

**TRANSLATION INVARIANT REALIZABILITY PROBLEM
ON THE d -DIMENSIONAL LATTICE:
AN EXPLICIT CONSTRUCTION**

E. CAGLIOTI¹, M. INFUSINO², T. KUNA³

ABSTRACT. We consider a particular instance of the truncated realizability problem on the d -dimensional lattice. Namely, given two functions $\rho_1(\mathbf{i})$ and $\rho_2(\mathbf{i}, \mathbf{j})$ non-negative and symmetric on \mathbb{Z}^d , we ask whether they are the first two correlation functions of a translation invariant point process. We provide an explicit construction of such a realizing process for any $d \geq 2$ when the radial distribution has a specific form. We also derive from this construction a lower bound for the maximal realizable density and compare it with the already known lower bounds.

1. INTRODUCTION

Let d be a positive integer. Given a point process $P = \{P_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^d}$ on the d -dimensional lattice, $P_{\mathbf{i}} \in \{0, 1\}$, whose distribution is described by the probability measure μ , we define the first and second order *correlation function* as follows

$$\begin{cases} \rho_1(\mathbf{i}) := \langle P_{\mathbf{i}} \rangle \\ \rho_2(\mathbf{i}, \mathbf{j}) := \langle P_{\mathbf{i}} P_{\mathbf{j}} \rangle - \rho_1(\mathbf{i}) \delta(\mathbf{i} - \mathbf{j}) \end{cases} ,$$

where $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$, δ is the Dirac delta function and $\langle \cdot \rangle$ denotes the expectation w.r.t. μ .

The *truncated realizability problem* addresses the inverse question. Namely, given two functions $\rho_1(\mathbf{i})$ and $\rho_2(\mathbf{i}, \mathbf{j})$ non-negative and symmetric for all $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$, does there exist a point process P for which these are the correspondent first and second order correlation functions? Clearly, the truncated realizability problem can be posed for any finite sequence of non-negative and symmetric functions $(\rho_k(\mathbf{i}_1, \mathbf{i}_2, \dots, \mathbf{i}_k))_{k=1}^n$. When the question is asked for a given infinite sequence $(\rho_k)_{k \in \mathbb{N}}$, then the problem is addressed as full realizability problem (see e.g. [15, 16] for a systematic study of the full realizability problem for point processes and [7, 8] for recent developments).

In the following we will consider the important special case of *translation invariant point processes*, which actually contains all the essential difficulties of the problem. In this case the realizability problem asks if, for given $\rho \in \mathbb{R}^+$ and $g : \mathbb{Z}^d \rightarrow \mathbb{R}^+$ symmetric, there exists a translation invariant point process such that its first two correlation functions are given by

$$(1) \quad \begin{cases} \rho_1(\mathbf{i}) = \rho \\ \rho_2(\mathbf{i}, \mathbf{j}) = \rho^2 g(\mathbf{i} - \mathbf{j}) \end{cases} .$$

If such a process exists, then it is said to be *realizing* and the pair (ρ, g) is called *realizable* on \mathbb{Z}^d . Note that writing the second order correlation in this form is

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not an additional restriction beyond the assumption of translation invariance. The function g is known in classic fluid theory as *radial distribution*, [6].

The truncated realizability problem is in fact a longstanding problem in the classical theory of fluids (see e.g. [5, 19, 20]), but it has been investigated in many other contexts such as stochastic geometry [17], spatial statistics [3, 23], spatial ecology [18] and neural spike trains [1, 9], just to name a few. In particular, Stillinger, Torquato et al. developed fascinating applications in the study of heterogeneous materials and mesoscopic structures based on the solvability of the truncated realizability problem (see e.g. [4, 21, 22, 24, 25, 26]). A structural investigation of this problem was recently started in [13], where the authors identify the realizability problem as a particular instance of the infinite-dimensional truncated moment problem (see [2, 10, 11, 12, 14] for further recent developments about the truncated realizability problem for point processes). As far as we know, the only earlier reference about the truncated infinite-dimensional moment problem is [28].

