Mathematical analysis of the small oscillations of a bubble in a cylindrical liquid column under gravity zero

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Abstract: This paper deals with the mathematical study of the small oscillations of a system formed by a a cylindrical liquid column bounded by two parallel circular disks and an internal spherical bubble constitued by a barotropic gas, under zero gravity. From the equations of motion, the authors deduce a variational equation. Then the study of the small oscillations depends on the coerciveness of a hermitian form that appears in this equation. It is proved that this last problem is reduced to an auxiliary eigenvalue problem. A careful discussion shows that our problem is a classical vibration problem.

Key–Words: Small oscillations, water waves, gas dynamics, free boundaries, capillarity (surface tension), variational methods,

1 Introduction

The problem of the small oscillatons of an incompressible inviscid liquid under zero gravity, in which the surface tension determine the character of the motion, is very important in the experiments in space laboratories.

This problem has been sudied by numerous researchers [(1), (2), (14), (15), (22), (16), (11), (17)]. The authors have made their contributions in a few papers [(3)-(9), (21)].

In this paper, they study the small motions of a system formed by a cilindrical liquid column bounded by two parallel circular disks, the liquid being anchored at the rim of the disk, and an internal spherical bubble constitued by a barotropic gas under zero gravity.

From the equations of motion in linear theory, they deduce a variational formulation of the problem.

The study of the spectrum depends on the coerciveness of a hermitian sesquilinear form that appears in the variational equation. The authors prove that the last problem is reduced to an auxiliary eigenvalue problem. From a careful discussion, the authors show that the problem is of the small oscillations of the system is a classical vibration problem.

In this paper the researches started in [(9)] are finished.

2 Position of the problem

In the absence of gravity, in the equilibrium position, with respect to orthogonal axes $Oxyz$, the system is formed by (see fig.):

1) an **incompressible inviscid liquid** occupying the domain Ω defined by $x^2+y^2 \leq b^2,\ x^2+y^2+z^2 \geq a^2, -h \leq z \leq z\ (a < b)$.

This domani is bounded by two rigid disks C and C' $(C : z = h, x^2 + y^2 \leq b^2;\ C' : z = -h, x^2 + y^2 \leq b^2)$, the boundaries of which are denoted by c and c', the cilindrical surface S defined by $x^2 + y^2 = b^2, -h \leq z \leq z$ and the spherical surface S_0 defined by $x^2 + y^2 + z^2 = a^2$;

2) a **barotropic gas** which occupies the domain $\Omega_0 : x^2 + y^2 + z^2 \leq a^2$.

We denote by $\vec{n}$ and $\vec{n}_0$ the unit vectors normal to S and S_0 directed to the exterior of S and S_0.

We introduce the cilindrical coordinates (r, θ, z) and the spherical coordinates $(R, \theta, \psi)(0 \leq \theta \leq 2\pi;\ 0 \leq \psi \leq \pi)$.

At the instant t, S (resp. S_0) occupies the position S_t (resp. S_{0t}) defined by equation $r = b + \zeta(\theta, z, t)$ (resp. $R = a + \zeta_0(\theta, \psi, t)$). ζ and ζ_0 and their derivatives will be considered as quantities of the first order with respect to amplitude of the oscillations.

We denote by α (resp.β) the surface tension on S_0 (resp. S).

We suppose that the liquid is anchored at the rims c and c' of the disks C ans C', so that we have

$(1)\ \ \zeta = 0 \quad \text{on}\ c\ \text{and}\ c'.$

3 Equations of motion

We study the problem in the framework of the linear theory: for all details see **ge**.

1) At first, we consider the **liquid**.

If ρ is its constant density, P the pressure, $\vec{u}$ the displacement of a particle with respect to its position at the equilibrium, we have

$$\left.\begin{array}{l}(2)\ \ \rho\,\ddot{\vec{u}} = -\vec{grad}\,P\ (\text{Euler's equation}),\\ div\ \dot{\vec{u}} = 0\ (\text{incompressibility}),\end{array}\right\}\text{in}\ \Omega$$

where $(\ddot{\vec{u}} = \frac{\partial^2\vec{u}}{\partial t^2})$. From the last equations, we deduce by integrating between the dates of the equilibrium and t

$(3)\ \ div\ \vec{u} = 0$ in Ω.

