Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 261 (2016) 4593–4630

Journal of Differential Equations

www.elsevier.com/locate/jde

Singularly perturbed control systems with noncompact fast variable ☆

Thuong Nguyen [a], Antonio Siconolfi [b,*]

[a] *Department of Mathematics, Quy Nhon University, 170 An Duong Vuong street, Quy Nhon city, Viet Nam*
[b] *Dipartimento di Matematica, Università degli Studi di Roma "La Sapienza", 00185 Roma, Italy*

Received 3 August 2015; revised 3 June 2016

Available online 9 August 2016

Abstract

We deal with a singularly perturbed optimal control problem with slow and fast variable depending on a parameter ε. We study the asymptotics, as ε goes to 0, of the corresponding value functions, and show convergence, in the sense of weak semilimits, to sub and supersolution of a suitable limit equation containing the effective Hamiltonian.

The novelty of our contribution is that no compactness condition is assumed on the fast variable. This generalization requires, in order to perform the asymptotic procedure, an accurate qualitative analysis of some auxiliary equations posed on the space of fast variable. The task is accomplished using some tools of Weak KAM theory, and in particular the notion of Aubry set.

MSC: 35B27; 49J15; 49L25

Keywords: Optimal control; Singular perturbation; Viscosity solutions; Hamilton–Jacobi–Bellman equations

1. Introduction

We study a singularly perturbed optimal control problem with a slow variable, say x, and a fast one, denoted by y, with dynamics depending on a parameter ε devoted to become infinitesimal.

☆ The work of the second author was partially supported by Programma Ricerca Scientifica Sapienza 2014.
* Corresponding author.
E-mail addresses: nguyenngocquocthuong@qnu.edu.vn (T. Nguyen), siconolf@mat.uniroma1.it (A. Siconolfi).

http://dx.doi.org/10.1016/j.jde.2016.07.005
0022-0396/

We are interested in the asymptotics, as ε goes to 0, of the corresponding value functions V^ε, depending on slow, fast variable and time, in view of proving convergence, in the sense of weak semilimits, to some functions independent of y, related to a limit control problem where y does not appear any more, at least as state variable.

More precisely, we exploit that the V^ε are solutions, in the viscosity sense, to a time-dependent Hamilton–Jacobi–Bellman equation of the form

$$u_t^\varepsilon + H\left(x, y, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}\right) = 0$$

and show that the upper/lower weak semilimit is sub/supersolution to a limit equation

$$u_t + \overline{H}(x, Du) = 0$$

containing the so-called effective Hamiltonian $\overline{H}$, obtained via a canonical procedure we describe below from the Hamiltonian of the approximating equations. We also show that initial conditions, i.e. terminal costs, are transferred, with suitable adaptations, to the limit. See Theorems 4.3, 4.4, which are the main results of the paper.

We tackle the subject through a PDE approach first proposed in this context by Alvarez–Bardi, see [1,2] and the booklet [3], in turn inspired by techniques developed in the framework of homogenization of Hamilton–Jacobi equations by Lions–Papanicolau–Varadhan and Evans, see [18,12,13]. The singular perturbation can be actually viewed as a relative homogenization of slow with respect to fast variable. In the original formulation, homogenization was obtained assuming periodicity in the underlying space plus coercivity of the Hamiltonian in the momentum variable.

Alvarez–Bardi keep periodicity in y, but do without coercivity, and assume instead bounded time controllability in the fast variable. A condition of this kind is indeed unavoidable, otherwise it cannot be expected to get rid of y at the limit, or even to get any limit. Another noncoercive homogenization problem, arising from turbulent combustion models, has been recently investigated with similar techniques in [19].

The novelty of our contribution is that we remove any compactness condition on the fast variable, and this requires major adaptations in the perturbed test function method, which is the core of the asymptotic procedure. We further comment on it later on.

Following a more classical control-theoretic approach, namely directly working on the trajectory of the dynamics, Arstein–Gaitsgory, see [7] and [5,6], have studied a similar model replacing in a sense periodicity by a coercivity condition in the cost, and allowing y to vary in the whole of $\mathbb{R}^M$, for some dimension M. Besides proving convergence, they also provide a thorough description of the limit control problem, in terms of occupational measures, see [6]. This is clearly a relevant aspect of the topic, but we do not treat it here.

Our aim is to recover their results adapting Alvarez–Bardi techniques. We assume, as in [7] and [5], coercivity of running cost, see **(H4)**, and a controllability condition, see **(H3)**, stronger than the one used in [1–3] and implying, see Lemma 2.9, coercivity of the corresponding Hamiltonian, at least in the fast variable. We do believe that our methods can also work under bounded time controllability, and so without any coercivity on H, but this requires more work, and the details have still to be fully checked and written down.

The focus of our analysis is on the associate cell problem, namely the one-parameter family of stationary equations, posed in the space of fast variable, obtained by freezing in H slow variable and momentum, say at a value (x_0, p_0). Its role, at least in the periodic case, is twofold:

it provides a definition of the effective Hamiltonian $\overline{H}$ at (x_0, p_0) as the minimum value of the parameter for which there is a subsolution (then also supersolutions or solutions do exist), the corresponding equation will be called critical in what follows, and critical sub/supersolutions play the crucial role of correctors in the perturbed test function method.

The absence of compactness calls into questions the very status of the critical value $\overline{H}(x_0, p_0)$ since, in contrast to what happens when periodicity is assumed, the existence of solutions does not characterize any more the critical equation, see Appendix A. Moreover critical sub/supersolutions must enjoy suitable additional properties, as explained below, to be effective in the asymptotic procedure.

The two issues are intertwined. By performing a rather accurate qualitative analysis of the cell problems, we show that (sub/super)solutions usable as correctors can be obtained only for the critical equation. We make essentially use for that of tools issued from weak KAM theory, and in particular of the capital notion of Aubry set. As far as we know, it is the first time that this methodology finds a specific application in singular perturbation or homogenization problems.

The geometric counterpart of coercivity in the cost functional is that the critical equation has a nonempty compact Aubry set for every fixed (x_0, p_0), see Lemma 3.8, which in turn implies existence of coercive solutions possessing a simple representation formula in terms of a related intrinsic metric, and bounded subsolutions as well, see Propositions 3.7, 3.9. Coercive solutions, up to modification depending on ε (see Subsection 3.3), are used in the upper semilimit part of the asymptotics, which is the most demanding point of the analysis.

The paper is organized as follows. In Section 2 we give some preliminary material and standing assumptions, we then study some relevant property of controlled dynamics and how they affect value functions. Approximating Hamilton–Jacobi–Bellman equations and limit problem are also defined. Section 3 is about cell problems and construction of distinguished critical sub/supersolutions to be used as correctors. Section 4 contains the main results. The appendix is devoted to review some basic facts of metric approach and Weak KAM theory for general Hamilton–Jacobi equations.

2. Setting of the problem

2.1. Notations and terminology

Given an Euclidean space, say to fix ideas $\mathbb{R}^N$, for some $N \in \mathbb{N}$, $x \in \mathbb{R}^N$ and $R > 0$ we denote by $B(x, R)$ the open ball centered at x with radius R. Given $B \subset \mathbb{R}^N$, we indicate by $\overline{B}$, $\operatorname{int} B$, its closure and interior, respectively. Given subsets B, C, and a scalar λ, we set

$$B + C = \{x + y \mid x \in B,\ y \in C\}$$
$$\lambda B = \{\lambda x \mid x \in B\}.$$

We make precise that in all Hamilton–Jacobi equations we will consider throughout the paper the term (sub/super)solution must be understood in the viscosity sense.

Given an upper semicontinuous (resp. lower semicontinuous) $u : \mathbb{R}^N \to \mathbb{R}$, we say that a function ψ is supertangent (resp. subtangent) to a u at some point x_0 if it is of class C^1, $u = v$ at x_0 and

$$\psi \geq u \ \text{(resp. } \psi \leq u\text{)}, \quad \text{locally at } x_0.$$

If strict inequalities hold in the above formula then ψ will be called strict supertangent (resp. subtangent).

Given a sequence of locally equibounded functions $u_n : \mathbb{R}^M \to \mathbb{R}$, the upper weak semilimit (resp lower weak semilimit) is defined via the formula

$$(\limsup{}^{\#} u_n)(x) = \sup\{\limsup_n u_n(x_n) \mid x_n \to x\}$$

$$(\text{resp.} \quad (\liminf{}_{\#} u_n)(x) = \inf\{\liminf_n u_n(x_n) \mid x_n \to x\}).$$

If u is a locally bounded function and we take in the above formula the sequence u_n constantly equal to u then we get through upper (resp. lower) weak semilimit the upper (resp. lower) semicontinuous envelope of u, denoted by $u^{\#}$ (resp. $u_{\#}$). It is minimal (resp. maximal) upper (resp. lower) semicontinuous function greater (resp. less) than or equal to u.

2.2. Assumptions

We assume that the slow variable, usually denoted by x, lives in $\mathbb{R}^N$ and the fast variable y in $\mathbb{R}^M$, for given positive integers N, M. We denote by A the control set, by $f : \mathbb{R}^N \times \mathbb{R}^M \times A \to \mathbb{R}^N$, $g : \mathbb{R}^N \times \mathbb{R}^M \times A \to \mathbb{R}^M$ the controlled vector fields related to slow and fast dynamics, respectively. We also have a running cost $\ell : \mathbb{R}^N \times \mathbb{R}^M \times A \to \mathbb{R}$ and a terminal cost $u_0 : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$. We call, as usual, control a measurable trajectory defined in $[0, +\infty)$ taking values in A. We require:

(H1) Control set: A is a compact subset of some Euclidean space;

(H2) Controlled dynamics: There is a constant $L_0 > 0$ with

$$|f(x_1, y_1, a) - f(x_2, y_2, a)| \le L_0 (|x_1 - x_2| + |y_1 - y_2|)$$

$$|g(x_1, y_1, a) - g(x_2, y_2, a)| \le L_0 (|x_1 - x_2| + |y_1 - y_2|)$$

for any (x_i, y_i), $i = 1, 2$ in $\mathbb{R}^N \times \mathbb{R}^M$ and $a \in A$; we assume in addition that $|f|$ is bounded with upper bound denoted by Q_0;

(H3) Total controllability: For any compact set $K \subset \mathbb{R}^N \times \mathbb{R}^M$ there exists $r = r(K) > 0$ such that

$$B(0, r) \subset \overline{\text{co}}\, g(x, y, A) \qquad \text{for } (x, y) \in K,$$

where $g(x, y, A) = \{g(x, y, a) \mid a \in A\}$;

(H4) Running cost: ℓ is continuous in $\mathbb{R}^N \times \mathbb{R}^M \times A$, and for any compact set $B \subset \mathbb{R}^N$

$$\lim_{|y| \to +\infty} \min_{(x,a) \in B \times A} \ell(x, y, a) = +\infty; \tag{1}$$

(H5) Terminal cost: u_0 is continuous and bounded from below in $\mathbb{R}^N \times \mathbb{R}^M$. To simplify notations, $-Q_0$, see **(H2)**, is also taken as lower bound of u_0 in $\mathbb{R}^N \times \mathbb{R}^M$.

Taking into account Assumption **(H5)**, we define

$$\overline{u}_0(x) = \inf_{y \in \mathbb{R}^M} u_0(x, y) \qquad \text{for any } x \in \mathbb{R}^N. \tag{2}$$

This function is apparently upper semicontinuous, and will play the role of initial condition in the limit equation we get in the asymptotic procedure.

Remark 2.1. Due to Relaxation Theorem plus Filippov Implicit Function Lemma, see for instance [4,11], the integral trajectories of the differential inclusion

$$\dot{\zeta} \in \overline{\mathrm{co}}\, g(x, \zeta, A) \qquad \text{for } x \text{ fixed in } \mathbb{R}^N,$$

are locally uniformly approximated in time by solutions to

$$\dot{\eta} = g(x, \eta, \alpha) \qquad \text{for some control } \alpha. \tag{3}$$

By iteratively applying this property to a concatenation of a sequence of curves of (3) for infinitesimal times, we derive local bounded time controllability for fast dynamics, namely, given R_1, R_2 positive, there is $T_0 = T_0(R_1, R_2)$ such that for any y_1, y_2 in $B(0, R_1)$, $x \in B(0, R_2)$, we can find a trajectory η of (3) joining y_1 to y_2 in a time $T \leq T_0$.

2.3. Controlled dynamics

For any $\varepsilon > 0$, any control α, the controlled dynamics is defined as

$$\begin{cases} \dot{\xi}(t) = \varepsilon\, f(\xi(t), \eta(t), \alpha(t)) \\ \dot{\eta}(t) = \quad g(\xi(t), \eta(t), \alpha(t)) \end{cases} \tag{CD_ε}$$

Notice that if ξ, η are solutions to (CD_ε) with initial data (x, y) then the trajectories

$$t \mapsto \xi(t/\varepsilon), \qquad t \mapsto \eta(t/\varepsilon)$$

are solutions to

$$\begin{cases} \dot{\xi}_0(t) = \quad f(\xi_0(t), \eta_0(t), \alpha(t/\varepsilon)) \\ \varepsilon\, \dot{\eta}_0(t) = g(\xi_0(t), \eta_0(t), \alpha(t/\varepsilon)) \end{cases} \tag{$\overline{CD}_\varepsilon$}$$

with the same initial data.

