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Homotopy abelian L_∞ algebras and splitting property

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Abstract: *We introduce a property for L_∞ algebras, which we call the* splitting property*, and prove that an L_∞ algebra has the splitting property if and only if it is homotopy abelian.*

Introduction

It is nowadays well understood, see [9, 15, 16, 19] and references therein, that every deformation problem over a field $\mathbb{K}$ of characteristic zero is controlled by some dg Lie algebra via the associated functor of Maurer-Cartan elements modulo Gauge equivalence (to our knowledge, this was first explicitly stated in the paper [21]), with quasi-isomorphic dg Lie algebras controlling the same deformation problem. This has been made precise recently in the works of Lurie [12] and Pridham [22], whereas some first results were obtained in Manetti's paper [18]: to do so, we have to extend the usual category of deformation functors (functors of Artin rings satisfying Schlessinger's conditions [23]) to a certain ∞-category of *derived* deformation functors, also called formal moduli problems, and the associated homotopy category is proved to be equivalent to the homotopy category of dg Lie algebras (with respect to the usual model category structure, where weak equivalences are quasi-isomorphisms). For several purposes, it is convenient to replace the category of dg Lie algebras with the equivalent (at the level of homotopy categories) category of L_∞ algebras, or strong homotopy Lie algebras [11]: the latter is not a model category anymore, but it carries a structure of category of fibrant objects in the sense of K. Brown [5], see [3, Section 5], thus an associated homotopy theory. For instance, even if we want to stick with dg Lie algebras, it is convenient to enlarge the class of morphisms by considering the L_∞ ones, since every morphism in the homotopy category is represented by an L_∞ morphism, whereas in general it is

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only represented by a zig-zag of dg Lie algebra morphisms. As another motivation, the minimal model of a controlling dg Lie algebra, which is in general only an L_∞ algebra, is strictly related to the Kuranishi space of a given deformation problem.

An L_∞ algebra is *homotopy abelian* if it is weakly equivalent to an L_∞ algebra with trivial brackets. These are precisely the L_∞ algebras such that the associated (derived) deformation problem is unobstructed, see for instance [18, Section 7]: for this reason, it is important to have criteria telling us whether a given L_∞ algebra is homotopy abelian. The purpose of this paper is to investigate one such criterium.

This is more suggestively explained in Kontsevich and Soibelman's geometric language, *cf.* [16]: recall that a formal pointed dg manifold is a dg cocommutative coaugmented coalgebra such that the underlying coalgebra is isomorphic to $S(V)$, the symmetric coalgebra over some graded space V. Then an L_∞ algebra structure on a space L is the same as the data of a formal pointed dg manifold and an isomorphism of the underlying coalgebra with $S(L[1])$. Accordingly, we may think of L_∞ algebras as formal pointed dg manifolds with a choice of coordinates. The tangent complex (at the marked point) of a formal pointed dg manifold is the space of primitives of the coalgebra structure with the induced differential. The dg Lie algebra of vector fields is the graded Lie algebra of coderivations with the induced dg Lie algebra structure. Finally, evaluation at the marked point induces a morphism of complexes from the vector fields to the tangent complex. We say that an L_∞ algebra has the *splitting property* if the associated evaluation morphism admits a dg right inverse (this is clearly a property of the associated formal pointed dg manifold, that is, of the L_∞ isomorphism class of the L_∞ algebra). Our main result, Theorem 2.4, is that an L_∞ algebra has the spitting property if and only if it is homotopy abelian.

As an application, we review with a different proof one of the main results from [2]. This is done in Theorem 2.9, where we show how to associate an L_∞ algebra to a dg Lie algebra and a pre-Lie product inducing the bracket: since it is easy to show that this L_∞ algebra has the splitting property, we can conclude that it is homotopy abelian. As a particular case of interest, this applies to the Kodaira-Spencer dg Lie algebra controlling the deformations of a Kähler manifold, together with the pre-Lie product induced by the choice of a Kähler metric via the $(1,0)$-part of the associated Chern connection, *cf.* Remark 2.10 for more details: in this case, the construction from Theorem 2.9 recovers the L_∞ algebra introduced by Kapranov in [13], showing in particular that the latter is homotopy abelian over the field of complex numbers.

1 – Review of $L_\infty[1]$ algebras

We work over a field $\mathbb{K}$ of characteristic zero and with cohomologically $\mathbb{Z}$-graded vector spaces $V = \oplus_{i\in Z} V^i$ (in particular, differentials raise the degree by one). Given a homogeneous element $v \in V$, its degree shall be denoted by $|v|$. Given an integer $n \in \mathbb{Z}$, we denote by $V[n]$ the space V with the degrees shifted by

n, $V[n]^i = V^{i+n}$, and by $s^{-n}\colon V \to V[n]\colon v \to s^{-n}v$ the natural degree $(-n)$ isomorphism. Given spaces V, W, we denote by $\mathrm{Hom}(V,W) = \oplus_{i\in\mathbb{Z}} \mathrm{Hom}^i(V,W)$ the inner $\mathrm{Hom}(-,-)$ in the category of graded spaces. We denote by $V^{\otimes n}$ the n-th tensor power of V, that is, the tensor product of n-copies of V, and by $V^{\odot n}$, $V^{\wedge n}$ the n-th symmetric and exterior powers of V, that is, the spaces of coinvariants of $V^{\otimes n}$ under the symmetric (resp.: alternate) action of the symmetric group S_n, with the usual Koszul rule for twisting signs. By convention $V^{\otimes 0} = V^{\odot 0} = V^{\wedge 0} = \mathbb{K}$. Given homogeneous elements $v_1,\dots,v_n \in V$ and a permutation $\sigma \in S_n$, we shall denote by $\varepsilon(\sigma) = \varepsilon(\sigma; v_1,\dots,v_n)$ the corresponding Koszul sign. Finally, we denote by $S(p,q) \subset S_{p+q}$ the set of (p,q)-unshuffles, that is, permutations $\sigma \in S_{p+q}$ such that $\sigma(i) < \sigma(i+1)$ for $i \neq p$.

1.1 – Symmetric coalgebras and Nijenhuis-Richardson bracket

Given a graded space V, we denote by $S(V) = \oplus_{n\geq 0} V^{\odot n}$ the symmetric coalgebra over V: the cocommutative coproduct $\Delta\colon S(V) \to S(V)\otimes S(V)$ is the unshuffle coproduct

$$\Delta(v_1\odot\cdots\odot v_n)=\sum_{k=0}^{n}\sum_{\sigma\in S(k,n-k)}\varepsilon(\sigma)(v_{\sigma(1)}\odot\cdots\odot v_{\sigma(k)})\otimes(v_{\sigma(k+1)}\odot\cdots\odot v_{\sigma(n)}),$$

with the understanding that in the extremal cases $k = 0, n$ we replace the empty string by $1 \in \mathbb{K} = V^{\odot 0} \subset S(V)$. Recall that $S(V)$ is the cofree object over V in a certain category of coalgebras[1]: in particular, given graded spaces V and W, a morphism of graded coalgebras $F\colon S(V) \to S(W)$ is uniquely determined by its corestriction $pF = (f_1,\dots,f_n,\dots)\colon S(V) \to S(W) \to W$, where we denote by $p\colon S(W) \to W^{\odot 1} = W$ the natural projection and by $f_n\colon V^{\odot n} \to W$ the components of pF under the isomorphism $\mathrm{Hom}^0(S(V),W) = \prod_{n\geq 0}\mathrm{Hom}^0(V^{\odot n},W)$ (notice that we always have $F(1) = 1$, and thus $f_0 = 0$). We call the f_n the Taylor coefficients of F and $f_1\colon V \to W$ its linear part. Recall the following formal analog of the inverse function theorem, for a proof *cf.* [16].

