Research Article

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Regularizing effect for a system of Schrödinger–Maxwell equations

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Abstract: We prove some existence results for the following Schrödinger–Maxwell system of elliptic equations:

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + A\varphi|u|^{r-2}u = f, & u \in W_0^{1,2}(\Omega), \\ -\operatorname{div}(M(x)\nabla\varphi) = |u|^r, & \varphi \in W_0^{1,2}(\Omega). \end{cases}$$

In particular, we prove the existence of a finite energy solution (u,φ) if $r > 2^*$ and f does not belong to the "dual space" $L^{\frac{2N}{N+2}}(\Omega)$.

Keywords: Elliptic systems, Schrödinger–Maxwell equations, critical points

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1 Introduction

In the paper [1], Benci and Fortunato studied an eigenvalue problem for the Schrödinger operator, coupled with the electromagnetic field. Set in $\mathbb{R}^3$, this study lead to the Schrödinger–Maxwell system

$$\begin{cases} -\frac{1}{2}\Delta v + \psi v = \omega v, \\ -\Delta\psi = 4\pi v^2, \end{cases} \tag{1.1}$$

for which the existence of an increasing and divergent sequence of eigenvalues $\{\omega_n\}$ was established. Their result was proved using the fact that the solutions of (1.1) are critical points of an indefinite functional, unbounded both from above and below. In the subsequent paper [3], the related Dirichlet problem with a source term f was studied, that is,

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + A\varphi|u|^{r-2}u = f, & u \in W_0^{1,2}(\Omega), \\ -\operatorname{div}(M(x)\nabla\varphi) = |u|^r, & \varphi \in W_0^{1,2}(\Omega), \end{cases} \tag{1.2}$$

where $r > 1$, $A > 0$, Ω is an open bounded subset of $\mathbb{R}^N$ with $N > 2$, f belongs to $L^m(\Omega)$ with $m \geq \frac{2N}{N+2}$, and $M(x)$ is a symmetric measurable matrix such that

$$(M(x)\xi)\xi \geq \alpha|\xi|^2, \quad |M(x)| \leq \beta \tag{1.3}$$

for almost every x in Ω and every ξ in $\mathbb{R}^N$, with $0 < \alpha \leq \beta$.

As in the case of (1.1), solutions of (1.2) are critical points of an indefinite functional, and in [3] it is proved that if

$$\frac{2N}{N+2} \leq m \leq \frac{2Nr}{N+2+4r}, \quad r \geq \frac{N+2}{N-2},$$

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then there exists a solution (u, φ) in $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$. The fact that φ belongs to $W_0^{1,2}(\Omega)$ is interesting since under the assumption $r > \frac{N+2}{N-2}$, the right-hand side $|u|^r$ of the second equation does not belong to the "dual space" $L^{\frac{2N}{N+2}}(\Omega)$ (recall that $\frac{2N}{N+2} = (2^*)'$). Hence, the fact that φ belongs to $W_0^{1,2}(\Omega)$ does not follow from it being a solution of the second equation, but from the coupling between u and φ given by the system. In other words, there is a regularizing effect on the solution φ due to the fact it solves a system.

In this paper we improve some existence results of [3], always in the spirit of this regularizing effect. Indeed, we prove that there exist finite energy solutions also when the datum f does not belong to the dual space $L^{\frac{2N}{N+2}}(\Omega)$, and that one can obtain such solutions by taking data almost in $L^1(\Omega)$, under the assumption that the exponent r is large enough. Once again, in order to do that, we take advantage of the coupling between the two equations of the system.

Our strategy to prove such a result will be the following. In Section 2 we will prove that, in the case of bounded data f, a solution (u, φ) of (1.2) can be found as a saddle point of the functional

$$J(z, \eta) = \frac{1}{2} \int_\Omega M(x)\nabla z \nabla z - \frac{A}{2r} \int_\Omega M(x)\nabla \eta \nabla \eta + \frac{A}{r} \int_\Omega \eta^+ |z|^r - \int_\Omega fz,$$

defined on $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$. We will then use the solutions found in this case to build an approximating sequence $\{(u_n, \varphi_n)\}$ of solutions – corresponding to data f_n converging to f in $L^m(\Omega)$ – which will converge to a solution (u, φ) of (1.2). This and the summability properties of both u and φ, which are the main results of this paper, will be proved in Section 3. In the final section, Section 4, we will prove that the solution we find in Section 3 is still a saddle point of the functional J above, in a suitable sense (thanks to the use of T-minima, introduced in [3]).

2 Data in the dual space

Our first result (which was the starting point in [3]) deals with bounded data f. In this case, as stated in the introduction, one can find a solution (u, φ) of (1.2) as a saddle point of a suitable functional.

Proposition 2.1. *Let f be in $L^\infty(\Omega)$, and let $A > 0$ and $r > 1$. Then there exists a weak solution (u, φ) of (1.2). Furthermore, u and φ belong to $L^\infty(\Omega)$, $\varphi \geq 0$ and (u, φ) is a saddle point of the functional defined on $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ as*

$$J(z, \eta) = \begin{cases} \frac{1}{2} \int_\Omega M(x)\nabla z \nabla z - \frac{A}{2r} \int_\Omega M(x)\nabla \eta \nabla \eta + \frac{A}{r} \int_\Omega \eta^+ |z|^r - \int_\Omega fz & \text{if } \int_\Omega \eta^+ |z|^r < +\infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

Proof. Fix $\psi \in W_0^{1,2}(\Omega)$, and let $v = S(\psi)$ in $W_0^{1,2}(\Omega)$ be the unique minimum of

$$I_1(z) = J(z, \psi).$$

Note that such a minimum exists since I_1 is weakly lower semicontinuous and coercive on $W_0^{1,2}(\Omega)$. Evidently, v is the unique weak solution of the Euler–Lagrange equation

$$-\operatorname{div}(M(x)\nabla v) + A\psi^+ |v|^{r-2} v = f, \quad v \in W_0^{1,2}(\Omega). \tag{2.2}$$

Observe that, by the classical theory of elliptic equations with discontinuous coefficients and since $\psi^+ \geq 0$, we have

$$\|v\|_{W_0^{1,2}(\Omega)} \leq C_1 \|f\|_{L^\infty(\Omega)}, \quad \|v\|_{L^\infty(\Omega)} \leq C_1 \|f\|_{L^\infty(\Omega)}. \tag{2.3}$$

