AUTOMORPHISMS OF WONDERFUL VARIETIES

GUIDO PEZZINI

Abstract. Let G be a complex semisimple linear algebraic group, and X a wonderful G-variety. We determine the connected automorphism group $\mathrm{Aut}^0(X)$ and we calculate Luna's invariants of X under its action.

1. Introduction

Let X be a smooth complete algebraic variety over $\mathbb{C}$, and let $\mathrm{Aut}^0(X)$ be the connected component containing the identity of its automorphism group. It is well known that $\mathrm{Aut}^0(X)$ is an algebraic group, with Lie algebra equal to the global sections of the tangent bundle of X.

Let us consider this group in the framework of varieties endowed with an action of a connected reductive algebraic group G. The simplest case occurs when the action of G on X is transitive, i.e. we have a projective homogeneous space $X = G/P$ for P a parabolic subgroup of G. In this setting $\mathrm{Aut}^0(X)$ is known, and it is always semisimple; it is interesting to notice that in a few cases it strictly contains the image of G under the given homomorphism $G \to \mathrm{Aut}^0(X)$.

As soon as we abandon the transitivity hypothesis the situation gets much more complicated, even if we stick to quasi-homogeneous varieties, which means X having an open dense G-orbit. For example, in general $\mathrm{Aut}^0(X)$ need not be reductive. The case of *toric varieties* is known since the work [De70] by Demazure: here $\mathrm{Aut}^0(X)$ is completely determined using the fan of convex cones representing the toric variety X. The case of *regular varieties* is studied by Bien and Brion in [BB96]; one of their results is the structure of the Lie algebra of $\mathrm{Aut}^0(X)$ as a G-module.

More recently, Brion has considered the case of *wonderful varieties*, which are regular varieties with some extra hypotheses. In [Br07] he shows that $\mathrm{Aut}^0(X)$ is always semisimple in this case, that X is wonderful under its action too, and describes some aspects of the action of $\mathrm{Aut}^0(X)$ on X. These results have also useful consequences on the broader class of *spherical varieties*.

In this article we determine the automorphism groups $\mathrm{Aut}^0(X)$ for any wonderful variety X. In particular we show that "most often" the image of G is equal to the whole $\mathrm{Aut}^0(X)$, especially if the number of G-orbits on X is greater than 2. Our approach to describe these varieties uses their discrete invariants introduced by Luna in [Lu01]: they separate wonderful varieties as shown by Losev in [Lo07].

We describe in details all varieties such that $\mathrm{Aut}^0(X)$ strictly contains the image of G, and we determine Luna's invariants of X under the action of $\mathrm{Aut}^0(X)$.

Date: June 31, 2008.
2000 *Mathematics Subject Classification.* 14J50, 14L30, 14M17.

Acknowledgements. I would like to thank the anonymous referees, for many useful remarks and for pointing out an error in the last section of a previous version of this paper.

2. Invariants of wonderful varieties

2.1. **Definitions.** Throughout this paper, G will be a semisimple connected linear algebraic group over $\mathbb{C}$. In G we fix a Borel subgroup B, a maximal torus $T \subset B$, and we denote by S the corresponding set of simple roots. For simple groups, we will refer to the usual Bourbaki numbering of simple roots.

We will often denote by L a Levi subgroup of some parabolic subgroup $P \supseteq B$ of G. The choice of L is always supposed to be such that $B \cap L$ and $T \cap L$ are resp. a Borel subgroup and a maximal torus of L: in this way the simple roots of L with respect to $B \cap L$ and $T \cap L$ are naturally identified with a subset of S.

More generally, whenever we have two reductive groups $\tilde{G} \supset G$, the choices of Borel subgroups $\tilde{B}$, B and maximal tori $\tilde{T}$, T will always be such that $\tilde{B} \supset B$ and $\tilde{T} \supset T$.

Definition 2.1.1. [Lu01] A *wonderful G-variety* is an irreducible algebraic variety X over $\mathbb{C}$ such that:

(1) X is smooth and projective;
(2) G has an open (dense) orbit on X, and the complement is the union of (G-stable) prime divisors $X^{(1)}, \ldots, X^{(r)}$ which are smooth, with normal crossings, and satisfy $\bigcap_{i=1}^{r} X^{(i)} \neq \emptyset$;
(3) if $x, y \in X$ are such that $\left\{i \mid x \in X^{(i)}\right\} = \left\{j \mid y \in X^{(j)}\right\}$, then x and y lie on the same G-orbit.

We will also use the notation (G, X) instead of X only, to keep track of the group G we are considering. The number r of G-stable prime divisors is the *rank* of (G, X), and the divisors $X^{(i)}$ are called *boundary prime divisors.* Their union is denoted by $\partial(G, X)$, the *boundary* of (G, X).

Wonderful varieties include complete G-homogeneous spaces G/P, for P a parabolic subgroup, as the special case where the rank is zero. A wonderful variety X is always spherical, i.e. a Borel subgroup has an open dense orbit on X: see [Lu96]. The theory developed by Luna in [Lu01] defines the following "discrete invariants":

Definition 2.1.2. Let X be a wonderful G-variety. We define $\Xi(G, X)$ to be the lattice of B-eigenvalues (B-*weights*) of rational functions on X that are B-eigenvectors. This lattice has a basis $\Sigma(G, X)$, whose elements are called the *spherical roots* of X, defined as the set of weights of T acting on the quotient tangent space:

$$\frac{T_z X}{T_z(G \cdot z)}$$

where z is the unique point of X fixed by the Borel subgroup opposite of B with respect to T. We define $\Delta(G, X)$ to be the set of B-stable but not G-stable prime divisors on X, called *colors.* It is a finite set, and it is equipped with a map:

$$\rho_{G,X} \colon \Delta(G, X) \to \mathrm{Hom}_{\mathbb{Z}}(\Xi(G, X), \mathbb{Z})$$

defined as follows: if D is a color then $\rho_{G,X}(D)$ is a functional on $\Xi(G, X)$ taking on γ the value $\nu_D(f_\gamma)$, where ν_D is the discrete valuation on $\mathbb{C}(X)^*$ associated to D, and $f_\gamma \in \mathbb{C}(X)^*$ is a B-eigenvector whose B-eigenvalue is γ. This notion is well defined because B has an open orbit on X and therefore f_γ is unique up to the multiplication by a constant. It is also common to write this as a coupling: $\langle D, \gamma\rangle = \rho_{G,X}(D)(\gamma)$. Finally, we define $S^p(G,X)$ to be the set of simple roots associated to the parabolic subgroup $P(G,X) \supseteq B$ defined as the stabilizer of the open B-orbit of X.

It is immediate from the definitions that any intersection of n boundary prime divisors of X is again a wonderful variety, of rank $r-n$ (for any $n = 0, \dots, r$). Moreover, each spherical root γ_i can be naturally associated to a boundary prime divisor $X^{(i)}$ on X, namely the one such that the T-weights of the quotient:

$$\frac{T_z X^{(i)}}{T_z(G \cdot z)}$$

are precisely $\Sigma(G,X) \setminus \{\gamma_i\}$. A spherical root γ_i can also be associated to a rank 1 wonderful G-subvariety on X, namely:

$$X_{(i)} = \bigcap_{j \neq i} X^{(j)}.$$

This $X_{(i)}$ has γ_i as its unique spherical root.

Finally, γ_i is always a linear combination of simple roots of G with non-negative coefficients; the set of simple roots whose coefficient is non-zero is called the *support* of γ_i. The *support* of a set of spherical roots is defined as the union of the supports of its elements.

It is worth noticing that these invariants obey some rather strict conditions of combinatorial nature, as discussed by Luna in [Lu01]. In the rest of this paper we will sometimes use these conditions, although it will not be necessary to recall all the combinatorics that arises from the theory.

The last ingredient we need here is the following relation between colors and simple roots:

Definition 2.1.3. Let X be a wonderful G-variety, D one of its colors and α a simple root of G. We say that α *moves* D if D is not stable under the action of the minimal parabolic subgroup containing B and associated to the simple root α.

With this definition, $S^p(G,X)$ is precisely the set of simple roots moving no color.

