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Fractional spherical random fields

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ABSTRACT

In this paper we study the solutions of different forms of fractional equations on the unit sphere $\mathbf{S}_1^2 \subset \mathbf{R}^3$ possessing the structure of time-dependent random fields. We study the correlation structures of the random fields emerging in the analysis of the solutions of two kinds of fractional equations displaying (Theorem 1) a long-range behaviour and (Theorem 2) a short-range behaviour.

1. Introduction

We consider a Brownian motion on the sphere, say $B : t \mapsto \mathbf{S}_1^2$ and a real-valued random field $T : \mathbf{S}_1^2 \mapsto \mathbf{R}$. We study here the time-space dependent random field $X = T \circ B$. We restrict ourselves to Gaussian isotropic random fields T for which the expansion in terms of spherical harmonics holds (see Marinucci and Peccati, 2011 and the references therein). The explicit law of the Brownian motion on $\mathbf{S}_1^2$ was first obtained in Yosida (1949). For Brownian motion processes on $\mathbf{S}_1^d$, (see Karlin and Taylor, 1975, pag. 338).

Time-dependent random fields on the line or on arbitrary Euclidean spaces have been studied by several authors (see, for example, Kelbert et al., 2005; Angulo et al., 2008; Leonenko et al., 2011 and the references therein).

We study here the time-space dependent random fields on the sphere $\mathbf{S}_1^2$, governed by different stochastic differential equations.

We first study random fields emerging as solutions to the Cauchy problem

$$\begin{cases} \left(\gamma - \mathbb{D}_M + \dfrac{\partial^\beta}{\partial t^\beta}\right) X_t(x) = 0, \quad x \in \mathbf{S}_1^2,\ t > 0,\ 0 < \beta < 1,\ \gamma > 0 \\ X_0(x) = T(x), \end{cases} \tag{1.1}$$

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where $\mathbb{D}_M$ is a suitable differential operator defined below and $\frac{\partial^\beta}{\partial t^\beta}$ is the Dzerbayshan–Caputo fractional derivative. We are able to obtain the solution $X_t(x)$ of (1.1) and to show that its covariance function displays a long-memory behaviour.

We then consider the non-homogeneous fractional equation

$$(\gamma - \mathbb{D}_M)^\beta X(x) = T(x), \quad x \in \mathbf{S}^2_1,\ 0 < \beta < 1 \tag{1.2}$$

of which

$$\left(\gamma - \mathbb{D}_M - \varphi \frac{\partial}{\partial t}\right)^\beta X_t(x) = T_t(x), \quad x \in \mathbf{S}^2_1,\ t > 0,\ 0 < \beta < 1,\ \gamma > 0,\ \varphi \ge 0 \tag{1.3}$$

is the time-dependent extension. We obtain a solution to (1.3) which is a random field on the sphere with covariance function with a short-range dependence.

The couple $(B_t, T(x + B_t))$ describes a random motion on the unit-radius sphere with dynamics governed by fractional stochastic equations (1.1) and (1.3).

Random fields similar to those examined here are considered in the analysis of the cosmic microwave background radiation (CMB radiation). In this case, the correlation structure turns out to be very important as well as the angular power spectrum. The angular power spectrum plays a key role in the study of the corresponding random field. In particular, the high-frequency behaviour of the angular power spectrum is related to some anisotropies of the CMB radiation (see for example D'Ovidio, 2014; Marinucci and Peccati, 2011). Such relations have been also investigated in D'Ovidio and Nane (2014) where a coordinates change driven by a fractional equation has been considered.

The time-varying random fields studied here can be useful because the CMB radiation can be affected by some anomalies captured by the angular power spectrum. For a fixed t ($t = 1$ for instance), our models well describe different forms of the angular power spectrum. For $t > 0$, our evolution models well outline anomalies due to instrumental errors, for instance.

Diffusions on the sphere arise in several contexts. At the cellular level, diffusion is an important mode of transport of substances. The cell wall is a lipid membrane and biological substances like lipids and proteins diffuse on it. In general, biological membranes are curved surfaces. Spherical diffusions also crop up in the swimming of bacteria, surface smoothening in computer graphics (Bulow, 2004) and global migration patterns of marine mammals (Brillinger and and Stewart, 1998).

2. Preliminaries

2.1. Isotropic random fields on the unit-radius sphere

We introduce notations and some properties of isotropic random fields on the sphere (for a complete presentation see the book by Marinucci and Peccati (2011)). We consider the square integrable isotropic Gaussian random field

$$\{T(x);\ x \in \mathbf{S}^2_1\} \tag{2.1}$$

on the sphere $\mathbf{S}^2_1 = \{x \in \mathbf{R}^3 : |x| = 1\}$ for which

$$\mathbb{E}T(gx) = \mathbb{E}T(x) = 0,$$
$$\mathbb{E}T^2(gx) = \mathbb{E}T^2(x) = const$$
$$\mathbb{E}[T(gx_1)\,T(gx_2)] = \mathbb{E}[T(x_1)\,T(x_2)] \quad \text{for arbitrary } x_1, x_2 \in \mathbf{S}^2_1,$$

for all $g \in SO(3)$ where $SO(3)$ is the special group of rotations in $\mathbf{R}^3$. We will consider the spectral representation

$$T(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{l,m} \mathcal{Y}_{l,m}(x) = \sum_{l=0}^{\infty} T^l(x) \tag{2.2}$$