The kinematic conditions are:

$(4)\ \ u_z = 0$ for $z = \pm h, r \leq b$,

$(5)\ \ u_n|_S \overbrace{=}^{def} \vec{u}\cdot\vec{n}|_S = \zeta,$

$(6)\ \ \vec{u}\cdot\vec{n_0}|_{S_0} = \zeta_0.$

The dynamic condition on S_t is given by the Laplace law

$$P - p_a = -\beta\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

where p_a is the constant atmospheric pressure and R_1, R_2 are the principal radii of curvature of S_t.

The formula that gives the mean curvature of a surface that differs from a circular cylinder is well known [10] and we have a

$(7)\ P_{S_t} - p_a = -\beta[-\frac{1}{b} + \frac{1}{b^2}(\zeta_{\theta\theta} + \zeta) + \zeta_{zz}],$ $(\zeta_{\theta\theta} = \frac{\partial^2\zeta}{\partial\theta^2})$.

2) Now, we consider the **gas**.

If ρ^* is the density, P^* the pressure, $\vec{u_0}$ the displacement, we have in Ω_0 :

$(8)\ \ P^* = \mathcal{P}(\rho^*)\ \ $ (equation of state),

where $\mathcal{P}$ is an increasing function of ρ^*,

$(9)\ \ \rho^*\ddot{\vec{u_0}} = -\vec{grad}\ P^*\ $ (Euler's equation)

$(10)\ \ \dfrac{\partial\rho^*}{\partial t} + div(\rho^*\dot{\vec{u_0}}^*) = 0\ \ $ (continuity equation).

The kinematic condition is

$(11)\ \ \vec{u_0}\cdot\vec{n_0}|_{S_0} = \vec{u}\cdot\vec{n_0}|_{S_0} = \zeta_0.$

The Laplace law gives, using the formula giving the mean curvature of a surface that differs little form a sphere [(13)]

$(12)\ P^*_{S_{0t}} - P_{S_{0t}} = \alpha\Big[\frac{2}{a} - \frac{2\zeta_0}{a^2} -$ $\frac{1}{a^2}\left(\frac{1}{\sin^2\psi}\zeta_{0\theta\theta} + \frac{1}{\sin\psi}\frac{\partial}{\partial\psi}(\sin\psi\zeta_{0\psi})\right)\Big].$

We are to linearize the equations of motion of the gas.

We set

$$\rho^* = \rho_0 + \tilde{\rho} + ...\quad,\quad P^* = P_0 + p_0 + ...\quad,$$

where $\tilde{\rho}$ and p_0 are of the first order and the dots represent terms of more large order.

The continuity equation gives, after integration

$$\tilde{\rho} = -\rho_0\ div\ \dot{\vec{u}}_0$$

and the equations of state

$(13)\ \ p_0 = c_0^2\tilde{\rho} = -c_0^2\rho_0\ div\ \vec{u}_0,\ c_0^2 = \mathcal{P}'(\rho_0).$

The Euler's equation becomes

$$\rho_0\ \ddot{\vec{u}}_0 = -\vec{grad}\,p_0 = c_0^2\ \vec{grad}(div\ \vec{u}_0)\ \ \text{in}\ \Omega_0.$$

If we suppose that the conditions of Lagrange's theorem are satisfied, $\vec{u}_0$ is a gradient.

As the volume of the liquid is constant, we have

$(15)\ \displaystyle\int_S \zeta\ dS - \int_{S_0}\zeta_0\ dS_0 = 0.$

Finally introducing the dynamic pressures p_0 and p we obtain easily:

$(16)\ p|_S = -\frac{\beta}{b^2}(\zeta_{\theta\theta} + \zeta + \zeta_{zz}),$

$(17)\ p_0|_{S_0} - p|_{S_0} =$ $-\frac{\alpha}{a^2}\Big[2\zeta_0 + \frac{1}{\sin^2\psi}\zeta_{0\theta\theta} + \frac{1}{\sin\psi}\frac{\partial}{\partial\psi}\left((\sin\psi\zeta_{0\psi}\right)\Big].$

3.1 Remark

It is easy to see that the right-hand side of (17) is equal to zero if we replace ζ_0 by $\cos\psi, \sin\psi\cos\theta, \sin\psi\sin\theta$. If these functions are solutions of the problem, we must discard them because we are no longer in the framework of the linearization.