Consider now the functional

$$I_2(\eta) = J(v, \eta).$$

Since v belongs to $L^\infty(\Omega)$, $I_2(\eta)$ is finite for every η in $W_0^{1,2}(\Omega)$. Since $-I_2$ is both weakly lower semicontinuous and coercive, there exists a unique maximum $\zeta = T(v)$ of I_2 on $W_0^{1,2}(\Omega)$. Since $I_2(\zeta) \geq I_2(\zeta^+)$, we have

$$-\frac{A}{2r} \int_\Omega M(x)\nabla \zeta \nabla \zeta + \frac{A}{r} \int_\Omega \zeta^+ |v|^r \geq -\frac{A}{2r} \int_\Omega M(x)\nabla \zeta^+ \nabla \zeta^+ + \frac{A}{r} \int_\Omega (\zeta^+)^+ |v|^r,$$

and it is easy to prove from this inequality that $\zeta \geq 0$. Observe now that since $\zeta \geq 0$ is a maximum, we have

$$\begin{aligned} -\frac{A}{2r}\int_\Omega M(x)\nabla\zeta\nabla\zeta + \frac{A}{r}\int_\Omega \zeta|v|^r &= -\frac{A}{2r}\int_\Omega M(x)\nabla\zeta\nabla\zeta + \frac{A}{r}\int_\Omega \zeta^+|v|^r \\ &\geq -\frac{A}{2r}\int_\Omega M(x)\nabla\psi\nabla\psi + \frac{A}{r}\int_\Omega \psi^+|v|^r \\ &\geq -\frac{A}{2r}\int_\Omega M(x)\nabla\psi\nabla\psi + \frac{A}{r}\int_\Omega \psi|v|^r, \end{aligned}$$

so that ζ is a maximum on $W_0^{1,2}(\Omega)$ of

$$I_3(\eta) = -\frac{A}{2r}\int_\Omega M(x)\nabla\eta\nabla\eta + \frac{A}{r}\int_\Omega \eta|v|^r.$$

Hence, it is the unique weak solution of the Euler–Lagrange equation

$$-\operatorname{div}(M(x)\nabla\zeta) = |v|^r, \quad \zeta \in W_0^{1,2}(\Omega). \tag{2.4}$$

Recalling the estimates

$$\|\zeta\|_{W_0^{1,2}(\Omega)} \leq C_2\|v\|_{L^\infty(\Omega)}^r, \quad \|\zeta\|_{L^\infty(\Omega)} \leq C_2\|v\|_{L^\infty(\Omega)}^r \tag{2.5}$$

and (2.3), we have

$$\|\zeta\|_{W_0^{1,2}(\Omega)} \leq C_2\|f\|_{L^\infty(\Omega)}^r = R,$$

so that the ball of $W_0^{1,2}(\Omega)$ of radius R is invariant for the map $\zeta = T(S(\psi))$.

We are now going to prove that $T \circ S$ satisfies the assumptions of Schauder's fixed point theorem. Let $\{\psi_n\}$ be a sequence in $W_0^{1,2}(\Omega)$ that is weakly convergent to some ψ, and let $v_n = S(\psi_n)$. Since the sequence $\{v_n\}$ is bounded both in $W_0^{1,2}(\Omega)$ and in $L^\infty(\Omega)$ by (2.3), it follows that (up to subsequences, still denoted by $\{v_n\}$) it weakly converges to some function v in $W_0^{1,2}(\Omega)$, and strongly converges to the same function in $L^q(\Omega)$ for every $q > 1$. This fact implies that v is the solution of (2.2) with datum ψ, i.e., $v = S(\psi)$. Furthermore, since the sequence $\{|v_n|^r\}$ is strongly compact in (say) $L^2(\Omega)$, classical elliptic estimates imply that the sequence $\zeta_n = T(v_n)$ (which is bounded in $W_0^{1,2}(\Omega)$ and in $L^\infty(\Omega)$ by (2.5)) is strongly convergent in $W_0^{1,2}(\Omega)$ to some function ζ, which is the solution of (2.4) with datum v. That is, $\zeta = T(v)$, so that $\zeta = T(S(\psi))$. We have therefore proved that if $\{\psi_n\}$ is bounded in $W_0^{1,2}(\Omega)$, then one can extract from $\zeta_n = T(S(\psi_n))$ a subsequence which is strongly convergent in $W_0^{1,2}(\Omega)$, so that $T \circ S$ transforms bounded sets of $W_0^{1,2}(\Omega)$ into pre-compact sets of $W_0^{1,2}(\Omega)$. Furthermore, if ψ_n is strongly convergent to ψ in $W_0^{1,2}(\Omega)$ and we consider any subsequence $\{\zeta_{n_k}\}$ of $\zeta_n = T(S(\psi_n))$, then a sub-subsequence exists which is strongly convergent in $W_0^{1,2}(\Omega)$ to $\zeta = T(S(\psi))$. This latter fact follows from the uniqueness results for both (2.2) and (2.4). Therefore, since the limit does not depend on the subsequence extracted, the whole sequence $\{T(S(\psi_n))\}$ converges to $\zeta = T(S(\psi))$. Thus, we have also proved that $T \circ S$ is continuous, and this allows to apply Schauder's fixed point theorem.

Let φ be the fixed point of $T \circ S$. Observe that since $u = S(\varphi)$ is a minimum for I_1 and $\varphi = T(u) = T(S(\varphi))$ is a maximum of I_2, we have

$$J(u,\eta) \leq J(u,\varphi) \leq J(z,\varphi) \quad \text{for all } z \in W_0^{1,2}(\Omega) \text{ and all } \eta \in W_0^{1,2}(\Omega),$$

and so (u,φ) is a saddle point. Since $\varphi = T(S(\varphi))$, we have that (u,φ) is a weak solution of

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + A\varphi|u|^{r-2}u = f, & u \in W_0^{1,2}(\Omega), \\ -\operatorname{div}(M(x)\nabla\varphi) = |u|^r, & \varphi \in W_0^{1,2}(\Omega), \end{cases}$$

with the required regularity properties. □

Remark 2.2. The only place in the proof of the previous theorem where we used that f belongs to $L^\infty(\Omega)$ was to prove (2.3). However, such an estimate holds under the weaker assumption that f belongs to $L^m(\Omega)$ with $m > \frac{N}{2}$, thanks to the results of Stampacchia (see [8]). Therefore, the same existence result for (u,φ) can be proved under this weaker assumption.

To conclude this section, we recall the existence result proved in [3], thus completing the "picture" in the case of data in the dual space (which yield finite energy solutions).

Proposition 2.3. *Let f in $L^m(\Omega)$ with $m \geq \frac{2N}{N+2} = 2_*$. Then there exists a weak solution (u, φ) of (1.2), with u and φ in $W_0^{1,2}(\Omega)$.*

3 Regularizing effect

We now deal with the case of data not in the dual space, proving the main result of this paper. In this case the interplay among the two equations of the system will be crucial in order to obtain estimates. If $k \geq 0$, we define the functions

$$T_k(s) = \max(-k, \min(s, k)), \quad G_k(s) = s - T_k(s).$$

Let $\{f_n\}$ be a sequence of $L^\infty(\Omega)$ functions strongly convergent to f in $L^m(\Omega)$, $m \geq 1$, and such that

$$|f_n| \leq |f|. \tag{3.1}$$

Then, by Theorem 2.1, there exists a solution (u_n, φ_n) of the system

$$\begin{cases} -\operatorname{div}(M(x)\nabla u_n) + A\varphi_n |u_n|^{r-2}u_n = f_n, & u_n \in W_0^{1,2}(\Omega), \\ -\operatorname{div}(M(x)\nabla \varphi_n) = |u_n|^r, & \varphi_n \in W_0^{1,2}(\Omega), \end{cases} \tag{3.2}$$

with u_n and φ_n in $L^\infty(\Omega)$.

