The Stochastic Properties of ℓ^1-Regularized Spherical Gaussian Fields

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Abstract

Convex regularization techniques are now widespread tools for solving inverse problems in a variety of different frameworks. In some cases, the functions to be reconstructed are naturally viewed as realizations from random processes; an important question is thus whether such regularization techniques preserve the properties of the underlying probability measures. We focus here on a case which has produced a very lively debate in the cosmological literature, namely Gaussian and isotropic spherical random fields, and we prove that Gaussianity and isotropy are not conserved in general under convex regularization over a Fourier dictionary, such as the orthonormal system of spherical harmonics.

Keywords and Phrases: Random Fields, Spherical Harmonics, Gaussianity, Isotropy, Convex Regularization, Inpainting.

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1 Introduction

Let $T : M \to \mathbb{R}$ be a square integrable function on a manifold M, and assume that the following is observed:

$$T^{obs} := \mathcal{A}T + n \ , \tag{1}$$

where $\mathcal{A} : L^2(M) \to L^2(M)$ is a linear operator that can represent, for instance, a blurring convolution or a mask setting some values of the function T to zero, while $n : M \to \mathbb{R}$ denotes observational noise. Recovering T from observations on T^{obs} is a standard example of a linear inverse problem, and it is now classical to pursue a solution for (1) by means of convex/ℓ^1-regularization procedures. More precisely, we can proceed by postulating that the signal T can be sparsely represented in a given dictionary Ψ, e.g., $T = \Psi\alpha_0$ where the vector α_0 is assumed to be sparse in a suitable sense, and then solving the ℓ^1-regularized problem

$$\alpha^{reg} := \arg\min_{\alpha} \left\{ \lambda \left\| \alpha \right\|_{\ell^1} + \frac{1}{2} \left\| T^{obs} - \mathcal{A}\Psi\alpha \right\|^2_{L^2(S^2)} \right\} \ , \tag{2}$$

which can be viewed for instance as a form of Basis Pursuit Denoising [8] or a variation of the Lasso algorithm introduced in the statistical literature by [27]. Often the following alternative formulation is considered:

$$\alpha^{reg} := \arg\min_{\alpha} \{\|\alpha\|_{\ell^1}\} \text{ subject to } \left\|T^{obs} - \mathcal{A}\Psi\alpha\right\|_{L^2(S^2)} \leq \varepsilon \ , \tag{3}$$

for some $\varepsilon > 0$; it is known that there exist a bijection $\lambda \leftrightarrow \varepsilon$ such that (2) and (3) have the same solution [24]. Many authors have worked on related regularization problems over the last two decades - a very incomplete list includes [18], [9], [12], [15], [17], [29], see for instance [24], Chapter 7 for more references and a global overview. These results are also connected to the rapidly growing literature on compressive sensing, see, e.g., [10, 5, 7, 19, 20].

In many applied fields, it is customary to view T as the realization of a random field, and the reconstruction problems (2) and (3) are usually just the first steps before statistical data analysis (e.g., estimation and testing) is implemented. In other words, T is viewed as a random object on a probability space $(\Omega, \Im, P)$, $T(\omega, x) := T : \Omega \times M \to \mathbb{R}$; hence it becomes important to verify that $T^{reg} := \Psi\alpha^{reg}$, $T^{reg} : \Omega \times M \to \mathbb{R}$, is close to T in a meaningful probabilistic sense. For instance, let M be a homogeneous space of a compact group $\mathcal{G}$; a natural question is the following:

Problem 1 *Assume that the field T is Gaussian and isotropic, e.g., the probability laws of $T(.)$ and $T^g(.) = T(g.)$ are the same for all $g \in \mathcal{G}$. Is the random field T^{reg} Gaussian and isotropic?*

The scenario we have described fits very well, for instance, the current situation in the Cosmological literature, in particular in the field of Cosmic Microwave Background (CMB) data analysis. The latter can be viewed as a snapshot picture of the Universe at the so-called age of recombination, e.g. 3.7×10^5 years after the Big Bang (some 13 billion years ago); its observation has been made possible by satellite experiments such as WMAP [6] and Planck [21], which have raised an enormous amount of theoretical and applied interest. CMB is usually viewed as a single realization of a Gaussian isotropic random field on the sphere, e.g., $M = S^2$ and $\mathcal{G} = SO(3)$, the group of rotations in $\mathbb{R}^3$; observations are corrupted by observational noise and various forms of convolutions (e.g., instrumental beams, masked regions) and a number of efforts have been devoted to solving (1) under these circumstances. In this setting, algorithms such as (2) and (3) have been widely proposed, in some cases (see e.g., [1], [11, 24, 25] and the references therein) taking as a dictionary the orthonormal system of spherical harmonics $\{Y_{\ell m}\}$. As well-known, the latter are eigenfunctions of the spherical Laplacians $\Delta_{S^2} Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}$ and lead to the spectral representation

$$T(x) = \sum_{\ell=0}^{\infty} T_\ell(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x) \ .$$

Under Gaussianity and isotropy, this representation holds in the mean square sense and the random coefficients are Gaussian and independent with variance $E a_{\ell m} \overline{a}_{\ell' m'} = C_\ell \delta_\ell^{\ell'} \delta_m^{m'}$, the sequence $\{C_\ell\}$ representing the angular power spectrum (see for instance [16]). A very lively debate has then developed, to ascertain whether in this setting the solution to the issue raised in Problem (1) should allow for a positive or negative answer, see for instance [25], [26] and the references therein. In particular, the recent paper [13] provides from an astrophysical perspective some arguments and a large amount of numerical evidence to suggest that isotropy will not hold in general.

The purpose of this paper is to address this question from a mathematical point of view. To this aim, we will focus on idealistic circumstances where $\mathcal{A}$ is just the identity operator and noise n is set identically to zero, so that T and T^{obs} coincide. Of course, under these circumstances the inverse problem would not really arise: however for our aims these assumptions suffice, as we will show that even in this idealistic setting stochastic properties such as Gaussianity and isotropy are not preserved by regularization according to (2) or (3).

1.1 Statement of the main results

To establish our results, we shall first reformulate (2) and (3) in a form which is more directly amenable to stochastic analysis; in particular, we shall show that:

Proposition 2 *Let T be a Gaussian isotropic spherical random field, and denote by Ψ the spherical harmonic dictionary. Then for any given $\delta, \varepsilon > 0$, there exist a positive $\lambda = \lambda(\delta, \varepsilon)$ such that the solution*

$$\alpha^{reg} := \arg\min_{\alpha} \left\{ \lambda(\delta, \varepsilon) \|\alpha\|_{\ell^1} + \frac{1}{2} \|T - \Psi\alpha\|^2_{L^2(S^2)} \right\} \tag{4}$$

satisfies

$$\Pr\left\{ \|T - \Psi\alpha^{reg}\|_{L^2(S^2)} \leq \varepsilon \right\} \geq 1 - \delta \ . \tag{5}$$

The previous result is stating that for a suitable choice of λ the solution to (2) satisfies the constraint in (3) with probability arbitrarily close to one, so that the two problems can be seen as substantially equivalent in a stochastic setting. Let us now write

$$T^{reg}_{\delta,\varepsilon}(x) := \sum_{\ell m} a^{reg}_{\ell m}(\delta, \varepsilon) Y_{\ell m}(x) = \sum_{\ell} T^{reg}_{\ell;\delta,\varepsilon}(x) \ .$$

The main claim of this paper is the following

Theorem 3 *The random fields $T^{reg}_{\delta,\varepsilon}(.)$ are necessarily anisotropic and nonGaussian, for any (arbitrarily small but positive) values of δ, ε.*

To make this claim more concrete, we shall also focus on the normalized fourth-moment

$$\kappa_\ell(\theta, \phi) := \frac{E\{T^{reg}_\ell(\theta, \phi)^4\}}{(E\{T^{reg}_\ell(0, 0)^2\})^2} \ ,$$

which of course should be constant for all $(\theta, \phi) \in S^2$ under isotropy, and identically equal to 3 under Gaussianity. On the contrary, we will provide an analytic expression for the value of $\kappa_\ell(\theta, \phi)$ at the North Pole $N : (\theta, \phi) = (0, 0)$, as a function of the angular power spectrum C_ℓ and the penalization parameter $\lambda(\delta, \varepsilon)$. In particular, for the so-called complex-valued regularization procedure (to be defined below), we shall show that

Theorem 4 *As $\lambda/\sqrt{C_\ell} \to \infty$, we have*

$$\lim_{\lambda/\sqrt{C_\ell} \to \infty} \frac{\log k_\ell(0, 0)}{\lambda^2/C_\ell} = 1.$$

Because the sequence C_ℓ is summable, the previous result entails that the kurtosis of the field diverges exponentially at the North Pole as $\ell \to \infty$, showing an extremely nonGaussian behavior at high frequencies. Under the same setting, we shall show that

Theorem 5 *As $\lambda/\sqrt{C_\ell} \to \infty$, we have that*

$$\lim_{\ell \to \infty} \frac{\kappa_\ell(\theta, \phi)}{\kappa_\ell(0, 0) P^4_\ell(\cos\theta)} = 1 \ , \textit{ for all } (\theta, \phi) \in S^2 \ ,$$

$P_\ell(.)$ denoting the usual Legendre polynomial.

The latter result entails that the so-called trispectrum of the random field is not constant over the sphere, as required by isotropy, but it rather exhibits anisotropic oscillations. Under the so-called real-norm regularization procedure (to be defined later), the asymptotic behavior is slightly different, but anisotropy remains and the oscillations of the trispectrum can again be predicted analytically, see below.

One heuristic intuition behind these results can be summarized as follows. To understand the relationship between convex regularization and isotropy, it can be convenient to view a problem like (2) as resulting from the maximization of a Bayesian posterior distribution on the spherical harmonic coefficients $a_{\ell m}$, assuming a Laplacian/Exponential prior on these coefficients. We can now recall some earlier results from [2] (see also [3, 16, 4]), showing that a random field generated by sampling such independent non-Gaussian coefficients is necessarily anisotropic; it can then be natural to conjecture that this implicit anisotropy in the prior fields will persist in the regularized maps. However, while this interpretation led us to conjecture the results of this paper, it should be noted that it plays no role in the arguments that follow. We refer again to [13] for further discussion on these issues and for a large set of numerical results.

The plan of the paper is as follows: in Section 2, we discuss regularized estimates in a stochastic setting, and we establish Proposition 2; in Section 3, we prove that regularized fields with the spherical harmonics dictionary are necessarily anisotropic and nonGaussian, while in Section 4 the trispectra and their asymptotic behavior are studied. Some final remarks are collected in Section 5.

