LECTURES ON SPHERICAL AND WONDERFUL VARIETIES

GUIDO PEZZINI

Abstract. These notes contain an introduction to the theory of spherical and wonderful varieties. We describe the Luna-Vust theory of embeddings of spherical homogeneous spaces, and explain how wonderful varieties fit in the theory.

1. Introduction

These are the notes of a course given at the CIRM (Luminy), in occasion of the conference "Actions hamiltoniennes: invariants et classification", April 2009, and are intended as an introduction to the theory of spherical and wonderful varieties. Their goal is to explain the first main themes and provide some insight, mainly in the direction of the work in progress about the classification of spherical varieties.

The richness in examples is quite distinctive for this theory, and we illustrate here the details of some of the easiest ones. Other examples are found in the notes [Br09] of Michel Brion and [B09] of Paolo Bravi. On the other hand there exist already excellent references for many results hereby reported, especially [Br97] but also [K91] and [T06]. Therefore we skip or only sketch the proofs whenever they are easily found in the literature.

Prerequisites are basic notions of algebraic geometry and structure of linear algebraic groups; a general reference is the book [H75].

For simplicity, we work over the ground field $\mathbb{C}$ of complex numbers, although the entire theory of embeddings of spherical homogeneous spaces holds in any characteristic.

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General notations. In these notes G will be a reductive connected linear algebraic group, $B \subseteq G$ a fixed Borel subgroup and $T \subseteq B$ a maximal torus. We will denote by B_- the opposite Borel subgroup of B with respect to T. We will denote by $S = \{\alpha_1, \alpha_2, \ldots\}$ the associated set of simple roots, by $\alpha_1^\vee, \alpha_2^\vee, \ldots$ the corresponding coroots, and when G is semisimple we will denote by $\omega_1, \omega_2, \ldots$ the fundamental dominant weights.

If G is a classical group, we choose always B to be the set of upper triangular matrices in G and T the set of diagonal matrices in G. Unless otherwise stated, we implicitly define symplectic and orthogonal groups using bilinear forms such that this choice of B and T gives resp. a Borel subgroup and a maximal torus.

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If H is any affine algebraic group, we define the abelian group of its *set of characters*:

$$\Lambda(H) = \{\lambda\colon H \to \mathbb{C}^* \text{ homomorphism of algebraic groups}\},$$

and we will denote the group operation additively. As a particular case, we recall that $\Lambda(B)$ and $\Lambda(T)$ are canonically isomorphic, and they are free abelian groups of rank equal to $\operatorname{rank} G$.

We will also denote by H° the connected component containing the unit element $e \in H$, by $C(H)$ the center of H, by H^u the unipotent radical of H and by H^r the radical of H.

A homogeneous space for G is a couple (X, x) with X a homogeneous G-variety and $x \in X$, which is called the *base point*. The quotient G/H for H a closed subgroup is naturally a homogeneous space, with base point H.

2. Spherical varieties

2.1. **Introduction and toric varieties.**

Definition 2.1.1. A G-variety X is *spherical* if it is normal, and B has an open orbit on X.

Let $x \in X$ be a point in the open B-orbit. Then its G-orbit Gx is open in X; if we denote by H the stabilizer of x in G, then Gx is equivariantly isomorphic to the homogeneous space G/H. We also say that X is an *embedding* of the homogeneous space G/H (a slightly more precise definition is needed and will be given in §2.5). On the other hand, when a quotient G/H is a spherical variety, we will say that H is a *spherical subgroup*. In this case the B-orbit of $H \in G/H$ is open if and only if BH is an open subset of G; we will say that H is B-spherical if this occurs.

Definition 2.1.2. Let X be a spherical G-variety. We denote by X°_G its open G-orbit, and by X°_B its open B-orbit.

The study of spherical G-varieties can be divided into two big steps:

(1) Fix a spherical subgroup $H \subseteq G$, and study all the embeddings X of G/H.
(2) Study all possible spherical subgroups $H \subseteq G$.

In this paper, we will discuss the first step. A general theory of embeddings of homogeneous spaces has been developed by Luna and Vust in [LV83] in great generality, using essentially discrete valuations. For a spherical homogeneous space, this theory can be reformulated in a quite effective way, leading to results that have strong formal analogies with the theory of toric varieties.

Example 2.1.1. Let $G = \mathrm{SL}_2$, and $H = T$. Then H is the stabilizer of a point in $G/H = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathrm{diag}(\mathbb{P}^1)$, where G acts diagonally (and linearly on each copy of $\mathbb{P}^1$). The Borel subgroup B has an open orbit, namely the set of couples $(p, q) \in \mathbb{P}^1 \times \mathbb{P}^1$ where $p \neq q$ and both are different from the unique point $[1, 0]$ fixed by B in $\mathbb{P}^1$.

As we will check in §2.5, the homogeneous space G/H admits only two embeddings: the trivial embedding $X = G/H$, and $X = \mathbb{P}^1 \times \mathbb{P}^1$.

Example 2.1.2. Let again $G = \mathrm{SL}_2$, but take $H = U$ the set of unipotent upper triangular matrices (see [Br09, Example 1.12.3]). Then G/H is equivariantly isomorphic to $\mathbb{C}^2 \setminus \{(0, 0)\}$, where G acts linearly. The Borel subgroup B has an open orbit, namely all points $(x, y) \in \mathbb{C}^2$ with $y \neq 0$.

Here G/H admits five different nontrivial embeddings (with obvious action of G extending the one on G/H):

(1) $X_1 = \mathbb{C}^2$,
(2) $X_2 = \mathbb{P}^2$, viewed as $\mathbb{C}^2 \cup$ (the line at infinity) (i.e. $\mathbb{P}^2 = \mathbb{P}(\mathbb{C}^2 \oplus \mathbb{C})$),
(3) $X_3 = \mathbb{P}^2 \setminus \{(0,0)\}$, where we mean $(\mathbb{C}^2 \setminus \{(0,0)\}) \cup$ (the line at infinity),
(4) $X_4 =$ the blowup of X_1 in $(0,0)$,
(5) $X_5 =$ the blowup of X_2 in $(0,0)$.

As an introduction to the techniques we will use, let us review very briefly the basics of the theory of toric varieties. We refer to [F93] for a complete tractation.

Definition 2.1.3. A *toric variety* for the torus $T = (\mathbb{C}^*)^n$ is a normal variety X where T acts faithfully with an open orbit.

The faithfulness is only convenient and not essential, one can indeed pass to the quotient of T by the kernel of the action.

Definition 2.1.3 is a special case of Definition 2.1.1, where $G = B = T$. In other words a toric variety is an embedding of the spherical homogeneous space $T/\{e\}$.

The theory begins describing the affine toric varieties for T: such an X is completely described by its ring of regular functions $\mathbb{C}[X]$. The T-action induces a structure of rational T-module on $\mathbb{C}[X]$, which is itself a T-submodule of the ring $\mathbb{C}[T]$ of regular functions on T.

We know that $\mathbb{C}[T]$ contains all irreducible T-modules exactly once, and each is 1-dimensional (see also [Br09, Examples 2.3.2 and 2.11.1]). The choice of $\mathbb{C}[X]$ inside $\mathbb{C}[T]$ is therefore equivalent to the choice of its irreducible submodules, and this is also equivalent to the choice of which *characters* of T extend to regular functions on X. Loosely speaking, this gives a correspondence:

$$\mathbb{C}[X] \longleftrightarrow \text{ a convex polyhedral cone } \sigma \subset \Lambda(T)$$

A further step is done by looking at the vector space $N(T) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda(T), \mathbb{Q})$, and taking the dual convex cone $\sigma^\vee$, defined to be the set of functionals that are non-negative on σ:

$$\mathbb{C}[X] \longleftrightarrow \sigma \subset \Lambda(T) \longleftrightarrow \sigma^\vee \subset N(T).$$

Before stating a precise theorem, let us introduce a little twist in this procedure. Although being completely equivalent, this will look non-standard compared to the usual approach. Let X be an affine toric variety; each irreducible T-submodule of $\mathbb{C}[X]$ is 1-dimensional, so it is made of T-eigenvectors: we consider their common T-eigenvalue (again a character). We then associate to $\mathbb{C}[X]$ all the T-eigenvalues obtained in this way.

This is non-standard, in the following sense. If a character $\lambda \in \Lambda(T)$ extends to a regular function on X, then it is a T-eigenvector of $\mathbb{C}[X]$. But its T-eigenvalue, also called its T-*weight*, is $(-\lambda)$, thanks to the fact that the action on $\mathbb{C}(T)$ induced by left translation is given by:

$$(t\lambda)(\bullet) = \lambda(t^{-1}\bullet) = \lambda(t^{-1})\,\lambda(\bullet) = (-\lambda)(t)\,\lambda(\bullet).$$

This produces a correspondence where the convex cone $\sigma^\vee$ is the opposite of the cone usually considered in the theory of toric varieties. For example, the "standard" $\mathbb{C}^n$, where $(\mathbb{C}^*)^n$ acts linearly with weights $\lambda_1, \ldots, \lambda_n$ (where $\lambda_i(t_1, \ldots, t_n) = t_i$), will be represented here by:

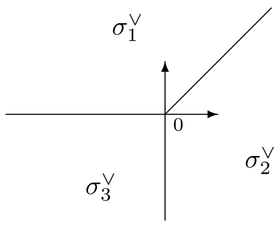
$$\sigma^\vee = \left(\mathrm{span}_{\mathbb{Q}_{\geq 0}}\{-\lambda_1, \ldots, -\lambda_n\}\right)^\vee \subset N(T)$$

which is exactly the opposite of the usual convention.

Theorem 2.1.1. *Let us fix an algebraic torus T.*

(1) *Any toric T-variety is the union of affine T-stable open sets, which are themselves toric T-varieties.*
(2) *In a toric T-variety each orbit closure is the intersection of T-stable subvarieties of codimension 1.*
(3) *The above correspondence induces a bijection between the affine toric T-varieties (up to T-equivariant isomorphism) and the strictly convex polyhedral cones in $N(T)$.*
(4) *As a consequence, toric T-varieties (up to T-equivariant isomorphism) are classified by families of strictly convex polyhedral cones in $N(T)$, called* fans, *that satisfy the following properties:*
    - *any face of a cone of the fan belongs to the fan;*
    - *the intersection of two cones of the fan is always a face of each.*

Example 2.1.3. Let us consider $(\mathbb{C}^*)^3$ acting linearly on $\mathbb{C}^3$ with weights $\lambda_1, \lambda_2, \lambda_3$. It induces naturally an action of the quotient $T = (\mathbb{C}^*)^3/\mathrm{diag}(\mathbb{C}^*)$ on $\mathbb{P}^2$ in such a way that $\mathbb{P}^2$ is a toric T-variety. A cover with affine T-stable open sets is given by $U_i = \{[x_1, x_2, x_3] \mid x_i \neq 0\}$ for $i = 1, 2, 3$. Taking $\mu_1 = \lambda_1 - \lambda_3$ and $\mu_2 = \lambda_2 - \lambda_3$ as a basis for $\Lambda(T)$, and setting $\sigma_i^\vee$ the cone in $N(T)$ associated to U_i, the fan of $\mathbb{P}^2$ is:



Before getting back to spherical varieties we make a few remarks about the differences arising when we replace the torus T with a reductive group G, possibly non-abelian. The basic observation is that T is both reductive and solvable, whereas if G is non-abelian we have two different groups: a reductive one (G) and a solvable one (B). This induces significant differences in the theory; more precisely, Theorem 2.1.1 as stated above is false for the general (i.e. spherical) case:

- Part 1 does not hold, as shown by the example of SL_2 acting linearly on $\mathbb{P}^1$.
- Part 2 does not hold, take for example SL_2 acting linearly on $\mathbb{C}^2$.
- No analog of parts 3 and 4 can hold without substantial modification, see e.g. Example 2.1.1 where there is only one non-trivial embedding.

