PLANE QUARTICS: THE UNIVERSAL MATRIX OF BITANGENTS

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Abstract. Aronhold's classical result states that a plane quartic can be recovered by the configuration of any Aronhold systems of bitangents, i.e. special 7-tuples of bitangents such that the six points at which any sub-triple of bitangents touches the quartic do not lie on the same conic in the projective plane. Lehavi (cf. [?]) proved that a smooth plane quartic can be explicitly reconstructed from its 28 bitangents; this result improved Aronhold's method of recovering the curve. In a 2011 paper [?] Plaumann, Sturmfels and Vinzant introduced an 8×8 symmetric matrix that parametrizes the bitangents of a nonsingular plane quartic. The starting point of their construction is Hesse's result for which every smooth quartic curve has exactly 36 equivalence classes of linear symmetric determinantal representations. In this paper we tackle the inverse problem, i.e. the construction of the bitangent matrix starting from the 28 bitangents of the plane quartic, and we provide a Sage script intended for computing the bitangent matrix of a given curve.

1. Introduction

As a consequence of the well known Plücker formulas the number of the bitangents to a non singular curve of degree d in the projective plane is given by the expression $\frac{1}{2}d(d-2)(d^2-9)$. The main properties of the bitangents have been deeply investigated by geometers since the late nineteenth century, particularly with reference to the first non trivial case, namely the case of degree 4. Aronhold's classical result states that a plane quartic can be recovered by the configuration of any of the 288 7-tuples of bitangents such that the six points at which any sub-triple of bitangents touches the quartic do not lie on the same conic in the projective plane (cf. [?]); these 7-tuples of bitangents are known as Aronhold systems. Caporaso and Sernesi proved in [?] that the general plane quartic is uniquely determined by its 28 bitangents; furthermore, they extended this result to general canonical curves of genus $g \geq 4$ (cf. [?]). In [?] Lehavi proved that a non singular plane quartic can be reconstructed from its 28 bitangents and derived an explicit formula for the curve. These results have actually improved Aronhold's characterization of the curve; in fact, the knowledge of both the bitangents and their contact points on the curve is necessary to recover the curve by Aronhold's method, whereas the sole configuration of the bitangents is enough to describe the geometry of the plane quartic by virtue of Caporaso and Sernesi's result. In a 2011 paper [?] Plaumann, Sturmfels and Vinzant defined a 8×8 symmetric matrix to parametrize the bitangents of a nonsingular plane quartic. Their construction resorts to Hesse's classical result for which every smooth quartic curve has exactly 36 equivalence classes of linear symmetric determinantal representations, each corresponding to three quadrics in $\mathbb{P}^3$ that intersect in eight points (cf. [?]). Once such a representation is chosen, the quartic can be described as the curve of the degenerate quadrics of the net generated by the three quadrics corresponding to the representation. Since each line of the net that joins two of the eight intersection points is a bitangent of the curve, the bitangent matrix is consequently defined.

We will briefly recall this construction (see [?] for more details and historical notes).
Let $\mathbb{P}^3 = \mathbb{P}(W)$ where W is a 4-dimensional vector space, and let

$$X = \{x_1, \dots, x_8\} = Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^3$$

be an unordered set of 8 distinct points in $\mathbb{P}^3$, which are complete intersection of three quadrics. If the net

$$\Lambda_X := |H^0(\mathbb{P}^3, \mathcal{I}_X(2))| = \langle Q_1, Q_2, Q_3 \rangle$$

contains no quadric of rank ≤ 2, then X is called a *regular Cayley octad* and Λ_X is called a *regular net*. If we denote by $C_X = C \subset \Lambda_X$ the curve of the degenerate quadrics of the regular net Λ_X and by Δ the quartic hypersurface of the singular quadrics in the linear system $|H^0(\mathbb{P}^3, \mathcal{O}(2))| \cong \mathbb{P}^9$ of all the quadrics of $\mathbb{P}^3$, then $C = \Lambda_X \cap \Delta$; hence C is a plane quartic, known as the *Hesse curve* of the net. After choosing a basis $\{Q_1, Q_2, Q_3\}$ of $H^0(\mathbb{P}^3, \mathcal{I}_X(2))$, the net can be identified to $\mathbb{P}^2$ by means of the homogeneous coordinates $z = (z_1, z_2, z_3)$ as follows

$$(z_1, z_2, z_3) \longleftrightarrow Q(z) = z_1 Q_1 + z_2 Q_2 + z_3 Q_3$$

and the curve C has equation $\det(A(z)) = 0$, where $A(z)$ is the symmetric 4×4 matrix of the coefficients of the generic quadric $Q(z)$. Since the net is regular, the Hesse curve of the net is nonsingular and $\det(A(z)) = 0$ is one of its 36 determinantal representations.
For any $1 \leq i < j \leq 8$, let $L_{ij} = \langle x_i, x_j \rangle$ the line joining the points x_i and x_j of the Cayley octad. These are 28 lines L_{ij} in $\mathbb{P}^3$ that are in correspondence to the 28 bitangents of C in $\mathbb{P}^2$, as any of the 28 distinct equations

$$x_i^t A(z) x_j = 0, \tag{1.1}$$

actually defines one of the bitangents to the curve (cf. Theorem 6.3.5 [?]). This construction can be reversed, as any non singular plane quartic can be regarded as the Hesse curve of a suitable net. A non singular plane quartic is uniquely determined by the seven bitangents $b_1, \dots b_7$ of an Aronhold system, and therefore by seven points in $\mathbb{P}^2$, which can be put in correspondence to seven points $p_1, \dots, p_7$ in $\mathbb{P}^3$; this is canonically done by means of the so called *Gale transform* (see [?] for details). The set of all the quadrics in $\mathbb{P}^3$ that contain these seven points $p_1, \dots, p_7$ is a regular net Λ, generated by three quadrics Q_1, Q_2, Q_3, that intersect in $p_1, \dots, p_7$ and in another point p; hence the non singular plane quartic is the Hesse curve associated with the regular net Λ and $X = \{p_1, \dots p_7, p\}$ is the corresponding Cayley octad determined by the chosen bitangents.
As concerns the Aronhold systems of C, a notable method to reconstruct one of them from the net Λ_X is provided by the Steiner embedding (see [?] and [?] for details). The singular points of the quadrics in Λ_X describe a curve Γ in $\mathbb{P}^3$, known as the *Steiner curve* of the net. As the net is regular, this curve turns out to be a smooth curve of degree 6. More precisely, there exists an even theta characteristic θ such that the map $f : C \to \mathbb{P}^3$ sending a point $p \in C$ to the singular point of the corresponding quadric $Q(p)$ in $\mathbb{P}^3$, is defined by the complete linear series $|K + \theta|$. In particular, the map f is an isomorphism on the image $f(C) = \Gamma$, and a bijection is defined between classes of regular nets of quadrics in $\mathbb{P}^3$ up to projective equivalences and isomorphism classes of smooth curves of genus 3 associated with a fixed even theta characteristic. If an order $x_1, \dots, x_8$ is chosen for the eight points of the regular Cayley octad X of the corresponding net Λ_X, the projection from

the point x_8 onto $\mathbb{P}^2$ sends $x_1, \ldots, x_7$ to a set of points $y_1, \ldots, y_7$ and sends the Steiner curve Γ to a sextic with seven double points at $y_1, \ldots, y_7$. The images of the exceptional curves blown up from these points $y_1, \ldots, y_7$ are the seven bitangents corresponding to the lines L_{i8} joining x_i and x_8; this set of bitangents is actually an Aronhold system for the curve C.

The algorithm described in [?] is meant to compute the matrix $A(z)$ for a non singular plane quartic C, described by the equation:

$$f(z_1, z_2, z_3) = c_{400}z_1^4 + c_{310}z_1^3z_2 + c_{301}z_1^3z_3 + c_{220}z_1^2z_2^2 + c_{211}z_1^2z_2z_3 + \cdots + c_{004}z_3^4,$$

where c_{ijk} are the 15 coefficients of the quartic. A determinantal representation of C is obtained in terms of three suitable 4×4 symmetric matrices as follows:

$$f(x, y, z) = \det(z_1A_1 + z_2A_2 + z_3A_3) \equiv \det(A(z)),$$

the matrices A_1, A_2 and A_3 being associated with three quadrics Q_1, Q_2 and Q_3 in $\mathbb{P}^3$. The algorithm determines $A(z)$ from the bitangents of the curve. A 4×4 matrix V is built after a choice of three bitangents among the $\binom{28}{3} = 3276$ possible choices. As the determinant of V is identically null whenever the triple of bitangents is not a sub-triple of an Aronhold system, the algorithm needs to start from a triple of bitangents that is contained in an Aronhold system; the number of these triples is 2016 and the corresponding triples of gradients of odd theta functions are called azygetic, while the other $3276 - 2016 = 1260$ triples are known as syzygetic. Once an azygetic triple is fixed, the matrix $A(z)$ is given by the adjoint of V divided by f^2. These 2016 determinantal representations factorize into 36 equivalence classes (two representations $A(z)$ and $A'(z)$ are equivalent whenever they are conjugated under the action of GL_4, i.e $A'(z) = U^tA(z)U$, for $U \in \mathrm{GL}_4$).
The bitangent matrix originates from $A(z)$ as follows. Choosing homogeneous coordinates for the eight points of the Cayley octad naturally leads to define the 8×4 matrix:

$$\mathtt{X} := \begin{pmatrix} x_{10} & x_{11} & x_{12} & x_{13} \\ x_{20} & x_{21} & x_{22} & x_{23} \\ \vdots & \vdots & \vdots & \vdots \\ x_{80} & x_{81} & x_{82} & x_{83} \end{pmatrix},$$

and consequently the 8×8 symmetric matrix:

$$L_X(z) := \mathtt{X}\ A(z)\ {}^t\mathtt{X}. \tag{1.2}$$

Clearly, $L_X(z)$ is a matrix of rank 4 with zero entries on the main diagonal. The 28 entries of $L_X(z)$ off the main diagonal are linear forms in z that define the bitangents of C, as seen in (**??**). Notice that the determinantal representations given by each of the $\binom{8}{4} = 70$ principal 4×4 minors of the matrix represent the same quartic and lie in the same equivalence class (cf. Remark 6.3.7 of [?]). Once a representation is determined, the other 35 inequivalent representations can be obtained by acting on the corresponding Cayley octad with a Cremona transformation. Each of the $2016 = 56 \cdot 36$ azygetic triples appears as a product of the corresponding bitangents in exactly one of the $\binom{8}{3} = 56$ principal 3×3 minor of one of the 36 inequivalent bitangent matrices; thus these minors are parametrized by azygetic triples.

