New bounds on the strength of some restrictions of Hindman's Theorem⋆

Lorenzo Carlucci[1] carlucci@di.uniroma1.it, Leszek Aleksander Kołodziejczyk[2] lak@mimuw.edu.pl, Francesco Lepore[1] leporefc@gmail.com, and Konrad Zdanowski[3] k.zdanowski@uksw.edu.pl

[1] Department of Computer Science, University of Rome I,
[2] Institute of Mathematics, University of Warsaw,
[3] Faculty of Mathematics and Natural Sciences, Cardinal Stefan Wyszyński University in Warsaw

Abstract. We prove upper and lower bounds on the effective content and logical strength for a variety of natural restrictions of Hindman's Finite Sums Theorem. For example, we show that Hindman's Theorem for sums of length at most 2 and 4 colors implies ACA_0. An emerging *leitmotiv* is that the known lower bounds for Hindman's Theorem and for its restriction to sums of at most 2 elements are already valid for a number of restricted versions which have simple proofs and better computability- and proof-theoretic upper bounds than the known upper bound for the full version of the theorem. We highlight the role of a sparsity-like condition on the solution set, which we call apartness.

1 Introduction and Motivation

The Finite Sums Theorem by Neil Hindman [15] (henceforth denoted HT) is a celebrated result in Ramsey Theory stating that for every finite coloring of the positive integers there exists an infinite set such that all the finite non-empty sums of distinct elements from it have the same color. Thirty years ago Blass, Hirst and Simpson proved in [2] that *all* computable instances of HT have *some* solutions computable in $\emptyset^{(\omega+1)}$ and that for *some* computable instances of HT *all* solutions compute $\emptyset'$. In terms of Reverse Mathematics, they showed that $\mathsf{ACA}_0^+ \vdash \mathsf{HT}$ and that $\mathsf{RCA}_0 \vdash \mathsf{HT} \to \mathsf{ACA}_0$ (see [21,17] for the definition of these systems). Both bounds hold for the particular case of colorings in two colors. Closing the gap between the upper and lower bound is one of the major open problems in Computable and Reverse Mathematics (see, e.g., [20]).

Blass advocated the study of restrictions of Hindman's Theorem in which a bound is put on the length (i.e., number of distinct terms) of sums for which

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monochromaticity is guaranteed [1], conjecturing that the complexity of Hindman's Theorem grows as a function of the length of sums. Recently Dzhafarov, Jockusch, Solomon and Westrick showed (see Corollary 3.4 in [12]) that the known $\emptyset'$ (ACA_0) lower bound on Hindman's Theorem holds for the restriction to sums of at most 3 terms (with no repetitions, as is the case throughout the paper), and 3 colors (henceforth denoted by $\mathsf{HT}^{\leq 3}_3$). They also established that the restriction to sums of at most 2 terms, and 2 colors (denoted $\mathsf{HT}^{\leq 2}_2$), is unprovable in RCA_0 (Corollary 2.3 in [12]) and implies SRT^2_2 (the Stable Ramsey's Theorem for pairs and 2 colors) over $\mathsf{RCA}_0 + B\Sigma^0_2$ (Corollary 2.4 in [12]) . This prompted the first author to look into direct combinatorial reductions yielding, e.g., a direct implication from $\mathsf{HT}^{\leq 2}_5$ to the Increasing Polarized Ramsey's Theorem for pairs of Dzhafarov and Hirst [11], which is strictly stronger than SRT^2_2 (see Section 4 for details).

It should be stressed that no upper bound other than the $\emptyset^{(\omega+1)}$ (ACA^+_0) upper bound on the full Finite Sums Theorem is known to hold for the restrictions of the theorem to sums of length (i.e., number of terms) ≤ 2 or ≤ 3. It is indeed a long-standing open question in Combinatorics whether the latter restrictions admit a proof that does not establish the full Finite Sums Theorem (see, e.g., [16], Question 12). On the other hand, Hirst investigated in [18] an apparently slight variant of the Finite Sums Theorem and proved it *equivalent* to $B\Sigma_2$. This prompted the first author to investigate versions of HT for which an upper bound better than $\emptyset^{(\omega+1)}$ (ACA^+_0) could be established, while retaining as strong a lower bound as possible. In [4] (resp. [3]) such restrictions were isolated and proved to attain the known lower bounds for HT (resp. $\mathsf{HT}^{\leq 2}_2$), while being provable from ACA_0 (resp. RT^2_2).

