

Classification of Algebraic Function Fields with Class Number One

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Abstract

In this paper we prove that there are exactly eight function fields, up to isomorphism, over finite fields with class number one and positive genus. This classification was already suggested, although not completely proved, in a previous work about this topic (see Stirpe [7]).

1 Introduction

The problem of the determination of algebraic function fields over finite fields with class number one was already treated in the paper of Leitzel, Madan and Queen [2]. But, in Stirpe [7], one more example is given, showing that the previous classification is incomplete. In this paper we give the full list of function fields with class number one and positive genus hence their classification is now complete.

The list of all function fields with class number one and positive genus is given in the following theorem.

Theorem 1.1. Let K be a function field over the finite field \mathbb{F}_q of genus $g > 0$ with class number one. Then K is isomorphic to a function field $\mathbb{F}_q(x, y)$ defined by one of the following equations:

- (i) $y^2 + y + x^3 + x + 1 = 0$, with $g = 1$ and $q = 2$;
- (ii) $y^2 + y + x^5 + x^3 + 1 = 0$, with $g = 2$ and $q = 2$;
- (iii) $y^2 + y + (x^3 + x^2 + 1)/(x^3 + x + 1) = 0$, with $g = 2$ and $q = 2$;
- (iv) $y^4 + xy^3 + (x^2 + x)y^2 + (x^3 + 1)y + x^4 + x + 1 = 0$, with $g = 3$ and $q = 2$;

2010 *Mathematics Subject Classification*: Primary 11R29; Secondary 11R37.
Key words and phrases: Class numbers, Class Field Theory.

- (v) $y^4 + (x^3 + x + 1)y + x^4 + x + 1 = 0$, with $g = 3$ and $q = 2$;
- (vi) $y^2 + 2x^3 + x + 1 = 0$, with $g = 1$ and $q = 3$;
- (vii) $y^2 + y - x^3 + \alpha = 0$, with $g = 1$ and $q = 4$, where α is a generator of the multiplicative group \mathbb{F}_4^* ;
- (viii) $y^5 + (x^4 + x + 1)y^3 + (x^4 + x + 1)y^2 + (x^6 + x^5 + x^4 + x^3 + 1)y + x^4 + x + 1 = 0$, with $g = 4$ and $q = 2$.

Function fields (i-vii) are already given in [2] but in that paper the authors wrongly claimed to have ruled out the only other possibility of curves of genus 4 over \mathbb{F}_2 . Madan and Queen also showed in [3] that any other example of function field with class number one and positive genus should be a function field of genus 4 with exactly one place of degree 4 and without places of smaller degree. A function field with this properties was later found in [7] but the author did not prove that such example is unique up to isomorphism. This function field is listed in (viii) and we prove uniqueness in Section 3 giving a definitive proof of Theorem 1.1. The proof is elementary and requires only basic facts about function fields and Class Field Theory. The necessary background is given in the next section.

In the fourth section we show another proof that is a complete version of the argument in [2]. The interested reader can also find a simplified argument in Qibin Shen and Shuhui Shi [4].

2 Background and definitions

Given two extensions of function field K and L of $\mathbb{F}_q(x)$, we assume that they are embedded in a same algebraic closure of $\mathbb{F}_q(x)$.

Let K/F be a separable extension of function fields. Let n be the degree $[K : F]$ and let g_K and g_F be the genus of K and F , respectively. We assume that the constant fields of F and K are the same. Then the genera of F and K are related by the Hurwitz Genus Formula:

$$(2.1) \quad 2g_K - 2 = n(2g_F - 2) + \deg \text{Diff}(K/F),$$

where $\text{Diff}(K/F) = \sum_P d_P P$ is the different of K/F and the sum runs over the ramified places of K . The reader can see Stichtenoth [5, Chapter 3] for definitions and main results. For our purposes we need only to know that the sum is actually a finite sum and the coefficients d_P are positive integers satisfying the following inequality, known as Dedekind's Different Theorem

(see [5, Theorem 3.5.1]):

$$(2.2) \quad d_P \geq e_P - 1,$$

where e_P is the ramification index of P in K/F and equality holds if and only if the characteristic p of the finite field does not divide e_P .

