Pseudo-Analysis: some measures of general information

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***Abstract*—We study and solve a system of functional equations coming from the theory of general information.**

I. Introduction

The pseudo-analysis was introduced by Pap and his colleagues [18], [19], [17] and it has had and still has many applications in different fields. It involves the definition of two operations called "pseudo-addition" $\oplus$ and "pseudo-multiplication $\odot$ which generalize the addition and multiplication of classical analysis. In some cases the pseudo-operations are defined by using a function g, called *generator function*: for this reason we speack about $G-$calculus (generalized calculus).

This research takes inspiration from the axiomatic theory of the general information measures defined without probability [14], [15], [11]. The basic problem is to express the information $J\Big(A\bigcup B\Big)$ as a suitable function Φ of $J(A)$ and $J(B)$. The most important equation derives from a type of $J-$independence property obtained by using a pseudo-addition

The properties of the function Φ are translated into a system of functional equations: we shall present some classes of solutions.

After some preliminaries in Sect.2, in Sect.3 we present the definition of $J-$independence property in pseudo-analysis and its application to information measure for the union of two disjont sets. Moreover, we give the system which expresses the property of the union. The solutions of the system are considered in Sect.4 and in Sect.5 we present the conclusions.

II. Preliminary notions

The *pseudo-addition* $\oplus$ is a binary function [8]

$$\oplus : [0,M]^2 \longrightarrow [0,M], M \in (0,+\infty]$$

which is strictly increasing (with respect to $\leq$), commutative, associative, with 0 as neutral element.

From now on, the pseudo-addition $\oplus_g$ will be defined by a continuous and bijective function $g : [0,M] \longrightarrow [0,+\infty]$ called *generator function*

$$a \oplus_g b = g^{-1}(g(a)+g(b)), \quad \text{with } g(0)=0. \tag{1}$$

So the pseudo-addition satisfies the following properties:

$(G1)$ $a \leq a', b \leq b' \Longrightarrow a \oplus_g b \leq a' \oplus_g b'$
$(G2)$ $x \oplus_g y = y \oplus_g x$
$(G3)$ $(a \oplus_g b) \oplus_g c = a \oplus_g (b \oplus_g c)$
$(G4)$ $x \oplus_g 0 = x$
$(G5)$ $a_n \longrightarrow a, b_n \longrightarrow b, \Longrightarrow a_n \oplus_g b_n \longrightarrow a \oplus_g b.$

Now, we recall the definition of general information measure [14], [15], [11].

Let Ω be an abstract space, $\mathcal{A} \subset \mathcal{P}(\Omega)$ a $\sigma-$algebra of subsets of Ω, so $(\Omega, \mathcal{A})$ is measurable space.

Measure $J(\cdot)$ *of general information* is a mapping

$$J(\cdot) : \mathcal{A} \longrightarrow [0,+\infty]$$

such that

(i) $A' \supset A \Longrightarrow J(A') \leq J(A) \quad \forall A, A' \in \mathcal{A},$
(ii) $J(\emptyset) = +\infty, J(\Omega) = 0.$

Moreover, given a subfamily $\mathcal{K} \subset \mathcal{A}$, a definition of J-independence has been proosed:

two sets $K, K' \in \mathcal{K}, \mathrm{K} \neq \mathrm{K}'$ with $K \cap K' \neq \emptyset$ are J-*Independent* (independent with respect to an information J [13], [16], [6], [3], [4]) if the couple (K, K') satisfies:

(iii) $J(K \cap K') = J(K) + J(K').$

This property has been generalized by the Authors in [21].

III. The J-independence property in pseudo-analysis

This paragraph is devoted to introduce the J- independence property in pseudo-analysis. Then this definition will be applied to calculate the general information of the union of two disjoint sets.

A. Definition of J-independence property in pseudo-analysis

As in [21], we proose that that K, K' are $J-$*independent in pseudo-analysis* if the couple (K, K') satisfies:

$$J(K \cap K') = J(K) \oplus_g J(K'), \tag{2}$$

$$K, K' \in \mathcal{K}, \mathrm{K} \neq \mathrm{K}', \mathrm{K} \cap \mathrm{K}' \neq \emptyset.$$

When the generator function of $\oplus_g$ is a linear function, then (2) coincides with (iii).