where

$$a_{l,m} = \int_{\mathbf{S}^2} T(x) \mathcal{Y}^*_{l,m}(x) \lambda(dx), \quad -l \le m \le l,\ l \ge 0 \tag{2.3}$$

are the Fourier random coefficients of T and $\mathcal{Y}_{l,m}(x) = (-1)^m \mathcal{Y}^*_{l,-m}(x)$. We denote by $\mathcal{Y}^*_{l,m}(x)$ the conjugate of $\mathcal{Y}_{l,m}(x)$. The spherical harmonics $\mathcal{Y}_{l,m}(\vartheta, \varphi)$ are defined as

$$\mathcal{Y}_{l,m}(\vartheta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} Q_{l,m}(\cos\vartheta) e^{im\varphi}, \quad 0 \le \vartheta \le \pi,\ 0 \le \varphi \le 2\pi$$

for $l \ge 0$ and $|m| \le l$ where, for $m \ge 0$,

$$Q_{l,m}(z) = (-1)^m (1 - z^2)^{\frac{m}{2}} \frac{d^m}{dz^m} Q_l(z), \quad |z| < 1,\ |m| \le l,\ \text{and } Q_{l,m}(z) = 0,\ m > l$$

are the associated Legendre functions and Q_l are the Legendre polynomials with Rodrigues representation, for $l \geq 0$,

$$Q_l(z) = \frac{1}{2^l l!} \frac{d^l}{dz^l}(z^2-1)^l, \quad |z| < 1.$$

The convergence in (2.2) must be meant in the sense that

$$\lim_{L\to\infty} \mathbb{E}\left[\int_{\mathbf{S}_1^2} \left(T(x) - \sum_{l=0}^{L}\sum_{m=-l}^{+l} a_{l,m}\, \mathcal{Y}_{l,m}(x)\right)^2 \lambda(dx)\right] = 0 \tag{2.4}$$

where $\lambda(dx)$ is the Lebesgue measure on the sphere $\mathbf{S}_1^2$. For the sake of clarity we observe that for all $x \in \mathbf{S}_1^2$ and $0 \leq \vartheta \leq \pi$, $0 \leq \varphi \leq 2\pi$:

$$\lambda(dx) = \lambda(d\vartheta, d\varphi) = d\varphi\, d\vartheta\ \sin\vartheta$$

and

$$x = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta).$$

We shall use the following notation

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} = \sum_{lm}$$

and write $f(x)$ instead of $f(\vartheta, \varphi)$ when no confusion arises.

The random coefficients (2.3) are zero-mean Gaussian complex random variables such that (Baldi and Marinucci, 2007)

$$\mathbb{E}[a_{l,m}\, a^*_{l',m'}] = \delta_l^{l'}\delta_m^{m'}\mathbb{E}|a_{l,m}|^2 \tag{2.5}$$

where

$$\mathbb{E}|a_{l,m}|^2 = C_l, \quad l \geq 0 \tag{2.6}$$

is the angular power spectrum of the random field T which under the assumption of Gaussianity fully characterizes the dependence structure of T. Clearly, δ_a^b is the Kronecker symbol.

We remind that the spherical harmonics solve

$$\Delta_{\mathbf{S}_1^2}\mathcal{Y}_{l,m} = -\mu_l\,\mathcal{Y}_{l,m}, \quad l \geq 0,\ |m| \leq l \tag{2.7}$$

where $\mu_l = l(l+1)$ and

$$\Delta_{\mathbf{S}_1^2} = \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)$$

is the spherical Laplace operator or Laplace–Beltrami operator.

In view of (2.5), the covariance function of $T(x)$ writes

$$\mathbb{E}[T(x)T(y)] = \sum_{lm} C_l \mathcal{Y}_{l,m}(x)\mathcal{Y}^*_{l,m}(y) = \sum_l C_l \frac{2l+1}{4\pi} Q_l(\langle x, y\rangle) \tag{2.8}$$

where in the last step we used the addition formula for spherical harmonics

$$\sum_{m=-l}^{+l} \mathcal{Y}_{l,m}(y)\mathcal{Y}^*_{l,m}(x) = \frac{2l+1}{4\pi} Q_l(\langle x, y\rangle) \tag{2.9}$$

and the inner product

$$\langle x, y\rangle = \cos d(x,y) = \cos\vartheta_x\cos\vartheta_y + \sin\vartheta_x\sin\vartheta_y\cos(\varphi_x - \varphi_y)$$

where $d(x, y)$ is the spherical distance between the points x, y.

2.2. Subordinators and fractional operators

Let $F(t)$, $t \geq 0$ be a Lévy subordinator with characteristic function

$$\mathbb{E}e^{i\xi F(t)} = e^{-t\Phi(\xi)} = e^{-t\left(ib\xi + \int_0^\infty \left(e^{i\xi y}-1\right)M(dy)\right)}, \quad \xi \in \mathbb{R} \tag{2.10}$$

where $b \geq 0$ is the drift and $M(\cdot)$ is the Lévy measure on $\mathbf{R}_+ \setminus \{0\}$ satisfying the conditions:

$$\int_0^\infty (y \wedge 1) M(dy) < \infty \quad \text{and} \quad M(-\infty, 0) = 0.$$

The Laplace transform of the law of a subordinator $F(t), t > 0$ defined above, can be written as

$$\mathbb{E}e^{-\xi F(t)} = e^{-t\Psi(\xi)} = e^{t\Phi(i\xi)} = e^{-t\left(b\xi + \int_0^\infty (1-e^{-\xi y}) M(dy)\right)}, \quad \xi \geq 0 \tag{2.11}$$

where $\Psi(\xi)$ is known as Laplace exponent of F. If $F(t), t \geq 0$, is the β-stable subordinator, then $\Psi(\xi) = \xi^\beta, \beta \in (0, 1)$. Hereafter, we assume $b = 0$.