Nevertheless, it is possible to generalize the notions and definitions we have seen so far, and prove a somewhat similar classification for the embeddings of a fixed spherical homogeneous space. Roughly speaking, one approaches the problems arising with Theorem 2.1.1 by:

- using covers of quasi-projective open sets, instead of affine ones;
- using B-stable codimension 1 subvarieties, instead of G-stable ones;
- adding further structure to $N(T)$ and to the fans of convex cones.

2.2. **Invariants and G-stable subvarieties.** From now on G/H will denote a spherical homogeneous space; unless otherwise stated, X will denote a spherical variety whose open orbit is G-isomorphic to G/H.

Definition 2.2.1.

(1) We define the set of rational functions on X that are B-*eigenvectors* (or B-semi-invariant):
$$\mathbb{C}(X)^{(B)} = \big\{ f \in \mathbb{C}(X) \setminus \{0\} \mid b\,f = \chi(b)\,f \;\forall b \in B, \text{ where } \chi\colon B \to \mathbb{C}^* \big\}.$$
(2) For any $f \in \mathbb{C}(X)^{(B)}$ the associated B-*weight* χ is a character of B and it is also denoted by χ_f.
(3) The set of all B-weights of rational functions on X is denoted by:
$$\Lambda(X) = \big\{ \chi_f \mid f \in \mathbb{C}(X)^{(B)} \big\}.$$
It is a free abelian group; its rank is by definition the *rank* of X.
(4) For any $\chi \in \Lambda(X)$, the functions $f \in \mathbb{C}(X)^{(B)}$ such that $\chi = \chi_f$ are all proportional by a scalar factor. We will denote by f_χ one of them, which is intended as defined up to a multiplicative scalar.
(5) We define the $\mathbb{Q}$-vector space:
$$N(X) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q}).$$

Part 4 is an immediate consequence of the fact that X is spherical: if f_1, f_2 have weight χ then f_1/f_2 is B-invariant, therefore constant on X.

All the above objects defined for X are obviously invariant under G-equivariant birational maps. In other words, they depend only on the open G-orbit G/H of X.

Example 2.2.1. (Rank 0.) Let $G = \mathrm{SL}_{n+1}$. Then $X = \mathbb{P}^n$ with the linear action of G is a spherical homogeneous space G/H, where H is the stabilizer of a point in $\mathbb{P}^n$ (a maximal parabolic subgroup). Any Grassmannian of subspaces of $\mathbb{P}^n$ is spherical, and more generally any complete homogeneous space for any reductive group G:
$$X = G/P.$$
with P a parabolic subgroup of G. To check that G/P is spherical, let us suppose that $P \supseteq B_-$. Then, the Bruhat decomposition implies that the B-orbit BP/P is open in G/P.

The varieties $X = G/P$ admit no non-constant B-semi-invariant rational function. Such an f must be invariant under the unipotent radical B^u of B, but B^uP/P is open in X too, hence f is constant. In other words $\mathbb{C}(G/P)^{(B)} = \mathbb{C}^*$ and $\Lambda(G/P)$, $N(G/P)$ are trivial.

Example 2.2.2. (Rank 1.) Let $G = \mathrm{SL}_2$ and $H = T$. The homogeneous space G/H is spherical and has rank 1; if we consider the B-stable open set:
$$\big\{([x,1],[y,1] \mid x \neq y\big\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathrm{diag}(\mathbb{P}^1) \cong G/H$$
then the function $(x-y)^{-1}$ is in $\mathbb{C}(G/H)^{(B)}$ and its weight is equal to α_1. It is easy to show that $\Lambda(G/H) = \mathbb{Z}\alpha_1$.

Example 2.2.3. Again with $G = \mathrm{SL}_2$, we can take $H = U$ to be the set of unipotent upper triangular matrices. Then:
$$G/H \cong \mathbb{C}^2 \setminus \{(0,0)\}.$$

where G acts lineary on $\mathbb{C}^2$. If x and y are the coordinates on $\mathbb{C}^2$ then the function y is a B-eigenvector with weight ω_1. This implies $\Lambda(G/H) = \mathbb{Z}\omega_1 = \Lambda(T)$.

Example 2.2.4. Take now $G = \mathrm{SL}_{n+1}$ with $n > 1$ and $H = \mathrm{GL}_n$. The homogeneous space:

$$G/H \cong \left\{(p, V) \in \mathbb{P}^n \times \mathrm{Gr}(n, \mathbb{C}^{n+1}) \mid p \notin \mathbb{P}(V)\right\}$$

is spherical and affine. If $e_1, \ldots, e_{n+1}$ is the canonical basis of $\mathbb{C}^{n+1}$, we can take H to be the stabilizer of $([e_1], \mathrm{span}\{e_2, \ldots, e_{n+1}\})$. Consider the quotient map $\pi\colon G \to G/H$; the pull-back of any $f \in \mathbb{C}(G/H)^{(B)}$ from G/H to G is a rational function on G, it is invariant under the right translation action of H and B-semi-invariant under the left translation.

Define $f_1 \in \mathbb{C}[G]$ to be the lower right $n \times n$ minor and f_2 the lower left matrix entry. Then $f_1 f_2$ is the pull-back from G/H of a function in $\mathbb{C}(G/H)^{(B)}$. Its B-weight is $\sigma = \alpha_1 + \ldots + \alpha_n$, and we have $\Lambda(G/H) = \mathbb{Z}\sigma$.

Example 2.2.5. (Rank 2.) Let $G = \mathrm{SL}_3$ acting on the space of smooth conics in $\mathbb{P}^2$, with the action induced by linear coordinate change on $\mathbb{P}^2$. The stabilizer of a conic q is $H \cong \mathrm{SO}_3 \cdot C(\mathrm{SL}_3)$, and the homogeneous space G/H is spherical. We choose q so that $H = \mathrm{SO}_3 \cdot C(\mathrm{SL}_3)$ where SO_3 is defined in the standard way as $\{A^{-1} = {}^tA\}^\circ$. Then define the two following functions on SL_3:

$$\begin{aligned} f_1(A) &= \text{the lower right } 2 \times 2 \text{ minor of } A \cdot {}^tA, \\ f_2(A) &= \text{the lower right matrix entry of } A \cdot {}^tA. \end{aligned}$$

Then the functions f_1/f_2^2 and f_2/f_1^2 descend to B-semi-invariant functions on G/H, with weights resp. $2\alpha_1$ and $2\alpha_2$. These two weights are a basis of $\Lambda(G/H)$.

Example 2.2.6. (Higher rank.) Let $G = \mathrm{SL}_{n+1} \times \mathrm{SL}_{n+1}$ and $H = \mathrm{diag}(\mathrm{SL}_{n+1})$; see also [Br09, Example 2.11.3]. The homogeneous space G/H is spherical and isomorphic to SL_{n+1} where G acts simultaneously with left and right multiplication, i.e. $(g, h)\, x = gxh^{-1}$. We choose T to be the couples of diagonal matrices and:

$$B = \{(b_1, b_2) \in G \mid b_1 \text{ is upper triangular, } b_2 \text{ is lower triangular}\}$$

Define $f_i \in \mathbb{C}[G]$ by:

$$f(a, b) = \text{the upper left } (i \times i)\text{-minor of } ba^{-1}$$

for $i = 1, \ldots, n$. It follows that f_i descends to a function on G/H for all i. Let us denote by $\omega_1, \ldots, \omega_n$ and $\omega'_1, \ldots, \omega'_n$ the fundamental dominant weights of the two copies of SL_{n+1}. We have that the B-weight of f_i is $(\omega_i, -\omega'_i)$. It follows easily that $\Lambda(G/H)$ can be identified with $\Lambda(B_{\mathrm{SL}_{n+1}})$ where $B_{\mathrm{SL}_{n+1}}$ is our standard choice of a Borel subgroup of SL_{n+1}.

We can reproduce this example with any connected reductive group H and $G = H \times H$.

Example 2.2.7. Let G be any semisimple group and set $H = B^u$. Then G/H is spherical, again as a consequence of the Bruhat decomposition. The ring $\mathbb{C}[G]$ has the well known $G \times G$-module structure:

$$\mathbb{C}[G] = \bigoplus V \otimes V^*$$

where the sum is taken over all irreducible G-modules V (see [Br09, Lemma 2.2]). It immediately follows that for each dominant weight ω there exists on G exactly one regular function (up to scalar multiplication) that is B^u-invariant on the right and B-semi-invariant on the left, with weight ω. It is not difficult to deduce that $\Lambda(G/H) = \Lambda(T)$.

Let us now discuss the first basic results of geometric nature. The first theorem and its corollary hold in full generality for actions of G on normal varieties.

Theorem 2.2.1. *Let X be a normal G-variety, Y a G-orbit. Then there exists an open set U of X containing Y, G-stable, and isomorphic to a G-stable subvariety of $\mathbb{P}(V)$ for some finite dimensional rational G-module V.*

Proof. (sketch, see [KKLV89, Theorem 1.1]) Let U_0 be an open affine set intersecting Y. Then $X \setminus U_0$ has pure codimension 1, let us call D the effective divisor on X which is the sum of the irreducible components of $X \setminus U_0$.