In this paper we tackle the inverse problem, i.e. the construction of the bitangent matrix starting from the 28 bitangents of the plane quartic. As shown by Riemann, the equations of the 28 bitangents $b_m(\tau, z)$ of the plane quartic C correspond to the 28 gradients of the odd theta functions, as they are related to the first term of the Taylor expansion of the theta functions with odd characteristics $\theta_m(\tau, z)$. To find an expression for the bitangent matrix in terms of Riemann theta functions, we need to start the construction by a suitable 8×8 matrix. Since azygetic triples occur in the defintion of the bitangent matrix, we will resort to Aronhold systems to build such a matrix. For any fixed even characteristic, there exist eight corresponding Aronhold sets (see Section **??**), ($288 = 36 \cdot 8$), which are obtained by translation from a chosen one. These 7-tuples of odd characteristics and the even one will be the rows of the matrix, whose 3×3 principal minors will contain 56 distinct azygetic triples ($2016 = 36 \cdot 56$). Thus we will work with the following bitangent matrix:

$$\mathcal{M} := \begin{pmatrix} 0 & b_{77} & b_{64} & b_{51} & b_{46} & b_{23} & b_{15} & b_{32} \\ b_{77} & 0 & b_{13} & b_{26} & b_{31} & b_{54} & b_{62} & b_{45} \\ b_{64} & b_{13} & 0 & b_{35} & b_{22} & b_{47} & b_{71} & b_{56} \\ b_{51} & b_{26} & b_{35} & 0 & b_{17} & b_{72} & b_{44} & b_{63} \\ b_{46} & b_{31} & b_{22} & b_{17} & 0 & b_{65} & b_{53} & b_{74} \\ b_{23} & b_{54} & b_{47} & b_{72} & b_{65} & 0 & b_{36} & b_{11} \\ b_{15} & b_{62} & b_{71} & b_{44} & b_{53} & b_{36} & 0 & b_{27} \\ b_{32} & b_{45} & b_{56} & b_{63} & b_{74} & b_{11} & b_{27} & 0 \end{pmatrix},$$

see Section **??** for the explanation of the meaning of the indices.

The rank of this matrix is generally equal to eight, which means one has to determine suitable coefficients c_{ij} in such a way the matrix $(c_{ij} b_{ij})$ has rank four. The aim of this paper is to determine uniquely these coefficients up to congruences by diagonal matrices, once an even characteristic m and a compatible Aronhold set of characteristics (i.e a level two structure of the moduli space of principally polarized abelian varieties of genus 3) are given. As multilinear algebra techniques are not sufficient to determine these coefficients, we will also need to properly use Riemann's relations and Jacobi's formula.

The coefficients will turn out to be modular functions holomorphic along the locus of the period matrices of smooth plane quartics (cf. Theorem **??**). Hence we get a deeper result, as we actually obtain a "universal " matrix of bitangents that varies holomorphically as the period matrix varies in the open set, given by the non hyperelliptic locus, of the the moduli space of principally polarized abelian variety with level two structure.

We also devised a Sage script to compute the bitangent matrix of a curve, once the curve is given as input. The details concerning the script, which makes use of the valuable package Abelfunctions [**?**, **?**], can be found at the end of Section **??**.

2. Aknowledgments

We are grateful to Giorgio Ottaviani and Edoardo Sernesi for explaining to us the construction of the bitangent matrix and the beautiful geometry related to plane quartics. The first three sections of this paper have been strongly influenced by discussions that the senior author had with them. We are also grateful to Igor V. Dolgachev for pointing out mistakes in the first version of the paper. Finally, we would like to thank Chris Swierczewski for helping us work with the package Abelfunctions and Alessandra Seghini for system support.

3. Aronhold systems.

In this section we introduce Aronhold systems of bitangents and Aronhold sets of theta characteristics and recall some basic facts about characteristics and the action of the symplectic group on them. A comprehensive exposition of this subject can be found in Solmon's classic treatise [**?**].

The next definition is of central importance in the geometry of plane quartics. Let C be a nonsingular plane quartic.

Definition 3.1. *A 7-tuple $\{\ell_1, \dots, \ell_7\}$ of bitangent lines to C is called an* Aronhold system of bitangents *if for each triple ℓ_i, ℓ_j, ℓ_k the six points of contact of $\ell_i \cup \ell_j \cup \ell_k$ with C are not on a conic.*

Not all 7-tuples of bitangents are Aronhold systems. There are exactly 288 Aronhold systems among the $\binom{28}{7}$ 7-tuples of bitangents of C (for more details we refer to [**?**] and [**?**]). Denote by θ_i the effective half-canonical divisor such that $2\theta_i = C \cap \ell_i$ (i.e. θ_i, or $\mathcal{O}(\theta_i)$, is an odd theta-characteristic). The condition that $\{\ell_1, \dots, \ell_7\}$ is an Aronhold system is equivalent to the condition that for each triple of pairwise distinct indices i, j, k we have

$$|2K - \theta_i - \theta_j - \theta_k| = \emptyset$$

and replacing $2K$ by $2\theta_i + 2\theta_j$ the condition is seen to be equivalent to the following: $\theta_i + \theta_j - \theta_k$ is an even theta-characteristic for each $i \neq j \neq k$. This can be taken as another definition of Aronhold system (cf. also Definition **??**).

Definition 3.2. An Aronhold set *on a non-hyperelliptic curve C of genus 3 is a 7-tuple $\{\theta_1, \dots, \theta_7\}$ of distinct odd theta-characteristics such that $\theta_i + \theta_j - \theta_k$ is an even theta-characteristic for each triple of pairwise distinct indices i, j, k.*

We have a purely combinatoric interpretation of the above description.

A characteristic m is a column vector in $\mathbb{Z}^{2g}$, with m' and m'' as first and second entry vectors. If we set:

$$e(m) = (-1)^{{}^t m' m''}, \tag{3.1}$$

then m is called even or odd according as $e(m) = 1$ or -1. For any triplet m_1, m_2, m_3 of characteristics we set

$$e(m_1, m_2, m_3) = e(m_1)e(m_2)e(m_3)e(m_1 + m_2 + m_3). \tag{3.2}$$

A sequence $m_1, \dots, m_r$ of characteristics is essentially independent if for any choice of an even number of indeces between 1 and r the sum of the corresponding characteristics is not congruent to $0 \bmod 2$.

The unique action of $\Gamma_g = \mathrm{Sp}(2g, \mathbb{Z})$ on the set of characteristics mod 2 keeping invariant (**??**), (**??**) and the condition of being essentially independent is defined by

$$\sigma \cdot m := \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} m' \\ m'' \end{pmatrix} + \begin{pmatrix} diag(C^t D) \\ diag(A^t B) \end{pmatrix}.$$

Henceforward, we often shall consider characteristics with 0 and 1 as entries. In this situation a special role is played by sequences of characteristics that form a fundamental system, defined as follows.

Definition 3.3. *A sequence of* $2g+2$ *characteristics in* $\mathbb{F}_2^{2g}$ *is a fundamental system if all its sub-triples are azygetic, i.e*

$$e(m_i, m_j, m_k) = -1,$$

for all indices $1 \leq i < j < k \leq 2g+2$.

Fundamental systems exist and are all conjugate under an extension of Γ_g by translations, we refer to [?] and [?] for details. The number of odd characteristics in a fundamental system is always congruent to $g \bmod 4$. So when $g = 3$ we have fundamental systems with 3 or 7 odd characteristics.

From now on we fix our attention on genus 3 case.

Definition 3.4. *Let*

$$m_0, n_1, n_2, \ldots, n_7$$

be a fundamental system with one even characteristic, m_0*, and seven odd characteristics. In this case*

$$n_1, n_2, \ldots, n_7$$

is called an Aronhold set and necessarily $m_0 = \sum_{i=1}^{7} n_i$.

There are exactly $288 = 36 \cdot 8$ such systems, hence each even characteristic appears exactly in eight such fundamental systems. We remark that the ordered set of fundamental systems are

$$288 \cdot 7! = 36 \cdot 8! = |\mathrm{Sp}(6, \mathbb{F}_2)|.$$

Concerning these fundamental systems with a fixed even characteristic m_0, we have the following Lemma.

Lemma 3.5. *A fundamental system* $m_0, n_1, n_2, \ldots, n_7$ *determines the remaining 7 via translations.*

Proof. The other seven fundamental systems can be obtained translating the initial fundamental system $m_0, n_1, n_2, \ldots, n_7$ with $m_0 + n_i$ with $i = 1, \ldots, 7$. □

Remark 3.6. *We observe that the* 8×8 *matrix*

$$\begin{pmatrix}
m_0 & n_1 & n_2 & \ldots & n_7 \\
n_1 & m_0 & (m_0+n_1)+n_2 & \ldots & (m_0+n_1)+n_7 \\
n_2 & (m_0+n_2)+n_1 & m_0 & \ldots & (m_0+n_2)+n_7 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
n_7 & (m_0+n_7)+n_1 & \ldots & \ldots & m_0
\end{pmatrix}$$

is symmetric in the symbols. It is unique once we fix a row, up to permutation of rows (and corresponding symmetric permutation of columns). Here is an explicit example (we use a row notation):

$$\begin{pmatrix}
[000,000] & [111,111] & [110,100] & [101,001] & [100,110] & [010,011] & [001,101] & [011,010]\\
[111,111] & [000,000] & [001,011] & [010,110] & [011,001] & [101,100] & [110,010] & [100,101]\\
[110,100] & [001,011] & [000,000] & [011,101] & [010,010] & [100,111] & [111,001] & [101,110]\\
[101,001] & [010,110] & [011,101] & [000,000] & [001,111] & [111,010] & [100,100] & [110,011]\\
[100,110] & [011,001] & [010,010] & [001,111] & [000,000] & [110,101] & [101,011] & [111,100]\\
[010,011] & [101,100] & [100,111] & [111,010] & [110,101] & [000,000] & [011,110] & [001,001]\\
[001,101] & [110,010] & [111,001] & [100,100] & [101,011] & [011,110] & [000,000] & [010,111]\\
[011,010] & [100,101] & [101,110] & [110,011] & [111,100] & [001,001] & [010,111] & [000,000]
\end{pmatrix}.$$

Notice that 36 essentially different matrices can be built in such a way, each of them corresponding to one of the 36 even characteristics.

