

GLOBAL RESULTS FOR EIKONAL HAMILTON-JACOBI EQUATIONS ON NETWORKS

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ABSTRACT. We study a one-parameter family of Eikonal Hamilton-Jacobi equations on an embedded network, and prove that there exists a unique critical value for which the corresponding equation admits global solutions, in a suitable viscosity sense. Such a solution is identified, via an Hopf-Lax type formula, once an admissible trace is assigned on an *intrinsic boundary*. The salient point of our method is to associate to the network an *abstract graph*, encoding all of the information on the complexity of the network, and to relate the differential equation to a *discrete functional equation* on the graph. Comparison principles and representation formulae are proven in the supercritical case as well.

1. INTRODUCTION

Over the last years there has been an increasing interest in the study of the Hamilton-Jacobi Equation on networks and related questions. These problems, in fact, involve a number of subtle theoretical issues and have a great impact in the applications in various fields, for example to data transmission, traffic management problems, etc... While locally – *i.e.*, on each branch of the network (*arcs*) –, the study reduces to the analysis of 1-dimensional problems, the main difficulties arise in matching together the information “converging” at the *junction* of two or more arcs, and relating the *local* analysis at a junction with the *global* structure/topology of the network.

In this article, we provide a thorough discussion of the above issues in the case of Eikonal type Hamilton-Jacobi equations on embedded networks (in \mathbb{R}^n or on a Riemannian manifold, see Remark 3.1). We show that there exists a unique critical value for which the corresponding equation admits global solutions, and extend most of the results known in the continuous setting for the critical and supercritical case.

The main rationale behind our approach consists in neatly distinguishing between the local problem on the arcs and the global analysis on the network. While the former can be solved by means of (classical) 1-dimensional viscosity techniques, the latter is definitely more engaging.

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Our novel idea is to tackle it by associating to the network an *abstract graph*, encoding all of the information on the complexity of the network, and to relate the problem to a *discrete functional equation* on the graph. This allows us to pursue a global analysis of the equation – that goes beyond what happens at a single juncture –, as well as to prove uniqueness and comparison principles in a simpler way. To the best of our knowledge, this is the first time that comparison type results are obtained in the network setting by completely bypassing the difficulties involved in the Crandall-Lions doubling variable method, in favor of a more direct analysis of a discrete equation.

In addition to this, by exploiting the simple geometry of the abstract graph we are able to identify an intrinsic boundary – the *Aubry set* – on which admissible traces can be assigned in order to get unique critical solutions on the whole network; these solutions can be represented by means of Hopf–Lax type formulae. In the supercritical case we get existence and uniqueness of solutions, on any open subset of the network, continuously extending admissible data prescribed on the complement.

Let us point out that the problem of formulating boundary problems on the network and accordingly determining “natural” subsets on which to assign boundary data is a subtle issue, yet not well settled in the literature; we believe that our approach helps clarify this matter, at least in the class of equations that we are considering.

The notions of viscosity solution and subsolution that we adopt are very natural in this setting (see Definitions 3.6 and 3.7). More specifically, the tests we use at vertices are classical in viscosity solutions theory and consist in (unilateral) state constraint type boundary conditions, introduced by Soner [27] to study control problems with constraints. In this regard, the notion of solution requires that at each vertex the state constraint condition holds for at least one arc ending there: it does not require other mixing conditions (on the vertices) between equations defined on different incident arcs.

Very recently, the same notion of solution has been also considered by Lions and Souganidis in [22] to deal with one dimensional junction-type problems for non convex discounted Hamilton-Jacobi equations and study its well-posedness (*i.e.*, comparison principle and existence). Global solutions on networks, however, are not therein studied.

As far as subsolutions are concerned, we only ask that they are continuous on the network and are (viscosity) subsolutions to the equation on the interior of each arc: no extra conditions are required on vertices. These assumptions are the minimal requirements that one needs to ask and, at a first sight, it might seem surprising that they are sufficient to develop a significant global theory. However, the validity of this approach is supported, among other things, by the fact that the notion of solutions can be recovered in terms of maximal subsolution attaining a specific value at a given point (vertex or internal point); see Theorem 7.1.

We also wish to point out that our hypothesis both on the topology of network and the Hamiltonians are very general. As far as the network is concerned, we only ask it to be made up by finite arcs and connected: hence, it may well include multiple connections between different vertices, as well as the presence of loops.

The Hamiltonians are assumed continuous in both variables, quasiconvex and coercive in the first order variable on any arc. Hamiltonians on different arcs are independent one from the others and no compatibility conditions at the vertices are required. See subsection 3.2 for more details.

We are confident that this very same set of ideas can be successfully applied to a broad range of other problems: for example, to the study of the *discounted* Hamilton-Jacobi equation on networks or to prove *homogenization* results for the Hamilton-Jacobi equation on periodic networks (also known as *topological crystals*). We plan to address these and other questions in a future work (in preparation).

1.1. Previous related literature. There is a huge amount of literature related to differential equations on networks, or others non-regular geometric structures (ramified/stratified spaces), in various contexts: hyperbolic problems, traffic flows, evolutionary equations, (regional) control problems, Hamilton-Jacobi equations, etc... An exhaustive description of the state of the art in all of these areas would go well beyond the aims of this paper; just to mention a few noteworthy items: [1, 4, 5, 8, 9, 10, 11, 15, 16, 18, 19, 20, 22, 24, 25, 26, 27]. See also references therein.

A model similar to ours has been previously considered by Camilli and Schieborn in [26], however just in the supercritical case and under some restriction on the topology of the network. In comparison with their hypothesis, we do not require continuity of the Hamiltonians at the vertices (and accordingly, no mixed conditions on the test functions at the vertices) and we do not ask a-priori existence of a regular strict subsolution.

Other relevant recent contributions are [22] (that we have already mentioned above) and [18]. In particular, the latter is a substantial work – whose point of view and techniques are rather different from ours – in which Imbert and Monneau attempt to recover the doubling variable method to their setting, by introducing an extra parameter (the flux limiter), a companion equation (the junction condition) and by using special vertex test functions. See also other related works by the same authors and collaborators [15, 20, 19].

Our analysis of the discrete functional equation is based on ideas and techniques inspired by the so-called *weak KAM theory*, firstly developed by Fathi [12] for the study of Tonelli Hamiltonian systems on closed manifolds (see also [28]). Developing a similar approach in the discrete setting is very natural and has been already exploited in several other works. In [6, 7], for example, a discretization of weak KAM theory was applied to investigate the properties of optimal transport maps; a more systematic development of a discrete weak

KAM theory for *cost functions* was described by Zavidovique in [31, 32] (see also [11]). In particular, [32] shares ideas similar to ours, although our setting has the peculiarity of this interplay between the discrete structure and the embedded network.

From a more dynamical systems point of view, a discrete analogue of Aubry-Mather theory and weak KAM theory was also discussed in [17] (see [29] for a recent related work).

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2. PRELIMINARIES ON GRAPH THEORY

In this section we recall some basic material on the theory of abstract graphs and on functions defined on them. For a more detailed presentation of these and other related topics, we refer the interested readers, for instance, to [30].

2.1. Abstract graphs. A (abstract) graph $\mathbf{X} = (\mathbf{V}, \mathbf{E})$ is an ordered pair of sets \mathbf{V} and \mathbf{E} , which are called, respectively, *vertices* and (directed) *edges*, plus two functions:

$$o : \mathbf{E} \longrightarrow \mathbf{V}$$

and

$$\begin{aligned} \bar{\cdot} : \mathbf{E} &\longrightarrow \mathbf{E} \\ e &\longmapsto \bar{e}, \end{aligned}$$

with the latter assumed to be a fixed-point-free involution, namely satisfying

$$\bar{\bar{e}} \neq e \quad \text{and} \quad \bar{\bar{e}} = e \quad \text{for any } e \in \mathbf{E}.$$

We give the following geometric picture of the setting: $o(e)$ is the *origin* (initial vertex) of e and \bar{e} its *reversed* edge, namely the same edge but with the opposite orientation. Analogously we define

$$t(e) = o(\bar{e})$$

the *terminal* vertex of e . The following compatibility condition holds true

$$t(\bar{e}) = o(\bar{\bar{e}}) = o(e).$$

We say that e links $o(e)$ to $t(e)$, observe that it might well happen that $o(e) = t(e)$, and in this case e will be called a *loop*. An edge is also said to be incident on $o(e)$ and $t(e)$. Two vertices are called *adjacent* if there is an edge linking them or, in other terms, if there is an edge incident on both of them.

We say that the graph is *finite* if the set \mathbf{E} , and consequently \mathbf{V} , has a finite number of elements. We denote by $|\mathbf{V}|$, $|\mathbf{E}|$ the number of vertices and edges.

We define a *path* to be a finite sequence of concatenated edges, namely $\xi = (e_1, \dots, e_M) = (e_i)_{i=1}^M$ satisfying

$$t(e_j) = o(e_{j+1}) \quad \text{for any } j = 1, \dots, M-1.$$

We set $o(\xi) = o(e_1)$, $t(\xi) = t(e_M)$, and call them the initial and final vertex of the path. We say that ξ links $o(\xi)$ to $t(\xi)$, we also say that ξ is incident on some vertex if there is some edge composing the path incident on it.

Given two paths ξ , η , we say that ξ is contained in η , mathematically $\xi \subset \eta$, if the edges of ξ make up a subset of the edges of η . If such a subset is proper, we say that ξ is *properly contained* in η . If $t(\xi) = o(\eta)$, we denote by $\xi \cup \eta$ the path obtained via concatenation of ξ and η .

We call a path a *loop* or a *cycle* if $o(\xi) = t(\xi)$. A path without repetition of vertices except possibly the initial and terminal ones will be called *simple*, in other terms $\xi = (e_i)_1^M$ is simple if

$$t(e_i) = t(e_j) \Rightarrow i = j,$$

or if there are no cycles properly contained in ξ . Note that there are finitely many simple paths in a finite graph.

A graph is called *connected* if any two vertices are linked by some path. All of the graphs we will consider hereafter are understood to be connected and finite.

Given $x \in \mathbf{V}$, we set

$$(1) \quad \mathbf{E}_x = \{e \in \mathbf{E} \mid o(e) = x\},$$

which we call \mathbf{E}_x the *star centered at x*; it should be considered as a sort of tangent space to the graph at x . The cardinality of \mathbf{E}_x is called the *degree* (or *valence*) of the vertex x .

2.2. Functions on graphs. In the following we will be interested in functions defined on abstract graphs. It is useful to introduce the following notions.

We define:

- the 0-cochain group $C^0(\mathbf{X}, \mathbb{R})$ as the space of functions from \mathbf{V} to \mathbb{R} . This space play the role of functions on the graph.
- The 1-cochain group $C^1(\mathbf{X}, \mathbb{R})$ as the space of functions from \mathbf{E} to \mathbb{R} , the compatibility condition $\omega(\bar{e}) = -\omega(e)$. This space plays the role of 1-forms on the graph. From now on we will indicate the reverse edge \bar{e} by $-e$ and we will consider the pairing $\langle \omega, e \rangle := \omega(e)$.

The relation between $C^0(\mathbf{X}, \mathbb{Z})$ and $C^1(\mathbf{X}, \mathbb{Z})$ can be expressed in terms of the so-called *coboundary operator*, or *differential*, $d : C^0(\mathbf{X}, \mathbb{Z}) \rightarrow C^1(\mathbf{X}, \mathbb{Z})$, which is defined for any

$f \in C^0(\mathbf{X}, \mathbb{Z})$ and $e \in \mathbf{E}$ as

$$df(e) := f(t(e)) - f(o(e)).$$

We can embed these spaces with the standard topology. A notion of convergence on the cochain spaces is given via

$$\begin{aligned} f_n \longrightarrow f &\iff f_n(x) \longrightarrow f(x) && \text{for any } x \in \mathbf{V} \\ \omega_n \longrightarrow \omega &\iff \omega_n(e) \longrightarrow \omega(e) && \text{for any } e \in \mathbf{E}. \end{aligned}$$

A sequence f_n is said *equibounded* if

$$|f_n(x)| \leq \beta \quad \text{for any } x \in \mathbf{V}, \text{ some } \beta > 0;$$

similarly ω_n is said equibounded if

$$|\langle \omega_n, e \rangle| \leq \beta \quad \text{for any } e \in \mathbf{E}, \text{ some } \beta > 0.$$

It is clear that any equibounded sequences f_n, ω_n are convergent, up to subsequences.

We directly deduce from the above definitions:

Proposition 2.1. *Let f_n, f be in $C^0(\mathbf{X}, \mathbb{R})$*

- i) *if $f_n \longrightarrow f$, then $df_n \longrightarrow df$;*
- ii) *if df_n is equibounded and the sequence $f_n(x_0)$ is bounded for some vertex x_0 , then f_n is convergent, up to subsequences.*

3. SETTING

In this section we first explain our setting, namely what is an *embedded network* and what we mean by *Hamiltonian* on a network. Then we introduce the class of *Hamilton-Jacobi equations* on a network we are interested in, and specify the notions of solutions and subsolutions.

3.1. Embedded networks. An *embedded network*, or *continuous graph*, is a subset $\Gamma \subset \mathbb{R}^N$ of the form

$$\Gamma = \bigcup_{\gamma \in \mathcal{E}} \gamma([0, 1]) \subset \mathbb{R}^N,$$

where \mathcal{E} is a finite collection of regular simple oriented curves, called *arcs* of the network, that we assume, without any loss of generality, parameterized on $[0, 1]$. We denote by \mathcal{E}^* the subset of arcs γ which are closed, namely with $\gamma(0) = \gamma(1)$.

Remark 3.1. Our setting can be easily extended to the case in which Γ is embedded in a Riemannian manifold (M, g) , for example by means of Nash embedding theorem [23].

Observe that on the support of any arc γ , we also consider the inverse parametrization defined as

$$\tilde{\gamma}(s) = \gamma(1 - s) \quad \text{for } s \in [0, 1].$$

We call $\tilde{\gamma}$ the *inverse arc* of γ . We assume

$$(2) \quad \gamma((0, 1)) \cap \gamma'((0, 1)) = \emptyset \quad \text{whenever } \gamma \neq \gamma', \gamma \neq \tilde{\gamma}'.$$

We call *vertices* the initial and terminal points of the arcs, and denote by \mathbf{V} the sets of all such vertices. Note that (2) implies that

$$\gamma((0, 1)) \cap \mathbf{V} = \emptyset \quad \text{for any } \gamma \in \mathcal{E}.$$

We assume that the network is connected, namely given two vertices there is a finite concatenation of arcs linking them.

The network Γ inherits a *geodesic distance*, denoted by d_Γ , from the Euclidean metric of \mathbb{R}^N . Hence, hereafter the notions of continuity and Lipschitz continuity, when referred to functions defined on Γ , must be understood with respect to such distance and the induced topology.

We can also consider a *differential structure* on Γ by defining the tangent space at any $x \in \Gamma \setminus \mathbf{V}$ as

$$T_\Gamma(x) = \{ \lambda \dot{\gamma}(t) \mid \lambda \in \mathbb{R}, \gamma \in \mathcal{E}, t \in (0, 1) \text{ and } x = \gamma(t) \}$$

and the cotangent space $T_\Gamma^*(x)$ as the dual space $(T_\Gamma(x))^*$; namely, it is the set of linear functionals $p : T_\Gamma(x) \rightarrow \mathbb{R}$.