Let P be a place of F and let P_1, P_2, \dots, P_r be the places of K over P . We denote by $e(P_i|P)$ and $f(P_i|P)$, respectively, the ramification index and the inertia degree of P_i/P for $i = 1, 2, \dots, r$. The integers $e(P_i|P)$ and $f(P_i|P)$ are related by the following equality (see [5, Theorem 3.1.11])

$$(2.3) \quad \sum_{i=1}^r f(P_i|P)e(P_i|P) = n.$$

We also need the following lemma.

Lemma 2.1 (Abhyankar's Lemma). Let K_1/F and K_2/F be separable extensions of function fields and let p be the characteristic of F . Let $K = K_1K_2$ be the compositum of K_1 and K_2 . Let Q be a place of K and let P be the place $Q \cap F$. We denote by P_1 and P_2 , respectively, the places $Q \cap K_1$ and $Q \cap K_2$ and by e_1 and e_2 the ramification index of P_1 and P_2 in K_1/F and K_2/F . We assume that p does not divide both integers e_1 and e_2 . Then the ramification index e_Q of Q/P in K/F is given by

$$e_Q = \text{lcm}(e_1, e_2).$$

Proof. See [5, Theorem 3.9.1]. □

Finally, we recall a result of Class Field Theory. Let \mathfrak{f} be a place of K and S be a non-empty set of places of K not containing \mathfrak{f} . We denote by $K_S^{\mathfrak{f}}$ the maximal abelian function field extension of K such that $K_S^{\mathfrak{f}}/K$ is unramified outside \mathfrak{f} and the points of S are totally split. It is known that $K_S^{\mathfrak{f}}$ is a finite extension. The reader can find more details in Auer [1]. The following result is a direct consequence of Stirpe [6, Remark 3.3] concerning enumeration of cyclic ray class field extensions with given conductor \mathfrak{f} .

Proposition 2.2. Let \mathfrak{m} be a place of K of degree t . Let d be a divisor of $\frac{h_K}{q-1}(q^t - 1)$, where h_K is the class number of K . Then there are exactly d cyclic extensions of K of degree d with conductor dividing \mathfrak{m} and constant field \mathbb{F}_q .

3 Uniqueness in the genus 4 case

From now on, we denote by L the function field listed in Theorem 1.1 part (viii). The function field L has genus 4 and class number one by [7]. Let K denote a function field over the finite field \mathbb{F}_2 of genus 4 and class number one, we want to prove that K is isomorphic to L . In [3] the zeta function of K has been computed and the authors show that K has exactly three places of degree 5, one place of degree 4 and no places of smaller degree.

First, we consider the degree $d = [K : \mathbb{F}_2(x)]$. The following Lemma allows us to assume that $d = 5$.

Lemma 3.1. Let K be a function field with the constant field \mathbb{F}_2 and with genus 4 and class number one. Then there is an element $x \in K$ such that the degree $d = [K : \mathbb{F}_2(x)]$ is equal to 5.

Proof. Let Q be a place of K of degree 5. By Riemann-Roch Theorem

$$l_K(Q) \geq \deg Q - g_K + 1 = 2.$$

Let $x \in K$ be a non constant element in $L_K(Q)$. Then the pole divisor $(x)_\infty$ of x is equal to Q and so (see [5, Theorem 1.4.11])

$$d = [K : \mathbb{F}_2(x)] = \deg Q = 5.$$

□

From now on, we set $x \in K$ as above and we assume $d = [K : \mathbb{F}_2(x)] = 5$. Since $v_Q(x) \equiv 1 \pmod{2}$, then x is a separating element for K/\mathbb{F}_2 and $K/\mathbb{F}_2(x)$ is a separable extension. See [5, Theorem 3.10.2, part a].

By the Hurwitz Genus Formula we can compute the degree of the different of $K/\mathbb{F}_2(x)$

$$(3.1) \quad \deg \text{Diff}(K/\mathbb{F}_2(x)) = 2g_K - 2 + 2d = 16.$$

By Dedekind's Different Theorem $d_Q \geq e_Q - 1$ for any ramified place Q of K but equality holds only if e_Q is odd so $d_Q \geq 2$. It follows that $\sum_Q \deg Q \leq 8$, but there is only one place of K of degree 4 and all other places have higher degree. It follows that there is only one ramified place in $K/\mathbb{F}_2(x)$. We denote by Q such a place of K and by P the place $Q \cap \mathbb{F}_2(x)$.

