

On the regularity set and angular integrability for the Navier–Stokes equation

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Abstract We investigate the size of the regular set for suitable weak solutions of the Navier–Stokes equation, in the sense of Caffarelli–Kohn–Nirenberg [2]. We consider initial data in weighted Lebesgue spaces with mixed radial-angular integrability, and we prove that the regular set increases if the data have higher angular integrability, invading the whole half space $\{t > 0\}$ in an appropriate limit. In particular, we obtain that if the L^2 norm with weight $|x|^{-\frac{1}{2}}$ of the data tends to 0, the regular set invades $\{t > 0\}$; this result improves Theorem D of [2].

Keywords Navier–Stokes · Angular integrability

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1 Introduction and main results

We consider the Cauchy problem for the Navier–Stokes equation on $[0, \infty) \times \mathbb{R}^3$

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \Delta u = -\nabla P \\ \nabla \cdot u = 0 \\ u(0, x) = u_0(x). \end{cases} \quad (1.1)$$

describing a viscous incompressible fluid in the absence of external forces, where as usual u is the velocity field of the fluid and P the pressure, and the initial data

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satisfy the compatibility condition $\nabla \cdot u_0 = 0$. We use the same notation for the norm of scalar, vector or tensor quantities:

$$\|P\|_{L^2} := (\int P^2 dx)^{\frac{1}{2}}, \quad \|u\|_{L^2}^2 := \sum_j \|u_j\|_{L^2}^2, \quad \|\nabla u\|_{L^2}^2 := \sum_{j,k} \|\partial_k u_j\|_{L^2}^2 \quad (1.2)$$

and we write simply $L^2(\mathbb{R}^3)$ instead of $[L^2(\mathbb{R}^3)]^3$, or $\mathcal{S}'(\mathbb{R}^3)$ instead of $[\mathcal{S}'(\mathbb{R}^3)]^3$ and so on. Regularity of the global weak solutions constructed in [17], [21] is a notorious open problem, although many partial results exist.

The case of small data is well understood. In the proofs of well posedness for small data, the equation is regarded as a linear heat equation perturbed by a small nonlinear term $(u \cdot \nabla)u$, and the natural approach is a fixed point argument around the heat propagator. More precisely, one rewrites the problem in integral form

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(s) ds \quad \text{in } [0, \infty) \times \mathbb{R}^3 \quad (1.3)$$

where \mathbb{P} is the Leray projection

$$\mathbb{P}f := f - \nabla \Delta^{-1}(\nabla \cdot f), \quad (1.4)$$

namely the projection onto the subspace of the L^2 divergence free vector fields, and then the Picard iteration scheme is defined by

$$u_1 := e^{t\Delta}u_0, \quad u_n := e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u_{n-1} \otimes u_{n-1})(s) ds. \quad (1.5)$$

Once the velocity is known the pressure can be recovered at each time via the representation formula $P = -\Delta^{-1} \nabla \otimes \nabla (u \otimes u)$. Small data results fit in the following abstract framework.

Proposition 1.1 ([20]) *Let $X \subset \bigcap_{s < \infty} L_t^2 L_{loc,x}^2((0, s) \times \mathbb{R}^3)^4$ be a Banach space such that the bilinear form*

$$B(u, v) := \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes v)(s) ds \quad (1.6)$$

is bounded from $X \times X$ to X :

$$\|B(u, v)\|_X \leq C_X \|u\|_X \|v\|_X.$$

Moreover, let $X_0 \subset \mathcal{S}'(\mathbb{R}^3)$ be a normed space such that $e^{t\Delta} : X_0 \rightarrow X$ is bounded:

$$\|e^{t\Delta} f\|_X \leq A_{X_0, X} \|f\|_{X_0}.$$

Then for every data u_0 such that $\|u_0\|_{X_0} < 1/4C_X A_{X_0, X}$ the sequence u_n is Cauchy in X and converges to a solution u of the integral equation (1.3). The solution satisfies

$$\|u\|_X \leq 2A_{X_0, X} \|u_0\|_{X_0}.$$

¹ The space L_{loc}^2 consists of the functions that are uniformly locally square-integrable (see [20] Definition 11.3). The operator (1.6) is well-defined on $\bigcap_{s < \infty} L_t^2 L_{loc,x}^2((0, s) \times \mathbb{R}^3) \times \bigcap_{s < \infty} L_t^2 L_{loc,x}^2((0, s) \times \mathbb{R}^3)$. We refer to [20], Chapter 11, for more details.