We present new results along these lines of research. In Section 3 we prove an ACA_0 lower bound for $\mathsf{HT}^{\leq 2}_4$, and an equivalence with ACA_0 for some principles from [4]. In Section 4 we establish combinatorial implications from other restrictions of Hindman's Theorem to the Increasing Polarized Ramsey's Theorem for Pairs. These reductions imply unprovability-in-WKL_0 results and also yield strong computable reducibility of IPT^2_2 to some Hindman-type theorem. We highlight the role of a sparsity-like condition on the solution set which we call the apartness condition, which is crucial in earlier work ([15,12,3,4]).

2 Restricted Hindman and the Apartness Condition

Let us fix some notation. For technical convenience and to avoid trivial cases we will deal with colorings of the positive integers. We use $\mathbf{N}$ to denote the positive integers. If $a \in \mathbf{N}$ and B is a set we denote by $FS^{\leq a}(B)$ (resp. $FS^{=a}(B)$) the set of non-empty sums of at most (resp. exactly) a-many distinct elements from B. More generally, if A and B are sets we denote by $FS^{A}(B)$ the set of all sums of j-many distinct terms from B, for all $j \in A$. By $FS(B)$ we denote $FS^{\mathbf{N}}(B)$. We use the notation $X = \{x_1, x_2, \dots\}_<$ to indicate that $x_1 < x_2 < \dots$. Let us recall the statement of Hindman's Finite Sums Theorem [15].

Definition 1 (Hindman's Finite Sums Theorem). HT *is the following assertion: For every coloring* $f : \mathbf{N} \to k$ *there exists an infinite set* $H \subseteq \mathbf{N}$ *such that* $FS(H)$ *is monochromatic for* f.

We define below two restrictions of Hindman's Theorem that will feature prominently in the present paper. We then discuss a sparsity-like condition that will be central to our results.

2.1 Hindman's Theorem with bounded-length sums

The following principles were discussed in [1] (albeit phrased in terms of finite unions instead of sums) and first studied from the perspective of Computable and Reverse Mathematics in [12].

Definition 2 (Hindman's Theorem with bounded-length sums). *Fix* $n, k \geq 1$.

1. $\mathsf{HT}^{\leq n}_k$ *is the following principle: For every coloring* $f : \mathbf{N} \to k$ *there exists an infinite set* $H \subseteq \mathbf{N}$ *such that* $FS^{\leq n}(H)$ *is monochromatic for* f.
2. $\mathsf{HT}^{=n}_k$ *is the following principle: For every coloring* $f : \mathbf{N} \to k$ *there exists an infinite set* $H \subseteq \mathbf{N}$ *such that* $FS^{=n}(H)$ *is monochromatic for* f.

The principle $\mathsf{HT}^{\leq 2}_2$ is the topic of a long-standing open question in Combinatorics: Question 12 of [16] asks whether there exists a proof of $\mathsf{HT}^{\leq 2}_2$ that does not also prove the full Finite Sums Theorem. On the other hand, the principle $\mathsf{HT}^{=2}_2$ easily follows from Ramsey's Theorem for pairs: given an instance $f : \mathbf{N} \to 2$ of $\mathsf{HT}^{=2}_2$, define $g : [\mathbf{N}]^2 \to 2$ by setting $g(x, y) := f(x+y)$. A solution for Ramsey's Theorem for pairs for g is a solution for $\mathsf{HT}^{=2}_2$ for f.

Dzhafarov, Jockusch, Solomon and Westrick recently proved in [12] that $\mathsf{HT}^{\leq 3}_3$ implies ACA_0 over RCA_0 (Corollary 3.4 of [12]) and that $\mathsf{HT}^{\leq 2}_2$ implies SRT^2_2 (the Stable Ramsey's Theorem for pairs) over $\mathsf{RCA}_0 + B\Sigma^0_2$ (Corollary 2.4 of [12]).[4]

The first author proved that $\mathsf{HT}^{\leq 2}_5$ implies IPT^2_2 (the Increasing Polarized Ramsey's Theorem for pairs) over RCA_0 (see [5]).