Lemma 3.2. The ramified place P in $K/\mathbb{F}_2(x)$ has degree 4 and the function field K is a covering of degree 5 of $\mathbb{F}_2(x)$ totally ramified at P .

Proof. By equality (3.1) the place Q over P satisfies $d_Q \deg Q = 16$. It follows that either $d_Q = 2$ and $\deg Q = 8$ or $d_Q = 4$ and $\deg Q = 4$.

We show that the first case cannot occur. We assume that $d_Q = 2$, hence by Dedekind's Theorem $e_Q \leq d_Q + 1 = 3$ so either $e_Q = 2$ or $e_Q = 3$. We denote by $u \in \mathbb{F}_2[x]$ the irreducible monic polynomial with zero divisor Z_u equal to P .

STEP 1 (Case $e_Q = 3$). We consider the extension of constant field $\mathbb{F}_4(x)/\mathbb{F}_2(x)$. The place P has even degree because $\deg Q = 8$, hence it splits in two places P_1 and P_2 . We consider a Kummer extension $E/\mathbb{F}_4(x)$ of degree 3 of $\mathbb{F}_4(x)$ totally ramified at P_i for $i = 1, 2$ and unramified outside P_1 and P_2 (see [5, Proposition 3.7.3]). We can define explicitly E as $E = \mathbb{F}_4(x, t)$ where t is such that $t^3 + u = 0$ (the interested reader can see [5, Corollary 3.7.4]). Now, we consider the compositum EK . By Abhyankar's Lemma the extensions of function fields $EK/K\mathbb{F}_4$ and EK/E are both unramified and the constant field of EK is \mathbb{F}_4 because

$$(3.2) \quad [EK : K\mathbb{F}_4] = [E : \mathbb{F}_4(x)] = 3,$$

is coprime to

$$(3.3) \quad [EK : E] = [K\mathbb{F}_4 : \mathbb{F}_4(x)] = 5.$$

We can compute the genus of EK in two ways by substitution of (3.2) and (3.3) in the Hurwitz Formula. We obtain that

$$2g_{EK} - 2 = [EK : K\mathbb{F}_4](2g_K - 2) = 3 \cdot 6 = 18,$$

and, similarly,

$$2g_{EK} - 2 = [EK : E](2g_E - 2) = 5(2g_E - 2).$$

It follows that

$$5(2g_E - 2) = 18$$

and we get a contradiction.

STEP 2 (Case $e_Q = 2$). The proof is similar to the previous case although is more complicated because the extension $K/\mathbb{F}_2(x)$ is wild. We consider an Artin-Schreier extension $A/\mathbb{F}_2(x)$ of degree 2 of $\mathbb{F}_2(x)$ ramified at P and unramified outside P . We can define explicitly A as $A = \mathbb{F}_2(x, t)$ with $t^2 + t + u^{-1} = 0$ (see [5, Remark 3.7.9, part a]). By classical theory of Artin-Schreier extensions, the exponent of the different $d_{P'}$ of the place P' of A over P is equal to $(p - 1)(m_P + 1) = 2$ (see [5, Proposition 3.7.8, part c]),

where $m_P = -v_P(u^{-1}) = 1$. Again we consider the compositum AK . As in Step 1 the constant function field of AK is \mathbb{F}_2 because

$$[AK : K] = [A : \mathbb{F}_2(x)] = 2,$$

is coprime to

$$[AK : A] = [K : \mathbb{F}_2(x)] = 5.$$

By the Hurwitz Formula we get

$$(3.4) \quad 2g_{AK} - 2 = [AK : K](2g_K - 2) + \sum_{T|P} d(T|T_K) \deg T,$$

and

$$(3.5) \quad 2g_{AK} - 2 = [AK : A](2g_A - 2) + \sum_{T|P} d(T|T_A) \deg T,$$

where the sums run over the places T of AK over P and where T_K is the place $T \cap K$ and $T_A = T \cap A$. Since the ramification indices in $K/\mathbb{F}_2(x)$ and in $A/\mathbb{F}_2(x)$ are equal, then the ramification indices $e(T|T_A)$ and $e(T|T_K)$ are equal too. Now, we prove that the coefficients $d(T|T_A)$ and $d(T|T_K)$ are also equal for any place T of AK over P . By the transitivity of the different (see [5, Corollary 3.4.12 b]))