Lemma 2.1.1. [Lu01] *Let α be a simple root moving a color D of a wonderful G-variety X. Suppose that α is not contained in the support of any spherical root of X. Then D is moved only by α, and α moves only D; moreover, the functional $\rho_{G,X}(D)$ is equal to $\alpha^\vee|_{\Xi(G,X)}$, where $\alpha^\vee$ is the coroot associated to α.*

Other links between functionals associated to colors and simple roots moving them will be found in section 4.

Definition 2.1.4. Let X be a wonderful G-variety. We denote by $\mathbb{A}(G,X)$ the set of colors of X moved by simple roots which are also spherical roots, and we call the triple $(S^p(G,X), \Sigma(G,X), \mathbb{A}(G,X))$ the *spherical system* of X.

Results in Losev's paper [Lo07] directly imply the following:

Theorem 2.1.1. [Lo07] *If two wonderful G-varieties X_1, X_2 have the same spherical system (where $\mathbb{A}(G,X_i)$ is considered just as an abstract finite set endowed with the application $\rho_{G,X_i}|_{\mathbb{A}(G,X_i)}$, $i = 1,2$), then they are G-equivariantly isomorphic.*

2.2. **Subvarieties, products and parabolic inductions.** Definitions and results in this subsection are a part of Luna's theory developed in [Lu01]: we will omit here all the proofs.

Let X be a wonderful G-variety of rank r, and consider a G-stable irreducible closed subvariety Y of codimension k. Any such Y is always wonderful, and equal to the intersection of k border prime divisors of X:

$$Y = \bigcap_{j=1}^{k} X^{(i_j)}$$

for distinct $i_1, \ldots, i_k$ in $\{1, \ldots, r\}$. We can describe the spherical system of Y in terms of that of X. First of all, $S^p(G,Y) = S^p(G,X)$. Then, the spherical roots $\Sigma(G,Y)$ are exactly $\Sigma(G,X) \setminus \{\gamma_{i_1}, \ldots, \gamma_{i_k}\}$. Finally, the set $\mathbb{A}(G,Y)$ is equal to $\mathbb{A}(G,X)$ minus all colors moved only by simple roots in $\Sigma(G,X) \setminus \Sigma(G,Y)$.

Definition 2.2.1. A wonderful G-variety X is a *product* if $G = G_1 \times G_2$ and $X = X_1 \times X_2$ where X_i is a wonderful G_i-variety for $i = 1, 2$. If X is not a product, we say it is *indecomposable*.

A wonderful variety is a product exactly when its associated data is a product, in the following sense:

Definition 2.2.2. The spherical system $(S^p(G,X), \Sigma(G,X), \mathbb{A}(G,X))$ is a *product* if $G = G_1 \times G_2$, correspondingly $S = S_1 \cup S_2$ with $S_1 \perp S_2$, and we have:

- $S^p(G,X) = S^p_1 \cup S^p_2$,
- $\Sigma(G,X) = \Sigma_1 \cup \Sigma_2$,
- $\mathbb{A}(G,X) = \mathbb{A}_1 \cup \mathbb{A}_2$,
- for all $D \in \mathbb{A}_1$, $\rho_{G,X}(D)$ is zero on Σ_2 and for all $D \in \mathbb{A}_2$, $\rho_{G,X}(D)$ is zero on Σ_1;

where we define: $S^p_i = S^p(G,X) \cap S_i$, $\Sigma_i = \{\gamma \in \Sigma(G,X) \mid \operatorname{supp}\gamma \subseteq S_i\}$, and $\mathbb{A}_i$ is the set of colors in $\mathbb{A}(G,X)$ moved only by simple roots in S_i $(i = 1, 2)$.

A very simple example of product is where some factor has rank 0; about this case, definition 2.2.2 immediately implies the following:

Lemma 2.2.1. *Let X be a wonderful G-variety, and suppose that $G = G_1 \times G_2$. Then X is a product $X_1 \times X_2$ where X_1 is a rank 0 wonderful G_1-variety if and only if* $\operatorname{supp}\Sigma(G,X)$ *doesn't contain any simple root of G_1.*

Let X be a wonderful G-variety, and suppose that the stabilizer H of a point in the open G-orbit is such that $R(Q) \subseteq H \subseteq Q$ for some proper parabolic subgroup Q of G, where $R(Q)$ is the radical of Q. Then X is isomorphic to $G \times_Q Y$ where Y is a Q-variety where the radical $R(Q)$ acts trivially. Moreover, Y turns out to be wonderful under the action of $Q/R(Q)$, thus also under the action of L a Levi subgroup of Q. Here $G \times_Q Y$ is defined as the quotient $(G \times Y)/\sim$ where $(g,x) \sim (gq, q^{-1}x)$ for all $q \in Q$.

Definition 2.2.3. Such a wonderful variety $X \cong G \times_Q Y$ is said to be a *parabolic induction* of Y by means of Q. A wonderful variety which is not a parabolic induction is said to be *cuspidal*.

On the combinatorial side, this corresponds to the following:

Definition 2.2.4. The spherical system $(S^p(G,X), \Sigma(G,X), \mathbb{A}(G,X))$ is said to be *cuspidal* if $\operatorname{supp}\Sigma(G,X) \cup S^p(G,X) = S$.

If X has no rank 0 factor then it is cuspidal if and only if $\operatorname{supp}\Sigma(G,X) = S$.

Non-cuspidal wonderful varieties are often "ignored" due to the fact that the G-action on X is completely determined by the L-action on Y, and the spherical system $(S^p(G,X), \Sigma(G,X), \mathbb{A}(G,X))$ is equal to $(S^p(L,Y), \Sigma(L,Y), \mathbb{A}(L,Y))$, of course up to the identification of the simple roots of L with a subset of S.

On the other hand, the whole sets of colors of X and Y are different: the set $\Delta(L,Y)$ is in natural bijection with a proper subset of $\Delta(G,X)$. The remaining colors of X are in bijection with the simple roots of G which are not simple roots of L: any such simple root α is associated to a color D of X, in such a way that α and D behave as in lemma 2.1.1.

In this article we won't leave aside non-cuspidal varieties: the automorphism groups $\operatorname{Aut}^0(X)$ and $\operatorname{Aut}^0(Y)$ may behave quite differently.

3. Automorphism groups

3.1. **Main theorem.** In his article [Br07], Brion proves a number of results about the automorphism groups of wonderful varieties. In particular:

Theorem 3.1.1. [Br07] *Let X be a wonderful G-variety, and $\tilde{G}$ be any closed connected subgroup of* $\operatorname{Aut}^0(X)$ *containing the image of G. Then:*

1. *$\tilde{G}$ is a semisimple group, and X is wonderful under its action;*
2. *the colors of X under the action of $\tilde{G}$ and under the image of G are the same (if we fix in $\tilde{G}$ a Borel subgroup containing the image of B);*
3. *the boundary prime divisors of X under the action of the full group* $\operatorname{Aut}^0(X)$ *are precisely the* fixed divisors, *i.e. by definition those boundary prime divisors $X^{(i)}$ (under the action of G) such that $\langle D, \gamma_i \rangle < 0$ for some color D.*

The term *fixed prime divisor* should not be confused with the notion of a color moved by a simple root. The latter regards B-stable prime divisors that lie on (more precisely, intersect) the open G-orbit of X. Instead, the fixed prime divisors are always contained in the boundary: they are exactly the divisors that are stable under any group of automorphisms of X containing G.

Here is our main result:

Theorem 3.1.2. *Let X be a wonderful G-variety, and let us suppose that G acts faithfully. Then* $\operatorname{Aut}^0(X) = G$*, except for the indecomposable varieties listed in subsections 3.2, 3.4, 3.5, 3.6, and products of wonderful varieties involving at least one of such cases.*

The theorem follows from corollary 3.3.1, and subsections 3.2, 3.4, 3.5, 3.6. In these subsections we also determine the boundary and the invariants of X as a wonderful variety with respect to the action of $\operatorname{Aut}^0(X)$, for all X where $\operatorname{Aut}^0(X) \neq G$. For many of these varieties, explicit geometrical descriptions are provided.

