x) - V(x) \right| \widehat{\phi_{p_j}^{\mathrm{m}}}^0(x)^2 dx \\ & + \int_{\widehat{A_{p_j}^{\mathrm{m}}}^0 \cap \{|x| > R\}} V(x) \widehat{\phi_{p_j}^{\mathrm{m}}}^0(x)^2 dx \\ & + \int_{\widehat{F_{p_j}^{\mathrm{m}}}^0 \cap \{|x| > R\}} V_{0,p_j}^{\mathrm{m}}(x) \widehat{\phi_{p_j}^{\mathrm{m}}}^0(x)^2 dx \\ & + \int_{\widehat{T_{p_j}^{\mathrm{m}}}^0 \cap \{|x| > R\}} V_{0,p_j}^{\mathrm{m}}(x) \widehat{\phi_{p_j}^{\mathrm{m}}}^0(x)^2 dx \\ = \quad & I_j \;+\; II_j \;+\; III_j \;+\; IV_j, \end{aligned}$$

where the set $\widehat{F_{p_j}^{\mathrm{m}}}^0$ is as in (5.30) while the set $\widehat{T_{p_j}^{\mathrm{m}}}^0$ is the scaling of the remaining set $A_{p_j}^{\mathrm{m}} \setminus B_{0,p_j}^{\mathrm{m}}$ with respect to the same scaling parameter M_{0,p_j}^{m}. Namely

$$\widehat{T_{p_j}^{\mathrm{m}}}^0 := (M_{0,p_j}^{\mathrm{m}})^{\frac{p_j-1}{2}} \left(A_{p_j}^{\mathrm{m}} \setminus B_{0,p_j}^{\mathrm{m}} \right).$$

Then

$$I_j = \int_{\widehat{A_{p_j}^{\mathrm{m}}}^0 \cap \{|x| \le R\}} \left| V_{0,p_j}^{\mathrm{m}}(x) - V(x) \right| |x|^2 \frac{\widehat{\phi_{p_j}^{\mathrm{m}}}^0(x)^2}{|x|^2} dx$$

$$
\begin{aligned}
&\le \sup_{B_R(0)} \left| V^{\mathrm{m}}_{0,p_j}(x) - V(x) \right| R^2 \int_{\mathbb{R}^N} \frac{\widehat{\phi^{\mathrm{m}}_{p_j}}^0(x)^2}{|x|^2} dx \\
&\overset{(5.27)}{=} \sup_{B_R(0)} \left| V^{\mathrm{m}}_{0,p_j}(x) - V(x) \right| R^2 \overset{(5.24)}{\le} \varepsilon
\end{aligned}
$$

for j sufficiently large.
Observe that the radial function $|x| \mapsto V(x)|x|^2 \to 0$ has a unique maximum for $|x| = N(N-2)$, hence by our choice of R in (5.69)

$$
\sup_{\{|x|>R\}} (V(x)|x|^2) \overset{(5.69)}{\le} V(R)R^2 \le \frac{N^2(N+2)(N-2)}{R^2} \overset{(5.69)}{\le} \varepsilon
$$

and so, for any $j \in \mathbb{N}$:

$$
\begin{aligned}
II_j &= \int_{\widehat{A^{\mathrm{m}}_{p_j}}^0 \cap \{|x|>R\}} V(x)|x|^2 \frac{\widehat{\phi^{\mathrm{m}}_{p_j}}^0(x)^2}{|x|^2} dx \\
&\le \sup_{\{|x|>R\}} (V(x)|x|^2) \int_{\widehat{A^{\mathrm{m}}_{p_j}}^0 \cap \{|x|>R\}} \frac{\widehat{\phi^{\mathrm{m}}_{p_j}}^0(x)^2}{|x|^2} dx \\
&\le \varepsilon \int_{\mathbb{R}^N} \frac{\widehat{\phi^{\mathrm{m}}_{p_j}}^0(x)^2}{|x|^2} dx \\
&\overset{(5.27)}{=} \varepsilon.
\end{aligned}
$$