Lemma 1.1. *A morphism $F\colon S(V) \to S(W)$ of graded coalgebras is an isomorphism (resp.: monomorphism, epimorphism) if and only if such is its linear part $f_1\colon V \to W$.*

We shall denote by $\mathrm{CE}(V) := \mathrm{Coder}(S(V))$ the graded Lie algebra of coderivations of $S(V)$: again, corestriction induces an isomorphism of graded spaces

$$\mathrm{CE}(V) \xrightarrow{\cong} \mathrm{Hom}(S(V),V) = \prod_{n\geq 0}\mathrm{Hom}(V^{\odot n},V)\colon Q \to pQ = (q_0,q_1,\dots,q_n,\dots)$$

[1] Namely, the counital, coaugmented locally conilpotent cocommutative coalgebras, *cf.* [16].

and we call the $q_n\colon V^{\odot n}\to V$ the Taylor coefficients of the coderivation Q, and $q_0\colon \mathbb{K}\to V$, $q_1\colon V\to V$ its constant and linear part respectively. We say that a coderivation is constant (resp.: linear) if only the constant (resp.: linear) Taylor coefficent is non-trivial. The inverse $\operatorname{Hom}(S(V),V)\to\operatorname{CE}(V)$ sends $(q_0,\dots,q_n,\dots)$ to the coderivation $Q\colon S(V)\to S(V)$

$$Q(v_1\odot\cdots\odot v_n)=\sum_{k=0}^{n}\sum_{\sigma\in S(k,n-k)}\varepsilon(\sigma)q_k(v_{\sigma(1)}\odot\cdots\odot v_{\sigma(k)})\odot v_{\sigma(k+1)}\odot\cdots\odot v_{\sigma(n)},$$

where once again we replace the empty string with $1\in V^{\odot 0}\subset S(V)$: in particular, $Q(1)=q_0(1)\in V^{\odot 1}\subset S(V)$. Given coderivations $Q,R\in\operatorname{CE}(V)$, we shall denote by $Q\bullet R\in\operatorname{CE}(V)$ the only coderivation with corestriction $pQR\colon S(V)\xrightarrow{R}S(V)\xrightarrow{Q}S(V)\xrightarrow{p}V$: in Taylor coefficients

$$p(Q\bullet R)=pQR=\left(q_1\bullet r_0, q_2\bullet r_0+q_1\bullet r_1,\dots,\sum_{k=0}^{n}q_{n+1-k}\bullet r_k,\dots\right),$$

where $-\bullet-\colon \operatorname{Hom}(V^{\odot n+1-k},V)\otimes\operatorname{Hom}(V^{\odot k},V)\to\operatorname{Hom}(V^{\odot n},V)$ is defined, according to the previous formulas, by

$$\begin{aligned}&(q_{n+1-k}\bullet r_k)(v_1\odot\cdots\odot v_n)\\&=\sum_{\sigma\in S(k,n-k)}\varepsilon(\sigma)q_{n+1-k}(r_k(v_{\sigma(1)}\odot\cdots\odot v_{\sigma(k)})\odot v_{\sigma(k+1)}\odot\cdots\odot v_{\sigma(n)}).\end{aligned}$$

It is not hard to check directly that $-\bullet-$ is a right pre-Lie product on $\operatorname{CE}(V)$, *cf.* Remark 2.8, which we call the *Nijenhuis-Richardson product*: the associated Lie bracket $[Q,R]=Q\bullet R-(-1)^{|Q||R|}R\bullet Q$ on $\operatorname{CE}(V)$, which we call the *Nijenhuis-Richardson bracket*, clearly coincides with the usual commutator of coderivations, *cf.* [21]. Finally, we shall denote by $\overline{\operatorname{CE}}(V)\subset\operatorname{CE}(V)$ the graded (right pre-)Lie subalgebra of coderivations Q with vanishing constant part $q_0=0$: this may be identified with the graded (right pre-)Lie algebra of coderivations of the reduced symmetric coalgebra $\overline{S}(V)=\oplus_{n\geq1}V^{\odot n}$ over V (with the reduced coproduct $\overline{\Delta}\colon\overline{S}(V)\to\overline{S}(V)\otimes\overline{S}(V)$ defined as before, but where the sum is taken from $k=1$ to $n-1$). Denoting by $i\colon\overline{\operatorname{CE}}(V)\to\operatorname{CE}(V)$ the inclusion and by $\operatorname{ev}(1)\colon\operatorname{CE}(V)\to V\colon Q\to q_0(1)$ evaluation at $1\in S(V)$, we have the following exact sequence of graded spaces

$$0\longrightarrow\overline{\operatorname{CE}}(V)\xrightarrow{\ i\ }\operatorname{CE}(V)\xrightarrow{\operatorname{ev}(1)}V\longrightarrow 0\,. \tag{1.1}$$

Given $v\in V$, we denote by $\sigma_v\colon\mathbb{K}\to V$ the map defined by $\sigma_v(1)=v$, and by $s_0(v):=(\sigma_v,0,\dots,0,\dots)$ the corresponding constant coderivation. Obviously

$s_0(-)\colon V\to \mathrm{CE}(V)$ is a right inverse to ev(1): furthermore, the image $s_0(V)\subset \mathrm{CE}(V)$ is an abelian Lie subalgebra, as follows immediately by the previous formula, which show more in general the following

LEMMA 1.2. *Given $v\in V$ and $Q=(q_0,\dots,q_n,\dots)\in \mathrm{CE}(V)$, the coderivation $[Q,s_0(v)]\in \mathrm{CE}(V)$ is explicitly given in Taylor coefficients by*

$$p[Q,s_0(v)]=([q_1,\sigma_v],\dots,[q_{n+1},\sigma_v],\dots)\,,$$
$$[q_{n+1},\sigma_v](v_1\odot\cdots\odot v_n)=q_{n+1}(v\odot v_1\odot\cdots\odot v_n)\quad \forall n\geq 0.$$

As a particular case, we notice that given $f\colon V\to V$, seen as a linear coderivation in $\mathrm{CE}(V)$, we have $[f,s_0(v)]=s_0(f(v))$ for all $v\in V$.