4 Variational formulation of the problem

1) We define the space of the kinematically admissive displacements $\vec{\tilde{u}}$ and $\vec{\tilde{u}}_0$. The functions are sufficiently smooth and verify:

$$div\ \vec{\tilde{u}} = 0\ \ \text{in}\ \ \Omega;\quad \tilde{\vec{u}}_z\ \ \text{on}\ \ z = \pm h, r \leq b;$$

we write $\tilde{u}_n|_S = \tilde{\zeta}$ with $\tilde{\zeta} = 0$ for $z = \pm h$; $\tilde{u}_{0n_0}|_{S_0} = \tilde{u}_{n_0}|_{S_0}$ and $\tilde{u}_{0n_0}|_{S_0} = \zeta_0$; moreover $\displaystyle\int_S \tilde{\zeta}\, dS = \int_{S_0} \tilde{\zeta}_0\, dS_0$.

This will started more precisely in what follows.

From the equation (2) and (14) and Green's formula. we obtain

$$\int_\Omega \rho \ddot{\vec{u}} \cdot \bar{\tilde{\vec{u}}}\, d\Omega + \int_{\Omega_0} \rho_0 \ddot{\vec{u}}_0 \cdot \bar{\tilde{\vec{u}}}_0 d\Omega_0 =$$

$$-\int_{\Omega_0} c_0^2\, \rho_0\, div\, \vec{u}_0\;\; div\, \bar{\tilde{\vec{u}}}_0\, d\Omega_0$$

$$-\int_S p|_S \bar{\tilde{u}}_n|_S\, dS - \int_{S_0} (p_0|_{S_0} p|_{S_0}) \bar{\tilde{u}}_{n_0}|_{S_0}\, dS_0,$$

and using (16) and (17):

$$\int_\Omega \rho \ddot{\vec{u}} \cdot \bar{\tilde{\vec{u}}}\, d\Omega + \int_{\Omega_0} \rho_0 \ddot{\vec{u}}_0 \cdot \bar{\tilde{\vec{u}}}_0\, d\Omega_0 +$$

$$\int_{\Omega_0} c_0^2\, \rho_0\, div\, \vec{u}_0\;\; div\, \bar{\tilde{\vec{u}}}_0\, d\Omega_0 - \frac{\beta}{b^2} \int_S (\zeta_{\theta\theta} + \zeta + \zeta_{zz}) \bar{\tilde{\zeta}} dS -$$

$$-\frac{\alpha}{a^2} \int_{S_0} \Big[2\zeta_0 + \frac{1}{\sin^2\psi} \zeta_{0\theta\theta} +$$

$$\frac{1}{\sin\psi} \frac{\partial}{\partial\psi} \Big(\sin\psi \zeta_{0\psi} \Big) \Big] \bar{\tilde{\zeta}} dS_0 = 0.$$

Integrating by parts the last two integrals, we obtain the *formal variational equation* of the problem:

$$(18)\;\; \int_\Omega \rho \ddot{\vec{u}} \cdot \bar{\tilde{\vec{u}}}\, d\Omega + \int_{\Omega_0} \rho_0 \ddot{\vec{u}}_0 \cdot \bar{\tilde{\vec{u}}}_0\, d\Omega_0 +$$

$$c_0^2\, \rho_0 \int_{\Omega_0} div\, \vec{u}_0\;\; div\, \bar{\tilde{\vec{u}}}_0\, d\Omega_0 +$$

$$\frac{\beta}{b^2} \int_S \Big(\zeta_\theta \bar{\tilde{\zeta}}_\theta + b^2 \zeta_z \bar{\tilde{\zeta}}_z - \zeta \bar{\tilde{\zeta}} \Big) dS +$$

$$\frac{\alpha}{a^2} \int_{S_0} \Big(\frac{1}{\sin^2\psi} \zeta_{0\theta} \bar{\tilde{\zeta}}_{0\theta} + \zeta_{0\psi} \bar{\tilde{\zeta}}_{0\psi} - 2\zeta_0 \bar{\tilde{\zeta}}_0 \Big)\, dS_0 = 0.$$

It is easy to see that the last two terms of (18) is the *virtual work of the surface tension forces*.