By our choice of R in (5.69) we may also apply Lemma 5.8 getting, for j large enough:

$$
III_j = \int_{\widehat{F^{\mathrm{m}}_{p_j}}^0 \cap \{|x|>R\}} V^{\mathrm{m}}_{0,p_j}(x) \widehat{\phi^{\mathrm{m}}_{p_j}}^0(x)^2 dx \le \varepsilon
$$

In order to estimate the term IV_j we need all the results about the function $f^{\mathrm{m}}_{p_j}$ defined in (5.37). To this purpose let us observe that the number K_ε in (5.54) has been chosen so that both Lemma 5.9 and Lemma 5.10 hold. Moreover since

$$
A^{\mathrm{m}}_{p_j} \setminus B^{\mathrm{m}}_{0,p_j} = \cup_{i=1}^{m-1} B^{\mathrm{m}}_{i,p_j} = \left\{ y \in \mathbb{R}^N : \ r^{\mathrm{m}}_{1,p_j} < |y| < 1 \right\}, \tag{5.70}
$$

it follows that

$$
\widehat{T^{\mathrm{m}}_{p_j}}^0 = (M^{\mathrm{m}}_{0,p_j})^{\frac{p_j-1}{2}} \left(\cup_{i=1}^{m-1} B^{\mathrm{m}}_{i,p_j} \right) = \left\{ x \in \mathbb{R}^N : \ r^{\mathrm{m}}_{1,p_j} (M^{\mathrm{m}}_{0,p_j})^{\frac{p_j-1}{2}} < |x| < (M^{\mathrm{m}}_{0,p_j})^{\frac{p_j-1}{2}} \right\}
$$

where by the property $(\mathcal{A}^{\mathrm{m}}_1)$ (which holds true by Proposition 3.8) one has

$$
r^{\mathrm{m}}_{1,p_j} (M^{\mathrm{m}}_{0,p_j})^{\frac{p_j-1}{2}} > R, \quad \text{for } j \text{ sufficiently large.} \tag{5.71}
$$

As a consequence, for j sufficiently large, we have:

$$
\begin{aligned}
IV_j \quad &= \quad \int_{\widehat{T^{\mathrm{m}}_{p_j}}^0 \cap \{|x|>R\}} V^{\mathrm{m}}_{0,p_j}(x) \widehat{\phi^{\mathrm{m}}_{p_j}}^0(x)^2 dx \\
&\overset{(5.71)}{=} \int_{\widehat{T^{\mathrm{m}}_{p_j}}^0} V^{\mathrm{m}}_{0,p_j}(x) \widehat{\phi^{\mathrm{m}}_{p_j}}^0(x)^2 dx \\
&\overset{(5.37)}{=} p_j \int_{A^{\mathrm{m}}_{p_j} \setminus B^{\mathrm{m}}_{0,p_j}} f^{\mathrm{m}}_{p_j}(y) \frac{\phi_{p_j}(y)^2}{|y|^2} dy \\
&\overset{(5.70)}{=} p_j \sum_{i=1}^{m-1} \int_{B^{\mathrm{m}}_{i,p_j}} f^{\mathrm{m}}_{p_j}(y) \frac{\phi_{p_j}(y)^2}{|y|^2} dy.
\end{aligned} \tag{5.72}
$$

Let K_ε be as in Lemma 5.10 and let us define the sets $G^{\mathrm{m}}_{i,p_j,\varepsilon} := G^{\mathrm{m}}_{i,p_j,K}$ with $K = K_\varepsilon$, $i = 1\ldots, m-1$, where $G^{\mathrm{m}}_{i,p_j,K}$ is as in (5.38). Let us also consider the set $H^{\mathrm{m}}_{i,p_j,\varepsilon}$, $i = 1\ldots, m-1$, introduced in (5.43), by Lemma 5.9 and 5.10

$$\emptyset \neq (G^{\mathrm{m}}_{i,p_j,\varepsilon} \cup H^{\mathrm{m}}_{i,p_j,\varepsilon}) \subset B^{\mathrm{m}}_{i,p_j}.$$