In this paper, we will show how to explicitly construct a point process on the d -dimensional lattice with $d \geq 2$ such that, for given $\alpha \geq 0$, (1) holds for certain values of ρ and for $g = g^{(\alpha)}$ defined as follows:

$$(2) \quad g^{(\alpha)}(\mathbf{x}) := \begin{cases} 0 & \text{if } \mathbf{x} = 0 \\ \alpha & \text{if } |\mathbf{x}| = 1 \\ 1 & \text{if } |\mathbf{x}| > 1 \end{cases} .$$

Explicit constructions of point processes realizing this lattice problem in the case $d = 1$ were provided in [12, Appendix 1]. Such a problem has been extensively studied for the case $\alpha = 0$ by Stillinger and Torquato in [21] (see also [4, 25]). The function $g^{(0)}$ describes a model with on-site and nearest neighbour exclusion and with no correlation for pairs of sites separated by two or more lattice spacings.

From [12, Section 1], we know that for a fixed α the set of realizable densities ρ is an interval $[0, \bar{\rho}_\alpha(d)]$ with $0 < \bar{\rho}_\alpha(d) \leq 1$. Moreover, in [12] the authors discuss:

- (i) certain general methods which, when applied to (2), yield lower bounds for $\bar{\rho}_\alpha(d)$ in any dimension d .
- (ii) concrete upper and lower bounds for $\bar{\rho}_\alpha(1)$. In particular, the lower bounds improve those obtained from the general methods (i).

Our d -dimensional construction combined with the one-dimensional lower bounds (ii) provides a lower bound for $\bar{\rho}_\alpha(d)$ for any $d \geq 2$ and any $\alpha \geq 0$. We will briefly compare this with the lower bound obtained from the general methods (i). We also follow techniques from [12] to get an upper bound for $\bar{\rho}_\alpha(d)$.

2. AN EXPLICIT REALIZING TRANSLATION INVARIANT POINT PROCESS ON \mathbb{Z}^d

In the following, we explicitly construct a point process $P = \{P_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^d}$ on the d -dimensional lattice with $d \geq 2$ such that, for given $\alpha \geq 0$, the following hold for certain values of ρ (depending on α and on d):

$$(3) \quad \langle P_{\mathbf{i}} \rangle = \rho$$

and

$$(4) \quad \langle P_{\mathbf{i}} P_{\mathbf{j}} \rangle = \begin{cases} \rho & \text{if } \mathbf{i} = \mathbf{j} \\ \alpha \rho^2 & \text{if } |\mathbf{i} - \mathbf{j}| = 1 \\ \rho^2 & \text{if } |\mathbf{i} - \mathbf{j}| > 1 \end{cases} ,$$

that is, the radial distribution is given by (2).

2.1. Construction in dimension 2.

In order to build such a process on \mathbb{Z}^2 we start from a realizing one-dimensional process achieving density γ . Namely, given $\alpha \geq 0$, we consider a point process $\{A_i\}_{i \in \mathbb{Z}}$, $A_i \in \{0, 1\}$, defined on the one-dimensional lattice and such that for some $\gamma > 0$ we have

$$\langle A_i \rangle = \gamma,$$

and

$$\langle A_i A_j \rangle = \begin{cases} \gamma & \text{if } i = j \\ \alpha\gamma^2 & \text{if } |i - j| = 1 \\ \gamma^2 & \text{if } |i - j| > 1 \end{cases} .$$

We denote a process of this kind by $BP\gamma$ that stays for *basic process with density γ* . As pointed out in the introduction, there exists a good number of constructions of realizing processes in the one-dimensional case, see e.g. [12, Appendix 1]. The results in the one-dimensional case relevant to our investigation (in particular the range where γ can vary) are recalled in Section 3, which is devoted to the discussion of the maximal realizable density in any dimension.