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2 ℓ^1-Regularized Random Fields

As motivated in the Introduction, we wish to consider the ℓ_1 minimization problem

$$\{a_{\ell m}^{reg}\} = \arg \min_{\{a_{\ell m}\}} \left\{ \lambda \sum_{\ell m} |a_{\ell m}| + \frac{1}{2} \left\| T^{obs} - \sum_{\ell m} a_{\ell m} Y_{\ell m} \right\|_{L^2(S^2)}^2 \right\} , \tag{6}$$

where as usual

$$T^{obs} = \sum_{\ell m} a_{\ell m}^{obs} Y_{\ell m} . \tag{7}$$

and $\|.\|_{L^2(S^2)}$ denotes the L^2 -norm for functions on the sphere , e.g.,

$$\left\| T^{obs} - \sum_{\ell m} a_{\ell m} Y_{\ell m} \right\|_{L^2(S^2)}^2 = \int_{S^2} \left| T^{obs} - \sum_{\ell m} a_{\ell m} Y_{\ell m} \right|^2 dx = \sum_{\ell m} \left| a_{\ell m}^{obs} - a_{\ell m} \right|^2 .$$

In equations (6) and (7), we take as usual $\{Y_{\ell m}\}$ to denote complex-valued spherical harmonics, so that $Y_{\ell m} = (-1)^m \overline{Y}_{\ell ,-m}$, the bar denoting complex conjugations, and $|.|$ the complex modulus

$|a_{\ell m}| := \sqrt{[\mathrm{Re}(a_{\ell m})]^2 + [\mathrm{Im}(a_{\ell m})]^2}$; we label this case as the *complex-valued regularization scheme*. As an alternative, an orthonormal expansion into a real-valued basis can be obtained by simply taking

$$T^{obs} = \sum_{\ell m} a_{\ell m}^{obs;\mathcal{R}} Y_{\ell m}^{\mathcal{R}} \ , \tag{8}$$

where $a_{\ell 0}^{obs;\mathcal{R}} = a_{\ell 0}^{obs}$, $Y_{\ell 0}^{\mathcal{R}} = Y_{\ell 0}$,

$$a_{\ell m}^{obs;\mathcal{R}} = \sqrt{2}\mathrm{Re}(a_{\ell m}^{obs}) \text{ for } m > 0 \ , \ a_{\ell m}^{\mathcal{R}} = -\sqrt{2}\mathrm{Im}(a_{\ell,-m}^{obs}) \text{ for } m < 0 \ ,$$

and

$$Y_{\ell m}^{\mathcal{R}} = \sqrt{2}\mathrm{Re}(Y_{\ell m}) \text{ for } m > 0 \ , \ Y_{\ell m}^{\mathcal{R}} = \sqrt{2}\mathrm{Im}(Y_{\ell,-m}) \text{ for } m < 0 \ .$$

We are then led to the *real-valued regularization scheme*

$$\left\{a_{\ell m}^{reg*}\right\} = \arg\min_{\{a_{\ell m}\}} \left\{ \lambda \sum_{\ell m} \left|a_{\ell m}^{\mathcal{R}}\right| + \frac{1}{2} \sum_{\ell m} \left| a_{\ell m}^{obs;\mathcal{R}} - a_{\ell m}^{\mathcal{R}} \right|^2 \right\} \ , \tag{9}$$

where $|.|$ is standard absolute value for real numbers. We shall consider both schemes in what follows.

The following two lemmas are standard, but nevertheless we report their straightforward proofs for completeness. We shall use below the standard polar coordinates for complex-valued random variables

$$a_{\ell m} = \rho_{\ell m} \exp(i\psi_{\ell m}) \ ,$$

$$\rho_{\ell m} := \sqrt{[(\mathrm{Re}(a_{\ell m})]^2 + [\mathrm{Im}(a_{\ell m})]^2} \ , \ \psi_{\ell m} := \arctan \frac{\mathrm{Re}(a_{\ell m})}{\mathrm{Im}(a_{\ell m})} \ .$$

Also, we denote by $|x|_+$ the positive part of the real number x.

Lemma 6 *If T^{obs} is Gaussian and isotropic, we have that for $m \neq 0$, $a_{\ell m}^{obs} = \rho_{\ell m}^{obs} \exp(i\psi_{\ell m}^{obs})$, where $\psi_{\ell m}^{obs} \sim U[0, 2\pi]$ and the density of $\rho_{\ell m}^{obs}$ is given by*

$$\Pr\left\{\rho_{\ell m}^{obs} \le R\right\} = \int_0^R f_{\rho;\ell}(r)dr \ ; \ f_{\rho;\ell}(r) = 2\frac{r}{C_\ell}\exp\{-\frac{r^2}{C_\ell}\} \ .$$

Proof. It suffices to notice that

$$\Pr\left\{\rho_{\ell m}^{obs} \le R\right\} = \Pr\left\{\left(\rho_{\ell m}^{obs}\right)^2 \le R^2\right\}$$

$$= \Pr\left\{\frac{1}{2}\frac{[\mathrm{Re}(a_{\ell m}^{obs})]^2 + [\mathrm{Im}(a_{\ell m}^{obs})]^2}{C_\ell/2} \le \frac{R^2}{C_\ell}\right\}$$

$$= \Pr\left\{\frac{\chi_2^2}{2} \le \frac{R^2}{C_\ell}\right\} = 1 - \exp(\frac{R^2}{C_\ell}) \ ,$$

where χ_2^2 is a Chi-squared with 2 degrees of freedom. Whence the result follows from differentiation. ■

Lemma 7 *The solution to (6) is provided by*

$$a_{\ell m}^{reg} := \mathrm{Re}(a_{\ell m}^{reg}) + i\mathrm{Im}(a_{\ell m}^{reg}) \ ,$$

where, for $\ell = 1, ..., \ell_{\max}$ *and* $m = -\ell, ..., \ell$ *we have*

$$\mathrm{Re}(a_{\ell m}^{reg}) = \left|\rho_{\ell m}^{obs} - \lambda\right|_+ \cos\psi_{\ell m}^{obs} \ , \ \mathrm{Im}(a_{\ell m}^{reg}) = \left|\rho_{\ell m}^{obs} - \lambda\right|_+ \sin\psi_{\ell m}^{obs} \ .$$

Proof. We can rewrite (6) as

$$\begin{aligned}\{a_{\ell m}^{reg}\} &= \arg\min_{\{a_{\ell m}\}} \left\{ \lambda \sum_{\ell m} |a_{\ell m}| + \frac{1}{2} \left\| T^{obs} - \sum_{\ell m} a_{\ell m} Y_{\ell m} \right\|_{L^2(S^2)}^2 \right\} \\ &= \arg\min_{\{a_{\ell m}\}} \left\{ \lambda \sum_{\ell m} |a_{\ell m}| + \frac{1}{2} \sum_{\ell m} \left| a_{\ell m}^{obs} - a_{\ell m} \right|^2 \right\}. \qquad (10)\end{aligned}$$

We can hence rewrite

$$\begin{aligned}&\frac{1}{2}\sum_{\ell m} \left|a_{\ell m}^{obs} - a_{\ell m}\right|^2 + \lambda \sum_{\ell m} |a_{\ell m}| \\ &= \frac{1}{2}\sum_{\ell m} \left\{ [\mathrm{Re}(a_{\ell m}^{obs}) - \mathrm{Re}(a_{\ell m})]^2 + [\mathrm{Im}(a_{\ell m}^{obs}) - \mathrm{Im}(a_{\ell m})]^2 \right\} + \lambda \sum_{\ell m} |a_{\ell m}| \\ &= \sum_{\ell m} v_{\ell m} \ ,\end{aligned}$$

where

$$\begin{aligned}v_{\ell m} &= \frac{1}{2}(\rho_{\ell m}^{obs})^2 \cos^2\psi_{\ell m}^{obs} + \frac{1}{2}\rho_{\ell m}^2 \cos^2\psi_{\ell m} - \rho_{\ell m}^{obs}\rho_{\ell m} \cos\psi_{\ell m}^{obs} \cos\psi_{\ell m} \\ &+ \frac{1}{2}(\rho_{\ell m}^{obs})^2 \sin^2\psi_{\ell m}^{obs} + \frac{1}{2}\rho_{\ell m}^2 \sin^2\psi_{\ell m} - \rho_{\ell m}^{obs}\rho_{\ell m} \sin\psi_{\ell m}^{obs} \sin\psi_{\ell m} + \lambda\rho_{\ell m} \\ &= \frac{1}{2}(\rho_{\ell m}^{obs})^2 + \frac{1}{2}\rho_{\ell m}^2 - \rho_{\ell m}^{obs}\rho_{\ell m} \cos(\psi_{\ell m}^{obs} - \psi_{\ell m}) + \lambda\rho_{\ell m} \ .\end{aligned}$$

It is obvious that for any value of $\rho_{\ell m}^{obs}$, $v_{\ell m}$ is minimized at $\psi_{\ell m}^{obs} = \psi_{\ell m}$; we are then led to the following optimization problem:

$$\min_{\{\rho_{\ell m}\}} \sum_{\ell m} \phi(\rho_{\ell m}^{obs}, \rho_{\ell m}; \lambda) \ ,$$

where

$$\phi(\rho_{\ell m}^{obs}, \rho_{\ell m}; \lambda) = \frac{1}{2}(\rho_{\ell m}^{obs})^2 + \frac{1}{2}\rho_{\ell m}^2 - \rho_{\ell m}^{obs}\rho_{\ell m} + \lambda\rho_{\ell m} \ .$$

Now it is standard calculus to show that, for $\lambda > \rho_{\ell m}^{obs}$

$$\frac{d\phi}{d\rho_{\ell m}} = \rho_{\ell m} + \lambda - \rho_{\ell m}^{obs} > 0 \ ,$$

while for $\lambda \leq \rho_{\ell m}^{obs}$

$$\frac{d\phi}{d\rho_{\ell m}} = \rho_{\ell m} + \lambda - \rho_{\ell m}^{obs} = 0 \Longleftrightarrow \rho_{\ell m} = \rho_{\ell m}^{obs} - \lambda \ .$$

The solution now follows immediately, given the global convexity of the function $\phi(.)$. ■

Remark 8 *The previous Lemma provides a simple generalization of the very well-known fact that soft-thresholding provides the solution to (2) when the dictionary is represented by an orthonormal basis of real valued functions. In particular, for (9) the solution is immediately seen to be given by*

$$a_{\ell m}^{reg*} = sign(a_{\ell m}^{obs;\mathcal{R}}) \left| |a_{\ell m}^{obs;\mathcal{R}}| - \lambda \right|_{+} \ .$$

It is important to note that the solution for the coefficient corresponding to $m = 0$ is exactly the same for both regularization schemes.

The next result shows that, in the simplified circumstances we are considering and for a suitable choice of the penalization parameter λ, the reconstruction error can be made arbitrarily small, with probability arbitrarily close to one. For finite variance fields we have $E\{T^2\} = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell < \infty$. To enforce this condition and for notational convenience, in what follows we shall assume that for all ℓ, $0 < C_\ell \leq K\ell^{-\alpha}$, for some $K > 0$ and $\alpha > 2$. This condition is minimal and fulfilled for instance by all physically relevant models for CMB radiation.

Proposition 9 *Under the above conditions, for all $\delta, \varepsilon > 0$ there exists a positive $\lambda = \lambda(\delta, \varepsilon)$ such that*

$$T_{\delta,\varepsilon}^{reg}(x) = \sum_{\ell m} a_{\ell m}^{reg}(\delta, \varepsilon) Y_{\ell m}(x)$$

$$\{a_{\ell m}^{reg}(\delta,\varepsilon)\} = \arg \min_{\{a_{\ell m}\}} \left\{ \lambda(\delta,\varepsilon) \sum_{\ell m} |a_{\ell m}| + \frac{1}{2} \sum_{\ell m} \left| a_{\ell m}^{obs} - a_{\ell m} \right|^2 \right\}$$

and the solution satisfies

$$\Pr \left\{ \left\| T^{obs} - T_{\delta,\varepsilon}^{reg} \right\|_{L^2(S^2)} < \varepsilon \right\} \geq 1 - \delta \ .$$

The same result holds when the real-valued regularization scheme is adopted.