2.2 The Apartness Condition

We discuss a property of the solution set – which we call the apartness condition – that is crucial in Hindman's original proof and in the proofs of the $\emptyset'$ (ACA_0) lower bounds in [2,12,4]. We use the following notation: Fix a base $t \geq 2$. For $n \in \mathbf{N}$ we denote by $\lambda_t(n)$ the least exponent of n written in base t, by $\mu_t(n)$ the largest exponent of n written in base t, and by $i_t(n)$ the coefficient of the least term of n written in base t. Our results are in terms of 2-apartness except in one case (Lemma 1 below) where we have to use 3-apartness for technical reasons. We will drop the subscript when clear from context.

[4] The principle $B\Sigma^0_2$ is used in the proof of the implication Corollary 2.4 in [12], as indicated in the final version of Dzhafarov et al. paper – our reference [12].

Definition 3 (Apartness Condition). *Fix $t \geq 2$. We say that a set $X \subseteq \mathbf{N}$ satisfies the t-apartness condition (or* is t-apart*) if for all $x, x' \in X$, if $x < x'$ then $\mu_t(x) < \lambda_t(x')$.*

Note that the apartness condition is inherited by subsets. In Hindman's original proof 2-apartness can be ensured (Lemma 2.2 in [15]) by a simple counting argument (Lemma 2.2 in [14]), under the assumption that we have a solution to the Finite Sums Theorem, i.e. an infinite H such that $FS(H)$ is monochromatic. For a Hindman-type principle P, let "P *with t-apartness*" denote the corresponding version in which the solution set is required to satisfy the t-apartness condition.

As will be observed below, it is significantly easier to prove lower bounds on P with t-apartness than on P in all the cases we consider. Moreover, for *all* restrictions of Hindman's Theorem for which a proof is available that does not also establish the full theorem, the t-apartness condition (for $t > 1$) can be guaranteed by construction (see, e.g., [3,4]). This is the case, e.g., for the principle $\mathsf{HT}^{=2}_2$: the proof from Ramsey's Theorem for pairs sketched above yields t-apartness for any $t > 1$ simply by applying Ramsey's Theorem relative to an infinite t-apart set. In *some* cases the apartness condition can be ensured at the cost of increasing the number of colors. This is the case of $\mathsf{HT}^{\leq n}_k$ as illustrated by the next lemma. The idea of the proof is from the first part of the proof of Theorem 3.1 in [12], with some needed adjustments.

Lemma 1 (RCA_0). *For all $n \geq 2$, for all $d \geq 1$, $\mathsf{HT}^{\leq n}_{2d}$ implies $\mathsf{HT}^{\leq n}_d$ with 3-apartness.*

Proof. We work in base 3. Let $f : \mathbf{N} \to d$ be given. Define $g : \mathbf{N} \to 2d$ as follows.

$$g(n) := \begin{cases} f(n) & \text{if } i(n) = 1, \\ d + f(n) & \text{if } i(n) = 2. \end{cases}$$

Let H be an infinite set such that $FS^{\leq n}(H)$ is homogeneous for g of color k. For $h, h' \in FS^{\leq n}(H)$ we have $i(h) = i(h')$. Then we claim that for each $m \geq 0$ there is at most one $h \in H$ such that $\lambda(h) = m$. Suppose otherwise, by way of contradiction, as witnessed by $h, h' \in H$. Then $i(h) = i(h')$ and $\lambda(h) = \lambda(h')$. Therefore $i(h + h') \neq i(h)$, but $h + h' \in FS^{\leq n}(H)$. Contradiction. Therefore we can computably obtain a 3-apart infinite subset of H. ◻

3 Restricted Hindman and Arithmetical Comprehension

We prove a new ACA_0 lower bound and a new ACA_0 equivalence result for restrictions of Hindman's Theorem. The lower bound proof is in the spirit of the proof by Blass, Hirst and Simpson that Hindman's Theorem implies ACA_0 – on which the proof of Theorem 3.1 of [12] is also based – with extra care to work with sums of length at most two. The upper bound proof is in the spirit of [4].

3.1 $\mathsf{HT}_4^{\leq 2}$ implies ACA_0

We show that $\mathsf{HT}_4^{\leq 2}$ implies ACA_0 over RCA_0. This is to be compared with Corollary 2.3 and Corollary 3.4 of [12], showing, respectively, that $\mathsf{RCA}_0 \nvdash \mathsf{HT}_2^{\leq 2}$ and that $\mathsf{RCA}_0 \vdash \mathsf{HT}_3^{\leq 3} \rightarrow \mathsf{ACA}_0$. Blass, towards the end of [1], states without giving details that inspection of the proof of the $\emptyset'$ lower bound for HT in [2] shows that these bounds are true for the restriction of the Finite Unions Theorem to unions of at most two sets.[5] Note that the Finite Unions Theorem has a built-in apartness condition. Blass indicates in Remark 12 of [1] that things might be different for restrictions of the Finite Sums Theorem, as those considered in this paper. Also note that the proof of Theorem 3.1 in [12], which stays relatively close to the argument in [2], requires sums of length 3.