Now one can show that D is Cartier on a G-stable neighborhood of Y (using the same approach as in [Br97, proof of Proposition 2.2]), therefore we can suppose that D is Cartier on X. For $m \in \mathbb{N}$ big enough, the invertible sheaf $\mathcal{O}(mD)$ is G-linearizable (see [KKLV89, Proposition 2.4]); choose such an m so that $f_1, \ldots, f_n \in H^0(X, \mathcal{O}(mD))$, where $f_1, \ldots, f_n$ are generators of the $\mathbb{C}$-algebra $\mathbb{C}[U_0]$. Set $W =$ the G-submodule generated by $f_1, \ldots, f_n$ in $H^0(X, \mathcal{O}(mD))$; the associated rational map $X \dashrightarrow \mathbb{P}(W^*)$ is G-equivariant and biregular on GU_0, and we can set $U = GU_0$. □

Corollary 2.2.1. *Let X be a normal G-variety with only one closed orbit Y. Then X is quasi-projective.*

Proof. Let U be the quasi-projective G-stable open set as in Theorem 2.2.1. Its complementary $X \setminus U$ is closed and G-stable, therefore either it is empty, or it contains a closed G-orbit different from Y. The claim follows. □

Corollary 2.2.2. *Let X be a normal G-variety and Y an irreducible G-stable closed subvariety. Then B-semi-invariant rational functions on Y can be extended to B-semi-invariant rational functions on X, obtaining $\Lambda(X) \supseteq \Lambda(Y)$.*

Proof. (sketch, see [Br97, proof of Proposition 1.1]) Suppose at first that X is affine. If $f \in \mathbb{C}(Y)^{(B)}$, then the B-module:

$$\left\{ q \in \mathbb{C}[Y] \mid qf \in \mathbb{C}[Y] \right\}$$

is non trivial, thus contains a B-eigenvector. It follows that f is the quotient of two functions in $\mathbb{C}[Y]^{(B)}$. Then, the induced G-equivariant map $\mathbb{C}[X] \to \mathbb{C}[Y]$ is surjective, and one can lift all B-semi-invariant regular functions from Y to X, obtaining $\Lambda(X) \supseteq \Lambda(Y)$.

If X is not affine, we can use Theorem 2.2.1 to reduce the problem to a quasi-projective $X \subseteq \mathbb{P}(V)$, and then apply the affine case on the cones over X and Y in V. □

We end this Section with an important finiteness result.

Theorem 2.2.2. *Let X be a spherical G-variety. Then both G and B have a finite number of orbits on X. Moreover, if Y is a G-stable closed subvariety of X, then Y is spherical.*

Proof. The finiteness of the number of G-orbits and the last assertion is proved for affine spherical varieties in [Br09, Theorem 2.14]. It is not difficult to extend these two claims to the case where $X \subseteq \mathbb{P}(V)$ is quasi-projective, using the cone over X inside V equipped with the action of $G \times \mathbb{C}^*$, where $\mathbb{C}^*$ acts by scalar multiplication on V (see also [Br09, Lemma 2.17 and Proposition 2.18]). Theorem 2.2.1 then assures the two claims for any spherical variety.

The finiteness of the number of B-orbits follows from the claim on G-orbits together with [Br09, Theorem 2.15]. □

2.3. **Local structure.** Let X be a spherical variety and Y a closed G-orbit. We are interested, in analogy with differentiable actions of compact Lie groups, in a local structure theorem of X around Y. One could hope for some result describing a G-invariant neighbourhood of Y with nice properties such as admitting a retraction onto Y, but examples show that this is not possible (see e.g. [Br97, §1.4]).

However, we can restrict the problem to the action of a parabolic subgroup of G; in this case a result on the local structure is possible. This is very useful to attack geometric problems on X (equipped with the action of G) by reducing them to some smaller variety M, spherical under the action of a smaller reductive group $L \subset G$.

Example 2.3.1. Let $G = \mathrm{SL}_2$ acting on $X = \mathbb{P}^1 \times \mathbb{P}^1$ with (diagonal) linear action, and consider the closed G-orbit $Z = \mathrm{diag}(\mathbb{P}^1)$. The orbit Z does not admit any G-stable neighbourhood except for the whole X, but let us consider the action of the Borel subgroup B. There exist two B-stable prime divisors other than Z:

$$D^+ = \mathbb{P}^1 \times \big\{[1,0]\big\}, \qquad D^- = \big\{[1,0]\big\} \times \mathbb{P}^1;$$

and there exists one B-stable affine open set intersecting Z, which we will call $X_{Z,B}$, namely:

$$X_{Z,B} = X \setminus (D^+ \cup D^-) = \big\{([x,1],[y,1])\big\} \cong \mathbb{A}^2$$

The action of B on $X_{Z,B}$ is particularly simple. First of all, the action of the unipotent radical B^u is simply the translation in the direction of the line $\{x = y\} \subset \mathbb{A}^2$, which is by the way $Z \cap X_{Z,B}$. Now we consider the maximal torus $T \subset B$: its action is also simple. It is just the scalar action on the whole $\mathbb{A}^2$ with weight α_1.

It is useful to notice that $X_{Z,B}$ splits into the product of $Z \cap X_{Z,B}$, which is B-stable, and a "section" M, i.e. a T-stable closed subvariety. Here one can take for M any other line through the origin, e.g. $\{x + y = 0\} \subset \mathbb{A}^2$.

The following local structure theorem can be stated in many different forms; here we report a version which is best suited for our needs.

Definition 2.3.1. For any spherical G-variety X and any closed G-orbit $Y \subset X$, we define the open set:

$$X_{Y,B} = X \setminus \bigcup D$$

where the union is taken over all B-stable prime divisors D that do not contain Y.

Theorem 2.3.1. *Let X be a spherical G-variety and $Y \subset X$ a closed G-orbit. Then:*

1. *The set $X_{Y,B}$ is affine, B-stable, and is equal to $\big\{x \in X \mid \overline{B \cdot x} \supseteq Y\big\}$.*
2. *If Y is the only closed G-orbit, then the B-stable prime divisors not containing it are all Cartier and generated by global sections.*

(3) *Define the parabolic subgroup $P \supseteq B$ to be the stabilizer of $X_{Y,B}$, and choose a Levi subgroup L of P. Then there exists an affine L-stable L-spherical closed subvariety M of $X_{Y,B}$ such that the action morphism:*

$$P^u \times M \to X_{Y,B}$$

is a P-equivariant isomorphism. Here the action of P on $P^u \times M$ is defined as $p \cdot (v, m) = (ulvl^{-1}, lm)$ where $p = ul \in P$ and $u \in P^u$, $l \in L$. Moreover we have $\Lambda(X) = \Lambda(M)$: in particular X and M have same rank.

Proof. We only sketch the strategy of the proof of the decomposition in (3) in the special (not necessarily spherical) case where $X = \mathbb{P}(V)$ is the projective space of a simple non-trivial G-module V, and we take for $X_{Y,B}$ the complementary in $\mathbb{P}(V)$ of the unique B-stable hyperplane U. All parts of the theorem follow from this (see [Br97, Proposition 2.2 and Theorem 2.3]).

So let $X = \mathbb{P}(V)$ and Y be its unique closed G-orbit. Consider the open set $Y_\circ = Y \setminus U$ of Y: it is isomorphic to P^u. The stabilizers of the points in $Y_\circ$ are exactly the Levi subgroups of P. Let us choose $m \in Y_\circ$ such that its stabilizer is L.

The action of L on $\mathbb{C}m \subseteq V$ is the multiplication by a scalar, which is not constantly $= 1$. Therefore the tangent space $\mathrm{T}_m(Gm)$ is actually an L-submodule of V. Since L is reductive, the module V admits a decomposition:

$$V = \mathrm{T}_m(Gm) \oplus W$$

where W is a complementary L-submodule. Set:

$$M = \mathbb{P}(\mathbb{C}m \oplus W) \setminus U.$$

This is evidently an L-stable affine subvariety of $\mathbb{P}(V) \setminus U$, and it is possible to prove that $\mathbb{P}(V) \setminus U$ is equivariantly isomorphic to $P^u \times M$. □

Corollary 2.3.1. *A spherical variety has rank* 0 *if and only if it is a complete homogeneous space G/P for some parabolic subgroup P.*

Proof. The "if" part has been proved in Example 2.2.1. For the converse, see [Br97, Corollary 1.4.1]. □

Corollary 2.3.2. *In the hypotheses of Theorem 2.3.1, the variety X is smooth if and only if M is.*

The following proposition and its corollary are straightforward.

Proposition 2.3.1. *Let X be a spherical G-variety and $Y \subset X$ a closed G-orbit. Define the set $X_{Y,G}$ to be $G\,X_{Y,B}$. Then:*

$$X_{Y,G} = \left\{ x \in G \mid \overline{G \cdot x} \supseteq Y \right\}.$$

This set is open, G-stable, and Y is its unique closed G-orbit.

Definition 2.3.2. A spherical G-variety X is *simple* if it contains a unique closed G-orbit.

Corollary 2.3.3. *Any spherical G-variety admits a cover by open G-stable simple spherical G-varieties.*

Example 2.3.2. We consider the spherical homogeneous space $G/H = \mathrm{SL}_{n+1}/\mathrm{SL}_n$. It can be described as:

$$G/H = \big\{(v, V) \in \mathbb{C}^{n+1} \times \mathrm{Gr}(n, \mathbb{C}^{n+1}) \mid v \notin V\big\}.$$

We define the embedding $X \supset G/H$ to be the closure of X inside $\mathbb{P}^{n+1} \times \mathrm{Gr}(n, \mathbb{C}^{n+1})$, where we consider $\mathbb{P}^{n+1}$ as $\mathbb{C}^{n+1}$ plus the hyperplane $\mathbb{P}^n_\infty$ at infinity. This X has two closed G-orbits: Y, given by the condition $v = 0$, and Z, given by the condition $v \in \mathbb{P}^n_\infty \cap \overline{V}$ (with obvious meaning).

The open set $X_{Y,G}$ is given by the condition $v \notin \mathbb{P}^n_\infty$, and $X_{Z,G}$ is simply $X \setminus Y$.

The first steps of our analogy with toric varieties can now be made more precise. Our last statements suggest that the sets $X_{Y,B}$, being open B-stable and affine, are good candidates for a description of their rings of functions $\mathbb{C}[X_{Y,B}]$ using the B-weights of their B-eigenvectors. On the other hand the sets $X_{Y,G}$, i.e. in general the simple spherical varieties, are good candidates to play here the role of affine toric varieties.

2.4. **Discrete valuations.** We approach now the additional structures on $N(X)$ that are required for the classification of embeddings. We use discrete valuations on the field of rational functions of G/H; the strategy is to relate this notion, which is in some sense quite abstract, to the invariants we have already defined.

Definition 2.4.1. A map $\nu\colon \mathbb{C}(X)^* \to \mathbb{Q}$ is a *discrete valuation* on X if, for all $f_1, f_2 \in \mathbb{C}(X)^*$:

(1) $\nu(f_1 + f_2) \geq \min\{\nu(f_1), \nu(f_2)\}$ whenever $f_1 + f_2 \in \mathbb{C}(X)^*$;
(2) $\nu(f_1 f_2) = \nu(f_1) + \nu(f_2)$;
(3) $\nu(\mathbb{C}^*) = \{0\}$;
(4) the image of ν is a discrete subgroup of $\mathbb{Q}$.

Definition 2.4.2.