The theorem can also be "a posteriori" reformulated in a different form, using certain smooth morphisms with connected fibers between wonderful varieties. This reformulation (theorem 4.2.1) has a nicer statement, evidencing the fact that wonderful varieties of rank 1 play a crucial role whenever we have an automorphism group bigger than G.

On the other hand, apparently it cannot be proven without appealing to theorem 3.1.2; this will be investigated in section 4.

Let us make a last remark about parabolic inductions and full automorphism groups: the following lemma will be useful in subsequent proofs.

Lemma 3.1.1. *Let (G, X) be a wonderful variety, and suppose that it is a parabolic induction obtained from a cuspidal one, call it (L, Y). Let γ be a spherical root of (L, Y), let α be a simple root of G but not of L, and suppose that α is non-orthogonal to some simple root of $\operatorname{supp}\gamma$. Then the border prime divisor of (G, X) associated to γ is fixed.*

Proof. The spherical root γ is a linear combination of the simple roots of its support, with positive coefficients. On the other hand, α is associated to some color D of (G, X), in such a way that $\rho_X(D) = \alpha^\vee$: see the end of subsection 2.2. This implies that D is non-positive on any simple root, and hence it is strictly negative on γ. □

3.2. **Rank 0.** Let X be a wonderful G-variety of rank 0, i.e. a homogeneous space G/P where P is a parabolic subgroup of G. Let us also suppose that G acts faithfully, hence in particular G is adjoint. In [De77] Demazure shows that $\mathrm{Aut}^0(G/P)$ is always equal to G, except for a few cases called *exceptional*. If G is simple, the only exceptional cases are:

- $\mathbf{1}_{rk=0}$ $G = \mathsf{PSp}_{2n}$ $(n \geq 2)$, $P =$ the stabilizer of a point of $\mathbb{P}^{2n-1}$: here $\mathrm{Aut}^0(X) = \mathsf{PSL}_{2n}$ and $G/P \cong \mathrm{Aut}^0(X)/P' \cong \mathbb{P}^{2n-1}$ where P' is the stabilizer in $\mathrm{Aut}^0(X)$ of a point of $\mathbb{P}^{2n-1}$;
- $\mathbf{2}_{rk=0}$ $G = \mathsf{PSO}_{2n+1}$ $(n \geq 2)$, $P =$ the stabilizer of an isotropic n-dimensional subspace of $\mathbb{C}^{2n+1}$: if we choose a suitable extension to $\mathbb{C}^{2n+2}$ of the symmetric bilinear form defining PSO_{2n+1}, then $\mathrm{Aut}^0(X) = \mathsf{PSO}_{2n+2}$ and P'=the stabilizer of an isotropic $(n+1)$-dimensional subspace of $\mathbb{C}^{2n+2}$; the quotients $\mathrm{Aut}^0(X)/P'$ and G/P are isomorphic via $F \mapsto F \cap \mathbb{C}^{2n+1}$;
- $\mathbf{3}_{rk=0}$ $G = \mathsf{G}_2$, $P =$ the stabilizer of $[v] \in \mathbb{P}(V)$, where v is a primitive vector in the irreducible 7-dimensional G_2-module V; the latter can be seen as the complexification of the real vector space of pure octonions: here $\mathrm{Aut}^0(X) = \mathsf{PSO}_7$ and it stabilizes the norm of octonions.

If G is not simple, then the exceptional cases occur exactly when one of the simple components G_i of G (and the corresponding parabolic subgroup P_i of G_i) appears as one of the three cases above.

Now let $X = G/P$ be exceptional, with G simple: X is a wonderful variety of rank zero both under $\mathrm{Aut}^0(X)$ and G. Hence $\Xi(G, X) = \Xi(\mathrm{Aut}^0(X), X) = \{0\}$ and $\Sigma(G, X) = \Sigma(\mathrm{Aut}^0(X), X) = \emptyset$, thus $\mathbb{A}(G, X) = \mathbb{A}(\mathrm{Aut}^0(X), X) = \emptyset$; the whole set of colors $\Delta(\mathrm{Aut}^0(X), X)$ is of course in bijection with $S \setminus S^p(\mathrm{Aut}^0(X), X)$. The only invariant to be calculated is $S^p(\mathrm{Aut}^0(X), X)$:

$\mathbf{1}_{rk=0}$	$G = \mathsf{PSp}_{2n}$ $(n \geq 2)$	$S^p(G, X) = \{\alpha_2, \ldots, \alpha_n\}$
	$\mathrm{Aut}^0(X) = \mathsf{PSL}_{2n}$	$S^p(\mathrm{Aut}^0(X), X) = \{\alpha_2, \ldots, \alpha_{2n-1}\}$
$\mathbf{2}_{rk=0}$	$G = \mathsf{PSO}_{2n+1}$ $(n \geq 2)$	$S^p(G, X) = \{\alpha_1, \ldots, \alpha_{n-1}\}$
	$\mathrm{Aut}^0(X) = \mathsf{PSO}_{2n+2}$	$S^p(\mathrm{Aut}^0(X), X) = \{\alpha_1, \ldots, \alpha_n\}$
$\mathbf{3}_{rk=0}$	$G = \mathsf{G}_2$	$S^p(G, X) = \{\alpha_2\}$
	$\mathrm{Aut}^0(X) = \mathsf{PSO}_7$	$S^p(\mathrm{Aut}^0(X), X) = \{\alpha_2, \alpha_3\}$

It will be useful to notice that in all three cases $S^p(G, X)$ contains all simple roots of G except for only one.

3.3. **Borders and automorphisms.** It is natural to ask how "strong" is the border as an invariant to study the full automorphism group. The rank 0 exceptional varieties show that we can have different groups acting with same border (empty in this case), and thus same orbits, on the same wonderful variety. This can happen in higher rank too, as we will see in this subsection; however as soon as one of the groups is $\mathrm{Aut}^0(X)$ the picture is quite simple.

In the proof of the following useful proposition, we will refer to the classification of cuspidal indecomposable rank 1 ([Ah83], [HS82], [Br89]) and rank 2 wonderful varieties ([Wa96]). A list of all of them, including their invariants, can be found in [Wa96].

Proposition 3.3.1. *Let X be a wonderful G-variety where G acts faithfully, and suppose $\partial(G, X) = \partial(\mathrm{Aut}^0(X), X)$. Then either $G = \mathrm{Aut}^0(X)$, or X is a product where at least one of the factors is one of the exceptional rank 0 cases listed in subsection 3.2.*

Proof. The case of rank 0 is already done in subsection 3.2. So we suppose X of rank at least 1, and let us suppose that $G \neq \mathrm{Aut}^0(X)$.

The closed G-orbit Z of X is the intersection of all G-boundary divisors. Since $\partial(G, X) = \partial(\mathrm{Aut}^0(X), X)$, Z is also the closed $\mathrm{Aut}^0(X)$-orbit. The adjoint groups of G and of $\mathrm{Aut}^0(X)$ are obviously different, and it is not difficult to see that they both act faithfully on Z.

In other words (G, Z) is exceptional in the sense of subsection 3.2. So either G is simple and (G, Z) is $\mathbf{1}_{rk=0}$, $\mathbf{2}_{rk=0}$ or $\mathbf{3}_{rk=0}$, or G splits into a product $G = G_1 \times G_2$, and correspondingly $Z = Z_1 \times Z_2$, where (G_1, Z_1) is one of the three cases above (actually, the adjoint group of G_1, but this is irrelevant in what follows). For simplicity, let us say that G always splits in this way, with G_2 possibly trivial.

If (G, X) itself splits as a product $(G_1, X_1) \times (G_2, X_2)$, where (G_1, X_1) has rank zero, then $X_1 = Z_1$ and we are done: (G_1, X_1) will be an exceptional rank 0 factor of X.

Let us suppose that (G, X) doesn't split as such a product: our aim is to show that this leads to an absurd. The idea is that the closed orbit (G, Z), with the exceptional factor (G_1, Z_1), forces which spherical roots are allowed to have support on simple roots of G_1. As a consequence, very strong conditions on the spherical system are imposed, leading to a contradiction to the fact that $\partial(G, X) = \partial(\mathrm{Aut}^0(X), X)$.