We write the transition density of a Brownian motion on the unit sphere (see Yosida, 1949) as follows

$$\begin{aligned} Pr\{x + B_t \in dy\}/dy &= Pr\{B_t \in dy \mid B_0 = x\}/dy \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} e^{-t\mu_l} \mathcal{Y}_{l,m}(y) \mathcal{Y}^*_{l,m}(x) \\ &= \sum_l e^{-t\mu_l} \frac{2l+1}{4\pi} Q_l(\langle x, y \rangle) \end{aligned} \tag{2.12}$$

where we used (2.9). Furthermore, we shall write

$$P_t f(x) = \mathbb{E} f(x + B_t) = \int_{\mathbf{S}_1^2} f(y) Pr\{x + B_t \in dy\} \tag{2.13}$$

where $P_t f(x)$ is the solution to the initial-value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_{\mathbf{S}_1^2} u, & x \in \mathbf{S}_1^2, \; t > 0 \\ u(x, 0) = f(x) \end{cases} \tag{2.14}$$

for a measurable function $f(x), \; x \in \mathbf{S}_1^2$.

Let f be a square integrable function on the unit sphere, that is $f \in L^2(\mathbf{S}_1^2)$. We define the following operator

$$\mathbb{D}_M f(x) := \int_0^\infty (P_t f(x) - f(x)) \, M(dt) \tag{2.15}$$

where, from (2.12) and (2.13), we have that

$$P_t f(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} e^{-t\mu_l} \mathcal{Y}_{l,m}(x) f_{l,m} \tag{2.16}$$

and $f_{l,m}$ are the Fourier coefficients of f.

Let $s > 0$. We introduce the Sobolev space (Aubin, 1998, pages 35–45)

$$H^s(\mathbf{S}_1^2) = \left\{ f \in L^2(\mathbf{S}_1^2) : \sum_{l=0}^{\infty} (2l+1)^{2s} f_l < \infty \right\} \tag{2.17}$$

where

$$f_l = \sum_{|m| \leq l} |f_{l,m}|^2 = \sum_{|m| \leq l} \left| \int_{\mathbf{S}_1^2} f(x) \mathcal{Y}^*_{l,m}(x) \lambda(dx) \right|^2, \quad l = 0, 1, 2, \ldots.$$

We now present the following property of the operator $\mathbb{D}_M$.

Proposition 1. *Let Ψ be the symbol of a subordinator introduced in (2.11). Let $f \in H^s(\mathbf{S}_1^2)$ and $s > 4$. Then,*

$$\mathbb{D}_M f(x) = -\sum_{l=0}^{\infty} \sum_{m=-l}^{+l} f_{l,m} \mathcal{Y}_{l,m}(x) \Psi(\mu_l) \tag{2.18}$$

where

$$f_{l,m} = \int_{\mathbb{S}_1^2} f(x) \mathcal{Y}^*_{l,m}(x) \lambda(dx)$$

are the Fourier coefficients of f.

Proof. The series (2.18) converges absolutely and uniformly. Indeed, $f_l < l^{-2s}$ with $s > 4$ (being $f \in H^s(\mathbf{S}_1^2)$), $\|Y_{l,m}\|_\infty < l^{1/2}$ (see Varshalovich et al., 2008) and $\Psi(\mu_l) \leq l^2$ and thus, by considering that

$$\sum_m |f_{l,m}| \leq \left(\sum_m |f_{l,m}|^2\right)^{\frac{1}{2}} (2l+1)^{\frac{1}{2}} = \sqrt{(2l+1) f_l} < C\, l^{-s+\frac{1}{2}}$$

for some positive constant C, we get the claim. □

The operator (2.15) can be rewritten as

$$\mathbb{D}_M f(x) = \int_{\mathbf{S}_1^2} (f(y) - f(x)) \widehat{J}(x, y) \lambda(dy) \tag{2.19}$$

where λ is the Lebesgue measure on $\mathbf{S}_1^2$ and

$$\widehat{J}(x, y) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} Q_l(\langle y, x\rangle) \widehat{\Psi}(\mu_l)$$

with $\widehat{\Psi}(\mu) = \int_0^\infty e^{-s\mu} M(ds)$ if the integral exists. Indeed we can write

$$\begin{aligned}
\mathbb{D}_M f(x) &= \int_0^\infty (P_s f(x) - f(x))\, M(ds) \\
&= \int_0^\infty \mathbb{E}\left[(f(x + B_s) - f(x))\right] M(ds) \\
&= \int_0^\infty \int_{\mathbf{S}_1^2} (f(y) - f(x))\, Pr\{x + B_s \in dy\} M(ds) \\
&= \int_{\mathbf{S}_1^2} (f(y) - f(x)) \widehat{J}(x, y) \lambda(dy)
\end{aligned}$$

where

$$\begin{aligned}
\widehat{J}(x, y)\lambda(dy) &= \int_0^\infty Pr\{x + B_s \in dy\} M(ds) \\
&= \lambda(dy) \sum_l \frac{2l+1}{4\pi} Q_l(\langle y, x\rangle) \int_0^\infty e^{-s\mu_l} M(ds) \\
&= \lambda(dy) \sum_l \frac{2l+1}{4\pi} Q_l(\langle y, x\rangle) \widehat{\Psi}(\mu_l).
\end{aligned}$$