2) Let us introduce the Hilbert spaces $H_0^1(S)$,

$$\tilde{H}^1(S_0) = \Big\{ \zeta_0 \in L^2(S_0), \zeta_{0\psi}, \frac{\zeta_{0\theta}}{\sin\psi} \in L^2(S_0) \Big\},$$

$$\mathcal{H}_1 = \Big\{ \mathrm{Z} = \begin{pmatrix} \zeta \\ \zeta_0 \end{pmatrix} \in \mathrm{H}_0^1(\mathrm{S}) \oplus \tilde{\mathrm{H}}^1(\mathrm{S}_0);$$

$$\int_S \zeta dS - \int_{S_0} \zeta_0 dS_0 = 0 \Big\},$$

equipped with the norm

$$\Big\| Z \Big\|_{H^1} = \Big[\int_S \Big(\zeta_\theta^2 + b^2 \zeta_z \Big) dS +$$

$$\int_{S_0} \Big(\frac{1}{\sin^2\psi} \zeta_{0\theta}^2 + \zeta_{0\psi}^2 - 2\zeta_0^2 \Big)\, dS_0 \Big]^{1/2}.$$

Recalling that $\vec{u}$ and $\vec{u_0}$ are gradient, we introduce now the space V :

$$V = \Big\{ U = \begin{pmatrix} \vec{u} \\ \vec{u_0} \end{pmatrix};\; \vec{u} = \vec{grad}\ \varphi, \tilde{\varphi} \in$$

$\tilde{H}^1(\Omega) \overset{def}{\widehat{=}} \{\varphi \in H^1(\Omega),\ \int_\Omega \varphi\, d\Omega;\ div\, \vec{u} = 0\}$;
$u_n|_S \in H_0^1(S),\ u_n = 0$ for $\pm h$; $\vec{u_0} = \vec{grad}\ \varphi_0$; $\tilde{\varphi_0} \in \tilde{H}_1(\Omega_0)$; $div\ \vec{u_0} \in L^2(\Omega_0)$; $u_{0n_0}|_{S_0} = u_{n_0}|_{S_0} \in \tilde{H}^1(S_0)$, $\int_S u_n|_S dS - \int_{S_0} u_{0n_0}|_{S_0} dS_0 = 0 \Big\}$

equipped with the norm defined by

$$\Big\| U \Big\|_V^2 = \int_\Omega \rho |\vec{u}|^2 d\Omega + ||u_n|_S||^2_{H_0^1(S)} + \int_{\Omega_0} \rho |\vec{u_0}|^2 d\Omega_0$$

$$+ \int_{\Omega_0} |div\ \vec{u_0}|^2 d\Omega_0 + ||u_{0n_0}|_{S_0}||^2_{\tilde{H}^1(S_0)},$$

and the space H, completion of V for the norm associated to the scalar product

$$(U, \tilde{U})_H = \int_\Omega \rho \vec{u} \cdot \bar{\tilde{\vec{u}}} d\Omega + \int_{\Omega_0} \rho \vec{u_0} \cdot \bar{\tilde{\vec{u}}}_0 d\Omega_0.$$

Denoting by $M(Z, \tilde{Z})$ the last two terms of (18), we obtain the *precise variational equation* of the problem:

$$(19) (\ddot{U}, \tilde{U})_H + \rho_0 c_0^2 \int_{\Omega_0} div\ \vec{u} \cdot \vec{u_0} d\Omega_0 + M(Z, \tilde{Z}) = 0$$

$\forall\, \tilde{U} \in V$.

5 Study of the sesquilinear form $M(Z, \tilde{Z})$

1) This form is obviously *hermitian* and *continuous* on $\mathcal{H}_1$.

In order to study its coercivness, we use a method that we can find in the book 16, so that we will sketch the proof.