First of all, thanks to lemma 2.2.1, there exists at least one spherical root, call it γ_1, whose support contains some simple root of G_1. Recall its associated rank 1 wonderful G-subvariety:

$$X_{(1)} = \bigcap_{j \neq 1} X^{(j)}.$$

It has only one spherical root, namely γ_1, and of course the same closed orbit Z as X.

Like any wonderful variety of rank 1, $(G, X_{(1)})$ can be obtained from a cuspidal, indecomposable rank 1 variety $(L, X'_{(1)})$ using parabolic inductions and products by rank 0 wonderful varieties.

Our first claim is that L contains the factor G_1, from which it follows that Z_1 is also a factor of Z', the closed orbit of $X'_{(1)}$. To prove the claim, observe that

$\operatorname{supp}\gamma_1$ contains some simple root of G_1: this shows that G_1 intersects non-trivially L.

Moreover, $S^p(G_1, Z_1)$ (and thus $S^p(G, X)$) contains all simple roots of G_1 except one, call it α. We already know that the set S^p can never contain a connected component of the support of a spherical root, this follows at once from the classification of rank 1 wonderful varieties. We conclude that $\alpha \in \operatorname{supp}\gamma_1$.

If L doesn't contain G_1, then there is some simple root of G_1 moving some color and outside the support of any spherical root, as described by the behaviour of colors under parabolic induction. This is an absurd, so we have proven our first claim.

Now we look at the list in [Wa96] of cuspidal indecomposable rank 1 wonderful varieties. Our $(L, X'_{(1)})$ must appear there, with closed orbit (L, Z'), and where the latter has the exceptional factor (G_1, Z_1) equal to $\mathbf{1}_{rk=0}$, $\mathbf{2}_{rk=0}$ or $\mathbf{3}_{rk=0}$. With these conditions, the only candidates for $(L, X'_{(1)})$ are cases 10, 11, 13, 14 of [Wa96]. More precisely, cases 10 or 11 could occur if $(G_1, Z_1) = \mathbf{2}_{rk=0}$, and cases 13 or 14 if $(G_1, Z_1) = \mathbf{3}_{rk=0}$. No case could occur if $(G_1, Z_1) = \mathbf{1}_{rk=0}$. Notice that for all these four candidates we have $G_1 = L$.

Our second claim is that $(L, X'_{(1)}) = (G_1, X'_{(1)})$ is a factor of (G, X). This follows immediately from the classification in rank 2 of [Wa96]: it shows that the rank 1 cases 10, 11, 13, 14 can appear inside some higher rank variety only if they are factors of it.

Finally, in all the cases 10, 11, 13, 14 the border divisor is not fixed. But $(G_1, X'_{(1)})$ is a factor of (G, X), so at least one of the border divisors of (G, X) is not fixed: this is absurd since we were supposing that $\partial(G, X) = \partial(\mathrm{Aut}^0(X), X)$. □

As we said, the proposition fails if we replace $\mathrm{Aut}^0(X)$ with some connected group $\tilde{G}$ strictly between $\mathrm{Aut}^0(X)$ and G: the above proof suggests the rank 1 cases 10, 11, 13, 14 of [Wa96] as counterexamples.

If (G, X) is one of them, then X is homogeneous under $\mathrm{Aut}^0(X)$, but there exists an intermediate connected group $G \subset \tilde{G} \subset \mathrm{Aut}^0(X)$ such that X is wonderful both under G and under $\tilde{G}$ with the same orbits. However, a proof similar to proposition 3.3.1 shows that these are the only indecomposable wonderful varieties (regardless of the rank, provided it is ≥ 1) having this property.

Let us see these cases in detail. For (G, X) equal to cases 10, 11, 13, 14, we have that $(\tilde{G}, X)$ is equal resp. to cases 5D, 6D, 7B, 8B. More precisely:

11-6D : $X = \mathbb{P}^7$, $\partial(G, X) = \partial(\tilde{G}, X)$ is a smooth quadric, and:

$$\begin{aligned} G &= \mathsf{PSO}_7 & \tilde{G} &= \mathsf{PSO}_8 \\ S^p(G, X) &= \{\alpha_1, \alpha_2\} & S^p(\tilde{G}, X) &= \{\alpha_2, \alpha_3, \alpha_4\} \\ \Sigma(G, X) &= \{\alpha_1 + 2\alpha_2 + 3\alpha_3\} & \Sigma(\tilde{G}, X) &= \{2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\} \\ \mathbb{A}(G, X) &= \emptyset & \mathbb{A}(\tilde{G}, X) &= \emptyset \end{aligned}$$

14-8B : $X = \mathbb{P}^6$, $\partial(G, X) = \partial(\tilde{G}, X)$ is a smooth quadric, and:

$$\begin{aligned} G &= \mathsf{G}_2 & \tilde{G} &= \mathsf{PSO}_7 \\ S^p(G, X) &= \{\alpha_2\} & S^p(\tilde{G}, X) &= \{\alpha_2, \alpha_3\} \\ \Sigma(G, X) &= \{4\alpha_1 + 2\alpha_2\} & \Sigma(\tilde{G}, X) &= \{2\alpha_1 + 2\alpha_2 + 2\alpha_3\} \\ \mathbb{A}(G, X) &= \emptyset & \mathbb{A}(\tilde{G}, X) &= \emptyset \end{aligned}$$

Case 10-5D (resp. 13-7B) is a smooth quadric of dimension 7 (resp. 6), a 2:1 cover of case 11-6D (resp. 14-8B). The group $\tilde{G}$ is SO_8 (resp. PSO_7) and all the invariants are the same of case 11-6D (resp. 14-8B), except that the spherical root is always one half of the spherical root of case 11-6D (resp. 14-8B), under both the actions of G and $\tilde{G}$.

Corollary 3.3.1. *If a wonderful variety (G, X) is a product of two wonderful varieties (G_1, X_1), (G_2, X_2), then $\mathrm{Aut}^0(X) = \mathrm{Aut}^0(X_1) \times \mathrm{Aut}^0(X_2)$.*

Proof. The wonderful variety $(\mathrm{Aut}^0(X_1) \times \mathrm{Aut}^0(X_2), X)$ is of course the product of $(\mathrm{Aut}^0(X_1), X_1)$ and $(\mathrm{Aut}^0(X_2), X_2)$. All boundary prime divisors of $(\mathrm{Aut}^0(X_1), X_1)$ and of $(\mathrm{Aut}^0(X_2), X_2)$ are fixed (theorem 3.1.1), and from the analysis of the invariants associated to a product this implies that all boundary prime divisors of $(\mathrm{Aut}^0(X_1) \times \mathrm{Aut}^0(X_2), X)$ are fixed. In other words, we have $\partial(\mathrm{Aut}^0(X), X) = \partial(\mathrm{Aut}^0(X_1) \times \mathrm{Aut}^0(X_2), X)$. Now our corollary is an easy consequence of proposition 3.3.1 applied to the wonderful variety $(\mathrm{Aut}^0(X_1) \times \mathrm{Aut}^0(X_2), X)$. □

This corollary allows us to deal, from now on, with indecomposable varieties only.

3.4. **Rank 1.**

Proposition 3.4.1. *Let X be an indecomposable wonderful G-variety of rank 1 where G acts faithfully. Then $G = \mathrm{Aut}^0(X)$ if and only if X is not cuspidal, or $G = \mathsf{G}_2$ and X is the only rank 1 wonderful G-variety having spherical root $\alpha_1 + \alpha_2$ (case 15 of* [Wa96]*).*

Proof. Theorem 3.1.1 implies that if a rank 1 variety (G, X) has a non-fixed border prime divisor, then $\mathrm{Aut}^0(X)$ strictly contains G, and X is homogeneous under the action of $\mathrm{Aut}^0(X)$. This regards all the cuspidal indecomposable rank 1 varieties of [Wa96] except one: case 15 (in the notations of the cited paper). On the other hand if (G, X) is equal to case 15, it is not homogeneous under $\mathrm{Aut}^0(X)$ thanks to the same theorem: its border prime divisor is fixed, hence $\partial(G, X) = \partial(\mathrm{Aut}^0(X), X)$ and proposition 3.3.1 assures that $\mathrm{Aut}^0(X) = G$ in this case.

