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AN EFFICIENT FILTERED SCHEME FOR SOME FIRST ORDER TIME-DEPENDENT HAMILTON–JACOBI EQUATIONS*

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Abstract. We introduce a new class of "filtered" schemes for some first order nonlinear Hamilton–Jacobi equations. The work follows recent ideas of Froese and Oberman [*SIAM J. Numer. Anal.*, 51 (2013), pp. 423–444] and Oberman and Salvador [*J. Comput. Phys.*, 284 (2015), pp. 367–388] for steady equations. Here we mainly study the time-dependent setting and focus on fully explicit schemes. Furthermore, specific corrections to the filtering idea are also needed in order to obtain high-order accuracy. The proposed schemes are not monotone but still satisfy some ϵ-monotone property. A general convergence result together with a precise error estimate of order $\sqrt{\Delta x}$ are given (Δx is the mesh size). The framework allows us to construct finite difference discretizations that are easy to implement and high-order in the domain where the solution is smooth. A novel error estimate is also given in the case of the approximation of steady equations. Numerical tests including evolutive convex and nonconvex Hamiltonians, and obstacle problems are presented to validate the approach. We show with several examples how the filter technique can be applied to stabilize an otherwise unstable high-order scheme.

Key words. Hamilton–Jacobi equation, high-order schemes, ϵ-monotone scheme, viscosity solutions, error estimates

AMS subject classifications. 65M06, 65M12, 35F21, 35F25

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1. Introduction.

In this work, our aim is to develop high-order and convergent schemes for first order Hamilton–Jacobi (HJ) equations of the following form:

$$\partial_t v + H(x, \nabla v) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \tag{1.1}$$

$$v(0, x) = v_0(x), \quad x \in \mathbb{R}^d. \tag{1.2}$$

Basic assumptions on the Hamiltonian H and the initial data v_0 will be introduced in the next section to guarantee existence and uniqueness in the framework of weak solutions in the viscosity sense. It is well known that, in the one dimensional case, there is a strong link between HJ equations and scalar conservation laws. Namely, the viscosity solution of the evolutive HJ equation is the primitive of the entropy solution of the corresponding hyperbolic conservation law with the same Hamiltonian. There are several schemes developed for hyperbolic conservation laws (see [17, 18, 10, 16]). Most of the numerical ideas to solve hyperbolic conservation laws can be extended to HJ equations. Well-known high-order essentially nonoscillatory (ENO) schemes have been introduced by Harten et al. in [19] for hyperbolic conservation laws and then extended to HJ equations by Osher and Shu [24]. ENO schemes have been shown

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to have high-order accuracy and have been quite successful in many applications, although a precise general convergence result is still missing. Despite the fact that there is no convergence proof of ENO schemes toward the viscosity solution of (1.1) in the general case, convergence results may hold for related schemes, e.g., MUSCL schemes, as has been proved by Lions and Souganidis in [22]. Convergence results of some nonmonotone schemes have also been shown in particular cases [7]. Another interesting result has been proved by Fjordholm, Mishra, and Tadmor [14]; they have shown that ENO interpolation is stable but the stability result is not sufficient to conclude total variation boundedness (TVB) of the ENO reconstruction procedure. In [13], a conjecture related to the weak total variation property for ENO schemes is given. Finally, let us also mention [9], where weighted ENO (WENO) schemes have been applied to HJ equations; the convergence proof of the scheme relies also on the work of Ferretti [12], where higher than first order schemes are proposed in a semi-Lagrangian (SL) setting, yet with restrictive conditions on the mesh steps.

In this paper we give a very simple way to construct high-order schemes in a convergent framework. It is known (by Godunov's theorem) that a monotone scheme can be at most first order. Therefore, it is necessary to look for nonmonotone schemes. The difficulty is then to combine nonmonotonicity of the scheme and convergence toward the viscosity solution of (1.1) and also to obtain error estimates. In our approach we will adapt a general idea of Froese and Oberman [15] that was presented for stationary second order HJ equations and based on the use of a "filter" function. The idea was also used to treat some stationary first order HJ equations in Oberman and Salvador [23].

Here we focus mainly on the case of the time-dependent first order HJ equation (1.1). As suggested in [15], the scheme can be adapted to solve steady HJ equations by using a fixed point approach. The schemes are written in explicit time marching form ("fully explicit" schemes), which is well adapted to time-dependent equations, while the setting of [15] or [23] can be better adapted to solve stationary equations. Let us emphasize that it is our experience that a direct application of the idea of [15] or [23] in the time-dependent setting, even if leading to convergent schemes, does not lead to second order schemes in general (a similar filtering idea was mentioned, for instance, in Osher and Shu [24, Remark 2.2], and see also Remark 2.6). One aim of the present work is to explain in more detail some adaptations that were needed in order to numerically achieve the second order convergence, at least for most examples tested.

We use the same discontinuous filter function as in [23] for which the filtered scheme is still an "ϵ-monotone" scheme (see (2.12)). In our case we justify the use of this discontinuous filter to obtain a second order numerical behavior of the scheme in the L^∞ norm. It is our experience that using instead the continuous filter initially introduced in [15] leads to only first order behavior. (However, in the case of steady equations (see Example 6 in section 3), it is our experience that both filters give very similar results).

Furthermore, when using a central finite difference scheme together with the filtering idea, we introduce a limiting process that is needed in order to obtain high-order accuracy and that is made precise in the case of front-propagation models. This limiting process was not needed in [15, 23] for the treatment of steady equations. Without the limiting process the scheme may switch back to first order after a few time steps (see, for instance, Example 2 in section 3). Moreover, the filtered scheme (2.6) needs the use of a filtering parameter (hereafter denoted "ϵ") that must be chosen in order to switch between the high-order scheme and the monotone scheme in a convenient way. A natural upper bound for the parameter is given in [15, 23], of order $O(\sqrt{\Delta x})$. We give here a similar upper bound that is justified theoretically to ensure an error estimate of order $O(\sqrt{\Delta x})$. However, in our case we furthermore give a lower bound on this parameter and some precise indications as to how to fix the parameter depending on the data. In our simulations, we will use $\epsilon = c_1 \Delta x$, where c_1 is a constant dependent on the second derivative of the data in order to numerically obtain a high-order behavior, and therefore our choice is slightly different from that of [23].

The approach also allows us to obtain new error estimates for a filtered scheme for general time-dependent HJ equations, of order $O(\sqrt{\Delta x})$, where Δx is the spatial mesh size, under a standard CFL condition on the time step (this result is new compared to the works [15, 23]). This is similar to the Crandall–Lions error estimate for monotone schemes [10], because the scheme can be written as a perturbation of a monotone scheme. In a quite similar way, we also adapt the argument to the case of steady equations in order to obtain a new and general error estimate for the approximation by filtered schemes (in [23] an error estimate is given only in a particular case).

Let us mention also the work [5] for steady equations where some ϵ-monotone SL schemes are studied.

The present work is a first step in using the filtering idea for time-dependent equations. It remains to improve the choice of the parameter ϵ, and the limiting process is detailed here in one dimension but not in two. The second order behavior is not obtained in some particular cases (involving nonconvex Hamiltonian functions). This is the subject of ongoing work, particularly concerning front-propagation models. However, we emphasize that in most cases we observe second order behavior with a relatively simple scheme, together with provable convergence and error estimates.

The paper is organized as follows. In section 2, we present the schemes and give main convergence results. Section 3 is devoted to the numerical tests on several academic examples to illustrate our approach in one and two dimensional cases. A test on nonlinear steady equations, as well as an evolutive "obstacle" HJ equation in the form of $\min(u_t + H(x, u_x), u - g(x)) = 0$ for a given function g, are also included in this section. Finally, in Appendix A we recall the definition of a simple second order ENO scheme, and in Appendix B we prove a comparison lemma.

2. Definitions and main results.

2.1. Setting of the problem. Let us denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^d$ ($d \geq 1$). The following classical assumptions will be considered in the remainder of this paper:

(A1) v_0 is a Lipschitz continuous function; i.e., there exist $L_0 > 0$ such that for every $x, y \in \mathbb{R}^d$,

$$|v_0(x) - v_0(y)| \leq L_0 |x - y|. \tag{2.1}$$

(A2) $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies, for all $R \geq 0$, $\exists C_R \geq 0$, for all $p, q, x, y \in \mathbb{R}^d$ such that $|p|, |q| \leq R$,

$$|H(y, p) - H(x, p)| \leq C_R(1 + |p|)|y - x| \tag{2.2}$$

and

$$|H(x, q) - H(x, p)| \leq C_R(1 + |x|)|q - p|. \tag{2.3}$$

Under assumptions (A1) and (A2) there exists a unique viscosity solution for (1.1) (by the same arguments as in Ishii [21]). Furthermore, v is locally Lipschitz continuous on $[0, T] \times \mathbb{R}^d$.

For clarity of presentation we focus on the one dimensional case and consider the following simplified problem:

$$v_t + H(x, v_x) = 0, \quad x \in \mathbb{R},\ t \in [0, T], \tag{2.4}$$

$$v(0, x) = v_0(x), \quad x \in \mathbb{R}. \tag{2.5}$$

2.2. Construction of the filtered scheme. Let $\tau = \Delta t > 0$ be a time step (in the form of $\tau = \frac{T}{N}$ for some $N \geq 1$) and $\Delta x > 0$ be a space step. A uniform mesh in time is defined by $t_n := n\tau$, $n \in [0, \dots, N]$, and a uniform mesh in space is defined by the nodes $x_j := j\Delta x$, $j \in \mathbb{Z}$.

The construction of a filtered scheme needs three ingredients:

- a monotone scheme, denoted S^M,
- a high-order scheme, denoted S^A, and
- a bounded "filter" function, denoted $F : \mathbb{R} \to \mathbb{R}$.

The high-order scheme need not be convergent or stable; the letter A stands for "arbitrary order," following [15]. For a start, S^M will be based on a finite difference scheme. Later we will also propose a definition of S^M based on an SL scheme.