Proof. Note that

$$\begin{aligned} E \left\| T^{obs} - T^{reg} \right\|_{L^2(S^2)}^2 &= \sum_{\ell m} E|a_{\ell m}^{obs} - a_{\ell m}^{reg}(\lambda)|^2 \\ &= \sum_{\ell m} E\{|a_{\ell m}^{obs}|^2 \mathbb{I}(\left| a_{\ell m}^{obs} \right| \leq \lambda)\} + \lambda^2 \sum_{\ell m} E\mathbb{I}(\left| a_{\ell m}^{obs} \right| > \lambda) \ . \end{aligned}$$

Now fix ℓ^* such that

$$\sum_{\ell > \ell^*} (2\ell+1) C_\ell \leq \frac{\varepsilon}{4} \ ,$$

and note that

$$\sum_{\ell=1}^{\infty} \sum_m E\{|a_{\ell m}^{obs}|^2 \mathbb{I}(\left| a_{\ell m}^{obs} \right| \leq \lambda)\} \leq \sum_{\ell=1}^{\ell^*} \sum_m E\{|a_{\ell m}^{obs}|^2 \mathbb{I}(\left| a_{\ell m}^{obs} \right| \leq \lambda)\} + \frac{\varepsilon}{4}$$

where

$$\sum_{\ell=1}^{\ell^*} \sum_m E\{|a_{\ell m}^{obs}|^2 \mathbb{I}(\left| a_{\ell m}^{obs} \right| \leq \lambda)\} \quad \leq \quad \lambda^2 \sum_{\ell=1}^{\ell^*} \sum_m E\{\mathbb{I}(\left| a_{\ell m}^{obs} \right| \leq \lambda)\}$$

$$
\begin{aligned}
&= \lambda^2 \sum_{\ell=1}^{\ell^*} \big[\Pr\{|a_{\ell 0}^{obs}| \le \lambda\} + \sum_{m\neq 0} \Pr\{|a_{lm}^{obs}| \le \lambda\} \big] \\
&= \lambda^2 \sum_{\ell=1}^{\ell^*} \big[\int_{-\lambda}^{\lambda} \frac{1}{\sqrt{2\pi C_\ell}} \exp\{-\frac{u^2}{2C_\ell}\} du + \sum_{m\neq 0} \int_0^{\lambda} \frac{2u}{C_\ell} \exp\{-\frac{u^2}{C_\ell}\} du \big] \\
&= \lambda^2 \sum_{\ell=1}^{\ell^*} \big[\int_{-\lambda}^{\lambda} \frac{1}{\sqrt{2\pi C_\ell}} \exp\{-\frac{u^2}{2C_\ell}\} du + \sum_{m\neq 0} \big(1 - \exp\{-\frac{\lambda^2}{C_\ell}\}\big) \big] \\
&\le \lambda^2 \sum_{\ell=1}^{\ell^*} \big[\int_{-\lambda}^{\lambda} \frac{1}{\sqrt{2\pi C_\ell}} du + \sum_{m\neq 0} \frac{\sqrt{2}\lambda}{\sqrt{\pi C_\ell}} \big] \\
&\le \lambda^2 \sum_{\ell=1}^{\ell^*} \frac{(2\ell+1)\sqrt{2}\lambda}{\sqrt{\pi C_\ell}} \le \frac{\lambda^3 \sqrt{2}}{\sqrt{\pi C_\ell}^*} (2\ell^* + \ell^{*2}) \le \frac{\varepsilon}{4} \ ,
\end{aligned}
$$

provided that $C_\ell^* := \min_{\ell=1,..,\ell^*} C_\ell$ and

$$
\lambda^3 \le \frac{\varepsilon \sqrt{\pi C_\ell^*}}{4\sqrt{2}(2\ell^* + \ell^{*2})} \ .
$$

Let Erfc be the complementary error function defined by $\mathrm{Erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp\{-x^2\} dx$. Since for $x > 0$, $\mathrm{Erfc}(x)$ is bounded by $\mathrm{Erfc}(x) \le \frac{2}{\sqrt{\pi}} \frac{1}{x + \sqrt{x^2 + \frac{4}{\pi}}} \exp\{-x^2\} \le \exp\{-x^2\}$, to bound the second term, we note that

$$
\begin{aligned}
\lambda^2 \sum_{\ell m} E\{\mathbb{I}(\left|a_{\ell m}^{obs}\right| > \lambda)\} &= \lambda^2 \sum_{\ell} \big[\Pr\{|a_{\ell,0}^{obs}| > \lambda\} + \sum_{m\neq 0} \Pr\{|a_{\ell m}^{obs}| > \lambda\} \big] \\
&= \lambda^2 \sum_{\ell} \big[2 \int_{\lambda}^{\infty} \frac{1}{\sqrt{2\pi C_\ell}} \exp\{-\frac{u^2}{2C_\ell}\} du + \sum_{m\neq 0} \int_{\lambda}^{\infty} \frac{2u}{C_\ell} \exp\{-\frac{u^2}{C_\ell}\} du \big] \\
&= \lambda^2 \sum_{\ell} \big[\mathrm{Erfc}(\frac{\lambda}{\sqrt{2C_\ell}}) + \sum_{m\neq 0} \exp\{-\frac{\lambda^2}{C_\ell}\} \big] \\
&= \lambda^2 \sum_{\ell} (2\ell+1) \exp\{-\frac{\lambda^2}{2C_\ell}\},
\end{aligned}
$$

now, for a fixed $\ell^+ > 1$, we write

$$
\lambda^2 \sum_{\ell m} E\{\mathbb{I}(\left|a_{\ell m}^{obs}\right| > \lambda)\} \le \lambda^2 (2\ell^+ + {\ell^+}^2) + \lambda^2 \sum_{\ell > \ell^+} (2\ell+1) \exp\{-\frac{\lambda^2}{2C_\ell}\}.
$$

Here we apply the integral test to the remainder of the series; since $f(\ell) = (2\ell+1)\exp\{-\frac{\lambda^2}{2C_\ell}\}$ for $C_\ell \le K\ell^{-\alpha}$, $\alpha > 2$, is a positive and monotonically decreasing function for all $\ell \ge 1$, we have

$$
\sum_{\ell > \ell^+} (2\ell+1) \exp\{-\frac{\lambda^2 \ell^\alpha}{2}\} \le \int_{\ell^+}^{\infty} (2x+1) \exp\{-\frac{\lambda^2 x^\alpha}{2}\} dx
$$

$$\leq \int_0^\infty (2x+1)\exp\{-\frac{\lambda^2 x^2}{2}\}dx = \frac{4+\lambda\sqrt{2\pi}}{2\lambda^2} \text{ for all } \lambda \geq 0 \ , \ \alpha > 2 \ .$$

Therefore

$$\lambda^2 \sum_{\ell m} E\{\mathbb{I}(\left|a_{\ell m}^{obs}\right| > \lambda)\} \leq \lambda^2(2\ell^+ + \ell^{+2}) + \lambda^2 \frac{4+\lambda\sqrt{2\pi}}{2\lambda^2} \leq \frac{\varepsilon}{2} \ ,$$

provided that we take λ such that

$$\lambda \leq \min\left\{\sqrt{\frac{\varepsilon}{4(2\ell^+ + \ell^{+2})}}, \left(\frac{\varepsilon}{2} - 4\right)\frac{1}{\sqrt{2\pi}}, \sqrt[3]{\frac{\varepsilon\sqrt{\pi C_\ell^*}}{4\sqrt{2}(2\ell^* + \ell^{*2})}}\right\}.$$

The proof for the real-valued regularization scheme is entirely analogous. ■

The previous result is straightforward, but it has some important consequences for the interpretation of the results to follow in the next Sections. In particular, it entails that the presence of nonGaussianity and anisotropy after convex regularization is not due to poor approximation properties of the reconstructed maps. The regularized fields we shall deal with can indeed be viewed as solutions to the optimization problem: for $\delta, \varepsilon > 0$,

$$\{a_{\ell m}^{reg}(\delta,\varepsilon)\} := \arg\min_{\{a_{\ell m}\}} \left\{\lambda(\delta,\varepsilon)\sum_{\ell m}|a_{\ell m}| + \sum_{\ell m}\left|a_{\ell m}^{obs} - a_{\ell m}\right|^2\right\}$$

where $\lambda(\delta, \varepsilon)$ is such that

$$\Pr\left\{\left\|T^{obs} - T_{\delta,\varepsilon}^{reg}\right\|_{L^2(S^2)} > \varepsilon\right\} \leq \delta \ .$$

We shall show that even for T Gaussian and δ, ε arbitrary small (but positive), $T_{\delta,\varepsilon}^{reg}$ exhibits nonGaussian statistics which diverge to infinity at the highest frequencies.

3 Anisotropy and NonGaussianity

Let us write as before

$$T^{reg} = \sum_\ell T_\ell^{reg} = \sum_{\ell m} a_{\ell m}^{reg} Y_{\ell m} \ , \ T^{reg*} = \sum_\ell T_\ell^{reg*} = \sum_{\ell m} a_{\ell m}^{reg*} Y_{\ell m}^{\mathcal{R}} \ ,$$

e.g., T^{reg}, T^{reg*} represent, respectively the ℓ^1-regularized maps under the complex and real-valued optimization schemes. For the discussion to follow, we need to recall briefly the following result:

Theorem 10 *(See Ref. [2]) Assume the spherical harmonic coefficients $\{a_{\ell m}\}$ of an isotropic random field are independent for $\ell = 1, 2, \ldots$ and $m = 0, 1, \ldots, \ell$. Then they are necessarily Gaussian.*

This result was established in [2], see also [3], [4] for extensions to homogeneous spaces of more general compact groups and [16], Theorem 6.12 for a proof. An obvious consequence is that a sequence of independent, but nonGaussian, random coefficients $\{a_{\ell m}\}$ will necessarily lead to anisotropic random fields.

We are now in the position to state and prove the first result of this paper; here and in what follows, we use $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^x \exp\{-\frac{1}{2}u^2\}du$ to denote the cumulative distribution function of a standard Gaussian variable.

Theorem 11 *Let T^{obs} be a Gaussian and isotropic spherical random field. Then the fields T^{reg}, T^{reg*} are necessarily nonGaussian and anisotropic. In particular, in the complex-valued regularization scheme we have*

$$E\left\{a_{\ell 0}^{reg}(\lambda)^2\right\} = \gamma_0(\frac{\lambda}{\sqrt{C_\ell}}) := C_\ell \left\{(1+\frac{\lambda^2}{C_\ell})2(1-\Phi(\frac{\lambda}{\sqrt{C_\ell}})) - \exp(-\frac{\lambda^2}{2C_\ell})\frac{\lambda}{\sqrt{C_\ell}}\sqrt{\frac{2}{\pi}}\right\} ,$$

while for $m \neq 0$

$$E\{|a_{\ell m}^{reg}(\lambda)|^2\} = \gamma_1(\frac{\lambda}{\sqrt{C_\ell}}) := C_\ell \left\{\exp(-\frac{\lambda^2}{C_\ell}) - \frac{\lambda}{\sqrt{C_\ell}}\sqrt{\pi}2(1-\Phi(\frac{\sqrt{2}\lambda}{\sqrt{C_\ell}}))\right\} . \tag{11}$$

Moreover, for all $m \neq 0$, we have that

$$-\lim_{\lambda/\sqrt{C_\ell}\to\infty} \frac{2C_\ell}{\lambda^2} \log \frac{E\{|a_{\ell m}^{reg}|^2\}}{E\{|a_{\ell 0}^{reg}|^2\}} = 1 . \tag{12}$$

and

$$\lim_{\lambda/\sqrt{C_\ell}\to\infty} \frac{E\left\{T_\ell^{reg}(\theta,\phi)^2\right\}}{E\left\{T_\ell^{reg}(0,0)^2\right\} P_\ell^2(\cos\theta)} = 1 .$$

Finally, in the real-valued regularization scheme

$$E\left\{a_{\ell 0}^{reg*}(\lambda)^2\right\} = E\{\left|a_{\ell m}^{reg*}(\lambda)\right|^2\} = \gamma_0(\frac{\lambda}{\sqrt{C_\ell}}) ,$$

for all $m = -\ell, ..., \ell$.