In the next section we will show how this matrix can be used to build the bitangent matrix by means of gradients of odd theta functions.

4. Theta Functions

We intend to give an explicit analytic expression for the bitangents. The main tool will be theta functions. We denote by $\mathcal{H}_g$ the *Siegel upper half-space*, i.e. the space of complex symmetric $g \times g$ matrices with positive definite imaginary part. An element $\tau \in \mathcal{H}_g$ is called a *period matrix*, and defines the complex abelian variety $X_\tau := \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. The group $\Gamma_g := \mathrm{Sp}(2g, \mathbb{Z})$ acts on $\mathcal{H}_g$ by automorphisms. For

$$\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z})$$

the action is $\gamma \cdot \tau := (a\tau + b)(c\tau + d)^{-1}$. The quotient of $\mathcal{H}_g$ by the action of the symplectic group is the moduli space of principally polarized abelian varieties (ppavs): $\mathcal{A}_g := \mathcal{H}_g/\mathrm{Sp}(2g, \mathbb{Z})$. The case $g = 1$ is special and in the following we will always assume $g > 1$.

We define the *level* subgroups of the symplectic group to be

$$\Gamma_g(n) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_g \,|\, \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n \right\},$$

$$\Gamma_g(n, 2n) := \left\{ \gamma \in \Gamma_g(n) \,|\, \mathrm{diag}(a^t b) \equiv \mathrm{diag}(c^t d) \equiv 0 \bmod 2n \right\}.$$

The corresponding *level moduli spaces of ppavs* are denoted $\mathcal{A}_g(n)$ and $\mathcal{A}_g(n, 2n)$, respectively.

A holomorphic function $F : \mathcal{H}_g \to \mathbb{C}$ is called a *modular form of weight k with respect to* $\Gamma \subset \Gamma_g$ if

$$F(\gamma \cdot \tau) = \det(c\tau + d)^k F(\tau), \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ \forall \tau \in \mathcal{H}_g.$$

More generally, let $\rho : \mathrm{GL}(g, \mathbb{C}) \to \mathrm{End}\, V$ be some representation. Then a map $F : \mathcal{H}_g \to V$ is called a *ρ- or V-valued modular form*, or, if there is no ambiguity in the choice of ρ, simply a *vector-valued modular form*, with respect to $\Gamma \subset \Gamma_g$ when

$$F(\gamma \cdot \tau) = \rho(c\tau + d)F(\tau), \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ \forall \tau \in \mathcal{H}_g.$$

For $m', m'' \in \mathbb{Z}^g$ and $z \in \mathbb{C}^g$ we define the *theta function with characteristic* $m = \left[\begin{smallmatrix} m' \\ m'' \end{smallmatrix}\right]$ to be

$$\vartheta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, z) := \vartheta_m(\tau, z) := \sum_{p \in \mathbb{Z}^g} \exp \pi i \left[\left(p + \frac{m'}{2}, \tau(p + \frac{m'}{2})\right) + 2\left(p + \frac{m'}{2}, z + \frac{m''}{2}\right)\right],$$

where $(\cdot, \cdot)$ in the argument of the exponential function denotes the usual scalar product. Here we list some properties of the theta functions. First, we observe that

$$\vartheta_{m+2n}(\tau, z) = (-1)^{{}^t m' n''} \vartheta_m(\tau, z), \qquad n \in \mathbb{Z}^{2g}.$$

Hence, the theta functions with characteristics can be parametrized by 2^{2g} vector columns m', m'' with m' and m'' thought as entries in $\{0,1\}^g$. Note that these are the roots of the canonical bundle. The preceding formula is called *reduction formula*. Henceforward, we will refer to such characteristics as *reduced characteristics* and to the corresponding theta functions as theta functions with half integral characteristics; clearly all the properties stated in Section **??** also hold in this case. Then, we recall the behavior of the theta functions under a change of sign of the z variable:

$$\vartheta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, -z) = e(m) \vartheta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, z).$$

The following formula shows that adding a so called *half period*, $\tau\frac{m'}{2} + \frac{m''}{2}$, to the argument z actually permutes the functions with half integral characteristics (see [**?**] or [**?**]):

$$\vartheta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, z) = \exp\left(\pi i \left[\left(\frac{m'}{2}, \tau\frac{m'}{2}\right) + 2\left(\frac{m'}{2}, z + \frac{m''}{2}\right)\right]\right) \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau, z + \tau\frac{m'}{2} + \frac{m''}{2}).$$

The *reduced characteristic* m is called *even* or *odd* depending on whether the scalar product $m' \cdot m'' \in \mathbb{Z}_2$ is zero or one and the corresponding theta function is even or odd in z, respectively. The number of even (resp. odd) theta characteristics is $2^{g-1}(2^g+1)$ (resp. $2^{g-1}(2^g-1)$). The transformation law for theta functions under the action of the symplectic group is (see [**?**]):

$$\vartheta \begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, z) = \phi(m', m'', \gamma, \tau, z) \det(c\tau + d)^{1/2} \vartheta \left[({}^t\gamma^{-1}) \begin{pmatrix} m' \\ m'' \end{pmatrix}\right] (\gamma \cdot \tau, (c\tau + d)z),$$

where ϕ is some complicated explicit function, and the action of ${}^t\gamma^{-1}$ on characteristics is taken modulo integers. It is further known (see [**?**], [**?**]) that for $\gamma \in \Gamma_g(4,8)$ we have $\phi|_{z=0} = 1$, while ${}^t\gamma^{-1}$ acts trivially on the characteristics m. Thus the theta functions valued at $z = 0$, namely the so called *theta constants*, are modular forms of weight one half with respect to $\Gamma_g(4,8)$, Henceforward, we will denote them by the symbol θ_m.

The group $\Gamma_g(2)/\Gamma_g(4,8)$ acts on the set of theta constants θ_m by certain characters whose values are fourth roots of the unity, as shown in [**?**]. The action of $\Gamma_g/\Gamma_g(2)$ on the set of theta with half integral characteristics is by permutations. Since the group $\Gamma_g(1,2)$ fixes the null characteristic, it acts on θ_0 by a multiplier.

All odd theta constants with half integral characteristics vanish identically, as the corresponding theta functions are odd functions of z, and thus there are $2^{g-1}(2^g+1)$ non-trivial theta constants.

Differentiating the theta transformation law above with respect to different z_i and then evaluating at $z=0$, we see that for $\gamma \in \Gamma_g(4,8)$ and $m = \left[\begin{smallmatrix} m' \\ m'' \end{smallmatrix}\right]$ odd

$$\frac{\partial}{\partial z_i}\vartheta\begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau,z)|_{z=0} = \det(c\tau+d)^{1/2}\sum_j (c\tau+d)_{ij}\frac{\partial}{\partial z_j}\vartheta\begin{bmatrix} m' \\ m'' \end{bmatrix}(\gamma\cdot\tau,(c\tau+d)z)|_{z=0},$$

in other words the gradient vector $\mathrm{grad_z}\,\vartheta\begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau,0)$ is a $\mathbb{C}^g$-valued modular form with respect to $\Gamma_g(4,8)$ under the representation $\rho(A) = (\det A)^{1/2}A$, for $A \in \mathrm{GL}(g,\mathbb{C})$.

The set of all even theta constants defines the map

$$I\!P\mathrm{Th} : \mathcal{A}_\mathrm{g}(4,8) \to \mathrm{P}^{2^{\mathrm{g}-1}(2^\mathrm{g}+1)-1}, \qquad \bar\tau \mapsto [\cdots,\theta_\mathrm{m}(\tau),\cdots],$$

with $\bar\tau \in A_g(4,8) = \mathcal{H}_g/\Gamma_g(4,8)$ and τ a representative of the equivalence class $\bar\tau$. It is known that the map $I\!P\mathrm{Th}$ is injective, see [?] and references therein. Considering the set of gradients of all odd theta functions at zero gives the map

$$\mathrm{grTh} : \mathcal{H}_\mathrm{g} \to (\mathbb{C}^\mathrm{g})^{\times 2^{\mathrm{g}-1}(2^\mathrm{g}-1)}, \qquad \tau \mapsto \mathrm{grTh}(\tau) := \left\{\cdots,\mathrm{grad_z}\vartheta\begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau,0),\cdots\right\}_{\mathrm{all\ odd\,m}},$$

which due to modular properties descends to the quotient

$$I\!P\mathrm{grTh} : \mathcal{A}_\mathrm{g}(4,8) \to (\mathbb{C}^\mathrm{g})^{\times 2^{\mathrm{g}-1}(2^\mathrm{g}-1)}/\rho(\mathrm{GL}(\mathrm{g},\mathbb{C})),$$

where $\mathrm{GL}(g,\mathbb{C})$ acts simultaneously on all $\mathbb{C}^g$'s in the product by ρ.

The image of $I\!P\mathrm{grTh}$ actually lies in the Grassmannian,

$$I\!P\mathrm{grTh} : \mathcal{A}_\mathrm{g}(4,8) \to \mathrm{Gr}_\mathbb{C}(\mathrm{g},\, 2^{\mathrm{g}-1}(2^\mathrm{g}-1))$$

of g-dimensional subspaces in $\mathbb{C}^{2^{g-1}(2^g-1)}$. The Plücker coordinates of this map are modular forms of weight $\frac{g}{2}+1$ and have been extensively studied, see [?, ?, ?]. Moreover, in [?], it is implicitly proved the following proposition.

Proposition 4.1. *The map*

$$I\!P\mathrm{grTh} : \mathcal{A}_3(4,8) \to \mathrm{Gr}_\mathbb{C}(3,\, 28)$$

is injective.

In genus 3 case the evaluation at zero of $\vartheta\begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau,z)$ and of the gradients have a significative meaning. In fact, as consequence of Riemann singularity theorem the following proposition holds.

Proposition 4.2. *Let τ be a jacobian matrix, then it is the period matrix of a hyperelliptic jacobian if and only if there exist an even characteristic m with $\vartheta\begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau,0)=0$.*

Let τ be the period matrix of a non-hyperelliptic jacobian (i.e the jacobian of a plane quartic), then for all odd characteristics m the gradient vector $\mathrm{grad_z}\vartheta\begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau,0)$ parametrizes the bitangents of the plane quartic.