REMARK 1.3. For future reference, we briefly recall the definition of the tensor coalgebra and the Gerstenhaber bracket. The former is $T(V)=\oplus_{\geq 0}V^{\otimes n}$ with the usual deconcatenation coproduct

$$\Delta(v_1\otimes\cdots\otimes v_n)=\sum_{k=0}^{n}(v_1\otimes\cdots\otimes v_k)\otimes(v_{k+1}\otimes\cdots\otimes v_n).$$

As before, corestriction induces an isomorphism of graded spaces $\mathrm{Coder}(T(V))\cong \prod_{n\geq 0}\mathrm{Hom}(V^{\otimes n},V)\colon Q\to pQ=(q_0,\dots,q_n,\dots)$. The usual commutator of coderivations, called in this case the *Gerstenhaber bracket*, is induced by a right pre-Lie product $-\bullet-$, called the *Gerstenhaber product*, *cf.* [8]: this is explicitly described in Taylor coefficients by

$$-\bullet-\colon \mathrm{Hom}(V^{\otimes i},V)\otimes \mathrm{Hom}(V^{\otimes j},V)\to \mathrm{Hom}(V^{\otimes i+j-1},V)\colon f\otimes g\to f\bullet g,$$
$$(f\bullet g)(v_1\otimes\cdots\otimes v_{i+j-1})=\sum_{k=0}^{i-1}\pm_K f(v_1\otimes\cdots\otimes v_k\otimes g(v_{k+1}\otimes\cdots\otimes v_{k+j})\otimes\cdots\otimes v_{i+j-1}),$$

where $\pm_K$ is the appropriate Koszul sign, namely, $\pm_K=(-1)^{|g|(|v_1|+\cdots+|v_k|)}$.

1.2 – L_∞ and $L_\infty[1]$ algebras

DEFINITION 1.4. An $L_\infty[1]$-algebra structure $(V,Q)=(V,q_1,\dots,q_n,\dots)$ on a graded space V is a dg coalgebra structure $Q\in\overline{\mathrm{CE}}^1(V)$, $[Q,Q]=0$, on $S(V)$ with vanishing constant part. An $L_\infty[1]$-morphism

$$F=(f_1,\dots,f_n,\dots)\colon (V,q_1,\dots,q_n,\dots)\to(W,r_1,\dots,r_n,\dots)$$

of $L_\infty[1]$ algebras is a morphism of dg coalgebras $F\colon(S(V),Q)\to(S(W),R)$. An $L_\infty[1]$ morphism F is strict if $f_n=0$ for $n\geq 2$.

Given an $L_\infty[1]$ algebra structure $Q=(q_1,\ldots,q_n,\ldots)$ on a graded space V, the linear part q_1 is a differential on V. Furthermore, the exact sequence (1.1) becomes an exact sequence of complexes

$$0 \longrightarrow (\overline{\mathrm{CE}}(V),[Q,-]) \xrightarrow{\;i\;} (\mathrm{CE}(V),[Q,-]) \xrightarrow{\mathrm{ev}(1)} (V,q_1) \longrightarrow 0\,. \tag{1.2}$$

Definition 1.5. We call $(\mathrm{CE}(V),[Q,-])$ (resp.: $(\overline{\mathrm{CE}}(V),[Q,-])$) the (resp.: reduced) Chevalley-Eilenberg complex of (V,Q) with coefficients in itself, and we denote its cohomology by $H_{\mathrm{CE}}(V,V)$ (resp.: $H_{\overline{\mathrm{CE}}}(V,V)$). We call (V,q_1) the tangent complex of (V,Q), and we denote its cohomology by $H(V)$.

Given an $L_\infty[1]$ morphism $F\colon (V,Q)\to(W,R)$, its linear Taylor coefficient $f_1\colon (V,q_1)\to(W,r_1)$ is a dg morphism between the tangent complexes.

Definition 1.6. We say that F is a weak equivalence if $H(f_1)\colon H(V)\to H(W)$ is an isomorphism.

Remark 1.7. The category of $L_\infty[1]$ algebras and $L_\infty[1]$ morphisms is isomorphic, via décalage, to the one of L_∞ algebras and L_∞ morphisms as defined, for instance, in [11]. Given graded spaces V, W and an integer $n\geq 1$, décalage is the degree $(n-1)$-isomorphism

$$\begin{gathered}\mathrm{d\acute{e}c}\colon \mathrm{Hom}(V^{\otimes n},W)\to\mathrm{Hom}(V[1]^{\otimes n},W[1])\colon f\to\mathrm{d\acute{e}c}(f),\\ \mathrm{d\acute{e}c}(f)(s^{-1}v_1\otimes\cdots\otimes s^{-1}v_n)=(-1)^{n|f|+\sum_{k=1}^{n}(n-k)|v_k|}s^{-1}f(v_1\otimes\cdots\otimes v_n).\end{gathered}$$

It is well known, and in any case easy to verify, that it turns graded anti-symmetric maps into graded symmetric ones, that is, it restricts to a degree $(n-1)$-isomorphism

$$\mathrm{d\acute{e}c}\colon \mathrm{Hom}(V^{\wedge n},W)\to\mathrm{Hom}(V[1]^{\odot n},W[1])\colon f\to\mathrm{d\acute{e}c}(f).$$

Given a dg Lie algebra $(L,d,[-,-])$, there is an induced $L_\infty[1]$ algebra structure $Q=(q_1,q_2,0,\ldots,0,\ldots)=(\mathrm{d\acute{e}c}(d),\mathrm{d\acute{e}c}([-,-]),0,\ldots,0,\ldots)$ on the desuspension $L[1]$. More generally, an L_∞ algebra structure on a graded space L is the datum of a differential $d\colon L\to L$ and degree $(2-n)$ graded anti-symmetric brackets $[-,\ldots,-]\colon L^{\wedge n}\to L$, for all $n\geq 2$, such that $(\mathrm{d\acute{e}c}(d),\ldots,\mathrm{d\acute{e}c}([-,\ldots,-]),\ldots)$ are the Taylor coefficents of an $L_\infty[1]$ algebra structure on $L[1]$. Similarly, an L_∞ morphism $F=(f_1,\ldots,f_n,\ldots)\colon L\to M$ is the datum of degree $(1-n)$ graded anti-symmetric maps $f_n\colon L^{\wedge n}\to M$, $n\geq 1$, such that $(\mathrm{d\acute{e}c}(f_1),\ldots,\mathrm{d\acute{e}c}(f_n),\ldots)$ are the Taylor coefficents of an $L_\infty[1]$ morphism $L[1]\to M[1]$.

We close this section by recalling the homotopy transfer theorem and the theory of the minimal model.

Definition 1.8. A contraction from a complex (V, q_1) to a complex (W, r_1) is the data of dg morphisms $\imath_1\colon W \to V$, $\pi_1\colon V \to W$ such that $\pi_1\imath_1 = \mathrm{id}_W$ and a homotopy $K \in \mathrm{Hom}^{-1}(V, V)$ between $\imath_1\pi_1$ and id_V, $\imath_1\pi_1 - \mathrm{id}_V = q_1K + Kq_1$, such that $\pi_1K = K^2 = K\imath_1 = 0$.

Theorem 1.9. *Given an $L_\infty[1]$ algebra (V, Q) and a contraction $\imath_1\colon W \to V$, $\pi_1\colon V \to W$, $K \in \mathrm{Hom}^{-1}(V, V)$ from (V, q_1) to a complex (W, r_1), there is an induced $L_\infty[1]$ algebra structure $(W, r_1, \ldots, r_n, \ldots)$ on W with linear part r_1, and induced $L_\infty[1]$ morphisms $\imath_\infty = (\imath_1, \ldots, \imath_n, \ldots)\colon (W, R) \to (V, Q)$ and $\pi_\infty = (\pi_1, \ldots, \pi_n, \ldots)\colon (V, Q) \to (W, R)$ with linear parts $\imath_1$ and π_1 respectively.*

For a proof of the previous important theorem, we refer to [4, 16]. We remark that there are explicit recursive formulas, as well as formulas in terms of sums over rooted trees, for the induced $L_\infty[1]$ algebra structure and morphisms (although the computation might be hard in general).