Proof. By assumption, the input coefficients $\{a_{\ell m}\}$, are Gaussian and independent. The inpainted coefficients can be written $a_{\ell m}^{reg} = j(a_{\ell m};\lambda)$, where the function $j(.;\lambda)$ is nonlinear; it follows immediately that they are independent and nonGaussian. Hence the fields T^{reg}, T^{reg*} are necessarily anisotropic, in view of Theorem 10. Focussing on $m = 0$, we have in particular

$$\Pr\left\{a_{\ell 0}^{reg} = 0\right\} = p_\ell(\lambda) := \int_{-\lambda}^{\lambda} \frac{1}{\sqrt{2\pi C_\ell}} \exp(-\frac{x^2}{2C_\ell})dx > 0$$

so that the distribution of $a_{\ell 0}^{reg} = a_{\ell 0}^{reg}(\lambda)$ is given by the mixture

$$p_\ell(\lambda)\delta_0 + (1-\frac{p_\ell(\lambda)}{2})\Phi^+(.;\lambda,C_\ell) + (1-\frac{p_\ell(\lambda)}{2})\Phi^-(.;\lambda,C_\ell) ,$$

where $\Phi^+(.;\lambda,C_\ell)$ is the distribution of a Gaussian random variable with mean $-\lambda$ and conditioned to be positive, and likewise $\Phi^-(.;\lambda,C_\ell)$ is the distribution of a Gaussian random variable with mean λ and conditioned to be negative. It is simple to see that we have

$$E\{a_{\ell 0}^{reg}(\lambda)\} = 0 ,$$

and

$$E\left\{a_{\ell 0}^{reg}(\lambda)^2\right\} = \frac{2}{\sqrt{2\pi C_\ell}} \int_0^\infty y^2 \exp\left\{-\frac{(y+\lambda)^2}{2C_\ell}\right\} dy$$

$$
\begin{aligned}
&= \frac{2}{\sqrt{2\pi C_\ell}} \int_\lambda^\infty (x-\lambda)^2 \exp\left\{-\frac{x^2}{2C_\ell}\right\} dx \\
&= \frac{2}{\sqrt{2\pi C_\ell}} \int_\lambda^\infty (x^2 - 2\lambda x + \lambda^2) \exp\left\{-\frac{x^2}{2C_\ell}\right\} dx \\
&= \frac{2}{\sqrt{\pi}} C_\ell \int_\lambda^\infty \frac{x}{\sqrt{2C_\ell}} \exp\left\{-\frac{x^2}{2C_\ell}\right\} d\frac{x^2}{2C_\ell} - \frac{4}{\sqrt{2\pi}} \sqrt{C_\ell} \int_\lambda^\infty \lambda \exp\left\{-\frac{x^2}{2C_\ell}\right\} d\frac{x^2}{2C_\ell} \\
&\quad + \frac{2}{\sqrt{\pi}} \lambda^2 \int_\lambda^\infty \exp\left\{-\frac{x^2}{2C_\ell}\right\} d\frac{x}{\sqrt{2C_\ell}} \\
&= \frac{2C_\ell}{\sqrt{\pi}} \Gamma(\frac{3}{2}; \frac{\lambda^2}{2C_\ell}) - \frac{4\sqrt{C_\ell}}{\sqrt{2\pi}} \lambda \exp\left\{-\frac{\lambda^2}{2C_\ell}\right\} + 2\lambda^2 \left\{1 - \Phi(\frac{\lambda}{\sqrt{C_\ell}})\right\} \\
&= 2C_\ell \left\{ \frac{1}{\sqrt{\pi}} \Gamma(\frac{3}{2}; \frac{\lambda^2}{2C_\ell}) - \frac{2}{\sqrt{2\pi}} \frac{\lambda}{\sqrt{C_\ell}} \exp\left\{-\frac{\lambda^2}{2C_\ell}\right\} + \frac{\lambda^2}{C_\ell} \left\{1 - \Phi(\frac{\lambda}{\sqrt{C_\ell}})\right\} \right\} ,
\end{aligned}
$$

where

$$
\Gamma(p; c) = \int_c^\infty x^{p-1} \exp(-x) dx
$$

denotes the incomplete Gamma function. Now using

$$
\Gamma(\frac{3}{2}; c) = \sqrt{c} e^{-c} + \frac{1}{2} \sqrt{\pi} \operatorname{Erfc}\left(\sqrt{c}\right) \ , \operatorname{Erfc}(u) := 2(1 - \Phi(\sqrt{2}u)) \ , \tag{13}
$$

the previous expression can be further developed to obtain

$$
\begin{aligned}
E\left\{a_{\ell 0}^{reg}(\lambda)^2\right\} &= 2C_\ell \left\{ \frac{1}{\sqrt{\pi}} \frac{\lambda}{\sqrt{2C_\ell}} \exp\{-\frac{\lambda^2}{2C_\ell}\} + \frac{1}{2} \operatorname{Erfc}\left(\frac{\lambda}{\sqrt{2C_\ell}}\right) - \frac{2}{\sqrt{2\pi}} \frac{\lambda}{\sqrt{C_\ell}} \exp\left\{-\frac{\lambda^2}{2C_\ell}\right\} + \frac{\lambda^2}{C_\ell} \left\{1 - \Phi(\frac{\lambda}{\sqrt{C_\ell}})\right\} \right\} \\
&= 2C_\ell \left\{ 1 - \Phi(\frac{\lambda}{\sqrt{C_\ell}}) - \sqrt{\frac{1}{2\pi}} \frac{\lambda}{\sqrt{C_\ell}} \exp\left\{-\frac{\lambda^2}{2C_\ell}\right\} + \frac{\lambda^2}{C_\ell} \left\{1 - \Phi(\frac{\lambda}{\sqrt{C_\ell}})\right\} \right\} \\
&= C_\ell \left\{ (1 + \frac{\lambda^2}{C_\ell}) 2(1 - \Phi(\frac{\lambda}{\sqrt{C_\ell}})) - \exp(-\frac{\lambda^2}{2C_\ell}) \frac{\lambda}{\sqrt{C_\ell}} \sqrt{\frac{2}{\pi}} \right\} .
\end{aligned}
$$

It can be checked easily that $\lim_{\lambda \to 0} E\left\{a_{\ell 0}^{reg}(\lambda)^2\right\} = C_\ell$, as expected. Similarly, by moving to polar coordinates it is easy to see that we have

$$
\begin{aligned}
E\{|a_{\ell m}^{reg}(\lambda)|^2\} &= \frac{1}{2\pi} \int_0^{2\pi} \int_\lambda^\infty (r-\lambda)^2 \frac{r}{C_\ell/2} \exp(-\frac{r^2}{C_\ell}) dr d\varphi \\
&= C_\ell \int_\lambda^\infty (\frac{r-\lambda}{\sqrt{C_\ell}})^2 \frac{2r}{C_\ell} \exp(-\frac{r^2}{C_\ell}) dr \\
&= C_\ell \int_\lambda^\infty (\frac{r-\lambda}{\sqrt{C_\ell}})^2 \exp(-\frac{r^2}{C_\ell}) d\frac{r^2}{C_\ell} \\
&= C_\ell \left\{ \int_{\lambda^2/C_\ell}^\infty u \exp(-u) du + \frac{\lambda^2}{C_\ell} \int_{\lambda^2/C_\ell}^\infty \exp(-u) du \right\} \\
&\quad + C_\ell \left\{ -2 \frac{\lambda}{\sqrt{C_\ell}} \int_{\lambda^2/C_\ell}^\infty \sqrt{u} \exp(-u) du \right\} .
\end{aligned} \tag{14}
$$

Now using (13) and

$$\Gamma(2;c) = \int_c^\infty u \exp(-u) du = e^{-c} + ce^{-c}, \tag{15}$$

we have

$$\begin{aligned}
E\{|a_{\ell m}^{reg}(\lambda)|^2\} &= C_\ell \left\{ \exp(-\frac{\lambda^2}{C_\ell}) + \frac{\lambda^2}{C_\ell}\exp(-\frac{\lambda^2}{C_\ell}) + \frac{\lambda^2}{C_\ell}\exp(-\frac{\lambda^2}{C_\ell}) \right\} \\
&\quad -2C_\ell \left\{ \frac{\lambda^2}{C_\ell}\exp(-\frac{\lambda^2}{C_\ell}) + \frac{1}{2}\sqrt{\pi}\mathrm{Erfc}\left(\frac{\lambda}{\sqrt{C_\ell}}\right)\frac{\lambda}{\sqrt{C_\ell}} \right\} \\
&= C_\ell \left\{ \exp(-\frac{\lambda^2}{C_\ell}) - \sqrt{\pi}\mathrm{Erfc}\left(\frac{\lambda}{\sqrt{C_\ell}}\right)\frac{\lambda}{\sqrt{C_\ell}} \right\} \\
&= C_\ell \left\{ \exp(-\frac{\lambda^2}{C_\ell}) - \frac{\lambda}{\sqrt{C_\ell}}\sqrt{\pi}2(1-\Phi(\frac{\sqrt{2}\lambda}{\sqrt{C_\ell}})) \right\} \ .
\end{aligned}$$

Hence we have

$$\begin{aligned}
\frac{E\{|a_{\ell m}^{reg}|^2\}}{E\{|a_{\ell 0}^{reg}|^2\}} &= \frac{\int_\lambda^\infty (r-\lambda)^2 2\frac{r}{C_\ell}\exp(-\frac{r^2}{C_\ell})dr}{\frac{2}{\sqrt{2\pi C_\ell}}\int_\lambda^\infty (x-\lambda)^2 \exp\left\{-\frac{x^2}{2C_\ell}\right\}dx} \\
&= \frac{C_\ell\int_\lambda^\infty (\frac{r}{\sqrt{C_\ell}} - \frac{\lambda}{\sqrt{C_\ell}})^2 2\frac{r}{\sqrt{C_\ell}}\exp(-\frac{r^2}{C_\ell})d\frac{r}{\sqrt{C_\ell}}}{\frac{2}{\sqrt{2\pi}}C_\ell\int_\lambda^\infty (\frac{x}{\sqrt{C_\ell}} - \frac{\lambda}{\sqrt{C_\ell}})^2 \exp\left\{-\frac{x^2}{2C_\ell}\right\}d\frac{x}{\sqrt{C_\ell}}} \\
&= \frac{\int_{\lambda/\sqrt{C_\ell}}^\infty (u - \frac{\lambda}{\sqrt{C_\ell}})^2 u\exp(-u^2)du}{\frac{1}{\sqrt{2\pi}}\int_{\lambda/\sqrt{C_\ell}}^\infty (u - \frac{\lambda}{\sqrt{C_\ell}})^2 \exp\left\{-\frac{u^2}{2}\right\}du} \\
&\le K_\varepsilon \exp(-\frac{\lambda^2}{2C_\ell}(1-\varepsilon))\frac{\int_{\lambda/\sqrt{C_\ell}}^\infty (u - \frac{\lambda}{\sqrt{C_\ell}})^2 \exp(-\frac{u^2}{2})du}{\int_{\lambda/\sqrt{C_\ell}}^\infty (u - \frac{\lambda}{\sqrt{C_\ell}})^2 \exp\left\{-\frac{u^2}{2}\right\}du} \\
&= K_\varepsilon \exp(-\frac{\lambda^2}{2C_\ell}(1-\varepsilon)) \ ,
\end{aligned}$$

for some constant $K_\varepsilon > 0$, any $\varepsilon > 0$, because $u\exp(-\frac{\varepsilon u^2}{2}) \le K_\varepsilon$ for all $u \ge \frac{\lambda}{\sqrt{C_\ell}}$. Now

$$\begin{aligned}
&- \lim_{\lambda/\sqrt{C_\ell}\to\infty} \frac{2C_\ell}{\lambda^2}\log\frac{E\{|a_{\ell m}^{reg}|^2\}}{E\{|a_{\ell 0}^{reg}|^2\}} \\
&= - \lim_{\lambda/\sqrt{C_\ell}\to\infty} \frac{2C_\ell}{\lambda^2}\log K_\varepsilon - \lim_{\lambda/\sqrt{C_\ell}\to\infty} \frac{2C_\ell}{\lambda^2}\log\exp(-\frac{\lambda^2}{2C_\ell}(1-\varepsilon)) \\
&= (1-\varepsilon) \ ,
\end{aligned}$$

and because ε is arbitrary, the first result follows. To conclude, it is then sufficient to note that

$$\frac{E\{T_\ell^{reg}(\theta,\phi)^2\}}{E\{T_\ell^{reg}(0,0)^2\}P_\ell^2(\cos\theta)} = \frac{\sum_m E\{|a_{\ell m}^{reg}|^2\}\,|Y_{\ell m}(\theta,\phi)|^2}{\frac{2\ell+1}{4\pi}E\{(a_{\ell 0}^{reg})^2\}P_\ell^2(\cos\theta)}$$

$$= 1 + \frac{\sum_{m \neq 0} E\{|a_{\ell m}^{reg}|^2\} |Y_{\ell m}(\theta, \phi)|^2}{\frac{2\ell+1}{4\pi} E\{(a_{\ell 0}^{reg})^2\} P_\ell^2(\cos\theta)} \to 1 \ ,$$

because

$$\frac{\sum_{m \neq 0} E\{|a_{\ell m}^{reg}|^2\} |Y_{\ell m}(\theta, \phi)|^2}{\frac{2\ell+1}{4\pi} E\{(a_{\ell 0}^{reg})^2\} P_\ell^2(\cos\theta)} \leq 2\ell \max_{m \neq 0} \frac{E\{|a_{\ell m}^{reg}|^2\} |Y_{\ell m}(\theta, \phi)|^2}{\frac{2\ell+1}{4\pi} E\{(a_{\ell 0}^{reg})^2\} P_\ell^2(\cos\theta)} \to 0 \ ,$$

as $\lambda/\sqrt{C_\ell} \to \infty$. The proof in the real-valued scheme is analogous. ■