Given a trajectory ξ, η of (CD_ε) with initial data (x, y) and control α, for some $\varepsilon > 0$, and $T > 0$, we deduce from standing assumptions and Grönwall Lemma, the following basic estimates:

$$|\xi(t) - x| \leq Q_0\, T \qquad \text{for } t \in [0, T/\varepsilon]. \tag{4}$$

If ζ satisfies

$$\dot{\zeta} = g(x, \zeta, \alpha) \quad \zeta(0) = y,$$

then

$$|\eta(T) - \zeta(T)| \le \int_0^T |g(\xi, \eta, \alpha) - g(x, \zeta, \alpha)| \, ds \tag{5}$$

$$\le L_0 \int_0^T \big(|\xi - x| + |\eta - \zeta|\big) \, ds \le L_0 \, \varepsilon \, Q_0 \, T^2 \, e^{L_0 T}.$$

Finally

$$|\eta(T) - y| \le \int_0^T |g(\xi, \eta, \alpha) - g(\xi, y, \alpha)| \, ds + \int_0^T |g(\xi, y, \alpha)| \, ds \tag{6}$$

$$\le L_0 \, R \, T \, e^{L_0 T},$$

where R is an upper bound of $|g|$ in $B(x, \varepsilon T) \times \{y\} \times A$, and similarly

$$|\eta(T) - y| \le \int_0^T |g(\xi, \eta, \alpha) - g(\xi, \eta(T), \alpha)| \, ds + \int_0^T |g(\xi, \eta(T), \alpha)| \, ds \tag{7}$$

$$\le L_0 \, R' \, T \, e^{L_0 T},$$

where R' is an upper bound of $|g|$ in $B(x, \varepsilon T) \times \{\eta(T)\} \times A$.

By using bounded time controllability condition, we further get:

Lemma 2.2. *Given R_1, R_2 positive, $x \in B(0, R_1)$, y, z in $B(0, R_2)$, there is, for any ε, a trajectory $(\xi_\varepsilon, \eta_\varepsilon)$ of (CD_ε), starting at (x, y) and a time T_ε with*

$$T_0(R_1, R_2) < T_\varepsilon < 3\, T_0(R_1, R_2) \tag{8}$$

such that

$$|\eta_\varepsilon(T_\varepsilon) - z| = \mathrm{O}(\varepsilon).$$

The quantity $T_0(\cdot, \cdot)$ is as in Remark 2.1.

Proof. By controllability condition, see Remark 2.1, there is a control α and a trajectory ζ with

$$\dot{\zeta} = g(x, \zeta, \alpha) \qquad \text{for a suitable } \alpha \tag{9}$$

starting at y and reaching z in a time $T_\varepsilon \le T_0(R_1, R_2)$. Up to adding a cycle passing through z, and satisfying (9) for some control, we can assume T_ε to satisfy (8). Note that such a cycle does exist again in force of the controllability condition. We then take, for any ε, the trajectories $(\xi_\varepsilon, \eta_\varepsilon)$ of (CD_ε) starting at (x, y) corresponding to the same control α, and invoke (5) to get the assertion. □

We derive:

Proposition 2.3. *Given a bounded set B of $\mathbb{R}^N \times \mathbb{R}^M$ and $S > 0$, there exists a bounded subset $B_0 \supset B$ such that for any initial data in B and any ε, we can find a trajectory of (CD_ε) lying in B_0 as $t \in [0, S/\varepsilon]$.*

Proof. We fix $(x, y) \in B$. By (4), we can find R_1, R_2 such that $B \subset B(0, R_1) \times B(0, R_2)$, and the first component ξ of any trajectory (ξ, η) of (CD_ε), for any ε, starting at (x, y) is contained in $B(0, R_1)$. We write T_0 for $T_0(R_1, R_2)$. Clearly, it is enough to establish the assertion for ε small.

By applying Lemma 2.2 with ε suitably small and $z = 0$, we find a time T_ε and a trajectory $(\xi_\varepsilon, \eta_\varepsilon)$ of (CD_ε) such that $(\xi_\varepsilon(T_\varepsilon), \eta_\varepsilon(T_\varepsilon)) \in B(0, R_1) \times B(0, R_2)$. Taking into account that the time T_ε is estimated from above and below by a positive quantity, see (8), we can iterate the procedure and get by concatenation of the curves so obtained, a trajectory (ξ_0, η_0) in $[0, t_0/\varepsilon]$, starting at (x, y), with the crucial property that there are times $\{t_i\}$, $i = 1, \cdots k$, for some index k, in $[0, S/\varepsilon]$ such that

for any $t \in [0, S/\varepsilon]$, there is t_i with $|t - t_i| \leq 3\, T_0$;
$\eta_\varepsilon(t_i) \in B(0, R_2)$ for any i.

We derive as $t \in \left[0, \frac{S}{\varepsilon}\right]$

$$|\xi_\varepsilon(t) - x_0| < Q_0\, S \tag{10}$$

$$|\eta_\varepsilon(t)| \leq R_2 + 3\, P\, T_0 \tag{11}$$

with constant P solely depending, see (6), upon R_1, R_2, $T_0(R_1, R_2)$. This proves the assertion. □

The next result is a strengthened version of Lemma 2.2 stating that the approximation of a value of the fast variable by a trajectory of the fast dynamics can be realized in any predetermined suitably large time. To establish it, we need exploiting total controllability assumption **(H3)** in its full extent. The lemma will be used in the proof of Theorem 4.4.

Lemma 2.4. *Given $x \in \mathbb{R}^N$, y, z in $\mathbb{R}^M$, and $S > 0$ suitably large, there is, for any ε, a trajectory $(\overline{\xi}_\varepsilon, \overline{\eta}_\varepsilon)$ of (CD_ε), starting at (x, y) such that*

$$|\overline{\eta}_\varepsilon(S) - z| = \mathrm{O}(\varepsilon).$$

Proof. We fix R_1, R_2 such that $x \in B(0, R_1)$, and y, z are in $B(0, R_2)$. We take S with $S > 3\, T_0(R_1, R_2)$. By applying Lemma 2.2, we find $T_\varepsilon < 3\, T_0(R_1, R_2) < S$ and, for any ε, a curve $(\xi_\varepsilon, \eta_\varepsilon)$ of (CD_ε) starting at (x, y) with

$$|\eta_\varepsilon(T_\varepsilon) - z| = \mathrm{O}(\varepsilon).$$

By iterating the procedure, if necessary, as in the proof of Lemma 2.2, we can extend it to an interval $[0, S_\varepsilon]$, with $S - S_\varepsilon < T_\varepsilon$, still getting

$$|\eta_\varepsilon(S_\varepsilon) - z| = \mathrm{O}(\varepsilon). \tag{12}$$

By **(H3)** and Relaxation Theorem, see Remark 2.1, we find a control β and a trajectory ζ_ε satisfying

$$\dot{\zeta}_\varepsilon = g(\xi_\varepsilon(S_\varepsilon), \zeta_\varepsilon, \beta) \quad \zeta_\varepsilon(0) = \eta_\varepsilon(S_\varepsilon)$$

with

$$|\zeta_\varepsilon(t) - \eta_\varepsilon(S_\varepsilon)| = \mathrm{O}(\varepsilon) \qquad \text{for } t \in [0, S - S_\varepsilon]. \tag{13}$$

Owing to (5), the trajectory $(\xi^0_\varepsilon, \eta^0_\varepsilon)$ of (CD_ε) starting at $(\xi_\varepsilon(S_\varepsilon), \eta_\varepsilon(S_\varepsilon))$, with control β satisfies

$$|\eta^0_\varepsilon(S - S_\varepsilon) - \zeta_\varepsilon(S - S_\varepsilon)| = \mathrm{O}(\varepsilon). \tag{14}$$

By concatenation of η_ε and η^0_ε, we finally get, in force of (12), (13), (14), a trajectory satisfying the assertion. □

2.4. Minimization problems and value functions

We consider for any $(x, y) \in \mathbb{R}^N \times \mathbb{R}^M$, $t > 0$, $\varepsilon > 0$, the optimization problems

$$\inf_\alpha \varepsilon \int_0^{\frac{t}{\varepsilon}} \ell\big(\xi_\varepsilon, \eta_\varepsilon, \alpha\big)\, \mathrm{d}s + u_0\left(\xi_\varepsilon\left(\frac{t}{\varepsilon}\right), \eta_\varepsilon\left(\frac{t}{\varepsilon}\right)\right) \tag{15}$$

with ξ_ε, η_ε are solutions to (CD_ε) in $[0, +\infty)$, issued from the initial datum (x, y). Or equivalently with the change of variables $r = \varepsilon\, s$

$$\inf_\alpha \int_0^t \ell\big(\xi^0_\varepsilon, \eta^0_\varepsilon, \alpha\big)\, \mathrm{d}r + u_0(\xi^0_\varepsilon(t), \eta^0_\varepsilon(t))) \tag{16}$$

with ξ^0_ε, η^0_ε are solutions to $(\overline{CD}_\varepsilon)$ in $[0, +\infty)$, issued from (x, y). We denote by V^ε the corresponding value functions, namely the functions associating to any initial datum (x, y) and time t the infimum of the functional in (15)/ (16). They are apparently continuous with respect to all arguments.

Remark 2.5. Looking at the form of the above minimization problem, we understand that coercivity assumption **(H4)** plus **(H5)** plays the role of a compactness condition for the fast variable, inasmuch as it implies that the trajectories of the fast dynamics realizing the value function, up to some small constant, lie in a compact subset of $\mathbb{R}^M$. This fact will be crucial in the asymptotic analysis.

We derive from Proposition 2.3:

Proposition 2.6. *The value functions V^ε are locally equibounded.*

Proof. Let C be a bounded set of $\mathbb{R}^N \times \mathbb{R}^M \times [0, +\infty)$, and $(x_0, y_0, t_0) \in C$. Thanks to Proposition 2.3, there are for any ε trajectories (ξ_0, η_0), we drop the dependence on ε to ease notations, of (CD_ε) starting at (x_0, y_0), and contained in a bounded set of $\mathbb{R}^N \times \mathbb{R}^M$ solely depending on C. By using the formulation (15) of the minimization problem, we get

$$V^\varepsilon(x_0, y_0, t_0) \le \varepsilon \int_0^{\frac{t_0}{\varepsilon}} \ell(\xi_0(s), \eta_0(s), \alpha(s))\, ds + u_0(\xi_0(t_0/\varepsilon), \eta_0(t_0/\varepsilon)).$$

Since the integrand in the above formula and u_0 are bounded independently of ε, we obtain the equiboundedness from above of the V^ε.

We now consider any trajectory (ξ, η) of (CD_ε) starting (x_0, y_0) and corresponding to a control β. By (4), $\xi(t)$ lies in a compact subset K of $\mathbb{R}^N$, only depending on C, for $t \in [0, t_0/\varepsilon]$, and by coercivity assumption **(H4)**, there is a constant P_0 with

$$\ell(x, y, a) \ge P_0 \qquad \text{for any } (x, y, a) \in K \times \mathbb{R}^M \times A. \tag{17}$$

Since $-Q_0$ is a lower bound of u_0 in $\mathbb{R}^N \times \mathbb{R}^M$, see **(H5)**, this implies

$$\varepsilon \int_0^{\frac{t_0}{\varepsilon}} \ell(\xi(s), \eta(s), \beta(s))\, ds + u_0(\xi(t_0/\varepsilon), \eta(t_0/\varepsilon)) \ge \tag{18}$$

$$\varepsilon \frac{t_0}{\varepsilon} P_0 + u_0(\xi(t_0/\varepsilon), \eta(t_0/\varepsilon)) \ge P_0 t_0 - Q_0.$$

Being (ξ, η) an arbitrary trajectory with initial point (x_0, y_0), the above inequality shows the claimed local equiboundedness from below of value functions. □

The previous result allows us to define $\limsup^\# V^\varepsilon$, $\liminf_\# V^\varepsilon$, these functions will be denoted by $\overline{V}$, $\underline{V}$, respectively, in what follows. The next proposition shows that they only depend on time and slow variable, at least for positive times.

Proposition 2.7. *We have*

$$(\liminf_\# V^\varepsilon)(x_0, y_0, t_0) = (\liminf_\# V^\varepsilon)(x_0, z_0, t_0) =: \underline{V}(x_0, t_0)$$
$$(\limsup^\# V^\varepsilon)(x_0, y_0, t_0) = (\limsup^\# V^\varepsilon)(x_0, z_0, t_0) =: \overline{V}(x_0, t_0)$$

for any $x_0 \in \mathbb{R}^N$, y_0, z_0 *in* $\mathbb{R}^M$ *and* $t_0 > 0$.