The filtered scheme S^F is then defined as

$$u_j^{n+1} \equiv S^F(u^n)_j := S^M(u^n)_j + \epsilon\tau F\left(\frac{S^A(u^n)_j - S^M(u^n)_j}{\epsilon\tau}\right), \tag{2.6}$$

where $\epsilon = \epsilon_{\tau,\Delta x} > 0$ is a parameter satisfying

$$\lim_{(\tau,\Delta x)\to 0} \epsilon = 0. \tag{2.7}$$

More hints on the choice of ϵ will be given later.

The scheme is initialized in the standard way, i.e.,

$$u_j^0 := v_0(x_j) \quad \forall j \in \mathbb{Z}. \tag{2.8}$$

Now we specify some requirements on S^M, S^A, and the function F.

Definition of the monotone finite difference scheme S^M. Following Crandall and Lions [10], we consider a finite difference scheme written as $u^{n+1} = S^M(u^n)$ with

$$S^M(u^n)(x) := u^n(x) - \tau\ h^M(x, D^-u^n(x), D^+u^n(x)), \tag{2.9}$$

with

$$D^{\pm}u(x) := \pm\frac{u(x \pm \Delta x) - u(x)}{\Delta x},$$

where h^M corresponds to a monotone numerical Hamiltonian that will be made precise below. We will denote also $S^M(u^n)_j := S^M(u^n)(x_j)$. Therefore, the scheme also reads, for all $j \in \mathbb{Z}$, for all $n \geq 0$,

$$u_j^{n+1} := u_j^n - \tau\ h^M(x_j, D^-u_j^n, D^+u_j^n), \quad D^{\pm}u_j^n := \pm\frac{u_{j\pm1}^n - u_j^n}{\Delta x}. \tag{2.10}$$

(A3) *Assumptions on* S^M. We will use the following assumptions throughout this paper:

(i) h^M is a Lipschitz continuous function.

(ii) (*Consistency*) for all x, for all p, $h^M(x, p, p) = H(x, p)$.

(iii) (*Monotonicity*) for any functions u, v, $u \leq v \Longrightarrow S^M(u) \leq S^M(v)$.

In practice condition (A3)(iii) is required only at mesh points, and the condition reads

$$u_j \le v_j \ \forall j \quad \Rightarrow \quad S^M(u)_j \le S^M(v)_j \ \forall j. \tag{2.11}$$

At this stage, we notice that under condition (A3) the filtered scheme is "ϵ-monotone" in the sense that

$$u_j \le v_j \ \forall j \quad \Rightarrow \quad S^F(u)_j \le S^F(v)_j + \epsilon\tau \, \|F\|_{L^\infty} \ \forall j \tag{2.12}$$

with $\epsilon \to 0$ as $(\tau, \Delta x) \to 0$. This implies the convergence of the scheme by the Barles–Souganidis convergence theorem (see [3, 1]).

Remark 2.1. Under assumption (i), the consistency property (ii) is equivalent to that, for any $v \in C^2([0,T] \times \mathbb{R})$, there exists a constant $C_M \ge 0$ independent of Δx such that

$$\left| h^M(x, D^-v(x), D^+v(x)) - H(x, v_x) \right| \le C_M \Delta x \|\partial_{xx} v\|_\infty. \tag{2.13}$$

The same statement holds true if (2.13) is replaced by the following consistency error estimate:

$$\begin{aligned} \mathcal{E}_{S^M}(v)(t,x) &:= \left| \frac{v(t+\tau,x) - S^M(v(t,.))(x)}{\tau} - \big(v_t(t,x) + H(x, v_x(t,x))\big) \right| \\ &\le C_M \bigg(\tau \|\partial_{tt} v\|_\infty + \Delta x \|\partial_{xx} v\|_\infty \bigg). \end{aligned} \tag{2.14}$$

Remark 2.2. Assuming (i), it is easily shown that the monotonicity property (iii) is equivalent to that $h^M = h^M(x, p^-, p^+)$ satisfies, a.e. $(x, p^-, p^+) \in \mathbb{R}^3$,

$$\frac{\partial h^M}{\partial p^-} \ge 0, \quad \frac{\partial h^M}{\partial p^+} \le 0 \tag{2.15}$$

(also denoted $h^M = h^M(\cdot, \uparrow, \downarrow)$) and the CFL condition

$$\frac{\tau}{\Delta x} \left(\frac{\partial h^M}{\partial p^-}(x, p^-, p^+) - \frac{\partial h^M}{\partial p^+}(x, p^-, p^+) \right) \le 1. \tag{2.16}$$

When using finite difference schemes, it is assumed that the CFL condition (2.16) is satisfied, and that can be written equivalently in the form

$$c_0 \frac{\tau}{\Delta x} \le 1, \tag{2.17}$$

where c_0 is a constant independent of τ and Δx.

Example 2.1. Let us consider the Lax–Friedrichs numerical Hamiltonian

$$h^{M,LF}(x, p^-, p^+) := H\left(x, \frac{p^- + p^+}{2}\right) - \frac{c_0}{2}(p^+ - p^-), \tag{2.18}$$

where $c_0 > 0$ is a constant. The scheme is consistent; it is furthermore monotone under the conditions $\max_{x,p} |\partial_p H(x,p)| \le c_0$, and $c_0 \frac{\tau}{\Delta x} \le 1$.

Definition of the high-order scheme S^A. We consider an iterative scheme of "high order" in the form $u^{n+1} = S^A(u^n)$, written as

$$S^A(u^n)(x) = u^n(x) - \tau h^A(x, D^{k,-}u^n(x), \ldots, D^-u^n(x), D^+u^n(x), \ldots, D^{k,+}u^n(x)), \tag{2.19}$$

where h^A corresponds to a "high-order" numerical Hamiltonian, and $D^{\ell,\pm}u(x) := \pm\frac{u(x\pm\ell\Delta x)-u(x)}{\Delta x}$ for $\ell = 1,\dots,k$. To simplify the notation we may write (2.19) in the more compact form

$$S^A(u^n)(x) = u^n(x) - \tau h^A\big(x, D^{\pm}u^n(x)\big) \tag{2.20}$$

even if there is a dependency on ℓ in $(D^{\ell,\pm}u^n(x))_{\ell=1,\dots,k}$.

(A4) *Assumptions on* S^A. We will use the following assumptions:

(i) h^A is a Lipschitz continuous function.

(ii) (*High-order consistency*) There exists $k \geq 2$ for all $\ell \in [1,\dots,k]$, for any function $v = v(t,x)$ of class $C^{\ell+1}$, and there exists $C_{A,\ell} \geq 0$,

$$\mathcal{E}_{S^A}(v)(t,x) := \left|\frac{v(t+\tau,x) - S^A(v(t,.))(x)}{\tau} - \big(v_t(t,x) + H(x, v_x(t,x))\big)\right| \tag{2.21}$$

$$\leq C_{A,\ell}\bigg(\tau^\ell \|\partial_t^{\ell+1}v\|_\infty + \Delta x^\ell \|\partial_x^{\ell+1}v\|_\infty\bigg). \tag{2.22}$$

Here v_x^ℓ denotes the ℓth derivative of v w.r.t. x.

Remark 2.3. The high-order consistency implies, for all $\ell \in [1,\dots,k]$, and for $v \in C^{\ell+1}(\mathbb{R})$,

$$\left|h^A(x,\dots,D^-v, D^+v,\dots) - H(x,v_x)\right| \leq C_{A,\ell}\|\partial_x^{\ell+1}v\|_\infty \Delta x^\ell.$$

Example 2.2 (centered scheme). A typical example with $k = 2$ is obtained with the centered total variation diminishing (TVD) approximation in space and the Runge–Kutta second order scheme in time (or Heun scheme)

$$S_0(u^n)_j := u_j^n - \tau H\bigg(x_j, \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\bigg) \tag{2.23a}$$

and

$$S^A(u) := \frac{1}{2}(u + S_0(S_0(u))). \tag{2.23b}$$

Of course there is no reason for the centered scheme to be stable (as will be shown in the numerical section). Using a filter will help stabilize the scheme.

A similar example with $k = 3$ can be obtained with any third order finite difference approximation in space and the TVD-RK3 scheme in time [16].

Remark 2.4. The filtering idea is straightforward to generalize to higher dimensions. For instance, in two dimensions, considering the PDE

$$u_t + H((x,y), u_x, u_y) = 0,$$

assuming the grid $x_{i,j} = (x_i, y_j)$ is uniform in both variables, with space steps Δx and Δy, with associated scheme values $u_{i,j}^n$, the monotone Lax–Friedrichs scheme is to take

(2.24)

$$h^{M,LF}((x,y), p^-, p^+, q^-, q^+) := H\bigg((x,y), \frac{p^- + p^+}{2}, \frac{q^- + q^+}{2}\bigg) - \frac{c_x}{2}(p^+ - p^-) - \frac{c_y}{2}(q^+ - q^-)$$

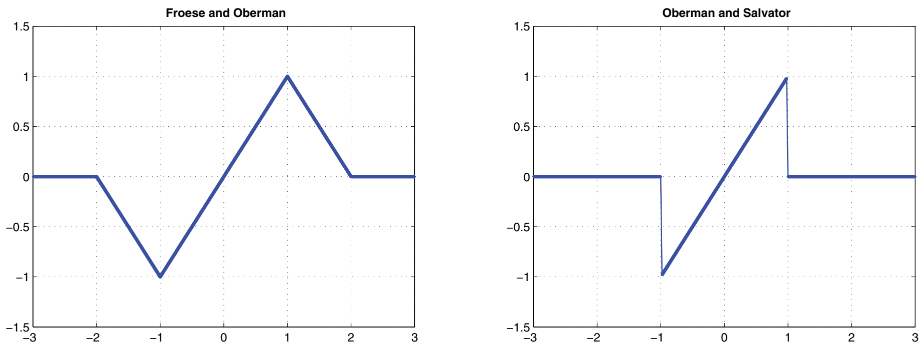

FIG. 1. *Froese and Oberman's filter (left). Oberman and Salvador's filter (right).*

with $\max_{x,p,q} |\partial_p H((x,y),p,q)| \le c_x$, $\max_{x,p,q} |\partial_q H((x,y),p,q)| \le c_y$, and $c_x \frac{\tau}{\Delta x} + c_y \frac{\tau}{\Delta y} \le 1$ (local bounds for c_x, c_y can also be used; see, for instance, [24]). The centered scheme is based on S^A as in (2.23b), and

$$S_0(u^n)_{i,j} := u^n_{i,j} - \tau H\left((x_i,y_j), \frac{u^n_{i+1,j} - u^n_{i-1,j}}{2\Delta x}, \frac{u^n_{i,j+1} - u^n_{i,j-1}}{2\Delta y}\right). \tag{2.25}$$

Definition of the filter function F. We recall that Froese and Oberman's filter function (see Figure 1) used in [15] is

$$\tilde{F}(x) = sign(x)\max(1 - ||x| - 1|, 0) = \begin{cases} x, & |x| \le 1, \\ 0, & |x| \ge 2, \\ -x+2, & 1 \le x \le 2, \\ -x-2, & -2 \le x \le -1. \end{cases} \tag{2.26}$$

In the present work we define the filter function as follows:

$$F(x) := x1_{|x|\le 1} = \begin{cases} x & \text{if } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.27}$$

This is the same filter function that Oberman and Salvador used in [23] for solving steady equations.