We will say that a function $f : \Gamma \rightarrow \mathbb{R}$ is of class $C^1(\Gamma \setminus V)$ if it is continuous in Γ and

$$t \mapsto f(\gamma(t)) \text{ is of class } C^1 \text{ in } (0, 1) \text{ for any } \gamma \in \mathcal{E}.$$

For such a function we define $D_\Gamma f(x)$, where $x = \gamma(t_0)$ for some $\gamma \in \mathcal{E}$ and $t_0 \in (0, 1)$, as the unique covector in $T_\Gamma^*(x)$ satisfying

$$(D_\Gamma f(x), \dot{\gamma}(t_0)) = \left. \frac{d}{dt} f(\gamma(t)) \right|_{t=t_0},$$

where (\cdot, \cdot) denotes the pairing between covectors and vectors.

Notice that this definition is invariant for a change of parametrization from γ to $\tilde{\gamma}$.

We can associate to any continuous network Γ an abstract graph $\mathbf{X} = (\mathbf{V}, \mathbf{E})$ with the same vertices of the network and edges corresponding to the arcs. More precisely, we consider an abstract set \mathbf{E} with a bijection

$$(3) \quad \Psi : \mathbf{E} \rightarrow \mathcal{E}.$$

This induces maps $o : \mathbf{E} \rightarrow \mathbf{V}$, $\bar{\cdot} : \mathbf{E} \rightarrow \mathbf{E}$ via

$$o(e) = \Psi(e)(0) \quad \text{and} \quad \bar{e} = \Psi^{-1}(\widetilde{\Psi(e)}),$$

satisfying the properties in the definition of graph. Intuitively, in the passage from the embedded network to the underlying abstract graph \mathbf{X} , the arcs become *immaterial* edges.

3.2. Hamiltonians on networks. A Hamiltonian on a network Γ is a collection of Hamiltonians $\mathcal{H} = \{H_\gamma\}_{\gamma \in \mathcal{E}}$, where

$$\begin{aligned} H_\gamma : [0, 1] \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (s, p) &\longmapsto H_\gamma(s, p) \end{aligned}$$

satisfies

$$(4) \quad H_{\bar{\gamma}}(s, p) = H_\gamma(1 - s, -p) \quad \text{for any } \gamma \in \mathcal{E}$$

Notice that we are not assuming any periodicity on H_γ when γ is a closed curve.

We require any H_γ to be:

- (H γ 1) continuous in (s, p) ;
- (H γ 2) coercive in p ;
- (H γ 3) quasiconvex in p , with

$$\text{Int}(\{p \mid H_\gamma(x, p) \leq a\}) = \{p \mid H_\gamma(x, p) < a\} \quad \text{for any } a \in \mathbb{R},$$

where $\text{Int}(\cdot)$ denotes the interior of a set.

We point out that, throughout the paper, the term (sub)solution to Hamilton–Jacobi equations involving the H_γ 's, must be understood in the viscosity sense, see for example [2, 3] for a comprehensive treatment of viscosity solutions theory.

We set for any $\gamma \in \mathcal{E}$

$$(5) \quad a_\gamma := \max_{s \in [0, 1]} \min_{p \in \mathbb{R}} H_\gamma(s, p)$$

$$(6) \quad c_\gamma := \min\{a : H_\gamma = a \text{ admits periodic subsolutions}\}.$$

By periodic subsolution, we mean subsolution to the equation in $(0, 1)$ taking the same value at the endpoints.

Remark 3.2. The definition of c_γ is indeed well-posed. In fact, given $\gamma \in \mathcal{E}$, because of the compactness of $[0, 1]$, we can choose a large enough to have

$$H(s, 0) \leq a \quad \text{any } s \in (0, 1).$$

This shows that any constant function is a subsolution and, consequently, the set in the definition of c_γ is non-empty. It is also bounded from below since for $a < a_\gamma$ the corresponding equation does not admit subsolutions and, therefore, it does not admit periodic ones. Finally, by basic stability properties in viscosity solution theory, there exists a periodic subsolution at the level c_γ , which justifies the minimum appearing in the definition. We will essentially use c_γ for $\gamma \in \mathcal{E}^*$, but in principle the definition and the above considerations hold for any γ .

We stress that

$$a_\gamma \leq c_\gamma \quad \text{for any } \gamma \in \mathcal{E}.$$

We further define

$$(7) \quad a_0 := \max \left\{ \max_{\gamma \in \mathcal{E} \setminus \mathcal{E}^*} a_\gamma, \max_{\gamma \in \mathcal{E}^*} c_\gamma \right\}.$$

We require a further condition:

(H γ 4) given any $\gamma \in \mathcal{E}$ with $a_\gamma = a_0$, the map $s \mapsto \min_{p \in \mathbb{R}} H_\gamma(s, p)$ is constant in $[0, 1]$.

Remark 3.3. The main role of **(H γ 4)** is to ensure uniqueness of solutions to the Dirichlet problem associated to the equation $H_\gamma = a_\gamma$, at least for the γ 's with $a_\gamma = a_0$. The uniqueness property for such kind of problems holds in general when the equation admits a strict subsolution, which is not the case at the level a_γ . The relevant consequence of condition **(H γ 4)** is that the family of subsolutions to $H_\gamma = a_\gamma$ reduces to a singleton, up to additive constants, see Proposition 5.3.

Notice finally that condition **(H γ 4)** is automatically satisfied if the H_γ 's are independent of the state variable.

3.3. The Eikonal Hamilton–Jacobi equation on networks. We define a notion of subsolution and solution to an equation of the form

$$(\mathcal{H}Ja) \quad \mathcal{H}(x, Du) = a \quad \text{on } \Gamma.$$

where $a \in \mathbb{R}$. This notation synthetically indicates the family (for γ varying in \mathcal{E}) of Hamilton–Jacobi equations

$$(HJ_\gamma a) \quad H_\gamma(s, (u \circ \gamma)') = a \quad \text{on } (0, 1).$$

We start by recalling some terminology of viscosity solutions theory.

Definition 3.4. Given a continuous function w in $[0, 1]$ and a function $\varphi \in C^1([0, 1])$, we say that:

- φ is *supertangent* to w at $s \in (0, 1)$ if

$$w = \varphi \text{ at } s \quad \text{and} \quad w \geq \varphi \text{ in } (s - \delta, s + \delta) \text{ for some } \delta > 0.$$

The notion of *subtangent* is given by just replacing \geq by \leq in the above formula.

- φ is a *constrained subtangent* to w at 1 if

$$w = \varphi \text{ at } 1 \quad \text{and} \quad w \geq \varphi \text{ in } (1 - \delta, 1) \text{ for some } \delta > 0.$$

A similar notion, with obvious adaptations, can be given at $t = 0$.

Definition 3.5. Given a continuous function w in $[0, 1]$, a point $s_0 \in \{0, 1\}$, we say that it satisfies *the state constraint boundary condition* for $(HJ_\gamma a)$ at s_0 if

$$H_\gamma(s_0, \varphi'(s_0)) \geq a$$

for any φ that is a constrained C^1 subtangent to w at s_0 .

Definition 3.6. We say that $u : \Gamma \rightarrow \mathbb{R}$ is *subsolution* to $(\mathcal{H}Ja)$ if

- i) it is continuous on Γ ;
- ii) $s \mapsto u(\gamma(s))$ is subsolution to $(HJ_\gamma a)$ in $(0, 1)$ for any $\gamma \in \mathcal{E}$.

We say that u is *solution* to $(\mathcal{H}Ja)$ if

- i) it is continuous;
- ii) $s \mapsto u(\gamma(s))$ is solution of $(HJ_\gamma a)$ in $(0, 1)$ for any $\gamma \in \mathcal{E}$;
- iii) for every vertex x there is at least one arc γ , having x as terminal point, such that $u(\gamma(s))$ satisfies the state constraint boundary condition for $(HJ_\gamma a)$ at $s = 1$.

Compare also this definition with the one in [22]. As far as we know, the idea of imposing a supersolution condition on just one arc incident to a given vertex, first appeared in [26].

We do not provide a notion of supersolution. This could be done straightforwardly but we will not need it in the remainder of the paper.

Definition 3.7. Given an open (in the relative topology) subset $\Gamma' \subset \Gamma$, we say that a continuous function $u : \Gamma \rightarrow \mathbb{R}$ is *solution to $(\mathcal{H}Ja)$ in Γ'* , if for any $x \in \Gamma' \setminus \mathbf{V}$, $x = \gamma(s_0)$ with $\gamma \in \mathcal{E}$, $s_0 \in (0, 1)$, the usual viscosity solution condition holds true for $u \circ \gamma$ at s_0 . If instead $x \in \Gamma' \cap \mathbf{V}$, we require condition iii) in Definition 3.6 to hold.

Remark 3.8. The definition of (sub)solutions on Γ requires $u \circ \gamma$ to be a (sub)solution of the corresponding equation in $(0, 1)$ on any arc γ . If, in particular γ is a closed curve, we must have in addition $u(\gamma(0)) = u(\gamma(1))$. This explains why on any arc $\gamma \in \mathcal{E}^*$ we are solely interested in periodic (sub)solutions, namely (sub) solutions in $(0, 1)$ taking the same value at 0 and 1. This also explains the role of c_γ .

Given a continuous function u defined in $[0, 1]$, it is apparent that a C^1 function φ is supertangent (resp. sub-tangent) to u at $s_0 \in (0, 1)$ if and only if $\tilde{\varphi}(s) := \varphi(1 - s)$ is supertangent (resp. sub-tangent) to $s \mapsto u(1 - s)$ at $1 - s_0$. Taking into account (4), we derive the following result.

Proposition 3.9. *Given an arc γ , a function $u(s)$ is subsolution (resp. solution) to $(HJ_\gamma a)$ if and only if $s \mapsto u(1 - s)$ is subsolution (resp. solution) to the the same equation with $H_{\tilde{\gamma}}$ in place of H_γ .*

It is not difficult to see that Lipschitz-continuity of subsolutions on any arc, coming from the coercivity condition in **(H γ 2)**, implies Lipschitz-continuity in Γ with respect to the geodesic distance. We provide a proof in Appendix A for reader's convenience.

Proposition 3.10. *The family of subsolutions to $(HJ_\gamma a)$, provided it is not empty, is equiLipschitz continuous on Γ with respect to the geodesic distance d_Γ .*

We derive from the previous result plus basic properties of viscosity solutions the existence of the maximal subsolution attaining a given value at a given point of the network.

Proposition 3.11. *Let a be such that the equation $(\mathcal{H}Ja)$ admits subsolution in Γ . Given $y \in \Gamma$, $\alpha \in \mathbb{R}$, the function*

$$w(x) = \max\{u(x) \mid \text{subsolution to } (\mathcal{H}Ja) \text{ with } u(y) = \alpha\}$$

is still a subsolution.

4. STRATEGY OF THE PROOF

The remaining of the article consists in the proof of our results on existence, uniqueness and regularity of global (sub)solutions to the Eikonal Hamilton-Jacobi equation on Γ . For the reader's convenience, a summary of all of our main results will be detailed in section 8.

Before starting, we believe it might be useful to provide here an outline of the forthcoming discussion.

In section 5, we focus on the *local* problem on each arc of the network. Namely, for each $\gamma \in \mathcal{E}$ we study the existence of (sub)solutions to the 1-dimensional Eikonal Hamilton-Jacobi equation $(HJ_\gamma a)$ with boundary conditions. In particular:

- We show that under suitable *admissibility conditions* on the boundary data, see (17), there exists a unique solution and we provide a representation formula (Proposition 5.5).
- We derive a characterization of condition iii) in Definition 3.6 in terms of this representation formula (Proposition 5.6).

In section 6 we concentrate on the global aspects of the problem.

- We introduce a *discrete functional equation* ($\mathcal{DFE}a$) on the abstract graph \mathbf{X} and provide the corresponding notions of solutions and subsolutions. The crucial result linking solutions to this equation and solutions to ($\mathcal{DFE}a$) is proven in Proposition 6.2.
- In (30) we define *Mañé critical value* $c(\mathcal{H})$. We first prove that this is the unique value for which solutions to the discrete functional equation may exist (Proposition 6.5), and then that the critical equation ($\mathcal{DFE}c$) admits indeed solutions (Theorem 6.16).
- In (39) and (40) we define the *Aubry set* $\mathcal{A}_{\mathbf{X}}^*$ and the *projected Aubry set* $\mathcal{A}_{\mathbf{X}}$, which are non-empty (Proposition 6.20). We prove in Proposition 6.21 that $\mathcal{A}_{\mathbf{X}}$ is a uniqueness set and provide a Hopf–Lax type representation formula for the solutions to ($\mathcal{DFE}c$) in terms of its values on $\mathcal{A}_{\mathbf{X}}$.

The supercritical case will be discussed in parallel to the critical one (see Proposition 6.3, Proposition 6.6 and Theorem 6.23).

Finally, in section 7 we switch our attention back to the immersed network:

- We prove in Theorem 7.1 that the notion of solution can be recovered in terms of maximal subsolution attaining a specific value at a given point.
- We introduce the analogue of the Aubry set on the network, we show in Theorem 7.5 that all critical subsolutions are of class C^1 on it and they all have the same differential on this set.
- We show the existence of C^1 critical subsolutions that are strict outside of the Aubry set (Theorem 7.6).
- We provide representation formulae and uniqueness results with traces that are not necessarily defined on vertices (Theorem 7.9).

5. LOCAL PART: THE EIKONAL HAMILTON-JACOBI EQUATION WITH BOUNDARY CONDITIONS ON ARCS

In this section we focus on a single arc γ and study the family of equations ($HJ_{\gamma}a$) in $(0, 1)$ plus suitable boundary conditions. We assume

$$a \geq a_0 = \max \left\{ \max_{\gamma \in \mathcal{E} \setminus \mathcal{E}^*} a_{\gamma}, \max_{\gamma \in \mathcal{E}^*} c_{\gamma} \right\}.$$

Our aim is to find admissible conditions on boundary data at $s = 0$ and $s = 1$ to get solutions of the corresponding Dirichlet problem, to show uniqueness of such solutions and, finally, to provide a characterization of maximal subsolutions taking a given value at $s = 0$ via state constraint boundary conditions.

We need specific results when γ is a closed curve because in this case we are solely interested to periodic (sub)solutions, as explained in Remark 3.8. We address the issue

in Subsection 5.3. In the first subsections 5.1 and 5.2 we will not distinguish between γ closed or not, and provide an unified presentation of the material.

The results are not new, we write down nevertheless the one-dimensional representation formulae, which are easy to handle and allows a direct and simplified treatment of the matter. We recall that, due to coercivity and quasiconvexity assumptions, all subsolutions to $(HJ_\gamma a)$ are Lipschitz-continuous in $[0, 1]$, and, in addition the notion of viscosity and a.e. subsolution are equivalent. Also notice that the subsolution property is not affected by addition of constants.