We set

$$\lambda = \underbrace{inf}_{Z \in \mathcal{H}^1} \Big(\frac{\frac{\beta}{b^2} \int_S (\zeta_\theta^2 + b^2 \zeta_z^2) dS}{\frac{\beta}{b^2} \int_S \zeta^2 dS + \frac{\alpha}{a^2} \int_{S_0} 2\zeta_0^2 dS_0} +$$

$$\frac{\frac{\alpha}{a^2} \int_{S_0} (\frac{1}{\sin^2\psi} \zeta_{0\theta}^2 + \zeta_{0\psi}^2)\, dS_0}{\frac{\beta}{b^2} \int_S \zeta^2 dS + \frac{\alpha}{a^2} \int_{S_0} 2\zeta_0^2 dS_0} \Big)$$

We can prove that there exists $\hat{Z} \in \mathcal{H}_1$ such that λ is the value of the ratio for $\zeta = z\hat{e}ta$ and $\lambda > 0$.

By defintion of λ, we have $\frac{\beta}{b^2}\int_S(\zeta_\theta^2+b^2\zeta_z^2)dS+\frac{\alpha}{a^2}\int_{S_0}(\frac{1}{\sin^2\psi}\ \zeta_{0\theta}^2+\zeta_{0\psi}^2)\ dS_0$

$$-\lambda\Big[\frac{\beta}{b^2}\int_S\zeta^2dS+\frac{\alpha}{a^2}\int_{S_0}2\zeta_0 dS_0\Big]\geq 0\ \forall\ Z\in\mathcal{H}_1.$$

Setting $Z=\hat{Z}+\varepsilon\delta Z,\epsilon\in[-\infty,+\infty],\delta Z=\begin{pmatrix}\delta\zeta\\ \delta\zeta_0\end{pmatrix}\in\mathcal{H}_1$, we can see that the inequality is possible for each $\varepsilon\in[-\infty,+\infty]$ only if the coefficient of 2ε is equal to zero

$\forall\delta\zeta\in H_0^1(S)$, $\forall\delta\zeta_0\in\tilde{H}_0^1(S_0)$, $\int_S\zeta d\zeta-\int_{S_0}\zeta_0 dS_0=0$.

Introducing the multiplier μ associated to the last condition, we obtain

$$\int_S\Big\{\frac{\beta}{b^2}\Big[(\hat{\zeta_\theta}\delta\zeta_\theta+b^2\hat{\zeta}_z\delta\zeta_z-\lambda\hat{\zeta}\delta\zeta\Big]+\mu\delta\zeta\Big\}dS$$
$$+\int_{S_0}\Big\{\frac{\alpha}{a^2}\Big[\frac{1}{\sin^2\psi}\ \hat{\zeta_{0\theta}}\delta\zeta_{0\theta}+\hat{\zeta_{0\psi}}\delta\zeta_{0\psi}-2\lambda\hat{\zeta}_0\delta\zeta_0\Big]$$
$$-2\mu\delta\zeta_0\Big\}\ dS_0=0\ \forall\delta\zeta\in\mathcal{H}_0^1(\mathrm{S}),\forall\delta\zeta_0\in\hat{\mathcal{H}}^1(\mathrm{S}_0).$$

Since $\mathcal{D}(\mathrm{S})\subset\mathcal{H}_0^1(\mathrm{S})$, we have

$$\frac{\beta}{b^2}\left(\hat{\zeta_{\theta\theta}}+b^2\hat{\zeta_{zz}}+\lambda\hat{\zeta}\right)-\mu=0\text{ in }\mathcal{D}'(\mathrm{S}),$$

and by virtue of a Schwartz theorem on the elliptic equation 18, in the classical sense.

By virtue of known results concerning the Laplace-Beltrami operator on the sphere [8], we obtai the classical equation

$$\frac{\alpha}{a^2}\Big[\frac{1}{\sin^2\psi}\ \hat{\zeta_{0\theta\theta}}+\frac{1}{\sin\psi}\frac{\partial}{\partial\psi}(\sin\psi\hat{\zeta_{0\psi}})+2\lambda\hat{\zeta_0}\Big]+2\mu=0.$$