$$d(T|P) = e(T|T_A)d(T_A|P) + d(T|T_A) = e(T|T_K)d(T_K|P) + d(T|T_K),$$

follows that

$$d(T|T_K) = d(T|T_A),$$

because

$$e(T|T_K) = e(T|T_A)$$

and

$$d(T_K|P) = d(T_A|P) = 2.$$

By equations (3.4) and (3.5) we get a contradiction because

$$[AK : A] = 5$$

does not divide

$$[AK : K](2g_K - 2) = 12.$$

It follows that the only possible case is $d_Q = 4$ and $\deg Q = 4$.

Finally, we show that $e_Q = 5$. If the ramification is wild we can only have $e_Q = 2$ or $e_Q = 4$. As in Step 2 above, one can show that $e_Q \neq 2$ by

considering $A = \mathbb{F}_2(x, t)$ with $t^2 + t + u^{-3} = 0$. We assume that $e_Q = 4$. By equality (2.3) we see that $f_Q < 2$, therefore $f_Q = 1$ and, similarly, any other place P' of K over P is not partially inert, i.e. $f_{P'} = 1$. But there is only one place of K of degree 4, so P' should be at least partially inert, contradiction. Hence $e_Q \neq 4$. Therefore the ramification must be tame, but in this case the equality holds in equation (2.3) and we get $e_Q = 5$ and $\deg Q = \deg P = 4$. \square

Proof of Theorem 1.1. Consider the ramified place P under Q in $\mathbb{F}_2(x)$. In the following, without loss of generality we can assume that P is the place $\mathfrak{f} = (x^4 + x + 1)$. In fact, if P is the place $(x^4 + x^3 + 1)$, we can use an isomorphism of the rational function field, namely $\sigma(x) = 1/x$, to send P in \mathfrak{f} . Similarly, if $P = (x^4 + x^3 + x^2 + x + 1)$ we can choose $\sigma(x) = 1/(x + 1)$ and again $\sigma(P) = \mathfrak{f}$. As before we denote by L the function field listed in 1.1 part (viii). One can show that $L/\mathbb{F}_2(x)$ is a cyclic extension (see [7]), totally ramified in \mathfrak{f} with ramification index equal to 5. We show that $K/\mathbb{F}_2(x)$ is also a Galois extension.

We consider the compositum KL . By Abhyankar's Lemma 2.1 the field extension $KL/\mathbb{F}_2(x)$ is partially ramified in \mathfrak{f} with ramification index equal to 5. It follows that KL/K is unramified. But the class number of K is one by hypothesis and KL/K is a Galois extension because $L/\mathbb{F}_2(x)$ is Galois hence $KL \subseteq K\mathbb{F}_{2^5}$. It follows that either $K = L$ or $KL = K\mathbb{F}_{2^5}$. In the first case we are done, in the second one we get

$$L\mathbb{F}_{2^5} = K\mathbb{F}_{2^5}.$$

It follows that the Galois closure of K is contained in $K\mathbb{F}_{2^5}$. The Galois group $H = \text{Gal}(\mathbb{F}_{2^5}/\mathbb{F}_2)$ is canonically embedded in the Galois group $G = \text{Gal}(K\mathbb{F}_{2^5}/\mathbb{F}_2(x))$ and it is normal. By Galois Theory, K is a normal extension with Galois group G/H and degree

$$[K : \mathbb{F}_2(x)] = |G/H| = 25/5 = 5.$$

By Class Field Theory, K is a ray class field extension over $\mathbb{F}_2(x)$ of degree 5 and genus 4 with conductor \mathfrak{f} . By Proposition 2.2 there are only five of such extensions. These five function fields are explicitly constructed in [7] but three of them are subfields of the ray class field extensions $K_S^\mathfrak{f}/\mathbb{F}_2(x)$ with S given by a rational place of $\mathbb{F}_2(x)$: these function fields have rational places and so their class number is greater than one by Madan and Queen [3, Theorem 2]. The last two ray class fields are contained in the ray class field extensions $K_S^\mathfrak{f}/\mathbb{F}_2(x)$, where S is a place of $\mathbb{F}_2(x)$ of degree 7 given