B. Application

Now, we want to apply the formula (2) in the case of the union of two disjoint sets.

We shall suppose that the information $J\left(A\bigcup B\right)$ depends only on $J(A)$ and $J(B)$, for this reason we set

$$J\left(A\bigcup B\right) = \Phi\left(J(A), J(B)\right), \tag{3}$$

where $\Phi : H \longrightarrow [0, +\infty]$, is a continuous function with $H = \{(x, y) : \exists\ A, B \in \mathcal{A}, A \cap B = \emptyset, x = J(A), y = J(B), x, y \in (0, +\infty]\}$. Later, the properties of the function Φ will be specified.

In the setting of pseudo-analysis, we suppose that there exists a set $K \in \mathcal{K}$ which is J-independent by A, B and $A \cup B$: from (2) it is

$$J\left(A\bigcap K\right) = J(A) \oplus_g J(K) \text{ and} \tag{4}$$

$$J\left(B\bigcap K\right) = J(B) \oplus_g J(K);$$

on the other hand

$$J\left(\left(A\bigcup B\right)\bigcap K\right) = J\left(A\bigcup B\right) \oplus_g J(K). \tag{5}$$

By distributivity property

$$J\left(\left(A\bigcup B\right)\bigcap K\right) = J\left(\left(A\bigcap K\right)\bigcup\left(B\bigcap K\right)\right);$$

taking into account (3), (4) and (5), we get

$$\Phi\left(J(A), J(B)\right) \oplus_g J(K) = \tag{6}$$

$$\Phi\left(J(A) \oplus_g J(K), J(B) \oplus_g J(K)\right).$$

C. The function Φ

In order to study (6), by (3), we recall that the function Φ is commutative, associative, strictly increasing (with respect $\le$), moreover $\Phi\left(J(A), J(B)\right) \le J(A) \bigwedge J(B)$.

Setting $x = J(A), y = J(B), z = J(C), t = J(K), x' = J(A')$ with $x, y, z, t, x' \in (0, +\infty]$, the function Φ satisfies some properties, mentioned above, that lead to solve the following system:

$$\begin{cases} [p1]\ \Phi(x, y) \le x \wedge y, \\ [p2]\ x' \le x \longrightarrow \Phi(x', y) \le \Phi(x, y)\ \forall y \text{ (monotonicity)}, \\ [p3]\ \Phi(x, y) = \Phi(y, x) \ \text{ (commutativity)}, \\ [p4]\ \Phi\left(\Phi(x, y), z\right) = \Phi\left(x, \Phi(y, z)\right) \ \text{ (associativity)}, \\ [p5]\ \Phi\left(x, y\right) \oplus_g t = \Phi\left(x \oplus_g t, y \oplus_g t\right) \\ \qquad\qquad \text{(compatibility equation)}. \end{cases}$$

If g is a linear function, the system above reduces to the form studied in [2], [5], [9], [10], [12].

IV. Solutions of the system

In this paragraph we present some classes of solutions of the system: we distinguish the idempotent case and the not idempotent case.

A. Idempotent case

We recall the following ([22])

Definition 1: An element x^* is called an idempotent element for any function Φ if

$$\Phi(x^*, x^*) = x^* \tag{7}$$

In order to look for a solution of the system $[p1] - [p5]$, we shall proceed on step by step.

From now on, we shall suppose that $\forall t > 0$ there exists $K \in \mathcal{K}$, independent by $A, B, A \cup B$, such that $t = J(K)$.

Lemma 2: Let Φ be any function which satisfies $[p5]$. If there exists in $(0, +\infty]$ an idempotent element x^* for Φ, then every $x \neq x^*$ is again an idempotent element for Φ.
Proof. Let $x \ge x^*$. From $[p5]$, for $x = y = x^*$, it is

$$\Phi(x^* \oplus_g t, x^* \oplus_g t) = \Phi(x^*, x^*) \oplus_g t = x^* \oplus_g t;$$

that means that $x^* \oplus_g t$ is an idempotent element for Φ. As $t > 0$, then $x^* \oplus_g t > x^*$ and every $x > x^*$ is an idempotent element for Φ.