The space X is usually called an *admissible (path) space*, while X_0 is called an *adapted space*. Many adapted spaces X_0 have been studied: L^3 [18], Morrey spaces [16], [33], Besov spaces [4], [14], [24] and several others. The largest space in which Picard iteration has been proved to converge is BMO^{-1} [19].

A crucial ingredient in the theory is symmetry invariance. The natural symmetry of the Navier–Stokes equation is the translation-scaling

$$u_0(x) \mapsto \lambda u_0(\lambda(x - x_0)), \quad \lambda \in \mathbb{R}^+ := (0, \infty), \quad x_0 \in \mathbb{R}^3,$$

and indeed all the spaces X_0 mentioned above are invariant for this transformations. On the other hand, in results of local regularity a role may be played by some spaces which are scaling but not translation invariant, like the weighted L^p spaces with norm

$$\| |x|^{1-\frac{3}{p}} u(x) \|_{L^p(\mathbb{R}^3)}.$$

When $p = 2$ this is the weighted L^2 space with norm $\| |x|^{-\frac{1}{2}} u(x) \|_{L^2}$, used in the classical regularity results of [2]. We recall a key definition from that paper.

Definition 1.2 A point $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^3$ is *regular* for a solution $u(t, x)$ of (1.1) if u is essentially bounded on a neighborhood of (t_0, x_0) . It follows that u is smooth in the space variables near (t_0, x_0) (see for instance [28]). A subset of $\mathbb{R}^+ \times \mathbb{R}^3$ is *regular* if all its points are regular.

The following result (Theorem D in [2]) applies to the special class of *suitable weak solutions*, see the beginning of Section 2 for the precise definition. The weak solutions constructed in [21] are actually suitable. We use the notation

$$\Pi_\alpha := \left\{ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 : t > \frac{|x|^2}{\alpha} \right\} \quad (1.7)$$

to denote the region above the paraboloid of aperture α (in the upper half space $t > 0$). Note that Π_α is increasing in α .

Theorem 1.3 (Caffarelli–Kohn–Nirenberg) *There exists a constant $\varepsilon_0 > 0$ such that the following holds. Let u be a suitable weak solution of Problem 1.1 with divergence free initial data $u_0 \in L^2(\mathbb{R}^3)$. If*

$$\| |x|^{-1/2} u_0 \|_{L^2(\mathbb{R}^3)}^2 = \varepsilon < \varepsilon_0 \quad (1.8)$$

then the set

$$\Pi_{\varepsilon_0 - \varepsilon} \equiv \left\{ (t, x) : t > \frac{|x|^2}{\varepsilon_0 - \varepsilon} \right\} \quad (1.9)$$

is regular for u .

The theorem states that if the weighted L^2 norm of the data is sufficiently small, then the solution is smooth above a certain paraboloid with vertex at the origin. If the size of the data tends to 0, the regular set increases and invades a limit set Π_{ε_0} , which is strictly contained in the half space $t > 0$.

It is reasonable to expect that the regular set actually invades the whole upper half space $t > 0$ when the size of the data tends to 0. This is indeed a special case of our main result, see Theorem 1.5 below and in particular Corollary 1.6.

However our main goal is a more general investigation of the influence on the regular set of additional angular integrability of the data. We measure angular regularity using the following mixed norms:

$$\|f\|_{L_{|x|}^p L_{\theta}^{\tilde{p}}} := \left(\int_0^{+\infty} \|f(\rho \cdot)\|_{L^{\tilde{p}}(\mathbb{S}^2)}^p \rho^2 d\rho \right)^{\frac{1}{p}}, \quad \|f\|_{L_{|x|}^{\infty} L_{\theta}^{\tilde{p}}} := \sup_{\rho > 0} \|f(\rho \cdot)\|_{L^{\tilde{p}}(\mathbb{S}^2)}. \quad (1.10)$$