Remark 12 *As a consequence of the previous Theorem, in the complex-valued regularization scheme the ratio* $E\,|a_{\ell 0}^{reg}|^2 / E\,|a_{\ell m}^{reg}|^2$ *diverges to infinity super-exponentially as* $C_\ell \to 0$, *and the covariance function is dominated by a single random coefficient, thus oscillating over the sphere as the square of a Legendre polynomial. For the real-valued algorithm, this is not the case, and the variance is constant; nevertheless, this field is still anisotropic, as confirmed by the analysis of higher order power spectra which we entertain in the next Sections.*

4 High-Frequency Behavior of Trispectra

4.1 The Trispectrum at the North Pole

A natural tool to explore non-Gaussian/anisotropic features of a spherical random fields is provided by the expected values of higher-order moments of its multipole components. For instance, fourth order moments lead to so-called trispectra, see [14, 16, 22] for properties and applications; for our aims, it suffices to recall that, under Gaussianity and isotropy, we should have

$$\frac{ET_\ell^4(x)}{(ET_\ell^2(x))^2} \equiv 3 \text{ , for all } x \in S^2, \ \ell = 0, 1, 2, ...$$

In the following result we show instead that the normalized trispectrum of convexly regularized fields diverges to infinity at the North Pole.

Theorem 13 *The normalized trispectrum of a convexly regularized random field at the North Pole is given by*

$$\frac{E\{T_\ell^{reg}(0,0)^4\}}{[E\{T_\ell^{reg}(0,0)^2\}]^2} = \frac{E\left\{a_{\ell 0}^{reg}(\lambda)^4\right\}}{[E\{a_{\ell 0}^{reg}(\lambda)^2\}]^2} = \sqrt{\frac{\pi}{2}} \psi(\frac{\lambda}{\sqrt{C_\ell}}) \ ,$$

where

$$\psi(\frac{\lambda}{\sqrt{C_\ell}}) := \frac{\int_0^\infty v^4 \exp\left\{-\frac{1}{2}\left[v + \frac{\lambda}{\sqrt{C_\ell}}\right]^2\right\} dv}{\left[\int_0^\infty v^2 \exp\left\{-\frac{1}{2}\left[v + \frac{\lambda}{\sqrt{C_\ell}}\right]^2\right\} dv\right]^2} \ .$$

The function $\psi(.)$ *is strictly positive and increasing, with*

$$\lim_{x \to 0} \psi(x) = 3 \ , \ \lim_{x \to \infty} \left[\frac{15}{4} x^3 \exp\left\{\frac{x^2}{2}\right\}\right]^{-1} \psi(x) = 1 \ .$$

Proof. Similarly to the proof of the previous Theorem and using the same notation, we have

$$
\begin{aligned}
E\left\{a_{\ell 0}^{reg}(\lambda)^4\right\} &= \frac{2}{\sqrt{2\pi C_\ell}}\int_\lambda^\infty (x-\lambda)^4 \exp\left\{-\frac{x^2}{2C_\ell}\right\}dx \\
&= \frac{2}{\sqrt{2\pi C_\ell}}\int_\lambda^\infty x^4 \exp\left\{-\frac{x^2}{2C_\ell}\right\}dx - 8\frac{1}{\sqrt{2\pi C_\ell}}\int_\lambda^\infty x^3\lambda \exp\left\{-\frac{x^2}{2C_\ell}\right\}dx \\
&+ 12\frac{1}{\sqrt{2\pi C_\ell}}\int_\lambda^\infty x^2\lambda^2 \exp\left\{-\frac{x^2}{2C_\ell}\right\}dx - 8\frac{1}{\sqrt{2\pi C_\ell}}\int_\lambda^\infty x\lambda^3 \exp\left\{-\frac{x^2}{2C_\ell}\right\}dx \\
&+ \frac{2}{\sqrt{2\pi C_\ell}}\int_\lambda^\infty \lambda^4 \exp\left\{-\frac{x^2}{2C_\ell}\right\}dx \\
&= \frac{2}{\sqrt{2\pi}}\sqrt{8}C_\ell^2\int_\lambda^\infty \frac{x^3}{\sqrt{8C_\ell^3}}\exp\left\{-\frac{x^2}{2C_\ell}\right\}d\frac{x^2}{2C_\ell} \\
&- 8\frac{1}{\sqrt{2\pi}}2\sqrt{C_\ell^3}\lambda\int_\lambda^\infty (\frac{x}{\sqrt{2C_\ell}})^2\exp\left\{-\frac{x^2}{2C_\ell}\right\}d\frac{x^2}{2C_\ell} \\
&+ 12\frac{1}{\sqrt{2\pi}}\sqrt{2}C_\ell\lambda^2\int_\lambda^\infty \frac{x}{\sqrt{2C_\ell}}\exp\left\{-\frac{x^2}{2C_\ell}\right\}d\frac{x^2}{2C_\ell} \\
&- 8\frac{\sqrt{C_\ell}}{\sqrt{2\pi}}\lambda^3\int_\lambda^\infty \exp\left\{-\frac{x^2}{2C_\ell}\right\}d\frac{x^2}{2C_\ell} + \frac{2}{\sqrt{\pi}}\lambda^4\int_\lambda^\infty \exp\left\{-\frac{x^2}{2C_\ell}\right\}d\frac{x}{\sqrt{2C_\ell}} \\
&= 2C_\ell^2\frac{\sqrt{8}}{\sqrt{2\pi}}\left\{\Gamma(\frac{5}{2};\frac{\lambda^2}{2C_\ell}) - \frac{4}{\sqrt{2}}\frac{\sqrt{2}\lambda}{\sqrt{2C_\ell}}\Gamma(2;\frac{\lambda^2}{2C_\ell}) + \frac{6}{\sqrt{2^2}}(\frac{\lambda}{\sqrt{C_\ell}})^2\Gamma(\frac{3}{2};\frac{\lambda^2}{2C_\ell})\right. \\
&\left. - \frac{\sqrt{2}\lambda^3}{C_\ell\sqrt{C_\ell}}\exp\left\{-\frac{\lambda^2}{2C_\ell}\right\} + \frac{4}{\sqrt{2^3}}\frac{\sqrt{2\pi}}{}(\frac{\lambda}{\sqrt{2C_\ell}})^4(1-\Phi(\frac{\lambda}{\sqrt{2C_\ell}}))\right\}.
\end{aligned}
$$

Observing that $\Gamma(1;c) = e^{-c}$, we obtain

$$
\begin{aligned}
&E\left\{a_{\ell 0}^{reg}(\lambda)^4\right\} \\
&= C_\ell^2\frac{4}{\sqrt{\pi}}\left\{\sum_{k=2}^{5}(-1)^{k+1}\left(\frac{\nu_\ell}{\sqrt{2}}\right)^{5-k}\binom{4}{5-k}\Gamma\left(\frac{k}{2};\left(\frac{\nu_\ell}{\sqrt{2}}\right)^2\right) + 2\sqrt{\pi}\left(\frac{\nu_\ell}{\sqrt{2}}\right)^4(1-\Phi(\frac{\nu_\ell}{\sqrt{2}})\right\},
\end{aligned}
$$

where $\nu_\ell := \frac{\lambda}{\sqrt{C_\ell}}$. Again, it is simple to check that

$$
\begin{aligned}
\lim_{\lambda/\sqrt{C_\ell}\to 0} E\left\{a_{\ell 0}^{reg}(\lambda)^4\right\} &= 2C_\ell^2\frac{\sqrt{8}}{\sqrt{2\pi}}\lim_{\lambda/\sqrt{C_\ell}\to 0}\Gamma(\frac{5}{2};\frac{\lambda^2}{2C_\ell}) \\
&= 2C_\ell^2\frac{\sqrt{8}}{\sqrt{2\pi}}\frac{3}{4}\sqrt{\pi} = 3C_\ell^2 ,
\end{aligned}
$$

as expected, because $\lim_{\lambda/\sqrt{C_\ell}\to 0} E\left\{a_{\ell 0}^{reg}(\lambda)^4\right\}/[E\left\{a_{\ell 0}^{reg}(\lambda)^2\right\}]^2 = 3$ provides the fourth moment of a standard Gaussian variable. Note also that

$$
\psi(\frac{\lambda}{\sqrt{C_\ell}}) = \frac{\int_0^\infty v^4\exp\left\{-\frac{1}{2}\left[v+\frac{\lambda}{\sqrt{C_\ell}}\right]^2\right\}dv}{\left[\int_0^\infty v^2\exp\left\{-\frac{1}{2}\left[v+\frac{\lambda}{\sqrt{C_\ell}}\right]^2\right\}dv\right]^2}
$$

$$= \exp\left\{\frac{1}{2}\frac{\lambda^2}{C_\ell}\right\} \frac{\int_0^\infty v^4 \exp\left\{-\frac{1}{2}\left[v^2 + \frac{2\lambda v}{\sqrt{C_\ell}}\right]\right\} dv}{\left[\int_0^\infty v^2 \exp\left\{-\frac{1}{2}\left[v^2 + \frac{2\lambda v}{\sqrt{C_\ell}}\right]\right\} dv\right]^2}$$

$$= \exp\left\{\frac{1}{2}\frac{\lambda^2}{C_\ell}\right\} \frac{-5\frac{\lambda}{\sqrt{C_\ell}} - (\frac{\lambda}{\sqrt{C_\ell}})^3 + \exp\{\frac{\lambda^2}{2C_\ell}\}\sqrt{\frac{\pi}{2}}(3 + 6\frac{\lambda^2}{C_\ell} + \frac{\lambda^4}{C_\ell^2})\mathrm{Erfc}(\frac{\lambda}{\sqrt{2C_\ell}})}{\left[-\frac{\lambda}{\sqrt{C_\ell}} + \exp\{\frac{\lambda^2}{2C_\ell}\}\sqrt{\frac{\pi}{2}}(1 + \frac{\lambda^2}{C_\ell})\mathrm{Erfc}(\frac{\lambda}{\sqrt{2C_\ell}})\right]^2},$$