Let us define two processes $B^{(1)} = \{B_{i_1, i_2}^{(1)}\}_{(i_1, i_2) \in \mathbb{Z}^2}$ and $B^{(2)} = \{B_{i_1, i_2}^{(2)}\}_{(i_1, i_2) \in \mathbb{Z}^2}$ on \mathbb{Z}^2 as follows. For a fixed $i_1 \in \mathbb{Z}$, the process $\{B_{i_1, i_2}^{(1)}\}_{i_2 \in \mathbb{Z}}$ is a $BP\gamma$ in i_2 . For any $i_1, j_1 \in \mathbb{Z}$ with $i_1 \neq j_1$, the processes $\{B_{i_1, i_2}^{(1)}\}_{i_2 \in \mathbb{Z}}$ and $\{B_{j_1, j_2}^{(1)}\}_{j_2 \in \mathbb{Z}}$ are independent. In particular, we have

$$(5) \quad \langle B_{i_1, i_2}^{(1)} B_{j_1, j_2}^{(1)} \rangle = \begin{cases} \gamma^2 & \text{if } i_1 \neq j_1 \\ \gamma & \text{if } i_1 = j_1 \text{ and } i_2 = j_2 \\ \alpha\gamma^2 & \text{if } i_1 = j_1 \text{ and } |i_2 - j_2| = 1 \\ \gamma^2 & \text{if } i_1 = j_1 \text{ and } |i_2 - j_2| > 1 \end{cases} .$$

In other words, the process $B^{(1)}$ can be seen as a sequence of vertical $BP\gamma$'s independent one from each other (see Figure 1 for an example).

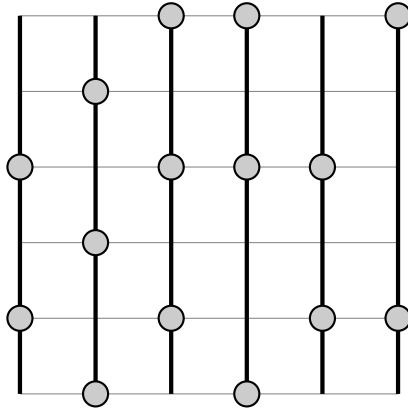


FIGURE 1. Example of process $B^{(1)}$ (with $\alpha = 0$)

Similarly, the process $B^{(2)}$ is defined as a sequence of horizontal $BP\gamma$'s independent one from each other (see Figure 2 for an example), i.e.

$$(6) \quad \langle B_{i_1, i_2}^{(2)} B_{j_1, j_2}^{(2)} \rangle = \begin{cases} \gamma^2 & \text{if } i_2 \neq j_2 \\ \gamma & \text{if } i_2 = j_2 \text{ and } i_1 = j_1 \\ \alpha\gamma^2 & \text{if } i_2 = j_2 \text{ and } |i_1 - j_1| = 1 \\ \gamma^2 & \text{if } i_2 = j_2 \text{ and } |i_1 - j_1| > 1 \end{cases}.$$

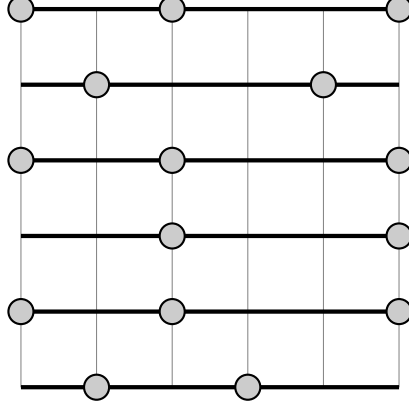


FIGURE 2. Example of process $B^{(2)}$ (with $\alpha = 0$)