Definition 1.10. An $L_\infty[1]$ algebra (H, R) is called minimal if $r_1 = 0$. Given an $L_\infty[1]$ algebra (V, Q), a minimal model is the datum of a minimal $L_\infty[1]$ algebra $(H, 0, r_2, \ldots, r_n, \ldots)$ and a weak equivalence $F\colon (H, R) \to (V, Q)$.

An explicit minimal model of (V, Q) can be constructed via homotopy transfer. Since we are working over a field, we may choose a direct sum decomposition $V = H \oplus B \oplus W$ such that the restriction of q_1 to W is an isomorphism $q_1\colon W \xrightarrow{\cong} B[1]$, and the restriction of q_1 to H is zero: in particular $H \cong H(V)$, and the inclusion $\imath_1\colon (H, 0) \to (V, q_1)$, the projection $\pi_1\colon (V, q_1) \to (H, 0)$ and the contracting homotopy $K\colon V \to B \xrightarrow{(q_1)^{-1}} W[-1] \to V[-1]$, where the first map is the projection and the last map is the inclusion, define a contraction from (V, q_1) to $(H, 0)$. Via homotopy transfer, there is an induced minimal $L_\infty[1]$ algebra structure $(H, 0, r_2, \ldots, r_n, \ldots)$ on H and a weak equivalence $\imath_\infty\colon (H, R) \to (V, Q)$ with linear part $\imath_1$.

Remark 1.11. There is also an induced $L_\infty[1]$ morphism $\pi_\infty\colon (V, Q) \to (H, R)$ with linear part π_1. According to Lemma 1.1, a weak equivalence between minimal $L_\infty[1]$ algebras has to be an $L_\infty[1]$ isomorphism: thus, given a second minimal model $F\colon H' \to V$ of V as in the previous definition, the composition $\pi_\infty F\colon H' \to V \to H$ is an $L_\infty[1]$ isomorphism. More in general, given a weak equivalence $V \to W$ between $L_\infty[1]$ algebras and minimal models H_V, H_W, constructed as before, the induced $H_V \to V \to W \to H_W$ is an $L_\infty[1]$ isomorphism. We conclude that the $L_\infty[1]$ isomorphism class of a minimal model is a homotopy invariant of an $L_\infty[1]$ algebra.

The inclusion $\imath_1\colon (H, 0) \to (V, q_1)$ extends to an isomorphism of complexes

$$g_1\colon (H, 0) \times (B \oplus W, q_1) \xrightarrow{\cong} (V, q_1).$$

Since $(B \oplus W, q_1)$ is acyclic, an obstruction theory argument proves the following structure theorem for $L_\infty[1]$ algebras, *cf.* [16], where the direct product of $L_\infty[1]$ algebras on the left hand side is defined in the obvious way.

THEOREM 1.12. *In the above situation, the morphism* $\imath_\infty\colon (H,R) \to (V,Q)$ *extends to an isomorphism* $G = (g_1,\dots,g_n,\dots)$ *of* $L_\infty[1]$ *algebras*

$$G\colon (H,0,r_2,\dots,r_n,\dots)\times(B\oplus W,q_1,0,\dots,0,\dots)\to(V,q_1,q_2,\dots,q_n,\dots).$$

1.3 – $L_\infty[1]$ algebras via higher derived brackets

Let $(M,d,[-,-])$ be a dg Lie algebra and $L \subset M$ a dg Lie subalgebra. The (desuspended) mapping cocone of the inclusion $i\colon L\to M$ is the complex $\mathrm{C}(i)[1] = L[1]\times M$, with the differential $r_1\colon\ \mathrm{C}(i)[1]\to\mathrm{C}(i)[1]\colon (s^{-1}l,m)\to(-s^{-1}(dl),dm-l)$. In the paper [7], there is shown how to define (via homotopy transfer) an $L_\infty[1]$ algebra structure $(\mathrm{C}(i)[1],r_1,\dots,r_n,\dots)$ so that $(\mathrm{C}(i)[1],R)$ is a model of the homotopy fiber if i in the homotopy category of $L_\infty[1]$ algebras: we refer to [7] for more details, and in particular to [7, Theorem 5.5] for explicit formulas.

Given a graded subspace $A\subset M$ such that $M = L\oplus A$, we shall denote by $P\colon M\to A$ the projection with kernel L and by $P^\perp := \mathrm{id}_M - P$. Since L is close under the differential, we see that $PdP^\perp = 0 \Rightarrow Pd = PdP$, and in particular that $Pd\colon A\to A$ is a differential on A. It is easy to exhibit a contraction $\imath_1\colon A\to\mathrm{C}(i)[1]$, $\pi_1\colon\ \mathrm{C}(i)[1]$, $K\colon\ \mathrm{C}(i)[1]\to\mathrm{C}(i)[1]$ from the mapping cocone $(\mathrm{C}(i)[1],r_1)$ to (A,Pd),

$$\imath_1(a) = (s^{-1}P^\perp(da),a),\quad \pi_1(s^{-1}l,m) = Pm,\quad K(s^{-1}l,m) = (s^{-1}P^\perp(m),0).$$

In particular, via homotopy transfer there is an induced $L_\infty[1]$ algebra structure on A and an induced $L_\infty[1]$ morphism $\imath_\infty\colon A\to\mathrm{C}(i)[1]$. These were explicitly computed in the paper [1] under the additional assumption that $A\subset M$ is a graded Lie subalgebra: in the particular case when $A\subset M$ is an abelian Lie subalgebra, we recover the $L_\infty[1]$ algebra structure induced via higher derived brackets as in Voronov's paper [25].

THEOREM 1.13. *In the above situation, if* $A\subset M$ *is an abelian Lie subalgebra, the* $L_\infty[1]$ *algebra structure* $(A,q_1 = Pd,q_2,\dots,q_n,\dots)$ *and the* $L_\infty[1]$ *morphism* $\imath_\infty = (\imath_1,\dots,\imath_n,\dots)\colon A\to\mathrm{C}(i)[1]$ *induced via homotopy transfer are explicitly given, for* $n\geq 2$, *by*

$$\begin{aligned} q_n(a_1\odot\cdots\odot a_n) &= P[\cdots[da_1,a_2],\dots,a_n],\\ \imath_n(a_1\odot\cdots\odot a_n) &= \left(s^{-1}P^\perp[\cdots[da_1,a_2]\cdots,a_n],\,0\right). \end{aligned}$$

For a proof, we refer to [1]. We point out that graded symmetry of the above q_n, $\imath_n$, follows from the hypothesis that A is abelian.