We begin with a result concerning the the positive and negative parts of u_n, and the truncates of φ_n; the proof is inspired by the techniques used in [7].

Lemma 3.1. *If (u_n, φ_n) is a solution of (3.2), then*

$$\int_\Omega M(x)\nabla T_1(\varphi_n)\nabla w \geq \int_{\{0\leq\varphi_n\leq 1\}} |u_n|^r w \tag{3.3}$$

for every w in $W_0^{1,2}(\Omega)$, $w \geq 0$.

Proof. Let $H_\varepsilon(s) = 1 - \frac{1}{\varepsilon} T_\varepsilon(G_1(s^+))$, and choose $H_\varepsilon(\varphi_n)w$ as test function in the second equation of (3.2), with w in $W_0^{1,2}(\Omega)$, $w \geq 0$. We have

$$-\frac{1}{\varepsilon} \int_{\{1\leq\varphi_n\leq 1+\varepsilon\}} M(x)\nabla\varphi_n\nabla\varphi_n w + \int_\Omega M(x)\nabla\varphi_n\nabla w H_\varepsilon(\varphi_n) = \int_\Omega |u_n|^r w H_\varepsilon(\varphi_n).$$

Since the first term is negative by (1.3), we can drop it, obtaining

$$\int_\Omega M(x)\nabla\varphi_n\nabla w H_\varepsilon(\varphi_n) \geq \int_\Omega |u_n|^r w H_\varepsilon(\varphi_n).$$

Letting ε tend to zero, we obtain (3.3). □

Our next result deals with a priori estimates on $\{u_n\}$.

Lemma 3.2. *Let $m > 1$ and $k \geq 0$. Then there exists $C_0 > 0$, independent of n and k, such that*

$$\int_\Omega |G_k(u_n)|^{m(r-1)} \leq C_0 \int_{\{|u_n|\geq k\}} |f|^m + C_0 \operatorname{meas}(\{|u_n| \geq k\}) \tag{3.4}$$

and

$$\Big(\int_\Omega |G_k(u_n)|^{m^{**}}\Big)^{\frac{1}{m^{**}}} \leq C_0 \Big(\int_{\{|u_n|\geq k\}} |f|^m\Big)^{\frac{1}{m}}, \tag{3.5}$$

In particular (choosing $k = 0$), $\{|u_n|^\rho\}$ is bounded in $L^1(\Omega)$ with $\rho = \max(m(r-1), m^{**})$.

Proof. Let $\gamma > -1$. First, we work with the first inequality. Let $\varepsilon > 0$, and choose $(|G_k(u_n)| + \varepsilon)^\gamma G_k(u_n)$ as test function. Note that since γ may be negative, we need to "add" the term with ε since the gradient of $|G_k(u_n)|^\gamma G_k(u_n)$ may be not defined when $G_k(u_n) = 0$, even though $|G_k(u_n)|^\gamma G_k(u_n)$ is well defined since $\gamma + 1 > 0$. After using (1.3) and (3.1), and dropping the positive term involving the principal part, we obtain

$$A \int_\Omega \varphi_n |u_n|^{r-1} (|G_k(u_n)| + \varepsilon)^\gamma |G_k(u_n)| \leq \int_\Omega |f| (|G_k(u_n)| + \varepsilon)^\gamma |G_k(u_n)|,$$

which implies, by letting ε tend to zero (recall that every u_n is a bounded function), that

$$A \int_\Omega \varphi_n |u_n|^{r-1} |G_k(u_n)|^{\gamma+1} \leq \int_\Omega |f| |G_k(u_n)|^{\gamma+1}.$$

Therefore, since $|G_k(u_n)| \leq |u_n|$ and $r > 1$, we have

$$\begin{aligned} A \int_{\{\varphi_n \geq 1\}} |G_k(u_n)|^{\gamma+r} &\leq A \int_\Omega \varphi_n |G_k(u_n)|^{\gamma+r} \\ &= A \int_\Omega \varphi_n |G_k(u_n)|^{r-1} |G_k(u_n)|^{\gamma+1} \\ &\leq A \int_\Omega \varphi_n |u_n|^{r-1} |G_k(u_n)|^{\gamma+1} \\ &\leq \int_\Omega |f| |G_k(u_n)|^{\gamma+1}. \end{aligned} \tag{3.6}$$

Now we work with the second equation in two different ways, according to the value of γ.

Suppose that $\gamma > 1$. Choose $w = |G_k(u_n)|^\gamma$ as test function in (3.3) to obtain

$$\gamma \int_\Omega M(x) \nabla T_1(\varphi_n) \nabla G_k(u_n) |G_k(u_n)|^{\gamma-2} G_k(u_n) \geq \int_{\{0 \leq \varphi_n \leq 1\}} |u_n|^r |G_k(u_n)|^\gamma \geq \int_{\{0 \leq \varphi_n \leq 1\}} |G_k(u_n)|^{\gamma+r}. \tag{3.7}$$

On the other hand, choosing $\gamma T_1(\varphi_n)(|G_k(u_n)| + \varepsilon)^{\gamma-2} G_k(u_n)$ as test function in the first equation of (3.2), by dropping the terms

$$\gamma \int_\Omega M(x) \nabla u_n \nabla [(|G_k(u_n)| + \varepsilon)^{\gamma-2} G_k(u_n)] T_1(\varphi_n)$$

and

$$A\gamma \int_\Omega \varphi_n |u_n|^{r-2} T_1(\varphi_n) (|G_k(u_n)| + \varepsilon)^{\gamma-2} u_n G_k(u_n),$$

which are positive, we get

$$\gamma \int_\Omega M(x) \nabla G_k(u_n) \nabla T_1(\varphi_n) (|G_k(u_n)| + \varepsilon)^{\gamma-2} G_k(u_n) \leq \gamma \int_\Omega |f| (|G_k(u_n)| + \varepsilon)^{\gamma-1}.$$