Let us define now the process $P = \{P_i\}_{i \in \mathbb{Z}^2}$ as

$$P_{i_1, i_2} := B_{i_1, i_2}^{(1)} B_{i_1, i_2}^{(2)},$$

(see Figure 3 for P constructed from the basic processes in Figures 1 and 2). Since the processes $B^{(1)}$ and $B^{(2)}$ are independent, we get

$$\langle P_{i_1, i_2} \rangle = \langle B_{i_1, i_2}^{(1)} B_{i_1, i_2}^{(2)} \rangle = \langle B_{i_1, i_2}^{(1)} \rangle \langle B_{i_1, i_2}^{(2)} \rangle = \gamma \cdot \gamma = \gamma^2.$$

Hence, (3) holds for $\rho = \gamma^2$. From the independence of $B^{(1)}$ and $B^{(2)}$, we also get

$$(7) \quad \langle P_{i_1, i_2} P_{j_1, j_2} \rangle = \langle B_{i_1, i_2}^{(1)} B_{i_1, i_2}^{(2)} B_{j_1, j_2}^{(1)} B_{j_1, j_2}^{(2)} \rangle = \langle B_{i_1, i_2}^{(1)} B_{j_1, j_2}^{(1)} \rangle \langle B_{i_1, i_2}^{(2)} B_{j_1, j_2}^{(2)} \rangle.$$

We can easily check, by using (5) and (6) in (7), that (4) holds for $\rho = \gamma^2$. In fact, we need to consider only the following four cases, because all the others are equivalent to these ones by symmetry.

a) If $i_1 = j_1$ and $i_2 = j_2$ then

$$\langle P_{i_1, i_2} P_{j_1, j_2} \rangle = \gamma \cdot \gamma = \gamma^2.$$

b) If $i_1 = j_1$ and $i_2 \neq j_2$ then

$$\langle P_{i_1, i_2} P_{j_1, j_2} \rangle = \langle B_{i_1, i_2}^{(1)} B_{j_1, j_2}^{(1)} \rangle \gamma^2.$$

Therefore:

- if $i_2 = j_2 + 1$ then $\langle B_{i_1, i_2}^{(1)} B_{j_1, j_2}^{(1)} \rangle = \alpha\gamma^2$ and so $\langle P_{i_1, i_2} P_{j_1, j_2} \rangle = \alpha\gamma^4$
- if $|i_2 - j_2| > 1$ then $\langle B_{i_1, i_2}^{(1)} B_{j_1, j_2}^{(1)} \rangle = \gamma^2$ and so $\langle P_{i_1, i_2} P_{j_1, j_2} \rangle = \gamma^4$.

c) If $i_1 = j_1 + 1$ and $|i_2 - j_2| > 1$ then

$$\langle P_{i_1, i_2} P_{j_1, j_2} \rangle = \langle B_{i_1, i_2}^{(1)} B_{j_1, j_2}^{(1)} \rangle \langle B_{i_1, i_2}^{(2)} B_{j_1, j_2}^{(2)} \rangle = \gamma^4.$$

d) If $|i_1 - j_1| > 1$ and $i_2 \neq j_2$ then

$$\langle P_{i_1, i_2} P_{j_1, j_2} \rangle = \langle B_{i_1, i_2}^{(1)} B_{j_1, j_2}^{(1)} \rangle \langle B_{i_1, i_2}^{(2)} B_{j_1, j_2}^{(2)} \rangle = \gamma^4.$$