Remark 4.3. *The equations of the bitangents are*

$$b_m(\tau,z) := \frac{\partial\vartheta\begin{bmatrix}m'\\m''\end{bmatrix}(\tau,z)}{\partial z_1}|_{z=0}z_1 + \frac{\partial\vartheta\begin{bmatrix}m'\\m''\end{bmatrix}(\tau,z)}{\partial z_2}|_{z=0}z_2 + \frac{\partial\vartheta\begin{bmatrix}m'\\m''\end{bmatrix}(\tau,z)}{\partial z_3}|_{z=0}z_3 = 0.$$

From now on, if it will be necessary, we will identify the gradient vectors with the bitangents

The following corollary is easily derived.

Corollary 4.4. *The hyperelliptic locus $\mathcal{I}_3 \subset \mathcal{A}_3$ is defined by the equation*

$$\prod_{m\, even} \vartheta\begin{bmatrix}m'\\m''\end{bmatrix}(\tau,0) = 0.$$

5. The bitangent matrix

This section will be entirely devoted to the description of the bitangent matrix related to the plane quartic. First of all, we will see how the language introduced in the previous section suitably translates the combinatorics described in Section **??** so as to let us build the matrix. Then, we will recall the properties of such a matrix and how it completely describes the geometry of the curve.

Using the language of Section **??**, the geometric condition defining an Aronhold system can be rephrased as a combinatorial condition, as $\theta_i + \theta_j - \theta_k$ is an even theta characteristic whenever $m_i + m_j + m_k$ is an even characteristic. Hence, we can use the statement in Remark **??** to build the 8×8 symmetric matrix $L_X(z)$ as defined in (**??**) in terms of Riemann theta functions. More precisely, using the notations in Remark **??** we set

$$M_X(z) := \begin{pmatrix} 0 & b_{n_1}(\tau,z) & \dots & b_{n_7}(\tau,z) \\ b_{n_1}(\tau,z) & 0 & \dots & b_{n_7+m_0+n_1}(\tau,z) \\ \dots & \dots & 0 & \dots \\ b_{n_7}(\tau,z) & b_{n_1+m_0+n_7}(\tau,z) & \dots & 0 \end{pmatrix},$$

and

$$M_i := \begin{pmatrix} 0 & \frac{\partial}{\partial z_i}\vartheta_{n_1}(\tau,z)|_{z=0} & \cdots & \frac{\partial}{\partial z_i}\vartheta_{n_7}(\tau,z)|_{z=0} \\ \frac{\partial}{\partial z_i}\vartheta_{n_1}(\tau,z)|_{z=0} & 0 & \cdots & \frac{\partial}{\partial z_i}\vartheta_{n_7+m_0+n_1}(\tau,z)|_{z=0} \\ \cdots & \cdots & 0 & \cdots \\ \frac{\partial}{\partial z_i}\vartheta_{n_7}(\tau,z)|_{z=0} & \frac{\partial}{\partial z_i}\vartheta_{n_1+m_0+n_7}(\tau,z)|_{z=0} & \cdots & 0 \end{pmatrix}.$$

The matrices

$$M_X(z) = z_1M_1 + z_2M_2 + z_3M_3$$

and $L_X(z)$ are strictly related, as we are going to prove in Proposition **??**. In both cases the entries of the matrices $M_X(z)$ and $L_X(z)$ are the bitangents to the canonical curves, but they are not uniquely determined, as the entries can differ by distinct proportionality factors; therefore, the rank of the matrix $M_X(z)$ is not necessarily equal to 4.

We will manipulate the matrix $M_X(z)$ and determine suitable coefficients c_{ij} in order that the matrix $(c_{ij}M_X(z)_{ij})$ has rank four; this will be done by using the action of the symplectic group, Riemann theta formula and Jacobi's derivative formula. We briefly recall them in the genus 3 case.

For any triple n_1, n_2, n_3 of odd characteristics we set

$$D(n_1, n_2, n_3)(\tau) := \mathrm{grad_z}\, \vartheta_{\mathrm{n_1}}(\tau, 0) \wedge \mathrm{grad_z}\, \vartheta_{\mathrm{n_2}}(\tau, 0) \wedge \mathrm{grad_z}\, \vartheta_{\mathrm{n_3}}(\tau, 0).$$

We recall when such nullwerte of jacobian of theta functions is a polynomial in the theta constants. The following statement holds, see [**?**].

Proposition 5.1. *Suppose that $g = 3$ and n_1, n_2, n_3 are odd characteristics distinct mod 2; then $D(n_1, n_2, n_3)$ is a polynomial in the theta constants if and only if n_1, n_2, n_3 form an azygetic triplet. Moreover, Jacobi's formula holds:*

$$D(n_1, n_2, n_3)(\tau) = \pm(\pi)^3 \theta_{m_1}\theta_{m_2}\theta_{m_3}\theta_{m_4}\theta_{m_5}(\tau),$$

with $m_1, \dots, m_5$ even characteristics and $n_1, n_2, n_3, m_1, \dots, m_5$ a uniquely determined fundamental system of characteristics.

Also, we briefly recall that Riemann's quartic addition theorem for theta constants with characteristic in genus three has the form

$$r_1 = r_2 + r_3,$$

where each r_i is a product of four theta constants with characteristics forming an even coset of a two-dimensional isotropic space. Such isotropic spaces are constructed by means of the symplectic form on $\mathbb{F}_2^6$ defined by

$$e(m, n) := (-1)^{m'^t n'' - m''^t n'}.$$

A full description of the curve C is provided by the matrix $L_X(z)$ defined in (**??**), as the following notable proposition states.

Proposition 5.2. *Let $L_X(z)$ be the matrix defined in* (**??**)*, then:*

- (i) *given the net Λ_X and a basis $\{Q_1, Q_2, Q_3\}$ of it, the matrix $L_X(z)$ is uniquely defined up to simultaneous multiplication by a constant factor of a row and the corresponding column and up to simultaneous permutations of rows and columns;*
- (ii) *the 28 entries of $L_X(z)$ outside the main diagonal are linear forms in z that define the bitangents of C;*
- (iii) *the seven bitangents on a given row (column) are elements of an Aronhold system. The 8 Aronhold systems represented by the rows (columns) of $L_X(z)$ are associated to the even theta characteristic on C defined by the net Λ_X;*
- (iv) *$L_X(z)$ has identically rank 4, and* any *of its 4×4 minors is a polynomial of degree 4 in z which defines C.*

Proof.

- (i) A change of the order of the eight points of the Cayley octad $x_1, \dots, x_8$ corresponds to a simultaneous permutation of rows and columns for the matrix. A change of the homogeneous coordinates of these points corresponds to a simultaneous multiplication by a constant factor. Hence, the statement follows.
- (ii) We refer to [**?**] for details (cf. Section 1, eq. (1.3)).

(iii) Note first that no three points of the Cayley octad $X = \{x_1, \dots, x_8\}$ lie on the same line, because the net Λ_X is regular. It follows that no four points of X are coplanar, because if $x_1, \dots, x_4$ belong to a plane H, they are contained in a pencil of conics of H. Then the restriction map $H^0(I\!P^3, \mathcal{I}_X(2)) \longrightarrow H^0(H, \mathcal{I}_{X\cap H}(2))$ has a kernel, thus contradicting the fact that Λ_X cannot contain a reducible quadric.
Fix $i \in \{1, \dots, 8\}$ and let $\Gamma \subset I\!P^3$ the Steiner curve of the quartic C. The projection of Γ from x_i onto $I\!P^2$ is a sextic with seven double points corresponding to the images of the remaining seven points of the Cayley octad. The images of the exceptional curves blown up from these double points are the seven bitangents corresponding to the lines $L_{ij} =< x_i, x_j >$, which are chords of Γ. Therefore, if we set $\{p_{ij}, q_{ij}\} := \Gamma \cap L_{ij}$, then $\theta_{ij} := p_{ij} + q_{ij}$, whenever $j \neq i$, is an odd theta characteristic. Assume that $\{\theta_{ij} : j \neq i\}$ is not an Aronhold system, then there exists a triple θ_{ij}, θ_{ik} and θ_{ih} such that $\theta_{ij} + \theta_{ik} - \theta_{ih} = \theta_{rs}$ for some odd theta characteristic θ_{rs}. On the other hand, since the points x_i, x_j and x_k are coplanar, then $\theta_{ij} + \theta_{ik} + \theta_{jk} \sim K + \theta$, where θ is the even theta characteristic defining the Steiner curve. Therefore, $\theta_{jk} + \theta_{ih} + \theta_{rs} \sim K + \theta$, which implies that the lines L_{jk} and L_{ih} are coplanar. This is an absurd statement since the points x_i, x_j, x_k and x_h cannot be coplanar.

(iv) The claim obviously follows by the way the matrix was defined.

□

6. Determining analytically the bitangent matrix

The aim of this Section is to determine analytically the bitangent matrix, thus obtaining a partial converse of the Proposition **??**. As a result, the following statement will be proved:

Theorem 6.1. *Let τ be the period matrix of the jacobian of a smooth plane quartic. Once an even characteristic and a corresponding Aronhold set of characteristics (i.e. a level 2 structure) are fixed, an 8×8 matrix $L(\tau, z)$ of rank four is uniquely determined up to congruences in such a way that its entries are proportional to the linear forms associated with the 28 bitangents. The equation of the corresponding plane quartic is thus obtained by taking the determinant of any minor of degree four of $L(\tau, z)$.*

We want to obtain an explicit expression for such a matrix $L(\tau, z)$. Our initial datum will be the 28 gradients of odd theta functions evaluated at $z = 0$, corresponding to the bitangents, and a chosen even characteristic m. In order to obtain a matrix congruent to $L_X(z)$ (cf. (**??**)), we have to determine the values of the functions $c_{n_i}(\tau)$ in the matrix:

$$\begin{pmatrix} 0 & c_{n_1}(\tau)b_{n_1}(\tau,z) & c_{n_2}(\tau)b_{n_2}(\tau,z) & \dots & c_{n_7}(\tau)b_{n_7}(\tau,z) \\ c_{n_1}(\tau)b_{n_1}(\tau,z) & 0 & c_{m+n_1+n_2}(\tau)b_{m+n_1+n_2}(\tau,z) & \dots & c_{m+n_1+n_7}(\tau)b_{m+n_1+n_7}(\tau,z) \\ c_{n_2}(\tau)b_{n_2}(\tau,z) & c_{m+n_1+n_2}(\tau)b_{m+n_1+n_2}(\tau,z) & 0 & \dots & c_{m+n_2+n_7}(\tau)b_{m+n_2+n_7}(\tau,z) \\ \dots & \dots & \dots & \dots & \dots \\ c_{n_7}(\tau)b_{n_7}(\tau,z) & c_{m+n_1+n_7}(\tau)b_{m+n_1+n_7}(\tau,z) & \dots & \dots & 0 \end{pmatrix}.$$

Up to permutations, there exist 8 Aronhold sets whose sum is the given characteristic m. The subgroup of the symplectic group that fixes the characteristic m permutes the eight Aronhold and also the elements of a fixed Aronhold set. Therefore we can focus on a chosen even characteristic and a corresponding Aronhold set; as already noted, the number of possible choices is exactly equal to $|Sp(6, \mathbb{F}_2)|$. To simplify our computation we assume $m = 0$, and choose as Aronhold set the one described in Remark **??**.