To ease notation, we write $H(s, p)$ instead of $H_\gamma(s, p)$, and accordingly we consider equation $(HJ_\gamma a)$ with H in place of H_γ . Moreover, we denote by H^\vee the Hamiltonian $H_{\tilde{\gamma}}$. We recall that the assumptions **(HJ1)**–**(HJ4)** are in force.

5.1. Setting of the local problem. We set for $s \in [0, 1]$

$$(8) \quad \sigma_a^+(s) = \max\{p \mid H(s, p) = a\}$$

$$(9) \quad \sigma_a^-(s) = \min\{p \mid H(s, p) = a\}.$$

If $a > a_\gamma$, we have by **(HJ3)**

$$(10) \quad (\sigma_a^-(s), \sigma_a^+(s)) = \{p \mid H(s, p) < a\} \quad \text{for } s \in [0, 1].$$

We deduce from assumption **(HJ4)** that if $a_\gamma = a_0$

$$(11) \quad \sigma_{a_\gamma}^+(s) = \sigma_{a_\gamma}^-(s) \quad \text{for any } s \in [0, 1].$$

Proposition 5.1. *The functions $s \mapsto \sigma_a^+(s)$, $s \mapsto \sigma_a^-(s)$ are continuous in $[0, 1]$ for any $a \geq a_\gamma$.*

Proof: It follows directly from the continuity and the coercivity of H that the function $s \mapsto \sigma_{a_\gamma}^+(s) = \sigma_{a_\gamma}^-(s)$ is continuous. If $a > a_\gamma$, the assertion follows from the fact that $\sigma_a^+(s)$, $\sigma_a^-(s)$ are univocally determined for any s by the conditions $H(s, \sigma_a^+(s)) = H(s, \sigma_a^-(s)) = a$ and, respectively, $\sigma_a^+(s) > \sigma_{a_\gamma}^+(s)$ or $\sigma_a^-(s) < \sigma_{a_\gamma}^-(s)$. \square

Notice that

$$(12) \quad u \text{ subsolution} \implies \sigma^-(s) \leq u'(s) \leq \sigma^+(s) \text{ for a.e. } s.$$

We introduce four relevant functions:

$$(13) \quad s \mapsto \int_0^s \sigma_a^+(t) dt$$

$$(14) \quad s \mapsto \int_0^s \sigma_a^-(t) dt$$

$$(15) \quad s \mapsto - \int_s^1 \sigma_a^-(t) dt$$

$$(16) \quad s \mapsto - \int_s^1 \sigma_a^+(t) dt.$$

Remark 5.2. According to (12), the function in (13) is the maximal (sub)solution to $(HJ_\gamma a)$ vanishing at $s = 0$, and the one in (14) the minimal (sub)solution vanishing at $s = 0$. Analogously, the function defined in (15) is the maximal (sub)solution vanishing at $s = 1$, and the one in (16) the minimal (sub)solution vanishing at $s = 1$. All of these functions are of class C^1 because of Proposition 5.1.

We remark that when we write *maximal (sub)solution* et similia, means that it is maximal in the class of subsolution to $(HJ_\gamma a)$ with a given property and it is, in addition, a solution to the equation.

If $a = a_\gamma$, it follows from (11) that all of the above functions coincide up to an additive constant. We can state the following result.

Proposition 5.3. *The (sub)solution to $(HJ_\gamma a)$, with $a = a_\gamma$ is unique up to additive constants.*

From the properties of the solutions in (13) and (14), we directly derive a necessary condition (*admissibility condition*) that two boundary data at 0 and 1 must satisfy in order to correspond to the values at the endpoints of a subsolution to $(HJ_\gamma a)$.

Lemma 5.4. *Assume that there is a subsolution to $(HJ_\gamma a)$ taking the values α and β at 0 and 1, then*

$$(17) \quad \int_0^1 \sigma_a^-(t) dt \leq \beta - \alpha \leq \int_0^1 \sigma_a^+(t) dt.$$

The above condition is actually also sufficient:

Proposition 5.5. *Given boundary data α, β , satisfying (17) the function w*

$$(18) \quad s \mapsto w(s) := \min \left\{ \alpha + \int_0^s \sigma_a^+(t) dt, \beta - \int_s^1 \sigma_a^-(t) dt \right\}$$

is the unique solution to $(HJ_\gamma a)$ taking the values α at $s = 0$, and β at $s = 1$.

The proof is in the Appendix A.

5.2. Maximal subsolutions. The main result of this section is:

Proposition 5.6. *Assume that w is a solution in $(0, 1)$ to $(HJ_\gamma a)$ for $a \geq a_\gamma$, continuously extended up to the boundary. If*

$$(19) \quad H(1, \varphi'(1)) \geq a \quad \text{for any } C^1 \text{ supertangent } \varphi \text{ to } w \text{ constrained to } [0, 1],$$

then w is the maximal (sub)solution taking the value $w(0)$ at 0. Namely:

$$(20) \quad w(s) = w(0) + \int_0^s \sigma_a^+(t) dt \quad \text{for } s \in [0, 1].$$

Conversely, if a solution w is of the form (20), then condition (19) holds true.

The proof is in the Appendix A.

We fix $s_0 \in (0, 1)$, by slightly generalizing the formulae provided in the previous result and arguing separately in the two subintervals $[0, s_0]$ and $[s_0, 1]$, we get:

Corollary 5.7. *Let $s_0 \in (0, 1)$. For any $\alpha \in \mathbb{R}$, the function*

$$s \mapsto \begin{cases} \alpha - \int_s^{s_0} \sigma_a^-(t) dt & \text{for } s \leq s_0 \\ \alpha + \int_{s_0}^s \sigma_a^+(t) dt & \text{for } s > s_0 \end{cases}$$

is the maximal subsolution to $(HJ_\gamma a)$ taking the value α at s_0 . It is in addition solution in $(0, 1) \setminus \{s_0\}$, but the solution property fails at s_0 , unless $a = a_\gamma$.

Remark 5.8. In the light of Proposition 3.9 and Remark 5.2, it is apparent that the maximal solution to $H^\vee = a$ vanishing at $s = 0$ is given by

$$s \mapsto - \int_{1-s}^1 \sigma_a^-(t) dt.$$

This function satisfies the state constraint boundary condition at $s = 1$.

5.3. Closed arcs. In this subsection we assume that γ is a closed curve. Keeping in mind Remark 3.8, we aim at showing the existence of periodic (sub)solution for any a or, in other terms, that periodic boundary conditions at $s = 0$ and $s = 1$ are admissible in the sense of (17)

Recall that $a \geq a_0 \geq c_\gamma$. We derive further information in the case where $a = a_0 = c_\gamma$. We will exploit the existence of periodic subsolutions at the level c_γ in $(0, 1)$, say, to fix ideas, vanishing at 0 and 1, as pointed out in Remark 3.2. These periodic subsolutions are *sandwiched* in between the function in (13) and the one in (14), according to Remark 5.2. We derive:

Lemma 5.9. *We have*

$$(21) \quad \int_0^1 \sigma_a^-(t) dt \leq 0 \leq \int_0^1 \sigma_a^+(t) dt,$$

and both the inequalities are strict if $a > c_\gamma$.

This in turn implies in view of (17)

Corollary 5.10. *There are periodic solutions to $(HJ_\gamma a)$ in $(0, 1)$.*

Moreover:

Proposition 5.11.

$$\min \left\{ - \int_0^1 \sigma_{c_\gamma}^-(t) dt, \int_0^1 \sigma_{c_\gamma}^+(t) dt \right\} = 0.$$

The proof is in the Appendix A.

From the previous result plus Proposition 5.6 and Remark 5.8, we derive the following.

Corollary 5.12. *Let $a = c_\gamma$ and $\alpha \in \mathbb{R}$; then, either the maximal solution to $H = a$ taking the value α at $s = 0$ or the maximal solution to $H^\vee = a$ taking the value α at $s = 0$ are periodic.*

In the final result of the section we provide a characterization for the maximal periodic subsolution taking a given value at $s_0 \in (0, 1)$. This corresponds, in the case of closed arcs, to Corollary 5.7.

Corollary 5.13. *Let $s_0 \in (0, 1)$, $\alpha \in \mathbb{R}$, we set*

$$\beta = \min \left\{ - \int_0^{s_0} \sigma_a^-(t) dt, \int_{s_0}^1 \sigma_a^+(t) dt \right\}.$$

- i) *The maximal periodic subsolution to $(HJ_\gamma a)$ taking the value α at s_0 , denoted by u , is uniquely determined by the condition of being solution of the equation in $(0, s_0)$ and $(s_0, 1)$ taking the values α at s_0 and $\alpha + \beta$ at 0 and 1.*

ii) If $\beta = -\int_0^{s_0} \sigma_a^-(t) dt$, then

$$(22) \quad u(s) = \alpha - \int_s^{s_0} \sigma_a^-(t) dt \quad \text{for } s \in [0, s_0].$$

If instead $\beta = \int_{s_0}^1 \sigma_a^+(t) dt$, then

$$(23) \quad u(s) = \alpha + \int_{s_0}^s \sigma_a^+(t) dt \quad \text{for } s \in [s_0, 1]$$

The proof is in the Appendix A.

5.4. From local to global. The subsequent step in our analysis will be to transfer the Hamilton–Jacobi equation from Γ to the underlying graph \mathbf{X} , where it will take the form of a discrete functional equation. In doing this, the relevant information we derive from the above study is the value at $s = 1$ of the maximal solution to $H = a$ vanishing at $s = 0$. It is given, in accordance with Proposition 5.6, by

$$\int_0^1 \sigma_a^+(t) dt.$$

Therefore, if $\gamma = \Psi(e)$ and $a \geq a_\gamma$, we define

$$(24) \quad \sigma_a(e) := \int_0^1 \sigma_a^+(t) dt.$$

(recall that $a \geq a_0 \geq c_\gamma$).

Accordingly, we have

$$(25) \quad \sigma_a(-e) := -\int_0^1 \sigma_a^-(t) dt.$$

If e is a loop, or equivalently $\gamma = \Psi(e)$ a closed curve, we summarize the information gathered in Propositions 5.9 and 5.11 as follows:

Proposition 5.14. *If e is a loop then $\sigma_a(e) > 0$ for $a > c_\gamma$ and $\min\{\sigma_{c_\gamma}(e), \sigma_{c_\gamma}(-e)\} = 0$.*

Moreover, we directly deduce from definition and (10) that

Lemma 5.15. *The function*

$$a \mapsto \sigma_a(e)$$

is continuous and strictly increasing in $[a_\gamma, +\infty)$.

6. GLOBAL PART: THE DISCRETE FUNCTIONAL EQUATION ON THE ABSTRACT GRAPH

In this section we push our analysis beyond the local existence of solutions to $(HJ_\gamma a)$ on each arc γ , and study the global existence of solutions to $(\mathcal{H}J a)$ on the whole network Γ .

Let us start by noticing that if we consider \mathbf{V} , the set of vertices of Γ , it is easy to check that any solution w to $(\mathcal{H}J a)$ has a well defined *trace* $u = w|_{\mathbf{V}}$ on \mathbf{V} , simply because of the continuity assumption. The following uniqueness result is straightforward. We provide a proof in Appendix A for reader's convenience.

Proposition 6.1. *Let u be a function defined on \mathbf{V} , then there exists at most one solution to $(\mathcal{H}J a)$ on Γ agreeing with u on \mathbf{V} .*

A converse property is by far more interesting, namely to find conditions on a function defined on \mathbf{V} in order to (uniquely) extend it on the whole network as solution to $(\mathcal{H}J a)$.

This issue – which is profoundly related to the global structure of the network – will be carefully addressed in this section.

More precisely, we study the problem of the admissibility, with respect to the equation $(\mathcal{H}J a)$, of a trace $g : \mathbf{V} \rightarrow \mathbb{R}$ defined on the global network and characterize all traces g that can be continuously extended to solutions to $(\mathcal{H}J a)$ on the whole Γ as solutions to an appropriate *discrete functional equation* on the underlying abstract graph $\mathbf{X} = (\mathbf{V}, \mathbf{E})$.

6.1. The discrete functional equation. Given $a \geq a_0$, the cochain $\sigma_a \in C^1(\mathbf{X}, \mathbb{R})$ is defined as in (24) where $e = \Psi^{-1}(\gamma)$ and Ψ has been defined in (3).

If we recall the admissibility condition introduced in (17) plus (24), (25), it is clear that the trace on \mathbf{V} of a function $g : \Gamma \rightarrow \mathbb{R}$ admissible for the equations on any arc satisfies

$$(26) \quad -\sigma_a(-e) \leq dg(e) = g(t(e)) - g(o(e)) \leq \sigma_a(e) \quad \text{for any } e \in \mathbf{E},$$

which in particular implies

$$g(x) \leq \min_{e \in \mathbf{E}_x} (g(t(e)) + \sigma_a(-e)) \quad \text{for } x \in \mathbf{V},$$

where \mathbf{E}_x denotes the star centered at x , as defined in (1).

Inspired by this, we introduce the following *Discrete Functional Equation*:

$$(DFEa) \quad u(x) = \min_{e \in \mathbf{E}_x} (u(t(e)) + \sigma_a(-e)) \quad \text{for } x \in \mathbf{V}.$$

Observe that the formulation of the discrete problem takes somehow into account the backward character of viscosity solutions.

A function v is solution to $(DFEa)$ in some subset \mathbf{V}' of \mathbf{V} if $(DFEa)$ holds true with v in place of u and $x \in \mathbf{V}'$.

A function $u : \mathbf{V} \rightarrow \mathbb{R}$ is a *subsolution* to $(DFEa)$ if

$$(27) \quad u(x) \leq \min_{e \in \mathbf{E}_x} (u(t(e)) + \sigma_a(-e)) \quad \text{for } x \in \mathbf{V}$$

or, equivalently, if for each $e \in \mathbf{E}$ we have

$$(28) \quad du(e) \leq \sigma_a(e)$$

which is equivalent to ask that $u(t(e)) \leq u(o(e)) + \sigma_a(e)$ for each $e \in \mathbf{E}$.

A subsolution is qualified as *strict*, if a strict inequality prevails in (27).

It is apparent that the property of being a solution or a subsolution is not affected by addition of additive constants.