From (5.72), for j sufficiently large, it then follows

$$\begin{aligned}
IV_j \quad &= \quad p_j \sum_{i=1}^{m-1} \int_{G^{\mathrm{m}}_{i,p_j,K_\varepsilon} \cup H^{\mathrm{m}}_{i,p_j,K_\varepsilon}} f^{\mathrm{m}}_{p_j}(y) \frac{\phi_{p_j}(y)^2}{|y|^2} dy \\
&\quad + p_j \sum_{i=1}^{m-1} \int_{\left\{\frac{1}{K_\varepsilon}(M^{\mathrm{m}}_{i,p_j})^{-\frac{p_j-1}{2}} \leq |y| \leq K_\varepsilon (M^{\mathrm{m}}_{i,p_j})^{-\frac{p_j-1}{2}}\right\}} f^{\mathrm{m}}_{p_j}(y) \frac{\phi_{p_j}(y)^2}{|y|^2} dy \\
&\overset{(5.14)+(3.61)}{\leq} \quad p_S \max_{y \in \bigcup_{i=1}^{m-1} (\overline{G^{\mathrm{m}}_{i,p_j,K_\varepsilon}} \cup \overline{H^{\mathrm{m}}_{i,p_j,K_\varepsilon}})} f^{\mathrm{m}}_{p_j}(y) \\
&\quad + p_S\, C \sum_{i=1}^{m-1} \int_{\left\{\frac{1}{K_\varepsilon}(M^{\mathrm{m}}_{i,p_j})^{-\frac{p_j-1}{2}} \leq |y| \leq K_\varepsilon (M^{\mathrm{m}}_{i,p_j})^{-\frac{p_j-1}{2}}\right\}} \frac{\phi_{p_j}(y)^2}{|y|^2} dy \\
&\overset{(*)}{\leq} \quad 5\,\varepsilon \;+\; 5\,C \sum_{i=1}^{m-1} \int_{\{|x| \in [\frac{1}{K_\varepsilon}, K_\varepsilon]\}} \frac{\widehat{\phi^{\mathrm{m}}_{p_j}}^{\,i}(x)^2}{|x|^2} dx,
\end{aligned}$$

where in $(*)$ we have used that $p_S \leq 5$ for any $N \geq 3$, we have estimated the first term by Lemma 5.9 and 5.10 and we have rescaled the second term. By collecting the estimates in I_j, II_j, III_j and IV_j we then have, for j sufficiently large:

$$\begin{aligned}
\int_{\widehat{A^+_{p_j}}} \left[V^{\mathrm{m}}_{0,p_j}(x) - V(x)\right] \widehat{\phi^{\mathrm{m}}_{p_j}}^{\,0}(x)^2 dx \quad &\leq \quad 8\,\varepsilon + 5\,C \sum_{i=1}^{m-1} \int_{\{|x| \in [\frac{1}{K_\varepsilon}, K_\varepsilon]\}} \frac{\widehat{\phi^{\mathrm{m}}_{p_j}}^{\,i}(x)^2}{|x|^2} dx \\
&\leq \quad 9\,\varepsilon.
\end{aligned}$$

where the last inequality follows by the assumption (5.54). Combining this result with (5.68) we have then proved that there exists $j_\varepsilon \in \mathbb{N}$ such that:

$$\widetilde{\beta}_1(m, p_j) \geq -(N-1) - 9\,\varepsilon\,, \qquad \text{for} \;\; j \geq j_\varepsilon,$$

namely we have obtained (5.56). □

Remark 5.11. *We stress that Proposition 5.1 does not hold in dimension* $N = 2$*, when* $p \to +\infty$*. Indeed in the* 2*-dimensional case and when* $m = 2$ *it is proved in* [13, Theorem 6.1] *that* $\lim_{p\to+\infty} \widetilde{\beta}_1(2,p) = -\frac{\ell^2+2}{2} < -1$*, for a number* $\ell > 0$ *which is explicitly computed.*