(1) Let D be a prime divisor on X. Then we denote by ν_D the associated discrete valuation.
(2) There is a naturally defined map:
$$\rho_X\colon \{\text{discrete valuations on } X\} \to N(X)$$
such that the image of ν is the functional $\rho_X(\nu)$ that takes the value $\nu(f_\chi)$ on $\chi \in \Lambda(X)$. This is well defined since f_χ is determined by χ up to a multiplicative constant. For a prime divisor D of X, we will also write for brevity $\rho_X(D)$ instead of $\rho_X(\nu_D)$; we will also drop the subscript "X" if no risk of confusion arises.
(3) A discrete valuation ν on X is *G-invariant* if $\nu(g \cdot f) = \nu(f)$ for all $f \in \mathbb{C}(X)^*$ and all $g \in G$. We denote by $\mathcal{V}(X)$ the set of G-invariant valuations on X.

Let us make a few remarks. If Y is a G-invariant prime divisor, then it is obvious that ν_Y is G-invariant. Also, if Z is any G-stable irreducible closed subvariety Z, let us define ν to be the valuation associated to an irreducible component of $\pi^{-1}(Z)$ where $\pi\colon \widetilde{X} \to X$ is the normalization of the blow-up of X along Z. The result is a discrete valuation with the following properties: $f \in \mathcal{O}_{X,Z}$ implies $\nu(f) \geq 0$, and $f|_Z = 0$ implies $\nu(f) > 0$. Whenever these conditions are satisfied for some ν, we say that ν has *center* on Z.

The last element we have to introduce for the classification of embeddings is the notion of the colors of a spherical variety.

Definition 2.4.3. A prime divisor D on X is called a *color* if it is B-stable but not G-stable. We denote by $\Delta(X)$ the set of all colors of X.

Again, all definitions here are invariant under G-equivariant birational maps, and thus depend only on G/H. This hold also for colors, if we accept the abuse of notation given by identifying the colors of G/H with their closures in X.

Example 2.4.1. Consider again the spherical homogeneous space $G/H = \mathrm{SL}_2/U$, which can be seen as $\mathbb{C}^2 \setminus \{(0,0)\}$. The unique color D is given by the equation $\{y = 0\}$, and we have:

$$\langle \rho(D), \omega_1 \rangle = \nu_D(f_{\omega_1}) = \nu_D(y) = 1.$$

We claim that $\mathcal{V}(\mathrm{SL}_2/U)$ is equal to the whole $N(\mathrm{SL}_2/U)$. Choose any $\eta \in \mathrm{N}(X)$, and define $\nu \colon \mathbb{C}[x,y]^* \to \mathbb{Q}$ by:

$$\nu(f) = \min\langle \eta, n\omega_1 \rangle$$

where the minimum is taken over all $n \in \mathbb{N}$ such that f has some part of degree n. This ν extends to a G-invariant valuation defined on $\mathbb{C}(\mathrm{SL}_2/U)$, and it satisfies $\rho(\nu) = \eta$.

The above example is a particular case of an interesting class of varieties.

Definition 2.4.4. A spherical variety with open orbit G/H is *horospherical* if $H \supseteq U$ where U is a maximal unipotent subgroup of G.

We will see in Corollary 3.2.1 that X is horospherical if and only if $\mathcal{V}(X) = N(X)$.

Example 2.4.2. If $X \supset G/H$ is the embedding $\mathbb{P}^1 \times \mathbb{P}^1 \supset \mathrm{SL}_2/T$, we have already seen its two colors D^+ and D^-, and its unique closed G-orbit $Z = \mathrm{diag}(\mathbb{P}^1)$. We also know a local equation $f(x,y) = (x-y)$ of Z in $X_{Z,B} = \{([x,1],[y,1]) \in \mathbb{P}^1 \times \mathbb{P}^1\}$, and obviously $f(x,y)$ is a B-eigenvector with weight $-\alpha_1$.

It follows that $\langle \rho(Z), \alpha_1 \rangle = -1$, and we have $\mathcal{V}(X) = \mathbb{Q}_{\geq 0}\nu_Z$. On the other hand, $f(x,y)$ has poles of order 1 along both colors, thus $\langle \rho(D^+), \alpha_1 \rangle = \langle \rho(D^-), \alpha_1 \rangle = 1$.

Example 2.4.3. If G/H is equal to the space of smooth conics for $G = \mathrm{SL}_3$, it is easy to see that there are two colors D_1, D_2. Their pull-backs on G have global equations resp. f_1 and f_2, in the notation of Example 2.2.5. A straightforward computation leads to:

$$\rho_{G/H}(D_1) = \frac{1}{2}\alpha_1^\vee|_{\Lambda(G/H)}, \quad \rho_{G/H}(D_2) = \frac{1}{2}\alpha_2^\vee|_{\Lambda(G/H)}$$

It is possible to prove that $\rho_{G/H}(\mathcal{V}(G/H))$ is equal to the negative Weyl chamber intersected with $\Lambda(G/H)$.

The discrete valuations on $\mathbb{C}(G/H)$ are the best-suited tool to describe any embedding of the homogeneous space G/H, and we will see that invariant ones will play a central role. The next theorem assures that there is no "loss of information" if we only calculate them on B-semi-invariant rational function, which are the core of our approach.

Theorem 2.4.1. *Let X be a spherical G-variety. Then the application ρ_X restricted to the set of G-invariant valuations:*

$$\rho_X|_{\mathcal{V}(X)} \colon \mathcal{V}(X) \to N(X)$$

is injective.

Proof. (sketch, see [Br97, Corollary 3.1.3]) Let ν be an invariant valuation: we want to prove that it depends only on its values at B-eigenvectors. It is possible to lift ν to a G-invariant (under left translation) discrete valuation defined on $\mathbb{C}(G)^*$, via the quotient $G \to G/H$ where G/H is the open G-orbit of X (we skip the proof of this lifting, see [Br97, Corollary 3.1.1]).

Let now f be a regular function on G and $W = \operatorname{span}\{gf \mid g \in G\}$ the G-submodule generated by f. We claim that:

$$\nu(f) = \min_{F \in W^{(B)}} \nu(F).$$

Indeed, for any B-eigenvector $F \in W$, we know that:

$$F = g_1 f + g_2 f + \ldots + g_n f$$

for some $g_1, \ldots, g_n \in G$. From invariance and the properties of discrete valuations it follows $\nu(F) \geq \nu(f)$.

On the other hand W is a G-module, therefore we can also generate it with elements of the form gF for $g \in G$ and F a B-eigenvector. It follows that:

$$f = g'_1 F_1 + g'_2 F_2 + \ldots + g'_m F_m$$

for $g'_1, \ldots, g'_m \in G$, and $F_1, \ldots, F_m \in W^{(B)}$. Our claim follows, and it is now easy to deduce that ν calculated on rational functions depends only on its values on $\mathbb{C}(G)^{(B)}$. It follows at once that the original valuation on X depends only on its values on $\mathbb{C}(X)^{(B)}$. □

We remark that ρ_X is not injective on all discrete valuations; see e.g. Example 2.4.2.

2.5. **Classification of embeddings.** Recall that we have fixed a spherical homogeneous space G/H, with the base point H.

We will say that (X, x) is an embedding of G/H if X is spherical, the G-orbit Gx is open in X, and H is the stabilizer of x. If no confusion arises, we will denote sometimes an embedding (X, x) simply by X.

Given two embeddings (X, x) and (X', x') of G/H, any G-equivariant map between X and X' will be assumed to send x to x'.

Now that we have all the necessary elements, we begin the classification of embeddings from the case of simple ones.

Definition 2.5.1. Let X be a simple spherical G-variety, with closed orbit Y. Then we define:

$$\mathcal{D}(X) = \big\{D \in \Delta(X) \mid D \supset Y\big\}.$$

Proposition 2.5.1. *Let (X, x) be a simple embedding of G/H.*

1. *Let $f \in \mathbb{C}(X)$. Then $f \in \mathbb{C}[X_{Y,B}]$ if and only if f is regular on the open B-orbit X°_B of X and $\nu_D(f) \geq 0$, where D runs through all G-invariant prime divisors of X and all elements of $\mathcal{D}(X)$.*
2. *Among simple embeddings of G/H, (X, x) is uniquely determined by $\mathcal{D}(X)$ and the valuations of the G-stable prime divisors of X.*

Proof. (sketch, see [Br97, Proposition 3.2.1]) From the definition of $X_{Y,B}$ we know that it contains X_B°; the complement $X_{Y,B} \setminus X_B^\circ$ is the union of the prime divisors in $\mathcal{D}(X)$ and of all the G-stable prime divisors of X. Since $X_{Y,B}$ is normal, any function is regular if it is regular in codimension 1, and the first part follows.

If (X, x) and (X', x') are simple embeddings of G/H with same $\mathcal{D}(X)$ and the same valuations of the G-stable prime divisors of X, then we know from the first part that $\mathbb{C}[X_{Y,B}] = C[X'_{Y,B}]$ as subrings of $\mathbb{C}(G/H)$. More precisely, the G-equivariant birational map $X \dashrightarrow X'$ sending x to x' induces an isomorphism between $X_{Y,B}$ and $X'_{Y,B}$. Since $GX_{Y,B} = X$ and $GX'_{Y,B} = X'$, the second part follows. □

We are ready to define the objects that classify simple embeddings. In what follows, we consider $\mathcal{V}(X)$ as a subset of $N(X)$ (via the map ρ_X).

We also consider the set of colors $\Delta(G/H)$ simply as an abstract set, equipped with a map $\rho_X|_{\Delta(G/H)}$ to $N(X)$ (also denoted simply by ρ). The reason is that we want to consider our invariants as combinatorial objects, just like the convex polyhedral cones of toric varieties.

Definition 2.5.2. Let (X, x) be a simple embedding of G/H. Define $\mathcal{C}(X) \subseteq N(X)$ to be the convex cone generated by $\rho_X(\mathcal{D}(X))$ and by all the G-invariant valuations associated to G-stable prime divisors of X. The couple $(\mathcal{C}(X), \mathcal{D}(X))$ is called the *colored cone* of X.

The "combinatorial counterpart" of this definition is the following:

Definition 2.5.3. A *colored cone* in $N(G/H)$ is a couple $(\mathcal{C}, \mathcal{D})$, where $\mathcal{C} \subseteq N(X)$ and $\mathcal{D} \subseteq \Delta(X)$, such that:

(1) the set $\mathcal{C}$ is a strictly convex polyhedral cone generated by $\rho_{G/H}(\mathcal{D})$ and a finite number of elements in $\mathcal{V}(G/H)$;
(2) the relative interior of $\mathcal{C}$ intersects $\mathcal{V}(G/H)$;
(3) we have $0 \notin \rho(\mathcal{D})$.

It is not difficult to show that the colored cone of an embedding satisfies the properties of Definition 2.5.3.

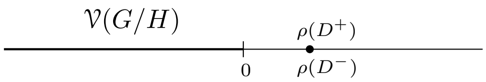
Theorem 2.5.1. *The map $(X, x) \mapsto (\mathcal{C}(X), \mathcal{D}(X))$ induces a bijection between simple embeddings of G/H (up to G-equivariant isomorphism) and colored cones in $N(G/H)$.*

Proof. See [Br97, Theorem 3.3]. □

It is convenient to make some remarks. First of all we recall that the G-equivariant isomorphisms between embeddings (X, x) and (X', x') are here required to send x to x': this is obviously necessary for the theorem to hold. Then, the injectivity of the map is essentially Proposition 2.5.1. Most of the next results on embeddings are proven using in the chart $X_{Y,B}$ and its ring of functions $\mathbb{C}[X_{Y,B}]$, in the same spirit of Proposition 2.5.1.