The idea of the present filter function is to keep the high-order scheme when $|h^A - h^M| \le \epsilon$ (because then $|S^A - S^M|/(\tau\epsilon) \le 1$ and $S^F = S^M + \tau\epsilon F(\frac{S^A - S^M}{\tau\epsilon}) \equiv S^A$), whereas $F = 0$ and $S^F = S^M$ if that bound is not satisfied; i.e., the scheme is simply given by the monotone scheme itself. Clearly the main difference is the discontinuity at $x = -1, 1$. As mentioned in [23], using a discontinuous filter may lead to difficulties when dealing with implicit schemes in order to show the existence of the numerical solution. However, here, we focus only on explicit schemes.

2.3. Convergence result. The following theorem gives several basic convergence results for the filtered scheme. Note that the high-order assumption (A4) will not be necessary to get the error estimates (i)–(ii). It will be used only to get a high-order consistency error estimate in the regular case (part (iii)). Globally the scheme will have just an $O(\sqrt{\Delta x})$ rate of convergence for Lipschitz continuous solutions because the jumps in the gradient prevent high-order accuracy on the kinks.

THEOREM 2.1. *Assume* (A1)–(A2), *and let* v_0 *be bounded. We assume also that* S^M *satisfies* (A3), *and* $|F| \le 1$. *Let* u^n *denote the filtered scheme* (2.6). *Let*

$v_j^n := v(t_n, x_j)$, where v is the exact solution of (2.4). Assume

$$0 < \epsilon \le c_0\sqrt{\Delta x} \tag{2.28}$$

for some constant $c_0 > 0$.

(i) *The scheme u^n satisfies the Crandall–Lions estimate*

$$\|u^n - v^n\|_\infty \le C\sqrt{\Delta x} \quad \forall\ n = 0, \dots, N, \tag{2.29}$$

for some constant C independent of Δx.

(ii) *(First order convergence for classical solutions.) If furthermore the exact solution v belongs to $C^2([0,T] \times \mathbb{R})$, and $\epsilon \le c_0\Delta x$ (instead of (2.28)), then, we have*

$$\|u^n - v^n\|_\infty \le C\Delta x, \quad n = 0, \dots, N, \tag{2.30}$$

for some constant C independent of Δx.

(iii) *(Local high-order consistency.) Assume that S^A is a high-order scheme satisfying (A4) for some $k \ge 2$. Let $1 \le \ell \le k$ and v be a $C^{\ell+1}$ function in a neighborhood of a point $(t, x) \in (0, T) \times \mathbb{R}$. Assume that*

$$(C_{A,1} + C_M)\left(\|v_{tt}\|_\infty\, \tau + \|v_{xx}\|_\infty\, \Delta x\right) \le \epsilon. \tag{2.31}$$

Then, for sufficiently small $t_n - t$, $x_j - x$, τ, Δx, it holds that

$$S^F(v^n)_j = S^A(v^n)_j,$$

and, in particular, a local high-order consistency error for the filtered scheme S^F holds:

$$\mathcal{E}_{S^F}(v^n)_j \equiv \mathcal{E}_{S^A}(v^n)_j = O(\Delta x^\ell)$$

(the consistency error $\mathcal{E}_{S^A}$ is defined in (2.21)).

Remark 2.5. In our simulations, in the time-dependent setting, we will use $\epsilon = c_1\Delta x$, where c_1 is a constant. It is natural to take $\epsilon \le c_0\sqrt{\Delta x}$ for the convergence of the scheme. It is also natural to take ϵ smaller than $c_1\Delta x$ so that when the solution is only C^2 differentiable we still have the estimate (ii) of Theorem 2.1. Finally, in order to switch to high order in a convenient way, we will see in section 2.6 that a lower bound for ϵ should also be of the form $c_1\Delta x$, with a specific estimate for the constant c_1. This choice is different from that of [23] for stationary equations, where ϵ of the order of $\sqrt{\Delta x}$ is used.

Proof of Theorem 2.1. (i) Let $w_j^{n+1} = S^M(w^n)_j$ be defined with the monotone scheme only, with $w_j^0 = v_0(x_j) = u_j^0$. By definition,

$$u_j^{n+1} - w_j^{n+1} = S^M(u^n)_j - S^M(w^n)_j + \epsilon\tau F(.).$$

Hence, by using the monotonicity of S^M,

$$\max_j |u_j^{n+1} - w_j^{n+1}| \le \max_j |u_j^n - w_j^n| + \epsilon\tau,$$

and by recursion, for $n \le N$,

$$\max_j |u_j^n - w_j^n| \le \epsilon n\tau \le T\epsilon.$$

On the other hand, by Crandall and Lions [10], an error estimate holds for the monotone scheme:

$$\max_j |w_j^n - v_j^n| \leq C\sqrt{\Delta x}$$

for some $C \geq 0$. By summing up the previous bounds, we deduce

$$\max_j |u_j^n - v_j^n| \leq C\sqrt{\Delta x} + T\epsilon,$$

and together with the assumption on ϵ, it gives the desired result.

(ii) Let $\mathcal{E}_j^n := \frac{v_j^{n+1} - S^M(v^n)_j}{\tau}$. If the solution is C^2 regular with bounded second order derivatives, then the consistency error is bounded by

$$|\mathcal{E}_j^n| \leq C_M(\tau + \Delta x). \tag{2.32}$$

Hence

$$\begin{aligned} |u_j^{n+1} - v_j^{n+1}| &= |S^M(u^n)_j - S^M(v^n)_j + \tau\mathcal{E}_j^n + \tau\epsilon F\big(.\big)| \\ &\leq \|u^n - v^n\|_\infty + \tau\|\mathcal{E}^n\|_\infty + \tau\epsilon. \end{aligned}$$

By recursion, for $n\tau \leq T$,

$$\|u^n - v^n\|_\infty \leq \|u^0 - v^0\|_\infty + T(\max_{0\leq k\leq N-1} \|\mathcal{E}^k\|_\infty + \epsilon).$$

Finally, by using the assumption on ϵ, the bound (2.32), and the fact that $\tau = O(\Delta x)$ (using CFL condition (2.17)), we get the desired result.

(iii) To prove that $S^F(v^n)_j = S^A(v^n)_j$, one has to check that

$$\frac{|S^A(v^n)_j - S^M(v^n)_j|}{\epsilon\tau} \leq 1$$

as $(\tau, \Delta x) \to 0$. By using the consistency error definitions,

$$\begin{aligned} \frac{|S^A(v^n)_j - S^M(v^n)_j|}{\tau} &= \bigg| \frac{v_j^{n+1} - S^M(v^n)_j}{\tau} + v_t(t_n, x_j) + H(x_j, v_x(t_n, x_j)) \\ &\qquad - \bigg(\frac{v_j^{n+1} - S^A(v^n)_j}{\tau} + v_t(t_n, x_j) + H(x_j, v_x(t_n, x_j)) \bigg) \bigg| \\ &\leq |\mathcal{E}_{S^M}(v^n)_j| + |\mathcal{E}_{S^A}(v^n)_j| \\ &\leq (C_{A,1} + C_M)(\tau\|v_{tt}\|_\infty + \Delta x\|v_{xx}\|_\infty). \end{aligned}$$

Hence the desired result follows. ∎

Remark 2.6 (other approaches). It is already known from the original work of Osher and Shu [24] that it is possible to modify an ENO scheme in order to obtain a convergent scheme. For instance, if $D^{\pm,A}u_j^n$ denotes a high-order finite difference derivative estimate (of ENO type), a projection on the first order finite difference derivative $D^\pm u_j^n$ can be used, up to a controlled error (see in particular Remark 2.2 of [24]):

$$\text{instead of } \ D^{\pm,A}u_j^n, \quad \text{use } P_{[D^\pm u_j^n, M\Delta x]}(D^{\pm,A}u_j^n),$$

where $P_{[a,b]}(y)$ is the projection defined by

$$P_{[a,b]}(y) := \begin{cases} y & \text{if } a-b \le y \le a+b, \\ a-b & \text{if } y \le a-b, \\ a+b & \text{if } y \ge a+b, \end{cases}$$

and $M > 0$ is some constant greater than the expected value $\frac{1}{2}|u_{xx}(t_n, x_j)|$. However, we emphasize that in our approach we do not consider a projection but a perturbation with a filter, which is slightly different. Indeed, by using a projection into an interval of the form $[a - M\Delta x,\, a + M\Delta x]$, where $a = D^{\pm}u_i^n$, numerical tests show that we may choose too often one of the extremal values $a \pm M\Delta x$ which then produces an overall too-big error (worse than using the first order finite differences).