Let $x < x^*$. Then, there exists $t > 0$ such that $x^* = x \oplus_g t$. Setting $x^* = x \oplus_g t$. in $[p4]$, we get

$$\Phi(x, y) \oplus_g t = \Phi(x^*, y \oplus_g t). \tag{8}$$

For $x = y$, the (8) becomes

$$\Phi(x, x) \oplus_g t = \Phi(x \oplus_g \tag{9}$$

But Φ verifies $[p4]$, so $\Phi(x, x) = x$: this means that every $x < x^*$ is an idempotent element for Φ. $\square$

Lemma 3: Let Φ be any function which satisfies $[p1, p2, p5]$. If there exists in $(0, +\infty]$ an idempotent element x^* for Φ then

$$\Phi(x, x^*) = x \wedge x^* \qquad \forall x \in (0, +\infty]. \tag{10}$$

Proof. Let $x \ge x^*$. For $[p1]$, $\Phi(x, x^*) \ge \Phi(x^*, x^*) = x^*$ and for $[p2], \Phi(x, x^*) \le x \wedge x^* = x^*$. Therefore, $\Phi(x, x^*) = x \wedge x^*$, for every $x \ge x^*$.

Now, let $x < x^*$. For Lemma (2) x is an idempotent element, then

$$x = \Phi(x, x) \le \Phi(x, x^*) \le x \wedge x^* = x.$$

We have proved that

$$\Phi(x, x^*) = x \wedge x^*.$$

$\square$

Now we are ready to give the main proposition.

Proposition 4: Let x^* be an idempotent element for Φ, which satisfies $[p1, p2, p5]$. Then the solution of the system $[p1 - p5]$ is

$$\Phi(x, y) = x \wedge y \qquad \forall x, y \in (0, +\infty). \tag{11}$$

Proof. It is easy to see that the function (11) satisfies $[p3]$ and $[p4]$. Moreover, the proof is a consequence of Lemmas (2) and (3) . $\square$

B. Not idempotent case

Now, we consider the case in which there not exists an idempotent element.

First of all, it is easy to see that there are all some classical solutions of the system.

C. Examples

Example 5: If the generator function g of the pseudo-addition $\oplus_g$ is a linear function, the system $[p_0 - p_5]$ admits as solutions [3]:

$$\Phi(x, y) = x \wedge y, \text{or}$$

$$\Phi(x, y) = -c \log \Big(\exp(-x/c) + (\exp -y/c) \Big), c > 0,$$

which is Shannon's information [20].

Example 6: Let $g(x)$ be any generator function of the pseudo-addition $\oplus_g$ and $k(x) = exp(-g(x)/c), c > 0$, the class of functions

$$\Psi_k(x, y) = k^{-1} \Big(k(x) + k(y) \Big) \tag{12}$$

satisfies the system $[p1 - p5]$ [3].

Now, we are going to find some classes of solutions of the system $[p1 - p5]$

Proposition 7: A class of solutions of the system $[p1 - p4]$ is

$$\Phi_h(x, y) = h^{-1} \Big(h(x) \oplus_g h(y) \Big) \tag{13}$$

where $h : [0, +\infty] \longrightarrow [0, +\infty]$ is any decreasing, continuous function with $h(0) = +\infty$ and $h(+\infty) = 0$ and $\oplus_g$ is the pseudo-addition defined by (1).

Proof. By the properties $(G1)$ and $(G2)$ any function Φ_h satisfies $[p2]$ and $[p3]$.

Moreover $gh(x) + gh(y) \geq gh(x)$ as $gh(y) > 0$, then

$$g^{-1}(gh(x) + gh(y)) \geq g^{-1}(gh(x)) = h(x) \Longleftrightarrow$$

$$h^{-1} \Big(h(x) \oplus_g h(y) \Big) \leq x.$$

In the same way $\Phi_h(x, y) \leq y$ and so $\Phi_h(x, y) \leq x \wedge y$, which is $[p1]$.