Our purpose is to obtain a matrix of rank four. Because of the action of the symplectic group, we can assume that the first four columns of the matrix are linearly independent and all the others are linear combinations of the first four columns: this will be our starting hypothesis. We recall that any set of four bitangents coming from an Aronhold set form a fundamental system of $\mathbb{P}^2$. We need seven bitangents $b_1, \dots b_7$ forming an Aronhold set; we, therefore, choose the bitangents described in Section **??**. Let $\mathcal{M}$ be the symmetric matrix:

$$\mathcal{M} := \begin{pmatrix} 0 & b_{77} & b_{64} & b_{51} & b_{46} & b_{23} & b_{15} & b_{32} \\ b_{77} & 0 & b_{13} & b_{26} & b_{31} & b_{54} & b_{62} & b_{45} \\ b_{64} & b_{13} & 0 & b_{35} & b_{22} & b_{47} & b_{71} & b_{56} \\ b_{51} & b_{26} & b_{35} & 0 & b_{17} & b_{72} & b_{44} & b_{63} \\ b_{46} & b_{31} & b_{22} & b_{17} & 0 & b_{65} & b_{53} & b_{74} \\ b_{23} & b_{54} & b_{47} & b_{72} & b_{65} & 0 & b_{36} & b_{11} \\ b_{15} & b_{62} & b_{71} & b_{44} & b_{53} & b_{36} & 0 & b_{27} \\ b_{32} & b_{45} & b_{56} & b_{63} & b_{74} & b_{11} & b_{27} & 0 \end{pmatrix},$$

where $b_{ij} := \sum_{k=1}^{3}(\frac{\partial}{\partial z_k}\vartheta_{ij}|_{z=0})z_k$, ϑ_{ij} being the theta function associated with the odd characteristic $\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right] := \left[\begin{smallmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{smallmatrix}\right]$, where $i = a_1 2^2 + a_2 2 + a_3$ and $j = b_1 2^2 + b_2 2 + b_3$. Since each equation $b_{ij} = 0$ defines a bitangent, the entries can be multiplied by a function which does not depend on the variables z_1, z_2, z_3, without changing the geometry.

To find out these 28 functions of the variable τ, we will resort to the following procedure. We will first determine the coefficients of suitable 5×5 principal minors so as to get symmetric matrices of rank 4; this will define some relations among the column vectors of the matrix. Then we will act on these minors properly so as to make their common entries equal. Finally we will use the resulting relations among the column vectors to determine the remaining coefficients of the matrix.

We first focus on the submatrix obtained by taking the first five rows and the first five columns. We need to compute λ_{ij} such that:

$$\operatorname{rank} \begin{pmatrix} 0 & \lambda_{77}b_{77} & \lambda_{64}b_{64} & \lambda_{51}b_{51} & \lambda_{46}b_{46} \\ \lambda_{77}b_{77} & 0 & \lambda_{13}b_{13} & \lambda_{26}b_{26} & \lambda_{31}b_{31} \\ \lambda_{64}b_{64} & \lambda_{13}b_{13} & 0 & \lambda_{35}b_{35} & \lambda_{22}b_{22} \\ \lambda_{51}b_{51} & \lambda_{26}b_{26} & \lambda_{35}b_{35} & 0 & \lambda_{17}b_{17} \\ \lambda_{46}b_{46} & \lambda_{31}b_{31} & \lambda_{22}b_{22} & \lambda_{17}b_{17} & 0 \end{pmatrix} = 4. \tag{6.1}$$

Note that $\mathrm{rk}(\mathrm{D^tAD}) = \mathrm{rk(A)}$ for any invertible diagonal matrix D. Therefore, the matrix in (**??**) is determined up to an invertible diagonal matrix which acts by congruence.

The condition of linear dependence on the vector columns V_i of the matrix in (**??**):

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \alpha_4 V_4 + \alpha_5 V_5 = 0$$

can be turned into:

$$\tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3 + \tilde{V}_4 - \tilde{V}_5 = 0 \tag{6.2}$$

when both the sides of the matrix are multiplied by the diagonal matrix diag $\left(\alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1}, \alpha_4^{-1}, -\alpha_5^{-1}\right)$. Hence, we can compute the coefficients λ_{ij} by demanding the condition (**??**) without any loss of generality. Note that whenever such an operation is performed again on the matrix, the diagonal matrix on the left will change the coefficients in (**??**).
On the first row (**??**) leads to:

$$\lambda_{77} b_{77} + \lambda_{64} b_{64} + \lambda_{51} b_{51} = \lambda_{46} b_{46},$$

which is equivalent to a linear system of three equations in the variables λ_{77}, λ_{64}, λ_{51}, λ_{46}:

$$\begin{cases} \lambda_{77}\partial_1\vartheta_{77}|_{z=0} + \lambda_{64}\partial_1\vartheta_{64}|_{z=0} + \lambda_{51}\partial_1\vartheta_{51}|_{z=0} = \lambda_{46}\partial_1\vartheta_{46}|_{z=0} \\ \lambda_{77}\partial_2\vartheta_{77}|_{z=0} + \lambda_{64}\partial_2\vartheta_{64}|_{z=0} + \lambda_{51}\partial_2\vartheta_{51}|_{z=0} = \lambda_{46}\partial_2\vartheta_{46}|_{z=0} \\ \lambda_{77}\partial_3\vartheta_{77}|_{z=0} + \lambda_{64}\partial_3\vartheta_{64}|_{z=0} + \lambda_{51}\partial_3\vartheta_{51}|_{z=0} = \lambda_{46}\partial_3\vartheta_{46}|_{z=0} \end{cases} \qquad \forall \tau \in \mathcal{H}_3, \tag{6.3}$$

where $\partial_k \vartheta_{ij}|_{z=0} := \frac{\partial}{\partial z_k}\vartheta_{ij}|_{z=0}$, with $k = 1, 2, 3$. The solution of (**??**) can be determined up to a constant:

$$\lambda_{77} = D(46, 64, 51), \quad \lambda_{64} = D(77, 64, 46), \quad \lambda_{51} = D(77, 46, 51), \quad \lambda_{46} = D(77, 64, 51).$$

Here, as in Section **??**, $D(l, m, n) := \det \frac{\partial(\vartheta_{\mathrm{l}}, \vartheta_{\mathrm{m}}, \vartheta_{\mathrm{n}})}{\partial \mathrm{z}_1 \partial \mathrm{z}_2 \partial \mathrm{z}_3}$. By repeating this procedure on each row we get the matrix:

$$M = \begin{pmatrix} 0 & D(46,64,51)b_{77} & D(77,46,51)b_{64} & D(77,64,46)b_{51} & D(77,64,51)b_{46} \\ D(31,13,26)b_{77} & 0 & D(77,31,26)b_{13} & D(77,13,31)b_{26} & D(77,13,26)b_{31} \\ D(22,13,35)b_{64} & D(64,22,35)b_{13} & 0 & D(64,13,22)b_{35} & D(64,13,35)b_{22} \\ D(17,26,35)b_{51} & D(51,17,35)b_{26} & D(51,26,17)b_{35} & 0 & D(51,26,35)b_{17} \\ D(17,31,22)b_{46} & D(46,17,22)b_{31} & D(46,31,17)b_{22} & D(46,31,22)b_{17} & 0 \end{pmatrix}.$$

Although this matrix is not symmetric, it can be turned into a symmetric one by multiplying it on the left by a suitable diagonal matrix (note that this operation does not change the rank). If we choose the matrix D_1:

$$D_1 := \mathrm{diag}\left(1, \frac{\mathrm{D}(46,64,51)}{\mathrm{D}(31,13,26)}, \frac{\mathrm{D}(77,46,51)}{\mathrm{D}(22,13,35)}, \frac{\mathrm{D}(77,64,46)}{\mathrm{D}(17,26,35)}, \frac{\mathrm{D}(77,64,51)}{\mathrm{D}(17,31,22)}\right),$$

we set $S_1' := D_1 M$ and we get:

$$S_1' = \begin{pmatrix} 0 & D(46,64,51)b_{77} & D(77,46,51)b_{64} & D(77,64,46)b_{51} & D(77,64,51)b_{46} \\ D(46,64,51)b_{77} & 0 & \frac{D(46,64,51)D(77,31,26)}{D(31,13,26)}b_{13} & \frac{D(46,64,51)D(77,13,31)}{D(31,13,26)}b_{26} & \frac{D(46,64,51)D(77,13,26)}{D(31,13,26)}b_{31} \\ D(77,46,51)b_{64} & \frac{D(77,46,51)D(64,22,35)}{D(22,13,35)}b_{13} & 0 & \frac{D(77,46,51)D(64,13,22)}{D(22,13,35)}b_{35} & \frac{D(77,46,51)D(64,13,35)}{D(22,13,35)}b_{22} \\ D(77,64,46)b_{51} & \frac{D(77,64,46)D(51,17,35)}{D(17,26,35)}b_{26} & \frac{D(77,64,46)D(51,26,17)}{D(17,26,35)}b_{35} & 0 & \frac{D(77,64,46)D(51,26,35)}{D(17,26,35)}b_{17} \\ D(77,64,51)b_{46} & \frac{D(77,64,51)D(46,17,22)}{D(17,31,22)}b_{31} & \frac{D(77,64,51)D(46,31,17)}{D(17,31,22)}b_{22} & \frac{D(77,64,51)D(46,31,22)}{D(17,31,22)}b_{17} & 0 \end{pmatrix}.$$

Thanks to the relations among the determinants induced by Jacobi's derivative formula [**?**], the matrix S'_1 is easily seen to be symmetric. Note that a different diagonal matrix D_i can be chosen for this operation in such a way that the matrix $S'_i := D_i M$ and the matrix M have the same entries on the i-th row. A straightforward computation proves that $D_i D_1^{-1} = c_i Id$ with a suitable c_i, hence $S'_i = c_i S'_1$.