Now we turn to the non-cuspidal case. Any non-cuspidal indecomposable rank 1 variety (G, X) is obtained by parabolic induction from a cuspidal one. The latter might be a product, however it will have only one cuspidal rank 1 factor, call it (L, X'). It is very easy to see from lemma 2.2.1 that in order to obtain an indecomposable (G, X), some simple root of G but not of L must be non-orthogonal to some simple root of L.

Since (L, X') is cuspidal and indecomposable of rank 1, $\mathrm{supp}\, \Sigma(L, X')$ is the whole set of simple roots of L. Lemma 3.1.1 assures in this case that the border prime divisor of (G, X) is fixed, and $\mathrm{Aut}^0(X) = G$ as a consequence of proposition 3.3.1. □

Let us describe $(\mathrm{Aut}^0(X), X)$ for all rank 1 varieties (G, X) that are homogeneous under $\mathrm{Aut}^0(X)$. In [Ah83] this is done for all X having an affine open G-orbit, so we will work on the remaining ones, namely cases 9B and 9C of [Wa96].

- 9B Here $G = \mathsf{PSO}_{2n+1}$ $(n \geq 2)$, $\Sigma(G, X) = \{\gamma_1 = \alpha_1 + \cdots + \alpha_n\}$, $S^p(G, X) = \{\alpha_2, \ldots, \alpha_{n-1}\}$, $\mathbb{A}(X) = \emptyset$. The generic stabilizer (described in [Wa96]) has a Levi component isomorphic to GL_n and the unipotent radical isomorphic to $\bigwedge^2\mathbb{C}^n$. If ω is the symmetric bilinear form on $\mathbb{C}^{2n+1}$ defining G, the open G-orbit of X is isomorphic to:

$$\left\{(E,F)\in \mathrm{Gr}(2n,\mathbb{C}^{2n+1})\times \mathrm{Gr}(n,\mathbb{C}^{2n+1}) \;\middle|\; \begin{array}{l}\omega|_E \text{ nondegenerate,}\\ F \text{ isotropic, } F\subset E\end{array}\right\}.$$

If we ignore the condition "$\omega|_E$ nondegenerate" we obtain the whole X. Therefore, with a suitable extension of ω to $\mathbb{C}^{2n+2}$, X is isomorphic to:

$$\left\{(E',F')\in \mathrm{Gr}(2n+1,\mathbb{C}^{2n+2})\times \mathrm{Gr}(n+1,\mathbb{C}^{2n+2}) \;\middle|\; \begin{array}{l}F' \text{ isotropic,}\\ F'\subset E'\end{array}\right\}$$

where the isomorphism is given by $(E',F')\mapsto (E'\cap\mathbb{C}^{2n+1},F'\cap\mathbb{C}^{2n+1})$. Now it is evident that X is homogeneous under $\mathrm{Aut}^0(X)=\mathsf{PSO}_{2n+2}$, and $S^p(\mathrm{Aut}^0(X),X)=\{\alpha_2,\ldots,\alpha_n\}$.

9C Here $G=\mathsf{PSp}_{2n}$ $(n\geq 2)$, $\Sigma_G(X)=\{\gamma_1=\alpha_1+2\alpha_2+\cdots+2\alpha_{n-1}+\alpha_n\}$, $S^p(G,X)=\{\alpha_3,\ldots,\alpha_n\}$, $\mathbb{A}(X)=\emptyset$. The generic stabilizer has a Levi component isogenous to $\mathsf{Sp}_{2n-2}\times\mathbb{C}^\times$ and the unipotent radical isomorphic to $\mathbb{C}$. If ω is the skew-symmetric bilinear form defining G, the open G-orbit of X is isomorphic to:

$$\left\{(E,F)\in \mathrm{Gr}(2,\mathbb{C}^{2n})\times \mathbb{P}(\mathbb{C}^{2n}) \;\middle|\; \begin{array}{l}E \text{ nonisotropic,}\\ F\subset E\end{array}\right\}.$$

If we ignore the condition "E nonisotropic" we obtain the whole X. So X is omogeneous under $\mathrm{Aut}^0(X)=\mathsf{PSL}_{2n}$, and we have $S^p(\mathrm{Aut}^0(X),X)=\{\alpha_3,\ldots,\alpha_{2n-1}\}$.

3.5. **Rank 2.**

Proposition 3.5.1. *Let X be an indecomposable G-wonderful variety of rank 2 where G acts faithfully. If X is cuspidal then $G=\mathrm{Aut}^0(X)$ except for the following cases:*

$\mathbf{1}_{rk=2}$ $G=\mathsf{PSL}_2\times\mathsf{PSp}_{2n}$ *($n\geq 2$), X is case 1 of type* C *of* [Wa96]*; in particular:*

$$\begin{aligned}S^p(G,X) &= \{\alpha'_3,\ldots,\alpha'_n\}\\ \Sigma(G,X) &= \{\gamma_1=\alpha_1+\alpha'_1,\gamma_2=\alpha'_1+2\alpha'_2+\cdots+2\alpha'_{n-1}+\alpha'_n\}\\ \mathbb{A}(G,X) &= \emptyset\end{aligned}$$

$\mathbf{2}_{rk=2}$ $G=\mathsf{PSp}_{2n}$ *($n\geq 2$), X is the first of the two cases 5 of type* C *of* [Wa96]*; in particular:*

$$\begin{aligned}S^p(G,X) &= \{\alpha_3,\ldots,\alpha_n\}\\ \Sigma(G,X) &= \{\gamma_1=2\alpha_1,\gamma_2=\alpha_1+2\alpha_2+\cdots+2\alpha_{n-1}+\alpha_n\}\\ \mathbb{A}(G,X) &= \emptyset\end{aligned}$$

$\mathbf{3}_{rk=2}$ $G=\mathsf{PSp}_{2n}$ *($n\geq 2$), X is the second of the two cases 5 of type* C *of* [Wa96]*; in particular:*

$$\begin{aligned}S^p(G,X) &= \{\alpha_3,\ldots,\alpha_n\}\\ \Sigma(G,X) &= \{\gamma_1=\alpha_1,\gamma_2=\alpha_1+2\alpha_2+\cdots+2\alpha_{n-1}+\alpha_n\}\\ \mathbb{A}(G,X) &= \{D_1^+,D_1^-\}\\ \rho_{G,X} &: \left\{\begin{array}{ll}\langle D_1^+,\gamma_1\rangle=1 & \langle D_1^-,\gamma_1\rangle=1\\ \langle D_1^+,\gamma_2\rangle=0 & \langle D_1^-,\gamma_2\rangle=0\end{array}\right.\end{aligned}$$

$\mathbf{4}_{rk=2}$ *$G = \mathsf{PSO}_9$, X is case 3 of type* B *of* [Wa96]*; in particular:*

$$\begin{aligned} S^p(G,X) &= \{\alpha_2, \alpha_3\} \\ \Sigma(G,X) &= \{\gamma_1 = \alpha_2 + 2\alpha_3 + 3\alpha_4, \gamma_2 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\} \\ \mathbb{A}(G,X) &= \emptyset \end{aligned}$$

If X is not cuspidal, then $G = \mathrm{Aut}^0(X)$ except for:

$\mathbf{5}_{rk=2}$ *any indecomposable non-cuspidal variety (G,X) obtained by parabolic induction from (L,Y), where we require that (L,Y) is a product (of possibly only one factor) having a rank 2 factor equal to the cuspidal case 1 above, and that $G = G_1 \times \mathsf{PSp}_{2n}$, i.e. the component PSp_{2n} of the group L is preserved when passing from L to G.*

Proof. Let us begin with the cuspidal case. Varieties $\mathbf{1}_{rk=2}$, $\mathbf{2}_{rk=2}$, $\mathbf{3}_{rk=2}$, $\mathbf{4}_{rk=2}$ are exactly the indecomposable cuspidal rank 2 varieties where some border divisor is not fixed, following the tables in [Wa96]. All other varieties satisfy $\partial(G,X) = \partial(\mathrm{Aut}^0(X),X)$ as a consequence of theorem 3.1.1, and $\mathrm{Aut}^0(X) = G$ as a consequence of proposition 3.3.1.