Eliminating λ and using the condition $\int_S\zeta d\zeta-\int_{S_0}\zeta_0 d\zeta_0=0$, we obtain

$$\mu=\frac{b^3}{k}\int_0^{2\pi}\Big[\hat{\zeta}_z(\theta,h)-\hat{\zeta}_z(\theta,-h)\Big]d\theta,\ \text{with}$$

$$k=2\pi(\frac{a^4}{\alpha}+\frac{2b^3h}{b}).$$

Finally, we obtain $\underbrace{inf}_{\mathcal{H}^1}$ *by solving the classical eigenvalues problem* $\mathcal{P}_\lambda$ *:*

(20)$\zeta_{\theta\theta}+b^2\zeta_{zz}+\lambda\zeta-\frac{b^5}{\beta k}\int_0^{2\pi}\Big[\hat{\zeta}_z(\theta,h)-\hat{\zeta}_z(\theta,-h)\Big]d\theta=$,

(21)$\frac{1}{\sin^2\psi}\ \zeta_{0\theta\theta}+\frac{1}{\sin\psi}\frac{\partial}{\partial\psi}(\sin\psi\zeta_{0\psi})+2\lambda\zeta_0+\frac{a^2b^3}{\alpha k}\int_0^{2\pi}\Big[\hat{\zeta}_z(\theta,h)-\hat{\zeta}_z(\theta,-h)\Big]d\theta=0,$

(22) $\zeta,\zeta_0 2\pi-\text{periodic in }\theta,$

(23) $\zeta(\theta,\pm h)=0,$

(24) $\int_S\zeta dS-\int_{S_0}\zeta_0 dS_0=0$

and λ is the smallest eigenvalue of the problem.

2) For solving the problem $\mathcal{P}_\lambda$, we use the method of separation of the variables.

We seek the solution in the form

$$\zeta=\Theta(\theta)\chi(z)\ ;\ \zeta_0=\Theta_0(\theta)\Psi_0(\psi).$$

We carry out in (20) e (21) and the conditions (23), (24) give

(25) $\chi(\pm h)=0,$

(26) $\int_0^{2\pi}\Theta(\theta)d\theta\cdot\int_h^h\chi(z)bdz=\int_0^{2\pi}\Theta_0(\theta)d\theta\cdot\int_0^\pi\Psi_0(\psi)a^2\sin\psi d\psi.$

The discussion is a little long, but not difficult, so we sketch it.

We must distinguish four case.

5.1 Case I $\int_0^{2\pi}\Theta(\theta)d\theta=0;\int_0^{2\pi}\Theta_0(\theta)d\theta=0$

1) At first, we have

$$\Theta''\chi+\Theta(b^2\chi''+\lambda\chi)=0$$

and consequently

$$\Theta(\theta)=A_n\cos n\theta+B_n\sin n\theta$$

$(A_n,B_n$ constants; $n=1,2,...)$

and the classical problem

$$\chi''+\frac{\lambda-n^2}{b^2}=0\ ;\ \chi(\pm h)=0\ (n=1,2,...).$$

We obtain easily for the problem $\mathcal{P}_\lambda$ the double eigenvalues

$$\lambda=n^2+\frac{k^2\pi^2b^2}{h^2}\ ,\ \lambda=n^2+\frac{(2k-1)^2\pi^2b^2}{4h^2}\ ,\ (k=1,2,...),$$

that are strictly greater than 1.

2) We have the equation

$$\Theta_0''\Psi_0+\Theta_0\Big[\sin\psi\frac{d}{d\psi}(sin\psi\Psi_0')+2\lambda\Psi_0\sin^2\psi\Big]=0$$

and consequently

$$\Theta_0(\theta)=A_{0n}\cos n\theta+B_{0n}\sin n\theta,(n=1,2,...)$$

and setting $\xi=\cos\psi$

$$\begin{cases}-\frac{d}{d\xi}\Big[(1-\xi^2)\Big]\frac{d\psi_0}{d\xi}]+\frac{n^2}{1-\xi^2}\psi_0=2\lambda\psi_0,\\ \psi_0\text{ regular in }\xi=\pm1,\end{cases}$$

or

$$\mathcal{L}_{\rm n}\Psi_0 = 2\lambda\Psi_0$$

where $\mathcal{L}_{\rm n}$ is the Legendre operator with index n [(10)].