Finally, we point out that $\mathcal{C}(X)$ might be already generated as a convex cone by its intersection with $\mathcal{V}(G/H)$: it is however necessary to keep track of $\mathcal{D}(X)$ in order to distinguish between different embeddings.

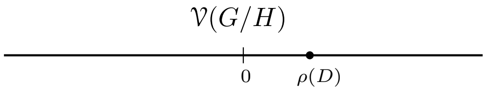
Example 2.5.1. Let G/H be the homogeneous space SL_2/T. Then the picture of $N(G/H)$, $\mathcal{V}(G/H)$ and $\Delta(G/H)$ is:



where $\rho(D^+) = \rho(D^-)$. We only have two possible colored cones (including the trivial one), and two simple embeddings:

(1) $(\mathcal{C}_1, \mathcal{D}_1) = (\{0\}, \varnothing)$, giving the trivial embedding G/H;
(2) $(\mathcal{C}_2, \mathcal{D}_2) = (\mathcal{V}(G/H), \varnothing)$, giving the embedding $\mathbb{P}^1 \times \mathbb{P}^1$.

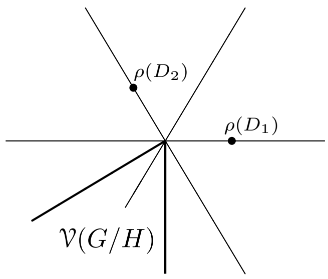
Example 2.5.2. Let G/H be the homogeneous space SL_2/U. Here the picture is the following:



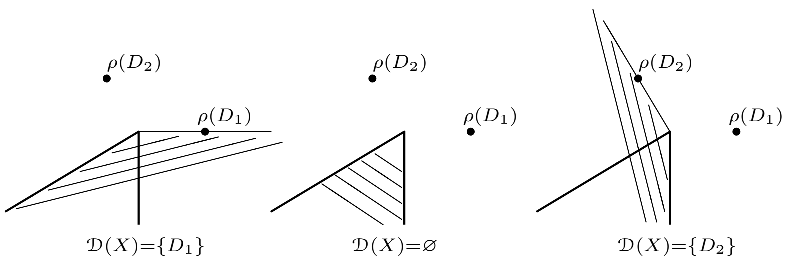
where D is the unique color of G/H. We have three nontrivial colored cones, which correspond to three nontrivial simple embeddings:

(1) $(\mathcal{C}_1, \mathcal{D}_1) = (\mathbb{Q}_{\leq 0}\rho(D), \varnothing)$, giving the embedding $G/H \cup$ (line at infinity);
(2) $(\mathcal{C}_2, \mathcal{D}_2) = (\mathbb{Q}_{\geq 0}\rho(D), \{D\})$, giving the embedding $\mathbb{C}^2 = G/H \cup \{(0,0)\}$;
(3) $(\mathcal{C}_3, \mathcal{D}_3) = (\mathbb{Q}_{\geq 0}\rho(D), \varnothing)$, giving the blowup of $\mathbb{C}^2$ in $(0,0)$.

Example 2.5.3. Let G/H be the homogeneous space $\mathrm{SL}_3/\mathrm{SO}_3 C(\mathrm{SL}_3)$ of smooth conics. Then we have:



Here there are infinitely many simple embeddings; some of them are:



The first one is the variety of all conics in $\mathbb{P}^2$, obviously an embedding of the space of smooth conics. More precisely, any nonsingular conic $\mathscr{C} \subset \mathbb{P}^2$ is defined by an homogeneous equation of degree 2 in three variables. The coefficients of the equation correspond to a point in $\mathbb{P}^5$, and we obtain in this way an identification of G/H with an open subset of $\mathbb{P}^5$. The closure of G/H, i.e. $\mathbb{P}^5$ itself, is the variety of conics (of points) in $\mathbb{P}^2$.

If we replace $\mathscr{C}$ with its dual $\mathscr{C}^* \subset (\mathbb{P}^2)^*$, we obtain a point in the dual projective space $(\mathbb{P}^5)^*$. The embedding $(\mathbb{P}^5)^*$ of G/H is the variety of conics of lines, and is represented by the third colored cone. The second colored cone corresponds to the closure in $\mathbb{P}^5 \times (\mathbb{P}^5)^*$ of the set of couples $(\mathscr{C}, \mathscr{C}^*)$: it is called the *variety of complete conics*, see Example 3.4.5 and [B09, Example 1.3].

The geometry of a simple embedding has many relations with the geometry of its colored fan. Nonetheless, the correspondence is not as well developed as for toric varieties; for example the characterization of smooth simple embeddings is in general more complicated.

The next result explains one of these relations, which is also useful for general embeddings.

Definition 2.5.4. A *face* of a colored cone $(\mathcal{C}, \mathcal{D})$ is a colored cone $(\mathcal{C}', \mathcal{D}')$ where $\mathcal{C}'$ is a face of $\mathcal{C}$ and $\mathcal{D}' = \mathcal{D} \cap \rho^{-1}(\mathcal{C}')$.

Proposition 2.5.2. *Let (X, x) be an embedding of G/H and Y one of its G-orbits. Then there is a bijection between the G-orbits of X containing Y in their closure and the faces of $(\mathcal{C}(X_{Y,G}), \mathcal{D}(X_{Y,G}))$, induced by associating to $Z \subseteq X$ the colored cone of the simple embedding $X_{Z,G}$.*

Proof. See [Br97, Proposition 3.4]. □

Finally, simple embeddings can be glued together to obtain general embeddings. As one would expect, this corresponds to the following:

Definition 2.5.5. A *colored fan* in $N(G/H)$ is a collection $\mathcal{F}$ of colored cones such that:

(1) any face of a colored cone of $\mathcal{F}$ is in $\mathcal{F}$;
(2) the relative interiors of the colored cones of $\mathcal{F}$ do not intersect.

Given an embedding (X, x) of G/H, we define its colored fan as:

$$\mathcal{F}(X) = \big\{\text{colored cones associated to } X_{Y,G} \text{ for any } G\text{-orbit } Y \text{ of } X\big\}.$$

Theorem 2.5.2. *The map $(X, x) \mapsto \mathcal{F}(X)$ induces a bijection between embeddings of G/H (up to G-equivariant isomorphism) and colored fans in $N(G/H)$.*

Proof. See [Br97, Theorem 3.4.1]. □

It is not difficult to prove the following last result:

Proposition 2.5.3. *An embedding (X, x) is complete if and only if the union of all $\mathcal{C}$ for $(\mathcal{C}, \mathcal{D})$ running through all colored cones in $\mathcal{F}(X)$ contains $\mathcal{V}(G/H)$.*

Example 2.5.4. We can now complete the list of embeddings for $G/H = \mathrm{SL}_2/T$ and $G/H = \mathrm{SL}_2/U$ we started in Examples 2.5.1 and 2.5.2. For the first one, it is clear that the two possible colored cones do not give rise to any other colored fan. Therefore SL_2/T does not admit any other embedding.

For SL_2/U, non-simple embeddings exist and are all complete. Precisely:

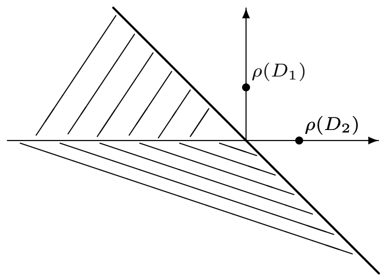
(1) $\mathcal{F} = \{(\{0\}, \varnothing), (\mathcal{C}_1, \mathcal{D}_1), (\mathcal{C}_2, \mathcal{D}_2)\}$, giving the embedding $\mathbb{P}^2$;
(2) $\mathcal{F} = \{(\{0\}, \varnothing), (\mathcal{C}_1, \mathcal{D}_1), (\mathcal{C}_3, \mathcal{D}_3)\}$, giving the blow up of $\mathbb{P}^2$ in the origin of $\mathbb{C}^2$.

Example 2.5.5. Let us describe the colored fan of an embedding X of $G/H = \mathrm{SL}_{n+1}/\mathrm{SL}_n$. The homogeneous space G/H was described in Example 2.3.2.

The lattice $\Lambda(G/H)$ has basis $\gamma_1 = \omega_1$ and $\gamma_2 = \omega_n$, which are the B-weights of the functions $f_1, f_2 \in \mathbb{C}[G]$ defined in Example 2.2.4 (both functions descend to functions in $\mathbb{C}[G/H]^{(B)}$). There are two colors D_1, D_2, where D_i is the set of zeros of f_i on G/H, and $(\rho(D_1), \rho(D_2))$ is the dual basis of (γ_2, γ_1).

Let X be the closure of G/H in the variety $\mathrm{Bl}_0(\mathbb{P}^{n+1}) \times \mathrm{Gr}(n, \mathbb{C}^{n+1})$ where $\mathbb{P}^{n+1}$ is considered as $\mathbb{C}^{n+1}$ plus the hyperplane at infinity, and the point $0 \in \mathbb{P}^{n+1}$ is just the origin of $\mathbb{C}^{n+1}$.

This embedding X is smooth, and its colored cones do not contain colors. Those of maximal dimension ($= 2$) give a subdivision of $\mathcal{V}(G/H)$:



2.6. **Functorial properties.** Let us discuss two functorial properties of the classification of embeddings. The first one is a generalization of dominant equivariant morphisms between toric embeddings of two given algebraic tori.

Let G/H and G/H' be two spherical homogeneous spaces with $H \subset H'$. Then the natural morphism:

$$\varphi \colon G/H \to G/H'$$

induces an injective homomorphism:

$$\varphi^* \colon \Lambda(G/H') \to \Lambda(G/H)$$

and a surjective linear map:

$$\varphi_* \colon N(G/H) \to N(G/H')$$

We define $\mathcal{C}_\varphi = \ker \varphi_*$. It is also possible to prove that $\varphi_*(\mathcal{V}(G/H)) = \mathcal{V}(G/H')$, using the lifting technique cited in the proof of Theorem 2.4.1.

Definition 2.6.1. We define $\mathcal{D}_\varphi$ to be the set of colors:

$$\mathcal{D}_\varphi = \left\{ D \in \Delta(G/H) \mid \overline{\varphi(D)} = G/H' \right\}$$

We remark that if a color D is not in $\mathcal{D}_\varphi$ then it is not difficult to show that $\overline{\varphi(D)}$ is a color of G/H'.