by $P_1 = (x^7 + x^4 + 1)$ or $P_2 = (x^7 + x^3 + 1)$. These ray class fields have class number one and they are isomorphic to each other. The first choice $S = P_1$ gives the function field L . The reader can see Stirpe [7, Remark 3], for more details and explicit computations. In both cases K is a ray class field extension of $\mathbb{F}_2(x)$ isomorphic to L . \square

4 Table of the original proof

In the proof given by Leitzel, Madan and Queen in [2, Section 2], an enumerative argument is given to prove that there are no function fields with class number one and genus 4. They prove that the field of genus 4 and class number one we are looking for, must be defined by a cubic and a quadric. They also prove that this pair must be one of the following:

$$\begin{aligned} & (C_1, Q_1 + L(k_1, k_2, k_3, k_4)^2), \\ & (C_2, Q_2 + L(k_1, k_2, k_3, k_4)^2), \\ & (C_3, Q_3 + L(k_1, k_2, k_3, k_4)^2), \\ & (C_4, Q_4 + L(k_1, k_2, k_3, k_4)^2), \end{aligned}$$

where

$$\begin{aligned} C_1 &= x_2^3 + x_1x_3^2 + x_4^3 + x_1^2x_3 + x_3x_4^2, \\ C_2 &= x_2^3 + x_1x_3^2 + x_2^2x_3 + x_2^2x_4 + x_1^3 + x_3^2x_4 + x_1^2x_2 + x_2x_4^2, \\ C_3 &= x_2^2x_3 + x_1x_4^2 + x_3^3 + x_3^2x_4 + x_1^2x_2 + x_4^3 + x_1^2x_3 + x_3x_4^2, \\ C_4 &= x_1^3 + x_1^2x_3 + x_1x_4^2 + x_2^2x_4 + x_2x_4^2 + x_3^3 + x_3x_4^2 + x_4^3, \\ Q_1 &= x_1x_2 + x_3x_4, \\ Q_2 &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_4, \\ Q_3 &= x_1x_3 + x_2x_3 + x_2x_4 + x_3x_4, \\ Q_4 &= x_1x_4 + x_2x_3 + x_3x_4 \end{aligned}$$

and where

$$L(k_1, k_2, k_3, k_4) = k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4,$$

is a homogeneous linear form in x_1, x_2, x_3, x_4 with $k_1, k_2, k_3, k_4 \in \{0, 1\}$. The list given at page 15 in [2] shows a list of only eight possible quadric surfaces and all of them are discarded by the authors case by case. This list is actually incomplete. In the following table, we write down the complete list and we show in the second column the rational points of lowest degree

that allows us to discard each case but one. This case not discarded, i.e. the pair $(C_2, Q_2 + L(1, 0, 1, 1)^2)$, defines a field isomorphic to that one of Theorem 1.1, case viii. This argument gives a second proof of Theorem 1.1.

The complete list, below, shows all the 64 quadric. The interested reader can also find a simplified proof in [4], where the number of cases to check is reduced by Qibin Shen and Shuhui Shi to 24 cases only. The authors also give minor simplification of the original paper [2].

Acknowledgements

The authors are grateful to Dinesh Thakur for his helpful comments and for letting us know about the paper of Qibin Shen and Shuhui Shi [4], where an argument similar to the one of Section 4 is given.