Letting ε tend to zero, we obtain

$$\gamma \int_\Omega M(x) \nabla G_k(u_n) \nabla T_1(\varphi_n) |G_k(u_n)|^{\gamma-2} G_k(u_n) \leq \gamma \int_\Omega |f| |G_k(u_n)|^{\gamma-1},$$

which, together with (3.7) and the fact that $|G_k(u_n)|^{\gamma-1} \leq |G_k(u_n)|^{\gamma+1} + 1$ (since $\gamma > 1$), implies that

$$\int_{\{0 \leq \varphi_n \leq 1\}} |G_k(u_n)|^{\gamma+r} \leq \gamma \int_\Omega |f| |G_k(u_n)|^{\gamma-1} \leq \gamma \int_\Omega |f| |G_k(u_n)|^{\gamma+1} + \gamma \int_{\{|u_n| \geq k\}} |f|. \tag{3.8}$$

Therefore, (3.6) and (3.8) give

$$\int_{\{\varphi_n \geq 1\}} |G_k(u_n)|^{\gamma+r} + \int_{\{0 \leq \varphi_n \leq 1\}} |G_k(u_n)|^{\gamma+r} \leq \Big(\frac{1}{A} + \gamma\Big) \int_\Omega |f| |G_k(u_n)|^{\gamma+1} + \gamma \int_{\{|u_n| \geq k\}} |f|,$$

that is,

$$\int_\Omega |G_k(u_n)|^{\gamma+r} \le \Big(\frac{1}{A}+\gamma\Big)\int_\Omega |f||G_k(u_n)|^{\gamma+1} + \gamma \int_{\{|u_n|\ge k\}} |f| \quad \text{for all } \gamma > 1. \tag{3.9}$$

Now we study the case $-1 < \gamma \le 1$, and choose $w = |G_k(u_n)|$ as test function in (3.3) to obtain

$$\int_\Omega M(x)\nabla T_1(\varphi_n)\nabla G_k(u_n)\,\mathrm{sgn}(G_k(u_n)) \ge \int_{\{0\le\varphi_n\le 1\}} |u_n|^r |G_k(u_n)| \ge \int_{\{0\le\varphi_n\le 1\}} |G_k(u_n)|^{r+1}. \tag{3.10}$$

Choosing $T_1(\varphi_n)\frac{1}{\varepsilon}T_\varepsilon(G_k(u_n))$ in the first equation of (3.2), and dropping two positive terms, we get

$$\int_\Omega M(x)\nabla G_k(u_n)\nabla T_1(\varphi_n)\frac{1}{\varepsilon}T_\varepsilon(G_k(u_n)) \le \int_{\{|u_n|\ge k\}} |f|,$$

so that, by letting ε tend to zero, we get

$$\int_\Omega M(x)\nabla G_k(u_n)\nabla T_1(\varphi_n)\,\mathrm{sgn}(G_k(u_n)) \le \int_{\{|u_n|\ge k\}} |f|,$$

which, together with (3.10), yields

$$\int_{\{0\le\varphi_n\le 1\}} |G_k(u_n)|^{r+1} \le \int_{\{|u_n|\ge k\}} |f|. \tag{3.11}$$

Since $\gamma \le 1$, we have

$$\int_{\{0\le\varphi_n\le 1\}} |G_k(u_n)|^{\gamma+r} \le \int_{\{0\le\varphi_n\le 1\}} |G_k(u_n)|^{r+1} + \mathrm{meas}(\{|u_n|\ge k\}) \le \int_{\{|u_n|\ge k\}} |f| + \mathrm{meas}(\{|u_n|\ge k\}),$$

which, together with (3.6), implies that

$$\int_\Omega |G_k(u_n)|^{\gamma+r} \le \frac{1}{A}\int_\Omega |f||G_k(u_n)|^{\gamma+1} + \int_{\{|u_n|\ge k\}} |f| + \mathrm{meas}(\{|u_n|\ge k\}). \tag{3.12}$$

Summing up the results of (3.9) and (3.12), for every $\gamma > -1$, we obtain

$$\int_\Omega |G_k(u_n)|^{\gamma+r} \le C_1\int_\Omega |f||G_k(u_n)|^{\gamma+1} + C_1\int_{\{|u_n|\ge k\}} |f| + C_1\,\mathrm{meas}(\{|u_n|\ge k\}).$$

Observe now that we have

$$\begin{aligned}\int_\Omega |f||G_k(u_n)|^{\gamma+1} &= \int_{\{|f|\le\delta|G_k(u_n)|^{r-1}\}} |f||G_k(u_n)|^{\gamma+1} + \int_{\{\delta|G_k(u_n)|^{r-1}<|f|\}} |f||G_k(u_n)|^{\gamma+1}\\ &\le \delta\int_\Omega |G_k(u_n)|^{\gamma+r} + C_\delta\int_{\{|u_n|\ge k\}} |f|^{\frac{\gamma+r}{r-1}}.\end{aligned}$$

We now choose γ such that $\frac{\gamma+r}{r-1} = m$. This implies that $\gamma > -1$, since $m > 1$, and that $\gamma + r = m(r-1)$. Thus, choosing δ such that $C_1\delta = \frac{1}{2}$, we have

$$\int_\Omega |G_k(u_n)|^{\gamma+r} \le \frac{1}{2}\int_\Omega |G_k(u_n)|^{\gamma+r} + C_2\int_{\{|u_n|\ge k\}} |f|^m + C_2\int_{\{|u_n|\ge k\}} |f| + C_2\,\mathrm{meas}(\{|u_n|\ge k\}),$$

so that

$$\int_\Omega |G_k(u_n)|^{m(r-1)} \le C_3\int_{\{|u_n|\ge k\}} |f|^m + C_3\,\mathrm{meas}(\{|u_n|\ge k\}),$$

which is (3.4).

To obtain (3.5), suppose that $\gamma > \frac{1}{2}$, and choose $(|G_k(u_n)| + \varepsilon)^{2\gamma-2} G_k(u_n)$ as test function in the first equation of (3.2) with $\varepsilon > 0$. Since

$$\nabla\big[(|G_k(u_n)| + \varepsilon)^{2\gamma-2} G_k(u_n)\big] = \nabla G_k(u_n)(|G_k(u_n)| + \varepsilon)^{2\gamma-3}\big[(2\gamma-1)|G_k(u_n)| + \varepsilon\big]$$

and

$$(2\gamma-1)|G_k(u_n)| + \varepsilon \geq \min(2\gamma-1, 1)(|G_k(u_n)| + \varepsilon),$$

we have, by dropping a positive term and using (1.3), that

$$\alpha \min(2\gamma-1, 1) \int_\Omega |\nabla G_k(u_n)|^2 (|G_k(u_n)| + \varepsilon)^{2\gamma-2} \leq \int_\Omega |f|(|G_k(u_n)| + \varepsilon)^{2\gamma-1}.$$