6. Proof of Theorem 1.1

Proof. Let u^{m}_p be a solution of (1.1) with $m \in \mathbb{N}^+$ nodal regions and $p \in (1, p_S)$. As explained in Section 5 we approximate the ball B by the annulus A_n choosing $n = n^{\mathrm{m}}_p$, where n^{m}_p is defined in (5.1), and we consider the radial weighted linear operators $\widetilde{L}^{\mathrm{m}}_{p,rad}$ defined in (5.3). The eigenvalues of $\widetilde{L}^{\mathrm{m}}_{p,rad}$, as in (5.2), are

$$\widetilde{\beta}_i(m,p), \quad \text{for any } i \in \mathbb{N}^+.$$

We also set $\widetilde{L}_p^{\mathrm{m}} := \widetilde{L_p^n}^{\mathrm{m}}$ for $n = n_p^{\mathrm{m}}$, where $\widetilde{L_p^n}^{\mathrm{m}}$ is the weighted operator defined in (4.7), whose eigenvalues we denote by

$$\widetilde{\mu}_i(m,p) := \widetilde{\mu_i^n}(m,p), \quad \text{for } n = n_p^{\mathrm{m}}, \quad \text{for any } i \in \mathbb{N}^+.$$

The number of negative eigenvalues of $\widetilde{L}_p^{\mathrm{m}}$ is then

$$\widetilde{k}_p(m) := \widetilde{k_p^n}(m), \quad \text{for } n = n_p^{\mathrm{m}},$$

where $\widetilde{k_p^n}(m)$ is as in (4.9).

By Proposition 4.3 to determine the Morse index $\mathsf{m}(u_p^{\mathrm{m}})$ is equivalent to counting the number $\widetilde{k}_p(m)$ of negative eigenvalues $\widetilde{\mu}_i(m,p)$ of the operator $\widetilde{L}_p^{\mathrm{m}}$. Hence we should show that

$$\widetilde{k}_p(m) = m + N(m-1) \quad \text{ for } p \text{ close to } p_S. \tag{6.1}$$

By (4.12) we have that

$$\widetilde{\mu}_j(m,p) \ = \ \widetilde{\beta}_i(m,p) \ + \ \lambda_k, \quad \text{for } i,j = 1,2,\ldots, \quad k = 0,1,\ldots \tag{6.2}$$

where λ_k are the eigenvalues of the Laplace-Beltrami operator $-\Delta_{S^{N-1}}$ on the unit sphere S^{N-1}, $N \geq 3$. As we already mentioned in (4.13)

$$\lambda_k = k(k+N-2) \ (\geq 0), \quad k = 0,1,\ldots$$

with multiplicity (see [6])

$$N_k - N_{k-2} \tag{6.3}$$

where N_h, $h \in \mathbb{Z}$, is defined in (4.15).

By (4.16) we already know that

$$\widetilde{k}_p(m) \geq m + N(m-1) \tag{6.4}$$

and that

$$\widetilde{\beta_1}(m,p) \leq \ldots \leq \widetilde{\beta}_m(m,p) < \ 0 \ \leq \widetilde{\beta}_{m+1}(m,p) \leq \ldots . \tag{6.5}$$

By (6.5), since $\lambda_k \geq 0$, it immediately follows that

$$\widetilde{\beta}_i(m,p) \ + \ \lambda_k, \ \geq \ 0 \qquad \forall\, i \geq m+1, \ \ \forall\, k \geq 0 \tag{6.6}$$

so that *all the eigenvalues* $\widetilde{\beta}_i(m,p)$ *with* $i \geq m+1$ *cannot produce any negative eigenvalue* $\widetilde{\mu_j}(m,p)$ by the formula (6.2).