The eigenfunctions are the Legendre functions $P_m^n(\xi)(m = n, n+1, ...)$ associated to the Legendre polinomials P_m.

For the problem $\mathcal{P}_\lambda$, we have for $n = 1, 2, ...$ the eigenfunctions

$$\cos n\theta P_{n+p}^n(\cos\psi)\ ,\ \sin n\theta P_{n+p}^n(\sin\psi)\ , p = 0, 1, 2, ...$$

with the double eigenvalues $\lambda = \frac{(n+p)(n+p+1)}{2}$.

We remark that, for $n = 1, p = 0$, we have $\lambda = 1$ and the eigenfunctions

$$\cos\theta P_1^1(\cos\psi)\ ,\ \sin\theta P_1^1(\sin\psi)\ ,$$

that we will obliged to discard by virtue of a provious remark.

The other eigenvalues are strictly grater than 1.

5.2 Case II. $\int_0^{2\pi}\Theta(\theta)d\theta \neq 0; \int_0^{2\pi}\Theta_0(\theta)d\theta = 0$

We find easily

$$\chi^{''} + \frac{\lambda}{b^2}\chi = 0\ ;\ \chi(\pm h) = 0\ ;\ \int_{-h}^{h}\chi(z)dz = 0$$

and

$$\Theta = \text{constant}.$$

For the problem $\mathcal{P}_\lambda$, we have the eigenvalues

$$\lambda = \frac{k^2\pi^2 b^2}{h^2}\ ,\ k = 1, 2...$$

The smallest eigenvalue is obtained for $k = 1$; it is $\frac{\pi^2 b^2}{h^2}$. It is greater tha 1 if $\frac{h}{b} < \pi$. Under this condition, all the other eigenvalues are strictly grater than 1.

It is easy to see that the problem for Ψ_0 and Θ_0 is the problem treated in the case I.

5.3 Case III.$\int_0^{2\pi}\Theta(\theta)d\theta = 0; \int_0^{2\pi}\Theta_0(\theta)d\theta \neq 0$

For Θ and χ, we obtain the problem treated in the case I.

We find easily Θ_0 =constant and for Ψ_0 the problem

$$\frac{1}{\sin\psi}\frac{d}{d\psi}(\sin\psi\Psi_0^{'}) + 2\lambda\Psi_0 = 0;$$

$$\Psi_0 \text{ regular for } \psi = 0, \psi = \pi.$$

Then, we have $\Psi_0 = P_m(\cos\psi)(m = 0, 1, 2, ...)$, where the $P_m(\xi)$ are the Lagendre polinomials, the eigenvalues being $m(m+1)$.

For the problem $\mathcal{P}_\lambda$, the eigenfunctions are $P_m(\cos\psi)$ and the eigenvalues $\lambda = \frac{m(m+1)}{2}$.

For $m = 0$, we have $P_0(\cos\psi) = 1$, that we must discar by virtue of a previuos remark.

The other eigenvalues are strictly greater than 1.

5.4 Case IV. $\int_0^{2\pi}\Theta(\theta)d\theta \neq 0; \int_0^{2\pi}\Theta_0(\theta)d\theta \neq 0$

We obtain easily Θ =constant and Θ_0 =constant. These constants being arbitrary, the condition (26) gives

$$\int_{-h}^{h}\chi(z)dz = 0\ ;\ \int_0^{\pi}\Psi_0(\psi)a^2\sin\psi d\psi = 0,$$

so that, for Θ, χ we find the problem of the case II and for Θ_0, Ψ_0 the problem of the case III.

Finally, by virtue of the properties of eigenfunctions that we have found, we have obtaind all the eigenvalues of the problem $\mathcal{P}_\lambda$.