Using the Sobolev embedding on the left, Hölder's inequality on the right, and then letting ε tend to zero, we have

$$\begin{aligned}\frac{\alpha \mathcal{S} \min(2\gamma-1, 1)}{\gamma^2}\Big(\int_\Omega |G_k(u_n)|^{2^*\gamma}\Big)^{\frac{2}{2^*}} &\leq \int_\Omega |f||G_k(u_n)|^{2\gamma-1} \\ &= \int_{\{|u_n|\geq k\}} |f||G_k(u_n)|^{2\gamma-1} \\ &\leq \Big(\int_{\{|u_n|\geq k\}} |f|^m\Big)^{\frac{1}{m}} \Big(\int_\Omega |G_k(u_n)|^{(2\gamma-1)m'}\Big)^{\frac{1}{m'}}.\end{aligned}$$

Choosing $\gamma = \frac{m^{**}}{2^*}$, and simplifying equal terms, yields

$$\Big(\int_\Omega |G_k(u_n)|^{m^{**}}\Big)^{\frac{1}{m^{**}}} \leq C_0 \Big(\int_{\{|u_n|\geq k\}} |f|^m\Big)^{\frac{1}{m}},$$

as desired. □

Remark 3.3. If $m = 1$, the estimate proved in [4] implies that the sequence $\{u_n\}$ is bounded in $L^s(\Omega)$ for every $s < \frac{N}{N-2}$. On the other hand, choosing $T_\varepsilon(u_n)/\varepsilon$ as test function in the first equation of (3.2), by observing that $u_n T_\varepsilon(u_n) = |u_n||T_\varepsilon(u_n)|$ and dropping a positive term, we have

$$A \int_{\{\varphi_n \geq 1\}} |u_n|^{r-1} \frac{|T_\varepsilon(u_n)|}{\varepsilon} \leq A \int_\Omega \varphi_n |u_n|^{r-1} \frac{|T_\varepsilon(u_n)|}{\varepsilon} \leq \int_\Omega f_n \frac{T_\varepsilon(u_n)}{\varepsilon} \leq \|f\|_{L^1(\Omega)}.$$

Letting ε tend to zero, we obtain

$$A \int_{\{\varphi_n \geq 1\}} |u_n|^{r-1} \leq \|f\|_{L^1(\Omega)}.$$

On the other hand, (3.11) with $k = 0$ implies that

$$\int_{\{0\leq\varphi_n\leq 1\}} |u_n|^{r-1} \leq \int_{\{0\leq\varphi_n\leq 1\}} |u_n|^{r+1} + \operatorname{meas}(\Omega) \leq \|f\|_{L^1(\Omega)} + \operatorname{meas}(\Omega),$$

so that we have

$$\int_\Omega |u_n|^{r-1} \leq \Big(1 + \frac{1}{A}\Big)\|f\|_{L^1(\Omega)} + \operatorname{meas}(\Omega).$$

Thus, since $r - 1 \geq \frac{N}{N-2}$ if $r \geq 2\frac{N-1}{N-2}$, we have that $\{u_n\}$ is bounded in $L^s(\Omega)$ for all $s < \frac{N}{N-2}$ if $1 < r < 2\frac{N-1}{N-2}$ and bounded in $L^{r-1}(\Omega)$ if $r \geq 2\frac{N-1}{N-2}$.

Now we recall that for the single equation

$$-\operatorname{div}(M(x)\nabla w) + a(x)w|w|^{r-2} = f, \quad a(x) \geq a_0 > 0,$$

it has been proved in [6] that w belongs to $W_0^{1,2}(\Omega)$ if f belongs to $L^m(\Omega)$ with $m \geq r'$.

In our case $a(x)$ is $\varphi(x)$ and we only know that $\varphi(x) \geq 0$. Nevertheless, we are able to prove (in the next result, the main of this paper) that the solution of the first equation belongs to $W_0^{1,2}(\Omega)$, under the same assumption $m \geq r'$, using the fact that we are dealing with a system.

Theorem 3.4. (A) *Let $r > 2^*$, and let f in $L^m(\Omega)$, with*

$$\frac{r}{r-1} = r' \leq m < \frac{2N}{N+2}.$$

Then there exists a weak solution (u, φ) of system (1.2), with u and φ in $W_0^{1,2}(\Omega)$.
(B) *Let $1 < r < 2^*$, and let f in $L^m(\Omega)$, with*

$$\max\Big(\frac{Nr}{N+2r}, 1\Big) < m < \frac{2N}{N+2}.$$

Then there exists a weak solution (u, φ) of system (1.2) with u in $W_0^{1,m^*}(\Omega)$, and φ in $W_0^{1,q}(\Omega)$ with

$$q = \begin{cases} 2 & \text{if } \frac{2Nr}{N+2+4r} \leq m < \frac{2N}{N+2}, \\ \frac{Nm}{Nr-2mr-m} & \text{if } \max(\frac{Nr}{N+2r}, 1) < m < \frac{2Nr}{N+2+4r}. \end{cases}$$

Remark 3.5. If $r = 2^*$, both cases of the previous theorem “collapse” to $m = \frac{2N}{N+2}$, and in this case the existence result is given by Proposition 2.3.

Proof. We begin with case (A): $r > 2^*$. Thanks to Lemma 3.2, the sequence $\{u_n\}$ is bounded in $L^{m(r-1)}(\Omega)$ (note that we have $m(r-1) > r-1 > \frac{N+2}{N-2} > 1$ for every $m > 1$), and in $L^{m^{**}}(\Omega)$. Since

$$m(r-1) \geq m^{**} \iff m \leq \frac{N}{2}\frac{r-2}{r-1}$$

and

$$\frac{N}{2}\frac{r-2}{r-1} > \frac{2N}{N+2} \quad \text{for every } r > 2^*,$$

we have $m(r-1) > m^{**}$ for every $r > 2^*$ and every $1 < m < \frac{2N}{N+2}$, so that the better estimate on $\{u_n\}$ is the boundedness in $L^{m(r-1)}(\Omega)$, $m > 1$.

Thus, $\{|u_n|^r\}$ is bounded in $L^{\frac{m}{r'}}(\Omega)$. In order to continue, we have to make a further restriction on m. Namely, $m \geq r'$, since we need the sequence $\{|u_n|^r\}$ to be bounded at least in $L^1(\Omega)$, in order to obtain estimates on $\{\varphi_n\}$ using the second equation of (3.2). Therefore, from now on, we are working with the assumption

$$r > 2^*, \quad r' \leq m < \frac{2N}{N+2}.$$

We choose now u_n as test function in the first equation of (3.2). Using (1.3), we have

$$\alpha \int_\Omega |\nabla u_n|^2 + A \int_\Omega \varphi_n |u_n|^r \leq \int_\Omega f_n u_n \leq \|f\|_{L^m(\Omega)} \|u_n\|_{L^{m'}(\Omega)}.$$