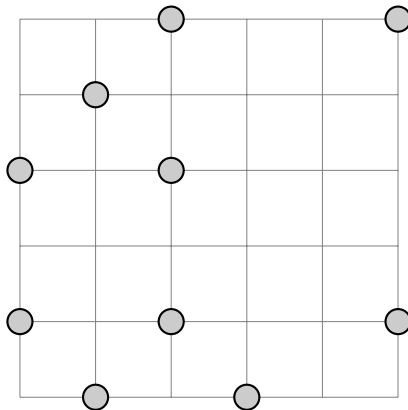


FIGURE 3. Process P constructed from the processes in Figure 1 and Figure 2

2.2. Construction in higher dimension.

The construction presented in the previous subsection easily generalizes to any dimension $d > 2$ by defining $P_{i_1, \dots, i_d} := B_{i_1, \dots, i_d}^{(1)} \cdots B_{i_1, \dots, i_d}^{(d)}$ where, for any fixed $i_2, \dots, i_d \in \mathbb{Z}$, $\{B_{i_1, \dots, i_d}^{(1)}\}_{i_1 \in \mathbb{Z}}$ is a $BP\gamma$ in the variable i_1 with density γ and similarly for the other variables. Therefore, the point process P on \mathbb{Z}^d defined as above satisfies (3) and (4) for $\rho = \gamma^d$.

3. BOUNDS FOR THE MAXIMAL REALIZABLE DENSITY

In this section, we will discuss the problem of estimating the maximal realizable density $\bar{\rho}_\alpha(d)$. In particular, we will show a general upper bound for any $d \geq 1$ using the technique introduced in [12] for $d = 1$. As for the lower bound, we will recall the results in [12, Appendix 1] for the one-dimensional case and combine them with the explicit construction proposed in Section 2 to produce a lower bound for $\bar{\rho}_\alpha(d)$ for any $d \geq 2$. We will compare this with the lower bound obtained by applying the general methods of [12] to the case when the radial distribution is given by (2).

3.1. Upper bounds for $\bar{\rho}_\alpha(d)$.

For $d \geq 1$ and $\alpha \geq 0$, the covariance matrix associated to a given pair $(\rho, g^{(\alpha)})$ realizable on \mathbb{Z}^d must be positive semidefinite. This is equivalent to the non-negativity of the corresponding infinite volume structure function \hat{S} on \mathbb{R}^d (for more details see e.g. [12, Section 2]):

$$\hat{S}(\mathbf{k}) := \rho + \rho^2 \sum_{\mathbf{x} \in \mathbb{Z}^d} e^{i\mathbf{k} \cdot \mathbf{x}} [g^{(\alpha)}(\mathbf{x}) - 1] \geq 0, \forall \mathbf{k} \in \mathbb{R}^d.$$

This leads to an explicit upper bound for the maximal realizable density $\bar{\rho}_\alpha(d)$. In fact, it is easy to see that for any $\mathbf{k} := (k_1, \dots, k_d) \in \mathbb{R}^d$ we get

$$\begin{aligned} \hat{S}(\mathbf{k}) &= \rho - \rho^2 + \rho^2 \sum_{\mathbf{x} \in \mathbb{Z}^d, |\mathbf{x}|=1} e^{i\mathbf{k} \cdot \mathbf{x}} (\alpha - 1) \\ &= \rho - \rho^2 + \rho^2 (\alpha - 1) \sum_{j=1}^d (e^{ik_j} + e^{-ik_j}) \\ &= \rho \left[1 - \rho \left(1 - 2(\alpha - 1) \sum_{j=1}^d \cos(k_j) \right) \right]. \end{aligned}$$

Then, using the non-negativity of \hat{S} on \mathbb{R}^d , we get that

$$\rho \leq \frac{1}{f_\alpha(k_1, \dots, k_d)}, \quad \forall (k_1, \dots, k_d) \in \mathbb{R}^d,$$

where $f_\alpha(k_1, \dots, k_d) := 1 - 2(\alpha - 1) \sum_{j=1}^d \cos(k_j)$. The best upper bound is then obtained for the points of \mathbb{R}^d where f_α attains the maximum. Hence, we have that

$$\bar{\rho}_\alpha(d) \leq \frac{1}{\max_{\mathbf{k} \in \mathbb{R}^d} f_\alpha(\mathbf{k})} =: R_F(\alpha, d).$$