3) Then, if Z belong to the space

$$\mathcal{H}_0 = \Big\{\mathrm{Z} \in \mathcal{H}_1; \int_{\mathrm{S}_0}\zeta_0\mathrm{P}_1(\cos\psi)\mathrm{dS}_0 = 0;$$

$$\int_{S_0}\zeta_0 P_1^1(\cos\psi)\cos\theta dS_0 = 0;$$

$$\int_{S_0}\zeta_0 P_1^1(\cos\psi)\sin\theta dS_0 = 0\Big\}$$

and if $\frac{h}{b} < \pi$, all the eingenvalues of the problem $\mathcal{P}_\lambda$ is strictly greaten then 1 and we have

$$\lambda_0 = \underbrace{inf}_{Z\in\mathcal{H}^0}\Big(\frac{\frac{\beta}{b^2}\int_S(\zeta_\theta^2 + b^2\zeta_z^2)dS}{\frac{\beta}{b^2}\int_S\zeta^2 dS + \frac{\alpha}{a^2}\int_{S_0}2\zeta_0^2 dS_0} +$$

$$\frac{\frac{\alpha}{a^2}\int_{S_0}(\frac{1}{\sin^2\psi}\ \zeta_{0\theta}^2 + \zeta_{0\psi}^2)\ dS_0}{\frac{\beta}{b^2}\int_S\zeta^2 dS + \frac{\alpha}{a^2}\int_{S_0}2\zeta_0^2 dS_0}\Big) > 1.$$

The, we can write, with $0 < \varepsilon < 1$:

$$M(Z, Z) \geq \varepsilon\Big[\frac{\beta}{b^2}\int_S(\zeta_\theta^2 + b^2\zeta_z^2)dS +$$

$$\frac{\alpha}{a^2}\int_{S_0}(\frac{1}{\sin^2\psi}\ \zeta_{0\theta}^2 + \zeta_{0\psi}^2)\ dS_0\Big] +$$

$$[(1-\varepsilon)\lambda_0 - 1]\Big[\frac{\beta}{b^2}\int_S\zeta^2 dS + \frac{\alpha}{a^2}\int_{S_0}2\zeta_0^2 dS_0\Big].$$

Choosing $0 < \varepsilon < \lambda_0^{-1}$, we see that $M(\cdot,\cdot)$ is coercive in $\mathcal{H}_0$.

6 The problem is a classical vibration problem

1) We introduce the space

$$V_0 = \Big\{ U = \begin{pmatrix} \vec{u} \\ \vec{u_0} \end{pmatrix} \in V;$$

$$\textstyle\int_{S_0} u_{0n_0}|_{S_0} \cdot \begin{cases} P_1(\cos\psi) \\ P_1^1(\cos\psi)\cos\theta \\ P_1^1(\cos\psi)\sin\theta \end{cases} dS_0 = 0 \Big\}$$

equipped with the hilbertean norm defined by

$$\left\| U \right\|_{V_0}^2 = \int_\Omega \rho|\vec{u}|^2 d\Omega + ||u_n|_S||^2_{H_0^1(S)} + \int_\Omega \rho_0|\vec{u_0}|^2 d\Omega_0$$

$$+ \int_{\Omega_0} |div\vec{u_0}|^2 d\Omega_0 + ||u_{0n_0}|_{S_0}||^2_{\tilde{H}^1(S)}$$

and the space H_0 completion of V_0 for the norm associated with the scalar product

$$(U, \tilde{U})_{H_0} = \int_\Omega \rho\vec{u} \cdot \bar{\tilde{\vec{u}}} d\Omega + \int_{\Omega_0} \rho_0 \vec{u_0} \cdot \bar{\tilde{\vec{u_0}}} d\Omega_0.$$

We set

$$a(U, \tilde{U}) = \rho_0 c_0^2 \int div\ \vec{u_0} \cdot div\ \bar{\tilde{\vec{u_0}}} d\Omega_0 + M(Z, \tilde{Z}).$$

The *final variational equation of the problem* is:

To find $U(\cdot) \in V_0$ such that

$$(27) (\ddot{U}, \tilde{U})_{H_0} + a(U, \tilde{U}) = 0 \ \ \forall \tilde{U} \in V_0.$$

2) Using the method introduced by [(17), p.65-68] we can prove, by means of litttle long, but analogous calculations, that we omit, that the problem is a *classical vibration problem*: there is a countable set of eigenvalues ω_n^2 such that

$0 < \omega_1^2 \leq \omega_2^2 \leq ... \leq \omega_n^2 \leq ...;\ \omega_n^2 \to \infty$ when $n \to \infty$,

and the eigenelements $\{U_n\}$ form an orthogonal basis in H_0.

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