Proposition 1 (RCA_0). *For any fixed $t \geq 2$, $\mathsf{HT}_2^{\leq 2}$ with t-apartness implies* ACA_0.

Proof. We write the proof for $t = 2$. Assume $\mathsf{HT}_2^{\leq 2}$ with 2-apartness and consider $f\colon \mathbf{N} \to \mathbf{N}$. We have to prove that the range of f exists.

For a number n, written as $2^{n_0} + \cdots + 2^{n_r}$ in base 2, with $n_0 < \cdots < n_r$, we call $j \in \{0, \ldots, r\}$ *important in n* if some value of $f{\restriction}[n_{j-1}, n_j)$ is below n_0. Here $n_{-1} = 0$. The coloring $c\colon \mathbf{N} \to 2$ is defined by

$$c(n) := \mathrm{card}\{j : j \text{ is important in } n\} \bmod 2.$$

By $\mathsf{HT}_2^{\leq 2}$ with 2-apartness, there exists an infinite set $H \subseteq \mathbf{N}$ such that H is 2-apart and $\mathrm{FS}^{\leq 2}(H)$ is monochromatic w.r.t. c. We claim that for each $n \in H$ and each $x < \lambda(n)$, $x \in \mathrm{rg}(f)$ if and only if $x \in \mathrm{rg}(f{\restriction}\mu(n))$. This will give us a Δ_1^0 definition of $\mathrm{rg}(f)$: given x, find the smallest $n \in H$ such that $x < \lambda(n)$ and check whether x is in $\mathrm{rg}(f{\restriction}\mu(n))$.

It remains to prove the claim. In order to do this, consider $n \in H$ and assume that there is some element below $n_0 = \lambda(n)$ in $\mathrm{rg}(f) \setminus \mathrm{rg}(f{\restriction}\mu(n))$. By the consequence of Σ_1^0-induction known as *strong Σ_1^0-collection* (see Exercise II.3.14 in [21], Thm I.2.23 and Definition I.2.20 in [13]), there is a number ℓ such that for any $x < \lambda(n)$, $x \in \mathrm{rg}(f)$ if and only if $x \in \mathrm{rg}(f{\restriction}\ell)$. By 2-apartness, there is $m \in H$ with $\lambda(m) \geq \ell > \mu(n)$. Write $n + m$ in base 2 notation,

$$n + m = 2^{n_0} + \cdots + 2^{n_r} + 2^{n_{r+1}} + \cdots + 2^{n_s},$$

where $n_0 = \lambda(n) = \lambda(n + m)$, $n_r = \mu(n)$, and $n_{r+1} = \lambda(m)$. Clearly, $j \leq s$ is important in $n + m$ if and only if either $j \leq r$ and j is important in n or $j = r + 1$; hence, $c(n) \neq c(n + m)$. This contradicts the assumption that $\mathrm{FS}^{\leq 2}(H)$ is monochromatic, thus proving the claim. □

[5] The Finite Unions Theorem states that every coloring of the finite non-empty sets of $\mathbf{N}$ admits an infinite and pairwise unmeshed family H of finite non-empty sets (sometimes called a block sequence) such that every finite non-empty union of elements of H is of the same color. Two finite non-empty subsets x, y of $\mathbf{N}$ are unmeshed if either $\max x < \min y$ or $\max y < \min x$. Note that Hindman's Theorem is equivalent to the Finite Unions Theorem only if the pairwise unmeshed condition is present.

Theorem 1 (RCA_0)**.** $\mathsf{HT}^{\leq 2}_4$ *implies* ACA_0.