Proof. We start by

Claim. *Given positive constants* R_1, R_2, S *we can determine* $P = P(R_1, R_2, S) > 0$ *such that for any* $\varepsilon > 0$, $x \in B(0, R_1)$, y, z *in* $B(0, R_2)$, $t \in [0, S]$ *there exist* x', x'', z', z'', t', t'', *depending on* ε, *with*

$$|x - x'| < \varepsilon P, \quad |z - z'| < \varepsilon P, \quad |t - t'| < \varepsilon P,$$
$$|x - x''| < \varepsilon P, \quad |z - z''| < \varepsilon P, \quad |t - t''| < \varepsilon P$$

such that

$$V^\varepsilon(x', z', t') < V^\varepsilon(x, y, t) + \varepsilon P$$
$$V^\varepsilon(x'', z'', t'') > V^\varepsilon(x, y, t) - \varepsilon P.$$

We fix ε. By controllability assumption (see Remark 2.1) z and y can be joined in a time T less than or equal to $T_0 = T_0(R_1, R_2)$ by a curve ζ satisfying

$$\dot{\zeta} = g(x, \zeta, \alpha) \qquad \text{for a suitable control } \alpha.$$

We consider the trajectory (ξ, η) of (CD_ε) with the same control α satisfying

$$\xi(T) = x \quad \text{and} \quad \eta(T) = y,$$

and set

$$x' = \xi(0) \quad \text{and} \quad z' = \eta(0).$$

By (4), (5), we get

$$|x' - x| < \varepsilon P_0 \tag{19}$$
$$|z' - z| < \varepsilon P_0 \tag{20}$$

for a suitable $P_0 > 0$. We select a trajectory (ξ_0, η_0) of (CD_ε) with initial datum (x, y), corresponding to a control β, such that

$$V^\varepsilon(x, y, t) \geq \varepsilon \int_0^{\frac{t}{\varepsilon}} \ell(\xi_0, \eta_0, \beta)\, ds + u_0\left(\xi_0\left(\frac{t}{\varepsilon}\right), \eta_0\left(\frac{t}{\varepsilon}\right)\right) - \varepsilon. \tag{21}$$

We set

$$t' = t + \varepsilon T, \tag{22}$$

by concatenation of α and β, ξ and ξ_0, η and η_0, we get a control γ and trajectory $(\overline{\xi}, \overline{\eta})$ of (CD_ε) starting at (x', z'), defined in $\left[0, \frac{t'}{\varepsilon}\right]$. We consequently have

$$V^\varepsilon(x', z', t') \leq$$
$$\varepsilon \int_0^{\frac{t'}{\varepsilon}} \ell(\overline{\xi}, \overline{\eta}, \gamma)\, ds + u_0\left(\overline{\xi}\left(\frac{t'}{\varepsilon}\right), \overline{\eta}\left(\frac{t'}{\varepsilon}\right)\right) =$$

$$\varepsilon \int_0^T \ell(\xi, \eta, \alpha)\, ds + \varepsilon \int_T^{\frac{t'}{\varepsilon}} \ell(\xi_0(s-T), \eta_0(s-T), \beta(s-T))\, ds +$$
$$u_0\left(\xi_0\left(\frac{t'}{\varepsilon}\right), \eta_0\left(\frac{t'}{\varepsilon}\right)\right).$$

By taking into account (5) and (21), we derive

$$V^\varepsilon(x', z', t') \le \varepsilon\, Q\, T_0 + V^\varepsilon(x, y, t) + \varepsilon \quad \text{for a suitable } Q > 0. \tag{23}$$

The first part of the claim is therefore proved taking into account (19), (20), (22), (23), and defining

$$P = \max\{P_0, T_0, Q\, T_0 + 1\}.$$

The estimates for x'', y'', z'', t'' can be obtained slightly modifying the above argument. We sketch the proof for reader's convenience. We denote by ζ' a curve joining y to z in a time $T' \le T_0$ and satisfying

$$\dot{\zeta}' = g(x, \zeta', \alpha) \qquad \text{for a suitable control } \alpha'.$$

We consider the trajectory (ξ', η') of (CD_ε) with the same control α' satisfying

$$\xi'(0) = x \quad \text{and} \quad \eta'(0) = y,$$

and set

$$x'' = \xi'(T') \quad \text{and} \quad z'' = \eta'(T').$$

As in the first part of the proof we get

$$|x'' - x| \le P_0\, \varepsilon$$
$$|z'' - z| \le P_0\, \varepsilon,$$

for a suitable P_0. We select a trajectory (ξ_0', η_0') of (CD_ε) with initial datum (x'', z''), corresponding to a control β', which is optimal for $V^\varepsilon(x'', z'', t - \varepsilon\, T')$ up to ε, namely

$$V^\varepsilon(x'', z'', t'') \ge \varepsilon \int_0^{\frac{t''}{\varepsilon}} \ell(\xi_0', \eta_0', \beta')\, ds + u_0\left(\xi_0'\left(\frac{t''}{\varepsilon}\right), \eta_0'\left(\frac{t''}{\varepsilon}\right)\right) - \varepsilon.$$

Here we are assuming ε so small that $t'' := t - \varepsilon\, T'$ is positive, this does not entail any limitation to the argument since we are interested to ε infinitesimal. From this point we go on as in the previous part.

We exploit the first part of the claim to show that for any pair of values y_0, z_0 of the fast variable, any $x_0 \in \mathbb{R}^N$, $t_0 > 0$

$$(\liminf_{\#} V^\varepsilon)(x_0, z_0, t_0) \le (\liminf_{\#} V^\varepsilon)(x_0, y_0, t_0), \tag{24}$$

which in turn implies by the arbitrariness of y_0, z_0, that $\liminf_{\#} V^\varepsilon$ independent of the fast variable. We consider ε_n, x_n, y_n, t_n converging to 0, x_0, y_0, t_0, respectively, with

$$\lim_n V^{\varepsilon_n}(x_n, y_n, t_n) = (\liminf_{\#} V^\varepsilon)(x_0, z_0, t_0).$$

Since all the x_n, y_n, and z_0, t_n are contained in compact subsets of $\mathbb{R}^N$, $\mathbb{R}^M$, $[0, +\infty)$, respectively, we can apply, for any given $n \in \mathbb{N}$, the claim to $\varepsilon = \varepsilon_n$, $x = x_n$, $y = y_n$, $z = z_0$, $t = t_n$ and get of x'_n, z'_n, t'_n with

$$|x_n - x'_n| < \varepsilon_n P, \quad |z_0 - z'_n| < \varepsilon_n P, \quad |t_n - t'_n| < \varepsilon_n P$$

and

$$V^{\varepsilon_n}(x'_n, z'_n, t'_n) < V^{\varepsilon_n}(x_n, y_n, t_n) + \varepsilon_n P$$

for a suitable P. Sending n to infinity we deduce

$$\liminf V^{\varepsilon_n}(x'_n, z'_n, t'_n) \le \lim V^{\varepsilon_n}(x_n, y_n, t_n) = (\liminf_{\#} V^\varepsilon)(x_0, z_0, t_0),$$

which implies (24) since $x'_n \to x_0$, $z'_n \to z_0$ and $t'_n \to t_0$.

The assertion relative to $\limsup^{\#} V^\varepsilon$ is obtained using the second part of the claim and slightly adapting the above argument. □

As a consequence of coercivity of running cost assumed in **(H4)** we deduce:

Proposition 2.8. *The value function V^ε satisfy for any ε, any compact subset K of $\mathbb{R}^N \times (0, +\infty)$*

$$\lim_{|y| \to +\infty} \min_{(x,t) \in K} V^\varepsilon(x, y, t) = +\infty.$$

Proof. We fix ε, we assume, without loosing any generality, that K is of the form $\widetilde{K} \times [S, T]$, where $\widetilde{K}$ is a compact subset of $\mathbb{R}^N$ and S, T are positive times. Given any $P > 0$, we can determine by **(H4)** a constant R such that the ball $B(0, R)$ of $\mathbb{R}^M$ satisfies

$$\ell(x, y, a) > P \qquad \text{for any } (x, a) \in \widetilde{K} \times A,\ y \in \mathbb{R}^M \setminus B(0, R). \tag{25}$$

Taking into account the estimate (7), we see that there exists $R_0 > R$ such that

$$\eta(t) \notin B(0, R) \qquad \text{for } t \in [0, T] \tag{26}$$

for any trajectory of (CD_ε) starting in $K_0 \times \left(\mathbb{R}^M \setminus B(0, R_0)\right)$. Given $\delta > 0$, we find, for any

$$(x, y, t) \in K_0 \times \left(\mathbb{R}^M \setminus B(0, R_0)\right) \times [S, T]$$

a trajectory (ξ_0, η_0) of (CD_ε), corresponding to a control α, starting at (x, y) with

$$V^\varepsilon(x, y, t) \geq \varepsilon \int_0^{\frac{t}{\varepsilon}} \ell(\xi_0, \eta_0, \alpha)\, \mathrm{d}s + u_0\left(\xi_0\left(\frac{t}{\varepsilon}\right), \eta_0\left(\frac{t}{\varepsilon}\right)\right) - \delta.$$

We deduce by (25), (26), **(H5)**

$$V^\varepsilon(x, y, t) \geq P\, S - Q_0 - \delta,$$

which gives the assertion, since P can be chosen as large as desired, and δ as small as desired. □

2.5. *HJB equations*

We define the Hamiltonian

$$H(x, y, p, q) = \max_{a \in A}\{-p \cdot f(x, y, a) - q \cdot g(x, y, a) - \ell(x, y, a)\}$$

The main contribution of Assumption **(H3)** is the following coercivity property on H:

Lemma 2.9. *For any given bounded set $C \subset \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^N$, we have*

$$\lim_{|q| \to +\infty} \min_{(x,y,p) \in C} H(x, y, p, q) = +\infty.$$

Proof. We denote by r the positive constant provided by **(H3)** in correspondence to the projection of C on the state variables space $\mathbb{R}^N \times \mathbb{R}^M$. We consequently have for (x, y) in such projection and $q \in \mathbb{R}^M$

$$\max\{q \cdot v \mid v \in g(x, y, A)\} = \max\{q \cdot v \mid v \in \overline{\mathrm{co}}\, g(x, y, A)\} \geq r\,|q|. \tag{27}$$

We take $(x, y, p) \in C$, and denote by a_0 an element in the control set such that $g(x, y, a_0)$ realizes the maximum in (27). We get from the very definition of H and (27)

$$H(x, y, p, q) \geq -|p|\,|f(x, y, a_0)| + r\,|q| - |\ell(x, y, a)| \qquad \text{for any } q.$$

When we send $|q|$ to infinity, all the terms in the right hand-side of the above formula stay bounded except $r\,|q|$. This gives the assertion. □

Given a bounded set B in $\mathbb{R}^N \times \mathbb{R}^M$, one can check by direct calculation that H satisfies

$$|H(x_1, y_1, p, q) - H(x_2, y_2, p, q)| \leq \tag{28}$$
$$L_0\,(|x_1 - x_2| + |y_1 - y_2|)(|p| + |q|) + \omega(|x_1 - x_2| + |y_1 - y_2|)$$

for any (x_1, y_1), (x_2, y_2) in B and $(p, q) \in \mathbb{R}^N \times \mathbb{R}^M$, where ω is an uniform continuity modulus of ℓ in $B \times A$ and L_0 is as in **(H2)**. We also have

$$|H(x, y, p_1, q_1) - H(x, y, p_2, q_2)| \leq \\ |f(x, y, a_0)|\,|p_1 - p_2| + |g(x, y, a_0)|\,|q_1 - q_2| \tag{29}$$

for any $(x, y) \in \mathbb{R}^N \times \mathbb{R}^M$, (p_1, q_1), (p_2, q_2) in $\mathbb{R}^N \times \mathbb{R}^M$, a suitable $a_0 \in A$.

We write, for any $\varepsilon > 0$, the family of Hamilton–Jacobi–Bellman problems

$$\begin{cases} u^\varepsilon_t + H\left(x, y, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}\right) = 0 \\ u^\varepsilon(x, y, 0) = u_0(x, y) \end{cases} \tag{HJ$_\varepsilon$}$$

It is well known that the value functions V^ε are solutions to (HJ$_\varepsilon$), even if not necessarily unique in our setting. However, due to the estimate (28), we have the following local comparison result (see for instance [9,10]):

Proposition 2.10. *Given a bounded open set B of $\mathbb{R}^N \times \mathbb{R}^M$ and times $t_2 > t_1$, let u, v be continuous subsolution and supersolution, respectively, of the equation in (HJ$_\varepsilon$). If $u \leq v$ in $\partial_p\big(B \times (t_1, t_2)\big)$ then $u \leq v$ in $B \times (t_1, t_2)$, where ∂_p stands for the parabolic boundary.*

We define the *effective Hamiltonian*

$$\overline{H}(x, p) = \inf\{b \in \mathbb{R} \mid H(x, y, p, Du) = b \text{ admits a subsolution in } \mathbb{R}^M\} \tag{30}$$

for any fixed $(x, p) \in \mathbb{R}^N \times \mathbb{R}^N$, where the equation appearing in the formula is solely in the fast variable y with slow variable x and corresponding momentum p frozen, and accordingly with $u(y)$ as unknown function. This quantity can be in principle infinite, however we will show in what follows that not only it is finite for any (x, p), but also that the infimum is actually a minimum.