Theorem 2.6.1. *Let (X, x) and (X', x') two embeddings resp. of G/H and G/H'. Then φ extends to a G-equivariant map $X \to X'$ if and only if $\mathcal{F}(X)$ dominates $\mathcal{F}(X')$, in the following sense: for each colored cone $(\mathcal{C}, \mathcal{D}) \in \mathcal{F}(X)$ there exists $(\mathcal{C}', \mathcal{D}') \in \mathcal{F}(X')$ such that $\varphi_*(\mathcal{C}) \subseteq \mathcal{C}'$ and $\varphi_*(\mathcal{D} \setminus \mathcal{D}_\varphi) \subseteq \mathcal{D}'$.*

Proof. See [Br97, Theorem 3.4.2]. □

Example 2.6.1. It is easy to check Theorem 2.6.1 for embeddings of Examples 2.5.1, 2.5.2, 2.5.4, and 2.5.3 assuming some familiarity with the variety of complete conics.

The embedding of Example 2.5.5 admits a G-equivariant surjective morphism to the unique non-trivial embedding X' of SL_{n+1}/GL_n. The variety X' can be described as:

$$X' = \mathbb{P}^n \times \mathrm{Gr}(n, \mathbb{C}^{n+1}),$$

and the morphism has an obvious definition which we leave to the Reader.

Our last theorem is a very useful result which describes all subgroups $H' \supseteq H$ of G such that H'/H is connected. It is also a generalization of the fact that connected subgroups of an algebraic torus T are described by linear subspaces of $N(T)$.

Definition 2.6.2. A *colored subspace* of $N(G/H)$ is a colored cone $(\mathcal{C}, \mathcal{D})$ such that $\mathcal{C}$ is a vector subspace of $N(G/H)$.

Theorem 2.6.2. *The map $H' \mapsto (\mathcal{C}_\varphi, \mathcal{D}_\varphi)$ is a bijection between the set of subgroups $H' \supseteq H$ of G such that H'/H is connected, and the set of colored subspaces of $N(G/H)$. For each such H', the map φ induces a bijection between $\Delta(G/H')$ and $\Delta(G/H) \setminus \mathcal{D}_\varphi$.*

Proof. See [Br97, Theorem 3.4.2]. □

3. Wonderful varieties

Wonderful varieties first appeared in the work of De Concini and Procesi [DP83], where they constructed special embeddings for symmetric homogeneous spaces $G/\mathrm{N}_G G^\theta$. Here G^θ is the subgroup of elements fixed by a given involution $\theta\colon G \to G$.

These embeddings shared very nice properties, and were used to approach in a new way classical problems in enumerative geometry: for example counting the number of quadrics simultaneously tangent to nine given quadrics in $\mathbb{P}^3$.

Their properties were then taken as axioms of the class of wonderful varieties, and apparently do not involve at all what we have seen in the previous Sections. In [Lu96] Luna showed that such varieties are always spherical, and we will see an alternative definition based upon the theory of embeddings.

We start with the properties of a more general family of spherical varieties, which is interesting on its own.

3.1. Toroidal embeddings.

Definition 3.1.1. An embedding (X, x) is *toroidal* if no color contains a G-orbit. Equivalently, if all colored cones $(\mathcal{C}, \mathcal{D}) \in \mathcal{F}(X)$ satisfy $\mathcal{D} = \varnothing$.

In the case of toroidal embeddings we can give a much stronger local structure theorem:

Theorem 3.1.1. *Let X be a toroidal embedding and $Y \subseteq X$ a closed G-orbit, let P be the stabilizer of $X_{Y,B}$ and choose a Levi subgroup $L \subseteq P$. Then, as in Theorem 2.3.1, there exists an affine L-stable L-spherical closed subvariety M of $X_{Y,B}$ such that the action morphism:*

$$P^u \times M \to X_{Y,B}$$

is a P-equivariant isomorphism. For any such M:

(1) *the action of (L, L) on M is trivial, therefore M is a toric variety under the action of a quotient of the torus $L/(L, L)$;*
(2) *the intersection with M induces a bijection between G-orbits in X and L-orbits in M.*

Proof. See [Br97, Proposition 2.4.1]. □

The proof of above theorem uses Theorem 2.3.1; one of the other ingredients is the following result, which is nice to single out:

Lemma 3.1.1. *Let H be a subgroup of G such that $G = BH$. Then H contains (G, G).*

Proof. We can suppose that G is semisimple, and it is not difficult to show that $G = BH$ implies $G = BH^\circ$. Consider the semisimple group $K = H^\circ/(H^\circ)^r$: since $G = BH^\circ$ then $G/B = H/B \cap H$. Hence the flag varieties of G and K are equal. Therefore the ranks of G and K are equal, being the ranks of the Picard group of the respective flag varieties, and they have the same number of roots (being the double of the dimensions of the respective flag varieties). It follows that $\dim G = \dim K$ and thus $G = K$. □

Obviously, Corollary 2.3.2 holds for toroidal varieties too. Here it is in some sense particularly relevant, due to the following:

Proposition 3.1.1. *Let X be a toroidal simple embedding, with colored cone $(\mathcal{C}_X, \varnothing)$. Then the convex cone associated to the affine toric variety M is $\mathcal{C}_X$.*

The Proposition follows at once from Theorem 3.1.1, and its importance comes from the fact that smoothness of toric varieties is easily checked on the associated convex cones.

3.2. **The cone of invariant valuations and canonical embeddings.**

Theorem 3.2.1. *Let X be a spherical variety. Then $\mathcal{V}(X)$ is a convex polyhedral cone which spans $N(X)$. Moreover, it can be defined as:*

$$\mathcal{V}(X) = \{\eta \in N(X) \mid \langle \eta, \sigma_i \rangle \leq 0,\ i = 1, \ldots, n\}$$

where $\sigma_1, \ldots, \sigma_n \in \Lambda(X)$ are linearly independent.

Proof. See [Br97, Theorem 4.1.1] for the first part of the Theorem. The proof of the second part is much more involved and is carried out in [Br90] where it rests ultimately on a case-by-case verification. See also [K96] for a proof with a different approach. □

Theorem 3.2.1 suggests the possible existence of a simple toroidal embedding which is in some sense "distinguished": the one associated to the colored cone $(\mathcal{V}(G/H), \varnothing)$. Its interest comes from the fact that it would only depend on G/H, without any further choice.

Unfortunately, the cone $\mathcal{V}(G/H)$ is not always strictly convex: namely, its linear part has a precise relationship with the group $(\mathrm{N}_G H/H)^\circ$.

Theorem 3.2.2.

(1) *There exists a choice of* $\sigma_1, \ldots, \sigma_n$ *as in Theorem 3.2.1, such that:*
$$\mathrm{N}_G H/H \cong \mathrm{Hom}\left(\frac{\Lambda(G/H)}{\mathrm{span}_{\mathbb{Z}}\{\sigma_1, \ldots, \sigma_n\}}, \mathbb{C}^*\right).$$
As a consequence, the group $\mathrm{N}_G H/H$ *is diagonalizable and its dimension is equal to the dimension of the linear part of* $\mathcal{V}(G/H)$.

(2) *We have a short exact sequence:*
$$0 \to \mathcal{V}(G/H) \to N(G/H) \to N(G/\mathrm{N}_G H) \to 0.$$

(3) *If* H *is* B*-spherical, the group* $\mathrm{N}_G H$ *is the stabilizer on the right of* $BH \subseteq G$*; we also have* $\mathrm{N}_G H = \mathrm{N}_G(H^\circ)$.

Proof. See [Br97, Theorem 4.3]. □

Corollary 3.2.1.

(1) *The cone* $\mathcal{V}(G/H)$ *is strictly convex if and only if* $\mathrm{N}_G H/H$ *is finite.*

(2) *The subgroup* H *contains a maximal unipotent subgroup of* G *if and only if* $\mathcal{V}(G/H) = N(G/H)$.

Proof. Part (1) follows immediately from Theorem 3.2.2. We prove part (2). Suppose that $\mathcal{V}(G/H) = N(G/H)$; from Theorem 3.2.2 it follows that $G/\mathrm{N}_G H$ has rank 0, in other words $\mathrm{N}_G H$ is a parabolic subgroup. The quotient $\mathrm{N}_G H/H$ is diagonalizable, therefore H contains a maximal unipotent subgroup of $\mathrm{N}_G H$ and thus of G.

Inversely, suppose that H contains a maximal unipotent subgroup U. We know that the normalizer of U is a Borel subgroup, hence $\mathcal{V}(G/U) = N(G/U)$. Then $\mathcal{V}(G/H)$ is a vector space too, as a consequence of §2.6. □

The two above results are easily verified on all the examples we have given so far.

Definition 3.2.1. If $\mathrm{N}_G H/H$ is finite then we define the *canonical embedding* of G/H to be the simple embedding associated to the colored cone $(\mathcal{V}(G/H), \varnothing)$.

It is immediate to check that if the canonical embedding exists, it is the unique simple toroidal complete embedding of G/H.

It exists if and only if H has finite index in its normalizer. We remark that thanks to Theorem 3.2.2 we know that $\mathrm{N}_G(\mathrm{N}_G H) = \mathrm{N}_G H$; as a consequence, a spherical homogeneous space of the form $G/\mathrm{N}_G H$ always admit a canonical embedding.

3.3. **Wonderful varieties.** Let us define wonderful varieties with the original definition, coming from the properties of the De Concini-Procesi compactifications of symmetric homogeneous spaces.

Definition 3.3.1. Let X be a G-variety. Then X is *wonderful* if:

(1) X is smooth and complete;

(2) X contains an open G-orbit X°_G, whose complementary is the union of smooth G-stable prime divisors $X^{(1)}, \ldots, X^{(r)}$, which have normal crossings and non-empty intersection;

(3) for all $x, y \in X$ we have:
$$Gx = Gy \Longleftrightarrow \{i \mid X^{(i)} \ni x\} = \{j \mid X^{(j)} \ni y\}.$$

The number r is by definition the *rank* of X, and the union of the G-stable prime divisors is called the *boundary* of X, denoted ∂X.

As an immediate consequence, on a wonderful variety X the G-orbits are exactly the sets:

$$\bigcap_{i\in I} X^{(i)} \setminus \bigcup_{j\notin I} X^{(j)}$$

for any subset $I \subseteq \{1, 2, \ldots, r\}$. So a wonderful variety has 2^r orbits, and only one is closed. Being simple and complete, a wonderful variety is always projective, thanks to Corollary 2.2.1.

We remark that Definition 3.3.1 does not assume X to be spherical. This is actually a consequence, as shown by Luna:

Theorem 3.3.1. *Any wonderful G-variety is spherical.*

Proof. See [Lu96]. □

We will give several examples of wonderful varieties at the end of §3.4, together with some relevant invariants.

We now show that wonderful varieties can be defined purely in terms of embeddings.