Remark 1.14. Composing $\imath_\infty$ with the projection $\mathrm{C}(i)[1] = L[1] \times M \to L[1]$ (which is a strict morphism of $L_\infty[1]$ algebras) yelds an $L_\infty[1]$ morphism $F = (f_1, \ldots, f_n, \ldots)\colon A \to L[1]$, $f_n(a_1 \odot \cdots \odot a_n) = s^{-1}P^{\perp}[\cdots[da_1, a_2]\cdots, a_n]$, fitting into a homotopy fiber sequence (*cf.* [6]) of $L_\infty[1]$ algebras

$$A \xrightarrow{F} L[1] \xrightarrow{i} M[1].$$

In particular, there is an associated long exact sequence in tangent cohomology, which may be identified (up to a shift in degrees) with the one associated to the short exact sequence of complexes

$$0 \to (L, d) \xrightarrow{i} (M, d) \xrightarrow{P} (A, Pd) \to 0.$$

A particular case when Theorem 1.13 applies is if we are given an $L_\infty[1]$ algebra (V, Q), and in the previous situation we take

$$M = (\mathrm{CE}(V), [Q, -], [-, -]), \quad L = (\overline{\mathrm{CE}}(V), [Q, -], [-, -]), \quad A = s_0(V),$$

where the bracket is the Nijenhuis-Richardson bracket and $s_0\colon V \to \mathrm{CE}(V)\colon v \to (\sigma_v, 0, \ldots, 0, \ldots)$ is defined as in Lemma 1.2. The $L_\infty[1]$ algebra structure on $s_0(V)$ induced via higher derived brackets pulls back to an $L_\infty[1]$ algebra structure on V. This is easily computed: the n-th higher derived bracket sends $s_0(v_1) \odot \cdots \odot s_0(v_n)$ to the constant part of $[\cdots[Q, s_0(v_1)]\cdots, s_0(v_n)]$, and according to Lemma 1.2 this is $s_0(q_n(v_1 \odot \cdots \odot v_n))$. We find that the induced $L_\infty[1]$ algebra structure on V is the original one (V, Q), in other words, every $L_\infty[1]$ algebra structure may be obtained via a higher derived brackets construction (this was already observed in [24]). The $L_\infty[1]$ morphism F as in Remark 1.14 can be computed using again Lemma 1.2, and turns out to be the natural $L_\infty[1]$ generalization of the adjoint morphism of a dg Lie algebra introduced in the paper [6].

Definition 1.15. Given an $L_\infty[1]$ algebra $(V, q_1, \ldots, q_n, \ldots)$, there is an $L_\infty[1]$ morphism $\mathrm{Ad}_\infty = (\mathrm{Ad}_1, \ldots, \mathrm{Ad}_k, \ldots)\colon V \to \overline{\mathrm{CE}}(V)[1]$, explicitly given by

$$\mathrm{Ad}_k(v_1 \odot \cdots \odot v_k) = s^{-1}\left(q_{1+k}(v_1 \odot \cdots \odot v_k \odot -), \cdots, q_{n+k}(v_1 \odot \cdots \odot v_k \odot -), \ldots\right)$$

fitting into a homotopy fiber sequence of $L_\infty[1]$ algebras

$$V \xrightarrow{\mathrm{Ad}_\infty} \overline{\mathrm{CE}}(V)[1] \xrightarrow{i} \mathrm{CE}(V)[1]\,.$$

We call Ad_∞ the $L_\infty[1]$ adjoint morphism of (V, Q).

2 – Homotopy abelian $L_\infty[1]$ algebras and splitting property

2.1 – Homotopy abelian $L_\infty[1]$ algebras

The aim of this section is to give several equivalent characterization of homotopy abelian $L_\infty[1]$ algebras.

Definition 2.1. An $L_\infty[1]$ algebra (V,Q) is abelian if $Q=(q_1,0,\ldots,0,\ldots)$ is a linear coderivation. An $L_\infty[1]$ algebra $(V,Q)=(V,q_1,\ldots,q_n,\ldots)$ is homotopy abelian if it is weakly equivalent to an abelian $L_\infty[1]$ algebra.

Given an $L_\infty[1]$ algebra (V,Q), recall the exact sequence of complexes from Definition 1.5

$$0 \longrightarrow (\overline{\mathrm{CE}}(V),[Q,-]) \xrightarrow{\ i\ } (\mathrm{CE}(V),[Q,-]) \xrightarrow{\mathrm{ev}(1)} (V,q_1) \longrightarrow 0\,.$$

Proposition 2.2. *The following are equivalent conditions:*

(1) *there is a dg right inverse* $s\colon (V,q_1)\to(\mathrm{CE}(V),[Q,-])$ *to* $\mathrm{ev}(1)$*;*
(2) *the induced* $H(\mathrm{ev}(1))\colon H_{\mathrm{CE}}(V,V)\to H(V)$ *is surjective;*
(3) *the induced* $H(i)\colon H_{\overline{\mathrm{CE}}}(V,V)\to H_{\mathrm{CE}}(V,V)$ *is injective;*
(4) *the induced* $H(\mathrm{Ad}_1)\colon H(V)\to H_{\overline{\mathrm{CE}}}(V,V)[1]$ *vanishes, where* Ad_1 *is the linear part of the adjoint* $L_\infty[1]$ *morphism* $\mathrm{Ad}_\infty\colon V\to\overline{\mathrm{CE}}(V)[1]$ *from Definition* 1.15.

Proof. (1)⇒(2) as $H(s)\colon H(V)\to H_{\mathrm{CE}}(V,V)$ will be a right inverse to $H(\mathrm{ev}(1))$. (2)⇔(3)⇔(4) by the long exact sequence in cohomology, which, as we already noticed in Remark 1.14, is the same as the one associated to the homotopy fiber sequence of $L_\infty[1]$ algebras

$$V \xrightarrow{\mathrm{Ad}_\infty} \overline{\mathrm{CE}}(V)[1] \xrightarrow{\ i\ } \mathrm{CE}(V)[1]\,.$$

Finally, assuming (4) we can find a primitive of $\mathrm{Ad}_1\colon x\to[Q,s_0(x)]-s_0(q_1(x))$ in the complex $\mathrm{Hom}(V,\overline{\mathrm{CE}}(V))$, where s_0 is as in Lemma 1.2, that is, we can find $\overline{s}\colon V\to\overline{\mathrm{CE}}(V)$ such that $[Q,\overline{s}(x)]-\overline{s}(q_1(x))=\mathrm{Ad}_1(x)=[Q,s_0(x)]-s_0(q_1(x))$, and then $s\colon V\to\mathrm{CE}(V)$, $s(x):=s_0(x)-\overline{s}(x)$, is a dg right inverse to $\mathrm{ev}(1)$. □

Definition 2.3. An $L_\infty[1]$ algebra $(V,q_1,\ldots,q_n,\ldots)$ has the splitting property if the equivalent conditions from the previous proposition are satisfied.

The main result of this paper is the following theorem.