here we use the classical asymptotic expansion of the complementary error function, i.e., for large x we have $\mathrm{Erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}}(1 - \frac{1}{2x^2} + \frac{3}{4x^4} + O(x^{-5}))$, then

$$\lim_{\lambda/\sqrt{C_\ell}\to\infty} \psi(\frac{\lambda}{\sqrt{C_\ell}}) \left[\exp\left\{\frac{1}{2}\frac{\lambda^2}{C_\ell}\right\} \frac{15}{4}\frac{\lambda^3}{C_\ell^{\frac{3}{2}}}\right]^{-1}$$

$$= \lim_{\lambda/\sqrt{C_\ell}\to\infty} \frac{-5\frac{\lambda}{\sqrt{C_\ell}} - (\frac{\lambda}{\sqrt{C_\ell}})^3 + \exp\{\frac{\lambda^2}{2C_\ell}\}\sqrt{\frac{\pi}{2}}(3 + 6\frac{\lambda^2}{C_\ell} + \frac{\lambda^4}{C_\ell^2})\mathrm{Erfc}(\frac{\lambda}{\sqrt{2C_\ell}})}{\frac{15}{4}\frac{\lambda^3}{C_\ell^{\frac{3}{2}}}\left[-\frac{\lambda}{\sqrt{C_\ell}} + \exp\{\frac{\lambda^2}{2C_\ell}\}\sqrt{\frac{\pi}{2}}(1 + \frac{\lambda^2}{C_\ell})\mathrm{Erfc}(\frac{\lambda}{\sqrt{2C_\ell}})\right]^2}$$

$$= \lim_{\lambda/\sqrt{C_\ell}\to\infty} \frac{-5\frac{\lambda}{\sqrt{C_\ell}} - (\frac{\lambda}{\sqrt{C_\ell}})^3 + \exp\{\frac{\lambda^2}{2C_\ell}\}\sqrt{\frac{\pi}{2}}(3 + 6\frac{\lambda^2}{C_\ell} + \frac{\lambda^4}{C_\ell^2})\frac{e^{-\frac{\lambda^2}{2C_\ell}}}{\frac{\lambda}{\sqrt{C_\ell}}\sqrt{\frac{\pi}{2}}}(1 - \frac{1}{\frac{\lambda^2}{C_\ell}} + \frac{3}{2\frac{\lambda^4}{C_\ell^2}})}{\frac{15}{4}\frac{\lambda^3}{C_\ell^{\frac{3}{2}}}\left[-\frac{\lambda}{\sqrt{C_\ell}} + \exp\{\frac{\lambda^2}{2C_\ell}\}\sqrt{\frac{\pi}{2}}(1 + \frac{\lambda^2}{C_\ell})\frac{e^{-\frac{\lambda^2}{2C_\ell}}}{\frac{\lambda}{\sqrt{C_\ell}}\sqrt{\frac{\pi}{2}}}(1 - \frac{1}{\frac{\lambda^2}{C_\ell}} + \frac{3}{2\frac{\lambda^4}{C_\ell^2}})\right]^2}$$

$$= \lim_{\lambda/\sqrt{C_\ell}\to\infty} \frac{3\frac{\lambda^5}{C_\ell^{\frac{5}{2}}}(3 + 5\frac{\lambda^2}{C_\ell})}{\frac{15}{4}\frac{\lambda^3}{C_\ell^{\frac{3}{2}}}(3 + 2\frac{\lambda^2}{C_\ell})^2} = 1.$$

■

Remark 14 *(Some Numerical Examples) It is instructive to provide some numerical evidence on the kurtosis of the multipole components at the North Pole, as a function of the penalization parameter λ and the angular power spectrum C_ℓ. It should be recalled that $\sum_\ell (2\ell + 1)C_\ell < \infty$ by finite variance, whence C_ℓ must decay at least as fast as $\ell^{-2-\tau}$, some $\tau > 0$, as $\ell \to \infty$. For instance, considering some physically realistic values for the power spectrum of CMB and a fixed penalization parameter $\lambda = 1$, we have*

ℓ	10	20	30	40	50	60	70	80	200
C_ℓ	48.20	13.7	7.17	4.8	3.7	2.9	2.4	2.1	0.76
κ_ℓ	3.50	4.08	4.65	5.19	5.65	6.21	6.76	7.22	15.39

4.2 Asymptotic Behavior of the Angular Trispectrum

Exploiting the computations developed so far, it is also possible to provide analytic expressions for the trispectra at the various multipoles, as follows. We start recalling the results we have earlier established on the moments of the spherical harmonic coefficients $\{a_{\ell m}\}$, under the complex and real-valued regularization schemes. More precisely, we have

- For all ℓ, m we have
$$E\{a_{\ell m}^{reg}\} = E\{a_{\ell m}^{reg*}\} = 0 \; ; \tag{16}$$
- For all $\ell, m = 0$
$$E\left\{(a_{\ell 0}^{reg})^2\right\} = E\left\{(a_{\ell 0}^{reg*})^2\right\} = \gamma_0(\frac{\lambda}{\sqrt{C_\ell}}) \; ;$$
- Under the complex-valued regularization scheme, for $m \neq 0$
$$E\{|a_{\ell m}^{reg}|^2\} = \gamma_1(\frac{\lambda}{\sqrt{C_\ell}}) \; ,$$
while in the real-valued framework
$$E\{(a_{\ell m}^{reg*})^2\} = \gamma_0(\frac{\lambda}{\sqrt{C_\ell}}) \; ;$$
- Finally for the fourth-order moments, for all ℓ
$$E\left\{(a_{\ell 0}^{reg})^4\right\} = E\{(a_{\ell 0}^{reg*})^4\} = \gamma_2(\frac{\lambda}{\sqrt{C_\ell}})$$
and for $m \neq 0$
$$E\{|a_{\ell m}^{reg}|^4\} = E\{(a_{\ell m}^{reg*})^4\} = \gamma_3(\frac{\lambda}{\sqrt{C_\ell}})$$
where
$$\gamma_2(\nu_\ell) := C_\ell^2 \frac{4}{\sqrt{\pi}} \left\{ \sum_{k=2}^{5} (-1)^{k+1} (\frac{\nu_\ell}{\sqrt{2}})^{5-k} \binom{4}{5-k} \Gamma(\frac{k}{2}; (\frac{\nu_\ell}{\sqrt{2}})^2) + 2\sqrt{\pi} (\frac{\nu_\ell}{\sqrt{2}})^4 (1 - \Phi(\frac{\nu_\ell}{\sqrt{2}})) \right\}$$
and
$$\gamma_3(\nu_\ell) = C_\ell^2 \sum_{k=0}^{4} (-1)^k \binom{4}{k} \nu_\ell^{4-k} \Gamma(\frac{k+2}{2}; \nu_\ell^2),$$

where $\nu_\ell = \frac{\lambda}{\sqrt{C_\ell}}$. For (16), we note first that it would be trivially true for an isotropic random field, but it requires to be checked under anisotropy. In any case, the proof is straightforward, indeed we have
$$\begin{aligned} E\{a_{\ell m}^{reg}\} &= E\left\{\rho_{\ell m}^{reg} \exp(i\psi_{\ell m}^{obs})\right\} \\ &= \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} r \exp(i\theta) f_{\rho;\ell}(r) dr d\theta \\ &= \frac{1}{2\pi} \int_0^\infty r f_{\rho;\ell}(r) \left\{ \int_0^{2\pi} \exp(i\theta) d\theta \right\} dr = 0 \; . \end{aligned}$$
A similar argument will actually cover any product of an odd number of spherical harmonic coefficients, because $\int_0^{2\pi} \exp(ik\theta) d\theta = 0$ for all non-zero integers k. We only need to study $E\{|a_{\ell m}^{reg}|^4\}$, for which we have
$$E\{|a_{\ell m}^{reg}|^4\} = \int_\lambda^\infty (r-\lambda)^4 2 \frac{r}{C_\ell} \exp(-\frac{r^2}{C_\ell}) dr$$

$$
\begin{aligned}
&= \int_{\lambda}^{\infty}(r-\lambda)^4 \exp(-\frac{r^2}{C_\ell})d\frac{r^2}{C_\ell} \\
&= C_\ell^2 \int_{\lambda^2/C_\ell}^{\infty}(u^2-4u^{3/2}\frac{\lambda}{\sqrt{C_\ell}}+6u\frac{\lambda^2}{C_\ell}-4\sqrt{u}\frac{\lambda^3}{\sqrt{C_\ell^3}}+\frac{\lambda^4}{C_\ell^2})\exp(-u)du \\
&= C_\ell^2\left\{\Gamma(3;\frac{\lambda^2}{C_\ell})-4\frac{\lambda}{\sqrt{C_\ell}}\Gamma(\frac{5}{2};\frac{\lambda^2}{C_\ell})+6\frac{\lambda^2}{C_\ell}\Gamma(2;\frac{\lambda^2}{C_\ell})-4\frac{\lambda^3}{\sqrt{C_\ell^3}}\Gamma(\frac{3}{2};\frac{\lambda^2}{C_\ell})+\frac{\lambda^4}{C_\ell^2}\exp(-\frac{\lambda^2}{C_\ell}))\right\} ,
\end{aligned}
$$

using repeatedly integration by parts on the incomplete Gamma function.

Remark 15 *It should be noted that, as expected,*

$$\lim_{\lambda\to 0} E\{|a_{\ell m}^{reg}|^2\} = C_\ell \ ,$$

and more generally

$$\lim_{C_\ell/\lambda^2\to\infty} \frac{1}{C_\ell} E\{|a_{\ell m}^{reg}|^2\} = 1 \ .$$

Moreover

$$\lim_{\lambda\to 0} E\{|a_{\ell m}^{reg}|^4\}/C_\ell^2 = 2 \ ,$$

again as expected, because in the limiting Gaussian case

$$
\begin{aligned}
E\{|a_{\ell m}|^4\} &= E\{[\mathrm{Re}(a_{lm})^2+\mathrm{Im}(a_{lm})^2]^2\} \\
&= E\{\mathrm{Re}(a_{lm})^4\}+E\{\mathrm{Im}(a_{lm})^4\}+2E\{\mathrm{Re}(a_{lm})^2\}E\{\mathrm{Im}(a_{lm})^2\} \\
&= \frac{3}{4}C_\ell^2+\frac{3}{4}C_\ell^2+\frac{2}{4}C_\ell^2 = 2C_\ell^2 \ .
\end{aligned}
$$

The previous result can be generalized as follows.

Proposition 16 *For all $p = 1, 2, 3, ...$ and for $m \neq 0$, we have*

$$E\left\{|a_{\ell m}^{reg}|^{2p}\right\} = C_\ell^p \sum_{k=0}^{2p}(-1)^k \binom{2p}{k}\nu_\ell^{2p-k}\Gamma(\frac{k+2}{2};\nu_\ell^2) \ ,$$

where $\binom{2p}{k} = \frac{(2p)!}{(2p-k)!k!}$ is the standard binomial coefficient.

Proof. The proof is identical to the previous arguments, and hence it is not repeated for brevity's sake. ■

An important consequence of these results is the following

Lemma 17 *We have that*

$$\lim_{C_\ell\to 0} \exp\left\{-\frac{\lambda^2}{C_\ell}\right\} \frac{E\{|a_{\ell m}^{reg}|^4\}}{\left[E\{|a_{\ell m}^{reg}|^2\}\right]^2} = 6 \ ,$$

whence $\frac{E\{|a_{\ell m}^{reg}|^4\}}{\left[E\{|a_{\ell m}^{reg}|^2\}\right]^2}$ diverges superexponentially.