By computing the maximum of f_α over \mathbb{R}^d , we get our upper bound

$$(8) \quad R_F(\alpha, d) = \frac{1}{1 + 2d|1 - \alpha|}.$$

As mentioned above, this technique was employed in [12, Appendix 1] to get $R_F(\alpha, 1)$. Furthermore, the authors provided another upper bound $R_Y(\alpha, 1)$ in the one-dimensional case by using the Yamada condition (see [27]). Note that

$$\begin{cases} R_Y(\alpha, 1) = R_F(\alpha, 1), & \text{if } \alpha = \frac{1}{2} \text{ or } \alpha = \frac{k \pm 1}{2k}, k \in \mathbb{N} \text{ or } \alpha \geq 1 \\ R_Y(\alpha, 1) < R_F(\alpha, 1) & \text{otherwise.} \end{cases}$$

3.2. Lower bounds for $\bar{\rho}_\alpha(d)$.

Applying [12, Theorem 3.2] for $g \equiv g^{(\alpha)}$ when $0 \leq \alpha < 1$ and [12, Theorem 5.1] for $G_2(\mathbf{x}, \mathbf{y}) = g^{(\alpha)}(\mathbf{y} - \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ such that $\mathbf{x} \neq \mathbf{y}$ when $\alpha \geq 1$, we get that

$$(9) \quad \bar{\rho}_\alpha(d) \geq r_A(\alpha, d) := \begin{cases} \frac{1}{e^{(2d+1-2d\alpha)}}, & \text{if } 0 \leq \alpha < 1, \\ \frac{1}{\alpha^{2d}}, & \text{if } \alpha \geq 1. \end{cases}$$

For $d = 1$, this lower bound has been improved in [12, Appendix 1] by explicitly constructing a translation invariant realizing process at some value of ρ and α . Let us summarize in one formula the lower bounds coming from the two main constructions considered in [12, Appendix 1]:

$$(10) \quad \bar{\rho}_\alpha(1) \geq \begin{cases} \frac{1}{(1+\sqrt{1-\alpha})^2}, & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ \frac{1}{1+\sqrt{2-2\alpha}}, & \text{if } \frac{1}{2} \leq \alpha \leq 1, \\ \frac{1}{2\alpha-1}, & \text{if } \alpha \geq 1. \end{cases}$$

In [2] a further explicit construction is provided for the case $\alpha = 0$, which slightly improves this lower bound to $\bar{\rho}_0(1) > 0.265$. In the same work also the upper bound is improved to $\bar{\rho}_0(1) < (326 - \sqrt{3115})/822 \approx 0.3287$. However, it remains an open problem to reduce the gap between lower and upper bounds for $\bar{\rho}_\alpha(1)$ for any $\alpha \geq 0$.

Exactly as in the one-dimensional case, also for $d \geq 2$, one can try to obtain better lower bounds than (9) by using explicit constructions. In the following, we will use the construction we proposed in Section 2 combined with the one-dimensional lower bound (10) to compute a new lower bound for $\bar{\rho}_\alpha(d)$, which we will briefly compare with (9).

If we apply the construction given for $d \geq 2$ in Section 2 starting with a basic process with density $\bar{\rho}_\alpha(1)$, then we get a point process on \mathbb{Z}^d which realizes the pair $((\bar{\rho}_\alpha(1))^d, g^{(\alpha)})$ for any $\alpha \geq 0$. This explicit construction guarantees that for any $\alpha \geq 0$,

$$\bar{\rho}_\alpha(d) \geq (\bar{\rho}_\alpha(1))^d.$$

Using the lower bounds (10) in the latter inequality, we directly have the following

$$(11) \quad \bar{\rho}_\alpha(d) \geq r_C(\alpha, d) := \begin{cases} \frac{1}{(1+\sqrt{1-\alpha})^{2d}}, & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ \frac{1}{(1+\sqrt{2-2\alpha})^d}, & \text{if } \frac{1}{2} \leq \alpha < 1, \\ \frac{1}{(2\alpha-1)^d}, & \text{if } \alpha \geq 1. \end{cases}$$