Proposition 3.3.1. *A spherical variety is wonderful if and only if it's the canonical embedding of its open G-orbit and this embedding is smooth. The ranks as wonderful variety and as spherical variety coincide.*

Proof. Let X be the smooth canonical embedding of G/H with rank r as a spherical variety: we prove it's wonderful of rank r. First of all it is simple and projective, and all its G-stable prime divisors contain its closed orbit Y. Consider the local structure of $X_{Y,B}$ as in Theorem 3.1.1. Then M is a smooth affine toric variety; it also has a fixed point because T has a fixed point in $X_{Y,B}$ lying on the closed G-orbit Y (the unique point fixed by B_-). It follows that $M \cong \mathbb{C}^r$ where r is the rank of G/H. Its dimension is correct because $\mathcal{V}(G/H)$ is strictly convex therefore its equations (as in Theorem 3.2.1) are exactly r.

The correspondence between G-orbits on X and T-orbits on Y implies that the G-stable prime divisors of X cut M along its r coordinate hyperplanes, so they are smooth with normal crossings. It also implies readily axiom (3) of Definition 3.3.1, since it is obviously true for M.

Now let X be a wonderful variety, of rank r. Then it is spherical, smooth, simple and projective, with open orbit G/H and closed orbit Y; it remains to show that it is toroidal, and its rank r' as spherical variety is equal to r. Let $(\mathcal{C}(X), \mathcal{D}(X))$ be its colored cone. Recall that ∂X denotes the set of the r prime G-stable divisors of X. Hence $\mathcal{C}(X)$ is generated as a convex cone by $\rho(\mathcal{D}(X))$ and $\rho(\partial X)$, and contains $\mathcal{V}(G/H)$. It also follows that $\mathcal{V}(G/H)$ is strictly convex and simplicial, defined by r' equations as in Theorem 3.2.1. Thus G/H admits a canonical embedding X'.

Consider now $\rho(\partial X)$ in view of the structure of the G-orbits of X. Proposition 2.5.2 and the orbit structure of X imply that any subset of $\rho(\partial X)$ generates a face F of $\rho(\partial X)$, in such a way that F is also a face of $\mathcal{C}(X)$ and its relative interior intersects $\mathcal{V}(G/H)$. In other words $\rho(\partial X)$ must be a simplicial cone equal to $\mathcal{C}(X)$ and to $\mathcal{V}(G/H)$. It follows also that $r = r'$.

Finally, we must prove that $\mathcal{D}(X) = \varnothing$. Consider the canonical embedding X' of G/H. The identity of G/H extends to a G-equivariant map $\varphi\colon X' \to X$, and applying Proposition 2.5.2, we see that φ induces a bijection between G-orbits of X and X'. In particular $\varphi^{-1}(Z) = Z'$, where Z and Z' are the closed G-orbits of resp. X and X'. The crucial observation is that this bijection preserves the dimensions, another easy consequence of the structure of orbits of X.

Suppose that there exists $D \in \mathcal{D}(X)$ and call D' the closure in X' of $D \cap X_G^\circ$. Then $D \supset Z$, but $D' \not\supset Z'$. We have that $\varphi(D') = D$, which implies that $(\varphi|_{D'})^{-1}(Z) = \varphi^{-1}(Z) \cap D' = Z' \cap D'$ has some irreducible component with same dimension as Z. This is in contradiction with $D' \not\supset Z'$. □

Hence a spherical homogeneous space admits at most one wonderful embedding: namely its canonical one, in the case it exists and is smooth.

The question whether a spherical homogeneous space admits a wonderful embedding is not yet completely settled. We have already seen the necessary condition that $\mathrm{N}_G H/H$ be finite, but this is not sufficient (see Example 3.4.3).

A sufficient condition is for example that $\mathrm{N}_G H = H$, as shown by a deep theorem of Knop (see [K96]). We skip its precise statement (see [B09, Theorem 3.9])

3.4. **Subvarieties.** From now on we assume that X is the wonderful embedding of a spherical homogeneous space G/H, of rank r.

Let Y be a G-stable closed subvariety of X. Definition 3.3.1 implies that Y is the intersection of some of the prime divisors:

$$Y = \bigcap_{i \in I} X^{(i)}$$

for some subset $I \subseteq \{1, 2, \ldots, r\}$, and any choice of I will produce such a subvariety. It also follows immediately that Y is wonderful, of rank $r - |I|$.

In other words, we have seen in the proof of Proposition 3.3.1 that the prime divisors $X^{(i)}$ "intersect like" the coordinate hyperplanes in $\mathbb{C}^r$; any intersection of some of them gives a wonderful subvariety.

Another consequence of the local structure of X is that the B-weights in the statement of Theorem 3.2.1 can be chosen to be a basis of the lattice $\Lambda(X)$. This choice gives an important invariant of X: its *spherical roots*. The T-equivariant isomorphism $M \cong \mathbb{C}^r$ shows that they can also be defined in the following alternative way:

Definition 3.4.1. Let $z \in X$ be the unique point fixed by B_-; it lies on $Z = Gz$ the unique closed G-orbit. Then consider the vector space:

$$\mathrm{T}_z X / \mathrm{T}_z Z$$

which is naturally a T-module. Its T-weights are called *spherical roots* of X, and their set is denoted by Σ_X.

Spherical roots are in bijection both with 1-codimensional G-stable closed subvarieties, and with $(r-1)$-codimensional ones. Indeed for any $\sigma \in \Sigma_X$ one can define a correspondence:

$$\sigma \longleftrightarrow X^\sigma \longleftrightarrow X_\sigma$$

where X^σ is the wonderful subvariety of rank $r-1$ uniquely determined by $\Sigma_{X^\sigma} = \Sigma_X \setminus \{\sigma\}$, and X_σ is the rank 1 wonderful subvariety uniquely determined by $\Sigma_{X_\sigma} = \{\sigma\}$.

At this point the set of spherical roots of any wonderful subvariety Y of X is clear. From our analysis follows immediately also $\Lambda(Y)$, which is equal to $\mathrm{span}_{\mathbb{Z}}\Sigma_Y$, and $\mathcal{V}(Y)$ which is given by equations $\langle\cdot,\sigma\rangle \leq 0$ for all $\sigma\in\Sigma_Y$. It remains the set of colors, for which it holds:

Proposition 3.4.1. *Let Y be any wonderful subvariety of X, and $D\in\Delta_X$. Then $D\cap Y$ is the union of some colors of Y. Furthermore, any color of Y is some irreducible component of $D\cap Y$ for some $D\in\Delta_X$.*

Proof. We must prove that $D\cap Y$ has no G-stable irreducible component, but this follows from the fact that D doesn't contain the unique closed G-orbit $Z\subseteq Y\subseteq X$. The second part can be proven using the same approach as Corollary 2.2.2, and we leave the details to the Reader. □

It is possible to determine precisely the irreducible components of $D\cap Y$ and the induced functionals on $\Lambda(Y)$, but we do not state here a precise result. It would require Lemmas 3.6.1 and 3.6.2, and a further analysis of the relationship between colors and spherical roots. We refer to [BL08, Proposition 1.2.3] and [Lu97, §3.5].

We now give some examples of wonderful varieties, together with their colors and spherical roots.

Example 3.4.1. The wonderful varieties of rank 0 are exactly the complete homogeneous spaces G/P for P a parabolic subgroup of G, see Corollary 2.3.1. Their colors are the closures of the Bruhat cells of codimension 1 in G/P.

Example 3.4.2. The group $G=\mathrm{SL}_2$ admits exactly 4 wonderful G-varieties (see also [Br09, Example 2.11.2]). They are:

(1) $X=\{\text{a point}\}$ with trivial G-action.
(2) $X=G/B\cong\mathbb{P}^1$, the flag variety of SL_2.
(3) $X=\mathbb{P}^1\times\mathbb{P}^1$, which has spherical root α_1.
(4) $X=\mathbb{P}^2=\mathbb{P}(\mathrm{Sym}^2(\mathbb{C}^2))$ where SL_2 acts linearly on $\mathbb{C}^2$. This variety can also be seen as the quotient of $\mathbb{P}^1\times\mathbb{P}^1$ by $(p,q)\sim(q,p)$. It has spherical root $2\alpha_1$, and only one color D with $\langle\rho(D),2\alpha_1\rangle=2$.

Example 3.4.3. The symmetric homogeneous spaces $G/H=H\times H/\mathrm{diag}(H)$ for H semisimple adjoint admit a wonderful compactification X, as shown in [DP83]. Denote by $\alpha_1,\ldots,\alpha_n$, $\alpha'_1,\ldots,\alpha'_n$ the simple roots of the two copies of H. Then X has spherical roots $\sigma_i=\alpha_i+\alpha'_i$ for $i=1,\ldots,n$. There are n colors, and the values of their functionals on the spherical roots are given by the Cartan matrix of H.

For PSL_2 there is an elementary description of X as the projective space of 2×2 matrices, i.e.:

$$X=\mathbb{P}(\mathscr{M}_{2\times 2}(\mathbb{C})),$$

with the action of $\mathrm{PSL}_2\times\mathrm{PSL}_2$ by left and right multiplication. The group SL_2 admits a wonderful compactification too, and it's the only non-adjoint simple group with this property. The compactification is:

$$X=\left\{ad-bc=t^2\right\}\subset\mathbb{P}(\mathscr{M}_{2\times 2}\oplus\mathbb{C}).$$

where $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathscr{M}_{2\times 2}$, $t\in\mathbb{C}$. The spherical root in this case is $\frac{1}{2}(\alpha_1+\alpha'_1)$.

Example 3.4.4. Let $G=\mathrm{SL}_{n+1}\times\mathrm{SL}_{m+1}$, and choose the Borel subgoup B to be the set of couples of upper triangular matrices. Call $\alpha_1,\ldots,\alpha_n$ the simple roots of

SL_{n+1}, and $\alpha'_1, \ldots, \alpha'_m$ those of SL_{m+1}. Define the variety:

$$X = \left\{ (E, F, M) \;\middle|\; \begin{array}{l} E \in \mathrm{Gr}(2, \mathbb{C}^{n+1}), \\ F \in \mathrm{Gr}(2, \mathbb{C}^{m+1}), \\ p \in \mathbb{P}(\mathrm{Hom}(E, F)) \end{array} \right\}.$$

The group G has a natural action on X, which is obvious on the coordinates E and F, and on $p = [M]$ is given by:

$$(g_1, g_2)p = [(g_1, g_2)M] = [g_2 \circ M \circ g_1^{-1}]$$

where $M \in \mathrm{Hom}(E, F)$ and $(g_1, g_2) \in G$.

The variety X is wonderful of rank 1, and its G-stable prime divisor is:

$$X^{(1)} = \{\mathrm{rank}\ M = 1\}.$$

The spherical root is $\sigma_1 = \alpha_1 + \alpha'_1$, and there are three colors:

$$\begin{array}{rcl} D_1 & = & \overline{\{[M] \notin D_1 \cup D_2,\ \ \pi(M) \text{ is upper triangular}\}}; \\ D_2 & = & \big\{E \cap \mathrm{span}\{e_1, \ldots, e_{n-1}\} \neq \{0\}\big\}; \\ D_3 & = & \big\{F \cap \mathrm{span}\{e'_1, \ldots, e'_{m-1}\} \neq \{0\}\big\}. \end{array}$$

where $(e_1, \ldots, e_{n+1})$ and $(e'_1, \ldots, e'_{m+1})$ are the canonical bases of $\mathbb{C}^{n+1}$ and $\mathbb{C}^{m+1}$, and $\pi(M) \in \mathrm{Hom}(\mathrm{span}\{e_n, e_{n+1}\}, \mathrm{span}\{e'_m, e'_{m+1}\})$ is induced by M in an obvious way using the projections along $\mathrm{span}\{e_1, \ldots, e_{n-1}\}$ and $\mathrm{span}\{e'_1, \ldots, e'_{m-1}\}$. The values of $\rho(D_1), \rho(D_2)$ and $\rho(D_3)$ on σ_1 are resp. $2, -1, -1$.