In the non-cuspidal case, let (G,X) be a non-cuspidal indecomposable rank 2 variety, obtained by parabolic induction from a cuspidal one, say (L,Y). We can always consider (L,Y) as a product, of possibly only one factor. Anyway, it has rank 2, like (G,X).

This means that the cuspidal indecomposable factors of (L,Y) of rank > 0 are either one of rank 2, or two of rank 1. In the latter case the situation is similar to the second part of the proof of proposition 3.4.1 (the "non-cuspidal part"): we discover in the same way that one of the two border divisors of (G,X) can be non-fixed only if (G,X) itself is a product where one of the factors has rank 1 and non-fixed border divisor. This is absurd, since we are supposing (G,X) indecomposable.

We are left with the case where (L,Y) has a rank 2 cuspidal indecomposable factor (L_1,Y_1). It is evident that if the border divisors of (L_1,Y_1) are all fixed, then the same happens for (G,X), so we may suppose that (L_1,Y_1) has at least one non-fixed divisor. Then (L_1,Y_1) is equal to $\mathbf{1}_{rk=2}$, $\mathbf{2}_{rk=2}$, $\mathbf{3}_{rk=2}$ or $\mathbf{4}_{rk=2}$.

For $\mathbf{2}_{rk=2}$, $\mathbf{3}_{rk=2}$ and $\mathbf{4}_{rk=2}$, the group L_1 is simple and the spherical root corresponding to the non-fixed divisor has support equal to the whole set of simple roots of the group. From lemma 3.1.1 we deduce that this divisor is non-fixed in (G,X) only if the factor L_1 of L is also a factor of G. So we have a decomposition $G = L_1 \times G_1$, where no spherical root of X has support on G_1: lemma 2.2.1 implies that (G,X) is either a product or equal to (L_1,Y_1), and this is absurd, (G,X) being non-cuspidal and indecomposable.

Therefore the only situation where (G,X) has a non-fixed divisor occurs if $(L_1,Y_1) = \mathbf{1}_{rk=2}$, where the non-fixed divisor corresponds to a spherical root whose support is the set of simple roots of Sp_{2n}. This divisor remains non-fixed in (G,X) if and only if the factor Sp_{2n} of L_1 is also a factor of G, and this completes the proof. □

Let us discuss the cuspidal varieties of the proposition above. In each case, only one border divisor is fixed: it is a rank 1 wonderful variety which is homogeneous under its automorphism group, and which coincides with the closed $\mathrm{Aut}^0(X)$-orbit on X. Moreover, it always appears in full details in subsection 3.4 as case 9B or 9C.

Using the knowledge of this divisor and its invariants under the action of its automorphism group, it is immediate to deduce the automorphism groups and the relative invariants of our rank 2 varieties. An explicit geometrical description is also possible for the first three varieties.

$\mathbf{1}_{rk=2}$ $\mathrm{Aut}^0(X) = \mathsf{PSL_2} \times \mathsf{PSL_{2n}}$, and X is the following variety:

$$X = \left\{ (E, M) \,\middle|\, \begin{array}{l} E \in \mathrm{Gr}(n, \mathbb{C}^{2n}), \\ M \in \mathbb{P}(\mathrm{Hom}(\mathbb{C}^2, E)) \end{array} \right\}$$

where $\mathrm{Hom}(\mathbb{C}^2, E)$ is the space of linear homomorphisms between a fixed $\mathbb{C}^2$ and the 2-dimensional space E. The action of G and of $\mathrm{Aut}^0(X)$ are defined in the same way. The factors $\mathsf{PSp_{2n}} \subset G$ and $\mathsf{PSL_{2n}} \subset \mathrm{Aut}^0(X)$ act in the usual way on the Grassmannian; an element (x, y) (where $x \in \mathsf{SL_2}$ and $y \in \mathsf{Sp_{2n}}$ or $\mathsf{SL_{2n}}$) acts on the coordinate "M" as:

$$\begin{array}{ccc} \mathbb{P}(\mathrm{Hom}(\mathbb{C}^2, E)) & \to & \mathbb{P}(\mathrm{Hom}(\mathbb{C}^2, yE)) \\ M = [f] & \mapsto & [y \circ f \circ x^{-1}]. \end{array}$$

The invariants under the action of $\mathrm{Aut}^0(X)$ are:

$$\begin{aligned} S^p(\mathrm{Aut}^0(X), X) &= \{\alpha'_3, \ldots, \alpha'_{2n-1}\} \\ \Sigma(\mathrm{Aut}^0(X), X) &= \{\gamma_1 = \alpha_1 + \alpha'_1\} \\ \mathbb{A}(\mathrm{Aut}^0(X), X) &= \emptyset \end{aligned}$$

$\mathbf{2}_{rk=2}$ $\mathrm{Aut}^0(X) = \mathsf{PSL_{2n}}$, $X = \mathrm{Bl}_{\mathrm{diag}(\mathbb{P}^{2n-1})}((\mathbb{P}^{2n-1} \times \mathbb{P}^{2n-1})/\sim)$, where $(x, y) \sim (y, x)$,

$$\begin{aligned} S^p(\mathrm{Aut}^0(X), X) &= \{\alpha_3, \ldots, \alpha_{2n-1}\} \\ \Sigma(\mathrm{Aut}^0(X), X) &= \{\gamma_1 = 2\alpha_1\} \\ \mathbb{A}(\mathrm{Aut}^0(X), X) &= \emptyset \end{aligned}$$

$\mathbf{3}_{rk=2}$ $\mathrm{Aut}^0(X) = \mathsf{PSL_{2n}}$, $X = \mathrm{Bl}_{\mathrm{diag}(\mathbb{P}^{2n-1})}(\mathbb{P}^{2n-1} \times \mathbb{P}^{2n-1})$,

$$\begin{aligned} S^p(\mathrm{Aut}^0(X), X) &= \{\alpha_3, \ldots, \alpha_{2n-1}\} \\ \Sigma(\mathrm{Aut}^0(X), X) &= \{\gamma_1 = \alpha_1\} \\ \mathbb{A}(\mathrm{Aut}^0(X), X) &= \{D_1^+, D_1^-\}, \quad \langle D_1^+, \gamma_1 \rangle = \langle D_1^-, \gamma_1 \rangle = 1 \end{aligned}$$

$\mathbf{4}_{rk=2}$ $\mathrm{Aut}^0(X) = \mathsf{PSO_{10}}$; here X under the action of $\mathrm{Aut}^0(X)$ is a parabolic induction from (L, Y), where $L = \mathsf{PSO_8}$ and (L, Y) is the rank 1 variety $\mathbb{P}^7$ (cf. rank 1 case 6D). The parabolic induction is given identifying L with a Levi part of the parabolic subgroup of $\mathsf{PSO_{10}}$ associated to the simple roots $\alpha_2, \alpha_3, \alpha_4, \alpha_5$. The invariants under the action of $\mathrm{Aut}^0(X)$ are:

$$\begin{aligned} S^p(\mathrm{Aut}^0(X), X) &= \{\alpha_2, \alpha_3, \alpha_4\} \\ \Sigma(\mathrm{Aut}^0(X), X) &= \{\gamma_1 = \alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5\} \\ \mathbb{A}(\mathrm{Aut}^0(X), X) &= \emptyset \end{aligned}$$

In the end, let us consider the non-cuspidal exception $\mathbf{5}_{rk=2}$. Here our (G, X) is a parabolic induction from (L, Y) where (L, Y) is the first cuspidal case, and $G = G_1 \times \mathsf{PSp}_{2n}$. Again, it is easy to prove that $\mathrm{Aut}^0(X) = G_1 \times \mathsf{PSL}_{2n}$ and the invariants behave, *mutatis mutandis*, in the same way as for $\mathbf{1}_{rk=2}$.