Following the present approach, we would rather advise using

$$\text{instead of } \; D^{\pm,A}u_j^n, \quad \text{the value } \tilde{P}_{[D^{\pm}u_j^n, M\Delta x]}(D^{\pm,A}u_j^n),$$

where $\tilde{P}_{[a,b]}(y)$ is defined by

$$\tilde{P}_{[a,b]}(y) := \begin{cases} y & \text{if } a-b \le y \le a+b, \\ a & \text{if } y \notin [a-b, a+b]. \end{cases}$$

Remark 2.7 (filtered SL scheme). Let us consider the case of

$$H(x,p) := \min_{b\in B} \max_{a\in A}\{-f(x,a,b).p - \ell(x,a,b)\}, \tag{2.33}$$

where $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ are nonempty compact sets (with $m, n \ge 1$), $f : \mathbb{R}^d \times A \times B \to \mathbb{R}^d$ and $\ell : \mathbb{R}^d \times A \times B \to \mathbb{R}$ are Lipschitz continuous w.r.t. x: $\exists L \ge 0$ for all $(a,b) \in A \times B$, for all x, y:

$$\max(|f(x,a,b) - f(y,a,b)|, |\ell(x,a,b) - \ell(y,a,b)|) \;\le\; L|x-y|. \tag{2.34}$$

(We notice that (A2) is satisfied for Hamiltonian functions such as (2.33).) Let $[u]$ denote the P^1-interpolation of u in dimension one on the mesh (x_j), i.e.,

$$x \in [x_j, x_{j+1}] \quad \Rightarrow \quad [u](x) := \frac{x_{j+1} - x}{\Delta x} u_j + \frac{x - x_j}{\Delta x} u_{j+1}. \tag{2.35}$$

Then a monotone SL scheme can be defined as follows:

$$S^M(u^n)_j := \min_{a\in A} \max_{b\in B} \Big([u^n]\big(x_j + \tau f(x_j,a,b)\big) + \tau \ell(x_j,a,b) \Big). \tag{2.36}$$

A filtered scheme based on SL can then be defined by (2.6) and (2.36). A convergence result as well as error estimates could also be obtained in this framework. (For error estimates for the monotone SL scheme, we refer the reader to [25, 11].)

2.4. Convergence result for a steady equation. Here we consider the special case of steady equations and show how the previous error estimates can be obtained in that case. We focus on the special case of

$$\lambda v + H(x, v_x) = 0, \quad x \in \mathbb{R}, \tag{2.37}$$

for some parameter $\lambda > 0$ (λ chosen large enough in order to have existence and uniqueness of Lipschitz continuous solutions for (2.37); see [2]).

For the scheme we can consider an approximation of the time-dependent equivalent $v_t + \lambda v + H(x, v_x) = 0$ in the form

$$\frac{u_j^{n+1} - u_j^n}{\tau} + \lambda u_j^n + h(x, D^- u_j^n, D^+ u_j^n) = 0, \quad j \in \mathbb{Z} \tag{2.38}$$

(together with some initialization of u_j^0), and consider the limit $n \to \infty$ (or $t \to \infty$). The filtered Hamiltonian h is of the form

$$h := h^M + \epsilon F\left(\frac{h^A - h^M}{\epsilon}\right),$$

where $|F| \leq 1$, and a monotone finite difference approximation is used for h^M (assumptions (A3) are satisfied for the operator $S^M(u)_i := u_i - \tau h^M(u)_i$, or equivalently, h^M is Lipschitz continuous and satisfies (2.15) and (2.16); see Remark 2.2), and a high-order numerical Hamiltonian h^A can be used.

For some given threshold $\eta > 0$, the scheme iterations are stopped when

$$\left\|\frac{u^{n+1} - u^n}{\tau}\right\|_\infty = \max_j \left|\frac{u_j^{n+1} - u_j^n}{\tau}\right| \;\leq\; \eta.$$

Equivalently, we can consider that we have computed values u_j^n such that

$$\max_j |\lambda u_j^n + h(x, D^- u_j^n, D^+ u_j^n)| \leq \eta. \tag{2.39}$$

The problem is to estimate the error $\|u^n - v\|_\infty := \max_j |u_j^n - v(x_j)|$, where v is the exact solution of (2.37) and u_j^n satisfies (2.39).

THEOREM 2.2. *Assume that u_i^n is linearly bounded, i.e., there exists some constant $K \geq 0$ such that $|u_i^n| \leq C(1 + |x_i|)$ for all $i \in \mathbb{Z}$. The following estimate holds for the filtered scheme:*

$$\|u^n - v\|_{L^\infty} \leq C(\eta + \epsilon + \sqrt{\Delta x}). \tag{2.40}$$

It is worth noting that, when η and ϵ are chosen to be of the order $O(\sqrt{\Delta x})$, all the terms on the right-hand of side of (2.40) give the same contribution, and we get an error bound of order $O(\sqrt{\Delta x})$.

Proof of Theorem 2.2. We first recall that if w^n is the solution of the monotone scheme,

$$\lambda w_j^n + h^M(x_j, D^- w_j^n, D^+ w_j^n) = 0, \quad j \in \mathbb{Z}, \tag{2.41}$$

then the Crandall–Lions [10] estimate

$$\max_j |w_j^n - v(x_j)| \leq C\sqrt{\Delta x} \tag{2.42}$$

holds for some constant $C \geq 0$. For the filtered scheme, since $|F(\cdot)| \leq 1$, the following holds:

$$\left|\lambda u_j^n + h^M(x_j, D^- u_j^n, D^+ u_j^n)\right| \leq \epsilon + \eta, \quad j \in \mathbb{Z}. \tag{2.43}$$

In particular,

$$\lambda u_j^n + h^M\left(x_j, \frac{u_j^n - u_{j-1}^n}{\Delta x}, \frac{u_{j+1}^n - u_j^n}{\Delta x}\right) \leq \eta + \epsilon, \quad j \in \mathbb{Z}.$$

Let $c := \frac{\eta+\epsilon}{\lambda}$ and $z_j := u_j^n - c$. The following also holds:

$$\lambda z_j + h^M\left(x_j, \frac{z_j^n - z_{j-1}^n}{\Delta x}, \frac{z_{j+1}^n - z_j^n}{\Delta x}\right) \le 0, \quad j \in \mathbb{Z}.$$

By using the comparison principle for the scheme (see Lemma B.1 of Appendix B), we obtain $z_j \le w_j^n$, and therefore

$$u_j^n - w_j^n \le c = \frac{\eta + \epsilon}{\lambda}, \quad j \in \mathbb{Z}.$$

In the same way one can prove that

$$u_j^n - w_j^n \ge -\frac{\eta + \epsilon}{\lambda}, \quad j \in \mathbb{Z}.$$

Therefore, $\|u^n - w^n\|_\infty \le \frac{\eta+\epsilon}{\lambda}$. Combining this with (2.42), we conclude the desired estimate. $\square$

2.5. Adding a limiter. The basic filtered scheme (2.6) is designed to be of high order where the solution is regular. However, in the case of front propagation, it can be observed that the filtered scheme may let small errors occur near extrema, when two possible directions of propagation occur in the same cell. This is the case, for instance, near a minimum for an eikonal equation. In order to improve the scheme near extrema, we propose introducing a limiter before beginning the filtering process. Limiting correction will be needed only at extrema.

Let us consider the case of front-propagation, i.e., an equation of type (2.4), now with

$$H(x, v_x) = \max_{a \in A} \big(f(x,a)v_x\big) \tag{2.44}$$

(i.e., no distributive cost in the Hamiltonian function).

In the one dimensional case, the cell centered in x_j may need a correction if there is a local minimum and if

$$\min_a f(x_j, a) \le 0 \quad \text{and} \quad \max_a f(x_j, a) \ge 0. \tag{2.45}$$

We decide to "mark" such cells. For a marked cell, the numerical solution should not go below the local minimum around the point; i.e., we want

$$u_j^{n+1} \ge u_{min,j} := \min(u_{j-1}^n, u_j^n, u_{j+1}^n), \tag{2.46}$$

and, in the same way, we want to impose that

$$u_j^{n+1} \le u_{max,j} := \max(u_{j-1}^n, u_j^n, u_{j+1}^n), \tag{2.47}$$

as would be the case in order to have L^∞ stability for an advection equation. If we consider the high-order scheme to be of the form $u_j^{n+1} = u_j^n - \tau h^A(u^n)$, then the limiting process amounts to saying that

$$h^A(u^n)_j \le h_j^{max} := \frac{u_j^n - u_{min,j}}{\tau}$$

and

$$h^A(u^n)_j \ge h_j^{min} := \frac{u_j^n - u_{max,j}}{\tau}.$$

This amounts to defining a limited $\bar{h}^A$ such that

$$\begin{cases} \bar{h}^A(u^n)_j := \min(\max(h^A(u^n)_j,\, h_j^{min}),\, h_j^{max}) & \text{if (2.45) holds at mesh point } x_j, \\ \bar{h}_j^A :\equiv h_j^A \text{ otherwise.} \end{cases}$$

Then the filtering process is the same, using $\bar{h}^A$ instead of h^A in the definition of S^F.

For two dimensional equations a similar limiter could be developed in order to make the scheme more efficient at singular regions. However, for the numerical tests of the next section (in two dimensions) we will simply limit the scheme by using an equivalent of (2.46)–(2.47). Hence, instead of the scheme value $u_{ij}^{n+1} = S^A(u^n)_{ij}$ for the high-order scheme, we will update the value by

$$u_{ij}^{n+1} = \min(\max(S^A(u^n)_{ij}, u_{ij}^{min}), u_{ij}^{max}), \tag{2.48}$$

where $u_{ij}^{min} = \min(u_{ij}^n, u_{i\pm1,j}^n, u_{i,j\pm1}^n)$ and $u_{ij}^{max} = \max(u_{ij}^n, u_{i\pm1,j}^n, u_{i,j\pm1}^n)$.