We can get a more convenient form for S'_1 acting by congruence with the diagonal matrix:

$$T_1 := \operatorname{diag}\left(1, \frac{\mathrm{D}(31,13,26)}{\mathrm{D}(46,64,51)\mathrm{D}(77,31,26)}, \frac{\mathrm{D}(22,13,35)}{\mathrm{D}(77,46,51)}, 1, 1\right).$$

Then we have:

$$S_1 := T_1 S'_1 T_1 = \begin{pmatrix} 0 & \frac{D(31,13,26)}{D(77,31,26)} b_{77} & D(22,13,35) b_{64} & D(77,64,46) b_{51} & D(77,64,51) b_{46} \\ \frac{D(31,13,26)}{D(77,31,26)} b_{77} & 0 & \frac{D(22,13,35)}{D(77,46,51)} b_{13} & \frac{D(77,13,31)}{D(77,31,26)} b_{26} & \frac{D(77,13,26)}{D(77,31,26)} b_{31} \\ D(22,13,35) b_{64} & \frac{D(22,13,35)}{D(77,46,51)} b_{13} & 0 & D(64,13,22) b_{35} & D(64,13,35) b_{22} \\ D(77,64,46) b_{51} & \frac{D(77,13,31)}{D(77,31,26)} b_{26} & D(64,13,22) b_{35} & 0 & \frac{D(77,64,46)D(51,26,35)}{D(17,26,35)} b_{17} \\ D(77,64,51) b_{46} & \frac{D(77,13,26)}{D(77,31,26)} b_{31} & D(64,13,35) b_{22} & \frac{D(77,64,46)D(51,26,35)}{D(17,26,35)} b_{17} & 0 \end{pmatrix}.$$

Likewise, the whole procedure can be repeated for the submatrices of $\mathcal{M}$ obtained by replacing the fifth column and row respectively with the sixth, the seventh and the eighth. Then we get the following symmetric matrices of rank 4:

$$S_2 := \begin{pmatrix} 0 & \frac{D(54,13,26)}{D(77,54,26)} b_{77} & D(47,13,35) b_{64} & D(77,64,23) b_{51} & D(77,64,51) b_{23} \\ \frac{D(54,13,26)}{D(77,54,26)} b_{77} & 0 & \frac{D(47,13,35)}{D(77,23,51)} b_{13} & \frac{D(77,13,54)}{D(77,54,26)} b_{26} & \frac{D(77,13,26)}{D(77,54,26)} b_{54} \\ D(47,13,35) b_{64} & \frac{D(47,13,35)}{D(77,23,51)} b_{13} & 0 & D(64,13,47) b_{35} & D(64,13,35) b_{47} \\ D(77,64,23) b_{51} & \frac{D(77,13,54)}{D(54,13,26)} b_{26} & D(64,13,47) b_{35} & 0 & \frac{D(77,64,23)D(51,26,35)}{D(72,26,35)} b_{72} \\ D(77,64,51) b_{23} & \frac{D(77,13,26)}{D(54,13,26)} b_{54} & D(64,13,35) b_{47} & \frac{D(77,64,23)D(51,26,35)}{D(72,26,35)} b_{72} & 0 \end{pmatrix},$$

$$S_3 := \begin{pmatrix} 0 & \frac{D(62,13,26)}{D(77,62,26)} b_{77} & D(71,13,35) b_{64} & D(77,64,15) b_{51} & D(77,64,51) b_{15} \\ \frac{D(62,13,26)}{D(77,62,26)} b_{77} & 0 & \frac{D(71,13,35)}{D(77,15,51)} b_{13} & \frac{D(77,13,62)}{D(77,62,26)} b_{26} & \frac{D(77,13,26)}{D(77,62,26)} b_{62} \\ D(71,13,35) b_{64} & \frac{D(71,13,35)}{D(77,15,51)} b_{13} & 0 & D(64,13,71) b_{35} & D(64,13,35) b_{71} \\ D(77,64,15 b_{51} & \frac{D(77,13,62)}{D(77,62,26)} b_{26} & D(64,13,71) b_{35} & 0 & \frac{D(77,64,15)D(51,26,35)}{D(44,26,35)} b_{44} \\ D(77,64,51) b_{15} & \frac{D(77,13,26)}{D(77,62,26)} b_{62} & D(64,13,35) b_{71} & \frac{D(77,64,15)D(51,26,35)}{D(44,26,35)} b_{44} & 0 \end{pmatrix},$$

$$S_4 := \begin{pmatrix} 0 & \frac{D(45,13,26)}{D(77,45,26)} b_{77} & D(56,13,35) b_{64} & D(77,64,32) b_{51} & D(77,64,51) b_{32} \\ \frac{D(45,13,26)}{D(77,45,26)} b_{77} & 0 & \frac{D(56,13,35)}{D(77,32,51)} b_{13} & \frac{D(77,13,45)}{D(77,45,26)} b_{26} & \frac{D(77,13,26)}{D(77,45,26)} b_{45} \\ D(56,13,35) b_{64} & \frac{D(56,13,35)}{D(77,32,51)} b_{13} & 0 & D(64,13,56) b_{35} & D(64,13,35) b_{56} \\ D(77,64,32) b_{51} & \frac{D(77,13,45)}{D(77,45,26)} b_{26} & D(64,13,56) b_{35} & 0 & \frac{D(77,64,32)D(51,26,35)}{D(63,26,35)} b_{63} \\ D(77,64,51) b_{46} & \frac{D(77,13,26)}{D(77,45,26)} b_{45} & D(64,13,35) b_{56} & \frac{D(77,64,32)D(51,26,35)}{D(63,26,35)} b_{63} & 0 \end{pmatrix}.$$

We can act by congruence on S_2, S_3 and S_4 using the diagonal matrices:

$$N_2 := \operatorname{diag}\left(\sqrt{\mathrm{A}}, \sqrt{\mathrm{A}}\frac{\mathrm{D(77,23,51)}}{\mathrm{D(77,46,51)}}, \frac{1}{\sqrt{\mathrm{A}}}\frac{\mathrm{D(22,13,35)}}{\mathrm{D(47,13,35)}}, \frac{1}{\sqrt{\mathrm{A}}}\frac{\mathrm{D(77,64,46)}}{\mathrm{D(77,64,23)}}, \frac{1}{\sqrt{\mathrm{A}}}\right), \qquad A := \frac{D(77,46,51)D(31,13,26)D(77,54,26)}{D(77,31,26)D(77,23,51)D(54,13,26)},$$

$$N_3 := \operatorname{diag}\left(\sqrt{\mathrm{B}}, \sqrt{\mathrm{B}}\frac{\mathrm{D(77,15,51)}}{\mathrm{D(77,46,51)}}, \frac{1}{\sqrt{\mathrm{B}}}\frac{\mathrm{D(22,13,35)}}{\mathrm{D(71,13,35)}}, \frac{1}{\sqrt{\mathrm{B}}}\frac{\mathrm{D(77,64,46)}}{\mathrm{D(77,64,15)}}, \frac{1}{\sqrt{\mathrm{B}}}\right), \qquad B := \frac{D(77,46,51)D(77,62,26)D(31,13,26)}{D(62,13,26)D(77,15,51)D(77,31,26)},$$

$$N_4 := \operatorname{diag}\left(\sqrt{\mathrm{C}}, \sqrt{\mathrm{C}}\frac{\mathrm{D(77,32,51)}}{\mathrm{D(77,46,51)}}, \frac{1}{\sqrt{\mathrm{C}}}\frac{\mathrm{D(22,13,35)}}{\mathrm{D(56,13,35)}}, \frac{1}{\sqrt{\mathrm{C}}}\frac{\mathrm{D(77,64,46)}}{\mathrm{D(77,64,32)}}, \frac{1}{\sqrt{\mathrm{C}}}\right), \qquad C = \frac{D(77,46,51)D(77,45,26)D(31,13,26)}{D(45,13,26)D(77,32,51)D(77,31,26)}.$$

Then S_1, $N_2S_2N_2$, $N_3S_3N_3$ and $N_4S_4N_4$ have the same entries on the common rows and columns; hence, we have the following 8×8 symmetric matrix:

$$\begin{pmatrix}
0 & \frac{D(31,13,26)}{D(77,31,26)}b_{77} & D(22,13,35)b_{64} & D(77,64,46)b_{51} & D(77,64,51)b_{46} & D(77,64,51)b_{23} & D(77,64,51)b_{15} & D(77,64,51)b_{32} \\
* & 0 & \frac{D(22,13,35)}{D(77,46,51)}b_{13} & \frac{D(77,13,31)}{D(77,31,26)}b_{26} & \frac{D(77,13,26)}{D(77,31,26)}b_{31} & \frac{1}{A}\frac{D(31,13,26)D(77,13,26)}{D(54,13,26)D(77,31,26)}b_{54} & \frac{1}{B}\frac{D(31,13,26)D(77,13,26)}{D(62,13,26)D(77,31,26)}b_{62} & \frac{1}{C}\frac{D(31,13,26)D(77,13,26)}{D(45,13,26)D(77,31,26)}b_{45} \\
* & * & 0 & D(64,13,22)b_{35} & D(64,13,35)b_{22} & \frac{1}{A}\frac{D(22,13,35)D(64,13,35)}{D(47,13,35)}b_{47} & \frac{1}{B}\frac{D(22,13,35)D(64,13,35)}{D(71,13,35)}b_{71} & \frac{1}{C}\frac{D(22,13,35)D(64,13,35)}{D(56,13,35)}b_{56} \\
* & * & * & 0 & \frac{D(77,64,46)D(51,26,35)}{D(17,26,35)}b_{17} & \frac{1}{A}\frac{D(77,64,46)D(51,26,35)}{D(72,26,35)}b_{72} & \frac{1}{B}\frac{D(77,64,46)D(51,26,35)}{D(44,26,35)}b_{44} & \frac{1}{C}\frac{D(77,64,46)D(51,26,35)}{D(63,26,35)}b_{63} \\
* & * & * & * & 0 & X_{65}b_{65} & X_{53}b_{53} & X_{74}b_{74} \\
* & * & * & * & * & 0 & X_{36}b_{36} & X_{11}b_{11} \\
* & * & * & * & * & * & 0 & X_{27}b_{27} \\
* & * & * & * & * & * & * & 0
\end{pmatrix},$$