We write the limit equation

$$u_t + \overline{H}(x, Du) = 0. \tag{$\overline{\text{HJ}}$}$$

3. Cell problems

The section is devoted to the analysis of the stationary Hamilton–Jacobi equations in $\mathbb{R}^M$ appearing in the definition of effective Hamiltonian, namely with slow variable and corresponding momentum frozen.

3.1. Basic analysis

We fix $(x_0, p_0) \in \mathbb{R}^N \times \mathbb{R}^N$, and set to ease notations

$$H_0(y, q) = H(x_0, y, p_0, q) \qquad \text{for any } (y, q) \in \mathbb{R}^M \times \mathbb{R}^M$$

$$\ell_0(y, a) = \ell(x_0, y, a) + p_0 \cdot f(x_0, y, a) \qquad \text{for any } (y, a) \in \mathbb{R}^M \times A$$

$$g_0(y, a) = g(x_0, y, a) \qquad \text{for any } (y, a) \in \mathbb{R}^M \times A$$

Given a control $\alpha(t)$, we consider the controlled differential equation in $\mathbb{R}^M$

$$\dot{\eta}(t) = g_0(\eta(t), \alpha(t)). \tag{31}$$

We directly derive from Lemma 2.9:

Lemma 3.1. *We have*

$$\lim_{|q|\to+\infty} \min_{y\in K} H_0(y, q) = +\infty$$

for any compact subset K of $\mathbb{R}^M$.

This result implies, according to Lemma A.1, that all subsolutions are locally Lipschitz-continuous, and allows adopting the metric method, see Appendix A, in the analysis of the cell equations. To ease notation, we set $c_0 = \overline{H}(x_0, p_0)$, also called the critical value of H_0, see (86). We will prove in Proposition 3.3 that c_0 is finite. We denote by Z, σ, S the corresponding sub-levels, support function and intrinsic distance, see Appendix A for the corresponding definitions. Same objects for a supercritical value b will be denoted by Z_b, σ_b, S_b.

To compare the metric and control-theoretic viewpoint, we notice

$$Z_b(y) = \{q \in \mathbb{R}^M \mid q \cdot (-g_0(y, a)) \le \ell_0(y, a) + b \text{ for any } a \in A\}$$

for any given supercritical $b \in \mathbb{R}$, namely $b \ge c_0$, and $y \in \mathbb{R}^M$. This implies that the support function $\sigma_b(y, \cdot)$ is the maximal subadditive positively homogeneous function $\rho : \mathbb{R}^M \to \mathbb{R}$ with

$$\rho(-g_0(y, a)) \le \ell_0(y, a) + b \qquad \text{for any } a \in A, \tag{32}$$

which somehow justifies the next equivalences.

Proposition 3.2. *Given a supercritical value b, the following conditions are equivalent:*

- **(i)** *u is a subsolution to $H_0 = b$;*
- **(ii)** *$u(y_2) - u(y_1) \le S_b(y_1, y_2)$ for any y_1, y_2;*
- **(iii)** *$u(y_1) - u(y_2) \le \int_0^T (\ell_0(\eta(t), \alpha(t)) + b)\, \mathrm{d}t$ for any y_1, y_2, time T, control α, any trajectory η of (31) with $\eta(0) = y_1$, $\eta(T) = y_2$.*

Proof. The equivalence **(i)** $\Longleftrightarrow$ **(ii)** is given in Proposition A.3 **(i)**, the equivalence **(i)** $\Longleftrightarrow$ **(iii)** is the usual characterization of subsolutions to Hamilton–Jacobi–Bellman equations in terms of suboptimality, see [8]. □

One advantage of the metric method is that any curve is endowed of a length, while integral cost functional is only defined on trajectories of the controlled dynamics. Also notice that there is a change of orientation between length and cost functional, that can be detected from (32) and comparison between items **(ii)** and **(iii)** in Proposition 3.2. This just depends on u_0 being terminal cost and initial condition in (HJ$_\varepsilon$), the discrepancy should be eliminated if (HJ$_\varepsilon$) were posed in $(-\infty, 0)$ and u_0 should consequently play the role of terminal condition and initial cost.

Proposition 3.3. *The critical value c_0 is finite.*

Proof. Owing to coercivity of ℓ and boundedness of f

$$H_0(y,0) = \max_{a \in A}\{-\ell_0(y,a)\} \to -\infty \quad \text{as } |y| \to +\infty$$

and consequently

$$H_0(y,0) < 0 \quad \text{outside some compact subset } K \text{ of } \mathbb{R}^M.$$

We set

$$b_0 = \max\left\{0, \max\{H_0(x,0) \mid x \in K\}\right\},$$

then the null function is subsolution to $H_0 = b_0$ in $\mathbb{R}^M$, and so $c_0 < +\infty$.

By controllability condition **(H3)**, we find a cycle η defined in $[0,T]$, for a positive T, solution to (31) for some control α. We put

$$R = \int_0^T \ell_0(\eta,\alpha)\,\mathrm{d}t,$$

and for $b < -\frac{R}{T}$ we get

$$\int_0^T (\ell_0(\eta,\alpha) + b)\,\mathrm{d}t < R - \frac{R}{T}T = 0.$$

The above cycle, repeated infinite times, gives a trajectory of (31) in $[0,+\infty)$, still denoted by η, such that

$$\int_0^\infty (\ell_0(\eta,\alpha) + b)\,\mathrm{d}t = -\infty. \tag{33}$$

If there were a subsolution u to $H_0 = b$ then

$$u(\eta(0)) - u(\eta(t_0)) \le \int_0^{t_0} (\ell_0(\eta(t),\alpha(t)) + b)\,\mathrm{d}t \quad \text{for any } t_0 > 0. \tag{34}$$

But the support of η is equal to $\eta([0,T])$ which is a compact subset of $\mathbb{R}^M$, so that the oscillation of u (which is locally Lipschitz continuous) on it is bounded. This shows that (33) and (34) are in contradiction. We then deduce that the equation $H_0 = b$ cannot have any subsolution, showing in the end that $c_0 > -\infty$. □

We deduce from standing assumptions a sign and a coercivity condition on the critical distances. To do that, we start selecting a compact set C of $\mathbb{R}^M$ with

$$H_0(y,0) = -\min_{a \in A} \ell_0(y,a) < c_0 - Q_0 \qquad \text{for any } y \in \mathbb{R}^M \setminus C, \tag{35}$$

where Q_0 is as in **(H2)**. This is possible since ℓ_0 is coercive. Further we set

$$K_0 = \left\{ y \mid d(y,C) \le \max_{C \times C} |S| \right\}. \tag{36}$$

Proposition 3.4. *The following properties hold true:*

(i) $\lim_{|y| \to +\infty} \inf_{y_0 \in K} S(y_0, y) = +\infty$ *for any compact set* $K \subset \mathbb{R}^M$*;*
(ii) $Z(y) \supset B(0,1)$ *for any* y *outside the compact set* K_0 *defined as in (36);*
(iii) $S(y_1, y_2) > 0$ *for any pair* y_1, y_2 *outside* K_0*.*

Proof. If $q \in \mathbb{R}^M$ satisfies

$$H_0(y,q) = c_0 \qquad \text{for some } y \text{ in } \mathbb{R}^M \setminus C, \tag{37}$$

where C is defined as in (35), then

$$c_0 = H_0(y,q) = \max_{a \in A}\{-g_0(y,a) \cdot q - \ell_0(y,a)\} \le Q_0\,|q| - \min_{a \in A} \ell_0(y,a)$$

and by the very definition of C

$$|q| \ge \frac{c_0 + \min_{a \in A} \ell_0(y,a)}{Q_0} > \frac{Q_0}{Q_0} = 1. \tag{38}$$

Since 0 is in the interior of $Z(y)$ by (35), we derive a stronger version of item **(ii)**, with C in place of K_0, which in turn implies

$$\frac{v}{|v|} \in Z(y) \qquad \text{for any } y \in \mathbb{R}^M \setminus C,\, v \in \mathbb{R}^M \text{ with } v \neq 0$$

and consequently

$$\sigma(y,v) \ge v \cdot \left(\frac{v}{|v|} \right) = |v| \qquad \text{for any } y \in \mathbb{R}^M \setminus C,\, v \in \mathbb{R}^M \text{ with } v \neq 0. \tag{39}$$

Next, we fix a compact set K and consider two points $y_1 \in K$, $y_2 \notin C$ and any curve ζ, defined in $[0,1]$, linking them. We distinguish two cases according on whether the intersection of ζ with C is nonempty or empty. In the first instance we set

$$t_1 = \min\{t \in [0,1] \mid \zeta(t) \in C\} \tag{40}$$

$$t_2 = \max\{t \in [0,1] \mid \zeta(t) \in C\}. \tag{41}$$

We denote by R an upper bound of $|S|$ in $C \times C$ and exploit (39) to get

$$\begin{aligned}\int_0^1 \sigma(\zeta,\dot\zeta)\,\mathrm{d}t &= \int_0^{t_1} \sigma(\zeta,\dot\zeta)\,\mathrm{d}t + \int_{t_1}^{t_2} \sigma(\zeta,\dot\zeta)\,\mathrm{d}t + \int_{t_2}^{1} \sigma(\zeta,\dot\zeta)\,\mathrm{d}t \\ &\geq |y_1 - \zeta(t_1)| + S(\zeta(t_1),\zeta(t_2)) + |y_2 - \zeta(t_2)| \\ &\geq -R + d(y_1, C) + d(y_2, C).\end{aligned} \tag{42}$$

If instead the curve ζ entirely lies outside C, we have by (39)

$$\int_0^1 \sigma(\zeta,\dot\zeta)\,\mathrm{d}t \geq |y_1 - y_2|. \tag{43}$$

In both cases we get item **(i)** sending y_2 to infinity and taking into account that y_1 has been arbitrarily chosen in K.

We finally see, looking at (42), (43), and slightly adapting the above argument that K_0, defined as in (36), satisfies item **(iii)**. □

Remark 3.5. Given a compact set $K \subset \mathbb{R}^M$, the same argument of Proposition 3.4 allows also proving

$$\lim_{|y|\to+\infty} \inf_{y_0\in K} S(y, y_0) = +\infty \tag{44}$$

Corollary 3.6. *For any bounded open set B there exists $R > 0$ such that if y_1, y_2 belong to B then all curves linking y_1 to y_2 with intrinsic length less than $S(y_1, y_2) + 1$ are contained in $B(0, R)$.*

Proof. We can assume without loosing generality that $B \supset K_0$, where K_0 is the set defined in (36). We set

$$P = \sup_{B\times B} |S|.$$

By Proposition 3.4 **(i)** there is R such that

$$\inf_{y_0\in B} S(y_0, y) > 2P + 2 \qquad \text{for } y \text{ with } |y| > R.$$

We claim that such an R satisfies the claim. In fact, assume by contradiction that there are y_1, y_2 in B and an 1-optimal curve ζ, defined in $[0, 1]$, for $S(y_1, y_2)$ not contained in $B(0, R)$. Let t_1 be a time in $(0, 1)$ with $\zeta(t_1) \notin B(0, R)$ and set

$$t_2 = \min\{t \in (t_1, 1) \mid \zeta(t) \in K_0 \subset B\}$$

then, taking into account Proposition 3.4

$$
\begin{aligned}
S(y_1, y_2) &\geq \int_0^1 \sigma(\zeta, \dot{\zeta})\, \mathrm{d}t - 1 \\
&= \int_0^{t_1} \sigma(\zeta, \dot{\zeta})\, \mathrm{d}t + \int_{t_1}^{t_2} \sigma(\zeta, \dot{\zeta})\, \mathrm{d}t + \int_{t_2}^{1} \sigma(\zeta, \dot{\zeta})\, \mathrm{d}t - 1 \\
&\geq S(y_1, \zeta(t_1)) + S(\zeta(t_1), \zeta(t_2)) + S(\zeta(t_2), y_2) - 1 \\
&\geq 2P + 2 - P - 1 = P + 1,
\end{aligned}
$$

which is in contrast with the very definition of P. □

3.2. Existence of special subsolutions and solutions

Here we show the existence of bounded critical subsolutions, and of coercive critical solutions.

Proposition 3.7. *There exists a bounded Lipschitz-continuous critical subsolution u, vanishing and strict outside the compact set K_0 defined as in (36).*

Proof. By Proposition 3.4, item **(iii)**

$$
S(y_1, y_2) \geq 0 \qquad \text{for any } y_1, y_2 \text{ in } \overline{\mathbb{R}^M \setminus K_0}, \tag{45}
$$

and consequently the null function is an admissible trace for subsolutions to $H_0 = c_0$ on $\overline{\mathbb{R}^M \setminus K_0}$ in the sense that (45) is the same inequality as in Proposition A.3 **(iii)** with the null function in place of w and S in place of S_b, consequently

$$
u(y) := \inf\{S(z, y) \mid z \in \overline{\mathbb{R}^M \setminus K_0}\}
$$

is a subsolution to $H_0 = c_0$ in $\mathbb{R}^M$ vanishing on $\overline{\mathbb{R}^M \setminus K_0}$. In addition

$$
H_0(y, Du) = H_0(y, 0) < c_0 - Q_0 \qquad \text{for } y \in \mathbb{R}^M \setminus K_0 \subset \mathbb{R}^M \setminus C
$$

by the very definition of C in (35). Since u is locally Lipschitz-continuous by Lemma 3.1 and vanishes outside a compact set, it is actually globally Lipschitz-continuous in $\mathbb{R}^M$. This fully shows the assertion. □

We denote by $\mathcal{A}_0$ the Aubry set of H_0, see Proposition A.5 for the definition. We have:

Lemma 3.8. *The Aubry set $\mathcal{A}_0$ is nonempty and contained in K_0, where K_0 is defined as in (36).*

Proof. We know from Proposition 3.7 that there is a critical subsolution which is strict outside K_0, so that by Proposition A.5 **(iii)** $\mathcal{A}_0 \subset K_0$. The point is then to show that the Aubry set is nonempty.