2.6. How to choose the parameter ϵ: A simplified approach. The scheme should switch to a high-order scheme when some regularity of the data is detected, and in that case we should have

$$\left|\frac{S^A(v) - S^M(v)}{\epsilon\tau}\right| \;=\; \left|\frac{h^A(\cdot) - h^M(\cdot)}{\epsilon}\right| \;\le\; 1.$$

In a region where a function $v = v(x)$ is regular enough, by using Taylor expansions, zero order terms in $h^A(x, D^\pm v)$ and $h^M(x, D^\pm v)$ vanish (they are both equal to $H(x, v_x(x)))$ and there remains an estimate of order Δx. More precisely, by using the high-order property (A4), we have

$$h^A(x_j, D^\pm v_j) = H(x_j, v_x(x_j)) + O(\Delta x^2).$$

On the other hand, by using Taylor expansions,

$$Dv_j^\pm = v_x(x_j) \pm \frac{1}{2} v_{xx}(x_j)\Delta x + O(\Delta x^2).$$

Hence, denoting $h^M = h^M(x, p^-, p^+)$, the following holds at points where h^M is regular:

$$h^M(x_j, Dv_j^-, Dv_j^+) = H(x_j, v_x(x_j)) + \frac{1}{2} v_{xx}(x_j)\left(\frac{\partial h_j^M}{\partial p^+} - \frac{\partial h_j^M}{\partial p^-}\right) + O(\Delta x^2).$$

Therefore,

$$|h^A(v) - h^M(v)| = \frac{1}{2}|v_{xx}(x_j)| \left|\frac{\partial h_j^M}{\partial p^+} - \frac{\partial h_j^M}{\partial p^-}\right| \Delta x + O(\Delta x^2).$$

Hence we will make the choice to take ϵ roughly such that

$$\frac{1}{2}|v_{xx}(x_j)| \left|\frac{\partial h_j^M}{\partial p^+} - \frac{\partial h_j^M}{\partial p^-}\right| \Delta x \le \epsilon \tag{2.49}$$

(where $h_j^M = h^M(x_j, v_x(x_j), v_x(x_j)))$. Therefore, if at some point x_j (2.49) holds, then the scheme will switch to the high-order scheme. Otherwise, when the expectations from h^M and h^A are different enough, the scheme will switch to the monotone scheme.

In conclusion, we have upper and lower bounds for the switching parameter ϵ:

- Choose $\epsilon \le c_0\sqrt{\Delta x}$ for some constant $c_0 > 0$ in order that the convergence and error estimate results hold (see Theorem 2.1).
- Choose $\epsilon \ge c_1\Delta x$, where c_1 is sufficiently large. This constant should be chosen roughly such that

$$\frac{1}{2}\|v_{xx}\|_\infty \left\| \frac{\partial h^M}{\partial u^+}(.,v_x,v_x) - \frac{\partial h^M}{\partial u^-}(.,v_x,v_x)\right\|_\infty \le c_1,$$

  where the range of values of v_x and v_{xx} can be estimated, in general, from the values of $(v_0)_x$, $(v_0)_{xx}$ and the Hamiltonian function H. Then the scheme is expected to switch to the high-order scheme, where the solution is regular.

3. Numerical tests. In this section we present several numerical tests in one and two dimensions. Unless otherwise indicated, the filtered scheme will refer to the scheme where the high-order Hamiltonian is the centered scheme in space (see Remark 2.2), with Heun (RK2) scheme discretization in time (see in particular (2.23a)–(2.23b)). Hereafter this scheme will be referred to as the "centered scheme." The monotone finite difference scheme and function h^M will be made precise for each example. The meshes for one dimensional and two dimensional tests are always Cartesian and uniform. For the filtered scheme, unless otherwise specified, the switching coefficient $\epsilon = 5\Delta x$ will be used. In practice we have numerically observed that taking $\epsilon = c_1\Delta x$ with c_1 sufficiently large does not much change the numerical results in the following tests. All the tested filtered schemes (apart from the obstacle equations) are tested within the convergence framework of the previous section, so in particular there is a theoretical convergence of order $\sqrt{\Delta x}$ under the usual CFL condition.

In the tests, the filtered scheme will be in general compared to a second order ENO scheme (for precise definition, see Appendix A) as well as the centered (a priori unstable) scheme without filtering. In most cases, the local error in the L^2 norm is computed in some subdomain D, which, at a given time t_n, corresponds to

$$e_{L^2_{loc}} := \left(\sum_{\{i,\ x_i\in D\}} \Delta x\ |v(t_n,x_i) - u_i^n|^2 \right)^{1/2}.$$

The first two numerical examples deal with one dimensional HJ equations, Examples 3 and 4 are concerned with two dimensional HJ equations, and the last three examples will concern a one dimensional steady equation and two nonlinear one dimensional obstacle problems.

Example 1: Eikonal equation. We consider the case of

$$v_t + |v_x| = 0, \quad t \in (0,T),\ x \in (-2,2), \tag{3.1}$$

$$v(0,x) = v_0(x) := \max(0, 1-x^2)^4, \quad x \in (-2,2). \tag{3.2}$$

In Table 1 (see also Figure 2), we compare the filtered scheme (with $\epsilon = 5\Delta x$) with the centered scheme and the ENO second order scheme, with CFL $= 0.37$ and terminal time $T = 0.3$ (where the CFL number corresponds to $c_0\frac{\tau}{\Delta x}$ with $c_0 = 1$ here). For the filtered scheme, the monotone Hamiltonian used is $h^M(x,v^-,v^+) := \max(v^-,-v^+)$. As expected, we observe that the centered scheme alone is unstable. On the other hand, the filtered and ENO schemes are numerically comparable in that case and second order convergent (the results are similar for the L^1 and the L^∞ errors).

TABLE 1
(Example 1 with initial data (3.2).) L^2 errors for filtered scheme, centered scheme, and ENO.

		Filtered ($\epsilon = 5\Delta x$)		Centered		ENO2	
M	N	L^2 error	order	L^2 error	order	L^2 error	order
40	9	7.51E-03	-	1.18E-01	-	1.64E-02	-
80	17	3.36E-03	1.16	1.14E-01	0.06	4.38E-03	1.91
160	33	8.02E-04	2.07	1.13E-01	0.00	1.19E-03	1.87
320	65	1.80E-04	2.16	1.13E-01	0.00	3.22E-04	1.89
640	130	4.53E-05	1.99	1.13E-01	0.00	8.22E-05	1.97

TABLE 2
(Example 1 with initial data (3.3).) L^2 errors for filtered scheme, centered scheme, and ENO.

		Filtered ($\epsilon = 5\Delta x$)		Centered		ENO2	
M	N	error	order	error	order	error	order
40	9	1.27E-02	-	2.03E-02	-	2.60E-02	-
80	17	3.17E-03	2.00	8.96E-03	1.18	8.00E-03	1.70
160	33	7.90E-04	2.01	1.06E-02	-0.24	2.50E-03	1.68
320	65	1.97E-04	2.00	1.26E-01	-3.57	7.80E-04	1.68
640	130	4.92E-05	2.00	1.06E+02	-9.71	2.44E-04	1.67

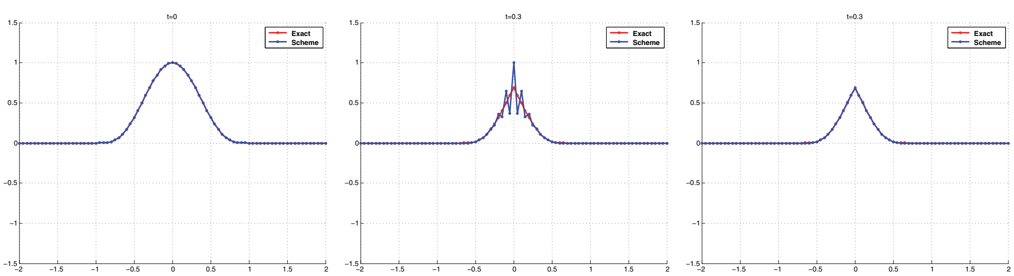

FIG. 2. *(Example 1.) With initial data (3.2) (left), and plots at time $T = 0.3$ with centered scheme (middle) and filtered scheme (right), using $M = 160$ mesh points.*

Then, in Table 2 (see also Figure 3), we consider the same PDE but with the following reversed initial data:

$$\tilde{v}_0(x) := -\max(0, 1 - x^2)^4, \quad x \in (-2, 2). \tag{3.3}$$

In that case the centered scheme alone is unbounded. The filtered scheme (with $\epsilon = 5\Delta x$) is second order. However, here, the limiter correction as described in section 2.5 was needed in order to get second order behavior. (More precisely, we first apply the limiter correction to the second order centered scheme, and then we apply the filter.) We also observe that the filtered scheme gives better results than the ENO scheme. We have also numerically tested the ENO scheme with the same limiter correction, but it does not improve the behavior of the ENO scheme alone.

In conclusion, this first example shows that the filtered scheme can stabilize an otherwise unstable scheme and that it can give the desired second order behavior.

Example 2: Burger's equation. In this example an HJ equivalent of the nonlinear Burgers equation is considered:

$$v_t + \frac{1}{2}|v_x|^2 = 0, \quad t > 0, \ x \in (-2, 2), \tag{3.4a}$$

$$v(0, x) = v_0(x) := \max(0, 1 - x^2), \quad x \in (-2, 2), \tag{3.4b}$$

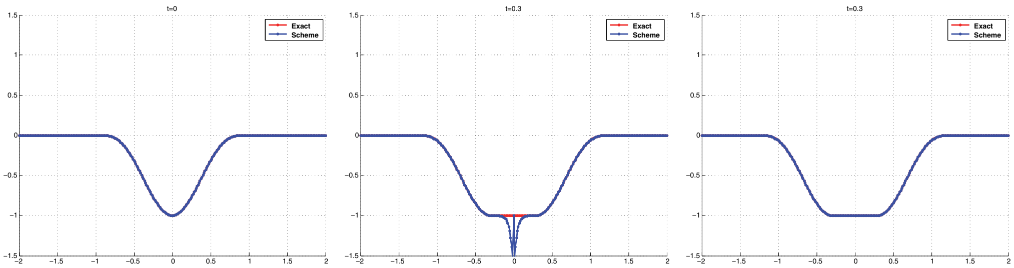

FIG. 3. *(Example 1.) With initial data (3.3) (left), and plots at time $T = 0.3$ with centered scheme (middle) and filtered scheme (right), using $M = 160$ mesh points.*

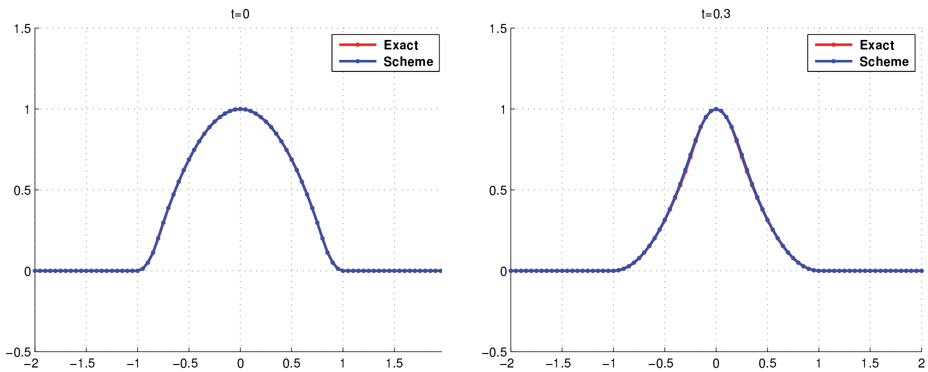

FIG. 4. *(Example 2.) Plots at $t = 0$ and $t = 0.3$ with the filtered scheme.*

TABLE 3
(Example 2.) L^2 errors for filtered scheme, centered scheme, and ENO second order scheme.