where the X_{ij} are to be determined in such a way that the rank of the matrix is equal to 4. For this purpose we note that we have determined the minors S_1, $N_2S_2N_2$, $N_3S_3N_3$ and $N_4S_4N_4$ by demanding precise relations among the eight vector columns V_i of the 8×8 matrix $\mathcal{M}$. If we set:

$$c_2 := \frac{D(46,64,51)D(77,31,26)}{D(31,13,26)}; \qquad c_3 := \frac{D(77,46,51)}{D(22,13,35)};$$

$$d_1 := \frac{1}{\sqrt{A}}; \quad d_2 := \frac{1}{\sqrt{A}}\frac{D(23,64,51)D(77,54,26)D(77,46,51)}{D(54,13,26)D(77,23,51)}; \quad d_3 := \sqrt{A}\frac{D(22,13,35)}{D(77,23,51)}; \quad d_4 := \sqrt{A}\frac{D(77,64,23)}{D(77,64,46)}; \quad d_6 := \sqrt{A};$$

$$e_1 := \frac{1}{\sqrt{B}}; \quad e_2 := \frac{1}{\sqrt{B}}\frac{D(15,64,51)D(77,62,26)D(77,46,51)}{D(62,13,26)D(77,15,51)}; \quad e_3 := \sqrt{B}\frac{D(77,15,51)}{D(22,13,35)}; \quad e_4 := \sqrt{B}\frac{D(77,64,15)}{D(77,64,46)}; \quad e_7 := \sqrt{B};$$

$$f_1 := \frac{1}{\sqrt{C}}; \quad f_2 := \frac{1}{\sqrt{C}}\frac{D(32,64,51)D(77,45,26)D(77,46,51)}{D(45,13,26)D(77,32,51)}; \quad f_3 := \sqrt{C}\frac{D(77,32,51)}{D(22,13,35)}; \quad f_4 := \sqrt{C}\frac{D(77,64,32)}{D(77,64,46)}; \quad f_8 := \sqrt{C};$$

then the following relations hold:

$$\begin{aligned}
V_1 + c_2V_2 + c_3V_3 + V_4 - V_5 &= 0, \\
d_1V_1 + d_2V_2 + d_3V_3 + d_4V_4 - d_6V_6 &= 0, \\
e_1V_1 + e_2V_2 + e_3V_3 + e_4V_4 - e_7V_7 &= 0, \\
f_1V_1 + f_2V_2 + f_3V_3 + f_4V_4 - f_8V_8 &= 0,
\end{aligned}$$

each respectively on the rows and the columns of the corresponding 5×5 minor.
In particular, the following relation holds on the first four rows:

$$\left(c_2 - \frac{d_2}{d_1}\right) V_2 + + \left(c_3 - \frac{d_3}{d_1}\right) V_3 + \left(1 - \frac{d_4}{d_1}\right) V_4 - V_5 + \frac{d_6}{d_1} V_6 = 0.$$

Hence, it can be used to compute X_{65}, by demanding it on the fifth row:

$$X_{65}^{(5)} = \frac{1}{A} \left(A \cdot D(77,23,51) - D(77,46,51)\right) \frac{D(22,31,17)D(64,13,35)}{D(65,31,17)D(22,13,35)},$$

Otherwise we can compute X_{65} by demanding the relation on the sixth row, we get:

$$X_{65}^{(6)} = \left(\frac{1}{A} - \frac{D(77,64,23)}{D(77,64,46)}\right) \frac{D(77,64,46)D(51,26,35)D(72,54,47)}{D(72,26,35)D(65,54,47)}.$$

Using Jacobi's formula to write these two expressions in terms of theta constants, we have:

$$\begin{aligned} X_{65}^{(5)} &= \pm\theta_{14}\theta_{33} \left(\frac{\theta_{00}\theta_{42}\theta_{57}\theta_{61}\theta_{70}}{\theta_{52}\theta_{75}}\right) \left(\frac{\theta_{02}\theta_{03}\theta_{24}\theta_{25}}{\theta_{41}\theta_{40}\theta_{66}\theta_{67}} - 1\right), \\ X_{65}^{(6)} &= \pm\theta_{06}\theta_{21} \left(\frac{\theta_{00}\theta_{42}\theta_{57}\theta_{61}\theta_{70}}{\theta_{40}\theta_{67}}\right) \left(1 - \frac{\theta_{03}\theta_{10}\theta_{24}\theta_{37}}{\theta_{52}\theta_{41}\theta_{66}\theta_{75}}\right). \end{aligned}$$

By virtue of the Riemann relations in genus 3:

$$\theta_{52}\theta_{75}\theta_{41}\theta_{66} - \theta_{03}\theta_{10}\theta_{24}\theta_{37} = \theta_{14}\theta_{07}\theta_{33}\theta_{20}, \qquad \theta_{40}\theta_{67}\theta_{41}\theta_{66} - \theta_{03}\theta_{02}\theta_{24}\theta_{25} = \theta_{06}\theta_{21}\theta_{07}\theta_{20}$$

the two expressions for X_{65} turn out to be equal:

$$X_{65}^{(5)} = \theta_{14}\theta_{33} \left(\frac{\theta_{00}\theta_{42}\theta_{57}\theta_{61}\theta_{70}}{\theta_{52}\theta_{75}}\right) \frac{\theta_{06}\theta_{21}\theta_{07}\theta_{20}}{\theta_{41}\theta_{40}\theta_{66}\theta_{67}} = \theta_{06}\theta_{21} \left(\frac{\theta_{00}\theta_{42}\theta_{57}\theta_{61}\theta_{70}}{\theta_{40}\theta_{67}}\right) \frac{\theta_{14}\theta_{07}\theta_{33}\theta_{20}}{\theta_{52}\theta_{41}\theta_{66}\theta_{75}} = X_{65}^{(6)}.$$

Likewise we get:

$$\begin{aligned} X_{53} &= \tfrac{1}{B} \left(BD(77,15,51) - D(77,46,51)\right) \tfrac{D(22,31,17)D(64,13,35)}{D(53,31,17)D(22,13,35)}; \\ X_{74} &= \tfrac{1}{C} \left(1 - C\tfrac{D(77,64,32)}{D(77,64,46)}\right) \tfrac{D(77,64,51)D(46,31,22)}{D(74,31,22)}; \\ X_{36} &= \tfrac{1}{B} \left(1 - \tfrac{D(54,13,26)D(77,23,51)D(15,64,51)D(77,62,26)}{D(23,64,51)D(77,54,26))D(77,15,51)D(62,13,26)}\right) \tfrac{D(77,64,51)D(23,47,72)}{D(36,47,72)}; \\ X_{11} &= \left(\tfrac{1}{C} - \tfrac{D(77,64,32)}{AD(77,64,23)}\right) \tfrac{D(23,54,47)D(77,64,51)}{D(11,54,47)}; \\ X_{27} &= \left(\tfrac{1}{C} - \tfrac{D(77,64,32)}{BD(77,64,15)}\right) \tfrac{D(15,62,71)D(77,64,51)}{D(27,62,71)}. \end{aligned}$$

Hence, each entry of the 8×8 symmetric matrix with rank equal to 4 is uniquely determined, up to congruences by diagonal matrices. In particular, we will get a suitable form for the matrix we have determined, by multiplying it on both sides by the diagonal matrix $\mathrm{diag}\,(1, \mathrm{D}(77,31,26), 1, 1, 1, 1, 1, 1)$:

$$\begin{pmatrix}
0 & D(31,13,26)b_{77} & D(22,13,35)b_{64} & D(77,64,46)b_{51} & D(77,64,51)b_{46} & D(77,64,51)b_{23} & D(77,64,51)b_{15} & D(77,64,51)b_{32} \\
* & 0 & \frac{D(22,13,35)D(77,31,26)}{D(77,46,51)}b_{13} & D(77,13,31)b_{26} & D(77,13,26)b_{31} & \frac{1}{A}\frac{D(31,13,26)D(77,13,26)}{D(54,13,26)}b_{54} & \frac{1}{B}\frac{D(31,13,26)D(77,13,26)}{D(62,13,26)}b_{62} & \frac{1}{C}\frac{D(31,13,26)D(77,13,26)}{D(45,13,26)}b_{45} \\
* & * & 0 & D(64,13,22)b_{35} & D(64,13,35)b_{22} & \frac{1}{A}\frac{D(22,13,35)D(64,13,35)}{D(47,13,35)}b_{47} & \frac{1}{B}\frac{D(22,13,35)D(64,13,35)}{D(71,13,35)}b_{71} & \frac{1}{C}\frac{D(22,13,35)D(64,13,35)}{D(56,13,35)}b_{56} \\
* & * & * & 0 & \frac{D(77,64,46)D(51,26,35)}{D(17,26,35)}b_{17} & \frac{1}{A}\frac{D(77,64,46)D(51,26,35)}{D(72,26,35)}b_{72} & \frac{1}{B}\frac{D(77,64,46)D(51,26,35)}{D(44,26,35)}b_{44} & \frac{1}{C}\frac{D(77,64,46)D(51,26,35)}{D(63,26,35)}b_{63} \\
* & * & * & * & 0 & \frac{1}{A}\left(A\cdot D(77,23,51)-D(77,46,51)\right)\frac{D(22,31,17)D(64,13,35)}{D(65,31,17)D(22,13,35)}b_{65} & \frac{1}{B}\left(BD(77,15,51)-D(77,46,51)\right)\frac{D(22,31,17)D(64,13,35)}{D(53,31,17)D(22,13,35)}b_{53} & \frac{1}{C}\left(1-C\frac{D(77,64,32)}{D(77,64,46)}\right)\frac{D(77,64,51)D(46,31,22)}{D(74,31,22)}b_{74} \\
* & * & * & * & * & 0 & \frac{1}{B}\left(1-\frac{D(54,13,26)D(77,23,51)D(15,64,51)D(77,62,26)}{D(23,64,51)D(77,54,26))D(77,15,51)D(62,13,26)}\right)\frac{D(77,64,51)D(23,47,72)}{D(36,47,72)}b_{36} & \left(\frac{1}{C}-\frac{D(77,64,32)}{AD(77,64,23)}\right)\frac{D(23,54,47)D(77,64,51)}{D(11,54,47)}b_{11} \\
* & * & * & * & * & * & 0 & \left(\frac{1}{C}-\frac{D(77,64,32)}{BD(77,64,15)}\right)\frac{D(15,62,71)D(77,64,51)}{D(27,62,71)}b_{27} \\
* & * & * & * & * & * & * & 0
\end{pmatrix}.$$