We argue by contradiction using a covering argument. If $\mathcal{A}_0 = \emptyset$, then we can associate by Proposition A.5 **(iii)** to any point $y \in K_0$ an open neighborhood B_y, a value $d_y < c_0$, and a critical subsolution w_y with

$$H_0(\cdot, Dw_y) \le d_y < c_0 \quad \text{in } B_y.$$

We extract a finite subcovering $\{B_1, \cdots, B_m\}$ corresponding to points $y_1, \cdots, y_m$ of K_0, and set

$$\begin{aligned} w_j &= w_{y_j} \\ d_j &= d_{y_j} \qquad \textit{for} \quad j = 1, \cdots, m. \end{aligned}$$

Then

$$\{B_0, B_1, \cdots, B_m\},$$

where $B_0 = \mathbb{R}^M \setminus K_0$, is an finite open cover of $\mathbb{R}^M$. We denote by u the critical subsolution constructed in Proposition 3.7 and set $d_0 = c_0 - Q_0$, so that

$$H_0(y, Du(y)) \le d_0 < c_0 \qquad \text{for any } y \in B_0.$$

We define

$$w = \lambda_0 \, u + \sum_{i=1}^{m} \lambda_j \, w_j,$$

where $\lambda_0, \lambda_1, \cdots, \lambda_m$ are positive coefficients summing to 1. We have by convexity of H_0

$$H_0(y, Dw(y)) \le \lambda_0 \, H_0(y, Du(y)) + \sum_{j=1}^{m} \lambda_j \, H_0(y, Dw_j(y)),$$

for a.e. $y \in \mathbb{R}^M$, and we derive

$$H_0(y, Dw(y)) \le \sum_{i \ne j} \lambda_i \, c_0 + \lambda_j \, d_j = (1 - \lambda_j) \, c_0 + \lambda_j \, d_j = c_0 + \lambda_j \, (d_j - c_0)$$

for a.e. $y \in B_j$, $j = 0, \cdots, m$. We set $\widetilde{d} = \max_j \lambda_j \, (d_j - c_0) < 0$ and conclude

$$H_0(y, Dw(y)) \le c_0 + \widetilde{d} < c_0 \quad \text{for a.e. } y \in \mathbb{R}^M,$$

which is impossible by the very definition of c_0. This gives by contradiction $\emptyset \ne \mathcal{A}_0 \subset K_0$, as desired. □

From the previous lemma and Proposition 3.4, item **(i)** we get:

Proposition 3.9. *All the functions $y \mapsto S(y_0, y)$, for $y_0 \in \mathcal{A}_0$, are coercive critical solutions.*

The previous line of reasoning can be somehow reversed. We proceed showing that the existence of coercive solutions, plus the coercivity of intrinsic distance, characterizes the critical equation and also directly implies that the Aubry set is nonempty, as made precise by the following result:

Proposition 3.10. *Assume that the equation*

$$H_0(y, Du) = b$$

admits a coercive solution in $\mathbb{R}^M$ *and limit relation* (44) *holds true with* S_b *in place of* S*, then* $b = c_0$ *and the corresponding Aubry set is nonempty.*

Proof. The argument is by contradiction. Let w be a coercive solution of the equation in object. If $b \neq c_0$ or $\mathcal{A}_0 = \emptyset$ then by Corollary A.6, Proposition A.7, there is, for any $R > 0$, an unique solution of the Dirichlet problem

$$\begin{cases} H_0(y, Du) = b & \text{in } B(0, R) \\ u = w & \text{on } \partial B(0, R) \end{cases}$$

which therefore must coincide with w, and

$$w(0) = w(z) + S_b(z, 0) \qquad \text{for any } R > 0, \text{ some } z \in \partial B(0, R). \tag{46}$$

Since we have assumed (44), with S_b in place of S, we have

$$\lim_{|z| \to +\infty} S_b(z, 0) = +\infty$$

and by assumption w is coercive. This shows that (46) is impossible, and concludes the proof. □

We derive:

Proposition 3.11. *The effective Hamiltonian* $\overline{H} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ *is continuous in both components and convex in* p.

Proof. It is easy to see using the continuity of H and the argument in the proof of Proposition 3.3 that $\overline{H}$ is locally bounded. We consider a sequence (x_n, p_n) converging to some (x, p), and assume that $\overline{H}(x_n, p_n)$ admits limit. We consider a sequence v_n of solutions to

$$H(x_n, y, p_n, Du) = \overline{H}(x_n, p_n)$$

of the form as in Proposition 3.9. By exploiting the continuity of H we see that the v_n are locally equiLipschitz-continuous, locally equibounded and equicoercive. They are consequently locally uniformly convergent, up to a subsequence, by Ascoli Theorem, with limit function, say w, locally Lipschitz-continuous and coercive. In addition, by basic stability properties of viscosity solutions theory, w satisfies

$$H(x, y, p, Dw) = \lim_n \overline{H}(x_n, p_n),$$

which implies by Proposition 3.10 that $\lim_n \overline{H}(x_n, p_n) = \overline{H}(x, p)$. This shows the claimed continuity of $\overline{H}$.

We see by the very definition of H that

$$H(x, y, \lambda p_1 + (1-\lambda) p_2, \lambda q_1 + (1-\lambda) q_2) \leq \lambda H(x, y, p_1, q_1) + (1-\lambda) H(x, y, p_2, q_2).$$

We derive from this that if u_i, $i = 1, 2$, satisfy $H(x, y, p_i, Du_i) \leq \overline{H}(x, p_i)$ in the viscosity sense, then

$$H(x, y, \lambda p_1 + (1-\lambda) p_2, \lambda Du_1 + (1-\lambda) Du_2) \leq \lambda \overline{H}(x, p_1) + (1-\lambda) \overline{H}(x, p_2),$$

which in turn implies

$$\overline{H}(x, \lambda p_1 + (1-\lambda) p_2) \leq \lambda \overline{H}(x, p_1) + (1-\lambda) \overline{H}(x, p_2)$$

as desired. □

3.3. Construction of a supersolution

We still keep (x_0, p_0) fixed. Starting from Proposition 3.9, we construct a supersolution of the cell problem which will play the role of corrector in Theorem 4.3. We denote by K_0 the set defined in (36). We fix $y_0 \in \mathcal{A}_0$; by the coercivity of $S(y_0, \cdot)$, see Proposition 3.9, there is a constant d such that

$$d + S(y_0, y) > 0 \qquad \text{for any } y \in \mathbb{R}^M. \tag{47}$$

We select a constant R_0 satisfying

$$B(0, R_0 - 3) \supset K_0 \tag{48}$$

$$R_0 - 3 \text{ satisfies Corollary 3.6 for a neighborhood of } y_0. \tag{49}$$

We aim at proving:

Theorem 3.12. *Let $U : \mathbb{R}^M \to \mathbb{R}$ be a function bounded from above in $\overline{B}(0, R_0)$ with*

$$U \leq 0 \qquad \text{in } B(0, R_0 - 1), \tag{50}$$

then there exists for any $\lambda > 0$, a locally Lipschitz-continuous supersolution w_λ of $H_0 = c_0$ in $\mathbb{R}^M$ with

$$U \leq \lambda w_\lambda \qquad \text{in } \overline{B}(0, R_0) \tag{51}$$

$$w_\lambda = d + S(y_0, \cdot) \qquad \text{in a neighborhood of } y_0. \tag{52}$$

To construct the supersolutions w_λ some preliminary steps are needed.
We define

$$M_0 = \max \left\{ \sup_{\overline{B}(0,R_0)} \frac{1}{\lambda} U, 1 \right\}.$$

We denote by $h_\lambda : [0, +\infty) \to [0, +\infty)$ a nondecreasing continuous function with

$$h_\lambda \equiv 1 \qquad \text{in } [0, R_0 - 3] \tag{53}$$

$$h_\lambda \equiv M_0 \qquad \text{in } [R_0 - 2, +\infty). \tag{54}$$

We introduce the length functional

$$\int_0^1 h_\lambda(|\xi|)\, \sigma(\xi, \dot{\xi})\, ds$$

for any curve ξ defined in $[0, 1]$, and denote by S^h the distance obtained by minimization of it among curves linking two given points, we drop dependence on λ to ease notations.

Lemma 3.13. *The function $S^h(y_0, \cdot)$ is a locally Lipschitz-continuous supersolution to $H_0 = c_0$ in $\mathbb{R}^M$, and coincides with $S(y_0, \cdot)$ in a neighborhood of y_0.*

Proof. The function h_λ, defined in (53), (54), satisfies $h_\lambda \geq 1$ and if $h_\lambda(|y|) > 1$ then by (53)

$$y \notin B(0, R_0 - 3) \supset K_0$$

so that by Proposition 3.4 **(ii)** $H_0(y, 0) < c_0$. We are thus in position to apply Proposition A.8, which directly gives the asserted supersolution property outside y_0, as well as the Lipschitz continuity. We also know by (49) and $h_\lambda \equiv 1$ in $B(0, R_0 - 3)$ that

$$S^h(y_0, \cdot) = S(y_0, \cdot) \qquad \text{in a neighborhood of } y_0,$$

and $S^h(y_0, \cdot)$ is solution to $H_0 = c_0$ on the whole space, by Proposition 3.9. This concludes the proof. □

By the very definition of S^h, we have:

$$S^h \geq S \qquad \text{in } \mathbb{R}^M \times \mathbb{R}^M. \tag{55}$$

We define

$$w_\lambda = d + S^h(y_0, \cdot) \tag{56}$$

where d, y_0 are as in (47).

Lemma 3.14. *The following inequalities hold true:*

$$\begin{aligned} w_\lambda &> 0 && \text{in } \mathbb{R}^M \\ w_\lambda &\geq M_0 && \text{in } \mathbb{R}^M \setminus B(0, R_0 - 1). \end{aligned}$$

Proof. From (55) and the definition of w_λ we derive

$$w_\lambda \geq d + S(y_0, \cdot)$$

and this in turn yields $w_\lambda > 0$ in $\mathbb{R}^M$ because of (47).

We fix $y \notin B(0, R_0 - 1)$, and consider any curve ζ defined in $[0, 1]$ linking y_0 to y. We set

$$t_1 = \max\{t \in [0, 1] \mid \zeta(t) \in B(0, R_0 - 2)\},$$

notice that

$$|\zeta(t_1) - y| > 1.$$

Owing to the above inequality, $w_\lambda > 0$, Proposition 3.4 item **(ii)**, the definition of h_λ, we have

$$\begin{aligned} d + \int_0^1 h_\lambda(|\zeta|)\, \sigma(\zeta, \dot{\zeta})\, \mathrm{d}t &= d + \int_0^{t_1} h_\lambda(|\zeta|)\, \sigma(\zeta, \dot{\zeta})\, \mathrm{d}t + \int_{t_1}^1 h_\lambda(|\zeta|)\, \sigma(\zeta, \dot{\zeta})\, \mathrm{d}t \\ &\geq w_\lambda(\zeta(t_1)) + \int_{t_1}^1 h_\lambda(|\zeta|)\, |\dot{\zeta}|\, \mathrm{d}t \\ &\geq w_\lambda(\zeta(t_1)) + M_0\, |y - \zeta(t_1)| > M_0. \end{aligned}$$

Taking into account the definition of w_λ and the fact that the curve ζ joining y_0 to $y \notin B(0, R_0 - 1)$ is arbitrary, we deduce from the above computation the desired inequality. □

Proof of Theorem 3.12. In view of Lemma 3.13, it is just left to show (51). It indeed holds true in $B(0, R_0 - 1)$ because of (50) and $w_\lambda > 0$. If $y \in \overline{B(0, R_0)} \setminus B(0, R_0 - 1)$, then by Lemma 3.14, we have

$$w_\lambda(y) \geq M_0 \geq \sup_{\overline{B}(0, R_0)} \frac{1}{\lambda} U \geq \frac{1}{\lambda} U(y). \quad \square$$