The crucial result linking the functional equation $(DFEa)$ to $(\mathcal{H}Ja)$ is:

Proposition 6.2. *Given $a \geq a_0$,*

- i) *any solution to $(DFEa)$ in \mathbf{V} can be (uniquely) extended to a solution of $(\mathcal{H}Ja)$ in Γ , conversely the trace on \mathbf{V} of any solution of $(\mathcal{H}Ja)$ in Γ is solution to $(DFEa)$;*
- ii) *any subsolution to $(DFEa)$ in \mathbf{V} can be extended to a subsolution of $(\mathcal{H}Ja)$ in Γ , conversely the trace on \mathbf{V} of any sub solution of $(\mathcal{H}Ja)$ in Γ is subsolution to $(DFEa)$.*

Proof: Assume that u solves $(DFEa)$. Let x, y be two adjacent vertices, e an edge with initial vertex x and final vertex y . We set $\gamma = \Psi(e)$ and consequently $\tilde{\gamma} = \Psi(-e)$, then $\gamma(0) = \tilde{\gamma}(1) = x$ and $\gamma(1) = \tilde{\gamma}(0) = y$. By the very definition of (sub)solution to $(DFEa)$, we have

$$\begin{aligned} u(\gamma(1)) - u(\gamma(0)) &\leq \sigma_a(e) \\ u(\gamma(1)) - u(\gamma(0)) &= u(\tilde{\gamma}(0)) - u(\tilde{\gamma}(1)) \geq -\sigma_a(-e). \end{aligned}$$

Taking into account (17), we derive that the values $u(\gamma(0)), u(\gamma(1))$ are admissible for $(HJ_\gamma a)$ in $(0, 1)$. We therefore deduce from Proposition 5.5 that there is an unique solution, say $w : [0, 1] \rightarrow \mathbb{R}$, to $(HJ_\gamma a)$ taking precisely these values at the boundary. We define

$$v(z) = w(\gamma^{-1}(z)) \quad \text{for } z \in \gamma((0, 1)).$$

Since $\gamma((0, 1)) = \tilde{\gamma}((0, 1))$, one needs to check that this definition is well-posed, performing the same construction for $\tilde{\gamma}$, but this is a direct consequence of Proposition 3.9.

So far, we have successfully checked conditions i), ii) in the definition of solution to $(\mathcal{H}Ja)$ (see Definition 3.6). It is left to show iii). Since u is a solution to $(DFEa)$, for any

$x \in \mathbf{V}$ there is an edge e_0 with x as terminal vertex such that

$$u(x) - u(o(e_0)) = \sigma_a(e_0).$$

Taking into account (24) and Proposition 5.6, we deduce that, for $\gamma = \Psi(e_0)$, $v \circ \gamma$ actually satisfies the state constraint boundary condition in iii) with respect to $(HJ_\gamma a)$.

Conversely, let u be a real function on \mathbf{V} which is the trace on Γ of a solution to $(\mathcal{H}Ja)$. It follows from the compatibility condition (17), and the notations (24)-(25), that u is a subsolution to $(DFEa)$, *i.e.*,

$$(29) \quad u(x) \leq \min_{e \in \mathbf{E}_x} (u(t(e)) + \sigma_a(-e)) \quad \text{for } x \in \mathbf{V}.$$

In order to show that it is a solution to $(DFEa)$, we need to prove that equality holds in (29) for every $x \in \mathbf{V}$. In fact, since u is the trace of a solution to $(\mathcal{H}Ja)$, then it follows from condition iii) in Definition 3.6, that for every vertex x there is at least one arc γ having x as terminal point, such that $u(\gamma(s))$ satisfies the state constraint boundary condition for $(HJ_\gamma a)$ at $s = 1$. In particular, in the light of Proposition 5.6, see (24), this implies that there exists e with $t(e) = x$, or in other terms $-e \in \mathbf{E}_x$, such that

$$u(x) - u(o(e)) = \sigma_a(e)$$

or equivalently

$$u(x) = u(t(-e)) + \sigma_a(e).$$

Hence, equality holds in (29), and this completes the proof of item i). Item ii) can be proven arguing along the same lines. \square

The same argument as in the above proof allows also showing the following:

Proposition 6.3. *Given $a \geq a_0$ and $\mathbf{V}' \subset \mathbf{V}$, a function $u : \mathbf{V} \rightarrow \mathbb{R}$ which is subsolution to $(DFEa)$ in \mathbf{V} and solution in $\mathbf{V} \setminus \mathbf{V}'$ can be (uniquely) extended to a function $v : \Gamma \rightarrow \mathbb{R}$ subsolution of $(\mathcal{H}Ja)$ in Γ and solution in $\Gamma \setminus \mathbf{V}'$. Conversely, the trace on \mathbf{V} of a function $v : \Gamma \rightarrow \mathbb{R}$, which is subsolution to $(\mathcal{H}Ja)$ in Γ and solution in $\Gamma \setminus \mathbf{V}'$, is a subsolution to $(DFEa)$ in \mathbf{V} and a solution in $\mathbf{V} \setminus \mathbf{V}'$.*

6.2. Existence of solutions to $(DFEa)$ and critical value. We want to introduce a notion of *critical value* for $(DFEa)$ and prove the existence of solutions.

Let us start by proving the following stability properties of solutions and subsolutions.

Proposition 6.4.

- i) *Let a_n be a sequence in \mathbb{R} converging to some a . Let u_n be subsolution to $(DFEa_n)$ for every n , with $u_n(x_0)$ bounded for some $x_0 \in \mathbf{V}$; then u_n converge, up to subsequences, to a subsolution to $(DFEa)$.*
- ii) *Let v_n be a sequence of solution to $(DFEa)$, for some $a \in \mathbb{R}$, with $v_n(x_0)$ bounded for some $x_0 \in \mathbf{V}$; then v_n converges, up to a subsequence, to a solution to $(DFEa)$.*

Proof: Owing to the definition of subsolution and Lemma 5.15, we see that

$$\langle du_n, e \rangle \leq \sigma_b(e) \quad \text{for every } e \in \mathbf{E},$$

where $b = \sup a_n$. This implies that the du_n 's are equibounded. We therefore get, exploiting the boundedness assumption on x_0 and Proposition 2.1 ii), that u_n is convergent, up to subsequences, to some u . In force of Lemma 5.15 we have

$$u(t(e)) - u(o(e)) - \sigma_a(e) = \lim_n (u_n(t(e)) - u_n(o(e)) - \sigma_{a_n}(e)) \leq 0$$

for any e , showing that u is subsolution to $(DFEa)$.

Let now v_n be a sequence of solutions to $(DFEa)$; because of the previous point, v_n converge, up to subsequences, to a subsolution v of the same equation. It is left to show that v is indeed a solution. Given $x \in \mathbf{V}$, we find $e_n \in \mathbf{E}_x$ with

$$v_n(t(e_n)) - v_n(x) - \sigma_a(-e_n) = 0.$$

Since the edges are finite, we deduce that there exists $e_0 \in \mathbf{E}_x$ such that

$$e_n = e_0 \quad \text{for infinitely many } n.$$

Up to extracting to a subsequence, passing to the limit as n goes to infinity, we obtain

$$v(t(e_0)) - v(x) - \sigma_a(-e_0) = 0,$$

which completes the proof. □

We define the *critical value* for $(DFEa)$ (also called *Mañé critical value*) as

$$(30) \quad c = c(\mathcal{H}) := \min\{a \geq a_0 \mid (DFEa) \text{ admits subsolutions}\}.$$

First of all, notice that it is well-defined. In fact, because of the coercivity of the H_γ 's, σ_a is strictly positive for every e , when a is large enough, so that any constant function is a subsolution to $(DFEa)$. This shows that c is finite. Note the minimum in the definition of c is justified by Proposition 6.4, showing the existence of critical subsolutions (namely, subsolutions to $(DFEa)$ with $a = c$).

The relevance of the critical value is apparent from the following result.

Proposition 6.5. *If there exists a solution to $(DFEa)$, then $a = c$.*

Proof: Clearly $a \geq c$, since every solution is also a subsolution. If $a > c$, then there exists a strict subsolution u to $(DFEa)$. Let us assume, by contradiction, that there exists also a solution v . Let x_0 be point at which $u - v$ achieves its maximum; then

$$(31) \quad v(x_0) - v(t(e)) \leq u(x_0) - u(t(e)) \quad \text{for any } e \in \mathbf{E}_{x_0}.$$

By the very definition of solution applied to v , there is $e_0 \in \mathbf{E}_{x_0}$ such that

$$v(x_0) = v(t(e_0)) + \sigma_a(-e_0).$$

We derive, taking into account (31),

$$u(x_0) \geq u(t(e_0)) + \sigma_a(-e_0),$$

which is in contrast with the very definition of strict subsolution. \square

We further deduce a uniqueness result in the supercritical case.

Proposition 6.6. *Let $a > c$, $\mathbf{V}' \subset \mathbf{V}$. For any given function u defined on \mathbf{V}' there is at most one solution v of $(DFEa)$ in $\mathbf{V} \setminus \mathbf{V}'$ agreeing with u on \mathbf{V}' .*

Proof: Assume by contradiction that there are two distinct solutions u_1, u_2 both satisfying the statement. Being $a > c$, we know that there is a strict subsolution w to $(DFEa)$. Therefore, given $\lambda \in (0, 1)$ we have

$$(32) \quad \lambda w(x) + (1 - \lambda) u_1(x) < \min_{e \in \mathbf{E}_x} (\lambda w(t(e)) + (1 - \lambda) u_1(t(e)) + \sigma_a(-e))$$

for any $x \in \mathbf{V} \setminus \mathbf{V}'$. Up to interchanging the roles of u_1 and u_2 , we can assume that $\max_{\mathbf{V}}(u_1 - u_2) > 0$, so that any maximizer is outside \mathbf{V}' . For λ sufficiently close to 0, we still have that $[\lambda w + (1 - \lambda) u_1] - u_2$ achieves its maximum in $\mathbf{V} \setminus \mathbf{V}'$. Let x_0 be one of these points of maximum; then, for every $e \in \mathbf{E}_{x_0}$ we have

$$[\lambda w(x_0) + (1 - \lambda) u_1(x_0)] - u_2(x_0) \geq [\lambda w(t(e)) + (1 - \lambda) u_1(t(e))] - u_2(t(e))$$

or

$$u_2(x_0) \leq u_2(t(e)) + \lambda w(x_0) + (1 - \lambda) u_1(x_0) - \lambda w(t(e)) - (1 - \lambda) u_1(t(e)).$$

Using (32) we can deduce

$$u_2(x_0) < \min_{e \in \mathbf{E}_{x_0}} (u_2(t(e)) - \sigma_a(-e))$$

in contrast with $x_0 \notin \mathbf{V}'$ and u_2 being solution to $(DFEa)$ in $\mathbf{V} \setminus \mathbf{V}'$. \square

Given $a \geq a_0$, we define for any path $\xi = (e_1, \dots, e_M) = (e_i)_{i=1}^M$

$$(33) \quad \sigma_a(\xi) = \sum_{i=1}^M \sigma_a(e_i),$$

and

$$(34) \quad S_a(x, y) := \inf\{\sigma_a(\xi) \mid \xi \text{ is a path linking } x \text{ to } y\}.$$

The following triangle inequality is a direct consequence of the definition

$$(35) \quad S_a(x, y) \leq S_a(x, z) + S_a(z, y) \quad \text{for any } x, y, z \text{ in } \mathbf{V}.$$

The next result starts unveiling the major role of cycles in the forthcoming analysis.

Lemma 6.7. $S_a \not\equiv -\infty$ if and only if

$$\sigma_a(\xi) \geq 0 \quad \text{for any cycle } \xi,$$

which is equivalent to say that $S_a(x, x) \geq 0$ for any $x \in \mathbf{V}$.

Proof: If $\sigma_a(\xi) < 0$ for some cycle ξ , then going through it several times, we deduce that $S_a \equiv -\infty$. Conversely, if $\sigma_a(\xi) \geq 0$ for any cycle ξ , then

$$S_a(x, x) \geq 0 \quad \text{for any } x \in \mathbf{V}$$

and therefore $S_a \not\equiv -\infty$. □

From the very definition of subsolution we derive the following result.

Proposition 6.8. A function u is a subsolution to $(\mathcal{D}FEa)$ if and only if

$$u(x) - u(y) \leq S_a(y, x) \quad \text{for any } x, y \in \mathbf{V}.$$

Proof: It follows easily from the definitions of subsolution in (28) and σ_a in (33) that

$$u(x) - u(y) \leq \sigma_a(\xi) \quad \text{for any path } \xi \text{ linking } y \text{ to } x.$$

Taking the minimum over all such paths, we get the inequality in the statement. The converse is trivial, observing that

$$S_a(o(e), t(e)) \leq \sigma_a(e) \quad \text{for every } e \in \mathbf{E}.$$

□

The previous result implies

Corollary 6.9. If $a \geq c$ then $S_a \not\equiv -\infty$.

Moreover:

Corollary 6.10. Given $a \geq c$, x, y in \mathbf{V} , there exists a simple path η with $o(\eta) = x$, $t(\eta) = y$ such that $\sigma_a(\eta) = S_a(x, y)$.

Proof: Let $\xi = (e_i)_{i=1}^M$ be any path linking x to y . If ξ is not simple there are indices $k > j$ such that $t(e_k) = t(e_j)$. We assume, to ease notations, that $k < M$, the case $k = M$ can be treated with straightforward modifications.

We have that $(e_i)_{i=j+1}^k$ is a cycle and the paths $(e_i)_{i=1}^j$, $(e_i)_{i=k+1}^M$ are concatenated. We get, according to Lemma 6.7 that

$$\sigma_a(\xi) = \sigma_a((e_i)_{i=1}^j) + \sigma_a((e_i)_{i=j+1}^k) + \sigma_a((e_i)_{i=k+1}^M) \geq \sigma_a((e_i)_{i=1}^j) + \sigma_a((e_i)_{i=k+1}^M)$$

and $(e_i)_{i=1}^j \cup (e_i)_{i=k+1}^M$ is still a path linking x to y . By iterating the above procedure, we remove all cycles properly contained in ξ and end up with a simple curve ξ_0 with $o(\xi_0) = x$,

$t(\xi_0) = y$ and $\sigma_a(\xi_0) \leq \sigma_a(\xi)$. This shows that $S_a(x, y)$ can be realized as the infimum of simple paths from x to y . Since there are finitely many of such paths, we get the assertion. \square

The condition in Corollary 6.9 is actually necessary and sufficient, as shown by the next result. In the proof we will use a form of the basic *Bellman optimality principle* adapted to our frame. It can be stated as follows: if $\xi = (e_i)_{i=1}^M$ is a path with

$$\sigma_a(\xi) = S_a(o(e), t(e))$$

and $1 \leq j < k \leq M$, then $\eta := (e_i)_{i=j}^k$ satisfies $\sigma_a(\eta) = S_a(o(e_j), t(e_k))$.

Proposition 6.11. *Assume $S_a \neq -\infty$. Given $y \in \mathbf{V}$, the function $u = S_a(y, \cdot)$ is solution to (DFEa) in $\mathbf{V} \setminus \{y\}$ and subsolution to (DFEa) in \mathbf{V} .*

Proof: The subsolution property comes from Proposition 6.8 and the triangle inequality (35). We proceed by showing that u is solution in $\mathbf{V} \setminus \{y\}$. Let $x \neq y$, then, by Corollary 6.10, there is a path $\xi = (e_i)_{i=1}^M$ linking y to x with

$$\sigma_a(\xi) = S_a(y, x).$$

By the Bellman optimality principle, the path $\eta := (e_i)_{i=1}^{M-1}$ satisfies

$$\sigma_a(\eta) = S_a(y, t(\eta)) = u(t(\eta)).$$

Consequently

$$u(x) = \sigma_a(\eta) + \sigma_a(e_M) = u(t(\eta)) + \sigma_a(e_M)$$

with $-e_M \in \mathbf{E}_x$. Hence

$$u(x) - u(t(-e_M)) = u(x) - u(t(\eta)) = \sigma_a(e_M).$$

This concludes the proof. \square

Using Proposition 6.8 and the triangle inequality (35), we also obtain

Corollary 6.12. *The function*

$$x \mapsto -S_c(x, y)$$

is a critical subsolution for any fixed $y \in \mathbf{V}$.