The idea of separating radial and angular regularity is not new; it proved useful especially in the context of Strichartz estimates and dispersive equations (see [5], [8], [13], [23], [26] [34]). The $L_{|x|}^p L_{\theta}^{\tilde{p}}$ scale includes the usual L^p norms when $\tilde{p} = p$:

$$\|u\|_{L_{|x|}^p L_{\theta}^p} = \|u\|_{L^p(\mathbb{R}^3)}. \quad (1.11)$$

Note also that for radial functions the value of \tilde{p} is irrelevant, in the sense that²

$$u \text{ radial} \Rightarrow \|u\|_{L_{|x|}^p L_{\theta}^{\tilde{p}}} \simeq \|u\|_{L^p(\mathbb{R}^3)} \quad \forall p, \tilde{p} \in [1, \infty] \quad (1.12)$$

while for generic functions we have only

$$\|u\|_{L_{|x|}^p L_{\theta}^{\tilde{p}}} \lesssim \|u\|_{L_{|x|}^p L_{\theta}^{\tilde{p}_1}} \quad \text{if } \tilde{p} \leq \tilde{p}_1. \quad (1.13)$$

With respect to scaling, the mixed radial-angular norm $L_{|x|}^p L_{\theta}^{\tilde{p}}$ behaves like L^p and in particular we have for all $\tilde{p} \in [1, \infty]$ and all $\lambda > 0$

$$\||x|^{\alpha} \lambda u_0(\lambda x)\|_{L_{|x|}^p L_{\theta}^{\tilde{p}}} = \||x|^{\alpha} u_0(x)\|_{L_{|x|}^p L_{\theta}^{\tilde{p}}}, \quad \text{provided } \alpha = 1 - \frac{3}{p}. \quad (1.14)$$

As a first application, we show that for initial data with small $\||x|^{\alpha} u_0\|_{L_{|x|}^p L_{\theta}^{\tilde{p}}}$ norm and $\tilde{p} \geq 2p/(p-1)$, the problem has a global smooth solution. As we prove in Section 2, this norm controls the $B_{q, \infty}^{-1+3/q}$ norm (for q large enough), and this space is embedded in BMO^{-1} , thus the existence part in Theorem 1.4 could be immediately deduced from the more general results in [4], [19], [24]. However, we are especially interested in the quantitative estimate (1.18), which will be a crucial tool for the proof of our main Theorem 1.5, so we prefer to give a more direct proof of Theorem 1.4 in Section 3.

Theorem 1.4 *Let $1 < p < 5$, $\tilde{p} \geq 2p/(p-1)$, $\alpha = 1 - 3/p$ and let $u_0 \in L_{|x|}^p \alpha_{p,d|} L_{\theta}^{\tilde{p}}$ be divergence free. Moreover, let*

$$\begin{aligned} \frac{2p}{p-1} \leq q < \infty & \text{ if } 1 < p \leq 2 \\ \frac{2p}{p-1} \leq q < \frac{3p}{p-2} & \text{ if } 2 \leq p < 3 \\ p < q < \frac{3p}{p-2} & \text{ if } 3 \leq p < 5 \end{aligned} \quad (1.15)$$

² As usual we write $A \lesssim B$ if there is a constant C independent of A, B such that $A \leq CB$ and $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.

and

$$\frac{2}{r} + \frac{3}{q} = 1. \quad (1.16)$$

There exists an $\bar{\varepsilon} = \bar{\varepsilon}(p, \tilde{p}, q) > 0$ such that, if

$$\| |x|^\alpha u_0 \|_{L^p_{|x|} L^{\tilde{p}}_\theta} < \bar{\varepsilon}, \quad (1.17)$$

Then Problem 1.3 has a unique global smooth solution u satisfying³

$$\|u\|_{L^r_t L^q_x} \leq \bar{C} \| |x|^\alpha u_0 \|_{L^p_{|x|} L^{\tilde{p}}_\theta} \quad (1.18)$$

for some constant $\bar{C} = \bar{C}(p, \tilde{p}, q)$ independent of u_0 .

In the following we shall need only the special case corresponding to the choice

$$p = 2, \quad \tilde{p} = 4, \quad q = 4. \quad (1.19)$$

Thus, using the notations

$$\varepsilon_1 := \bar{\varepsilon}(2, 4, 4), \quad C_1 := \bar{C}(2, 4, 4), \quad (1.20)$$

we see in particular that for all divergence free initial data with

$$\| |x|^{-1/2} u_0 \|_{L^2_{|x|} L^4_\theta} < \varepsilon_1 \quad (1.21)$$

there exists a unique global smooth solution u , which satisfies the estimate

$$\|u\|_{L^8_t L^4_x} \leq C_1 \| |x|^{-1/2} u_0 \|_{L^2_{|x|} L^4_\theta}. \quad (1.22)$$