4. Asymptotic analysis

We summarize the relevant output of the previous section in the following

Theorem 4.1. *We consider* $(x_0, p_0) \in \mathbb{R}^N \times \mathbb{R}^N$, *a constant* R_0 *satisfying* (48), (49), *a function* U *bounded from above in* $\overline{B}(0, R_0)$ *and less than or equal to zero in* $B(0, R_0 - 1)$, *any positive constant* λ. *Then the equation*

$$H(x_0, y, p_0, Du) = \overline{H}(x_0, p_0) \qquad \text{in } \mathbb{R}^M$$

admits a bounded Lipschitz-continuous subsolution and a locally Lipschitz-continuous supersolution, say w_λ, *satisfying* (51), (52)

We recall the notations $\overline{V} = \limsup^\# V^\varepsilon$, $\underline{V} = \liminf_\# V^\varepsilon$, where the V^ε are the value functions of problems (15)/(16). We consider a point $(x_0, t_0) \in \mathbb{R}^N \times (0, +\infty)$, and set

$$K_\delta = B(x_0, \delta) \times (t_0 - \delta, t_0 + \delta) \qquad \text{for } \delta < t_0. \tag{57}$$

We further consider a constant $R_0 > 0$ satisfying (48), (49). The next lemma, based on Theorem 3.12, will be of crucial importance. The entities $y_0 \in \mathcal{A}_0$ and d appearing in the statement are defined as in (47):

Lemma 4.2. *Let* ψ *be a strict supertangent to* $\overline{V}$ *at* (x_0, t_0) *such that* (x_0, t_0) *is the unique maximizer of* $\overline{V} - \psi$ *in* K_{δ_0}, *for some* $\delta_0 < t_0$. *Then, given any infinitesimal sequence* ε_j, *and* $\delta < \delta_0$, *we find a constant* $\rho_\delta > 0$ *and a family* w^j *of supersolutions to* $H(x_0, y, D\psi(x_0, t_0), Du) = \overline{H}(x_0, D\psi(x_0, t_0))$ *in* $\mathbb{R}^M$ *satisfying for* j *suitably large*

$$\varepsilon_j\, w^j \geq V^{\varepsilon_j} - \psi + \rho_\delta \qquad \text{in } \partial\big(K_\delta \times B(0, R_0)\big) \tag{58}$$

$$w^j = d + S(y_0, \cdot) \qquad \text{in a neighborhood } A_0 \text{ of } y_0, \tag{59}$$

where S *is the intrinsic critical distance, see Subsection 3.1, related to* $(x_0, D\psi(x_0, t_0))$.

Proof. By supertangency properties of ψ at (x_0, t_0), we find, for any $\delta < \delta_0$, a $\rho_\delta > 0$ with

$$\max_{\partial K_\delta} \big(\overline{V} - \psi\big) < -3\,\rho_\delta. \tag{60}$$

We fix a δ and define

$$U^\varepsilon(y) = \begin{cases} \max_{(x,t)\in\partial K_\delta} \{V^\varepsilon(x, y, t) - \psi(x, t) + \rho_\delta\} & \text{for } y \in B(0, R_0 - 1/2) \\ \max_{(x,t)\in K_\delta} \{V^\varepsilon(x, y, t) - \psi(x, t) + \rho_\delta\} & \text{for } y \in \mathbb{R}^M \setminus B(0, R_0 - 1/2). \end{cases}$$

Notice that the U^ε are continuous for any ε and locally equibounded, since the V^ε are locally equibounded in force of Proposition 2.6. To ease notations we set

$$U^j = U^{\varepsilon_j}.$$

Claim. *There is* $j_0 = j_0(R_0)$ *such that*

$$U^j \leq -\rho_\delta \qquad \text{in } B(0, R_0 - 1), \text{ for } j > j_0.$$

Were the claim false, there should be a subsequence y_j contained in $B(0, R_0 - 1)$ with

$$U^j(y_j) > -\rho_\delta.$$

The y_j converge, up to further extracting a subsequence, to some $\overline{y}$, and, being ε_j infinitesimal, we get

$$(\limsup{}^{\#} U^\varepsilon)(\overline{y}) \geq -\rho_\delta. \tag{61}$$

Moreover, there exists an infinitesimal sequence ε_i and elements z_i converging to $\overline{y}$ with

$$\lim_i U^{\varepsilon_i}(z_i) = (\limsup{}^{\#} U^\varepsilon)(\overline{y}),$$

at least for i large $z_i \in B(0, R_0 - 1/2)$, and by the very definition of U^ε in $B(0, R_0 - 1/2)$, we get

$$U^{\varepsilon_i}(z_i) = V^{\varepsilon_i}(x_i, z_i, t_i) - \psi(x_i, t_i) + \rho_\delta \qquad \text{for some } (x_i, t_i) \in \partial K_\delta,$$

up to extracting a subsequence, (x_i, t_i) converges to some $(\overline{x}, \overline{t}) \in \partial K_\delta$ so that by (60)

$$\begin{aligned}(\limsup{}^{\#} U^\varepsilon)(\overline{y}) &= \lim U^{\varepsilon_i}(z_i) = \lim \big[V^{\varepsilon_i}(x_i, z_i, t_i) - \psi(x_i, t_i) + \rho_\delta\big] \\ &\leq \overline{V}(\overline{x}, \overline{t}) - \psi(\overline{x}, \overline{t}) + \rho_\delta \leq -2\,\rho_\delta,\end{aligned}$$

which is in contradiction with (61). This ends the proof of the claim.

We are then in the position to apply Theorem 3.12 to any U^j, and get a supersolution w^j to $H(x_0, \cdot, D\psi(x_0, t_0), \cdot) = \overline{H}(x_0, D\psi(x_0, t_0))$, which satisfies, for $j > j_0$, the condition (59) and

$$\varepsilon_j\, w^j \geq U^j \qquad \text{in } \overline{B}(0, R_0).$$

Owing to the very definition of U^j, we derive from the latter inequality that

$$\varepsilon_j\, w^j(y) \geq V^{\varepsilon_j}(x, y, t) - \psi(x, t) + \rho_\delta$$

holds in

$$\partial K_\delta \times B(0, R_0) \cup K_\delta \times \partial B(0, R_0) = \partial\big(K_\delta \times B(0, R_0)\big).$$

This proves (58) and conclude the proof. □

We proceed establishing the asymptotic result for upper weak semilimit of the V^ε. The first part of the proof is a version, adapted to our setting, of perturbed test function method. We are going to use as correctors, depending on ε, the special supersolutions to cell equations constructed in Subsection 3.3 in the frame of Lemma 4.2. The argument of the second half of the proof concerning the behavior of the limit function at $t = 0$ makes a direct use of the material of Subsections 2.3, 2.4.

Theorem 4.3. *The function* $\overline{V} = \limsup^{\#} V^{\varepsilon}$ *is a subsolution to* (HJ) *satisfying*

$$\limsup_{\substack{(x,t)\to(x_0,0)\\ t>0}} \overline{V}(x,t) \leq \overline{u}_0(x_0) \qquad \textit{for any } x_0 \in \mathbb{R}^N. \tag{62}$$

Proof. Let (x_0, t_0) be a point in $\mathbb{R}^N \times (0, +\infty)$, and ψ a strict supertangent to $\overline{V}$ at (x_0, t_0) such that (x_0, t_0) is the unique maximizer of $\overline{V} - \psi$ in K_{δ_0}, for some $\delta_0 > 0$ (see (57) for the definition of K_δ).

By Proposition 2.7, we can find an infinitesimal sequence ε_j and (x_j, y_j, t_j) converging to (x_0, y_0, t_0), where y_0 is as in (47), with

$$\lim_j V^{\varepsilon_j}(x_j, y_j, t_j) = \overline{V}(x_0, t_0) = \psi(x_0, t_0). \tag{63}$$

We assume by contradiction

$$\psi_t(x_0, t_0) + \overline{H}(x_0, D\psi(x_0, t_0)) > 2\,\eta \tag{64}$$

for some positive η. We apply Lemma 2.9, about coercivity of H, to the bounded set

$$C := B(x_0, \delta_0) \times B(0, R_0) \times D\psi(K_{\delta_0}),$$

where R_0 satisfies (48), (49), and exploit that $\overline{H}$ is locally bounded to find $P > 0$ with

$$H(x, y, p, q) > \overline{H}(x, p) \qquad \text{for } (x, y, p) \in C,\, q \text{ with } |q| \geq P. \tag{65}$$

Applying the estimates (28) to $B(x_0, \delta_0) \times B(0, R_0)$ and (29), we find

$$\begin{aligned} &|H(x_0, y, D\psi(x_0, t_0), q) - H(x, y, p, q)| \leq \\ &L_0\,(|x - x_0|)(|D\psi(x_0, t_0)| + |q|) + \\ &\omega(|x - x_0|) + Q\,|D\psi(x_0, t_0) - p| \end{aligned} \tag{66}$$

for any $(x, y) \in B(x_0, \delta_0) \times B(0, R_0)$ and $(p, q) \in \mathbb{R}^N \times \mathbb{R}^M$, where ω is an uniform continuity modulus of ℓ in $B(x_0, \delta_0) \times B(0, R_0) \times A$, L_0 is as in **(H2)** and Q is an upper bound of $|f|$ in $B(x_0, \delta_0) \times B(0, R_0) \times A$.

Exploiting the continuity of $D\psi$, ψ_t, $\overline{H}$, we can determine, $\delta_0 > \delta > 0$ such that using (64), (66) with $q \in B(0, P)$ and p of the form $D\psi(x, t)$, we get

$$|H(x_0, y, D\psi(x_0, t_0), q) - H(x, y, D\psi(x, t), q)| < \eta \tag{67}$$

$$|D\psi(x, t) - D\psi(x_0, t_0)| < \eta \tag{68}$$

$$\psi_t(x, t) + \overline{H}(x, D\psi(x, t)) > 0 \tag{69}$$

for $(y, q) \in B(0, R_0) \times B(0, P)$, $(x, t) \in K_\delta$. By applying Lemma 4.2 to such a δ, we find a constant $\rho_\delta > 0$ and a family w^j of supersolutions to

$$H(x_0, y, p_0, D\psi(x_0, t_0), Du) = \overline{H}(x_0, D\psi(x_0, t_0)) \qquad \text{in } \mathbb{R}^M$$

with

$$\varepsilon_j\, w^j \geq V^{\varepsilon_j} - \psi + \rho_\delta \qquad \text{in } \partial\big(K_\delta \times B(0, R_0)\big) \tag{70}$$

$$w^j = d + S_0(y_0, \cdot) \qquad \text{in a neighborhood } A_0 \text{ of } y_0, \tag{71}$$

for j large enough, see (47) for the definition of d. We claim that the corrected test function $\psi + w^j$ satisfies

$$\psi_t(x,t) + H(x, y, D\psi(x,t), Dw^j) \geq 0$$

in $K_\delta \times B(0, R_0)$ in the viscosity sense. In fact, let ϕ be a subtangent to $\psi + w^j$ at some point $(x, y, t) \in K_\delta \times B(0, R_0)$, then

$$\phi_t(x, y, t) = \psi_t(x, t)$$
$$D_x\phi(x, y, t) = D\psi(x, t)$$

and so, to prove the claim, we have to show the inequality

$$\psi_t(x,t) + H(x, y, D\psi(x,t), D_y\phi(x, y, t)) \geq 0.$$

We have that

$$z \mapsto \phi(x, z, t)$$

is supertangent to w^j at y, which implies by the supersolution property of w^j

$$H(x_0, y, D\psi(x_0, t_0), D_y\phi(x, y, t)) \geq \overline{H}(x_0, D\psi(x_0, t_0))$$

If $|D_y\phi(x, y, t)| < P$ then by (64), (67) and (68)

$$\psi_t(x,t) + H(x, y, D\psi(x,t), D_y\phi(x, y, t)) \geq$$
$$\psi_t(x_0, t_0) - \eta + H(x_0, y, D\psi(x_0, t_0), D_y\phi(x, y, t)) - \eta \geq$$
$$\psi_t(x_0, t_0) + \overline{H}(x_0, D\psi(x_0, t_0)) - 2\eta \geq 0.$$

If instead $|D_y\phi(x, y, t)| \geq P$ then by (65), (69)

$$\psi_t(x,t) + H(x, y, D\psi(x,t), D_y\phi(x, y, t)) \geq$$
$$\psi_t(x,t) + \overline{H}(x, D\psi(x,t)) \geq 0.$$

The claim is then proved. For j large enough, the functions V^{ε_j}, $\psi + \varepsilon_j\, w^j - \rho_\delta$ are then subsolutions and supersolutions, respectively, to

$$u_t + H\left(x, y, D_x u, \frac{D_y u}{\varepsilon_j}\right) = 0$$

in $K_\delta \times B(0, R_0)$, then taking into account the boundary inequality (70), we can apply the comparison principle of Proposition 2.10 to the above equation to deduce

$$V^{\varepsilon_j} \le \psi + \varepsilon_j\, w^j - \rho_\delta \qquad \text{in } K_\delta \times B(0, R_0). \tag{72}$$

On the other side, let (x_j, y_j, t_j) be the sequence converging to (x_0, y_0, t_0) introduced in (63), then for j large $(x_j, y_j, t_j) \in K_\delta \times B(0, R_0)$, and $w^j(y_j) = d + S(y_0, y_j)$ by (71), so that

$$\lim_j \varepsilon_j\, w^j(y_j) = 0.$$

We therefore get

$$\lim_j \left[V^{\varepsilon_j}(x_j, y_j, t_j) - \psi(x_j, t_j) - \varepsilon_j\, w^j(y_j)\right] = \overline{V}(x_0, t_0) - \psi(x_0, t_0) = 0$$

which contradicts (72).