		Filtered ($\epsilon = 5\Delta x$)		Centered		ENO2	
M	N	error	order	error	order	error	order
40	9	2.06E-02	-	2.07E-02	-	2.55E-02	-
80	17	6.24E-03	1.73	6.24E-03	1.73	8.24E-03	1.63
160	33	1.85E-03	1.76	1.85E-03	1.76	2.81E-03	1.55
320	65	5.51E-04	1.74	5.51E-04	1.74	1.03E-03	1.45
640	130	1.68E-04	1.71	1.68E-04	1.71	3.74E-04	1.47

with Dirichlet boundary condition on $(-2, 2)$. The exact solution is known:

$$v(t,x) = \frac{(\max(0, 1 - |\bar{x}|))^2}{2t} 1_{\{t>\frac{1}{2}\}} + \frac{(1-2t)^2 - |x|^2}{1-2t} \; 1_{\{1 \geq |x| \geq 1-2t\}}.$$

In order to test high-order convergence, we have considered the smoother initial data, which is that obtained from (3.4) at time $t_0 := 0.1$, i.e.,

$$w_t + \frac{1}{2}|w_x|^2 = 0, \quad t > 0, \; x \in (-2, 2), \tag{3.5a}$$

$$w(0, x) := v(t_0, x), \quad x \in (-2, 2), \tag{3.5b}$$

with exact solution $w(t, x) = v(t + t_0, x)$. An illustration is given in Figure 4.

For the filtered scheme, the monotone Hamiltonian used is $h^M(x, p^-, p^+) := \frac{1}{2}(p^-)^2 \; 1_{p^->0} + \frac{1}{2}(p^+)^2 \; 1_{p^+<0}$. Errors are given in Table 3, using CFL $= 0.37$ and

TABLE 4
(Example 3.) Nonconvex Hamiltonian: Errors for the filtered scheme.

	L^1 error		L^2 error		L^∞ error	
M	error	order	error	order	error	order
10	1.91E-02	-	1.96E-02	-	3.28E-02	-
20	6.40E-03	1.58	7.13E-03	1.46	1.18E-02	1.47
40	1.89E-03	1.76	2.28E-03	1.64	5.52E-03	1.10
80	4.40E-04	2.10	5.25E-04	2.12	1.58E-03	1.81
160	8.96E-05	2.30	1.06E-04	2.31	2.97E-04	2.41
320	2.11E-05	2.09	2.41E-05	2.14	1.02E-04	1.54
640	4.99E-06	2.08	5.41E-06	2.16	1.88E-05	2.44

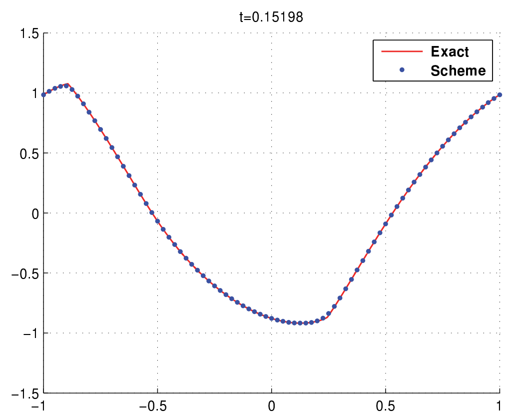

FIG. 5. *(Example 3.) Nonconvex Hamiltonian $H(p) = -cos(p + 1)$.*

terminal time $T = 0.3$. In conclusion, we observe numerically that the filtered scheme keeps the good behavior of the centered scheme (here stable and almost second order).

Example 3: Tests on nonconvex Hamiltonians. We now consider three cases with nonconvex Hamiltonians.

The first case is the same as in [24] with Hamiltonian $H(p) := -\cos(p + 1)$ and with periodic boundary conditions on $(-1, 1)$:

$$v_t - \cos(v_x + 1) = 0, \quad t > 0, \ x \in (-1, 1), \tag{3.6a}$$

$$v(0, x) = v_0(x) := -\cos(\pi x), \quad x \in (-1, 1). \tag{3.6b}$$

Here the monotone scheme is based on a Lax–Friedrichs Hamiltonian function as in (2.18) (here with constant $c_0 = 1$), while the high-order Hamiltonian is again the centered scheme and $\epsilon = 5\Delta x$. In Table 4 and Figure 5, the solution obtained with the filtered scheme is computed at time $T = 1.5/\pi^2$, when two singularities are present ($x_1 \simeq 0.895$ and $x_2 \simeq 0.245$). Local errors are shown for grid points x such that $|x - x_i| \geq 0.05$ for $i = 1, 2$, with CFL $= 0.31$. A reference solution is obtained using the filtered scheme with 10240 points (and the same for the next two cases). In this case the numerical results are almost second order in all norms L^1, L^2, and L^∞.

The second case is similar to the nonconvex Hamiltonian example of [20]:

$$v_t + \frac{1}{4}(v_x^2 - 1)(v_x^2 - 4) = 0, \quad t > 0, \ x \in (-1, 1), \tag{3.7a}$$

$$v(0, x) = v_0(x) := 2 - 2|x|, \quad x \in (-1, 1). \tag{3.7b}$$

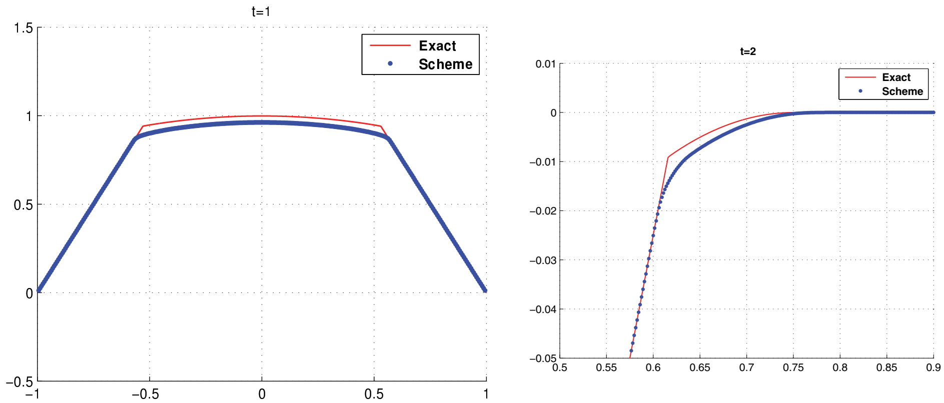

Fig. 6. *(Example 3.) Left: Solving (3.7) with the filtered scheme; plot at time $t = 1$ with $M = 320$ mesh points. Right: Solving (3.8) with the filtered scheme; plot at time $t = 2$ with $M = 640$ mesh points (zoom).*

Results are given in Figure 6 (left). In this case only first order convergence is numerically observed. This example is known to be more difficult for numerical schemes. However, the filtered scheme does converge toward the viscosity solution without the need of further corrections.

The third case is as in [8]:

$$v_t + H(v_x) = 0, \quad t > 0, \ x \in (0, 1), \tag{3.8a}$$

$$v(0, x) = v_0(x) := \min\left(x - \frac{1}{4}, 0\right), \quad x \in (0, 1), \tag{3.8b}$$

where $H(p) = \frac{1}{4}p(1-p)1_{p\le\frac{1}{2}} + (\frac{1}{2}p(p-1)+\frac{3}{16})1_{p\ge\frac{1}{2}}$, together with Neumann boundary conditions $u_x(0) = -1$ and $u_x(1) = 0$.

An illustration is given in Figure 6 (right). Again, in this case only first order convergence is numerically observed, but the filtered scheme does converge toward the viscosity solution.

It is rather difficult to give a general interpretation of the results in the nonconvex case. In the cases where we just have first order accuracy, further investigation is needed.

Example 4: Two dimensional rotation. We consider the following equation in two dimensions:

$$v_t - yv_x + xv_y = 0, \quad (x, y) \in \Omega, \ t > 0, \tag{3.9}$$

$$v(0, x, y) = v_0(x, y) := 0.5 - 0.5 \max\left(0, \frac{1 - (x-1)^2 - y^2}{1 - r_0^2}\right)^4, \tag{3.10}$$

where $\Omega := (-A, A)^2$ (with $A = 2.5$), $r_0 = 0.5$, and with Dirichlet boundary condition $v(t, x) = 0.5$ for $x \in \partial\Omega$. This initial condition is regular and such that the level set $v_0(x, y) = 0$ corresponds to a circle centered at $(1, 0)$ and of radius r_0. In this example the monotone numerical Hamiltonian is defined by

$$\begin{aligned} h^M(u_x^-, p_x^+, p_y^-, p_y^+) := {} & \max(0, f_1(a, x, y))p_x^- + \min(0, f_1(a, x, y))p_x^+ \\ & + \max(0, f_2(a, x, y))p_y^- + \min(0, f_2(a, x, y))p_y^+, \end{aligned} \tag{3.11}$$

TABLE 5
(Example 4.) Global L^2 errors for the filtered, centered, and second order ENO schemes (with CFL 0.37).