Furthermore, from (2.16), the operator (2.15) can be written as follows

$$\begin{aligned}
\mathbb{D}_M f(x) &= \int_0^\infty (P_s f(x) - P_0 f(x))\, M(ds) \\
&= \sum_{lm} f_{l,m} \mathcal{Y}_{l,m}(x) \int_0^\infty \left(e^{-s\mu_l} - 1\right) M(ds) \\
&= [\text{by (2.11)}] = -\sum_{lm} f_{l,m} \mathcal{Y}_{l,m}(x) \Psi(\mu_l) \\
&= -\sum_{lm} \left(\int_{\mathbf{S}_1^2} f(y) \mathcal{Y}^*_{l,m}(y) \lambda(dy)\right) \mathcal{Y}_{l,m}(x) \Psi(\mu_l) \\
&= -\int_{\mathbf{S}_1^2} f(y) \left(\sum_{lm} \Psi(\mu_l) \mathcal{Y}_{l,m}(x) \mathcal{Y}^*_{l,m}(y)\right) \lambda(dy) \\
&= -\int_{\mathbf{S}_1^2} f(y) J(x, y) \lambda(dy)
\end{aligned}$$

where

$$J(x, y) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \Psi(\mu_l) \mathcal{Y}_{l,m}(x) \mathcal{Y}^*_{l,m}(y) = \sum_{l=0}^{\infty} \Psi(\mu_l) \frac{2l+1}{4\pi} Q_l(\langle x, y\rangle) \tag{2.20}$$

(in the last step we have applied the addition formula (2.9)). The results above show that we are able to give two alternative forms of the operator $\mathbb{D}_M$.

We now define the following semigroup which will be useful in the sequel.

Definition 1. $\mathbb{P}_t := \exp(t\mathbb{D}_M)$ is the semigroup associated with (2.19) with symbol $\widehat{\mathbb{P}_t} = \exp(-t\Psi)$ where $-\Psi$ is the Fourier multiplier of $\mathbb{D}_M$.

3. Some fractional equations on the sphere

We recall the Dzerbayshan–Caputo (D–C) fractional derivative

$$\frac{\partial^\beta u}{\partial t^\beta}(x,t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\beta} \tag{3.1}$$

for $0 < \beta < 1, x \in \mathbb{R}, t > 0$, see, e.g. Meerschaert and Sikorskii (2012, p. 38). The D–C fractional derivative is related to the inverse of a stable subordinator, say $\mathfrak{L}_t^\beta$, in the sense that $u(x,t) = Pr\{\mathfrak{L}_t^\beta \in dx\}/dx$ solves the fractional equation $\frac{\partial^\beta u}{\partial t^\beta}(x,t) = -\frac{\partial u}{\partial x}(x,t)$. The inverse $\mathfrak{L}_t^\beta$ of a β-stable subordinator $\mathfrak{H}_t^\beta$ can be defined by the following relationship

$$Pr\{\mathfrak{L}_t^\beta < x\} = Pr\{\mathfrak{H}_x^\beta > t\}$$

for $x, t > 0$ (see, e.g. Meerschaert and Sikorskii, 2012, p. 101) and, we have that $\mathbb{E}e^{-\lambda \mathfrak{L}_t^\beta} = E_\beta(-\lambda t^\beta)$, $\lambda \geq 0$, where the one-parameter Mittag-Leffler function is defined as (see, e.g., Meerschaert and Sikorskii, 2012, p. 35)

$$E_\beta(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(\beta k+1)}, \quad x \in \mathbb{R},\ \beta > 0. \tag{3.2}$$

The random field T introduced in (2.2) is Gaussian and therefore its Fourier coefficients $a_{l,m}$ are independent, complex-valued, zero-mean Gaussian r.v.'s. Denote by $F(\mathfrak{L}_t^\beta)$ the subordinator with symbol Ψ time-changed by the inverse of a stable subordinator of order $\beta \in (0,1)$.

From now on we consider random fields of the form

$$\sum_{l=0}^\infty \sum_{m=-l}^{+l} a_{l,m}\, \mathcal{T}_l(t)\, \mathcal{Y}_{l,m}(x) \tag{3.3}$$

with

$$\mathcal{T}_l(t) = \mathbb{E}[\exp -\mu_l F(\mathcal{L}_t^\beta)], \quad l \geq 0,\ t \geq 0, \tag{3.4}$$

where the series (3.3) converges in $L^2(dP \times d\lambda)$ sense for all $t \geq 0$, that is

$$\lim_{L\to\infty} \mathbb{E}\left[\int_{\mathbf{S}_1^2} \left(X_t(x) - \sum_{l=0}^L \sum_{m=-l}^{+l} a_{l,m}\, \mathcal{T}_l(t)\, \mathcal{Y}_{l,m}(x)\right)^2 \lambda(dx)\right] = 0, \quad \forall t. \tag{3.5}$$

We now pass to the first theorem.