To prepare for our last result, we introduce the notations

$$\theta_1(\tilde{p}) := \left(\frac{2\tilde{p}-4}{4-\tilde{p}}\right)^{1-\tilde{p}/4}, \quad \theta_2(\tilde{p}) := \left(\frac{2\tilde{p}-4}{4-\tilde{p}}\right)^{1-\tilde{p}/2}, \quad \tilde{p} \in (2, 4). \quad (1.23)$$

It is easy to check that

$$\lim_{\tilde{p} \rightarrow 2^+} \theta_1 = 0, \quad \lim_{\tilde{p} \rightarrow 4^-} \theta_1 = 1, \quad (1.24)$$

$$\lim_{\tilde{p} \rightarrow 2^+} \theta_2 = 1, \quad \lim_{\tilde{p} \rightarrow 4^-} \theta_2 = 0. \quad (1.25)$$

Thus we may set $\theta_1(2) = 0$, $\theta_2(2) = 1$. We also define the norm

$$[u_0]_{\tilde{p}} := \| |x|^{-\frac{2}{\tilde{p}}} u_0 \|_{L^{\tilde{p}/2}_{|x|} L^{\tilde{p}}_\theta} \| |x|^{-\frac{1}{\tilde{p}}} u_0 \|_{L^{\tilde{p}}_x}. \quad (1.26)$$

Note the following facts:

³ Here and in the following we use the notation $\|f\|_{XYZ} := \| \| \|f\|_Z \|_Y \|_X$ for nested norms. When we write $\|u\|_{L^r_t L^q_x}$ we mean that the integration is extended to all $x \in \mathbb{R}^3$ and $t > 0$.

1. It is easy to construct initial data such that $[u_0]_{\tilde{p}}$ is arbitrarily small while $\|u_0\|_{BMO^{-1}}$ is arbitrarily large. Indeed, fix $\phi \in C_c^\infty(\mathbb{R}^3)$ and denote with $\phi_K(x) := \phi(x - K\xi)$ its translate in the direction ξ for some $|\xi| = 1$ and $K > 1$; we have obviously

$$\||x|^{-\frac{1}{\tilde{p}}}\phi_K\|_{L_x^{\tilde{p}}} \simeq K^{-\frac{1}{\tilde{p}}} \quad (1.27)$$

since the $L_x^{\tilde{p}}$ norm is translation invariant. On the other hand, if the support of ϕ is contained in a sphere $B(0, R)$, we have

$$\||x|^{-\frac{2}{\tilde{p}}}\phi_K\|_{L_{|x|}^{\tilde{p}/2}L_\theta^{\tilde{p}}}^{\tilde{p}/2} = \int_0^{+\infty} (\int_{\mathbb{S}^2} |\phi(\theta\rho - K\xi)|^{\tilde{p}} dS_\theta)^{\frac{1}{2}} \rho d\rho \lesssim \int_{K-R}^{K+R} K^{-1} \rho d\rho \simeq 1 \quad (1.28)$$

and we obtain

$$[\phi_K]_{\tilde{p}} \lesssim (1)^{\frac{\tilde{p}}{2}-1} (K^{-\frac{1}{\tilde{p}}})^{2-\frac{\tilde{p}}{2}} = K^{\frac{1}{2}-\frac{2}{\tilde{p}}}. \quad (1.29)$$

Thus, by the translation invariance of BMO^{-1} , we conclude that if $\tilde{p} \in [2, 4)$

$$[\phi_K]_{\tilde{p}} \rightarrow 0 \quad \text{while} \quad \|\phi_K\|_{BMO^{-1}} = \text{const} \quad \text{as} \quad K \rightarrow \infty. \quad (1.30)$$

2. In the limit cases $\tilde{p} = 2$ and $\tilde{p} = 4$ we have simply

$$[u_0]_2 = \||x|^{-1/2}u_0\|_{L_x^2}, \quad [u_0]_4 = \||x|^{-1/2}u_0\|_{L_{|x|}^2L_\theta^4} \quad (1.31)$$

and actually the $[\cdot]_{\tilde{p}}$ norm arises as an interpolation norm between the two cases (see (4.2), (4.3) and (4.7) below).