Proof. By Proposition 1 and Lemma 1. □

3.2 Equivalents of ACA_0

In [4], a family of natural restrictions of Hindman's Theorem was isolated such that each of its members admits a simple combinatorial proof, yet each member of a non-trivial sub-family implies ACA_0. The weakest principle of the latter kind considered in [4] is the following, called the Hindman-Brauer Theorem: Whenever $\mathbf{N}$ is 2-colored there is an infinite set $H \subseteq \mathbf{N}$ and *there exist* positive integers a, b such that $FS^{\{a,b,a+b,a+2b\}}(H)$ is monochromatic. It was proved in [4] that the Hindman-Brauer Theorem with 2-apartness is equivalent to ACA_0. We show that the same holds for the following apparently weaker principle.

Definition 4. $\mathsf{HT}^{\exists\{a,b\}}_2$ *is the following principle: For every coloring* $f : \mathbf{N} \to 2$ *there is an infinite set* $H \subseteq \mathbf{N}$ *and positive integers* $a < b$ *such that* $FS^{\{a,b\}}(H)$ *is monochromatic.*

Theorem 2. $\mathsf{HT}^{\exists\{a,b\}}_2$ *with 2-apartness is equivalent to* ACA_0 *over* RCA_0.

Proof. We first prove the upper bound. Given $c : \mathbf{N} \to 2$ let $g : [\mathbf{N}]^3 \to 8$ be defined as follows:

$$g(x_1, x_2, x_3) := \langle c(x_1), c(x_1 + x_2), c(x_1 + x_2 + x_3)\rangle.$$

Fix an infinite and 2-apart set $H_0 \subseteq \mathbf{N}$. By RT^3_8 relativized to H_0 we get an infinite (and 2-apart) set $H \subseteq H_0$ monochromatic for g. Let the color be $\sigma = (c_1, c_2, c_3)$, a binary sequence of length 3. Then, for each $i \in \{1, 2, 3\}$, g restricted to $FS^{=i}(H)$ is monochromatic of color c_i. Obviously for some positive integers a, b such that $a < b \leq 3$ it must be that $c_a = c_b$. Then $FS^{\{a,b\}}(H)$ is monochromatic of color c_a.

The lower bound is proved by a minor adaptation of the proof of Proposition 1. As the n in that proof take an a-term sum. Then take a $(b - a)$-term sum as the m. □

The same proof yields that the following Hindman-Schur Theorem with 2-apartness from [4] implies ACA_0: Whenever $\mathbf{N}$ is 2-colored there is an infinite 2-apart set H and *there exist* positive integers a, b such that $FS^{\{a,b,a+b\}}(H)$ is monochromatic. It was shown in [4] to be provable in ACA_0.

4 Restricted Hindman and Polarized Ramsey

In this section we establish new lower bounds for restricted versions of Hindman's Theorem, most of which do not imply ACA_0 and are therefore provably weaker than HT. Lower bounds are established by reduction to the Increasing Polarized Ramsey's Theorem for pairs [11]. In particular we obtain unprovability in WKL_0.

All proofs in the present section yield strongly computable reductions in the sense of [10], not just implications. P is *strongly computably reducible* to Q, written $\mathsf{P} \leq_{sc} \mathsf{Q}$, if every instance X of P computes an instance X^* of Q, such that if Y^* is any solution to X^* then there is a solution Y to X computable from Y^*.

Definition 5 (Increasing Polarized Ramsey's Theorem). *Fix $n, k \geq 1$. IPT^n_k is the following principle: For every $f : [\mathbf{N}]^n \to k$ there exists a sequence $(H_1, \dots, H_n)$ of infinite sets such that all increasing tuples $(x_1, \dots, x_n)$ in $H_1 \times \dots \times H_n$ have the same color under f. The sequence $(H_1, \dots, H_n)$ is called increasing polarized homogeneous (or increasing p-homogeneous) for f.*

Note that IPT^2_2 is strictly stronger than SRT^2_2. On the one hand, $\mathsf{RCA}_0 \vdash \mathsf{IPT}^2_2 \to \mathsf{D}^2_2$ by Proposition 3.5 of [11], and $\mathsf{RCA}_0 \vdash \mathsf{D}^2_2 \to \mathsf{SRT}^2_2$ by Theorem 1.4 of [8].[6] However, $\mathsf{RCA}_0 + \mathsf{SRT}^2_2 \nvdash \mathsf{IPT}^2_2$: Theorem 2.2 in [9] showed that there is a non-standard model of $\mathsf{SRT}^2_2 + B\Sigma^0_2$ having only low sets in the sense of the model. Lemma 2.5 in [11] can be formalized in RCA_0 and shows that no model of IPT^2_2 can contain only Δ^0_2 sets.[7]

4.1 $\mathsf{HT}^{=2}_2$ with 2-apartness implies IPT^2_2

We show that $\mathsf{HT}^{=2}_2$ with 2-apartness implies IPT^2_2 by a combinatorial reduction. This should be contrasted with the fact that no lower bounds on $\mathsf{HT}^{=2}_2$ without apartness are known.