Combining Corollary 6.9 and Proposition 6.11 we get

Corollary 6.13. *The distance $S_a \neq -\infty$ if and only if $a \geq c$.*

We further have

Proposition 6.14. *Given $y \in \mathbf{V}$, the function $x \mapsto S_a(y, x)$ is solution to (DFEa) if and only if there exists a cycle ξ incident on y with $\sigma_a(\xi) = 0$.*

Proof: (\implies) We will prove in Proposition 6.15 a more general property, namely that if the equation $(\mathcal{DFE}a)$ admits a solution, then there is a cycle ξ with $\sigma_a(\xi) = 0$.

(\impliedby) Assume the existence of a cycle, say $\xi = (e_i)_{i=1}^M$, with $\sigma_a(\xi) = 0$ incident on y . Up to relabelling the e_i 's, we can set $y = o(\xi) = t(\xi)$. We claim that $u := S_a(y, \cdot)$ is a solution on the whole \mathbf{V} . In force of Proposition 6.11, it is enough to prove the assertion at y . We have

$$0 \leq S_a(y, y) = u(y) \leq \sigma_a(e_M) + S_a(y, o(e_M)) \leq \sigma_a(\xi),$$

and since $\sigma_a(\xi) = 0$, all the inequalities in the above formula must indeed be equalities; in particular

$$u(y) - u(t(-e_M)) - \sigma_a(e_M) = u(y) - S_a(y, o(e_M)) - \sigma_a(e_M) = 0$$

with $-e_M \in \mathbf{E}_y$. This proves the claim. □

As announced, we complete the above proof by showing:

Proposition 6.15. *If the equation $(\mathcal{DFE}a)$ admits a solution, then there is a cycle ξ with $\sigma_a(\xi) = 0$.*

Proof: Let us assume that v is a solution to $(\mathcal{DFE}a)$. Take any $x \in \mathbf{V}$; by the definition of solution, we can find an edge e with terminal vertex x such that

$$v(x) - v(o(e)) = \sigma_a(e).$$

By iterating backward the procedure, we can construct for any M a path $\xi = (e_i)_{i=1}^M$ such that

$$(36) \quad v(t(e_j)) - v(o(e_k)) = \sigma_a\left((e_i)_{i=k}^j\right) \quad \text{for any } j \geq k.$$

Since the graph is finite, taking M large enough, we have that for suitable indices $j > k$, the path $(e_i)_{i=k}^j$ is a cycle, so that $v(t(e_j)) - v(o(e_k)) = 0$, and the relation (36) provides the assertion. □

The argument of the next proof is reminiscent of the one used for the existence of critical solutions of Hamilton–Jacobi equations in compact manifolds, see [14].

Theorem 6.16. *The critical equation $(\mathcal{DFE}c)$ admits solutions.*

Proof: We break the argument according to whether $c = a_0$ or $c > a_0$. Let us first discuss the first instance. If in addition $c = a_\gamma$ for some arc γ , and we set $e = \Psi^{-1}(\gamma)$, then we get from (11), (24), (25) that

$$\sigma_c(e \cup (-e)) = 0.$$

If instead $a_0 = c_\gamma$ for some closed arc γ of the network, then $e = \Psi^{-1}(\gamma)$ is a loop and we obtain by Remark 5.14

$$\sigma_c(e) = 0 \quad \text{or} \quad \sigma_c(-e) = 0.$$

In both cases, we infer the existence of a critical solution in the light of Proposition 6.14.

We proceed considering the case $c > a_0$. Let us assume by contradiction that there are no critical solutions. For any $y \in \mathbf{V}$, setting $u_y = S_c(y, \cdot)$, we can therefore find by Proposition 6.11 a positive constant δ_y with

$$(37) \quad \max_{e \in \mathbf{E}_y} (u_y(y) - u_y(t(e)) - \sigma_c(-e)) = -\delta_y.$$

We define $u = \sum_y \lambda_y u_y$, where the λ_y are positive coefficients summing to 1, and set

$$\delta = \min_y \lambda_y \delta_y.$$

Exploiting that all the u_y 's are subsolutions on the whole \mathbf{V} and using (37), we conclude that for any $e \in \mathbf{E}$

$$(38) \quad \begin{aligned} & u(t(e)) - u(o(e)) - \sigma_c(e) \\ &= \sum_{y \neq t(e)} \lambda_y (u_y(t(e)) - u_y(o(e)) - \sigma_c(e)) + \lambda_{t(e)} (u_{t(e)}(t(e)) - u_{t(e)}(o(e)) - \sigma_c(e)) \\ &\leq -\lambda_{t(e)} \delta_{t(e)} \leq -\delta. \end{aligned}$$

Owing to Lemma 5.15 and the fact that $c > a_0$, there is $a_0 < b < c$ with

$$\sigma_b(e) > \sigma_c(e) - \delta \quad \text{for every } e \in \mathbf{E};$$

then we deduce from (38) that

$$u(t(e)) - u(o(e)) - \sigma_b(e) \leq 0 \quad \text{for every } e.$$

This proves that u is a subsolution to $(DFEa)$ with $a = b$, which is impossible because $b < c$. Therefore the maximum in (37) must be 0 for some y_0 , which in turn implies that $S_c(y_0, \cdot)$ is a critical solution, as it was claimed. \square

Remark 6.17. Let u be a solution to $(DFEc)$. Let e be a loop with $o(e) = t(e) = x$, and $\gamma = \Psi(e)$ is hence a closed curve. If $c < c_\gamma$, then, according to Remark 5.14

$$0 = u(o(e)) - u(t(e)) < \sigma_c(e), \quad 0 = u(o(-e)) - u(t(-e)) < \sigma_c(-e)$$

which shows that neither e nor $-e$ realizes

$$\min_{e \in \mathbf{E}_x} (u(t(e)) + \sigma_a(-e)).$$

This in turn implies that the edge e , and consequently $-e$, can be removed from the edges of \mathbf{X} without affecting the status of solution for u or any other critical solution.

Things are different if $c = c_\gamma$ because in this case, see Remark 5.14,

$$0 = \min\{\sigma_c(e), \sigma_c(-e)\} = u(o(e)) - u(t(e)) = u(o(-e)) - u(t(-e)).$$

6.3. The Aubry set and some structural properties of solutions. Inspired by what discussed in the previous subsection, we introduce the following definition.

Definition 6.18. The *Aubry set* is defined as

$$(39) \quad \mathcal{A}_{\mathbf{X}}^* = \mathcal{A}_{\mathbf{X}}^*(\mathcal{H}) = \{e \in \mathbf{E} \mid \text{belonging to some cycle with } \sigma_c(\xi) = 0\}.$$

The *projected Aubry set* is given by

$$(40) \quad \mathcal{A}_{\mathbf{X}} = \mathcal{A}_{\mathbf{X}}(\mathcal{H}) = \{y \in \mathbf{V} \mid \exists \xi \text{ cycle incident on } y \text{ with } \sigma_c(\xi) = 0\}.$$

The projected Aubry set is partitioned in *static classes*, defined as the equivalence classes with respect to the relation

$$S_c(x, y) + S_c(x, y) = 0.$$

Equivalently x and y belong to the same static class if there is a cycle ξ with $\sigma_c(\xi) = 0$ incident on both of them; in particular, the whole cycle ξ is then contained in this static class.

Remark 6.19. Clearly, $x \in \mathcal{A}_{\mathbf{X}}$ if and only if $x = o(e) = t(e')$, for some e, e' in $\mathcal{A}_{\mathbf{X}}^*$; moreover, if $e \in \mathcal{A}_{\mathbf{X}}^*$, then $o(e)$ and $t(e)$ belong to $\mathcal{A}_{\mathbf{X}}$. The converse of this last property is not true because, for instance, if $e \in \mathcal{A}_{\mathbf{X}}^*$ then $-e$ might not belong to $\mathcal{A}_{\mathbf{X}}^*$. It is also possible to have a pair of adjacent vertices belonging to different static classes of $\mathcal{A}_{\mathbf{X}}$ linked by an edge not in $\mathcal{A}_{\mathbf{X}}^*$, or even vertices of the same static classes linked by multiple edges not all belonging to $\mathcal{A}_{\mathbf{X}}^*$.

We immediately derive from Proposition 6.15 and Theorem 6.16 the following result.

Lemma 6.20. *The Aubry sets are nonempty. Moreover*

$$\mathcal{A}_{\mathbf{X}} = \{y \in \mathbf{V} : S_c(y, y) = 0\} = \{y \in \mathbf{V} : S_c(y, \cdot) \text{ is solution to } (\mathcal{DFEc})\}.$$

We have a structural result on critical solutions. By admissible trace g on $\mathbf{V}' \subset \mathbf{V}$ (for the critical equation), we mean a function satisfying

$$(41) \quad g(x) - g(y) \leq S_c(y, x) \quad \text{for any } x, y \text{ in } \mathbf{V}'.$$

Theorem 6.21. *Given an admissible trace g on $\mathcal{A}_{\mathbf{X}}$, the unique solution to (\mathcal{DFEc}) taking the value g on $\mathcal{A}_{\mathbf{X}}$ is*

$$(42) \quad v(x) := \min\{g(y) + S_c(y, x) \mid y \in \mathcal{A}_{\mathbf{X}}\}.$$

In particular, $\mathcal{A}_{\mathbf{X}}$ represents a uniqueness set for the equation.

Proof: Taking into account (41) and the fact that $S_c(y, y) = 0$ for any $y \in \mathcal{A}_{\mathbf{X}}$, we deduce that g and v coincide on $\mathcal{A}_{\mathbf{X}}$. The function v is a subsolution in force of Proposition 6.8. Take $x_0 \in \mathbf{V}$, then

$$v(x_0) = g(y_0) + S_c(y_0, x_0) \quad \text{for some } y_0 \in \mathcal{A}_{\mathbf{X}}.$$

We know that the function

$$\psi(x) = g(y_0) + S_c(y_0, x)$$

is a critical solution, in addition x_0 is a maximizer of $v - \psi$ in \mathbf{V} , consequently

$$\psi(x_0) - \psi(t(e)) \leq v(x_0) - v(t(e)) \quad \text{for any } e \in \mathbf{E}_{x_0}.$$

Since ψ is critical solution, there is $e_0 \in \mathbf{E}_{x_0}$ with

$$0 = \psi(x_0) - \psi(t(e_0)) - \sigma_c(-e_0) \leq v(x_0) - v(t(e_0)) - \sigma_c(-e_0).$$

Since v is a subsolution, the inequality in the above formula must be an equality. This shows that v is a critical solution.

Assume now that w is another solution agreeing with g on $\mathcal{A}_{\mathbf{X}}$. Given any $x \in \mathbf{V}$, we construct, arguing as in Proposition 6.15, a path $\xi = (e_i)_{i=1}^M$ with $t(\xi) = x$ and such that

$$w(t(e_j)) - w(o(e_k)) = \sigma_c\left((e_i)_{i=k}^j\right) \quad \text{for any } j \geq k.$$

If M is sufficiently large, there must exist $j_0 \geq k_0$ such that $(e_i)_{i=k_0}^{j_0}$ is a cycle. We deduce that there are $y \in \mathcal{A}_{\mathbf{X}}$ and a path η linking y to x with

$$w(x) = w(y) + \sigma_c(\eta) \geq g(y) + S_c(y, x) \geq v(x).$$

Since the converse inequality holds true by Proposition 6.8, we get $w(x) = v(x)$. This ends the proof. \square

We record for later use an immediate consequence of the above result:

Corollary 6.22. *Given $\mathbf{V}' \subset \mathcal{A}_{\mathbf{X}}$, and an admissible trace g on it, the function*

$$(43) \quad v(x) := \min\{g(y) + S_c(y, x) \mid y \in \mathbf{V}'\}$$

is a solution to (DFEc) taking the value g on \mathbf{V}' .

We can also derive a representation formula for solutions at $a > c$ in some subset of \mathbf{V} . To help understanding the next statement, we recall that $S_a(x, x) > 0$ for any $x \in \mathbf{V}$ whenever $a > c$.

Theorem 6.23. *Let $a > c$, $\mathbf{V}' \subset \mathbf{V}$. Let g be a function defined on \mathbf{V}' satisfying (41) with S_a in place of S_c , then the function*

$$v(x) = \begin{cases} g(x) & \text{if } x \in \mathbf{V}' \\ \min\{g(y) + S_a(y, x) \mid y \in \mathbf{V}'\} & \text{if } x \notin \mathbf{V}' \end{cases}$$

is the unique solution to $(DFEa)$ in $\mathbf{V} \setminus \mathbf{V}'$ agreeing with g on \mathbf{V}' . It is in addition subsolution on the whole of \mathbf{V} .

Proof: We claim that

$$(44) \quad v(z) - v(x) \leq S_a(x, z) \quad \text{for any } z, x \text{ in } \mathbf{V}.$$

The property is true by assumption if both z, x are in \mathbf{V}' , if instead z, y are in $\mathbf{V} \setminus \mathbf{V}'$ we have

$$v(z) - v(x) \leq g(y) + S_a(y, z) - g(y) - S_a(y, x) \leq S_a(x, z),$$

where $y \in \mathbf{V}'$ is optimal for $v(x)$ and we have exploited the triangle inequality (35). If $z \notin \mathbf{V}', x \in \mathbf{V}'$, then (44) directly comes from the very definition of v . Finally, if $z \in \mathbf{V}', x \notin \mathbf{V}'$, we denote by y an optimal element in \mathbf{V}' and use the triangle inequality to write

$$v(z) - v(x) = g(z) - g(y) - S_a(y, x) \leq S_a(y, z) - S_a(y, x) \leq S_a(x, z).$$

This concludes the proof of claim (44) and therefore shows, according to Proposition 6.8, that v is a subsolution in \mathbf{V} . Taking into account that $S_a(y, \cdot)$ is solution in $\mathbf{V} \setminus \mathbf{V}'$, we also get, arguing as in Theorem 6.21, that v is solution in $\mathbf{V} \setminus \mathbf{V}'$. Uniqueness follows from Proposition 6.6. \square

7. BACK TO THE NETWORK

In this section we switch our attention back to the network Γ , or in other terms, we give again visibility, besides the vertices, to the interior points of the arcs. We combine the global information gathered on the abstract graph with the outputs of the local analysis on the arcs of the network. We define an appropriate notion of Aubry set and provide a PDE characterization of its points.

Exploiting the richer (differentiable) structure of Γ , we establish, on the basis of our findings in the previous section, some regularity properties for critical subsolutions and solutions. This will generalize what is known for the continuous case in the framework of Weak KAM theory, see for example [12]. Finally, we give specific uniqueness results and representation formulae for solutions on the network.

7.1. Subsolutions and solutions on Γ . The next result shows, as pointed out already in the Introduction, how the notion of solution to $(\mathcal{H}Ja)$ can be recovered from the notion of subsolution. The relevance of the issue is that the latter just requires the usual subsolution property on any arc and continuity at the junctures. The argument significantly illustrates the interplay between the immersed network and underlying abstract graph.