Note that:

- if $0 \leq \alpha < \frac{1}{2}$ then $r_C(\alpha, d) \leq r_A(\alpha, d)$
- if $\alpha \geq 1$ then $r_A(\alpha, d) \leq r_C(\alpha, d)$
- if $\frac{1}{2} \leq \alpha < 1$ then the relation between the two bounds depends on the dimension d . Actually, for each $d \geq 2$ there exists $\alpha_C(d) \in [\frac{1}{2}, 1]$ such that $r_A(\alpha, d) \leq r_C(\alpha, d)$ for any $\alpha_C(d) \leq \alpha < 1$.

The comparison between the lower bounds $r_C(\alpha, d)$ and $r_A(\alpha, d)$ is illustrated in Figure 4 for $d = 2, \dots, 6$ and for $0 \leq \alpha < 1$.

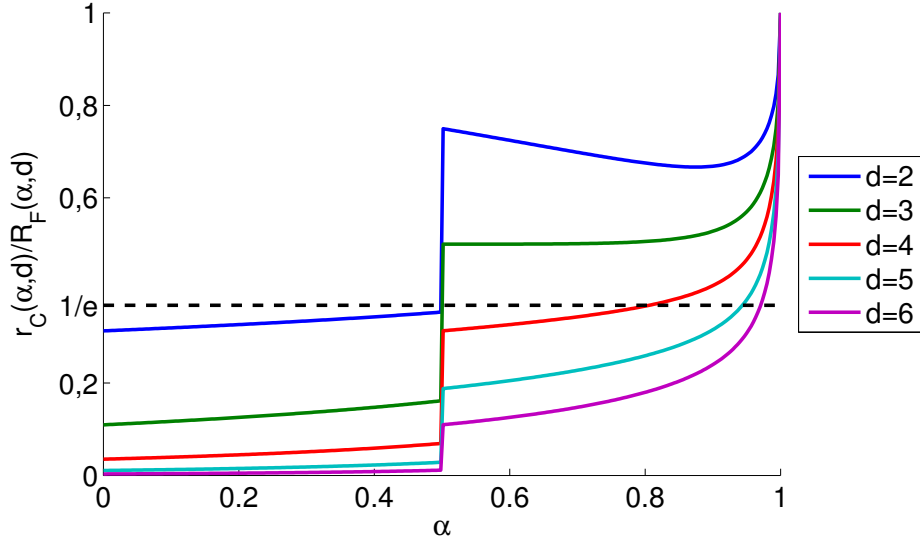


FIGURE 4. Comparison between the lower bounds $r_C(\alpha, d)$ and $r_A(\alpha, d)$ plotted relatively to the upper bound $R_F(\alpha, d)$ as functions of α with $0 \leq \alpha < 1$. The coloured lines correspond to $\frac{r_C(\alpha, d)}{R_F(\alpha, d)}$ for $d = 2, \dots, 6$ and the dotted line to $\frac{r_A(\alpha, d)}{R_F(\alpha, d)}$ for any d . For the definitions of $R_F(\alpha, d)$, $r_A(\alpha, d)$, $r_C(\alpha, d)$ see (8), (9), (11), respectively.

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¹ *Emanuele Caglioti*

DIPARTIMENTO DI MATEMATICA,
SAPIENZA UNIVERSITÀ DI ROMA,
00185, ROME, ITALY.

E-mail address: `caglioti@mat.uniroma1.it`

² *Maria Infusino*

FACHBEREICH MATHEMATIK UND STATISTIK,
UNIVERSITÄT KONSTANZ,
78457 KONSTANZ, GERMANY.

E-mail address: `maria.infusino@uni-konstanz.de`

³ *Tobias Kuna*

DEPARTMENT OF MATHEMATICS AND STATISTICS,
UNIVERSITY OF READING,
RG66AX, READING, UK.

E-mail address: `t.kuna@reading.ac.uk`