Next we analyze the contribution given by the last negative eigenvalue $\widetilde{\beta}_m(m,p)$. Observe that $\lambda_1 = N-1$ and, by Proposition 4.4, $\widetilde{\beta}_m(m,p) > -(N-1)$, hence we get

$$\widetilde{\beta}_m(m,p) \ + \ \lambda_k \ > \ 0, \quad \ \forall\, k \geq 1. \tag{6.7}$$

On the other side, from (6.5) and observing that $\lambda_0 = 0$, we have that

$$\widetilde{\beta}_m(m,p) \ + \lambda_0 \ = \ \widetilde{\beta}_m(m,p) \ < \ 0. \tag{6.8}$$

Hence, by (6.2), (6.8) gives *one negative eigenvalue of* $\widetilde{L}_p^{\mathrm{m}}$*, which is radial and simple*, since by (6.3) it follows that λ_0 has multiplicity one. Furthermore, because of (6.7), this eigenvalue is *the only* negative eigenvalue obtained by summing $\widetilde{\beta}_m(m,p)$ with the eigenvalues of $-\Delta_{S^{N-1}}$

Then (6.1) is obviously proved in the case $m = 1$.

In the case $m \geq 2$ we need to study the remaining negative eigenvalues $\widetilde{\beta}_i(m,p)$, $i = 1, \ldots, m-1$ and, since there are exactly m radial simple negative eigenvalues of $\widetilde{L}_p^{\mathrm{m}}$, we have to prove that they produce exactly $N(m-1)$ negative *nonradial* eigenvalues $\widetilde{\mu_j}(m,p)$ by the formula (6.2) (counted with their multiplicity).

Since by Proposition 5.1 and Corollary 5.2 we have

$$\liminf_{p \to p_S} \widetilde{\beta}_i(m,p) \geq -(N-1), \qquad \text{for any } \; i = 1, \ldots, m-1 \tag{6.9}$$

and observing that $\lambda_k \geq 2N > N-1$ for all $k \geq 2$, it follows that for p sufficiently close to p_S

$$\widetilde{\beta}_i(m,p) \; + \; \lambda_k \; > \; 0, \qquad \text{for any } i = 1, \ldots, m-1, \;\; \text{for all } k \geq 2. \tag{6.10}$$

By (6.10) and the estimate (6.4) we immediately have that for p close to p_S

$$\widetilde{\beta}_i(m,p) + \lambda_1 < 0, \qquad \text{for any } i = 1, \ldots, m-1. \tag{6.11}$$

Indeed, since there are exactly m radial simple negative eigenvalues of $\widetilde{L}_p^{\mathrm{m}}$, by (6.4) there must be at least $N(m-1)$ negative *nonradial* eigenvalues of $\widetilde{L}_p^{\mathrm{m}}$ (counted with their multiplicity). By (6.10), for p close to p_S, these nonradial eigenvalues must be obtained by the formula (6.2) for $i = 1, \ldots, m-1$ and $k = 1$ (for $k = 0$ only radial eigenvalues may be constructed). Hence, observing that the multiplicity of λ_1 is N (by (6.3)), we deduce that, if (6.11) does not hold, then (6.4) cannot be satisfied.

In conclusion by (6.10) and (6.11), for p close to p_S there are exactly $N(m-1)$ *negative nonradial eigenvalues of* $\widetilde{L}_p^{m}$, counted with their multiplicity, given by

$$\widetilde{\beta}_i(m,p) \; + \; \lambda_1 \; < \; 0, \qquad i = 1, \ldots, m-1. \tag{6.12}$$

This proves (6.1) and ends the proof of Theorem 1.1.

□

Remark 6.1. *We point out that, combining* (6.9) *with* (6.11) *and observing that* $\lambda_1 = -(N-1)$*, we also get*

$$\lim_{p \to p_S} \widetilde{\beta}_i(m,p) = -(N-1), \qquad \forall\, i = 1, \ldots, m-1, \tag{6.13}$$

as anticipated in Remark 5.3.

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