Theorem 7.1. *Let $a \geq c$ and $y \in \Gamma$, then the maximal subsolution to $(\mathcal{H}Ja)$ attaining a given value at y is solution in $\Gamma \setminus \{y\}$.*

Proof: We can assume $y \in \Gamma \setminus \mathbf{V}$ otherwise the assertion is a consequence of Propositions 6.8, 6.11 and Proposition 6.3 with $\mathbf{V}' = \{y\}$. It is not restrictive to take 0 as value assigned at y . We therefore denote by v the maximal subsolution vanishing at y , see Proposition 3.11. We select $\gamma \in \mathcal{E}$ such that $y = \gamma(s_0)$ for some $s_0 \in (0, 1)$, and set $e = \Psi^{-1}(\gamma)$. We first assume that γ is not a closed arc. Since v must be in particular subsolution in the arc γ , we have by Corollary 5.7

$$\begin{aligned} v(\gamma(1)) &\leq \int_{s_0}^1 \sigma_a^+(t) dt =: \beta \\ v(\gamma(0)) &\leq - \int_0^{s_0} \sigma_a^-(t) dt =: \alpha, \end{aligned}$$

where σ_a^+ , σ_a^- are defined as in (8), (9). The maximal admissible trace g , in the sense of (41), on $\mathbf{V}' := \{o(e), t(e)\}$ dominated by α at $o(e) = \gamma(0)$, and β at $t(e) = \gamma(1)$ is

$$\begin{aligned} \alpha^* &:= \min\{\alpha, \beta + S_a(t(e), o(e))\} \\ \beta^* &:= \min\{\alpha, \beta + S_a(o(e), t(e))\}. \end{aligned}$$

According to Proposition 6.8, Theorem 6.23 and Corollary 6.22, the function $w : \mathbf{V} \rightarrow \mathbb{R}$ defined as

$$w(x) = \begin{cases} \alpha^* & \text{if } x = o(e) \\ \beta^* & \text{if } x = t(e) \\ \min\{\alpha^* + S_a(o(e), x), \beta^* + S_a(t(e), x)\} & \text{if } x \neq o(e) \text{ and } x \neq t(e) \end{cases}$$

is the maximal subsolution to $(DFEa)$ on \mathbf{V} agreeing with α^* , β^* at the vertices of e . It is in addition solution in $\mathbf{V} \setminus \{\gamma(0), \gamma(1)\}$. By Proposition 6.3 it can thus be extended to a subsolution of $(\mathcal{H}Ja)$ in Γ , denoted by \bar{w} , which is in addition solution in $\Gamma \setminus \{\gamma(0), \gamma(1)\}$. The function \bar{w} is the maximal subsolution to $(\mathcal{H}Ja)$ taking the values α^* , β^* on the vertices of γ , but it does not necessarily vanish at y . We have in any case

$$(45) \quad v \leq \bar{w} \quad \text{in } \Gamma.$$

To complete the proof, we need to suitably adjust \bar{w} inside γ in order to attain the value 0 at y . To this end, we proceed by showing that the boundary data α^* , 0 and 0, β^* are admissible, in the sense of (17), for $(HJ_\gamma a)$ restricted to the subintervals $[0, s_0]$ and $[s_0, 1]$, respectively. In fact,

$$(46) \quad \alpha^* \leq \alpha = - \int_0^{s_0} \sigma_a^-(t) dt,$$

and if a strict inequality prevails in the above formula, we get

$$(47) \quad \alpha^* = \int_{s_0}^1 \sigma_a^+(t) dt + S_a(t(e), o(e)).$$

Let us consider a cycle in \mathbf{X} of the form $\xi \cup e$, where ξ be a path in linking $t(e)$ to $o(e)$ with $\sigma_a(\xi) = S_a(t(e), o(e))$, see Corollary 6.10. Then $\sigma_a(\xi \cup e) \geq 0$ and consequently

$S_a(t(e), o(e)) \geq -\sigma_a(e)$. By plugging this relation in (47) and recalling the definition of $\sigma_a(e)$, we get

$$(48) \quad \alpha^* \geq \int_{s_0}^1 \sigma_a^+(t) dt - \int_0^1 \sigma_a^+(t) dt = - \int_0^{s_0} \sigma_a^+(t) dt.$$

By combining (46), (48) we have

$$\int_0^{s_0} \sigma_a^-(t) dt \leq -\alpha^* \leq \int_0^{s_0} \sigma_a^+(t) dt,$$

proving the claimed admissibility property in $[0, s_0]$. A straightforward modification of the previous argument shows the same in $[s_0, 1]$. Thus, there exists a function u on $\gamma([0, 1])$ uniquely determined by requiring $u \circ \gamma$ to be solution to $(HJ_\gamma a)$ in $(0, s_0)$ and $(s_0, 1)$, and in addition to take the values α^* , 0 , β^* at $\gamma(0)$, y , $\gamma(1)$, respectively. This is also the maximal subsolution of $(HJ_\gamma a)$ in $(0, 1)$ taking such values at the boundary points and at $s = s_0$. The function

$$\bar{w}(x) = \begin{cases} \bar{w} & \text{in } \Gamma \setminus \gamma[0, 1] \\ u & \text{in } \gamma[0, 1] \end{cases}$$

is subsolution to $(\mathcal{H}Ja)$ in Γ and by the maximality property of u on γ and (45)

$$v \leq \bar{w} \quad \text{in } \Gamma,$$

which immediately implies $v = \bar{w}$.

The function v is by construction solution to $(\mathcal{H}Ja)$ in $\Gamma \setminus \{\gamma(0), y, \gamma(1)\}$. Moreover, taking into account Remark 5.2 and Proposition 5.6 applied to the subinterval $[0, s_0]$, we see that if $\bar{w}(\gamma(0)) = \alpha$ then \bar{w} satisfies condition iii) in definition of solution to $(\mathcal{H}Ja)$ at $\gamma(0)$ with respect to the arc $\tilde{\gamma}$. If instead $\bar{w}(o(e)) = \alpha + S_a(t(e), o(e))$ then again condition iii) of definition of solution is satisfied with respect to some arc different from γ , $\tilde{\gamma}$ because of Propositions 6.11 and 6.3. Similarly, we prove that v is solution at $\gamma(1)$. This concludes the proof if γ is not a closed arc.

If instead γ is a closed arc, then we indicate by w the maximal periodic subsolution of $(HJ_\gamma a)$ in $(0, 1)$ vanishing at $s = s_0$, see Corollary 5.13. Arguing as in the first part of the proof, we see that the maximal subsolution v to $(\mathcal{H}Ja)$ in Γ vanishing at y is given by

$$v(x) = \begin{cases} w(\gamma^{-1}(x)) & \text{in } \gamma([0, 1]) \\ w(\gamma(0)) + S_a(\gamma(0), x) & \text{in } \Gamma \setminus \gamma([0, 1]). \end{cases}$$

Taking into account the representation formulae for w provided in item ii) of Corollary 5.13 and arguing again as in the first part of the proof, we show that v is solution to $(\mathcal{H}Ja)$ in $\Gamma \setminus \{y\}$, as it was claimed. \square

7.2. Aubry set in Γ . We define the Aubry set \mathcal{A}_Γ on the network as

$$(49) \quad \mathcal{A}_\Gamma := \{x \in \mathbb{R}^N \mid x = \Psi(e)(t) \text{ for some } e \in \mathcal{A}_{\mathbf{X}}^*, t \in [0, 1]\}.$$

One could also consider a lift of \mathcal{A}_Γ to the tangent bundle $T\Gamma$, as in continuous case. For example, this could be useful to study the analogues in this setting of Mather's measures, Mather sets, minimal average actions, etc. (see for example [12, 28] for precise definitions); this discussion, however, would go beyond our current objectives, so we decided to postpone it to a future investigation.

Remark 7.2. We point out for later use that the support of an arc γ belongs to \mathcal{A}_Γ if and only if $\gamma = \Psi(e)$ and at least one between e or $-e$ is in $\mathcal{A}_{\mathbf{X}}^*$.

The first lemma regards subsolutions to the critical equation on \mathbf{X} . Briefly, it says that – analogously to what happens in the continuous case, see [12] – the differential of a critical subsolution is prescribed on the Aubry set and that critical subsolutions are never strict on the Aubry set. On the other hand, it is always possible to find critical subsolutions that are strict outside the Aubry set. This will be used in the next subsection to obtain the same results on networks. See Theorems 7.5, 7.6.

Lemma 7.3. *Given a subsolution u to (DFEc), one has*

$$(50) \quad \langle du, e \rangle = \sigma_a(e) \quad \text{for any } e \in \mathcal{A}_{\mathbf{X}}^*.$$

Furthermore, there exists a subsolution w to (DFEc) with

$$(51) \quad \langle dw, e \rangle < \sigma_a(e) \quad \text{for any } e \in \mathbf{E} \setminus \mathcal{A}_{\mathbf{X}}^*.$$

Proof: Let u be a critical subsolution and assume for purposes of contradiction that

$$\langle du, \bar{e} \rangle < \sigma_a(\bar{e}) \quad \text{for some } \bar{e} \in \mathcal{A}_{\mathbf{X}}^*.$$

By the very definition of Aubry set, we can find a cycle $\xi = (e_i)_{i=1}^M$ such that $\bar{e} = e_j$ for some $j = 1, \dots, M$ and $\sigma_c(\xi) = 0$. Taking into account that u is a subsolution, we have

$$\langle du, e_i \rangle \leq \sigma_a(e_i) \quad \text{for } i \neq j \quad \text{and} \quad \langle du, e_j \rangle < \sigma_a(e_j).$$

This implies

$$0 = \sum_i \langle du, e_i \rangle < \sum_i \sigma_c(e_i) = \sigma_c(\xi) = 0,$$

which is impossible. We pass to the second part of the statement. We start constructing for any $e_0 \in \mathbf{E} \setminus \mathcal{A}_{\mathbf{X}}^*$ a critical subsolution u_{e_0} with

$$(52) \quad \langle du_{e_0}, e_0 \rangle < \sigma_a(e_0).$$

The argument will be organized taking into account the classification of edges in $\mathcal{A}_{\mathbf{X}}^*$ provided in Remark 6.19. If $t(e_0) \notin \mathcal{A}_{\mathbf{X}}$, then we set $u_{e_0} = S_c(t(e_0), \cdot)$, according to

Lemma 6.20, u_{e_0} is not a critical solution at $t(e_0)$ which implies (52). If $t(e_0) \in \mathcal{A}_{\mathbf{X}}$, we consider the critical subsolutions $S_c(t(e_0), \cdot)$ and $-S_c(\cdot, t(e_0))$, see Proposition 6.11 and Corollary 6.12. Taking into account the characterization of $\mathcal{A}_{\mathbf{X}}$ given in Lemma 6.20, we have

$$\begin{aligned} -S_c(t(e_0), o(e_0)) &= S_c(t(e_0), t(e_0)) - S_c(t(e_0), o(e_0)) \leq \sigma_c(e_0) \\ S_c(o(e_0), t(e_0)) &= -S_c(t(e_0), t(e_0)) + S_c(o(e_0), t(e_0)) \leq \sigma_c(e_0). \end{aligned}$$

If equality prevails in both above formulae, we get

$$S_c(o(e_0), t(e_0)) + S_c(t(e_0), o(e_0)) = 0$$

which is possible if and only if both $o(e_0)$, $t(e_0)$ are in the Aubry set and belong to the same static class. If this is not the case, we satisfy (52) up to choosing u_{e_0} equals to $S_c(t(e_0), \cdot)$ or $-S_c(\cdot, t(e_0))$. If instead the two vertices are in the same static class, we claim that

$$(53) \quad S_c(t(e_0), t(e_0)) - S_c(t(e_0), o(e_0)) = -S_c(t(e_0), o(e_0)) < \sigma_c(e_0).$$

In fact, we know, by the very definition of static class, that there is a path ξ linking $t(e_0)$ to $o(e_0)$ with all the edges belonging to $\mathcal{A}_{\mathbf{X}}^*$. Therefore, using Lemma 6.20 and the first part of the statement that we have just proven, applied to the critical subsolution $-S_c(\cdot, o(e_0))$, we have that

$$S_c(t(e_0), o(e_0)) = -S_c(o(e_0), o(e_0)) + S_c(t(e_0), o(e_0)) = \sigma_c(\xi).$$

Were (53) false, we should further have

$$0 = -S_c(t(e_0), o(e_0)) + S_c(t(e_0), o(e_0)) = \sigma_c(\xi \cup e_0)$$

and consequently $e_0 \in \mathcal{A}_{\mathbf{X}}^*$, which is impossible. Formula (52) is therefore satisfied with $u_{e_0} = S_c(t(e_0), \cdot)$. This completes the proof of (52).

We conclude arguing along the same lines of Theorem 6.16. Given $e \in \mathbf{E} \setminus \mathcal{A}_{\mathbf{X}}^*$, we denote by u_e a critical subsolution satisfying (52) with e in place of e_0 . We choose positive constants λ_e , for $e \in \mathbf{E} \setminus \mathcal{A}_{\mathbf{X}}^*$, summing to 1, and define a critical subsolution via

$$w = \sum_{e \in \mathbf{E} \setminus \mathcal{A}_{\mathbf{X}}^*} \lambda_e u_e.$$

Given $e_0 \in \mathbf{E} \setminus \mathcal{A}_{\mathbf{X}}^*$, we have

$$\langle dw, e_0 \rangle = \sum_{e \neq e_0} \lambda_e \langle du_e, e_0 \rangle + \lambda_{e_0} \langle du_{e_0}, e_0 \rangle < \sigma_c(e_0),$$

as we wished to prove. \square

We derive a PDE characterization of points in the Aubry set, generalizing a property of the continuous case.

Proposition 7.4. *The maximal subsolution to $(\mathcal{H}Jc)$ taking a given value at a point $y \in \Gamma$ is a critical solution on the whole network if and only if $y \in \mathcal{A}_\Gamma$.*

Proof: If $y \in \mathbf{V}$, the assertion comes from Lemma 6.20, we can then assume from now on that $y \in \Gamma \setminus \mathbf{V}$. We prescribe, without loss of generality, the value 0 at y , and denote by v the maximal subsolution vanishing at y , see Proposition 3.11. We denote by γ an arc whose support contains y .