Proof. It suffices to notice that

$$\begin{aligned}
\frac{E\{|a_{\ell m}^{reg}|^4\}}{\left[E\{|a_{\ell m}^{reg}|^2\}\right]^2} &= \frac{C_\ell^2 \int_\lambda^\infty (\frac{r-\lambda}{\sqrt{C_\ell}})^4 \ \frac{2r}{\sqrt{C_\ell}} \exp(-\frac{r^2}{C_\ell}) d \ \frac{r}{\sqrt{C_\ell}}}{\left\{C_\ell \int_\lambda^\infty (\frac{r-\lambda}{\sqrt{C_\ell}})^2 2 \frac{r}{\sqrt{C_\ell}} \exp(-\frac{r^2}{C_\ell}) d \frac{r}{\sqrt{C_\ell}}\right\}^2} \\
&= \frac{\int_{\lambda/\sqrt{C_\ell}}^\infty (u-\frac{\lambda}{\sqrt{C_\ell}})^4 2u \exp(-u^2) du}{\left\{2\int_{\lambda/\sqrt{C_\ell}}^\infty (u-\frac{\lambda}{\sqrt{C_\ell}})^2 u \exp(-u^2) du\right\}^2} \\
&= \frac{\int_0^\infty v^4 \ 2(v+\frac{\lambda}{\sqrt{C_\ell}}) \exp(-(v+\frac{\lambda}{\sqrt{C_\ell}})^2) dv}{\left\{2\int_0^\infty v^2 (v+\frac{\lambda}{\sqrt{C_\ell}}) \exp(-(v+\frac{\lambda}{\sqrt{C_\ell}})^2) dv\right\}^2} \\
&= \exp(\frac{\lambda^2}{C_\ell}) \frac{\int_0^\infty v^4 (v+\frac{\lambda}{\sqrt{C_\ell}}) \exp(-v^2 - 2\frac{\lambda v}{\sqrt{C_\ell}}) dv}{2\left\{\int_0^\infty v^2 (v+\frac{\lambda}{\sqrt{C_\ell}}) \exp(-v^2 - 2\frac{\lambda v}{\sqrt{C_\ell}}) dv\right\}^2} ,
\end{aligned}$$

where, by applying the expansion of the Erfc function $\frac{e^{-x^2}}{x\sqrt{\pi}}(1-\frac{1}{2x^2}+\frac{3}{4x^4}-\frac{15}{8x^6}+O(x^{-7}))$, for large x, we have

$$\begin{aligned}
&\lim_{\lambda/\sqrt{C_\ell}\to\infty} \frac{\int_0^\infty v^4 (v+\frac{\lambda}{\sqrt{C_\ell}}) \exp(-v^2 - 2\frac{\lambda v}{\sqrt{C_\ell}}) dv}{2\left[\int_0^\infty v^2 (v+\frac{\lambda}{\sqrt{C_\ell}}) \exp(-v^2 - 2\frac{\lambda v}{\sqrt{C_\ell}}) dv\right]^2} \\
= &\lim_{\lambda/\sqrt{C_\ell}\to\infty} \frac{1+\frac{\lambda^2}{C_\ell} - \frac{\sqrt{\pi}}{2}\frac{\lambda}{\sqrt{C_\ell}} \exp\{\frac{\lambda^2}{C_\ell}\}(3+2\frac{\lambda^2}{C_\ell}) \mathrm{Erfc}(\frac{\lambda}{\sqrt{C_\ell}})}{2\left[\frac{1}{2}(1-\exp\{\frac{\lambda^2}{C_\ell}\}\sqrt{\pi}\frac{\lambda}{\sqrt{C_\ell}} erfc(\frac{\lambda}{\sqrt{C_\ell}}))\right]^2} \\
= &\lim_{\lambda/\sqrt{C_\ell}\to\infty} \frac{1+\frac{\lambda^2}{C_\ell} - \frac{\sqrt{\pi}}{2}\frac{\lambda}{\sqrt{C_\ell}} \exp\{\frac{\lambda^2}{C_\ell}\}(3+2\frac{\lambda^2}{C_\ell}) \frac{e^{-\frac{\lambda^2}{C_\ell}}}{\frac{\lambda}{\sqrt{C_\ell}}\sqrt{\pi}}(1-\frac{1}{2\frac{\lambda^2}{C_\ell}}+\frac{3}{4\frac{\lambda^4}{C_\ell^2}}-\frac{15}{8x^6})}{2\left[\frac{1}{2}(1-\exp\{\frac{\lambda^2}{C_\ell}\}\sqrt{\pi}\frac{\lambda}{\sqrt{C_\ell}} \frac{e^{-\frac{\lambda^2}{C_\ell}}}{\frac{\lambda}{\sqrt{C_\ell}}\sqrt{\pi}}(1-\frac{1}{2\frac{\lambda^2}{C_\ell}}+\frac{3}{4\frac{\lambda^4}{C_\ell^2}}-\frac{15}{8x^6}))\right]^2} \\
= &\lim_{\lambda/\sqrt{C_\ell}\to\infty} \frac{24\frac{\lambda^6}{C_\ell{}^3}(15+4\frac{\lambda^2}{C_\ell})}{(15-6\frac{\lambda^2}{C_\ell}+4\frac{\lambda^4}{C_\ell{}^2})^2} = 6 \ .
\end{aligned}$$

■

In view of the previous results, it is simple to provide exact analytic expressions for the expected trispectra $E\{T_\ell\}$ under both regularization schemes, and to study their asymptotic behavior as the frequencies increase. We obtain

Theorem 18 *We have*

$$\begin{aligned}
E\left\{T_\ell^{reg}(\theta,\phi)^4\right\} &= \gamma_2(\frac{\lambda}{\sqrt{C_\ell}}) \left|Y_{\ell 0}(\theta,\phi)\right|^4 + \gamma_3(\frac{\lambda}{\sqrt{C_\ell}}) \sum_{m\neq 0} \left|Y_{\ell m}(\theta,\phi)\right|^4 \\
&+ 2\gamma_0(\frac{\lambda}{\sqrt{C_\ell}})\gamma_1(\frac{\lambda}{\sqrt{C_\ell}}) \left|Y_{\ell 0}(\theta,\phi)\right|^2 \left\{\frac{2\ell+1}{4\pi} - \left|Y_{\ell 0}(\theta,\phi)\right|^2\right\}
\end{aligned}$$

$$+\gamma_1^2(\frac{\lambda}{\sqrt{C_\ell}})\sum_{m'\neq m},\ m,m'\neq 0\left|Y_{\ell m}(\theta,\phi)\right|^2\left|Y_{\ell m'}(\theta,\phi)\right|^2 .$$

Likewise

$$E\left\{T_\ell^{reg*}(\theta,\phi)\right\}^4=\gamma_2(\frac{\lambda}{\sqrt{C_\ell}})\sum_m\left|Y_{\ell m}^{\mathcal{R}}(\theta,\phi)\right|^4+\gamma_0^2(\frac{\lambda}{\sqrt{C_\ell}})\sum_{m'\neq m}\left|Y_{\ell m}(\theta,\phi)\right|^2\left|Y_{\ell m'}(\theta,\phi)\right|^2 .$$

As $\lambda/\sqrt{C_\ell}\to\infty$*, the trispectrum is then asymptotic to*

$$\lim_{\lambda/\sqrt{C_\ell}\to\infty}\frac{E\left\{T_\ell^{reg}(\theta,\phi)^4\right\}}{E\left\{T_\ell^{reg}(0,0)^4\right\}P_\ell^4(\cos\theta)}=1\ .$$

For the real-valued regularization scheme we get

$$\lim_{\lambda/\sqrt{C_\ell}\to\infty}\frac{E\left\{T_\ell^{reg*}(\theta,\phi)^4\right\}}{E\{T_\ell^{reg*}(0,0)^4\}V_\ell(\theta,\phi)}=1,$$

where as $\ell\to\infty$

$$V_\ell(\theta,\phi)=\left(\frac{4\pi}{2\ell+1}\right)^2\sum_m(Y_{\ell,m}^{\mathcal{R}})^4\to\begin{cases}1, & \textit{for } (\theta,\phi)=(0,0),\\ 0\ \textit{a.e.}, & \textit{otherwise}\ .\end{cases}$$

Proof. Recall that $E\left\{T_\ell^{reg}(0,0)^4\right\}=E\{|a_{\ell 0}^{reg}|^4\}\left\{\frac{2\ell+1}{4\pi}\right\}^2$, and note that

$$\begin{aligned}
&E\left\{T_\ell^{reg}(\theta,\phi)^4\right\}\\
&=\sum_m E\{|a_{\ell m}^{reg}|^4\}\left|Y_{\ell m}(\theta,\phi)\right|^4+\sum_{m\neq m'}E\{|a_{\ell m}^{reg}|^2\}E\{|a_{\ell m'}^{reg}|^2\}\left|Y_{\ell m}(\theta,\phi)\right|^2\left|Y_{\ell m'}(\theta,\phi)\right|^2\\
&=\sum_{m\neq 0}E\{|a_{\ell m}^{reg}|^4\}\left|Y_{\ell m}(\theta,\phi)\right|^4+\sum_{m\neq m'}E\{|a_{\ell m}^{reg}|^2\}E\{|a_{\ell m'}^{reg}|^2\}\left|Y_{\ell m}(\theta,\phi)\right|^2\left|Y_{\ell m'}(\theta,\phi)\right|^2 \qquad (17)\\
&\quad+E\{(a_{\ell 0}^{reg})^4\}\left\{\frac{2\ell+1}{4\pi}\right\}^2P_\ell^4(\cos\theta)
\end{aligned}$$

whence it suffices to notice that the expected values in (17) are all of smaller order with respect to $E\{(a_{\ell 0}^{reg})^4\}$. Indeed from (12) it follows easily that

$$\lim_{\lambda/\sqrt{C_\ell}\to\infty}\frac{\sum_{m\neq m'}E\{|a_{\ell m}^{reg}|^2\}E\{|a_{\ell m'}^{reg}|^2\}}{E\left\{(a_{\ell 0}^{reg})^4\right\}}\leq\lim_{\lambda/\sqrt{C_\ell}\to\infty}\ (4\ell^2+2\ell)\max_{m\neq m'}\frac{E\{|a_{\ell m}^{reg}|^2\}E\{|a_{\ell m'}^{reg}|^2\}}{E\left\{(a_{\ell 0}^{reg})^4\right\}}=0\ .$$

Note also that

$$\begin{aligned}
E\{|a_{\ell m}^{reg}|^4\} &= \int_0^\infty r^4 f_{\rho_{\ell m}}(r)dr\\
&= \int_\lambda^\infty (r-\lambda)^4 2\frac{r}{C_\ell}\exp(-\frac{r^2}{C_\ell})dr\\
&= 2C_\ell^2\int_\lambda^\infty(\frac{r-\lambda}{\sqrt{C_\ell}})^4\frac{r}{\sqrt{C_\ell}}\exp(-\frac{r^2}{C_\ell})d\frac{r}{\sqrt{C_\ell}}
\end{aligned}$$

$$= \quad 2C_\ell^2 \int_{\lambda/\sqrt{C_\ell}}^{\infty} (u - \frac{\lambda}{\sqrt{C_\ell}})^4 u \exp(-u^2) du \ .$$

It follows that

$$\begin{aligned}
\frac{E\{|a_{\ell m}^{reg}|^4\}}{E\{(a_{\ell 0}^{reg})^4\}} &= \frac{2C_\ell^2 \int_{\lambda/\sqrt{C_\ell}}^{\infty} (u - \frac{\lambda}{\sqrt{C_\ell}})^4 u \exp(-u^2) du}{\frac{2}{\sqrt{2\pi}} C_\ell^2 \int_{\lambda/\sqrt{C_\ell}}^{\infty} (u - \frac{\lambda}{\sqrt{C_\ell}})^4 \exp\left\{-\frac{u^2}{2}\right\} du} \\
&\leq K \exp\left\{-\frac{\lambda^2(1-\varepsilon)}{2C_\ell}\right\}, \text{ any } \varepsilon > 0 \ ,
\end{aligned}$$

so that

$$\lim_{\lambda/\sqrt{C_\ell}\to\infty} \frac{E\{|a_{\ell m}^{reg}|^4\}}{E\left\{(a_{\ell 0}^{reg})^4\right\}} = 0 \ .$$