We proceed proving (62). We consider (x_n, t_n) converging to $(x_0, 0)$ such that $\overline{V}(x_n, t_n)$ admits limit. Our task is then to show

$$\lim_n \overline{V}(x_n, t_n) \le \overline{u}_0(x_0).$$

We find for any n an infinitesimal sequence ε^n_j and (x^n_j, y^n_j, t^n_j) converging to $(x_n, 0, t_n)$ with

$$\lim_j V^{\varepsilon^n_j}(x^n_j, y^n_j, t^n_j) = \overline{V}(x_n, t_n),$$

$0 \in \mathbb{R}^M$ is clearly an arbitrary choice, in view of Proposition 2.7. By applying a diagonal argument we find ε_n converging to 0 and (z_n, y_n, s_n) converging to $(x_0, 0, 0)$ with

$$\lim_n V^{\varepsilon_n}(z_n, y_n, s_n) = \lim_n \overline{V}(x_n, t_n) \tag{73}$$

$$\lim_n \frac{s_n}{\varepsilon_n} = +\infty. \tag{74}$$

Given $\delta > 0$, we denote by $\widetilde{y}$ a δ-minimizer of $y \mapsto u_0(x_0, y)$ in $\mathbb{R}^M$, see assumption **(H5)**. By applying Proposition 2.3, Lemma 2.4 and taking into account (74), we find for any n sufficiently large a trajectory (ξ_n, η_n) of (CD_ε), with $\varepsilon = \varepsilon_n$, corresponding to controls α_n and starting at (z_n, y_n), such that

$$(\xi_n, \eta_n) \text{ is contained in a compact subset independent of } n \text{ as } t \in [0, s_n/\varepsilon_n] \tag{75}$$

$$|\eta_n(s_n/\varepsilon_n) - \widetilde{y}| = \mathrm{O}(\varepsilon_n) \tag{76}$$

By using formulation (15) of minimization problem, we discover

$$V^{\varepsilon_n}(z_n, y_n, s_n) \leq \varepsilon_n \int_0^{\frac{s_n}{\varepsilon_n}} \ell(\xi_n(t), \eta_n(t), \alpha_n(t))\, \mathrm{d}t + u_0(\xi_n(s_n/\varepsilon_n), \eta_n(s_n/\varepsilon_n)),$$

where the integrand is estimated from above by a constant, say Q, independent of n, because of (75), therefore

$$V^{\varepsilon_n}(x_n, y_n, s_n) \leq Q\, s_n + u_0(\xi_n(s_n/\varepsilon_n), \eta_n(s_n/\varepsilon_n))$$

Owing to (4), (76), (73), and the fact that s_n is infinitesimal, we then get

$$\lim_n \overline{V}(x_n, t_n) = \lim_n V^{\varepsilon_n}(z_n, y_n, s_n) \leq u_0(x_0, \widetilde{y}) \leq \overline{u}_0(x_0) + \delta.$$

This concludes the proof because δ is arbitrary. □

The second main result concerns lower weak semilimit. Here we essentially exploit the existence of bounded Lipschitz-continuous subsolutions to cell equations established in Proposition 3.7 plus the coercivity of the V^ε proved in Proposition 2.8. The part of the proof about behavior of limit function at $t = 0$ is direct and not based on a PDE approach. We recall that $(\overline{u}_0)_\#$ stands for the lower semicontinuous envelope of $\overline{u}_0$, see Subsection 2.1 for definition.

Theorem 4.4. *The function* $\underline{V} = \liminf_\# V^\varepsilon$ *is a supersolution to* ($\overline{\mathrm{HJ}}$) *satisfying*

$$\liminf_{\substack{(x,t)\to(x_0,0)\\ t>0}} \underline{V}(x,t) \geq (\overline{u}_0)_\#(x_0) \qquad \textit{for any } x_0 \in \mathbb{R}^N. \tag{77}$$

Proof. Let (x_0, t_0) be a point in $\mathbb{R}^N \times (0, +\infty)$, and φ a strict subtangent to $\underline{V}$ at (x_0, t_0) such that (x_0, t_0) is the unique minimizer of $\underline{V} - \varphi$ in K_{δ_0}, for some $\delta_0 > 0$ (see (57) for the definition of K_δ). We assume by contradiction

$$\varphi_t(x_0, t_0) + \overline{H}(x_0, D\varphi(x_0, t_0)) < 0. \tag{78}$$

Given $\varepsilon > 0$, we can find by Proposition 2.8 about coercivity of value functions, $R_\varepsilon > 1$ satisfying

$$V^\varepsilon(x, y, t) > \sup_{K_{\delta_0}} \varphi + 1 \qquad \text{for } (x,t) \in K_{\delta_0},\ y \in \mathbb{R}^M \setminus B(0, R_\varepsilon). \tag{79}$$

We can also find, exploiting Proposition 3.7, a Lipschitz-continuous subsolution u to the cell problem

$$H(x_0, y, D\varphi(x_0, t_0), Du) = \overline{H}(x_0, D\varphi(x_0, t_0)) \qquad \text{in } \mathbb{R}^M \tag{80}$$

with

$$u(y) < 0 \qquad \text{for any } y \in \mathbb{R}^M. \tag{81}$$

By using estimate (28) on H, Lipschitz continuity of u, continuity of $\overline{H}$, $D\varphi$, φ_t and (78), (80) we can determine $0 < \delta < \delta_0$ such that $u + \varphi$ is subsolution to

$$w_t + H(x, y, D\varphi(x,t), Dw) = 0 \qquad \text{in } K_\delta \times \mathbb{R}^M.$$

Owing to strict subtangency property of φ, there is $1 > \rho > 0$ with

$$\underline{V} - \varphi > 2\rho \qquad \text{in } \partial K_\delta,$$

and, taking into account that $\underline{V}$ is the lower semilimit of the V^ε, we derive

$$V^\varepsilon - \varphi > \rho \qquad \text{in } \partial K_\delta \times B(0, R_\varepsilon)$$

for ε sufficiently small, which in turn implies by (81)

$$V^\varepsilon - \varphi - u > \rho \qquad \text{in } \partial K_\delta \times B(0, R_\varepsilon). \tag{82}$$

Owing to (79), (81), we also have

$$V^\varepsilon - \varphi - u > \rho \qquad \text{in } K_\delta \times \partial B(0, R_\varepsilon). \tag{83}$$

Since V^ε, $\varphi + \varepsilon u + \rho$ are supersolution and subsolution, respectively, to

$$w_t + H\left(x, y, D_x w, \frac{D_y w}{\varepsilon}\right) = 0$$

in $K_\delta \times B(0, R_0)$, the boundary conditions (82), (83) plus the comparison principle in Proposition 2.10 implies

$$V^\varepsilon \geq \varphi + \varepsilon u + \rho \qquad \text{in } K_\delta \times B(0, R_\varepsilon), \text{ for } \varepsilon \text{ small.} \tag{84}$$

On the other side, there is by Proposition 2.7 an infinitesimal sequence ε_j and a sequence (x_j, y_j, t_j) converging to $(x_0, 0, t_0)$ with

$$\lim_j V^{\varepsilon_j}(x_j, y_j, t_j) = \underline{V}(x_0, t_0)$$

and consequently

$$\lim_j \left[V^{\varepsilon_j}(x_j, y_j, t_j) - \varphi(x_j, t_j) - \varepsilon_j u(y_j)\right] = \overline{V}(x_0, t_0) - \varphi(x_0, t_0) = 0.$$

Taking into account that $R_\varepsilon > 1$ for any ε, and (x_j, y_j, t_j) are in $K_\delta \times B(0, 1)$ for j large, the last limit relation contradicts (84).

We proceed proving (77). We consider (x_n, t_n) converging to $(x_0, 0)$ such that $\underline{V}(x_n, t_n)$ admits limit, with the aim of showing

$$\lim_n \underline{V}(x_n, t_n) \geq (\overline{u}_0)_\#(x_0).$$

Arguing as in the final part of Theorem 4.3, we find an infinitesimal sequence ε_n and (z_n, y_n, s_n) converging to $(x_0, \widetilde{y}, 0)$, for some $\widetilde{y} \in \mathbb{R}^M$, with

$$\lim_n V^{\varepsilon_n}(z_n, y_n, s_n) = \lim_n \underline{V}(x_n, t_n).$$

We fix $\delta > 0$. Arguing as in second half of Proposition 2.6, see estimate (18), we determine a constant P_0 independent of n and trajectories (ξ_n, η_n) of the controlled dynamics starting at (z_n, y_n) with

$$V^{\varepsilon_n}(z_n, y_n, s_n) \geq P_0\, s_n + u_0(\xi_n(s_n/\varepsilon_n), \eta_n(s_n/\varepsilon_n)) - \delta \geq P_0\, s_n + \overline{u}_0(\xi_n(s_n/\varepsilon_n)) - \delta.$$

Since by the boundedness assumption on f

$$|\xi_n(s_n/\varepsilon_n) - z_n| \leq Q_0\, s_n,$$

we get at the limit

$$\lim_n \underline{V}(x_n, t_n) = \lim_n V^{\varepsilon_n}(z_n, y_n, s_n) \geq \liminf_n \overline{u}_0(\xi_n(s_n/\varepsilon_n)) - \delta \geq (\overline{u}_0)_\#(x_0) - \delta,$$

which gives the assertion since δ is arbitrary. $\square$

If $\overline{u}_0$ is continuous then we deduce from (62), (77) that

$$\limsup_{\substack{(x,t)\to(x_0,0)\\ t>0}} \overline{V}(x,t) \leq \liminf_{\substack{(x,t)\to(x_0,0)\\ t>0}} \underline{V}(x,t)$$

If, in addition, a comparison result holds for $(\overline{\mathrm{HJ}})$ we further obtain

$$\overline{V} \leq \underline{V} \qquad \text{in } (0,+\infty)\times\mathbb{R}^N$$

and so equality between $\overline{V}$ and $\underline{V}$, the converse inequality being obvious by the very definition of weak semilimits. In this case we thus have local uniform convergence of V^ε to $\overline{V} = \underline{V}$ in $\mathbb{R}^N \times \mathbb{R}^M \times (0, +\infty)$.

Appendix A. Facts from weak KAM theory

In this section we provide a reference frame for the weak KAM results we have exploited in Section 3 for the analysis of the Hamiltonian H_0. To comment on our approach, we recall that by freezing slow variables in H, we are able to define a new Hamiltonian H_0 which fulfills the basic assumption of Weak KAM theory. We derive from this some relevant consequences and properties which do not depend any more on the specific form of H_0, or its link with a control problem.

This is the reason why we transfer the above basic assumptions to an abstract Hamiltonian $F(y, q)$, and present Weak KAM facts in a general way. Some other considerations about the relationship between PDE, metric and control theoretical viewpoint are contained in Remark A.4.

Starting from F, we consider defined in $\mathbb{R}^M \times \mathbb{R}^M$ and the family of equations

$$F(y, Du) = b \qquad \text{in } \mathbb{R}^M, \text{ for } b \in \mathbb{R} \tag{85}$$

We assume F to satisfy

F is continuous in both variables;
F is convex in q;
$\lim_{|q|\to+\infty} \min_{y\in K} F(y, q) = +\infty$ for any compact subset K of $\mathbb{R}^M$.

We define the critical value of F as

$$c = \inf\{b \mid (85) \text{ has subsolutions in } \mathbb{R}^M\}. \tag{86}$$

Being the ambient space non-compact c can also be infinite. We assume in what follows

The critical value of F is finite.

We call supercritical a value b with $b \geq c$. By stability properties of viscosity (sub)solutions, subsolutions for the critical equation do exist. We derive from coercivity of F:

Lemma A.1. *Let b a supercritical value. The subsolutions to $F = b$ are locally equiLipschitz-continuous.*

We adopt the so-called metric method which is based on the definition of an intrinsic distance starting from the sublevels of the Hamiltonian for any supercritical value, see [14,16,17,15]. More precisely, the distance is obtained via minimization of a certain length functional, and the distances from any given initial point make up a class of fundamental subsolutions to (85) which will play a crucial role.

For any $b \geq c$ we set

$$Z_b(y) = \{q \mid F(y, q) \leq b\} \qquad y \in \mathbb{R}^M.$$

Owing to continuity, convexity and coercivity of F, we have:

Lemma A.2. *For any $b \geq c$, the multifunction $y \mapsto Z_b(y)$ takes convex compact values, it is in addition Hausdorff-continuous at any point y_0 where $\operatorname{int} Z_b(y_0) \neq \emptyset$ and upper semicontinuous elsewhere.*

We further set

$$\sigma_b(y, v) = \max\{q \cdot v \mid q \in Z_b(y)\} \qquad \text{for any } y, v \text{ in } \mathbb{R}^M,$$

namely the support function of $Z_b(y)$ at q, and define for any curve ξ defined in $[0, 1]$ the associated intrinsic length via

$$\int_0^1 \sigma_b(\xi, \dot{\xi})\, ds.$$

Notice that the above integral is invariant for orientation-preserving change of parameter, being the support function positively homogeneous and subadditive, as a length functional should be. Also notice that because of this invariance the choice of the interval [0, 1] is not restrictive. For any pair y_1, y_2 we define the intrinsic distance as

$$S_b(y_1, y_2) = \inf \left\{ \int_0^1 \sigma_b(\xi, \dot{\xi})\, \mathrm{d}s \mid \xi \ \text{ with } \xi(0) = y_1, \ \ \xi(1) = y_2 \right\}. \tag{87}$$

The intrinsic distance is finite for any supercritical value b.

