A survey on wonderful varieties

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Let G be a reductive connected complex algebraic group, and let X be a normal irreducible G-variety: X is *spherical* if it has a dense B-orbit where B is a Borel subgroup of G. This definition represents a sort of common generalization of many families studied in the litterature about reductive groups, such as toric varieties (where G is a complex torus), grassmannians and flag varieties, symmetric varieties. Spherical varieties have also links with the theory of hamiltonian actions on real symplectic manifolds, being the "algebraic analogue" of multiplicity free manifolds.

Wonderful varieties are a special class of spherical ones, and their definition comes from the properties of the compactifications of symmetric homogeneous spaces given by De Concini and Procesi in [8]. An irreductible G-variety X is *wonderful* if:

(1) X is smooth and projective;
(2) X has an open G-orbit X^0_G, whose complement is the union of prime divisors $X_1, \ldots, X_r$ which are smooth, with normal crossings, and $X_1 \cap \ldots \cap X_r \neq \emptyset$;
(3) $x, y \in X$ are on the same G-orbit if and only if $\{i \mid X_i \ni x\} = \{j \mid X_j \ni y\}$.

The theory of *embeddings*, developed by Luna and Vust in [15] and described in the spherical case in [10], shows that wonderful varieties are precisely those spherical varieties being smooth, projective, having only one closed G-orbit, and such that all B-stable prime divisors containing a G-orbit are also G-stable.

From this point of view one extracts from a wonderful variety X several discrete invariants. They come mainly from the action of B on X:

(1) the (finite) set Δ_X of all B-stable but not G-stable prime divisors, called *colors*;
(2) the B-weights of the rational functions on X being B-eigenvectors; these weights are a sublattice Ξ_X of the group of characters of T, a chosen maximal torus inside B;
(3) the B-weights appearing in the T-module $T_z(X)/T_z(G.z)$, where z is the unique fixed point of B_- (the opposite of B with respect to T); these are called the *spherical roots*, and are a basis (denoted Σ_X) of Ξ_X.
(4) the set of simple roots S^p_X associated to the stabilizer of the open B-orbit on X; this is a parabolic subgroup containing B.

These invariants are involved in a number of "tools" used to study these varieties, such as for example "generalizations" of the Cartan matrix and the open cell (as in the theory of reductive groups), of the little Weyl group (as for symmetric varieties). Colors here are considered as elements of an abstract set, each equipped with an associated element in $\mathrm{Hom}_{\mathbb{Z}}(\Xi_X, \mathbb{Q})$.

Wondeful varieties play a significative role in the classification of spherical varieties. The paper [13] shows this relation: it proves that if the triples of invariants

$(S^p_\bullet, \Sigma_\bullet, \Delta_\bullet)$ classify wonderful $G/Z(G)$-varieties for a given group G, then it is possible to classify all spherical G-varieties. The triple $(S^p_X, \Sigma_X, \Delta_X)$ is called the *spherical system* of X.

In the same paper Luna conjectured a set of axioms defining admissible triples; these axioms are inspired by known classification of varieties having rank 1 and 2 (see [1], [19]), the rank being the number r in the definition. Using these axioms, spherical systems are considered as combinatorial objects, and represented by diagrams attached to the Dynkin diagram of G. The standard conjecture is then:

Conjecture. *Spherical systems classify wonderful varieties.*

In [13] Luna proves the conjecture for all semisimple G of type A, Bravi and Pezzini for G of mixed type $\mathsf{A}-\mathsf{C}$ (partially) and $\mathsf{A}-\mathsf{D}$ in [4], Bravi for G simply laced ($\mathsf{A}-\mathsf{D}-\mathsf{E}$) in [2]. The conjecture for all G is still an open problem, although recently Losev has shown in [12] that spherical systems at least separate wonderful varieties.

A variety can be wonderful under the action of many different groups, with the same G-stable divisors or not. For example, symmetric rank 1 wonderful varieties are homogeneous under the action of a group bigger than G. In [6] Brion shows that for any wonderful variety X the full group $\mathrm{Aut}^0(X)$ is semisimple, and X is wonderful under its action too.

Other researches about the geometry of wonderful varieties are under development; for example the cohomology of line bundles. Here one can look for a generalization of the classical Borel-Weil theorem: the Picard group has a basis given by the classes of colors (see [5]), the global sections of a line bundle generated by global sections is not an irreducible G-module but it is multiplicity-free and it is completely described (see [5]), higher cohomology of line bundles generated by global sections is zero (see [7]), all ample line bundles are very ample (see [16]). A complete description of the cohomology of line bundles is accomplished only for varieties of *minimal rank*, i.e. $\operatorname{rank} X = \operatorname{rank} G - \operatorname{rank} H$, H being a generic stabilizer, by Tchoudjem (see [18]). Such varieties are also completely classified, by Ressayre (see [17]).

Another problem is to construct wonderful varieties in some projective space, and give their equations. The compactifications of De Concini and Procesi were found in the projective space of irreducible G-modules, but this can be done only when the stabilizers of all points of X are equal to their normalizers (see [16]). Moreover, Losev has shown in [11] that when the generic stabilizer H is equal to its normalizer, then the *Demazure embedding* produces a wonderful variety. The construction is the following: one considers the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of G and H, and the grassmannian $\mathrm{Gr}(\mathfrak{g}, \dim \mathfrak{h})$; the closure of the orbit $G \cdot [\mathfrak{h}]$ gives a variety G-isomorphic to X.

Wonderful varieties have been used in the recent theory of invariant Hilbert schemes developed by Alexeev and Brion. This scheme parametrizes G-stable affine subvarieties of a given G-module V; more precisely all subvarieties $Y \subseteq V$ whose coordinate ring $\mathbb{C}[Y]$ has the same structure as a G-module: this plays the role of the Hilbert polynomial for classical Hilbert schemes. In particular, if $\mathbb{C}[Y]$ has no multiplicities then Y is spherical, and $\mathbb{C}[Y]$ is simply described by the monoid of highest weights appearing in it.

Luna has worked (see [14]) on the case where $\mathbb{C}[Y]$ is the sum of one copy of each irreducible G-module: the so-called *model* varieties. He shows that for any G there exists a wonderful G-variety whose orbits parametrize G-isomorphism classes of such varieties.

Jansou has worked (see [9]) on the case where V^* is irreducible of highest weight λ and the monoid Γ of highest weights of $\mathbb{C}[Y]$ is $\mathbb{N} \cdot \lambda$. Here he obtains a similar result, and proves that in this case the invariant Hilbert scheme is a reduced point or an affine line, corresponding resp. to the cases where the wonderful variety involved has rank 0 or 1.

Bravi and Cupit-Foutou in [3] extended this result to any V, in the case where the monoid Γ is *saturated*, i.e. $\mathbb{Z}\Gamma \cap \Lambda^+ = \Gamma$, where Λ^+ is the set of dominant weights. Here the invariant Hilbert scheme will be an affine space, of dimension equal to the rank of associated wonderful variety.

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