Theorem 1. *Let us consider $\gamma \geq 0$ and $\beta \in (0,1)$. The solution in $L^2(dP \times d\lambda)$ to the fractional equation*

$$\left(\gamma - \mathbb{D}_M + \frac{\partial^\beta}{\partial t^\beta}\right) X_t(x) = 0, \quad x \in \mathbf{S}_1^2,\ t \geq 0 \tag{3.6}$$

with initial condition $X_0(x) = T(x)$ is a time-dependent random field on the sphere $\mathbf{S}_1^2$ written as

$$X_t(x) = \sum_{l=0}^\infty \sum_{m=-l}^{+l} a_{l,m} E_\beta\left(-t^\beta(\gamma + \Psi(\mu_l))\right) \mathcal{Y}_{l,m}(x) \tag{3.7}$$

where

$$a_{l,m} = \int_{\mathbf{S}_1^2} X_0(x) \mathcal{Y}_{l,m}^*(x) \lambda(dx). \tag{3.8}$$

Furthermore, the following representation holds

$$X_t(x) = \mathbb{E}\left[T(x + B(\gamma \mathfrak{L}_t^\beta + F(\mathfrak{L}_t^\beta))) \big| \mathfrak{F}_T\right] \tag{3.9}$$

where $\mathfrak{F}_T$ is the σ-field generated by $X_0 = T$.

Proof. First we notice that

$$-\frac{\partial}{\partial t}\mathbb{E}e^{-\xi(\gamma t+F(t))}\Big|_{t=0}=\xi\gamma+\Psi(\xi) \tag{3.10}$$

where $\mathbb{E}e^{-\xi(\gamma t+F(t))}$ coincides with (2.11) for $\gamma=b$. Indeed, we are dealing with the symbol Ψ of the subordinator F_t without drift. Furthermore, it is well-known that the Mittag-Leffler function E_β is the eigenfunction of the D–C fractional derivative, that is

$$\frac{\partial^\beta}{\partial t^\beta}E_\beta(-t^\beta\mu)=-\mu E_\beta(-t^\beta\mu),\quad \mu>0. \tag{3.11}$$

Assume that (3.7) holds true. From the fact that

$$\mathbb{D}_M\,\mathcal{Y}_{l,m}(x)=\int_0^\infty\left(P_s\mathcal{Y}_{l,m}(x)-\mathcal{Y}_{l,m}(x)\right)M(ds)$$

where $\mathcal{Y}_{l,m}(x)=(-1)^m\mathcal{Y}^*_{l,-m}(x)$ and

$$P_s\mathcal{Y}_{l,m}(x)=e^{-s\mu_l}\mathcal{Y}_{l,m}(x) \tag{3.12}$$

we obtain that

$$\begin{aligned}\mathbb{D}_M\,\mathcal{Y}_{l,m}(x)&=\int_0^\infty\left(e^{-s\mu_l}\mathcal{Y}_{l,m}(x)-\mathcal{Y}_{l,m}(x)\right)M(ds)\\&=\int_0^\infty\left(e^{-s\mu_l}-1\right)M(ds)\,\mathcal{Y}_{l,m}(x)\\&=-\Psi(\mu_l)\,\mathcal{Y}_{l,m}(x).\end{aligned}$$

Formula (3.12) can be obtained by considering that

$$\begin{aligned}P_s\mathcal{Y}_{l,m}(x)&=\mathbb{E}\mathcal{Y}_{l,m}(x+B_s)\\&=\sum_{l'm'}e^{-s\mu_{l'}}\mathcal{Y}^*_{l',m'}(x)\int_{\mathbf{S}^2_1}\mathcal{Y}_{l,m}(y)\mathcal{Y}_{l',m'}(y)\lambda(dy)\\&=\sum_{l'm'}e^{-s\mu_{l'}}\mathcal{Y}^*_{l',m'}(x)(-1)^{m'}\int_{\mathbf{S}^2_1}\mathcal{Y}_{l,m}(y)\mathcal{Y}^*_{l',-m'}(y)\lambda(dy)\\&=\sum_{l'm'}e^{-s\mu_{l'}}\mathcal{Y}^*_{l',m'}(x)(-1)^{m'}\delta_l^{l'}\delta_m^{-m'}=e^{-s\mu_l}\mathcal{Y}_{l,m}(x).\end{aligned}$$

Thus, we get that

$$(\gamma-\mathbb{D}_M)X_t(x)=\sum_{l=0}^\infty\sum_{m=-l}^{+l}a_{l,m}\left(\gamma+\Psi(\mu_l)\right)E_{\beta,1}\left(-t^\beta\gamma-t^\beta\Psi(\mu_l)\right)\mathcal{Y}_{l,m}(x)$$

and, from (3.11), we arrive at

$$\left(\frac{\partial^\beta}{\partial t^\beta}+\gamma-\mathbb{D}_M\right)X_t(x)=\sum_{l=0}^\infty\sum_{m=-l}^{+l}a_{l,m}\left(\frac{\partial^\beta}{\partial t^\beta}+\gamma+\Psi(\mu_l)\right)E_\beta\left(-t^\beta\gamma-t^\beta\Psi(\mu_l)\right)\mathcal{Y}_{l,m}(x)=0$$

term by term and therefore Eq. (3.6) is satisfied. This concludes the proof. □

Remark 1. Since

$$T(x)=\sum_{lm}a_{l,m}\mathcal{Y}_{l,m}(x) \tag{3.13}$$

we have that

$$P_tT(x)=\mathbb{E}[T(x+B_t)|\mathfrak{F}_T]=\sum_{lm}e^{-t\mu_l}a_{l,m}\mathcal{Y}_{l,m}(x)=T_t(x). \tag{3.14}$$