Theorem 3 (RCA_0)**.** $\mathsf{HT}^{=2}_2$ *with 2-apartness implies* IPT^2_2.

Proof. Let $f : [\mathbf{N}]^2 \to 2$ be given. Define $g : \mathbf{N} \to 2$ as follows.

$$g(n) := \begin{cases} 0 & \text{if } n = 2^m, \\ f(\lambda(n), \mu(n)) & \text{if } n \neq 2^m. \end{cases}$$

Note that g is well-defined since $\lambda(n) < \mu(n)$ if n is not a power of 2. Let $H = \{h_1, h_2, \dots\}_<$ witness $\mathsf{HT}^{=2}_2$ with 2-apartness for g. Let the color be $k < 2$. Let

$$H_1 := \{\lambda(h_{2i-1}) \,:\, i \in \mathbf{N}\},\ H_2 := \{\mu(h_{2i}) \,:\, i \in \mathbf{N}\}.$$

We claim that (H_1, H_2) is increasing p-homogeneous for f.

First observe that we have

$$\lambda(h_1) < \lambda(h_3) < \lambda(h_5) < \dots,$$

and

$$\mu(h_2) < \mu(h_4) < \mu(h_6) < \dots.$$

[6] Note that the latter result is not present in the diagram in [11]. D^2_2, defined in [7], is the following assertion: For every 0, 1-valued function $f(x, s)$ for which a $\lim_{s\to\infty} f(x, s)$ exists for each x, there is an infinite set H and a $k < 2$ such that for all $h \in H$ we have $\lim_{s\to\infty} f(h, s) = k$.

[7] We thank Ludovic Patey for pointing out to us the results implying strictness.

This is so because $\lambda(h_1) \leq \mu(h_1) < \lambda(h_2) \leq \mu(h_2) < \dots$ by the 2-apartness condition. Then we claim that $f(x_1, x_2) = k$ for every increasing pair $(x_1, x_2) \in H_1 \times H_2$. Note that $(x_1, x_2) = (\lambda(h_i), \mu(h_j))$ for some $i < j$ (the case $i = j$ is impossible by construction of H_1 and H_2). Then we have

$$k = g(h_i + h_j) = f(\lambda(h_i + h_j), \mu(h_i + h_j)) = f(\lambda(h_i), \mu(h_j)) = f(x_1, x_2),$$

since $FS^{=2}(H)$ is monochromatic for g with color k. This shows that (H_1, H_2) is increasing p-homogeneous of color k for f. ☐

The proof of Theorem 3 yields that $\mathsf{IPT}^2_2 \leq_{\mathrm{sc}} \mathsf{HT}^{=2}_2$ with 2-apartness, and, with minor adjustments, that $\mathsf{IPT}^2_2 \leq_{\mathrm{sc}} \mathsf{HT}^{\leq 2}_4$ (a self-contained proof appeared in [5]).

4.2 IPT^2_2 and the Increasing Polarized Hindman's Theorem

We define an (increasing) polarized version of Hindman's Theorem. We prove that its version for pairs and 2 colors with an appropriately defined notion of 2-apartness is equivalent to IPT^2_2.

Definition 6 ((Increasing) Polarized Hindman's Theorem). *Fix $n \geq 1$. PHT^n_2 (resp. IPHT^n_2) is the following principle: For every $f : \mathbf{N} \to 2$ there exists a sequence $(H_1, \dots, H_n)$ of infinite sets such that for some color $k < 2$, for all (resp. increasing) $(x_1, \dots, x_n) \in H_1 \times \dots \times H_n$, $f(x_1 + \dots + x_n) = k$.*

We impose a t-apartness condition on a solution $(H_1, \dots, H_n)$ of IPHT^n_2 by requiring that the union $H_1 \cup \dots \cup H_n$ is t-apart. We denote by "IPHT^n_2 with t-apartness" the principle IPHT^n_2 with this t-apartness condition on the solution set.