This variety is obtained from the wonderful compactification above defined of $\mathrm{PSL}_2 \times \mathrm{PSL}_2/\mathrm{diag}(\mathrm{PSL}_2)$ via a procedure called *parabolic induction*.

Example 3.4.5. The variety X of complete conics is the wonderful compactification of the homogeneous space $\mathrm{SL}_3/\mathrm{SO}_3 C(\mathrm{SL}_3)$. It has rank 2 and can be described as follows:

$$X = \{([A], [B]) \in \mathbb{P}(\mathscr{M}_{3\times 3}) \times \mathbb{P}(\mathscr{M}_{3\times 3}) \mid AB \in \mathbb{C} \cdot 1_{3\times 3}\}.$$

where $1_{3\times 3}$ is the unit (3×3)-matrix. The two G-stable divisors are:

$$\begin{array}{rcl} X^{(1)} & = & \{A \text{ non-invertible}\}, \\ X^{(2)} & = & \{B \text{ non-invertible}\}. \end{array}$$

The spherical roots are $\sigma_1 = 2\alpha_1$ and $\sigma_2 = 2\alpha_2$.

Example 3.4.6. Let $G = \mathrm{Sp}_{2a} \times \mathrm{Sp}_{2b}$, and call Ω, Ω' the bilinear forms on $\mathbb{C}^{2a}$ and $\mathbb{C}^{2b}$ corresponding to Sp_{2a} and Sp_{2b}. Call $\alpha_1, \ldots, \alpha_a$ the simple roots of Sp_{2a}, and $\alpha'_1, \ldots, \alpha'_b$ those of Sp_{2b}. Consider the variety X of Example 3.4.4 for $n + 1 = 2a$, $m + 1 = 2b$: G has a natural action on X defined in the same way as in Example 3.4.4.

Under this action X is wonderful of rank 3, with G-stable prime divisors:

$$\begin{array}{rcl} X^{(1)} & = & \{\mathrm{rank}\ M = 1\}, \\ X^{(2)} & = & \{\Omega|_{E_1} = 0\}, \\ X^{(3)} & = & \{\Omega'|_{E_2} = 0\}. \end{array}$$

The three spherical roots are:

$$\begin{aligned}\sigma_1 &= \alpha_1 + \alpha_1',\\ \sigma_2 &= \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n,\\ \sigma_3 &= \alpha_1' + 2\alpha_2' + \ldots + 2\alpha_{m-1}' + \alpha_m'.\end{aligned}$$

The 3 colors D_1, D_2, D_3 are the same as before, and their induced functionals have the following values on the spherical roots:

	σ_1	σ_2	σ_3
$\rho(D_1)$	2	0	0
$\rho(D_2)$	-1	1	0
$\rho(D_3)$	-1	0	1

3.5. **Morphisms.** We have seen in §2.6 that colored subspaces of $N(G/H)$ correspond to inclusions $H \subseteq H'$ such that the quotient H'/H is connected. It is also clear that if G/H admits a canonical embedding, then G/H' has this property too, and from Theorem 2.6.1 it follows that $G/H \to G/H'$ extends to a G-equivariant surjective map between the two canonical embeddings.

Since in this case $\mathcal{V}(G/H)$ is strictly convex, it is possible to simplify the correspondence between morphisms and colored subspaces, in the sense that we can state Theorem 2.6.2 purely in terms of the functionals associated to colors.

Definition 3.5.1. Let Δ' be a subset of Δ_X. We say that Δ' is *distinguished* if there exists a linear combination η of $\rho(D)$ for $D \in \Delta'$ with positive integer coefficients, such that $\langle \eta, \sigma \rangle \geq 0$ for all $\sigma \in \Sigma_X$.

Theorem 3.5.1. *Let X be a fixed wonderful variety. The map $\varphi \mapsto \mathcal{D}_\varphi$ (in the notations of §2.6) induces a bijection between G-equivariant surjective maps $\varphi\colon X \to X'$ (up to G-equivariant isomorphisms of X') with connected fibers where X' is the canonical embedding of some spherical homogeneous space G/H', and distinguished subsets of Δ_X.*

Proof. Consider a distinguished subset $\Delta' \subseteq \Delta_X$, and all possible linear combinations η as in the above Definition. For each such η we have $-\eta \in \mathcal{V}(G/H)$, and we can consider the face F of $\mathcal{V}(G/H)$ of maximal dimension whose relative interior contains some $-\eta$. If we let $\mathcal{C}$ be the subspace spanned by F and $\rho(\Delta')$, then $(\mathcal{C}, \Delta')$ is a colored subspace.

Viceversa, any map φ gives rise to a distinguished subset of colors $\mathcal{D}_\varphi$, and from the strict convexity of $\mathcal{V}(G/H)$ we deduce that if $\mathcal{D}_\varphi = \mathcal{D}_\psi$ for some other map ψ, then also $\mathcal{C}_\varphi = \mathcal{C}_\psi$. □

From §2.6 we know that the colors of X' are in bijection with $\Delta_X \setminus \Delta'$, and that:

$$N(G/H') = \frac{N(G/H)}{\mathcal{C}_\varphi}, \quad \Lambda(G/H') = \left\{\gamma \in \Lambda(G/H) \mid \langle \eta, \gamma \rangle = 0 \ \forall \eta \in \mathcal{C}_\varphi \right\}.$$

Also, the cone $\mathcal{V}(G/H')$ is defined inside $N(G/H')$ by equations which are in the monoid:

$$\left\{ \sum_{\sigma \in \Sigma_X} n(\sigma)\sigma \;\middle|\; n(\sigma) \in \mathbb{Z}_{\geq 0} \right\} \cap \Lambda(G/H').$$

The irreducible elements of the above monoid are a basis of $\Lambda(G/H')$ if and only if X' is smooth, i.e. if it's wonderful. We say in this case that Δ' is (*)-distinguished.

It is interesting to remark that no example is known of a distinguished but not (*)-distinguished subset of colors, and a conjecture of Luna states that none exists in general.

3.6. **The Cartan pairing of a wonderful variety.** The last topic of these notes is towards the classification of wonderful varieties. The invariants we have seen so far, namely the spherical roots and the colors, can be used to define some combinatorial object associated to a wonderful variety, called its *spherical system*. Luna conjectured in [Lu01] that these spherical systems classify all wonderful varieties for adjoint groups. The conjecture has been verified in many particular cases, and Losev also proved in [L07] the so called "uniqueness part", i.e. that these invariants separate wonderful varieties.

Luna also showed in [Lu01] that a proof of this conjecture would lead to a complete classification of spherical homogeneous spaces.

We explain now why the behaviour of spherical roots and colors has a "discrete" nature, and thus can be represented by combinatorial objects. The first observation is that for any G, the set of all possible spherical roots of wonderful G-varieties is finite and known, thanks to the classification of wonderful varieties of rank 1 in [A83], [HS82], [Br89].

Then, the functionals associated to colors do respect some rather strict conditions. We mention that the map ρ_X on colors can also be represented as a pairing:

$$c_X\colon \mathrm{span}_{\mathbb{Z}}\Delta_X \times \mathrm{span}_{\mathbb{Z}}\Sigma_X \to \mathbb{Z}$$

and this is called the *Cartan pairing* of X. The definition is motivated by the fact that for the wonderful compactification of an adjoint group G this pairing is represented by the Cartan matrix of G. We point out that Σ_X is always the set of simple roots of some root system (see [Br90] and [K96]), whereas the set of colors behaves in general in a much more complicated way than the coroots.

Definition 3.6.1. Let X be a wonderful G-variety, $D \in \Delta_X$, and α a simple root of G. Then we say that α *moves* D if $P_\alpha D \neq D$ where P_α is the minimal parabolic subgroup of G containing B and associated to α. We define $\Delta(\alpha)$ to be the set of colors moved by α.

Lemma 3.6.1. *For any simple root α, the set $\Delta(\alpha)$ has cardinality at most 2. More precisely, only one of the following cases can occur:*

(1) $\Delta(\alpha) = \varnothing$. *In this case α is among the simple roots associated to the stabilizer in G of X°_B.*
(2) $\Delta(\alpha) = \{D\}$ *and no multiple of α is in Σ_X. In this case:*
$$\rho(D) = \alpha^\vee|_{\Lambda(X)}.$$
(3) $\Delta(\alpha) = \{D\}$ *and some multiple of α is in Σ_X. In this case $2\alpha \in \Sigma_X$, and:*
$$\rho(D) = \frac{1}{2}\alpha^\vee|_{\Lambda(X)}.$$
(4) $\Delta(\alpha) = \{D^+, D^-\}$. *This case is equivalent to $\alpha \in \Sigma_X$, and we have:*
$$\rho(D^+) + \rho(D^-) = \alpha^\vee|_{\Lambda(X)}.$$

Proof. See [Lu97, §3.2 and §3.4]. □

Lemma 3.6.2. *Let α, β be simple roots. If $\alpha \perp \beta$ and $\alpha+\beta$ or $\frac{1}{2}(\alpha+\beta)$ is in Σ_X, then $\Delta(\alpha) = \Delta(\beta) = \{D\}$ and case (2) of the above Lemma occurs for both α and β. Otherwise, if α and β move the same color then $\alpha, \beta \in \Sigma_X$ and $|\Delta(\alpha) \cup \Delta(\beta)| = 3$.*

Proof. See [Lu01, Proposition 3.2]. □

The two above Lemmas can be proved using the technique of *localization* and then the known classifications of wonderful varieties of rank 1 and 2.

They are a crucial tool to reduce the combinatorial complexity of our invariants. Indeed, as soon as a color D is moved by simple roots which are not spherical roots, then its associated functional is immediately determined by Σ_X and the simple root(s) moving D. This almost allows us to "forget" about these colors, and focus only on the pairs of colors $\Delta(\alpha) = \{D^+, D^-\}$ where $\alpha \in \Sigma_X \cap S$.

So, in the end, the only relevant invariants are these particular colors, the set Σ_X, and the set of simple roots moving no color. These elements form the so-called *spherical system* of a wonderful variety.

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Departement Mathematik, Universität Erlangen-Nürnberg, Bismarckstrasse 1 $\frac{1}{2}$, 91054 Erlangen, Deutschland
E-mail address: pezzini@mi.uni-erlangen.de