This represents the solution to (3.6) with $\beta=1$, $\gamma=0$ and $\mathbb{D}_M=\Delta_{\mathbf{S}^2_1}$. From (3.9), for $\beta=1$, $\Psi(\xi)=\xi$, that is for the elementary subordinator $F(t)=t$ (and $\mathfrak{L}^1_t=t$) we have that

$$X_t(x)=\mathbb{E}\left[T(x+B(\gamma t+t))\middle|\mathfrak{F}_T\right]=\sum_{lm}a_{l,m}e^{-t(\gamma+1)\mu_l}\mathcal{Y}_{l,m}(x).$$

Theorem 2. *Let us consider $\gamma > 0, \varphi \geq 0$ and $\beta \in (0,1)$. A solution in $L^2(dP \times d\lambda)$ to the fractional equation*

$$\left(\gamma - \mathbb{D}_M - \varphi \frac{\partial}{\partial t}\right)^{\beta} X_t(x) = T_t(x), \quad x \in \mathbf{S}_1^2, \; t \geq 0 \tag{3.15}$$

where $T_t(x)$ is given in (3.14), is a time-dependent random field on the sphere $\mathbf{S}_1^2$ written as

$$X_t(x) = \sum_{lm} a_{l,m} e^{-t\mu_l} \left(\gamma + \varphi\mu_l + \Psi(\mu_l)\right)^{-\beta} \mathcal{Y}_{l,m}(x), \tag{3.16}$$

where $e^{-t\mu_l} a_{l,m}$ are the random coefficients involved in the representation (3.14) of the innovation process $T_t(x)$ in (3.14) in terms of spherical harmonics.

Proof. We have that

$$\begin{aligned} X_t(x) &= \left(\gamma - \mathbb{D}_M - \varphi \frac{\partial}{\partial t}\right)^{-\beta} T_t(x) \\ &= \int_0^\infty ds \, \frac{s^{\beta-1}}{\Gamma(\beta)} e^{s\varphi \frac{\partial}{\partial t} - s\gamma + s\mathbb{D}_M} T_t(x) \\ &= \int_0^\infty ds \, \frac{s^{\beta-1}}{\Gamma(\beta)} e^{s\varphi \frac{\partial}{\partial t} - s\gamma} \mathbb{P}_s T_t(x) \\ &= \int_0^\infty ds \, \frac{s^{\beta-1}}{\Gamma(\beta)} e^{-s\gamma} \mathbb{P}_s T_{t+\varphi s}(x) \end{aligned}$$

where we used the translation rule

$$e^{a \frac{\partial}{\partial z}} f(z) = f(z+a), \quad a \in \mathbf{R}$$

which holds for bounded continuous functions f on $(0, +\infty)$ (see, for example, formula (3.9) in D'Ovidio, 2015 and the references therein for details). From the fact that

$$\mathbb{P}_s \mathcal{Y}_{l,m}(x) = e^{-s\Psi(\mu_l)} \mathcal{Y}_{l,m}(x) \tag{3.17}$$

where $\mathbb{P}_s = \exp(s\mathbb{D}_M)$ we get that

$$\begin{aligned} X_t(x) &= \sum_{lm} a_{l,m} \left(\int_0^\infty ds \, \frac{s^{\beta-1}}{\Gamma(\beta)} e^{-s\gamma} e^{-(t+\varphi s)\mu_l} \mathbb{P}_s \mathcal{Y}_{l,m}(x)\right) \\ &= \sum_{lm} a_{l,m} \left(\int_0^\infty ds \, \frac{s^{\beta-1}}{\Gamma(\beta)} e^{-s\gamma} e^{-(t+\varphi s)\mu_l} e^{-s\Psi(\mu_l)}\right) \mathcal{Y}_{l,m}(x) \\ &= \sum_{lm} a_{l,m} e^{-t\mu_l} \left(\int_0^\infty ds \, \frac{s^{\beta-1}}{\Gamma(\beta)} e^{-s\gamma - s\varphi\mu_l - s\Psi(\mu_l)}\right) \mathcal{Y}_{l,m}(x) \\ &= \sum_{lm} a_{l,m} e^{-t\mu_l} \left(\gamma + \varphi\mu_l + \Psi(\mu_l)\right)^{-\beta} \mathcal{Y}_{l,m}(x) \end{aligned}$$

and this concludes the proof. □

We now examine the special case $\varphi = 0$.

Corollary 1. *Let $\beta \in (0, 1]$. The solution to*

$$(\gamma - \mathbb{D}_M)^{\beta} X(x) = T(x) \tag{3.18}$$

is written as

$$X(x) = \sum_{lm} a_{l,m} \left(\gamma + \Psi(\mu_l)\right)^{-\beta} \mathcal{Y}_{l,m}(x). \tag{3.19}$$

Proof. For $\beta \in (0, 1)$ we consider the following relation concerning the fractional power of operators (Bessel potential). For $f \in L^2(\mathbf{S}_1^2)$ we have that

$$(\gamma - \mathbb{D}_M)^\beta f(x) = \frac{\beta}{\Gamma(1-\beta)} \int_0^\infty \frac{ds}{s^{\beta+1}} \left(1 - e^{-s\gamma + s\mathbb{D}_M}\right) f(x)$$
$$= \frac{\beta}{\Gamma(1-\beta)} \int_0^\infty \frac{ds}{s^{\beta+1}} \left(f(x) - e^{-s\gamma} \mathbb{P}_s f(x)\right)$$