Here we note that the assumption $r' \leq m$ implies that $m' \leq m(r-1)$, therefore $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$ and

$$\int_\Omega \varphi_n |u_n|^r \leq C_1.$$

Choosing φ_n as test function in the second equation of (3.2), using (1.3) and the last estimate, yields

$$\alpha \int_\Omega |\nabla \varphi_n|^2 \leq \int_\Omega \varphi_n |u_n|^r \leq C_1,$$

so that also $\{\varphi_n\}$ is bounded in $W_0^{1,2}(\Omega)$. Thus, up to subsequences, (u_n, φ_n) converges to (u, φ) weakly in $(W_0^{1,2}(\Omega))^2$, strongly in $(L^p(\Omega))^2$ for every $p < 2^*$, and almost everywhere in Ω. In order to prove that (u, φ) is a solution of the system, we have to pass to the limit in the two nonlinear terms. However, the strong convergence in $L^p(\Omega)$ with $p < 2^* < r$, is not enough to pass to the limit, so that we will have to use again Lemma 3.2. Indeed, since $m(r-1) \geq r$ by the assumption $m \geq r'$, inequality (3.4) implies that

$$\int_\Omega |G_k(u_n)|^r \leq C_2\Big(\int_{\{|u_n|\geq k\}} |f|^m + \operatorname{meas}(\{|u_n| \geq k\})\Big)^{\frac{r}{m(r-1)}}. \tag{3.13}$$

Therefore, if E is a measurable subset of Ω, then we have

$$\begin{aligned}\int_E |u_n|^r &\leq 2^{r-1}\int_E |T_k(u_n)|^r + 2^{r-1}\int_E |G_k(u_n)|^r \\ &\leq 2^{r-1}k^r \operatorname{meas}(E) + 2^{r-1}\int_\Omega |G_k(u_n)|^r.\end{aligned} \tag{3.14}$$

Let $\varepsilon > 0$, and fix $k_0 > 0$ so that

$$2^{r-1}\int_\Omega |G_{k_0}(u_n)|^r \leq \frac{\varepsilon}{2} \quad \text{for all } n \in \mathbb{N}.$$

This can be done (using (3.13)) since f belongs to $L^m(\Omega)$, and the measure of $\{|u_n| \geq k\}$ is uniformly (in n) small if k is large by the a priori estimates on $\{u_n\}$. Once k_0 is fixed, choose meas(E) small enough so that

$$2^{r-1}k_0^r \operatorname{meas}(E) < \frac{\varepsilon}{2} \quad \text{for all } n \in \mathbb{N}.$$

Thus, by (3.14), the sequence $\{|u_n|^r\}$ is equiintegrable, and by Vitali's theorem we have

$$\lim_{n\to+\infty}\int_\Omega |u_n|^r = \int_\Omega |u|^r.$$

This is enough to pass to the limit in the second equation of (3.2), with test functions in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Choose now $\frac{1}{\varepsilon}T_\varepsilon(G_k(u_n))$ as test function in the first equation of (3.2). By dropping a positive term, and then letting ε tend to zero, we get

$$\int_{\{|u_n|\geq k\}} \varphi_n|u_n|^{r-1} \leq \frac{1}{A}\int_{\{|u_n|\geq k\}} |f_n| \leq \frac{1}{A}\int_{\{|u_n|\geq k\}} |f|. \tag{3.15}$$

Thus, if E is a measurable subset of Ω, then

$$\int_E \varphi_n|u_n|^{r-1} \leq \int_E \varphi_n|T_k(u_n)|^{r-1} + \int_{E\cap\{|u_n|\geq k\}} \varphi_n|u_n|^{r-1} \leq k^{r-1}\int_E \varphi_n + C_3\int_{\{|u_n|\geq k\}} |f|.$$

As before, for fixed $\varepsilon > 0$, we use the estimates on u_n, and the fact that f belongs to $L^1(\Omega)$, to choose $k_0 > 0$ such that

$$C_3\int_{\{|u_n|\geq k\}} |f| < \frac{\varepsilon}{2} \quad \text{for all } n \in \mathbb{N},$$

and then use the fact that φ_n is strongly compact in $L^1(\Omega)$ (by Rellich's theorem) to choose meas(E) small enough so that

$$k_0^{r-1}\int_E \varphi_n < \frac{\varepsilon}{2} \quad \text{for all } n \in \mathbb{N}.$$

Thus, the sequence $\{\varphi_n|u_n|^{r-1}\}$ is equiintegrable, so that, by Vitaly's theorem,

$$\lim_{n\to+\infty}\int_\Omega \varphi_n|u_n|^{r-1} = \int_\Omega \varphi|u|^{r-1}.$$

Now this convergence is enough to pass to the limit in the first equation of (3.2), with test functions in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

We turn now to the case (B): $1 < r < 2^*$. Note that if $r < 2^*$, then

$$\frac{N}{2}\frac{r-2}{r-1} < \frac{Nr}{N+2r} < \frac{2N}{N+2} < r'.$$

Using Lemma 3.2, we have that $\{|u_n|^\rho\}$ is bounded in $L^1(\Omega)$, with

$$\rho = \begin{cases} m^{**} & \text{if } m \geq \frac{N}{2}\frac{r-2}{r-1},\ m > 1, \\ m(r-1) & \text{if } 1 < m < \frac{N}{2}\frac{r-2}{r-1}. \end{cases}$$

In order to have that $\{|u_n|^r\}$ is bounded at least in $L^1(\Omega)$, we need $\rho \geq r$, so that

$$\begin{cases} \frac{m^{**}}{r} \geq 1 & \text{if } m \geq \frac{N}{2}\frac{r-2}{r-1},\ m > 1, \\ \frac{m(r-1)}{r} \geq 1 & \text{if } 1 < m < \frac{N}{2}\frac{r-2}{r-1}, \end{cases} \quad \text{that is,} \quad \begin{cases} \frac{Nr}{N+2r} \leq m < \frac{2N}{N+2},\ m > 1, \\ r' \leq m < \frac{N}{2}\frac{r-2}{r-1}. \end{cases}$$

However, since $r < 2^*$, we have $\frac{N}{2}\frac{r-2}{r-1} < r'$, so that the second inequality above is never satisfied. Thus, the range of possible values of r and m becomes

$$1 < r < 2^*, \qquad \frac{Nr}{N+2r} \leq m < \frac{2N}{N+2}, \qquad m > 1. \tag{3.16}$$

Since the summability of u_n is the one obtained using only the principal part of the operator in the first equation and not the lower order term, one cannot expect boundedness in $W_0^{1,2}(\Omega)$ for the sequence $\{u_n\}$. In other words, the regularizing effect of the second equation on u_n is lost in this case, and one only has the estimate proved in [5]. More precisely, $\{u_n\}$ is bounded in $W_0^{1,m^*}(\Omega)$. As for φ_n, we can use the second equation of (3.2), and the fact that $\{|u_n|^r\}$ is bounded in $L^s(\Omega)$, with $s = \frac{m^{**}}{r} \geq 1$ by (3.16). Again, by the theory of elliptic equations, we have the following cases (see [4, 5]):

(i) $\{\varphi_n\}$ is bounded in $W_0^{1,2}(\Omega)$ if $s \geq \frac{2N}{N+2}$,
(ii) $\{\varphi_n\}$ is bounded in $W_0^{1,s^*}(\Omega)$ if $1 < s < \frac{2N}{N+2}$,
(iii) $\{\varphi_n\}$ is bounded in $W_0^{1,q}(\Omega)$ for all $q < \frac{N}{N-1}$ if $s = 1$.