3.6. **Higher rank.**

Proposition 3.6.1. *Let X be an indecomposable G-wonderful variety of rank at least 3, and let us suppose that G acts faithfully. Then $G = \mathrm{Aut}^0(X)$ except for any variety (G, X) such that G has a factor PSp_{2n} with associated simple roots $\alpha_1, \ldots, \alpha_n$, and the invariants of X satisfy:*

$$\begin{aligned} \Sigma(G, X) &\ni \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n, \\ S^p(G, X) &\not\ni \alpha_1. \end{aligned}$$

Before proving the proposition, we remark that this kind of varieties appear also in rank 1, namely case 9C, and in rank 2, namely varieties $\mathbf{1}_{rk=2}$, $\mathbf{2}_{rk=2}$, $\mathbf{3}_{rk=2}$, $\mathbf{5}_{rk=2}$.

Proof. If $G \neq \mathrm{Aut}^0(X)$, then some of the border prime divisors of (G, X) is not fixed. Hence at least one of the rank 1 wonderful G-subvarieties must have a non fixed border prime divisor: call it $(G, X_{(1)})$. Since (G, X) is indecomposable, definition 2.2.2 implies directly that $(G, X_{(1)})$ is contained in some rank 2 wonderful G-subvariety having an indecomposable rank 2 factor.

Call this factor (G_1, Y): it has again a non-fixed border prime divisor, so it must appear in proposition 3.5.1. If (G_1, Y) is non-cuspidal, then (G, X) falls into the family $\mathbf{5}_{rk=2}$, therefore our conditions are satisfied.

Otherwise, we have to look at the classification of rank 2 wonderful varieties in [Wa96]. We find out that varieties $\mathbf{2}_{rk=2}$ and $\mathbf{4}_{rk=2}$ cannot be subvarieties (nor factors of subvarieties) of any indecomposable bigger variety, so (G_1, Y) is either $\mathbf{1}_{rk=2}$ or $\mathbf{3}_{rk=2}$: again, our conditions for (G, X) are satisfied.

On the other hand, the classification in rank 2 shows that if a variety satisfy our conditions, then the divisor associated to the spherical root $\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n$ cannot be fixed, and this finishes the proof. □

We can describe a bit more precisely these exceptions in rank > 2: the classification in rank 2 can be used easily to understand what are all wonderful varieties satisfying the conditions in proposition 3.6.1. They might be regarded as composed by three families:

$\mathbf{1}_{rk>2}$ any variety of rank > 2 satisfying the conditions of $\mathbf{5}_{rk=2}$;

$\mathbf{2}_{rk>2}$ any variety (G, X) such that: G has a simple factor PSp_{2n} with associated simple roots $\alpha_1, \ldots, \alpha_n$, and the invariants of X satisfy:

$$\Sigma(G, X) \supseteq \left\{ \begin{array}{l} \gamma_1 = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n, \\ \gamma_2 = \alpha_1, \\ \gamma_3 = \alpha \end{array} \right\}$$

$$\mathbb{A}(G, X) \ni D, \text{ such that } D \text{ is moved by both } \alpha_1 \text{ and } \alpha\ ,$$

for α some simple root of G not among $\alpha_1, \ldots, \alpha_n$.

$\mathbf{3}_{rk>2}$ the rank 3 variety (G, X) where $G = \mathsf{PSp}_{2n} \times \mathsf{PSp}_{2m}$ and X is uniquely determined by:

$$\Sigma(G, X) = \left\{ \begin{array}{l} \gamma_1 = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n_1-1} + \alpha_n, \\ \gamma_2 = \alpha'_1 + 2\alpha'_2 + \ldots + 2\alpha'_{n_2-1} + \alpha'_m, \\ \gamma_3 = \alpha_1 + \alpha'_1 \end{array} \right\}.$$

In this case, the variety X has a decription similar to $\mathbf{1}_{rk=2}$, cf. subsection 3.5:

$$X = \left\{ (E_1, E_2, M) \;\middle|\; \begin{array}{l} E_1 \in \mathrm{Gr}(2, \mathbb{C}^{2n}), \\ E_2 \in \mathrm{Gr}(2, \mathbb{C}^{2m}), \\ M \in \mathbb{P}(\mathrm{Hom}(E_1, E_2)) \end{array} \right\}$$

To conclude this section, we describe the varieties $(\mathrm{Aut}^0(X), X)$ for all these higher rank exceptions. Let (G, X) be one of them: the group G has one or more factors isomorphic to PSp,

$$G = G_1 \times \mathsf{PSp}_{2n_1} \times \mathsf{PSp}_{2n_2} \times \cdots \times \mathsf{PSp}_{2n_k},$$

where each PSp_{2n_i} (with simple roots $\alpha^i_1, \ldots, \alpha^i_{n_i}$) gives the prescribed spherical root $\alpha^i_1 + 2\alpha^i_2 + \ldots + 2\alpha^i_{n_i-1} + \alpha^i_{n_i}$. Like in the previous subsection, it is straightforward to show that $\mathrm{Aut}^0(X)$ will differ from G only in the factors PSp_{2n_i} which turn into PSL_{2n_i}, and that the spherical systems of X with respect to G and to $\mathrm{Aut}^0(X)$ will differ only accordingly to this change.

Precisely: let us call $\beta^i_1, \ldots, \beta^i_{2n_i-1}$ the simple roots of PSL_{2n_i}. Then, the spherical root $\alpha^i_1 + 2\alpha^i_2 + \ldots + 2\alpha^i_{n_i-1} + \alpha_{n_i}$ disappears in $\Sigma(\mathrm{Aut}^0(X), X)$; any other spherical root involving α^i_1 in the support (such as for example α^i_1 itself) remains the same, with β^i_1 taking the place of α^i_1. All other spherical roots of (G, X) appear unchanged in $(\mathrm{Aut}^0(X), X)$, and the sets $\mathbb{A}(G, X)$, $\mathbb{A}(\mathrm{Aut}^0(X), X)$ are exactly the same.

The set $S^p(\mathrm{Aut}^0(X), X)$ concides with $S^p(G, X)$ in what concerns the simple roots of G_1. Finally, $S^p(\mathrm{Aut}^0(X), X)$ "behaves" like $S^p(G, X)$ on the PSp factors of the group, in the sense that $S^p(G, X)$ contains all simple roots of PSp_{2n_i} except α^i_1 and α^i_2, and correspondingly $S^p(\mathrm{Aut}^0(X), X)$ contains all roots of PSL_{2n_i} except β^i_1 and β^i_2.

4. Morphisms

4.1. **The set of all colors.** It is possible to recover all colors and the values of the associated functionals, starting only from the spherical system of a wonderful variety (G, X). Following [Lu01] and its notations, we can identify each color in $\Delta(G, X) \setminus \mathbb{A}(G, X)$ with the simple roots it is moved by. This gives a disjoint union:

$$\Delta(G, X) = \mathbb{A}(G, X) \cup \Delta^{a'}(G, X) \cup \Delta^b(G, X)$$

The set $\Delta^{a'}(G, X)$ is in bijection with the set of simple roots α such that 2α is a spherical root. For such a color D, we have $\rho_{G,X}(D) = \frac{1}{2}\alpha^\vee|_{\Xi(G,X)}$. The set $\Delta^b(G, X)$ is in bijection with the following set:

$$\left(S \setminus \left(\Sigma(G, X) \cup \frac{1}{2}\Sigma(G, X) \cup S^p(G, X) \right) \right) \Big/ \sim$$

where $\alpha \sim \beta$ if $\alpha = \beta$, or if $\alpha \perp \beta$ and $\alpha + \beta \in \Sigma(G, X)$ (or $\frac{1}{2}(\alpha + \beta) \in \Sigma(G, X)$). For such a color D, the associated functional $\rho_{G,X}(D)$ is equal to $\alpha^\vee$, for α any representative of the $\sim$-equivalence class associated to D.

4.2. **Smooth G-equivariant morphisms.** The theory of spherical varieties is used in [Lu01] to study surjective G-equivariant morphisms with connected fibers between wonderful G-varieties.