We first assume that γ is not a closed curve. Taking into account Theorem 7.1, it is enough to show that v is solution at y if and only if $y \in \mathcal{A}_\Gamma$. Looking at the proof of Theorem 7.1, we see that the solution property at y is in turn equivalent to the following: the solution of $(HJ_\gamma c)$ in $(0, 1)$ taking the values $v(\gamma(0))$, $v(\gamma(1))$ at 0, 1, respectively, vanishes at $s = s_0$. In the light of Proposition 5.5, this boils down to show

$$(54) \quad \min\{v(\gamma(0)) + \mathbf{A}, v(\gamma(1)) - \mathbf{B}\} = 0,$$

where σ_c^+ , σ_c^- are defined as in (8), (9), respectively, and

$$\mathbf{A} = \int_0^{s_0} \sigma_c^+(t) dt \quad \mathbf{B} = \int_{s_0}^1 \sigma_c^-(t) dt.$$

Taking into account the proof of Theorem 7.1, we know that

$$(55) \quad v(\gamma(0)) = \min\{-\mathbf{D}, \mathbf{C} + S_c(\gamma(1), \gamma(0))\}$$

$$(56) \quad v(\gamma(1)) = \min\{\mathbf{C}, -\mathbf{D} + S_c(\gamma(0), \gamma(1))\}$$

where

$$\mathbf{C} = \int_{s_0}^1 \sigma_c^+(t) dt \quad \mathbf{D} = \int_0^{s_0} \sigma_c^-(t) dt.$$

Then

$$(57) \quad v(\gamma(0)) + \mathbf{A} = \begin{cases} \int_0^{s_0} [\sigma_c^+(t) - \sigma_c^-(t)] dt & \text{if } v(\gamma(0)) = -\mathbf{D} \\ \int_0^1 \sigma_c^+(t) dt + S_c(\gamma(1), \gamma(0)) & \text{if } v(\gamma(0)) = \mathbf{C} + S_c(\gamma(1), \gamma(0)) \end{cases}$$

and

$$(58) \quad v(\gamma(1)) - \mathbf{B} = \begin{cases} \int_{s_0}^1 [\sigma_c^+(t) - \sigma_c^-(t)] dt & \text{if } v(\gamma(1)) = \mathbf{C} \\ -\int_0^1 \sigma_c^-(t) dt + S_c(\gamma(0), \gamma(1)) & \text{if } v(\gamma(1)) = -\mathbf{D} + S_c(\gamma(0), \gamma(1)). \end{cases}$$

Exploiting the property that $\sigma_c(\xi) \geq 0$ for any cycle ξ in \mathbf{X} , we see that

$$S_c(\gamma(0), \gamma(1)) \geq -\sigma_c(-e) = \int_0^1 \sigma_c^-(t) dt$$

$$S_c(\gamma(1), \gamma(0)) \geq -\sigma_c(e) = -\int_0^1 \sigma_c^+(t) dt.$$

Equality holds in the first formula if and only if there is a cycle ξ with $-e \subset \xi$, $\sigma_c(\xi) = 0$, and in the second one if and only if there a cycle η with $e \subset \eta$, $\sigma_c(\eta) = 0$. We in addition have that

$$\int_0^{s_0} [\sigma_c^+(t) - \sigma_c^-(t)] dt = 0 \quad \text{or} \quad \int_{s_0}^1 [\sigma_c^+(t) - \sigma_c^-(t)] dt = 0$$

if and only if $c = a_\gamma$, and this case both e and $-e$ belong to $\mathcal{A}_{\mathbf{X}}^*$. In the light of the above remarks, (57), (58), we conclude that (54) holds if and only if $y \in \mathcal{A}_\Gamma$.

This concludes the proof when γ is not a closed arc. The argument for γ closed arc goes along the same lines just adapting the representation formulae for solutions of $(HJ_\gamma c)$ and taking into account Corollary 5.13. \square

7.3. Regularity results for critical subsolutions. We state and prove the main regularity results of this section. They can be considered as a generalization to the network setting of the results in [13].

Theorem 7.5. *Any critical subsolution $u : \Gamma \rightarrow \mathbb{R}$ is of class C^1 in $\mathcal{A}_\Gamma \setminus \mathbf{V}$, and all such subsolutions possess the same differential in $\mathcal{A}_\Gamma \setminus \mathbf{V}$.*

Proof: Let u be a critical subsolution on Γ and $\gamma = \Psi(e)$ an arc with $e \in \mathcal{A}_{\mathbf{X}}^*$. According to Lemma 7.3, formula (50)

$$u(\gamma(1)) - u(\gamma(0)) = \sigma_c(e),$$

therefore $u \circ \gamma$ is the maximal subsolution taking the value $u(\gamma(0))$ at $s = 0$ and, according to Proposition 5.6, has the form

$$u(\gamma(s)) = \int_0^s \sigma_c^+(t) dt,$$

where σ_c^+ is as in (8) with H_γ in place of H and c in place of a . We deduce that $s \mapsto u(\gamma(s))$ is of class C^1 for $t \in (0, 1)$ and for any $x = \gamma(t_0)$, with $t_0 \in (0, 1)$, the differential $D_\Gamma u(x)$ is uniquely determined among the elements of $T_\Gamma^*(x)$ by the condition

$$(D_\Gamma u(x), \dot{\gamma}(t_0)) = \left. \frac{d}{dt} u(\gamma(t)) \right|_{t=t_0} = \sigma_c^+(t_0).$$

This concludes the proof. \square

Moreover:

Theorem 7.6. *For any critical subsolution w on \mathbf{X} , there exists a critical subsolution u on Γ , with $w = u$ on \mathbf{V} , which is of class C^1 in $\Gamma \setminus \mathbf{V}$. There exists in addition a critical subsolution v on Γ of class $C^1(\Gamma \setminus \mathbf{V})$ satisfying*

$$\mathcal{H}(x, D_\Gamma v(x)) < c \quad \text{for } x \in \Gamma \setminus (\mathcal{A}_\Gamma \cup \mathbf{V}).$$

Proof: Let w be a critical subsolution in \mathbf{X} . Given any arc $\gamma = \Psi(e)$, we know, see Proposition 6.2, that $w(\gamma(0))$ and $w(\gamma(1))$ satisfy the compatibility condition (17), so that

$$(59) \quad w(\gamma(0)) + \int_0^1 \sigma_c^-(t) dt \leq w(\gamma(1)) \leq w(\gamma(0)) + \int_0^1 \sigma_c^+(t) dt,$$

where σ_c^+ , σ_c^- are defined as in (8), (9) with H_γ , c in place of H , a , respectively. We can therefore find $\lambda \in [0, 1]$ with

$$(60) \quad w(\gamma(1)) = w(\gamma(0)) + \int_0^1 [\lambda \sigma_c^-(t) + (1 - \lambda) \sigma_c^+(t)] dt,$$

and the function

$$(61) \quad s \mapsto w(\gamma(0)) + \int_0^s [\lambda \sigma_c^-(t) + (1 - \lambda) \sigma_c^+(t)] dt$$

is a subsolution of class C^1 to $H_\gamma = c$ in $(0, 1)$ taking the values $w(\gamma(0))$, $w(\gamma(1))$ at $s = 0$ and $s = 1$, respectively. This shows the first part of the assertion.

As far as the second claim is concerned, we proceed by taking a critical subsolution w satisfying (51). This implies that strict inequalities prevail in formula (59) whenever $\gamma = \Psi(e)$ with e , $-e$ not in $\mathcal{A}_\mathbf{X}^*$. The λ appearing in (60) can be consequently taken in $(0, 1)$, so that the function defined in (61) is a strict subsolution to $H_\gamma = c$. This concludes the proof in the light of Remark 7.2. \square

Remark 7.7. Notice that if we apply the procedure of first part of the previous result starting with a critical solution rather than a critical subsolution, then the property of being solution could be possibly false for the regularized function.

7.4. Representation formulae and uniqueness results on the network. In this section, we want to provide representation formulae and uniqueness results with traces that are not necessarily defined on vertices, but on a general subset of the network Γ . To this aim, we extend S_a , for $a \geq c$, from \mathbf{V} to the whole Γ defining a semidistance intrinsically related to \mathcal{H} and the level a . This is basically the same object introduced in [26]. We do not develop here any further the metric point of view, but just use it to establish an admissibility condition for data assigned on subsets of Γ , and provide representation formulae.

Given a portion of arc $\gamma|_{[s_1, s_2]}$, for $0 \leq s_1 \leq s_2 \leq 1$, we define

$$\ell_a \left(\gamma|_{[s_1, s_2]} \right) = \int_{s_1}^{s_2} (\sigma_a^+)^{\gamma}(t) dt,$$

where $(\sigma_a^+)^{\gamma}$ is defined as in (8). We get in particular, for the whole arc, the relation

$$(62) \quad \ell_a(\gamma) = \sigma_a(\Psi^{-1}(\gamma)) \quad \text{for any } \gamma \in \mathcal{E}.$$

We define ℓ_a for a curve on Γ given by a finite number of concatenated arcs or portions of arcs as the sum of the lengths of the arcs or portion of arcs making it up. We introduce the related geodesic (semi)distance on Γ via

$$(63) \quad S_a^\Gamma(x, y) = \min\{\ell_a(\xi) \mid \xi \text{ union of concatenated arcs linking } x \text{ to } y\}.$$

We deduce from the results on σ_a and (62) the following lemma.

Lemma 7.8.

- i) If $x \neq y$ are in \mathbf{V} , then $S_a(x, y) = S_a^\Gamma(x, y)$.
- ii) If ξ is a closed curve on Γ , then $\ell_a(\xi) \geq 0$.

It is easy to check that the maximal subsolution v to $(DFEa)$ vanishing at $y \in \Gamma$ given in Theorem 7.1 and Proposition 7.4 is

$$v(x) = S_a^\Gamma(y, x) \quad \text{for any } a \geq c, x \in \Gamma.$$

We derive, taking also into account Proposition 6.8, that for a continuous function $u : \Gamma \rightarrow \mathbb{R}$, the condition

$$(64) \quad u(x) - u(y) \leq S_a^\Gamma(y, x) \quad \text{for any pair } x, y \text{ in } \Gamma'$$

is necessary and sufficient for being subsolution to $(\mathcal{H}Ja)$. Given a function g defined on a subset Γ' of Γ , we therefore introduce the following admissibility condition for $(DFEa)$

$$(65) \quad g(x) - g(y) \leq S_a^\Gamma(x, y) \quad \text{for any } x, y \text{ in } \Gamma'.$$

We give in the next theorem a couple of examples of uniqueness results for solutions to $(DFEa)$, and corresponding representation formulae, one can obtain prescribing values on subsets not necessarily contained in \mathbf{V} . Further results are reachable following the same line. Similar formulae, even if for subsets of vertices and just in the supercritical case, have been already obtained in [26].

Theorem 7.9. *Let Γ' be a closed subset of Γ and g an admissible trace defined on it, in the sense of (65). We set*

$$v(x) = \min\{g(y) + S_a^\Gamma(y, x) \mid y \in \Gamma'\}.$$

- (i) CRITICAL CASE: *if $a = c$ and $\Gamma' \subset \mathcal{A}_\Gamma$ with*

$$(66) \quad \Gamma' \cap \gamma([0, 1]) \neq \emptyset \quad \text{for any } \gamma \text{ with } \Psi^{-1}(\gamma) \in \mathcal{A}_{\mathbf{X}}^*,$$

then v is the unique solution in Γ to $\mathcal{H}(x, Du) = c$ agreeing with g on Γ' .

- (ii) SUPERCRITICAL CASE: *If $a > c$, then v is uniquely characterized by the properties of being in $C(\Gamma, \mathbb{R})$, being solution of $(\mathcal{H}Ja)$ in $\Gamma \setminus \Gamma'$, and agreeing with g on Γ' .*

Proof: The solution property of v in both cases, in Γ and $\Gamma \setminus \Gamma'$ respectively, follows directly from being a subsolution in Γ , in force of (64), and satisfying the subgradient test as minimum of solutions, in Γ and $\Gamma \setminus \Gamma'$ respectively. In addition v is the maximal solution

(in Γ or $\Gamma \setminus \Gamma'$) agreeing with g on Γ' in force of Theorem 7.1, Proposition 7.4, and the admissibility condition (65).

Now, assume u to be another solution taking the value g on Γ' , by adapting the backward procedure explained in Proposition 6.15 and Theorem 6.21, we construct, for any $x \in \Gamma \setminus \Gamma'$, a curve ξ made up by concatenated arcs or portion of arcs starting at some point $y \in \Gamma'$ and ending at x with

$$u(x) = g(y) + \ell_a(\xi) \geq v(x).$$

In the critical case condition (66) plays a crucial role for this. The maximality property of v then implies that equality must hold in the above formula. This ends the proof. \square

8. SUMMARY OF THE MAIN RESULTS

In this final section we summarize our results for the Hamilton–Jacobi equation posed on the network.

Main Theorem. *Let Γ be an embedded network (finite, connected, possibly including loops and more arcs connecting two vertices) and let $\mathbf{X} = (\mathbf{V}, \mathbf{E})$ be the underlying abstract graph. Let us consider a Hamiltonian $\mathcal{H} = \{H_\gamma\}_{\gamma \in \mathcal{E}}$ on the network, satisfying conditions **(H γ 1)**–**(H γ 4)** for any $\gamma \in \mathcal{E}$ and let a_0 denote the value defined in (7). Then:*

I. GLOBAL SOLUTIONS:

- (i) **(EXISTENCE)** *There exists a unique value $c = c(\mathcal{H}) \geq a_0$ – called Mañé critical value – for which the equation $\mathcal{H}(x, Du) = c$ admits global solutions. In particular, these solutions are Lipschitz continuous on Γ .*
- (ii) **(UNIQUENESS)** *Let $\mathcal{A}_{\mathbf{X}} = \mathcal{A}_{\mathbf{X}}(\mathcal{H}) \subseteq \mathbf{V}$ be the (projected) Aubry set associated to \mathcal{H} and let $S_c : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$ be the function defined in (34). Then, given any admissible trace g on $\mathcal{A}_{\mathbf{X}}$, i.e., a function $g : \mathcal{A}_{\mathbf{X}} \rightarrow \mathbb{R}$ such that for every $x, y \in \mathcal{A}_{\mathbf{X}}$*

$$g(x) - g(y) \leq S_c(y, x),$$

there exists a unique global solution $u \in C(\Gamma, \mathbb{R})$ to $\mathcal{H}(x, Du) = c$ agreeing with g on $\mathcal{A}_{\mathbf{X}}$. Conversely, for any solution u to $\mathcal{H}(x, Du) = c$, the function $g = u|_{\mathcal{A}_{\mathbf{X}}}$ gives rise to admissible trace on $\mathcal{A}_{\mathbf{X}}$.

- (iii) **(HOPF–LAX TYPE FORMULA 1)** *Let $g : \mathcal{A}_{\mathbf{X}} \rightarrow \mathbb{R}$ be an admissible trace and $u \in C(\Gamma, \mathbb{R})$ the corresponding solution to $\mathcal{H}(x, Du) = c$. Then, on the support of any arc $\gamma \in \mathcal{E}$, u is given by*

$$u(\gamma(s)) = \min\{\mathbf{A}, \mathbf{B}\},$$

where

$$\mathbf{A} := \min\{g(y) + S_c(y, \gamma(0)) \mid y \in \mathcal{A}_{\mathbf{X}}\} + \int_0^s \sigma_c^+(t) dt$$

$$\mathbf{B} := \min\{g(y) + S_c(y, \gamma(1)) \mid y \in \mathcal{A}_{\mathbf{X}}\} - \int_s^1 \sigma_c^-(t) dt,$$

with $s \in [0, 1]$ and σ_c^+ , σ_c^- defined as in (8), (9) with H_γ in place of H .