Note that

$$\frac{2Nr}{N+2+4r} \leq m < \frac{2N}{N+2} \implies s \geq \frac{2N}{N+2},$$
$$\max\Big(\frac{Nr}{N+2r}, 1\Big) < m < \frac{2rN}{N+4r+2} \implies 1 < s < \frac{2N}{N+2},$$

and $s = 1$ if $m = \frac{Nr}{N+2r}$ and $m > 1$. Therefore, the following hold:

(i) $\{\varphi_n\}$ is bounded in $W_0^{1,2}(\Omega)$ if $\frac{2Nr}{N+2+4r} \leq m < \frac{2N}{N+2}$,
(ii) $\{\varphi_n\}$ is bounded in $W_0^{1,s^*}(\Omega)$ if $\max(\frac{Nr}{N+2r}, 1) < m < \frac{2Nr}{N+2+4r}$,
(iii) $\{\varphi_n\}$ is bounded in $W_0^{1,q}(\Omega)$ for all $q < \frac{N}{N-1}$ if $m = \frac{Nr}{N+2r}$, $m > 1$.

Since

$$s^* = \Big(\frac{m^{**}}{r}\Big)^* = \frac{Nm}{Nr - 2mr - m},$$

we have the desired estimates on φ_n.

In order to pass to the limit in the approximate equations, we can follow the same steps as in the case $r > 2^*$, using (3.5) for the second equation of (3.2), and (3.15) (which still holds) for the nonlinear term of the first one. □

Remark 3.6. Theorem 3.4 does not deal with the case $m = 1$ and $1 < r < \frac{N}{N-2}$. However, following the proof of (B) above, we can prove the following:

(i) the sequence $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$,
(ii) $\{\varphi_n\}$ is bounded in $W_0^{1,2}(\Omega)$ if $1 < r < \frac{N+2}{2(N-2)}$,
(iii) $\{\varphi_n\}$ is bounded in $W_0^{1,\rho}(\Omega)$ if $\frac{N+2}{2(N-2)} \leq r < \frac{N}{N-2}$ with $1 \leq \rho < \frac{N}{Nr-2r-1}$.

Note that the first case above is empty if $N \geq 6$ since $r > 1$.

4 Saddle points

In Proposition 2.1, we have proved that if f is bounded, then there exists a saddle point (u, φ) of the functional J defined in (2.1). Thus, the approximating solution (u_n, φ_n) of (3.2) can be found as a saddle point of the functional J_n defined as

$$J_n(v, \psi) = \frac{1}{2}\int_\Omega M(x)\nabla v \nabla v - \frac{A}{2r}\int_\Omega M(x)\nabla\psi\nabla\psi + \frac{A}{r}\int_\Omega \psi^+|v|^r - \int_\Omega f_n v, \tag{4.1}$$

in the sense that

$$J_n(u_n, \psi) \le J_n(u_n, \varphi_n) \le J_n(v, \varphi_n) \quad \text{for all } v \in W_0^{1,2}(\Omega) \text{ and all } \psi \in W_0^{1,2}(\Omega).$$

In this section, we study how the convergences on (u_n, φ_n), proved in the previous section, can be used in order to give a meaning to the concept of "saddle point of J" if the function f does not belong to $L^\infty(\Omega)$ (actually, if it does not belong to $L^{2_*}(\Omega)$). Note that in this case the functional J is not well defined, since the are two terms (with opposite signs) which can possibly be unbounded.

Recalling that (u_n, φ_n) is a saddle point for $J_n(v, \psi)$ and that φ_n is positive, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega M(x)\nabla u_n\nabla u_n - \frac{A}{2r}\int_\Omega M(x)\nabla\psi\nabla\psi + \frac{A}{r}\int_\Omega \psi^+|u_n|^r - \int_\Omega f_n u_n \\
&\quad\le \frac{1}{2}\int_\Omega M(x)\nabla u_n\nabla u_n - \frac{A}{2r}\int_\Omega M(x)\nabla\varphi_n\nabla\varphi_n + \frac{A}{r}\int_\Omega \varphi_n|u_n|^r - \int_\Omega f_n u_n \\
&\quad\le \frac{1}{2}\int_\Omega M(x)\nabla v\nabla v - \frac{A}{2r}\int_\Omega M(x)\nabla\varphi_n\nabla\varphi_n + \frac{A}{2r}\int_\Omega \varphi_n|v|^r - \int_\Omega f_n v
\end{aligned}$$

for every v and ψ in $W_0^{1,2}(\Omega)$, with v such that

$$\int_\Omega \varphi_n|v|^r < +\infty.$$

Splitting the above inequalities, simplifying equal terms, and rearranging them, we get

$$\frac{1}{2}\int_\Omega M(x)\nabla\varphi_n\nabla\varphi_n - \int_\Omega \varphi_n|u_n|^r \le \frac{1}{2}\int_\Omega M(x)\nabla\psi\nabla\psi - \int_\Omega \psi^+|u_n|^r$$

and

$$\frac{1}{2}\int_\Omega M(x)\nabla u_n\nabla u_n + \frac{A}{r}\int_\Omega \varphi_n|u_n|^r - \int_\Omega f_n u_n \le \frac{1}{2}\int_\Omega M(x)\nabla v\nabla v + \frac{A}{2r}\int_\Omega \varphi_n|v|^r - \int_\Omega f_n v.$$

Now we follow [3] and choose $\psi = \varphi_n - T_k(\varphi_n - \eta)$ with η in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $\eta \ge 0$ in the first inequality, and we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega M(x)\nabla\varphi_n\nabla\varphi_n + \int_\Omega [\varphi_n - T_k(\varphi_n - \eta)]^+|u_n|^r \\
&\quad\le \frac{1}{2}\int_\Omega M(x)\nabla[\varphi_n - T_k(\varphi_n - \eta)]\nabla[\varphi_n - T_k(\varphi_n - \eta)] + \int_\Omega \varphi_n|u_n|^r.
\end{aligned}$$