Proposition A.3. *Given $b \geq c$, we have*

- **(i)** *a function u is a subsolution to $F = b$ if and only if*

$$u(y_2) - u(y_1) \leq S_b(y_1, y_2) \qquad \text{for any } y_1,\ y_2;$$

- **(ii)** *for any fixed y_0, the function $y \mapsto S_b(y_0, y)$ is subsolution to $F = b$ in $\mathbb{R}^M$ and solution in $\mathbb{R}^M \setminus \{y_0\}$;*
- **(iii)** *Let C, w be a closed set of $\mathbb{R}^M$ and a function defined in C satisfying*

$$w(y_2) - w(y_1) \leq S_b(y_1, y_2) \qquad \text{for any } y_1,\ y_2 \text{ in } C$$

then the function

$$y \mapsto \inf\{w(z) + S_b(z, y) \mid z \in C\}$$

is subsolution to $F = b$ in $\mathbb{R}^M$, solution in $\mathbb{R}^M \setminus C$ and equal to w in C.

Remark A.4. In the main body of the paper we have followed a dynamic programming approach to the control models in object. That is, we have associated to a family of minimization problems defined on the trajectories of a controlled dynamics the corresponding Hamilton–Jacobi–Bellman equation. The procedure outlined in the appendix is somehow the inverse. For a given abstract Hamiltonian F and a value b, some minimization problems, see (87), are introduced to represent (sub)solutions of the stationary Hamilton–Jacobi equation $F = b$.

It is natural to wonder on whether the circle can be closed. Namely, if we start from (87), is it possible to interpret $F = b$ as the corresponding Hamilton–Jacobi–Bellman equations ? The answer is straightly positive for $b > c$; if b is instead equal to the critical value the picture is more involved since one should take into account the Aubry set.

Indeed, if $b > c$, then the equation $H = b$ admits a strict subsolution, say φ, which can be made smooth via local mollification plus partition of unity, see [17]. Up to passing from $F(x, p)$ to $F(x, p) - D\varphi(x)$, we can therefore assume, without loss of generality, that

$$F(x, 0) < b \qquad \text{for any } x \in \mathbb{R}^M.$$

This implies that

$$\sigma_b(x,p) > 0 \qquad \text{for any } x,\ p \neq 0$$

so that

$$Z_b^*(x) = \{q \mid \sigma_b(x) \leq 1\}$$

is a convex compact valued continuous set valued function. In addition any $p \neq 0$ belongs to $Z_b^*(x)$, up to multiplication of a suitable positive constant. We introduce the differential inclusion

$$\dot{\xi} \in -Z_b^*(x) \tag{88}$$

and deduce, using the positive homogeneity of the support function, that the definition of intrinsic distance can be rephrased as

$$S_b(y_1,y_2) = \inf\{T \mid \exists\, \xi \ \text{ with } \xi(T-\cdot) \text{ solution of (88) such that } \xi(0)=y_1,\ \ \xi(T)=y_2\}.$$

We therefore see that the class of fundamental subsolutions (and solution in $\mathbb{R}^M \setminus \{y\}$) $S_b(y,\cdot)$, for $y \in \mathbb{R}^N$, are the minimum time functions with dynamics (88) and target $\{y\}$. The corresponding Bellman Hamiltonian is

$$\widetilde{F}(x,p) = \max_{q \in Z_b^*(x)} \{p \cdot q - 1\}$$

It is a standard result of convex analysis that the 1-sublevels of $\widetilde{F}$ and the b-sublevels of F coincide. Therefore the two equations $\widetilde{F} = 1$, which is the Hamilton–Jacobi–Bellman equation of the above minimum time problem, and $F = b$ are equivalent, in the sense that they have the same solutions and subsolutions.

In contrast to what happens when the ambient space is compact, namely $F = b$ admits solutions in the whole space if and only if $b = c$, in the noncompact case instead there are solutions for any supercritical equation. It is in fact enough that the intrinsic length is finite, as always is the case for supercritical values, to get a solution.

The construction of such a solution is in fact quite simple. One considers a sequence y_n with $|y_n|$ diverging and the functions

$$u_n = S_b(y_n,\cdot) - S_b(y_n,0).$$

By Lemma A.1 and Proposition A.3 the u_n are solutions except at y_n, are locally equiLipschitz-continuous, and also equibounded, since they vanish at 0. They therefore converge, up to a subsequence, by Ascoli Theorem. Having swept away the bad (in the sense of Proposition A.3 **(ii)**) points y_n to infinity, but kept the solution property by stability properties of viscosity solutions under uniform convergence, we see that the limit function is indeed the sought solution of $F = b$.

We say that a function u is a strict subsolution to $F = b$ in some open set B if

$$F(x,Du) \leq b - \delta \qquad \text{for some } \delta > 0, \text{ in the viscosity sense in } B.$$

The points satisfying the equivalent properties stated in the following proposition, make up the so-called Aubry set, denoted by $\mathcal{A}$, see [17].

Proposition A.5. *Given $y_0 \in \mathbb{R}^M$, the following three properties are equivalent:*

(i) *there exists a sequence of cycles ξ_n passing through y_0 and defined in* $[0, 1]$ *with*

$$\inf_n \int_0^1 \sigma_c(\xi_n, \dot{\xi}_n)\, \mathrm{d}s = 0 \quad \textit{and} \quad \inf_n \int_0^1 |\dot{\xi}_n|\, \mathrm{d}s > 0;$$

(ii) *$y \mapsto S_c(y_0, y)$ is solution to $F = c$ in the whole of $\mathbb{R}^M$;*
(iii) *if a function u is a strict critical subsolution in a neighborhood of y_0, then u cannot be subsolution to $F = c$ in $\mathbb{R}^M$.*

Notice that, in contrast with the compact case, even if the critical value is finite, the Aubry set can be empty for Hamiltonian defined in $\mathbb{R}^M \times \mathbb{R}^M$. We derive from Proposition A.5 **(iii)** adapting the same argument of Lemma 3.8:

Corollary A.6. *Assume that the Aubry set is empty, then for any bounded open set B of $\mathbb{R}^M$, there is a critical subsolution which is strict in B.*

We record for later use:

Proposition A.7. *Let B, b be an open bounded set of $\mathbb{R}^M$, and a critical value, respectively. Assume that the equation $F = b$ admits a strict subsolution in B, and denote by w a subsolution of $F = b$ in $\mathbb{R}^M$. Then the Dirichlet problem*

$$\begin{cases} F(y, Du) = b & \textit{in } B \\ u = w & \textit{on } \partial B \end{cases}$$

admits an unique solution u given by the formula

$$u(y) = \inf\{w(z) + S_b(z, y) \mid z \in \partial B\}.$$

We now consider a supercritical value b and a function $h : \mathbb{R}^M \to \mathbb{R}$ with

$$h \geq 1 \quad \text{in } \mathbb{R}^M \quad \text{and} \quad h(y) > 1 \Rightarrow F(y, 0) \leq b\,. \tag{89}$$

We define for any curve ξ in $[0, 1]$ the length functional

$$\int_0^1 h(\xi)\, \sigma_b(\xi, \dot{\xi})\, ds$$

and denote by S_b^h the corresponding distance obtained as the infimum of lengths of curves joining two given points of $\mathbb{R}^M$. We have

Proposition A.8. *Let b, h be a supercritical value for F and a function satisfying (89), respectively, then $S_b^h(z_0, \cdot)$ is a locally Lipschitz-continuous supersolution to (85) in $\mathbb{R}^M \setminus \{z_0\}$, for any $z_0 \in \mathbb{R}^M$.*

Proof. We fix z_0. For any $(y, v) \in \mathbb{R}^M \times \mathbb{R}^M$, $h(y)\,\sigma_b(y, v)$ is the support function of the b-sublevel of the Hamiltonian

$$(y, q) \mapsto F\left(y, \frac{q}{h(y)}\right) \tag{90}$$

and S_b^h is the corresponding intrinsic distance. According to Proposition A.3 **(ii)**, $w := S_b^h(z_0, \cdot)$ is subsolution to (85) in $\mathbb{R}^M$, and supersolution in $\mathbb{R}^M \setminus \{z_0\}$, with F replaced by the Hamiltonian in (90). Since the Hamiltonian in (90) keeps the coercivity property of F, this implies that w is locally Lipschitz-continuous in force of Lemma A.1.

Taking into account the supersolution information on w, we consider a subtangent ψ to w at a point y. If $h(y) = 1$ then

$$F(y, D\psi(y)) = F\left(y, \frac{D\psi(y)}{h(y)}\right) \ge b. \tag{91}$$

If instead $h(y) > 1$ then by (89) and convex character of F

$$\begin{aligned} F\left(y, \frac{D\psi(y)}{h(y)}\right) &= F\left(y, \left(1 - \frac{1}{h(y)}\right) 0 + \frac{D\psi(y)}{h(y)}\right) \\ &\le \frac{1}{h(y)} F(y, D\psi(y)) + \left(1 - \frac{1}{h(y)}\right) b \end{aligned}$$

and consequently

$$\frac{1}{h(y)} F(y, D\psi(y)) \ge b - \left(1 - \frac{1}{h(y)}\right) b = \frac{1}{h(y)} b. \tag{92}$$

Formulas (91), (92) provide the assertion. □

References

[1] Olivier Alvarez, Martino Bardi, Viscosity solutions methods for singular perturbations in deterministic and stochastic control, SIAM J. Control Optim. 40 (2001).

[2] Olivier Alvarez, Martino Bardi, Singular perturbations of nonlinear degenerate parabolic PDEs: a general convergence result, Arch. Ration. Mech. Anal. 170 (2003).

[3] Olivier Alvarez, Martino Bardi, Ergodicity, Stabilization, and Singular Perturbations for Bellman–Isaacs Equations, Mem. Amer. Math. Soc., vol. 204, 2010.

[4] Jean-Pierre Aubin, Arrigo Cellina, Differential Inclusions, Springer-Verlag, Berlin, 1984.

[5] Arstein Zvi, On the Value Function of Singularly Perturbed Optimal Control Systems, 43rd IEEE Conference on Decision and Control, vol. 41, 2004.

[6] Arstein Zvi, Three lectures on: control of coupled fast and slow dynamics, SADCO Course 2012, http://uma.ensta-paristech.fr/itn-sadco/talks/ravello2012/ravello2012artstein.pdf.

[7] Arstein Zvi, Vladimir Gaitsgory, The value function of singularly perturbed control systems, Appl. Math. Optim. 41 (2000).

[8] Martino Bardi, Italo Capuzzo Dolcetta, Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations, Birkhäuser, Boston, 2008.

[9] Guy Barles, First order Hamilton–Jacobi equations and applications, CIME Course 2011, http://php.math.unifi.it/users/cime/Courses/2011/04/201142-Notes.pdf.

[10] Guy Barles, Solutions de viscosité des équations de Hamilton–Jacobi, Springer, Berlin, 1994.

[11] Frank H. Clarke, Optimization and Nonsmooth Analysis, Society for Industrial and Applied Mathematics, Philadelphia, 1990.

[12] Lawrence C. Evans, The perturbed test function method for viscosity solutions of nonlinear PDEs, Proc. Soc. Edinburgh 111 (1989).

[13] Lawrence C. Evans, Periodic homogenization of certain fully nonlinear partial differential equations, Proc. Roy. Soc. Edinburgh 120 (1992).

[14] Albert Fathi, Weak KAM theorem in Lagrangian dynamics, ghost book, http://www.math.ens.fr/~baladi/fathidea.pdf.

[15] Albert Fathi, Weak KAM from a PDE point of view: viscosity solutions of the Hamilton–Jacobi equation and Aubry set, CANPDE Course 2011, https://www.ceremade.dauphine.fr/~pbernard/enseignement/m2/fathi.pdf.

[16] Albert Fathi, Antonio Siconolfi, Existence of C1 critical subsolutions of the Hamilton–Jacobi equation, Invent. Math. 155 (2004).

[17] Albert Fathi, Antonio Siconolfi, PDE aspects of Aubry–Mather theory for quasiconvex Hamiltonians, Calc. Var. Partial Differential Equations 22 (2005).

[18] Pierre-Louis Lions, Gerge Papanicolau, S.R. Srinivasa Varadhan, Homogenization of Hamilton–Jacobi equations, unpublished preprint (circa 1986).

[19] Jack Xin, Yifeng Yu, Periodic homogenization of inviscid G-equation for incompressible flows, Commun. Math. Sci. 8 (2010).