(iv) (HOPF–LAX TYPE FORMULA 2): Let Γ' be a closed subset of Γ with

$$\Gamma' \cap \gamma([0, 1]) \neq \emptyset \quad \text{for any } \gamma \text{ with } \Psi^{-1}(\gamma) \in \mathcal{A}_{\mathbf{X}}^*.$$

For any admissible trace g on Γ' , in the sense of (65) with c in place of a , there exists a unique solution $u \in C(\Gamma, \mathbb{R})$ to $\mathcal{H}(x, Du) = c$ agreeing with g on Γ' , which is given by

$$u(x) = \min\{g(y) + S_c^\Gamma(y, x) \mid y \in \Gamma'\},$$

where $S_c^\Gamma(\cdot, \cdot)$ denotes the intrinsic (semi)distance defined in (63).

II. SUBSOLUTIONS:

- (i) (MAXIMAL SUBSOLUTIONS) For $a \geq c$, $y \in \Gamma$, the maximal subsolution to $(\mathcal{H}Ja)$ taking an assigned value at y is solution in $\Gamma \setminus \{y\}$.
- (ii) (PDE CHARACTERIZATION OF THE AUBRY SET) Let $\mathcal{A}_\Gamma = \mathcal{A}_\Gamma(\mathcal{H}) \subset \Gamma$ be the Aubry set on the network, as defined in (49). The maximal subsolution to $(\mathcal{H}Jc)$ taking a given value at a point $y \in \Gamma$ is a critical solution on the whole network if and only if $y \in \mathcal{A}_\Gamma$.
- (iii) (REGULARITY OF CRITICAL SUBSOLUTIONS) Any subsolution $v : \Gamma \rightarrow \mathbb{R}$ to $\mathcal{H}(x, Du) = c$ is of class $C^1(\Gamma \setminus \mathbf{V})$ and they all possess the same differential on $\mathcal{A}_\Gamma \setminus \mathbf{V}$. More specifically, if $x_0 \in \mathcal{A}_\Gamma$ and $x_0 = \gamma(s_0)$, for some $\gamma \in \mathcal{E}$ and $s_0 \in (0, 1)$, then its differential at x_0 is uniquely determined by the relation

$$(D_\Gamma v(x_0), \dot{\gamma}(s_0)) = \sigma_c^+(s_0),$$

where σ_c^+ was defined in (8), and therefore

$$v(\gamma(s)) = v(\gamma(0)) + \int_0^s \sigma_c^+(t) dt \quad \text{for any } s \in [0, 1].$$

We infer from this that any pair of critical subsolutions differs by a constant on the support of γ .

- (iv) (EXISTENCE OF C^1 CRITICAL SUBSOLUTIONS) *Given a function $g : \mathbf{V} \rightarrow \mathbb{R}$ such that*

$$g(x) - g(y) \leq S_c(y, x) \quad \forall x, y \in \mathbf{V},$$

there exists a critical subsolution v on Γ , with $v = g$ on \mathbf{V} , which is of class C^1 on $\Gamma \setminus \mathbf{V}$.

In addition, there exists a critical subsolution v of class $C^1(\Gamma \setminus \mathbf{V})$ satisfying

$$H_\gamma(s, Dv(\gamma(s))) < c$$

for all $s \in (0, 1)$ and $\gamma \in \mathcal{E}$ with $\gamma((0, 1)) \cap \mathcal{A}_\Gamma = \emptyset$.

- (v) (HOPF–LAX FORMULA FOR MAXIMAL SUPERCRITICAL SUBSOLUTIONS 1) *Let $a > c$ and $\mathbf{V}' \subset \mathbf{V}$. For any $g : \mathbf{V}' \rightarrow \mathbb{R}$ satisfying*

$$g(x) - g(y) \leq S_a(y, x) \quad \forall x, y \in \mathbf{V}',$$

where $S_a(\cdot, \cdot)$ was defined in (34), there exists a unique solution u to $\mathcal{H}(x, Du) = a$ in $\Gamma \setminus \mathbf{V}'$ agreeing with g on \mathbf{V}' ; in addition, u is also a subsolution to $\mathcal{H}(x, Du) = a$ on the whole Γ . In particular, on the support of any arc $\gamma \in \mathcal{E}$, u is given by:

$$u(\gamma(s)) = \min\{\mathbf{C}, \mathbf{D}\},$$

where

$$\mathbf{C} := \tilde{g}(\gamma(0)) + \int_0^s \sigma_a^+(t) dt$$

$$\mathbf{D} := \tilde{g}(\gamma(1)) - \int_s^1 \sigma_a^-(t) dt$$

$$\tilde{g}(x) := \begin{cases} g(x) & \text{if } x \in \mathbf{V}' \\ \min\{g(y) + S_a(y, x) \mid y \in \mathbf{V}'\} & \text{if } x \notin \mathbf{V}' \end{cases}$$

with $s \in [0, 1]$ and σ_a^+, σ_a^- defined as in (8), (9).

- (vi) (HOPF–LAX FORMULA FOR MAXIMAL SUPERCRITICAL SUBSOLUTIONS 2) *Let $a > c$ and Γ' be a closed subset of Γ . Let g be an admissible trace on Γ' , in the sense of (65), then there exists a unique solution $u \in C(\Gamma, \mathbb{R})$ to $(\mathcal{H}Ja)$ on $\Gamma \setminus \Gamma'$ agreeing with g on Γ' , which is given by*

$$u(x) = \min\{g(y) + S_a^\Gamma(y, x) \mid y \in \Gamma'\},$$

where $S_a^\Gamma(\cdot, \cdot)$ denotes the intrinsic (semi)distance defined in (63).

Proof:

- I. (i) Existence follows from Theorem 6.16 and Proposition 6.2; Lipschitz continuity from Proposition 3.10.
- (ii) This part is obtained by combining Proposition 6.2 and Theorem 6.21
- (iii) This representation formula was proven in Proposition 5.5.
- (iv) See Theorem 7.9 (i).

- II. (i) See Theorem 7.1.
- (ii) See Proposition 7.4.
- (iii) See Theorem 7.5.
- (iv) See Theorem 7.6.
- (v) These results are obtained by combining Proposition 6.3, Proposition 6.6, Theorem 6.23 and using the representation formula in Proposition 5.5.
- (vi) See Theorem 7.9 (ii).

□

APPENDIX A

Proof: (Proposition 3.10) Taking into account that for any $\gamma \in \mathcal{E}$ (which is a finite set) $w \circ \gamma$ is Lipschitz-continuous in $[0, 1]$, thanks to the coercivity condition **(H γ 2)**, we deduce that there exists $L > 0$ such that, for any given subsolution w

$$(67) \quad |w(\gamma(s_2)) - w(\gamma(s_1))| \leq L \ell \left(\gamma|_{[s_1, s_2]} \right) \quad \text{for all } \gamma \in \mathcal{E}, \text{ and } s_1 \leq s_2 \in [0, 1];$$

hereafter ℓ indicates the Euclidean length of curves in \mathbb{R}^N .

We proceed by considering x and y in Γ and a finite sequence of concatenated arcs $\gamma_1, \dots, \gamma_M$, for some index M , that realize the geodesic distance $d_\Gamma(x, y)$. More specifically, we assume that $x = \gamma_1(t_x)$, $y = \gamma_M(t_y)$ with t_x, t_y in $[0, 1]$ and that

$$d_\Gamma(x, y) = \ell \left(\gamma_1|_{[t_x, 1]} \right) + \sum_{i=2}^{M-1} \ell(\gamma_i) + \ell \left(\gamma_M|_{[0, t_y]} \right).$$

In the remainder of the proof we assume that $M > 2$ in order to ease the notation (the other cases can be treated analogously).

We deduce from (67) that

$$\begin{aligned}
|w(y) - w(x)| &\leq |w(\gamma_1(1)) - w_1(\gamma_1(t_x))| \\
&\quad + \sum_{i=2}^{M-1} |w(\gamma_i(1)) - w(\gamma_i(0))| + |w(\gamma_M(t_y)) - w_1(\gamma_M(0))| \\
&\leq L \left[\ell(\gamma_1|_{[t_x, 1]}) + \sum_{i=2}^{M-1} \ell(\gamma_i) + \ell(\gamma_M|_{[0, t_y]}) \right] \\
&= L d_\Gamma(x, y).
\end{aligned}$$

This concludes the proof. \square

Proof: (Proposition 5.5) We denote by w the function appearing in the statement. If $a = a_\gamma$, the assertion comes from (11) and Proposition 5.3. Instead, if $a > a_\gamma$, the function w is an a.e. subsolution, being the minimum of two C^1 (sub)solutions. Using a basic property in viscosity solutions theory, it is also a supersolution, as minimum of supersolutions. Moreover, $w(0) = \alpha$, $w(1) = \beta$ hold thanks to (17).

Finally, the function $s \mapsto \int_0^s \sigma_{a_\gamma}^+$ is a strict subsolution to $(HJ_\gamma a)$, and this implies by an argument going back to [21] that the Dirichlet problem with admissible data α, β is uniquely solved. \square

Proof: (Proposition 5.6) If $a = a_\gamma$, then, as already pointed out in Proposition 5.3, the solution is unique up to additive constants, hence it is automatically given by (20) once the value $w(0)$ is assigned.

Therefore, from now on we can assume that $a > a_\gamma$. By Proposition 5.5

$$w(s) = \min \left\{ w(0) + \int_0^s \sigma_a^+(t) dt, w(1) - \int_s^1 \sigma_a^-(t) dt \right\} \quad \text{for any } s.$$

We claim that if

$$(68) \quad w(s_0) = w(1) - \int_{s_0}^1 \sigma_a^-(t) dt$$

for some $s_0 \in (0, 1)$, then

$$w(s) = w(1) - \int_s^1 \sigma_a^-(t) dt \quad \text{for any } s \in (s_0, 1].$$

Assume by contradiction that there exists $s_1 > s_0$ such that

$$w(0) + \int_0^{s_1} \sigma_a^+(t) dt = w(0) + \int_0^{s_0} \sigma_a^+(t) dt + \int_{s_0}^{s_1} \sigma_a^+(t) dt < w(1) - \int_{s_1}^1 \sigma_a^-(t) dt;$$

this implies that

$$(69) \quad w(0) + \int_0^{s_0} \sigma_a^+(t) dt < w(1) - \int_{s_1}^1 \sigma_a^-(t) dt - \int_{s_0}^{s_1} \sigma_a^+(t) dt.$$

It is apparent that

$$\int_{s_0}^{s_1} \sigma_a^+(t) dt > \int_{s_0}^{s_1} \sigma_a^-(t) dt$$

and we can consequently deduce from (69) that

$$w(0) + \int_0^{s_0} \sigma_a^+(t) dt < w(1) - \int_{s_1}^1 \sigma_a^-(t) dt - \int_{s_0}^{s_1} \sigma_a^-(t) dt = w(1) - \int_{s_0}^1 \sigma_a^-(t) dt,$$

in contrast with (68). We assume, for purposes of contradiction, that (68) holds true for some $s_0 \in (0, 1)$. Since $a > a_\gamma$, we can take p_0 with $H(1, p_0) < a$. If w is not of the form (20), then, owing to the previous claim, we can fix s_0 in such a way that

$$w(s) = w(1) - \int_s^1 \sigma_a^-(t) dt \quad \text{and} \quad H(s, p_0) < a$$

for $s \in [s_0, 1]$. This implies

$$\varphi(s) := w(1) + p_0(s - 1) \leq w(1) - \int_s^1 \sigma_a^-(t) dt = w(s),$$

for $s \in [s_0, 1]$, and consequently φ is a constrained subgradient to w at 1 with

$$H(1, \varphi'(1)) = H(1, p_0) < 1,$$

contradicting (19). We deduce that w is of the form (20) showing the first part of the assertion.

Conversely, if w is of the form (20), then it is of class C^1 in $(0, 1)$ with $w'(s) = \sigma_a^+(s)$. Consequently, any constrained subgradient φ at $t = 1$ must satisfy

$$w(1) - \int_s^1 \varphi' dt = \varphi(s) \leq w(s) = w(1) - \int_s^1 \sigma_a^+ dt$$

for s sufficiently close to 1. This implies

$$\int_s^1 \varphi' dt \geq \int_s^1 \sigma_a^+ dt$$

and shows the existence of a sequence s_n contained in $(0, 1)$ and converging to 1 as n goes to infinity, with $\varphi'(s_n) \geq \sigma_a^+(s_n)$. Passing to the limit as n goes to infinity, we get $\varphi'(1) \geq \sigma_a^+(1)$. We deduce from this the inequality (19) and conclude the proof. \square

Proof: (Proposition 5.11) If $a = c_\gamma = a_\gamma$ then the integrals in (21) coincide in force of (11), then they must both vanish, and this shows the assertion. Assume now that $c_\gamma > a_\gamma$ and also assume for purposes of contradiction that strict inequalities prevail instead in (21). Then, we can find $\lambda \in (0, 1)$ with

$$\int_0^1 [\lambda \sigma_{c_\gamma}^+(t) + (1 - \lambda) \sigma_{c_\gamma}^-(t)] dt = 0.$$

Taking into account that $\sigma_{c_\gamma}^+(t) > \sigma_{c_\gamma}^-(t)$ for any t , this implies that

$$s \mapsto \int_0^s [\lambda \sigma_{c_\gamma}^+(t) + (1 - \lambda) \sigma_{c_\gamma}^-(t)] dt$$

is a strict periodic subsolution to $H = c_\gamma$. This is impossible by the very definition of c_γ . \square

Proof: (Corollary 5.13) The unique point to check is that the values $\alpha + \beta$ at $s = 0$ and α at $s = s_0$ are admissible, in the sense of (17), for $(HJ_\gamma a)$ in $(0, s_0)$, and the same holds true in $(s_0, 1)$ for the value α at $s = s_0$ and $\alpha + \beta$ at $s = 1$. The argument is the same for the two subintervals. We therefore focus on $[s_0, 1]$.

If $u(1) - u(s_0) = \beta = \int_{s_0}^1 \sigma^+ a(t) dt$ the compatibility property is immediate and the solution in $(s_0, 1)$ is given by (23), as asserted in item ii) of the statement. Let us instead assume

$$(70) \quad u(1) - u(s_0) = \beta = - \int_0^{s_0} \sigma^- a(t) dt < \int_{s_0}^1 \sigma^+ a(t) dt.$$

We have by Lemma 5.9 $\int_0^1 \sigma_a^-(t) dt \leq 0$ and consequently

$$u(1) - u(s_0) \geq \int_{s_0}^1 \sigma_a^-(t) dt.$$

The last inequality plus (5.13) shows the claimed admissibility property. This concludes the proof. \square

Proof: (Proposition 6.1) Let w be a solution to $(\mathcal{H}J a)$ with trace u on \mathbf{V} . We know by the very definition of solution that given any arc γ , then $w \circ \gamma$ is a solution to $H_\gamma = a$ in $(0, 1)$ taking the values $u(\gamma(0))$ and $u(\gamma(1))$ at 0 and 1, respectively. This implies that such boundary values are admissible with respect to H_γ , in the sense of formula (17) with H_γ in place of H . By the uniqueness property showcased in Proposition 5.5, the values of w on the support of γ are therefore uniquely determined by $u(\gamma(0))$, $u(\gamma(1))$ and H_γ . Since the arc γ has been arbitrarily chosen, we can hence conclude the asserted uniqueness. \square

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