We observe now that $\nabla[\varphi_n - T_k(\varphi_n - \eta)]$ is $\nabla\varphi_n$ where $|\varphi_n - \eta| > k$, and $\nabla\eta$ where $|\varphi_n - \eta| \le k$, and that, since $\eta \ge 0$, $[\varphi_n - T_k(\varphi_n - \eta)]^+ = \varphi_n - T_k(\varphi_n - \eta)$. Therefore, simplifying again equal terms, we have

$$\frac{1}{2}\int_{\{|\varphi_n - \eta| \le k\}} M(x)\nabla\varphi_n\nabla\varphi_n \le \frac{1}{2}\int_{\{|\varphi_n - \eta| \le k\}} M(x)\nabla\eta\nabla\eta + \int_\Omega T_k(\varphi_n - \eta)|u_n|^r.$$

Recall now that, under any assumption on f and r, we proved that u_n converges to u strongly in $L^r(\Omega)$, so that one can pass to the limit in the last term. Furthermore, using Lebesgue's theorem, one can easily pass to the limit in the second term. As for the first one, we remark that on the set $\{|\varphi_n - \eta| \leq k\}$ one has $|\varphi_n| \leq h$, where $h = k + \|\eta\|_{L^\infty(\Omega)}$. Therefore, one has

$$\int_{\{|\varphi_n-\eta|\leq k\}} M(x)\nabla\varphi_n\nabla\varphi_n = \int_{\{|\varphi_n-\eta|\leq k\}} M(x)\nabla T_h(\varphi_n)\nabla T_h(\varphi_n).$$

Recalling that $T_k(\varphi_n)$ weakly converges to $T_k(\varphi)$ in $W_0^{1,2}(\Omega)$ (since $\{\varphi_n\}$ or $\{T_h(\varphi_n)\}$ is bounded in $W_0^{1,2}(\Omega)$, depending on the assumptions on f and r), we can use weak lower semicontinuity to have that

$$\frac{1}{2}\int_{\{|\varphi-\eta|\leq k\}} M(x)\nabla\varphi\nabla\varphi \leq \frac{1}{2}\int_{\{|\varphi-\eta|\leq k\}} M(x)\nabla\eta\nabla\eta + \int_\Omega T_k(\varphi-\eta)|u|^r$$

for every η in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, $\eta \geq 0$.

Using the inequality for u_n, and choosing $v = u_n - T_k(u_n - w)$ with w in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, as before, we obtain

$$\frac{1}{2}\int_{\{|u_n-w|\leq k\}} M(x)\nabla u_n\nabla u_n + \frac{A}{r}\int_\Omega \varphi_n\big[|u_n|^r - |u_n - T_k(u_n-w)|^r\big] \leq \frac{1}{2}\int_{\{|u_n-w|\leq k\}} M(x)\nabla w\nabla w + \int_\Omega f_n T_k(u_n - w).$$

Observe now that since

$$\big||s|^r - |s-t|^r\big| \leq r(|s|+|t|)^{r-1}|t|,$$

we have

$$\big||u_n|^r - |u_n - T_k(u_n-w)|^r\big| \leq r(|u_n| + |T_k(u_n-w)|)^{r-1}|T_k(u_n-w)| \leq rk(|u_n|+k)|^{r-1}.$$

Recall now that we have proved, under any assumption on f and r, that $\varphi_n|u_n|^{r-1}$ strongly converges in $L^1(\Omega)$ to $\varphi|u|^{r-1}$. Therefore, Vitaly's theorem implies that

$$\lim_{n\to+\infty}\int_\Omega \varphi_n\big[|u_n|^r - |u_n - T_k(u_n-w)|^r\big] = \int_\Omega \varphi\big[|u|^r - |u-T_k(u-w)|^r\big].$$

Hence, recalling that $T_k(u_n)$ weakly converges to $T_K(u)$ in $W_0^{1,2}(\Omega)$, and observing, as before, that

$$\int_{\{|u_n-w|\leq k\}} M(x)\nabla u_n\nabla u_n = \int_{\{|u_n-w|\leq k\}} M(x)\nabla T_h(u_n)\nabla T_h(u_n),$$

where $h = k + \|w\|_{L^\infty(\Omega)}$, we have

$$\frac{1}{2}\int_{\{|u-w|\leq k\}} M(x)\nabla u\nabla u + \frac{A}{r}\int_\Omega \varphi\big[|u|^r - |u-T_k(u-w)|^r\big] \leq \frac{1}{2}\int_{\{|u-w|\leq k\}} M(x)\nabla w\nabla w + \int_\Omega f T_k(u-w)$$

for every w in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

We have therefore proved the following result. Note that (4.2) and (4.3) below state that both u and φ are T-minima of suitable functionals (see [2] for the definition of T-minimum of a functional).

Theorem 4.1. *Under the same assumptions on f and r made in Theorems 2.3 and 3.4, if (u_n, φ_n) is a saddle point of the functional $J_n(v,\psi)$ defined in (4.1), and if (u,φ) is the solution of (1.2) obtained as limit of (u_n,φ_n), then we have*

$$\frac{1}{2}\int_{\{|\varphi-\eta|\leq k\}} M(x)\nabla\varphi\nabla\varphi \leq \frac{1}{2}\int_{\{|\varphi-\eta|\leq k\}} M(x)\nabla\eta\nabla\eta + \int_\Omega T_k(\varphi-\eta)|u|^r \tag{4.2}$$

for every η in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, $\eta\geq 0$, and

$$\frac{1}{2}\int_{\{|u-w|\leq k\}} M(x)\nabla u\nabla u + \frac{A}{r}\int_\Omega \varphi\big[|u|^r - |u-T_k(u-w)|^r\big] \leq \frac{1}{2}\int_{\{|u-w|\leq k\}} M(x)\nabla w\nabla w + \int_\Omega f T_k(u-w) \tag{4.3}$$

for every w in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

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References

[1] V. Benci and D. Fortunato, An eigenvalue problem for the Schrödinger–Maxwell equations, *Topol. Methods Nonlinear Anal.* **11** (1998), 283–293.
[2] L. Boccardo, T–minima: An approach to minimization problems in $L^{1'}$, *Ric. Mat.* **49** (2000), 135–154.
[3] L. Boccardo, Elliptic systems of Schrödinger type in the spirit of Benci–Fortunato, *Adv. Nonlinear Stud.* **14** (2014), 469–478.
[4] L. Boccardo and T. Gallouët, Nonlinear elliptic and parabolic equations involving measure data, *J. Funct. Anal.* **87** (1989), 149–169.
[5] L. Boccardo and T. Gallouët, Nonlinear elliptic equations with right-hand side measures, *Comm. Partial Differential Equations* **17** (1992), 641–655.
[6] G. R. Cirmi, Regularity of the solutions to nonlinear elliptic equations with a lower order term, *Nonlinear Anal.* **25** (1995), 569–580.
[7] G. Dal Maso, F. Murat, L. Orsina and A. Prignet, Renormalized solutions of elliptic equations with general measure data, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **28** (1999), 741–808.
[8] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann. Inst. Fourier (Grenoble)* **15** (1965), 189–258.