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$Q_1 + L(k_1, k_2, k_3, k_4)^2$	Low degree point	$Q_3 + L(k_1, k_2, k_3, k_4)^2$	Low degree point
$x_1x_2 + x_3x_4$	(1 : 0 : 0 : 0)	$x_1x_3 + x_2x_3 + x_2x_4 + x_3x_4$	(1 : 0 : 0 : 0)
$x_1x_2 + x_3x_4 + x_4^2$	(1 : 0 : 0 : 0)	$x_1x_3 + x_2x_3 + x_2x_4 + x_3x_4 + x_4^2$	(1 : 0 : 0 : 0)
$x_1x_2 + x_3^2 + x_3x_4$	(1 : 0 : 0 : 0)	$x_1x_3 + x_2x_3 + x_3^2 + x_2x_4 + x_3x_4$	(1 : 0 : 0 : 0)
$x_1x_2 + x_3^2 + x_3x_4 + x_4^2$	(1 : 0 : 0 : 0)	$x_1x_3 + x_2x_3 + x_3^2 + x_2x_4 + x_3x_4 + x_4^2$	(1 : 0 : 0 : 0)
$x_1x_2 + x_2^2 + x_3x_4$	(1 : 0 : 0 : 0)	$x_1^2 + x_1x_3 + x_2x_3 + x_2x_4 + x_3x_4$	(1 : 0 : 0 : 0)
$x_1x_2 + x_2^2 + x_3x_4 + x_4^2$	(1 : 0 : 0 : 0)	$x_1^2 + x_1x_3 + x_2x_3 + x_2^2 + x_2x_4 + x_3x_4 + x_4^2$	(1 : 0 : 0 : 0)
$x_1x_2 + x_2^2 + x_3^2 + x_3x_4$	(1 : 0 : 0 : 0)	$x_1^2 + x_1x_3 + x_2x_3 + x_3^2 + x_2x_4 + x_3x_4$	(0 : 1 : 0 : 0)
$x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2$	(1 : 0 : 0 : 0)	$x_1^2 + x_1x_3 + x_2x_3 + x_3^2 + x_2x_4 + x_3x_4 + x_4^2$	(0 : 1 : 0 : 0)
$x_1^2 + x_1x_2 + x_3x_4$	(0 : 0 : 1 : 0)	$x_1^2 + x_1x_3 + x_2x_3 + x_2x_4 + x_3x_4$	(0 : 1 : 0 : 0)
$x_1^2 + x_1x_2 + x_3x_4 + x_4^2$	(0 : 0 : 1 : 0)	$x_1^2 + x_1x_3 + x_2x_3 + x_2x_4 + x_3x_4 + x_4^2$	(0 : 1 : 0 : 0)
$x_1^2 + x_1x_2 + x_3^2 + x_3x_4$	(1 : 0 : 1 : 0)	$x_1^2 + x_1x_3 + x_2x_3 + x_3^2 + x_2x_4 + x_3x_4$	(0 : 1 : 0 : 0)
$x_1^2 + x_1x_2 + x_3^2 + x_3x_4 + x_4^2$	(1 : 0 : 1 : 0)	$x_1^2 + x_1x_3 + x_2x_3 + x_3^2 + x_2x_4 + x_3x_4 + x_4^2$	(0 : 1 : 0 : 0)
$x_1^2 + x_1x_2 + x_2^2 + x_3x_4$	(0 : 0 : 1 : 0)	$x_1^2 + x_2^2 + x_1x_3 + x_2x_3 + x_2x_4 + x_3x_4$	(1 : 1 : 1 : 1)
$x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + x_4^2$	(0 : 0 : 1 : 0)	$x_1^2 + x_2^2 + x_1x_3 + x_2x_3 + x_2x_4 + x_3x_4 + x_4^2$	(0 : 0 : 1 : 1)
$x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4$	(1 : 0 : 1 : 0)	$x_1^2 + x_2^2 + x_1x_3 + x_2x_3 + x_3^2 + x_2x_4 + x_3x_4$	(0 : 0 : 1 : 1)
$x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2$	(1 : 0 : 1 : 0)	$x_1^2 + x_2^2 + x_1x_3 + x_2x_3 + x_3^2 + x_2x_4 + x_3x_4 + x_4^2$	(1 : 1 : 1 : 1)
$Q_2 + L(k_1, k_2, k_3, k_4)^2$	Low degree point	$Q_4 + L(k_1, k_2, k_3, k_4)^2$	Low degree point