THEOREM 2.4. *Given an $L_\infty[1]$-algebra (V,Q) the following are equivalent conditions:*

1. *(V,Q) is homotopy abelian;*
2. *the minimal model of (V,Q) has a trivial $L_\infty[1]$ algebra structure;*
3. *there is an $L_\infty[1]$ isomorphism*

$$F=(f_1=\mathrm{id}_V,f_2,\dots,f_n,\dots)\colon (V,q_1,0,\dots,0,\dots)\to(V,q_1,q_2,\dots,q_n,\dots)$$

with linear part the identity;

4. *(V,Q) has the splitting property;*
5. *The natural (Chevalley-Eilenberg) spectral sequence computing $H_{\mathrm{CE}}(V,V)$ degenerates at E_1.*

PROOF. We included item (5) to establish a connection with the paper [20], where there is proved (1)⇔(5): the most interesting aspect of this result is the fact, which is the main result from [20], that more in general degeneration at E_2 of the Chevalley-Eilenberg spectral sequence is equivalent to *formality* of the $L_\infty[1]$ algebra (V,Q), see [20] for more details. Recall that the Chevalley-Eilenberg spectral sequence is the cohomology spectral sequence associated to the exhaustive and complete filtration $F^p\,\mathrm{CE}(V)=\{R\in\mathrm{CE}(V)\ \text{ s.t. } r_i=0\ \text{ for } i<p\}$, $p\geq 0$, on the Chevalley-Eilenberg complex. It is well known that degeneration at E_1 of the spectral sequence is equivalent to injectivity of $H(F^{p+1}\,\mathrm{CE}(V))\to H(F^p\,\mathrm{CE}(V))$ for all $p\geq 0$, and since $H(F^0\,\mathrm{CE}(V))=H_{\mathrm{CE}}(V,V)$, $H(F^1\,\mathrm{CE}(V))=H_{\overline{\mathrm{CE}}}(V,V)$, this shows (5)⇒(4) (more precisely, this shows (5) ⇒ item (3) in Proposition 2.2). On the other hand, the implication (3)⇒(5) in the claim of the theorem is a straightforward consequence of invariance properties of the Chevalley-Eilenberg spectral sequence under $L_\infty[1]$ isomorphism, *cf.* [20, Proposition 5.5]. Having said so, we concentrate on the equivalence between the first four items: we should point out that only the equivalence (1)⇔(4) is actually interesting, while the other ones are rather straightforward (and in any case well known).

We show (1)⇒(2). According to Remark 1.11, a weak equivalence between (V,Q) and an abelian $L_\infty[1]$ algebra induces an $L_\infty[1]$ isomorphism between the minimal models: but the minimal model of an abelian $L_\infty[1]$ algebra has a trivial $L_\infty[1]$ algebra structure, and this property is stable under $L_\infty[1]$ isomorphisms[2].

The implication (2)⇒(3) follows from Theorem 1.12: the desired F is the composition of the $L_\infty[1]$ isomorphism

$$G\colon (H,0,0,\dots,0,\dots)\times(B\oplus W,q_1,0,\dots,0,\dots)\to(V,q_1,q_2,\dots,q_n,\dots)$$

[2] This is a trivial fact: notice however how *abelian* $L_\infty[1]$ algebra structures fail to be stable under $L_\infty[1]$ isomorphisms.

as in Theorem 1.12 and the isomorphism of complexes $(g_1)^{-1}$, seen as a strict $L_\infty[1]$ isomorphism

$$(g_1)^{-1}\colon (V, q_1, 0, \ldots, 0, \ldots) \to (H, 0, 0, \ldots, 0, \ldots) \times (B \oplus W, q_1, 0, \ldots, 0, \ldots).$$

To prove (3)⇒(4) we consider the induced isomorphism of dg Lie algebras (where the bracket is the Nijenhuis-Richardson bracket)

$$F - F^{-1}\colon (\mathrm{CE}(V), [q_1, -], [-, -]) \to (\mathrm{CE}(V), [Q, -], [-, -]).$$

A dg right inverse $s\colon V \to \mathrm{CE}(V)$ to $\mathrm{ev}(1)$ is given by $s(v) := F s_0(v) F^{-1}$, where $s_0\colon V \to \mathrm{CE}(V)$ is defined as in Lemma 1.2: in fact

$$[Q, s(v)] = [Q, F s_0(v) F^{-1}] = F[q_1, s_0(v)]F^{-1} = F s_0(q_1(v)) F^{-1} = s(q_1(v)).$$

Finally, to prove (4)⇒(1) we consider (recall the results from Section 1.3) the mapping cocone $(\mathrm{C}(i)[1], r_1, \ldots, r_n, \ldots)$ of the inclusion $i\colon \overline{\mathrm{CE}}(V) \to \mathrm{CE}(V)$. Given a dg right inverse $s\colon V \to \mathrm{CE}(V)$ to $\mathrm{ev}(1)$, there is an induced quasi-isomorphism of complexes $\widetilde{s}\colon V \to \mathrm{C}(i)[1] = \overline{\mathrm{CE}}(V)[1] \times \mathrm{CE}(V)\colon v \to (0, s(v))$, and since the explicit formulas from [7, Theorem 5.5] show that the higher Taylor coefficients r_n, $n \geq 2$, vanish on $\mathrm{Im}\,(\widetilde{s}^{\odot n}) \subset \mathrm{CE}(V)^{\odot n} \subset \mathrm{C}(i)[1]^{\odot n}$, we see that $\widetilde{s}\colon (V, q_1, 0, \ldots, 0, \ldots) \to (\mathrm{C}(i)[1], r_1, r_2, \ldots, r_n, \ldots)$ is a strict morphism of $L_\infty[1]$ algebras. The diagram of $L_\infty[1]$ algebras and weak equivalences

$$(V, Q) \xrightarrow{\imath_\infty} (\mathrm{C}(i)[1], R) \xleftarrow{\widetilde{s}} (V, q_1)\ ,$$

where $\imath_\infty$ is as in Theorem 1.13, implies that (V, Q) is homotopy abelian: in fact, we could also observe that the composition of $\widetilde{s}$ and the $L_\infty[1]$ morphism $\pi_\infty\colon (\mathrm{C}(i)[1], R) \to (V, Q)$ induced via homotopy transfer yelds an $L_\infty[1]$ isomorphism $F\colon (V, q_1) \to (V, Q)$ as in item (3) (although, in general it will be hard to compute π_∞ explicitly). □

Remark 2.5. In light of the results discussed in Section 1.3, the implication (4)⇒(1) from the above theorem is just a particular case of [10, Lemma 2.1].

It seems worthwhile to reformulate Theorem 2.4 in the particular case when $V = L[1]$ is the $L_\infty[1]$ algebra associated to a dg Lie algebra $(L, d, [-, -])$.