where, we recall that $\mathbb{P}_s f$ is the transition semigroup associated with the operator $\mathbb{D}_M$ and $u(x, t) = \mathbb{P}_t f(x)$ solves the Cauchy problem $(\frac{\partial}{\partial t} - \mathbb{D}_M)u(x, t) = 0$ with $u(x, 0) = f(x)$. Therefore, if we assume that there exists the following spectral representation for the solution X as a random function on $\mathbf{S}_1^2$,

$$X(x) = \sum_{lm} \hat{a}_{l,m} \mathcal{Y}_{l,m}(x) \tag{3.20}$$

then we can immediately write

$$(\gamma - \mathbb{D}_M)^\beta X(x) = \frac{\beta}{\Gamma(1-\beta)} \sum_{lm} \hat{a}_{l,m} \int_0^\infty \frac{ds}{s^{\beta+1}} \left(\mathcal{Y}_{l,m}(x) - e^{-s\gamma} \mathbb{P}_s \mathcal{Y}_{l,m}(x)\right)$$
$$= \frac{\beta}{\Gamma(1-\beta)} \sum_{lm} \hat{a}_{l,m} \int_0^\infty \frac{ds}{s^{\beta+1}} \left(1 - e^{-s\gamma} e^{-s\Psi(\mu_l)}\right) \mathcal{Y}_{l,m}(x)$$
$$= \sum_{lm} \hat{a}_{l,m} \left(\gamma + \Psi(\mu_l)\right)^\beta \mathcal{Y}_{l,m}(x).$$

Eq. (3.18) turns out to be satisfied only if

$$\hat{a}_{l,m} = a_{l,m} \left(\gamma + \Psi(\mu_l)\right)^{-\beta}.$$

On the other hand, by repeating the arguments of the proof of Theorem 2 we have that

$$X(x) = (\gamma - \mathbb{D}_M)^{-\beta} T(x)$$
$$= \int_0^\infty ds \frac{s^{\beta-1}}{\Gamma(\beta)} e^{-s\gamma + s\mathbb{D}_M} T(x)$$
$$= \int_0^\infty ds \frac{s^{\beta-1}}{\Gamma(\beta)} e^{-s\gamma} \mathbb{P}_s T(x)$$
$$= \sum_{lm} a_{l,m} \int_0^\infty ds \frac{s^{\beta-1}}{\Gamma(\beta)} e^{-s\gamma} e^{-s\Psi(\mu_l)} \mathcal{Y}_{l,m}(x)$$
$$= \sum_{lm} a_{l,m} \left(\gamma + \Psi(\mu_l)\right)^{-\beta} \mathcal{Y}_{l,m}(x).$$

This confirms result (3.19). □

We now study the covariance of the random fields introduced so far. Let us consider the representation

$$X_t(x) = \sum_{lm} a_{l,m} \mathcal{T}_l(t) \mathcal{Y}_{l,m}(x) = \sum_l \mathcal{T}_l(t) T^l(x) \tag{3.21}$$

already introduced in (3.3). We also recall that, for $x, y \in \mathbf{S}_1^2$,

$$\mathbb{E}[X_0(x)\, X_0(y)] = \sum_l \frac{2l+1}{4\pi} C_l\, Q_l(\langle x, y\rangle) = \mathbb{E}[T(x)\, T(y)]. \tag{3.22}$$

Furthermore,

$$\mathbb{E}[T(x)\, T(y)] = \sum_l \mathbb{E}[T^l(x)\, T^l(y)]. \tag{3.23}$$

This is due to the fact that the coefficients $a_{l,m}$ are uncorrelated over l.

Remark 2. Let us consider (3.3). We observe that

- for $X_t(x)$ as in Theorem 1,

$$\mathcal{T}_l(t) = E_\beta\left(-t^\beta(\gamma + \Psi(\mu_l))\right) \geq \frac{1}{1+\Gamma(1-\beta)t^\beta(\gamma+\Psi(\mu_l))}, \quad t \geq 0,\ l \geq 0. \tag{3.24}$$

  For this inequality, consult Simon (2014, Theorem 4).
- for $X_t(x)$ as in Theorem 2,

$$\mathcal{T}_l(t) = e^{-t\mu_l}\left(\gamma + \varphi\mu_l + \Psi(\mu_l)\right)^{-\beta} \leq e^{-tl^2}\left(\gamma + \varphi l^2 + \Psi(l^2)\right)^{-\beta}, \quad t \geq 0,\ l \geq 0 \tag{3.25}$$

- for the stationary case $X(x)$ of Corollary 1,

$$\mathcal{T}_l(t) = (\gamma + \Psi(\mu_l))^{-\beta} \leq \left(\gamma + \Psi(l^2)\right)^{-\beta}, \quad t \geq 0,\ l \geq 0. \tag{3.26}$$

Remark 3. Let $B(\tau_t) = B \circ \tau_t$ be a time-changed Brownian motion on the unit sphere where τ_t is a random time process. We refer to $B(\tau_t)$ as a coordinates change for the random field on the sphere T. From the previous results, we observe that

$$X_t(x) = \mathbb{E}[T_0(x + B(\tau_t))|\mathfrak{F}_T] = \mathbb{E}[T_{\tau_t}(x)|\mathfrak{F}_T], \quad x \in \mathbf{S}^2_1,\ t > 0 \tag{3.27}$$

where

$$T_t(x) = \sum_l e^{-t\mu_l} T^l(x), \quad x \in \mathbf{S}^2_1,\ t > 0. \tag{3.28}$$

Moreover,

$$\mathcal{T}_l(t) = \mathbb{E}e^{-\mu_l \tau_t}. \tag{3.29}$$

We can state the following result for which the spherical Brownian motions $x + B_t$, $y + B_t$ underlying $X_t(x) = T(x + B_t)$ and $X_t(y) = T(y + B_t)$ are assumed to be independent.