$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_4$	(0 : 0 : 1 : 0)	$x_2x_3 + x_1x_4 + x_3x_4$	(0 : 1 : 0 : 0)
$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_4 + x_4^2$	(0 : 0 : 1 : 0)	$x_2x_3 + x_1x_4 + x_3x_4 + x_4^2$	(0 : 1 : 0 : 0)
$x_1x_2 + x_1x_3 + x_3^2 + x_1x_4 + x_2x_4$	(0 : 0 : 0 : 1)	$x_2x_3 + x_3^2 + x_1x_4 + x_3x_4$	(0 : 1 : 0 : 0)
$x_1x_2 + x_1x_3 + x_3^2 + x_1x_4 + x_2x_4 + x_4^2$	(1 : 1 : 1 : 1)	$x_2x_3 + x_3^2 + x_1x_4 + x_3x_4 + x_4^2$	(0 : 1 : 0 : 0)
$x_1x_2 + x_2^2 + x_1x_3 + x_1x_4 + x_2x_4$	(0 : 0 : 1 : 0)	$x_2^2 + x_2x_3 + x_1x_4 + x_3x_4$	(1 : 1 : 1 : 1)
$x_1x_2 + x_2^2 + x_1x_3 + x_1x_4 + x_2x_4 + x_4^2$	(0 : 0 : 1 : 0)	$x_2^2 + x_2x_3 + x_1x_4 + x_3x_4 + x_4^2$	($\beta : 0 : \beta^3 : 1$)
$x_1x_2 + x_2^2 + x_1x_3 + x_3^2 + x_1x_4 + x_2x_4$	(0 : 0 : 0 : 1)	$x_2^2 + x_2x_3 + x_3^2 + x_1x_4 + x_3x_4$	(1 : 0 : α : 1)
$x_1x_2 + x_2^2 + x_1x_3 + x_3^2 + x_1x_4 + x_2x_4 + x_4^2$	(1 : 0 : 1 : 0)	$x_2^2 + x_2x_3 + x_3^2 + x_1x_4 + x_3x_4 + x_4^2$	(1 : 1 : 1 : 1)
$x_1^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_4$	(0 : 0 : 1 : 0)	$x_1^2 + x_2x_3 + x_1x_4 + x_3x_4$	(1 : 1 : 1 : 1)
$x_1^2 + x_1x_2 + x_1x_3 + x_3^2 + x_1x_4 + x_2x_4$	(0 : 0 : 1 : 0)	$x_1^2 + x_2x_3 + x_3^2 + x_1x_4 + x_3x_4$	(0 : 1 : 0 : 0)
$x_1^2 + x_1x_2 + x_1x_3 + x_3^2 + x_1x_4 + x_2x_4 + x_4^2$	(0 : 0 : 0 : 1)	$x_1^2 + x_2x_3 + x_3^2 + x_1x_4 + x_3x_4 + x_4^2$	(0 : 1 : 0 : 0)
$x_1^2 + x_1x_2 + x_2^2 + x_1x_3 + x_1x_4 + x_2x_4$	(0 : 0 : 0 : 1)	$x_1^2 + x_2x_3 + x_2^2 + x_1x_4 + x_3x_4$	(0 : α : 1 : 1)
$x_1^2 + x_1x_2 + x_2^2 + x_1x_3 + x_1x_4 + x_2x_4 + x_4^2$	(0 : 0 : 0 : 1)	$x_1^2 + x_2x_3 + x_2^2 + x_1x_4 + x_3x_4 + x_4^2$	(0 : 1 : 1 : 1)
$x_1^2 + x_1x_2 + x_2^2 + x_1x_3 + x_3^2 + x_1x_4 + x_2x_4$	(0 : 0 : 0 : 1)	$x_1^2 + x_2^2 + x_2x_3 + x_1x_4 + x_3x_4 + x_4^2$	(1 : 1 : 1 : 1)
$x_1^2 + x_1x_2 + x_2^2 + x_1x_3 + x_3^2 + x_1x_4 + x_2x_4 + x_4^2$	(1 : 1 : 1 : 1)	$x_1^2 + x_2^2 + x_2x_3 + x_3^2 + x_1x_4 + x_3x_4 + x_4^2$	(0 : α : 1 : 1)

Notation: we denote by $\alpha \in \mathbb{F}_4$ a solution of $x^2 + x + 1 = 0$ and by $\beta \in \mathbb{F}_8$ a solution of $x^3 + x + 1 = 0$.