Here it is enough to recall the case of smooth morphisms. Let Δ' be a subset of $\Delta(G,X)$, such that there exists a linear combination with positive integer coefficients:

$$\eta = \sum_{D\in\Delta'} n_D \rho_{G,X}(D)$$

such that $\eta(\gamma)\geq 0$ for all $\gamma\in\Sigma(G,X)$. Suppose moreover that whenever we have a spherical root γ with $\eta(\gamma)=0$, then $\langle D,\gamma\rangle = 0$ for all $D\in\Delta'$.

Then there exist a unique (up to G-isomorphism) wonderful G-variety $X_{\Delta'}$, and a unique surjective G-equivariant map $f_{\Delta'}\colon X\to X_{\Delta'}$ with connected fibers, such that:

$$\Delta' = \{D\in\Delta(G,X) \mid f_{\Delta'}(D) = X_{\Delta'}\}\,.$$

The morphism $f_{\Delta'}$ is smooth, and we call such a Δ' a *smooth distinguished* subset of colors. The invariants of $X_{\Delta'}$ are the same of X, except for the following changes:

(1) all spherical roots where η is positive disappear;
(2) all colors of Δ' disappear; this may cause obvious changes in the sets $\mathbb{A}$ and S^p.

A particular case occurs when Δ' contains only one element δ: it must take non-negative values on all spherical roots and we call *positive* such a color.

In [Lu01] the induced maps $f_{\{\delta\}}$ were studied in details in the case where $\delta\in\mathbb{A}(G,X)$ and where G has only components of type A: it was called a *projective fibration*. Without these restrictions on δ and G the analysis of *loc.cit.* cannot be always applied.

If we restrict ourselves to indecomposable varieties of rank at least 2, we can reformulate our main theorem 3.1.2 in the following way:

Theorem 4.2.1. *Let (G,X) be an indecomposable wonderful variety of rank at least 2. Then $\mathrm{Aut}^0(X)\neq G$ if and only if there exists a smooth distinguished subset of colors $\Delta'\subset\Delta(G,X)$ such that the variety $(G,X_{\Delta'})$ has a rank 1 factor which is homogeneous under its full automorphism group, and which has spherical root γ where all colors of Δ' take non-negative values (when γ is considered as a spherical root of X).*

Proof. The "if" part can be proven independently from theorem 3.1.2: it is an easy consequence of theorem 3.1.1, so let us begin with it.

Consider the variety $X_{\Delta'}$. The rank 1 factor in the hypothesis implies that one of the border divisors of $X_{\Delta'}$ is not fixed, in other words all colors of $X_{\Delta'}$ take non-negative values on the associated spherical root γ.

This border divisor corresponds to a border divisor of X, say $X^{(1)}$. But we have $\Delta(G,X)=\Delta(G,X_{\Delta'})\cup\Delta'$ and colors of Δ' are never negative on γ, so $X^{(1)}$ is not fixed and $\mathrm{Aut}^0(X)\neq G$.

Now we proceed to the "only if" part. Here we must use the proof of theorem 3.1.2 and find the required Δ' in each of the exceptions listed in subsections 3.5 and 3.6. For most of them, Δ' contains only one color δ, which is positive:

$\mathbf{1}_{rk=2}$ $\delta\in\Delta^b(G,X)$, moved by α_1 and α_1';
$\mathbf{2}_{rk=2}$ $\delta\in\Delta^{a'}(G,X)$, moved by α_1;
$\mathbf{3}_{rk=2}$ $\delta = D_1^+\in\mathbb{A}(G,X)$;
$\mathbf{4}_{rk=2}$ $\delta\in\Delta^b(G,X)$, moved by α_4;
$\mathbf{5}_{rk=2}$ as for $\mathbf{1}_{rk=2}$;
$\mathbf{1}_{rk>2}$ as for $\mathbf{1}_{rk=2}$;

$\mathbf{3}_{rk>2}$ $\delta \in \Delta^b(G,X)$, moved by α_1 and α_1'.

For the remaining case $\mathbf{2}_{rk>2}$ we have to take as Δ' the whole $\Delta(G,X)$ except the unique color moved by α_2.

It is easy to see that all these Δ' satisfy the required conditions, and this finishes the proof. □

In the hypotheses of the above theorem, let us consider the variety $(\mathrm{Aut}^0(X), X)$ and the set Δ'. It is easy to check case-by-case that Δ' is again smooth distinguished under the action of $\mathrm{Aut}^0(X)$, thus it defines a smooth $\mathrm{Aut}^0(X)$-equivariant map:

$$\tilde{f}_{\Delta'} \colon X \to \tilde{X}_{\Delta'}$$

with the same properties as $f_{\Delta'}$. Again a simple case-by-case proof shows that $\tilde{X}_{\Delta'} = X_{\Delta'}$, and that $\mathrm{Aut}^0(X_{\Delta'}) = \mathrm{Aut}^0(X)$. From the uniqueness property of $f_{\Delta'}$ and the fact that $\mathrm{Aut}^0(X) \supset G$, it follows also that $\tilde{f}_{\Delta'} = f_{\Delta'}$.

We may rephrase these conclusions in the following way: the group $\mathrm{Aut}^0(X_{\Delta'})$ is bigger than G, and all the elements of $\mathrm{Aut}^0(X_{\Delta'}) \setminus G$ lift to X via $f_{\Delta'}$.

It would be very interesting to have a direct proof of this lifting property of $f_{\Delta'}$. However this property fails if we try to replace the condition in theorem 4.2.1 with the weaker one "there exists Δ' such that $X_{\Delta'}$ has an automorphism group bigger than G": this is not enough to assure $\mathrm{Aut}^0(X) \neq G$, not even in the cases where Δ' contains only one color.

A counterexample is given by the rank 1 cuspidal case 15 of [Wa96], where $G = \mathsf{G}_2$ and:

$$\begin{aligned} S^p(G,X) &= \emptyset \\ \Sigma(G,X) &= \{\gamma_1 = \alpha_1 + \alpha_2\} \\ \mathbb{A}(G,X) &= \emptyset \end{aligned}$$

Here there are two colors $D_1, D_2 \in \Delta^b(G,X)$, moved resp. by α_1, α_2. We can choose $\delta = D_2$, so that $X_{\{\delta\}} = \mathbf{3}_{rk=0}$. Thus $\mathrm{Aut}^0(X_{\{\delta\}}) \neq G$, but proposition 3.4.1 says that $\mathrm{Aut}^0(X) = G$.

References

[Ah83] D.N. Ahiezer, *Equivariant completions of homogeneous algebraic varieties by homogeneous divisors*, Ann. Global Anal. Geom. **1** (1983), no. 1, 49–78.

[BB96] F. Bien, M. Brion, *Automorphisms and local rigidity of regular varieties*, Compositio Math. **104** (1996), no. 1, 1–26.

[Br89] M. Brion, *On spherical varieties of rank one*, CMS Conf. Proc. **10** (1989), 31–41.

[Br07] M. Brion, *The total coordinate ring of a wonderful variety*, J. Algebra **313** (2007), no. 1, 61–99.

[De70] M. Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona*, Ann. Sci. École Norm. Sup. **3**, (1970), no. 4, 507–588.

[De77] M. Demazure, *Automorphismes et déformations des variétés de Borel*, Invent. Math. **39** (1977), no. 2, 179–186.

[HS82] A. Huckleberry, D. Snow, *Almost-homogeneous Kähler manifolds with hypersurface orbits*, Osaka J. of Math. **19** (1982), 763–786.

[Lo07] I. Losev, *Uniqueness property for spherical homogeneous spaces*, preprint, arXiv:math/0703543.

[Lu96] D. Luna, *Toute variété magnifique est sphérique*, Transform. Groups **1** (1996), no. 3, 249–258.

[Lu01] D. Luna, *Variétés sphériques de type A*, Inst. Hautes Études Sci. Publ. Math. **94** (2001), 161–226.

[Wa96] B. Wasserman, *Wonderful varieties of rank two*, Transform. Groups **1** (1996), no. 4, 375–403.

Departement Mathematik, Universität Basel, Rheinsprung 21, 4051 Basel, Switzerland

E-mail address: Guido.Pezzini@unibas.ch