Theorem 4. *IPT^2_2 and IPHT^2_2 with 2-apartness are equivalent over RCA_0.*

Proof. We first prove that IPT^2_2 implies IPHT^2_2 with 2-apartness. Given $c : \mathbf{N} \to 2$ define $f : [\mathbf{N}]^2 \to 2$ in the obvious way setting $f(x, y) := c(x+y)$. Fix two infinite disjoint sets S_1, S_2 such that $S_1 \cup S_2$ is 2-apart. By Lemma 4.3 of [11], IPT^2_2 implies over RCA_0 its own relativization: there exists an increasing p-homogeneous sequence (H_1, H_2) for f such that $H_i \subseteq S_i$. Therefore $H_1 \cup H_2$ is 2-apart by construction. Let the color be $k < 2$. Obviously we have that for any increasing pair $(x_1, x_2) \in H_1 \times H_2$, $c(x_1 + x_2) = f(x_1, x_2) = k$. Therefore (H_1, H_2) is an increasing p-homogeneous pair for c.

Next we prove that IPHT^2_2 with 2-apartness implies IPT^2_2. Let $f : [\mathbf{N}]^2 \to 2$ be given. Define as usual $c : \mathbf{N} \to 2$ by setting $c(n) := f(\lambda(n), \mu(n))$ if n is not a power of 2 and $c(n) := 0$ otherwise. Let (H_1, H_2) be a 2-apart solution to IPHT^2_2 for c, of color $k < 2$. By, possibly, recursively thinning out H_1 and H_2 we can assume without loss of generality that $H_1 \cap H_2 = \varnothing$. Let $H_1 = \{h_1, h_2, \dots\}_<$ and $H_2 = \{h'_1, h'_2, \dots\}_<$. Then set $H_1^+ := \{\lambda(h) : h \in H_1\}$ and $H_2^+ := \{\mu(h) : h \in H_2\}$. We claim that (H_1^+, H_2^+) is a solution to IPT^2_2 for

f. Let $(x_1, x_2) \in H_1^+ \times H_2^+$ be an increasing pair. Then for some $h \in H_1$ and $h' \in H_2$, $\lambda(h) = x_1$ and $\mu(h') = x_2$. Also, since $H_1 \cup H_2$ is apart and $H_1 \cap H_2 = \emptyset$, it must be the case that $h < h'$. Therefore (h, h') is an increasing pair in $H_1 \times H_2$ and the following holds:

$$k = c(h + h') = f(\lambda(h + h'), \mu(h + h')) = f(\lambda(h), \mu(h')) = f(x_1, x_2).$$

□

4.3 Hindman's Theorem for Exactly Large Sums

We present here some preliminary results on a restriction of Hindman's Theorem to exactly large sums. A finite set $S \subseteq \mathbf{N}$ is *exactly large*, or $!\omega$-large, if $|S| = \min(S) + 1$. We denote by $[X]^{!\omega}$ the set of exactly large subsets of X and by $FS^{!\omega}(X)$ the set of positive integers that can be obtained as sums of terms of an exactly large subset of X. We call sums of this type *exactly large sums* (from X). Ramsey's Theorem for exactly large sums ($\mathsf{RT}_2^{!\omega}$) asserts that every 2-coloring f of the exactly large subsets of an infinite set $X \subseteq \mathbf{N}$ admits an infinite set $H \subseteq X$ such that f is constant on $[H]^{!\omega}$. It was studied in [6] and there proved equivalent to ACA_0^+. We introduce an analogue for Hindman's Theorem.

Definition 7 (Hindman's Theorem for Large Sums). $\mathsf{HT}_2^{!\omega}$ *denotes the following principle: For every coloring $c : \mathbf{N} \to 2$ there exists an infinite set $H \subseteq \mathbf{N}$ such that $FS^{!\omega}(H)$ is monochromatic under c.*

$\mathsf{HT}_2^{!\omega}$ (with t-apartness, for any $t > 1$) is a consequence of HT, but also admits an easy proof from $\mathsf{RT}_2^{!\omega}$. Given $c : \mathbf{N} \to 2$ just set $f(S) := c(\sum S)$, for S an exactly large set (to get t-apartness, restrict f to an infinite t-apart set). By results from [6] this reduction yields an upper bound of $\emptyset^{(\omega)}$ on $\mathsf{HT}_2^{!\omega}$.