As regards $[p4]$, by using $(G3)$

$$\Phi_h \Big(\Phi_h(x, y), z \Big) =$$

$$h^{-1} \Big\{ h\Phi_h(x, y) \oplus_g h(z) \Big\} =$$

$$h^{-1} \Big\{ hh^{-1} \Big[h(x) \oplus_g h(y) \Big] \oplus_g h(z) \Big\} =$$

$$h^{-1} \Big\{ h(x) \oplus_g \Big[h(y) \oplus_g h(z) \Big] \Big\} =$$

$$h^{-1} \Big\{ h(x) \oplus_g hh^{-1} \Big[h(y) \oplus_g h(z) \Big] \Big\} =$$

$$\Phi_h \Big(x, \Phi_h(y, z) \Big).$$

$\square$

Proposition 8: Let g be any increasing generator function of the pseudo-addition $\oplus_g$, solution of the Cauchy equation [1]

$$g(a + b) = g(a \cdot b), \tag{14}$$

then any function h of the class (13) satisfies also $[p5]$ if h is a decreasing solution of the same Cauchy equation (14).

Proof. First of all, taking into account (14), it results

$$\Phi_h(x, y) = h^{-1} \Big(h(x) \oplus_g h(y) \Big) = \tag{15}$$

$$h^{-1} \Big\{ g^{-1} \Big[gh(x) + gh(y) \Big] \Big\} =$$

$$h^{-1} \Big\{ g^{-1} g \Big(h(x)h(y) \Big) \Big\} = h^{-1} \Big\{ h(x)h(y) \Big\}.$$

Moreover, from (15),

$$\Phi_h(x, y) \oplus_g t = g^{-1} \Big\{ g\Phi(x, y) + g(t) \Big\} = \tag{16}$$

$$g^{-1} \Big\{ g \Big(\Phi_h(x, y) \cdot t \Big) \Big\} =$$

$$\Phi_h(x, y) \cdot t = \Big(h^{-1} \Big\{ h(x)h(y) \Big\} \Big) \cdot t.$$

on the other hand, as $x \oplus_g t = g^{-1}(g(x) + g(t)) = g^{-1}(g(x \cdot t)) = x \cdot t$, $y \oplus_g t = y \cdot t$, and $h(x \oplus_g t) \oplus_g h(y \oplus_g t) = g^{-1} \Big(gh(x \cdot t) + gh(y \cdot t) \Big) = g^{-1} \Big(g\big[h(x \cdot t)h(y \cdot t) \big] \Big) = h(x \cdot t)h(y \cdot t)$ it is

$$\Phi_h(x \oplus_g t, y \oplus_g t) = h^{-1} \Big(h(x \oplus_g t) \oplus_g h(y \oplus_g t) \Big) = \tag{17}$$

$$h^{-1} \Big(h(x \cdot t)h(y \cdot t) \Big).$$

Now, we apply the hypothesys on the function h to the expressions (16) and (17). We get

$$\Phi_h(x, y) \oplus_g t = \Big(h^{-1} \Big\{ h(x + y) \Big\} \Big) \cdot t.$$

$$\Phi_h(x \oplus_g t, y \oplus_g t) = h^{-1} \Big(h\big[(x \cdot t) + (y \cdot t) \big] \Big).$$

By applying again the condition (14) to the previous equalities, we get:

$$\Phi_h(x, y) \oplus_g t = \Big(h^{-1} \Big\{ h(x + y) \Big\} \Big) \cdot t = (x + y) \cdot t =$$

$$(x \cdot t) + (y \cdot t) = h^{-1} \Big(h\big[(x \cdot t) + (y \cdot t) \big] \Big) = \Phi_h(x \oplus_g t, y \oplus_g t)$$

the equation $[p5]$ of the given system is satisfied. $\square$

V. Conclusion

In this paper we have considered a generalization of $J-$ independence property for a general measure of information, applied to the union of two disjoint sets.

The properties of the union have been translated into a system of functional equations. We have found classes of solutions, which include those of the classical analysis.

Acknowledgement. This paper has been supported by GNFM of MIUR (Italy).

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