		Filtered		Centered		ENO	
Mx	Ny	L^2 error	order	L^2 error	order	L^2 error	order
20	20	5.05E-01	-	5.05E-01	-	6.99E-01	-
40	40	1.48E-01	1.77	1.48E-01	1.77	4.66E-01	0.58
80	80	3.77E-02	1.98	3.77E-02	1.98	2.04E-01	1.19
160	160	9.40E-03	2.00	9.40E-03	2.00	5.50E-02	1.89
320	320	2.34E-03	2.01	2.34E-03	2.01	1.29E-02	2.10

and the high-order scheme is the centered finite difference scheme in both spacial variables and RK2 in time. The filtered scheme is otherwise the same as (2.6). However, it is necessary to use a greater constant c_1 in the choice $\epsilon = c_1 \Delta x$ in order to get (global) second order errors. We have used here $\epsilon = 20\Delta x$. On the other hand, the CFL is $\mu := 0.37$, where μ is defined by

$$\mu := c_0 \left(\frac{\tau}{\Delta x} + \frac{\tau}{\Delta y} \right), \tag{3.12}$$

and where $c_0 = 2.5$ (an upper bound for the dynamics in the considered domain Ω).

Results are shown in Table 5 for terminal time $T := \pi/2$. Although the centered scheme is a priori unstable, in this example it is numerically stable and of second order. So we observe that the filtered scheme ϵ has a good behavior and is also of second order (the ENO scheme gives comparable results here).

Example 5: Eikonal equation. In this example we consider the eikonal equation

$$v_t + |\nabla v| = 0, \quad (x, y) \in \Omega,\ t > 0, \tag{3.13}$$

in the domain $\Omega := (-3, 3)^2$. The initial data is defined by

$$v_0(x, y) = 0.5 - 0.5 \max \left(\max \left(0, \frac{1 - (x-1)^2 - y^2}{1 - r_0^2}\right)^4,\ \max \left(0, \frac{1 - (x+1)^2 - y^2}{1 - r_0^2}\right)^4 \right).$$

The zero-level set of v_0 corresponds to two separate circles of radius r_0 which are centered in $A = (1, 0)$ and $B = (-1, 0)$, respectively. Dirichlet boundary conditions are used as the previous example.

The monotone Hamiltonian h^M used in the filtered scheme is in Lax–Friedrichs form:

$$\begin{aligned} h^M(x, p_1^-, p_1^+, p_2^-, p_2^+) = H\left(x, \frac{p_1^- + p_1^+}{2}, \frac{p_2^- + p_2^+}{2}\right) \\ -\frac{C_x}{2}(p_1^+ - p_1^-) - \frac{C_y}{2}(p_2^+ - p_2^-), \end{aligned} \tag{3.14}$$

where, here, $C_x = C_y = 1$. We use the CFL condition $\mu = 0.37$ as in (3.12). Also, the simple limiter (2.48) is used for the filtered scheme as described in section 2.5. It is needed in order to get good second order behavior at extrema of the numerical solution. The filter coefficient is set to $\epsilon = 20\Delta x$ as in the previous example.

Numerical results are given in Table 6, showing the global L^2 errors for the filtered scheme, the centered scheme, and a second order ENO scheme, at time $t = 0.6$. We observe that the centered scheme has some instabilities for fine mesh, while the filtered performs as expected (see Figure 7).

TABLE 6
(Example 5.) Global L^2 errors for filtered, centered, and second order ENO schemes.

		Filtered ($\epsilon = 20\Delta x$)		Centered		ENO2	
Mx	Ny	L^2 error	order	L^2 error	order	L^2 error	order
25	25	5.39E-01	-	6.00E-01	-	5.84E-01	-
50	50	1.82E-01	1.57	2.25E-01	1.41	2.11E-01	1.47
100	100	3.72E-02	2.29	8.46E-02	1.41	6.88E-02	1.62
200	200	9.36E-03	1.99	3.53E-02	1.26	2.02E-02	1.76
400	400	2.36E-03	1.99	1.36E-01	-1.95	5.98E-03	1.76

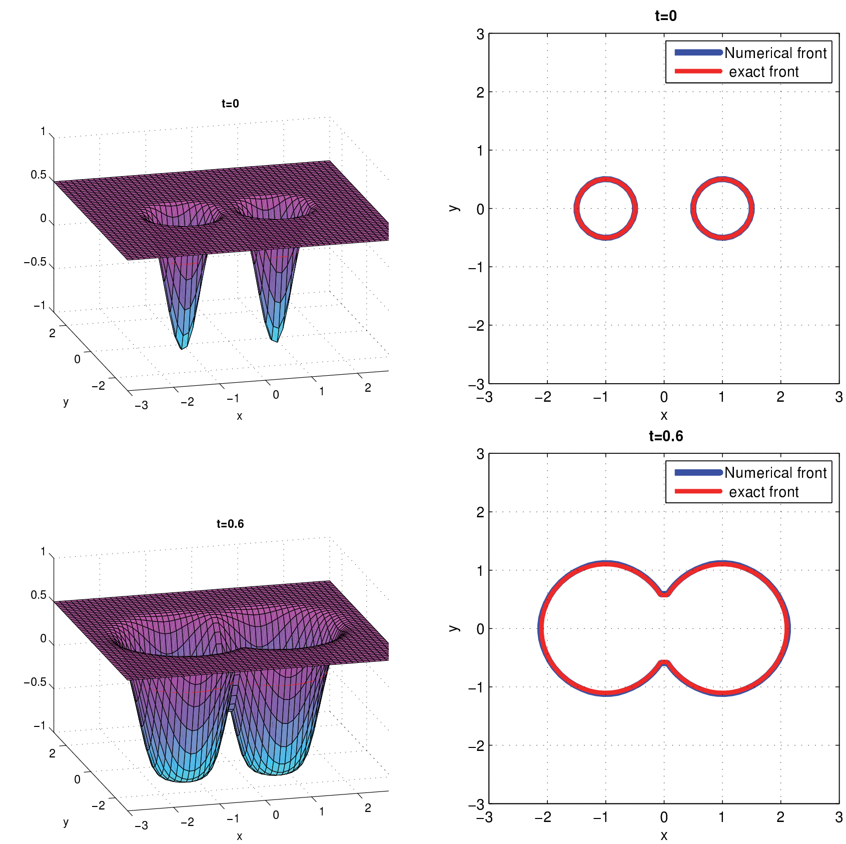

FIG. 7. *(Example 5.) Plots at times $t = 0$ (top) and $t = \pi/2$ (bottom) for the filtered scheme with $M = 50$ mesh points. The figures to the right represent the 0-level sets.*

Example 6: Steady eikonal equation. We consider a steady eikonal equation with Dirichlet boundary condition, which is taken from Abgrall [1]:

$$|v_x| = f(x), \qquad x \in (0, 1), \tag{3.15a}$$

$$v(0) = v(1) = 0, \tag{3.15b}$$

where $f(x) = 3x^2 + a$, with $a = \frac{1-2x_0^3}{2x_0-1}$ and $x_0 = \frac{\sqrt[3]{2}+2}{4\sqrt[3]{2}}$. An exact solution is known:

$$v(x) := \begin{cases} x^3 + ax, & x \in [0, x_0], \\ 1 + a - ax - x^3, & x \in [x_0, 1]. \end{cases} \tag{3.16}$$

TABLE 7
(Example 6.) Global L^2 errors for filtered scheme, compared with the centered (unstable) scheme, ENO scheme, and a filtered ENO scheme. Here with the discontinuous filter (2.27).

	Filtered		Centered	ENO		Filtered + ENO	
M	L^2 error	order	L^2 error	L^2 error	order	L^2 error	order
50	2.16E-03	-	NaN	3.98E-04	-	5.29E-03	-
100	7.14E-04	1.60	NaN	1.21E-04	1.73	1.35E-03	1.97
200	2.17E-04	1.72	NaN	3.99E-05	1.59	3.42E-04	1.98
400	6.32E-05	1.78	NaN	1.36E-05	1.55	8.61E-05	1.99
800	2.17E-05	1.54	NaN	4.75E-06	1.52	2.16E-05	2.00

TABLE 8
(Example 6.) Global L^2 errors for filtered scheme, compared with the filtered ENO scheme. Here with the continuous filter (2.26) of Froese and Oberman.

	Filtered		Filtered + ENO	
M	L^2 error	order	L^2 error	order
50	1.65E-03	-	3.06E-03	-
100	4.50E-04	1.88	8.10E-04	1.92
200	1.40E-04	1.68	2.07E-04	1.97
400	3.17E-05	2.15	5.25E-05	1.98
800	9.82E-06	1.69	1.32E-05	1.99

The steady solution for (3.15) can be considered as the limit $\lim_{t\to\infty} v(t,x)$, where v is the solution of the time marching corresponding form:

$$v_t + |v_x| = f(x), \qquad x \in (0,1),\ t > 0, \tag{3.17a}$$

$$v(t,0) = v(t,1) = 0, \quad t > 0. \tag{3.17b}$$

In this example the upwind monotone scheme is used:

$$h^M(.)_j := \frac{u_j^{n+1} - u_j^n}{\tau} - \max\left\{\frac{u_j^n - u_{j-1}^n}{\Delta x}, \frac{u_j^n - u_{j+1}^n}{\Delta x}\right\} - \tau f(x_j),$$

the high-order scheme will be the centered scheme, and the filtered scheme (2.6) will be used with $\epsilon = 5\Delta x$. The scheme is initialized with $u_i^0 = 0$. The iterations are stopped when the difference between two successive time steps is small enough or a fixed number of iterations has passed, i.e.,

$$\|u^{n+1} - u^n\|_{L^\infty} := \max_i |u_i^{n+1} - u_i^n| \le 10^{-6} \quad \text{or} \quad n \ge N_{max} := 5000. \tag{3.18}$$

As analyzed in [5] for ϵ-monotone schemes, for a given mesh step, even if the iterations do not converge (because of the nonmonotony of the scheme), the scheme values can be shown to be close to a fixed point after enough iterations.