Using the expression of the Jacobian determinant in terms of theta constants and the Riemann relations we get the matrix $L(\tau,z)$:

$$L(\tau,z)=\begin{pmatrix}
0 & D(31,13,26)b_{77} & D(22,13,35)b_{64} & D(77,64,46)b_{51} & D(77,64,51)b_{46} & D(77,64,51)b_{23} & D(77,64,51)b_{15} & D(77,64,51)b_{32} \\
* & 0 & \pm\frac{\theta_{60}}{\theta_{04}}D(77,31,26)b_{13} & D(77,13,31)b_{26} & D(77,13,26)b_{31} & \pm\left(\frac{\theta_{07}\theta_{10}\theta_{25}\theta_{61}\theta_{73}}{\theta_{04}\theta_{40}\theta_{52}\theta_{67}\theta_{75}}\right)D(77,23,32)b_{54} & \pm\left(\frac{\theta_{07}\theta_{10}\theta_{25}\theta_{57}\theta_{73}}{\theta_{04}\theta_{43}\theta_{52}\theta_{67}\theta_{76}}\right)D(77,15,32)b_{62} & \pm\left(\frac{\theta_{01}\theta_{16}\theta_{34}\theta_{70}\theta_{73}}{\theta_{04}\theta_{40}\theta_{43}\theta_{75}\theta_{76}}\right)D(77,23,32)b_{45} \\
* & * & 0 & D(64,13,22)b_{35} & D(64,13,35)b_{22} & \pm\left(\frac{\theta_{03}\theta_{14}\theta_{25}\theta_{60}\theta_{61}}{\theta_{04}\theta_{40}\theta_{41}\theta_{66}\theta_{67}}\right)D(64,23,32)b_{47} & \pm\left(\frac{\theta_{03}\theta_{14}\theta_{25}\theta_{57}\theta_{60}}{\theta_{04}\theta_{41}\theta_{50}\theta_{67}\theta_{76}}\right)D(64,15,32)b_{71} & \pm\left(\frac{\theta_{05}\theta_{12}\theta_{34}\theta_{60}\theta_{70}}{\theta_{04}\theta_{40}\theta_{50}\theta_{66}\theta_{76}}\right)D(64,23,32)b_{56} \\
* & * & * & 0 & \pm\left(\frac{\theta_{04}}{\theta_{55}}\right)D(51,26,35)b_{17} & \pm\left(\frac{\theta_{03}\theta_{10}\theta_{21}\theta_{61}}{\theta_{41}\theta_{52}\theta_{66}\theta_{75}}\right)D(51,23,32)b_{72} & \pm\left(\frac{\theta_{03}\theta_{10}\theta_{21}\theta_{57}}{\theta_{41}\theta_{43}\theta_{50}\theta_{52}}\right)D(51,15,32)b_{44} & \pm\left(\frac{\theta_{01}\theta_{12}\theta_{30}\theta_{70}}{\theta_{43}\theta_{50}\theta_{66}\theta_{75}}\right)D(51,23,32)b_{63} \\
* & * & * & * & 0 & \pm\left(\frac{\theta_{06}\theta_{07}\theta_{14}\theta_{20}\theta_{21}\theta_{33}}{\theta_{40}\theta_{41}\theta_{52}\theta_{66}\theta_{67}\theta_{75}}\right)D(22,31,17)b_{65} & \pm\left(\frac{\theta_{05}\theta_{07}\theta_{14}\theta_{16}\theta_{21}\theta_{30}}{\theta_{41}\theta_{43}\theta_{50}\theta_{52}\theta_{67}\theta_{76}}\right)D(22,31,17)b_{53} & \pm\left(\frac{\theta_{05}\theta_{16}\theta_{20}\theta_{33}\theta_{52}}{\theta_{21}\theta_{40}\theta_{50}\theta_{66}\theta_{76}}\right)D(46,31,22)b_{74} \\
* & * & * & * & * & 0 & \pm\left(\frac{\theta_{03}\theta_{07}\theta_{10}\theta_{14}\theta_{21}\theta_{25}\theta_{55}\theta_{60}\theta_{73}}{\theta_{40}\theta_{41}\theta_{43}\theta_{50}\theta_{52}\theta_{66}\theta_{67}\theta_{75}\theta_{76}}\right)D(22,31,17)b_{36} & \pm\left(\frac{\theta_{06}\theta_{24}\theta_{33}\theta_{42}\theta_{55}\theta_{57}\theta_{60}\theta_{61}\theta_{70}}{\theta_{40}\theta_{41}\theta_{43}\theta_{50}\theta_{52}\theta_{66}\theta_{67}\theta_{75}\theta_{76}}\right)D(77,23,32)b_{11} \\
* & * & * & * & * & * & 0 & \pm\left(\frac{\theta_{01}\theta_{05}\theta_{16}\theta_{12}\theta_{30}\theta_{34}\theta_{55}\theta_{60}\theta_{73}}{\theta_{40}\theta_{41}\theta_{43}\theta_{50}\theta_{52}\theta_{66}\theta_{67}\theta_{75}\theta_{76}}\right)D(22,31,17)b_{27} \\
* & * & * & * & * & * & * & 0
\end{pmatrix}.$$

Remark 6.2. *Note that each coefficient can be written as a product of at most* 8 *determinants over* 7 *determinants, although there seems not to be any canonical choice for such an expression.*
Take, for instance, the coefficient X_{65}*; the triples of even characteristics* $\{(06),(07),(14)\}$ *and* $\{(40),(41),(52)\}$ *extend to azygetic* 5*-tuples by means of the same pair* $\{(55),(70)\}$*, and the triples* $\{(20),(21),(33)\}$ *and* $\{(66),(67),(75)\}$ *extend to azygetic* 5*-tuples by means of the pair* $\{(34),(70)\}$*. Therefore, we can write:*

$$X_{65}=\pm\frac{\theta_{55}\theta_{70}}{\theta_{55}\theta_{70}}\left(\frac{\theta_{06}\theta_{07}\theta_{14}}{\theta_{40}\theta_{41}\theta_{52}}\right)\cdot\frac{\theta_{34}\theta_{70}}{\theta_{34}\theta_{70}}\left(\frac{\theta_{20}\theta_{21}\theta_{33}}{\theta_{66}\theta_{67}\theta_{75}}\right)D(22,31,17)=\pm\frac{D(11,53,72)D(11,13,74)}{D(11,15,72)D(11,13,32)}D(22,31,17).$$

In a similar way, we can get such an expression for X_{13} *and* X_{53}

$$X_{13}=\pm\frac{\theta_{60}}{\theta_{04}}D(77,31,26)=\frac{D(22,13,35)}{D(77,46,51)}D(77,31,26),$$
$$X_{53}=\pm\frac{\theta_{37}\theta_{61}}{\theta_{37}\theta_{61}}\left(\frac{\theta_{05}\theta_{07}\theta_{21}}{\theta_{41}\theta_{43}\theta_{67}}\right)\cdot\frac{\theta_{37}\theta_{70}}{\theta_{37}\theta_{70}}\left(\frac{\theta_{14}\theta_{16}\theta_{30}}{\theta_{50}\theta_{52}\theta_{76}}\right)D(22,31,17)=\pm\frac{D(26,36,65)D(26,27,74)}{D(23,26,36)D(26,27,32)}D(22,31,17).$$

and so on for each entry of the matrix.

The previous discussion proves Theorem **??**.

In order to determine the matrix $A(z)$ we can consider the minor obtained by taking the first 4 rows and columns of the matrix $L(\tau, z)$ divided for a suitable jacobian determinant, so we will get modular functions as coefficients, as stated in the following corollary.

Corollary 6.3. *Let τ be the period matrix of the jacobian of a smooth plane quartic, then the matrix $A(z)$ is congruent to the following matrix:*

$$Q(\tau, z) = \begin{pmatrix} 0 & \frac{D(31,13,26)}{D(77,31,26)}b_{77} & \frac{D(22,13,35)}{D(77,31,26)}b_{64} & \frac{D(77,64,46)}{D(77,31,26)}b_{51} \\ * & 0 & \frac{D(22,13,35)}{D(77,46,51)}b_{13} & \frac{D(77,13,31)}{D(77,31,26)}b_{26} \\ * & * & 0 & \frac{D(64,13,22)}{D(77,31,26)}b_{35} \\ * & * & * & 0 \end{pmatrix}.$$

Moreover

$$\det Q(\tau, z) = 0,$$

is an equation for the plane quartic.

A similar equation for the plane quartic has been obtained in [**?**], using the Riemann model for the quartic. In this case we do not need to require it, since it is granted from the structure of the bitangent matrix. We also remind that all the data of the matrix $Q(\tau, z)$ can be encoded from the first stage of the construction of the matrix $L(\tau, z)$, i.e. the 5×5 matrix S_1. In particular, each principal minor of order 4 of S_1 provides the same explicit equation for the quartic. Obviously, from a computational point of view, the bitangents can be easily computed if the equation of the quartic is known. A computer software can be used to get the values of the theta constants and of the gradients, thus immediately determining the 28 bitangents. This can be done, for example, by exploiting the package presented in [**?**] that can be freely downloaded [**?**]. We created a script to compute the 8×8 bitangent matrix $L(\tau, z)$ using our formula and the 4×4 matrix $Q(\tau, z)$ for a given curve, in our example for $f(x, y) = x^4 + y^4 + 1$. The script is developed using Sage (version 6.1.1) and the packages Sympy (version 0.7.4), Numpy (version 1.7.0) and Abelfunctions (version 0.1.0) and can be freely downloaded from the link `http://www.RSM.it`.

Instead, from a theoretical point of view, the matrix $L(\tau, z)$ explains the holomorphic variation of the matrix induced by the Cayley octad. Hence, we are interested in having expressions for the entries of this matrix in the form ${}^t v_m(\tau) z$ with the $v_m(\tau)$ explicit meromorphic vector valued modular forms. Since they are defined on $\mathcal{H}_3$ with respect to the representation $\rho = (\det A)^{5/2} A$, for $A \in \mathrm{GL}(3, \mathbb{C})$, the entries of the matrix are related to vector valued modular forms, i.e. meromorphic sections of vector bundles defined on the moduli space of principally polarized abelian variety with level two structure. Any other inequivalent bitangent matrix will be obtained by changing the even characteristic and considering the eight corresponding Aronhold sets.

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