It is then immediate to see that $\left[(17)/E\left\{T_\ell^{reg}(0,0)^4\right\}\right] \to 1$ as $\lambda/\sqrt{C_\ell} \to \infty$, whence our first result is established. By an analogous argument, it is easy to see that

$$\begin{aligned}
\frac{E\left\{T_\ell^{reg*}(\theta,\phi)^4\right\}}{E\{T_\ell^{reg*}(0,0)^4\}V_\ell(\theta,\phi)} &= \frac{\sum_m E\{\left|a_{\ell m}^{reg*}\right|^4\} \left|Y_{\ell m}^{\mathcal{R}}(\theta,\phi)\right|^4}{E\{T_\ell^{reg*}(0,0)^4\}V_\ell(\theta,\phi)} \\
&+ \frac{\sum_{m\neq m'} E\{\left|a_{\ell m}^{reg*}\right|^2\}E\{\left|a_{\ell m'}^{reg*}\right|^2\} \left|Y_{\ell m}^{\mathcal{R}}(\theta,\phi)\right|^2 \left|Y_{\ell m'}^{\mathcal{R}}(\theta,\phi)\right|^2}{E\{T_\ell^{reg*}(0,0)^4\}V_\ell(\theta,\phi)} \to 1 \ ,
\end{aligned}$$

so that we need only investigate the asymptotic behavior of

$$V_\ell(\theta,\phi) := \left\{\frac{4\pi}{2\ell+1}\right\}^2 \sum_{m=-\ell}^{\ell} (Y_{\ell,m}^{\mathcal{R}})^4 \ .$$

To this aim, we recall the following recent result by Sogge and Zelditch [23]; as $\ell \to \infty$

$$\frac{1}{2\ell+1} \int_{S^2} \sum_{m=-\ell}^{\ell} |Y_{\ell,m}(x)|^4 \, dx = o(\{\log \ell\}^{1/4}) \ .$$

Now of course

$$\begin{aligned}
\sum_{m=-\ell}^{\ell} |Y_{\ell,m}(x)|^4 &= \sum_{m=-\ell}^{\ell} \left\{|\mathrm{Re}(Y_{\ell,m}(x))|^2 + |\mathrm{Im}(Y_{\ell,m}(x))|^2\right\}^2 \\
&= \sum_{m=-\ell}^{\ell} \left\{|\mathrm{Re}(Y_{\ell,m}(x))|^4 + |\mathrm{Im}(Y_{\ell,m}(x))|^4 + 2\,|\mathrm{Re}(Y_{\ell,m}(x))|^2\,|\mathrm{Im}(Y_{\ell,m}(x))|^2\right\} \\
&\geq \frac{1}{2} \sum_{m=-\ell}^{\ell} (Y_{\ell,m}^{\mathcal{R}})^4 \ ,
\end{aligned}$$

from which we obtain immediately

$$\left\{\frac{4\pi}{2\ell+1}\right\}^2 \sum_{m=-\ell}^{\ell} (Y_{\ell,m}^{\mathcal{R}})^4 = o(\frac{\{\log \ell\}^{1/4}}{\ell}) \text{ for almost all } x \in S^2,$$

as claimed. ■

Remark 19 *The previous Theorem can be expressed in plain words as follows: under the complex-valued regularization scheme, after normalization, the trispectrum behaves asymptotically as the fourth power of the Legendre polynomial. In the real-valued case, the normalized trispectrum behaves as the averaged sum of the fourth-powers of (real-valued) spherical harmonics. The first result is heuristically explained considering that sparsity will enforce the choice of the single coefficient* $a_{\ell 0}^{reg}$ *more and more often,as* $\lambda/\sqrt{C_\ell} \to 0$*; in the latter case, each coefficient* $a_{\ell m}^{\mathcal{R}}$ *has the same probability to be selected, as they are all identically distributed: however in the limit at most one of them will be nonzero, so the trispectrum will reproduce the oscillations of a single (randomly chosen) functions* $Y_{\ell m}^{\mathcal{R}}$*. Note that as* $\ell \to \infty$*,* $P_\ell(\cos\theta) \to 0$ *for all* $\theta \neq 0, \pi$*, whence in both cases the trispectrum at the Poles has a dominating behavior with respect to almost all other directions.*

5 Some Concluding Remarks

In this paper, we have shown that convex regularization of spherical isotropic Gaussian fields with a Fourier dictionary does not preserve in general the Gaussianity and isotropy properties of the input random fields. We refer to [13] for more discussions form a physical point of view and ample numerical evidence to illustrate these claims, in a setting related to Cosmological data analysis.

In a nutshell, our arguments can be summarized as follows. The result of convex regularization is basically a form of soft-thresholding on the spherical harmonic coefficients $\{a_{\ell m}\}$. These coefficients are hence independent and nonGaussian, whence anisotropy follows. Indeed, this finding complements earlier results from [2, 3, 16, 4], entailing that independent coefficients in a spherical harmonic basis are necessarily Gaussian under isotropy, and therefore cannot be sparse in the usual meaning with which this concept is understood.

It seems hence quite natural to extend our results and to suggest that for Gaussian isotropic random fields defined on homogeneous spaces of noncommutative groups, sparsity cannot be imposed on the random coefficients of a Fourier basis. The crucial difficulty here is the choice of a Fourier basis as a sparsity dictionary in a noncommutative setting; in particular it should be noted that our arguments do not entail that anisotropy will arise when choosing, for instance, a wavelet frame as a dictionary. Likewise, no anisotropy would arise for homogeneous spaces of commutative groups: for instance, soft- or hard- thresholding the random Fourier coefficients of isotropic random fields on the circle does not make these random fields anisotropic. It is indeed the noncommutative manifold structure of the sphere, and in particular the multiplicity of eigenfunctions corresponding to the same eigenvalue, which brings in a conflict between independence and nonGaussianity, under isotropy assumptions; because the random spherical harmonics coefficients arising from convex regularization are independent and nonGaussian, anisotropy follows.

The paper has presented a number of characterizations of such anisotropy as a function of the angular power spectra of the input fields; the relevance of these findings for the areas where convex regularization is exploited as a preliminary step for spherical data analysis is going to be investigated elsewehere.

References

[1] Abrial, P., Moudden, Y., Starck, J., et al. (2007), Morphological Component Analysis and Inpainting over the Sphere: Application in Physics and Astrophysics, *Journal of Fourier Analysis and Applications,* 13, 66, 729-748.

[2] Baldi, P. and Marinucci, D. (2007) Some Characterizations of the Spherical Harmonics Coefficients for Isotropic Random Fields. *Statistics and Probability Letters* 77, 5, 490-496.

[3] Baldi, P., Marinucci, D. and Varadarajan, V.S.V. (2007) On the Characterization of Isotropic Gaussian Fields on Homogeneous Spaces of Compact Groups, *Electronic Communications in Probability*, Vol. 12, 291-302.

[4] Baldi, P. and Trapani, S. (2013) Fourier Coefficients of Invariant Random Fields on Homogeneous Spaces of Compact Groups, arxiv preprint 1304.5142

[5] Baraniuk, R., Cevher, V., and Wakin, M. (2010), Low-Dimensional Models for Dimensionality Reduction and Signal Recovery: A Geometric Perspective, *Proceedings of the IEEE*, 98, 959.

[6] Bennett, C. L. et al. (2012) Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Final Maps and Results, arXiv:1212.5225

[7] Candes, E., Romberg, J. and Tao, T. (2005) Stable Signal Recovery from Incomplete and Inaccurate Measurements. *Comm. Pure Appl. Math.,* 59, 1207-1223.

[8] Chen, S.S., Donoho, D.L., Saunders, M.A. (1999) Atomic Decomposition by Basis Pursuit, *SIAM Journal on Scientific Computing* 20, 1, 33-61

[9] Daubechies, I., Fornasier, M. and Loris, I. (2008) Accelerated Projected Gradient Method for Linear Inverse Problems with Sparsity Constraints, *Journal of Fourier Analysis and Applications,* 14, 5-6, pp.764-792

[10] Donoho, D. (2006), For Most Large Underdetermined Systems of Linear Equations the Minimal 1-norm Solution is Also the Sparsest Solution, *Communications on Pure and Applied Mathematics,* 59,797-829.

[11] Dupe, F.-X., Rassat, A., Starck, J.-L., and Fadili, M. J. (2011), Measuring the Integrated Sachs-Wolfe Effect, *Astronomy and Astrophysics,* 534.

[12] Efron, B., Hastie, T., Johnstone, I. and Tibshirani R. (2004) Least Angle Regression, *Annals of Statistics*, 32, 2, 407-499

[13] Feeney, S., Marinucci, D. McEwen, J., Peiris, H., Wandelt, B. and Cammarota, V. (2013), Sparse Inpainting and Isotropy, arXiv:1308.0602. *Submitted*

[14] Hu, W. (2001) The Angular Trispectrum of the CMB *Phys.Rev.* D64 (2001) 083005

[15] Fornasier, M. (2007) Domain Decomposition Methods for Linear Inverse Problems with Sparsity Constraints, *Inverse Problems,* 23, 2505-2526

[16] Marinucci, D. and Peccati, G. (2011) *Random Fields on the Sphere: Representations, Limit Theorems and Cosmological Applications,* Cambridge University Press, Cambridge.

[17] McEwen J.D., Puy G., Thiran J.P., Vandergheynst P., Ville D.V.D., Wiaux Y., (2013), Sparse Image Reconstruction on the Sphere: Implications of a New Sampling Theorem, *IEEE Trans. Image* Proc., 22, 6, 2275, arXiv:1205.1013

[18] Osborne, M.R., Presnell, B. and Turlach B.A. (2000) A New Approach to Variable Selection in Least Squares Problems, *IMA Journal of Numerical Analysis,* 20, 3, 389-403

[19] Rauhut, H. and Ward, R. (2012), Sparse Legendre expansions via ℓ^1-minimization, *Journal of Approximation Theory,* Vol.164, Issue 5.

[20] Rauhut, H. and Ward, R. (2012), Sparse Recovery for Spherical Harmonic Expansions, arXiv 1102.4097

[21] Planck Collaboration (2013) Planck 2013 results. I. Overview of Products and Scientific Results, arXiv 1303.5062

[22] Planck Collaboration (2013) Planck 2013 Results. XXIV. Constraints on primordial non-Gaussianity, arXiv:1303.5084 [pdf, other]

[23] Sogge, C. D. and Zelditch, S. (2012) Concerning the L^4 norms of typical eigenfunctions on compact surfaces, Recent Development in Geometry and Analysis, ALM 23, Int.Press, Beijing-Boston, 407-423, arXiv:1011.0215

[24] Starck, J.-L., Murtagh, F., and Fadili, M. (2010), *Sparse Image and Signal Processing,* Cambridge University Press, Cambridge.

[25] Starck, J.-L., Fadili, M. J., and Rassat, A. (2013), Low-l CMB Analysis and Inpainting, *Astronomy and Astrophysics*, 550, A15, 8

[26] Starck, J.-L., Donoho, D.L. , Fadili, M.J., Rassat, A. (2013), Sparsity and the Bayesian Perspective, arXiv:1302.2758

[27] Tibshirani, R. (1996) Regression Shrinkage and Selection by the Lasso, *Journal of Royal Statistical Society*, 58,1, 267-288

[28] D.A. Varshalovich, A.N. Moskalev and V.K. Khersonskii (1988). *Quantum Theory of Angular Momentum*, World Scientific Press.

[29] Wright, S., Nowak, R. and Figuereido, M. (2009) Sparse Reconstruction by Separable Approximation, *IEEE Transactions on Signal Processing* 57, 2479-2493