Results are given in Figure 8 and in Tables 7 and 8 for the L^2 errors and in Table 9 for the L^∞ errors. In the first two tables, we use the usual discontinuous filter (2.27) (as in [23]). In the second table, the same tests are performed with the continuous filter (2.26) (as in [15]). We observe that the combination of the filter and the ENO scheme shows second order behavior in all norms. The ENO scheme alone, even if it performs well, gives only first order results in the L^∞ norm.

Remark 3.1. The choice of the filter, for this steady equation, does not change much the numerical results. It is our experience that there is no significant difference between using the filter function (2.27) rather than (2.26) for steady equations.

Table 9

(Example 6.) Global L^∞ errors for filtered scheme, compared with the centered (unstable) scheme, ENO scheme, and a filtered ENO scheme. Here with the discontinuous filter (2.27).

	Filtered		Centered	ENO		Filtered + ENO	
M	L^∞ error	order	L^∞ error	L^∞ error	order	L^∞ error	order
50	6.04E-03	-	Nan	1.70E-03	-	5.70E-03	-
100	1.32E-03	2.20	Nan	7.89E-04	1.11	1.45E-03	1.97
200	3.02E-04	2.12	Nan	3.82E-04	1.05	3.64E-04	1.99
400	7.17E-05	2.08	Nan	1.88E-04	1.02	9.11E-05	2.00
800	2.07E-05	1.79	Nan	9.36E-05	1.01	2.27E-05	2.01

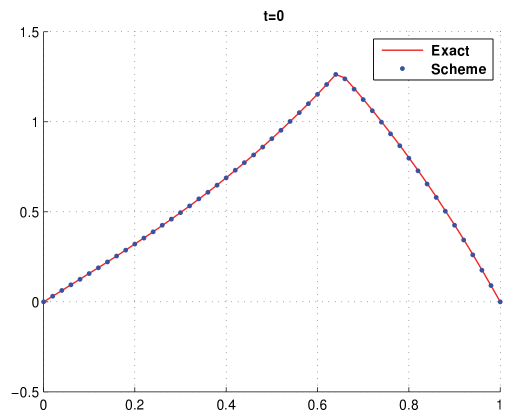

Fig. 8. *(Example 6.) Filtered scheme for a steady equation, with $M = 50$ mesh points.*

On the other hand, this is not the case for time-dependent equations and for the proposed schemes, where we have numerically observed that the continuous filter (switching back to first order behavior after several times steps) was not as good as the discontinuous filter. This motivates our choice to use the discontinuous filter.

Example 7: Eikonal equation with an obstacle. We consider an eikonal equation with an obstacle term, also taken from [4]:

$$\min(v_t + |v_x|, v - g(x)) = 0, \quad t > 0,\ x \in [-1, 1], \tag{3.19}$$

$$v_0(x) = 0.5 + \sin(\pi x), \quad x \in [-1, 1], \tag{3.20}$$

with periodic boundary condition on $(-1, 1)$ and $g(x) = \sin(\pi x)$. The exact solution is given by $v(t, x) = \max(\bar{v}(t, x), g(x))$, where $\bar{v}$ is the solution of the eikonal equation $v_t + |v_x| = 0$. The formula $\bar{v}(t, x) = \min_{y\in[x-t,x+t]} v_0(y)$ holds, which simplifies to

$$\bar{v}(t, x) := \begin{cases} v_0(x + t) & \text{if } x < -0.5 - t, \\ -0.5 & \text{if } x \in [-0.5 - t, -0.5 + t], \\ \min(v_0(x - t), v_0(x + t)) & \text{if } x \geq -0.5 + t. \end{cases} \tag{3.21}$$

Remark 3.2. For an obstacle problem of the form $\min(u_t + H(u_x), u - g(x)) = 0$, the finite difference scheme used here is

$$u_i^{n+1} := \max(u^{n+1,1},\ g(x_i)),$$

where $u^{n+1,1}$ is the usual filtered scheme for the HJ equation $u_t + H(u_x) = 0$ starting from the previous scheme values u^n. Such schemes for obstacle problems are well known (see, for instance, [6, 4]).

TABLE 10
(Example 7.) Filtered scheme and ENO scheme at time $t = 0.2$.

Errors	Filtered		ENO2	
M	error	order	error	order
40	3.74E-03	-	6.85E-03	-
80	6.26E-04	2.58	2.12E-03	1.69
160	1.13E-04	2.47	6.80E-04	1.64
320	2.26E-05	2.32	2.18E-04	1.64
640	5.50E-06	2.04	6.96E-05	1.65

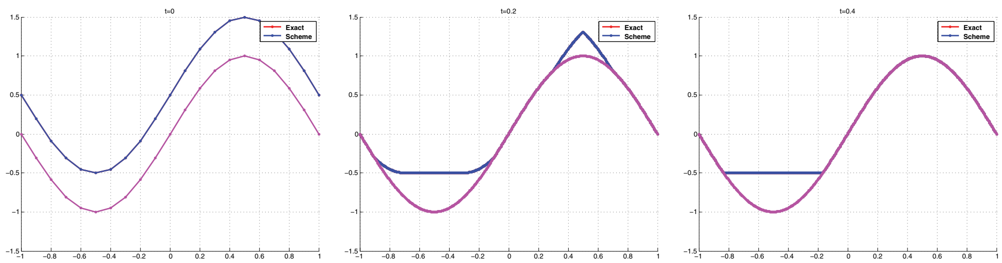

FIG. 9. *(Example 7.) Plots at times $t = 0$, $t = 0.2$, and $t = 0.4$. The dark line is the numerical solution, similar to the exact solution, and the light line is the obstacle function.*

Results are given in Table 10 for terminal time $T = 0.2$. Plots are also shown in Figure 9 for different times (for $t \geq \frac{1}{3}$ the solution remains unchanged).

Appendix A. An essentially nonoscillatory (ENO) scheme of second order. We recall here a simple ENO method of order two based on the work of Osher and Shu [24] for the HJ equation (the ENO method was designed by Harten et al. [19] for the approximation solution of nonlinear conservation laws).

Let m be the minmod function defined by

$$m(a,b) = \begin{cases} a & \text{if } |a| \leq |b|,\ ab > 0, \\ b & \text{if } |b| < |b|,\ ab > 0, \\ 0 & \text{if } ab \leq 0 \end{cases} \tag{A.1}$$

(other functions can be considered such as $m(a,b) = a$ if $|a| \leq |b|$ and $m(a,b) = b$ otherwise). Let $D^{\pm}u_j = \pm(u_{j\pm1} - u_j)/\Delta x$ and

$$D^2 u_j := \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}.$$

Then the right and left ENO approximations of the derivative can be defined by

$$\bar{D}^{\pm}u_j = D^{\pm}u_j \mp \frac{1}{2}\Delta x\ m(D^2u_j, D^2u_{j\pm1})$$

and the ENO (Euler forward) scheme by

$$S_0(u)_j := u_j - \tau h^M(x_j,\ \bar{D}^- u_j,\ \bar{D}^+ u_j).$$

The corresponding RK2 scheme can then be defined by $S(u) = \frac{1}{2}(u + S_0(S_0(u)))$.

Appendix B. Comparison lemma for the scheme values.

LEMMA B.1. *Assume that $u = (u_j)$ satisfies*

$$\lambda u_j + h^M\left(x_j, \frac{u_j - u_{j-1}}{\Delta x}, \frac{u_{j+1} - u_j}{\Delta x}\right) \leq 0, \quad j \in \mathbb{Z}, \tag{B.1}$$

and that $v = (v_j)$ *satisfies*

$$\lambda v_j + h^M\left(x_j, \frac{v_j - v_{j-1}}{\Delta x}, \frac{v_{j+1} - v_j}{\Delta x}\right) \geq 0, \quad j \in \mathbb{Z}, \tag{B.2}$$

as well as $u_j \leq K(1+|x_j|)$ *and* $v_j \geq -K(1+|x_j|)$ *for all* j*, for some constant* $K \geq 0$*. Then,*

$$u_j \leq v_j, \quad j \in \mathbb{Z}.$$

Proof. We give here a sketch of the proof of the lemma for the sake of completeness. Let $\alpha := \max_{j\in\mathbb{Z}} u_j - v_j$. Suppose that $\alpha > 0$ and that we want to obtain a contradiction. Assume that the index i is such that $u_i - v_i = \alpha$. Then, for all j, $u_j - v_j \leq u_i - v_i$, and therefore

$$\frac{u_i - u_{i-1}}{\Delta x} \geq \frac{v_i - v_{i-1}}{\Delta x}, \quad \text{and} \quad \frac{u_{i+1} - u_i}{\Delta x} \leq \frac{v_{i+1} - v_i}{\Delta x}. \tag{B.3}$$

Using $h^M = h^M(., \uparrow, \downarrow)$ and the inequalities (B.1), (B.2), and (B.3), the following holds:

$$\lambda v_i - \lambda u_i \geq h^M\left(x_i, \frac{u_i - u_{i-1}}{\Delta x}, \frac{u_{i+1} - u_i}{\Delta x}\right) - h^M\left(x_i, \frac{v_i - v_{i-1}}{\Delta x}, \frac{v_{i+1} - v_i}{\Delta x}\right) \tag{B.4}$$

$$\geq 0. \tag{B.5}$$

Hence $v_i - u_i \geq 0$, in contradiction with the fact that $\alpha = u_i - v_i > 0$.

In the case when $\alpha = \max_j(u_j - v_j)$ is not reached locally, we can consider a perturbation argument: for any given $\beta > 0$, for $\tilde{u}_j := u_j - \beta x_j^2$ and $\tilde{v}_j := v_j + \beta x_j^2$ we can show that $\tilde{u}_j \leq \tilde{v}_j$ using the previous arguments, so that finally $u_j - v_j \leq 2\beta x_j^2$, which concludes the desired result as $\beta \to 0$. ◻

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