Theorem 3. *For $x, y \in \mathbf{S}^2_1$, for all $g \in SO(3)$, we have that*

$$\mathbb{E}[X_t(gx)\, X_s(gy)] = \sum_l \frac{2l+1}{4\pi}\, C_l\, \mathcal{T}_l(t)\, \mathcal{T}_l(s)\, Q_l(\langle x, y\rangle), \quad t, s \geq 0. \tag{3.30}$$

Proof. First we observe that

$$\mathbb{E}[a_{l,m} a_{l',m'}] = (-1)^m \delta_l^{l'} \delta_{-m}^{m'} C_l \tag{3.31}$$

from the property $\mathcal{Y}_{l,m}(x) = (-1)^m \mathcal{Y}^*_{l,-m}(x)$ of the spherical harmonics. From the representation (3.21) we can write

$$\begin{aligned}
\mathbb{E}[X_t(x)\, X_s(y)] &= \sum_{lm}\sum_{l'm'} \mathbb{E}[a_{l,m} a_{l'm'}] \mathcal{T}_l(t) \mathcal{T}_{l'}(s) \mathcal{Y}_{l,m}(x) \mathcal{Y}_{l'm'}(y) \\
&= \sum_{lm} C_l\, \mathcal{T}_l(t) \mathcal{T}_l(s)\, \mathcal{Y}_{l,m}(x) \mathcal{Y}^*_{l,m}(y) \\
&= \sum_l \frac{2l+1}{4\pi} C_l\, \mathcal{T}_l(t) \mathcal{T}_l(s)\, Q_l(\langle x, y\rangle)
\end{aligned}$$

where $\mathcal{T}_l(t)$ is given as in (3.29) and we used the addition formula in order to arrive at $Q_l(\langle x, y\rangle)$. □

Remark 4. We can immediately see that the variance becomes

$$\mathbb{E}[X_t(gx)]^2 = \sum_l \frac{2l+1}{4\pi} C_l\, |\mathcal{T}_l(t)|^2, \quad \forall g \in SO(3). \tag{3.32}$$

We recall that C_l is the angular power spectrum of T and, is usually assumed to be $C_l \sim l^{-\gamma}$ with $\gamma > 2$ for large l to ensure summability (or $C_l \sim L(l)/l^\theta$, $\theta > 0$ where $L(\cdot)$ is a slowly varying function as $l \to \infty$). As Remark 2 shows we have the high-frequency behaviour also for $\mathcal{T}_l(t)$ in both the variable $t > 0$ and the frequency $l > 0$. The convergence of (3.32) therefore entails different correlation structures for the solutions $X_t(x)$ of the equations investigated so far.

We say that the zero mean process $X_t(x)$ exhibits a long range dependence if

$$\sum_{h=1}^{\infty} \mathbb{E}[X_{t+h}(x)X_t(y)] = \infty, \quad x, y \in \mathbf{S}^2_1. \tag{3.33}$$

Conversely, we say that X exhibits a short range dependence if the series (3.33) converges. The series (3.33) diverges if the dependence of the r.v.'s of the processes $X_t(\cdot)$ slowly decreases in time.

Remark 5. We write

$$\mathcal{K}_t(x, y) = \sum_{h\geq 1} \mathbb{E}[X_{t+h}(x)\, X_t(y)], \quad t \geq 0$$

for $x, y \in \mathbf{S}^2_1$. From the discussion above, we have that:

- write

$$g_l(t) = \frac{1}{1 + \Gamma(1-\beta)t^\beta(\gamma + \Psi(\mu_l))},$$

  for $X_t(x)$ as in Theorem 1,

$$\begin{aligned}
\mathcal{K}_t(x, y) &= \sum_{h\geq 1}\sum_{l\geq 0} \frac{2l+1}{2\pi} C_l E_\beta(-t^\beta(\gamma + \Psi(\mu_l)))E_\beta(-(t+h)^\beta(\gamma + \Psi(\mu_l)))Q_l(\langle x, y\rangle)\\
&\geq \sum_{h\geq 1}\sum_{l\geq 0} \frac{2l+1}{2\pi} C_l\, g_l(t)\, g_l(t+h)\, Q_l(\langle x, y\rangle)\\
&\geq \sum_{l\geq 0} \frac{2l+1}{2\pi} C_l g_l(t)\, Q_l(\langle x, y\rangle) \sum_{h\geq 1} g_l(t+h)\\
&= \infty,
\end{aligned}$$

  that is the random field exhibits a long-range dependence;
- for $X_t(x)$ as in Theorem 2,

$$\begin{aligned}
\mathcal{K}_t(x, y) &\leq \sum_{h\geq 1}\sum_{l\geq 0} \frac{2l+1}{4\pi} C_l\, e^{-2tl^2-hl^2} \left(\gamma + \varphi l^2 + \Psi(l^2)\right)^{-2\beta} Q_l(\langle x, y\rangle)\\
&\leq \sum_{l\geq 0} \frac{2l+1}{4\pi} \frac{C_l}{e^{l^2}-1} e^{-2tl^2} \left(\gamma + \varphi l^2 + \Psi(l^2)\right)^{-2\beta} Q_l(\langle x, y\rangle)\\
&< \infty,
\end{aligned}$$

  that is the random field has a short range dependence.

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