Corollary 2.6. *A dg Lie algebra $(L, d, [-, -])$ is homotopy abelian if and only if there exist degree $(-n)$ linear maps $\varphi_n\colon L \to \mathrm{Hom}(L^{\wedge n}, L)$, for all $n \geq 1$, with the property that for all $x, y \in L$*

$$d\,(\varphi_1(x)(y)) + \varphi_1(dx)(y) + (-1)^{|x|}\varphi_1(x)(dy) = [x, y],$$

and for all $x, y_1, \ldots, y_n \in L$, $n \geq 2$,

$$\begin{aligned}
&d\left(\varphi_n(x)(y_1\wedge\cdots\wedge y_n)\right) - (-1)^n\varphi_n(dx)(y_1\wedge\cdots\wedge y_n)\\
&\quad - (-1)^{n+|x|}\varphi_n(x)(dy_1\wedge\cdots\wedge y_n) - \cdots - (-1)^{n+|x|+\cdots+|y_{n-1}|}\varphi_n(x)(y_1\wedge\cdots\wedge dy_n)\\
&= \sum_{\sigma\in S(2,n-2)} \chi(\sigma)\varphi_{n-1}(x)\left([y_{\sigma(1)},y_{\sigma(2)}]\wedge\cdots\wedge y_{\sigma(n)}\right)\\
&\quad - (-1)^n \sum_{\sigma\in S(n-1,1)} \chi(\sigma)\left[\varphi_{n-1}(x)\left(y_{\sigma(1)}\wedge\cdots\wedge y_{\sigma(n-1)}\right), y_{\sigma(n)}\right],
\end{aligned}$$

where we denote by $\chi(\sigma) = \chi(\sigma; y_1, \ldots, y_n)$ *the alternate Koszul sign.*

Proof. We denote by $Q = (q_1, q_2, 0, \ldots, 0, \ldots) = (\text{déc}(d), \text{déc}([-,-]), 0, \ldots, 0, \ldots)$ the induced $L_\infty[1]$ algebra structure on $L[1]$. A family of degree $(-n)$ maps $\varphi_n\colon L \to \text{Hom}(L^{\wedge n}, L)$, $n \geq 1$, corresponds, under décalage (*cf.* Remark 1.7), to a family of degree zero maps $\nabla_n\colon L[1] \to \text{Hom}(L[1]^{\odot n}, L[1])$, $n \geq 1$: explicitly,

$$\begin{aligned}
&\nabla_n(s^{-1}x)(s^{-1}y_1\odot\cdots\odot s^{-1}y_n)\\
&= (-1)^{n|x|+(n-1)|y_1|+\cdots+|y_{n-1}|}s^{-1}\varphi_n(x)(y_1\wedge\cdots\wedge y_n).
\end{aligned}$$

A tedious but straightforward verification shows that the maps φ_n satisfy the relations in the claim of the proposition if and only if the maps ∇_n are the components of a dg right inverse

$$(L[1], q_1) \to (\text{CE}(L[1]), [Q,-])\colon s^{-1}x \to \left(\sigma_{s^{-1}x}, \nabla_1(s^{-1}x), \ldots, \nabla_n(s^{-1}x), \ldots\right)$$

to the evaluation $\text{ev}(1)\colon (\text{CE}(L[1]), [Q,-]) \to (L[1], q_1)$. □

2.2 – Formality of Kapranov's brackets on pre-Lie algebras

As an application of Theorem 2.4, we present a different proof of one of the main results from [2].

Definition 2.7 (Cf. [17]). A left pre-Lie algebra $(L, \triangleright)$ is a graded space L with a bilinear product $\triangleright\colon L^{\otimes 2} \to L$ such that the associator

$$A\colon L^{\otimes 3} \to L\colon x\otimes y\otimes z \to A(x,y,z) = (x\triangleright y)\triangleright z - x\triangleright(y\triangleright z)$$

is graded symmetric in the first two arguments, *i.e.*, $A(x,y,z) = (-1)^{|x||y|}A(y,x,z)$. In particular, this implies that the commutator $[-,-]\colon L^{\wedge 2} \to L$,

$$[x,y] = x\triangleright y - (-1)^{|x||y|}y\triangleright x = \nabla_x(y) - (-1)^{|x||y|}\nabla_y(x),$$

is a Lie bracket on L. We denote by $\nabla\colon L \to \mathrm{End}(L)\colon x \to \nabla_x(y) = x \triangleright y$ the left adjoint morphism: then the left pre-Lie identity is the same as

$$[\nabla_x, \nabla_y] = \nabla_{[x,y]} \qquad \text{for all } x, y \in L, \tag{2.1}$$

where on the right hand side we have the induced bracket on L and on the left hand side the usual commutator in $\mathrm{End}(L)$.

REMARK 2.8. A graded right pre-Lie algebra is defined similarly: in this case we require the associator to be graded symmetric in the *last* two arguments, or equivalently that the *right* adjoint morphism *anti*-commutes with the brackets.

THEOREM 2.9. *Let $(L, \triangleright)$ be a graded left pre-Lie algebra. Given a derivation $d \in \mathrm{Der}(L, [\cdot,\cdot])$ of the associated graded Lie algebra, the maps $\Phi(d)_n\colon L^{\otimes n} \to L$, $n \geq 1$, defined by the recursion*

$$\begin{cases} \Phi(d)_1 = d \\ \Phi(d)_2(x \otimes y) = -[d, \nabla_x](y) + \nabla_{dx}(y) \\ \Phi(d)_{n+1}(x \otimes y_1 \otimes \cdots \otimes y_n) = -[\Phi(d)_n, \nabla_x](y_1 \otimes \cdots \otimes y_n) \qquad \text{for } n \geq 2, \end{cases}$$

where on the right hand side we take the Gerstenhaber bracket in $\mathrm{Coder}(T(L))$ *(cf. Remark 1.3) and we consider $\nabla_x\colon L \to L$ as a linear coderivation, are all graded symmetric. The induced correspondence*

$$\Phi(-)\colon\ \mathrm{Der}(L, [-,-]) \to \overline{\mathrm{CE}}(L)\colon d \to \Phi(d) = (\Phi(d)_1, \ldots, \Phi(d)_n, \ldots)$$

is a morphism of graded Lie algebras, where we consider the commutator bracket on $\mathrm{Der}(L, [-,-])$ *and the Nijenhuis-Richardson bracket on* $\overline{\mathrm{CE}}(L)$. *Finally, given a dg Lie algebra structure $d \in \mathrm{Der}^1(L, [-,-])$, $[d,d] = 0$, on $(L, [-,-])$, $\Phi(d)$ is a homotopy abelian $L_\infty[1]$ algebra structure on L.*

PROOF. We can rewrite $\Phi(d)_2$ as

$$\Phi(d)_2(x \otimes y) = -[d, \nabla_x](y) + \nabla_{dx}(y) = -d(x \triangleright y) + d(x) \triangleright y + (-1)^{|x||d|} x \triangleright d(y).$$

In other words, $\Phi(d)_2$ measures how far is d from satisfying the Leibniz rule with respect to the pre-Lie product $\triangleright$. It is then cleat that

$$\Phi(d)_2(x \otimes y) - (-1)^{|x||y|}\Phi(d)_2(y \otimes x) = -d[x,y] + [dx, y] + (-1)^{|x||d|}[x, dy] = 0,$$

since d is supposed to be a derivation with respect to $[-,-]$. We can therefore consider $\Phi(d)_2$ as an element in $\mathrm{CE}(L)$.