Proposition 2 (RCA_0)**.** $\mathsf{HT}_2^{!\omega}$ *with 2-apartness implies* IPHT_2^2 *with 2-apartness.*

Proof. Let $f : \mathbf{N} \to 2$ be given, and let $H = \{h_0, h_1, h_2, \dots\}_<$ be an infinite 2-apart set such that $FS^{!\omega}(H)$ is monochromatic for f of color $k < 2$. Let $H_s = \{s_1, s_2, \dots\}_<$ be the 2-apart set whose elements are exactly large sums of consecutive elements from H. Let $H_t = \{t_1, t_2, \dots\}_<$ be the set of elements from H_s minus their largest term (when written as $!\omega$-sums). Note that distinct elements of H_s share no term, because H_s is 2-apart. Let $H_1 := H_t$ and let $H_2 := \{s_i - t_i : i \in \mathbf{N}\}$. Then (H_1, H_2) is a 2-apart solution for IPHT_2^2: □

From Proposition 3 we get that $\mathsf{HT}_2^{!\omega}$ with 2-apartness implies IPT_2^2. In particular it is unprovable in WKL_0. Other results on $\mathsf{HT}_2^{!\omega}$ have been proved by the third author in his BSc. Thesis. E.g., over RCA_0, $\mathsf{HT}_2^{!\omega}$ with 2-apartness implies $\forall n \mathsf{HT}_2^{=2^n}$, and $\mathsf{HT}_2^{!\omega}$ implies $\forall n \mathsf{PHT}_2^n$ (see Definition 6).

5 Conclusion

We contributed to the study of restricted versions of Hindman's Theorem by proving implications from (and equivalence of) some such restrictions to ACA_0 and to the Increasing Polarized Ramsey's Theorem for Pairs. Our results improve and integrate the recent results by Dzhafarov, Jockusch, Solomon and Westrick [12]. In many cases they confirm that the known lower bounds on Hindman's Theorem hold for restricted versions of Hindman's Theorem for which — contrary to the restrictions studied in [12] — the upper bound lies strictly below $\emptyset^{(\omega+1)}$ (most being consequences of ACA_0 or even of RT^2_2). This also complements the results of [3] and [4] and might be an indication that the known lower bounds for Hindman's Theorem are sub-optimal. We highlighted the role of the apartness condition on the solution set.

Table 1. Summary of results

Principles	Lower Bounds	Upper Bounds
$\mathsf{HT}^{\leq 2}_2$	$\mathsf{RCA}_0 \nvdash$ ([12])	$\emptyset^{(\omega+1)}$, ACA_0^+ ([2])
$\mathsf{HT}^{\leq 2}_2 + B\Sigma^0_2$	SRT^2_2 ([12])	$\emptyset^{(\omega+1)}$, ACA_0^+ ([2])
$\mathsf{HT}^{\leq 2}_2$ with 2-apartness	ACA_0 (Proposition 1)	$\emptyset^{(\omega+1)}$, ACA_0^+ ([2])
$\mathsf{HT}^{\leq 2}_4$	ACA_0 (Theorem 1)	$\emptyset^{(\omega+1)}$, ACA_0^+ ([2])
$\mathsf{HT}^{\exists\{a,b\}}_2$	?	$\emptyset'$, ACA_0 ([4])
$\mathsf{HT}^{\exists\{a,b\}}_2$ with 2-apartness	ACA_0 (Theorem 2)	$\emptyset'$, ACA_0 (Theorem 2)
$\mathsf{HT}^{=2}_2$	?	RT^2_2 (folklore)
$\mathsf{HT}^{=2}_2$ with 2-apartness	IPT^2_2 (Theorem 3)	RT^2_2 (folklore)
IPHT^2_2 with 2-apartness	IPT^2_2 (Theorem 4)	IPT^2_2 (Theorem 4)
$\mathsf{HT}^{!\omega}_2$	?	$\emptyset^{(\omega)}$, ACA_0^+ ([6])
$\mathsf{HT}^{!\omega}_2$ with 2-apartness	IPT^2_2 (Proposition 2)	$\emptyset^{(\omega)}$, ACA_0^+ ([6])

Note: We have improved some of the above results and obtained some new results. E.g., both $\mathsf{HT}^{=3}_2$ with 2-apartness and $\mathsf{HT}^{!\omega}_2$ imply ACA_0 over RCA_0. These and further results will be presented in an extended version of this paper.

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