The recursive definition implies that $\Phi(d)_3$ is graded symmetric in the last two arguments, so it suffices to show that it is also graded symmetric in the first two. Using graded symmetry of $\Phi(d)_2$ and Lemma 1.2, we can write

$$\Phi(d)_3(x \otimes y \otimes z) = -[\Phi(d)_2, \nabla_x](y \odot z) = -[[\Phi(d)_2, \nabla_x], \sigma_y](z)$$

where now the bracket is the Nijenhuis-Richardson one in CE(L). Graded symmetry of $\Phi(d)_3$ follows from the following computation in the graded Lie algebra CE(L), where we use the Jacobi identity, the recursive definition, Lemma 1.2 and the pre-Lie identity (2.1).

$$\begin{aligned}
&[[\Phi(d)_2,\nabla_x],\sigma_y]-(-1)^{|x||y|}[[\Phi(d)_2,\nabla_y],\sigma_x]\\
&\qquad=[\Phi(d)_2,[\nabla_x,\sigma_y]]+(-1)^{|x||y|}[[\Phi(d)_2,\sigma_y],\nabla_x]\\
&\qquad\quad-(-1)^{|x||y|}[\Phi(d)_2,[\nabla_y,\sigma_x]]-[[\Phi(d)_2,\sigma_x],\nabla_y]\\
&\qquad=[\Phi(d)_2,\sigma_{\nabla_x(y)}]+(-1)^{|x||y|}[\nabla_{dy}-[d,\nabla_y],\nabla_x]\\
&\qquad\quad-(-1)^{|x||y|}[\Phi(d)_2,\sigma_{\nabla_y(x)}]-[\nabla_{dx}-[d,\nabla_x],\nabla_y]\\
&\qquad=\nabla_{d\nabla_x(y)}-[d,\nabla_{\nabla_x(y)}]+(-1)^{|x||y|}[\nabla_{dy},\nabla_x]\\
&\qquad\quad-(-1)^{|x||y|}[[d,\nabla_y],\nabla_x]-(-1)^{|x||y|}\nabla_{d\nabla_y(x)}\\
&\qquad\quad+(-1)^{|x||y|}[d,\nabla_{\nabla_y(x)}]-[\nabla_{dx},\nabla_y]+[[d,\nabla_x],\nabla_y]\\
&\qquad=\nabla_{d[x,y]}-[d,\nabla_{[x,y]}]+(-1)^{|x||y|}\nabla_{[dy,x]}-\nabla_{[dx,y]}\\
&\qquad\quad+[[d,\nabla_x],\nabla_y]+(-1)^{|x||d|}[\nabla_x,[d,\nabla_y]]\\
&\qquad=\nabla_{d[x,y]-[dx,y]-(-1)^{|x||d|}[x,dy]}+[d,[\nabla_x,\nabla_y]-\nabla_{[x,y]}]=0\,.
\end{aligned}$$

Finally, suppose inductively that we have shown that $\Phi(d)_i$ is graded symmetric for all $1\le i<n$, $n\ge 4$. As before, by graded symmetry of $\Phi(d)_{n-1}$ and Lemma 1.2 we can write

$$\begin{aligned}
\Phi(d)_n(x\otimes y\otimes z_1\otimes\cdots\otimes z_{n-2})&=-[\Phi(d)_{n-1},\nabla_x](y\odot z_1\odot\cdots\odot z_{n-2})\\
&=-[[\Phi(d)_{n-1},\nabla_x],\sigma_y](z_1\odot\cdots\odot z_{n-2}).
\end{aligned}$$

The above computation can be repeated, and it actually becomes simpler, due to the simpler form of the recursion: we obtain

$$[[\Phi(d)_{n-1},\nabla_x],\sigma_y]-(-1)^{|x||y|}[[\Phi(d)_{n-1},\nabla_y],\sigma_x]=[\Phi(d)_{n-2},[\nabla_x,\nabla_y]-\nabla_{[x,y]}]=0.$$

This finishes the proof that the maps $\Phi(d)_n$ are graded symmetric. In particular, $\Phi(d)=(\Phi(d)_1,\ldots,\Phi(d)_n,\ldots)$ is a well defined element of $\overline{\mathrm{CE}}(L)$.

We denote by s the correspondence

$$s\colon L\to \mathrm{CE}(L),\qquad s(x)=(\sigma_x,\nabla_x,0,\ldots,0,\ldots).$$

We notice that $\Phi(d)\in\overline{\mathrm{CE}}(L)\subset \mathrm{CE}(L)$ could be characterized more compactly as the only coderivation such that

$$[\Phi(d),s(x)]=s(dx)\qquad \text{for all } x\in L. \tag{2.2}$$

In fact, expanding the above identity in Taylor coefficients and using Lemma 1.2 we exactly recover the recursive definition for the $\Phi(d)_n$ in the claim of the theorem. Given derivations $d_1, d_2 \in \mathrm{Der}(L, [-,-])$, equation (2.2) and the Jacoby identity show that

$$[[\Phi(d_1), \Phi(d_2)], s(x)] = s([d_1, d_2](x)) \qquad \text{for all } x \in L.$$

and by the above, this proves $[\Phi(d_1), \Phi(d_2)] = \Phi([d_1, d_2])$. Finally, it is now clear that if $d \in \mathrm{Der}^1(L, [-,-])$, $[d,d] = 0$, is a dg Lie algebra structure on $(L, [-,-])$ then $[\Phi(d), \Phi(d)] = \Phi([d,d]) = 0$, that is, $\Phi(d)$ is an $L_\infty[1]$ algebra structure on L, and it is also clear that it has the splitting property, hence it is homotopy abelian, since (2.2) exhibits $s\colon (L, d) \to (\mathrm{CE}(L), [\Phi(d), -])$ as an explicit dg right inverse to $\mathrm{ev}(1)$. □

Remark 2.10. As briefly mentioned in the introduction, the previous construction of $L_\infty[1]$ brackets has an interesting instance in Kähler geometry. We denote by $KS_X = \mathcal{A}^{0,*}(T_X)$ the Kodaira-Spencer dg Lie algebra of a Kähler manifold X, see [19]. Given a Kähler metric, there is unique connection (the *Chern connection*) on the tangent bundle T_X which is compatible with both the complex structure and the metric, see for instance [14, Proposition 4.9]. Denoting by ∇ the $(1,0)$-part of the Chern connection and by $\nabla_\alpha(\beta) \in \mathcal{A}^{0,p+q}(T_X)$, $\alpha \in \mathcal{A}^{0,p}(T_X)$, $\beta \in \mathcal{A}^{0,q}(T_X)$, the corresponding covariant derivative (combined with the wedge product of forms), then the Kähler property implies that ∇ is torsion free, see [14, Proposition 7.19], that is, the commutator $[\alpha, \beta] = \nabla_\alpha(\beta) - (-1)^{pq}\nabla_\beta(\alpha)$ is the usual bracket on KS_X. Knowing this fact, the identity (2.1) follows by holomorphic Cartan formulas and easy manipulations, see [2, Section 4] for more details: in other words, $\triangleright\colon KS_X \otimes KS_X \to KS_X$, $\alpha \triangleright \beta := \nabla_\alpha(\beta)$, is a left pre-Lie product inducing the usual bracket on KS_X. We are in the hypotheses of Theorem 2.9, with $d = \overline{\partial}$ the usual Dolbeault differential. Finally, there is proven in [2, Theorem 4.2] that the induced $L_\infty[1]$ algebra structure $\Phi(\overline{\partial})$ on $\mathcal{A}^{0,*}(T_X)$ coincides with the one introduced by Kapranov in the paper [13], showing in particular that the latter is homotopy abelian over the field $\mathbb{C}$ of complex numbers.

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