We can now state our main result, which interpolates between Theorems 1.3 and 1.4:

Theorem 1.5 *There exists a constant $\delta > 0$ such that the following holds. Let u be a suitable weak solution of Problem 1.1 with divergence free initial data $u_0 \in L^2(\mathbb{R}^3)$, and let $\tilde{p} \in [2, 4)$ and $M > 1$.*

If the norm $[u_0]_{\tilde{p}}$ of the initial data satisfies

$$\theta_1 \cdot [u_0]_{\tilde{p}} \leq \delta, \quad \theta_2 \cdot [u_0]_{\tilde{p}} \leq \delta e^{-4M^2} \quad (1.32)$$

then the set

$$\Pi_{M\delta} := \left\{ (t, x) : t > \frac{|x|^2}{M\delta} \right\} \quad (1.33)$$

is regular for u .

The result can be interpreted as follows. Since $\theta_2(\tilde{p}) \rightarrow 0$ as $\tilde{p} \rightarrow 4$, we can choose $\tilde{p} = \tilde{p}_M$ as a function of M in such a way that

$$e^{4M^2} \cdot \theta_2(\tilde{p}_M) \rightarrow 0 \quad \text{as} \quad M \rightarrow +\infty. \quad (1.34)$$

Then, since $\theta_1(\tilde{p}) \rightarrow 1$ as $\tilde{p} \rightarrow 4$, from the theorem it follows that, for all sufficiently large M ,

$$[u_0]_{\tilde{p}_M} < \delta \quad \Rightarrow \quad \Pi_{M\delta} \text{ is a regular set for } u. \quad (1.35)$$

In other words, if we take $M \rightarrow +\infty$ and the norm $[u_0]_{\tilde{p}_M}$ is less than δ , then the regular set invades the whole half space $t > 0$. Note that, as remarked above, the $[u_0]_{\tilde{p}_M}$ norm can be small even if the BMO^{-1} norm of u_0 is large.

Even in the special case $\tilde{p} = 2$, which is covered by Theorem D of [2], the result gives some new information on the regular set. Indeed, for $\tilde{p} = 2$ we have $\theta_1 = 0$, $\theta_2 = 1$, and we obtain:

Corollary 1.6 *There exists a constant $\delta > 0$ such that for any suitable weak solution u of Problem 1.1 with divergence free initial data $u_0 \in L^2(\mathbb{R}^3)$, and for every $M > 1$, if the initial data satisfy*

$$\| |x|^{-1/2} u_0 \|_{L^2(\mathbb{R}^3)} \leq \delta e^{-4M^2} \quad (1.36)$$

then the set $\Pi_{M\delta}$ is regular for u .

Thus, taking $M \rightarrow +\infty$, we see that if the weighted L^2 norm of the data is sufficiently small, then the regular set invades the whole half space $t > 0$, as claimed above.

The rest of the paper is organized as follows. In Section 2 we collect the necessary tools, in particular we recall the fundamental Caffarelli–Kohn–Nirenberg regularity criterion from [2]; in Section 3 we prove Theorem 1.4; Section 4 is devoted to the proof of Theorem 1.5.

2 Preliminaries

We recall some definitions from [2].

Definition 2.1 Let $u_0 \in L^2(\mathbb{R}^3)$. The couple (u, P) is a *suitable weak solution* of Problem 1.1 if⁴

1. (u, P) satisfies (1.1) in the sense of distributions;
2. $u(t) \rightarrow u_0$ weakly in L^2 as $t \rightarrow 0$;
3. for some constants E_0, E_1

$$\int_{\mathbb{R}^3} |u|^2(t) dx \leq E_0,$$

for all $t > 0$ and

$$\int \int_{\mathbb{R}^+ \times \mathbb{R}^3} |\nabla u|^2 dt dx \leq E_1;$$

4. for all non negative $\phi \in C_c^\infty([0, \infty) \times \mathbb{R}^3)$ and for all $t > 0$

$$\begin{aligned} & \int_{\mathbb{R}^3} |u|^2 \phi(t) + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi \\ & \leq \int_{\mathbb{R}^3} |u_0|^2 \phi(0) + \int_0^t \int_{\mathbb{R}^3} |u|^2 (\phi_t + \Delta \phi) + \int_0^t \int_{\mathbb{R}^3} (|u|^2 + 2P) u \cdot \nabla \phi. \end{aligned} \quad (2.1)$$

⁴ This definition of suitable weak solutions is appropriate to work with the initial datum u_0 . For more details compare the Sections 2 and 7 of [2].

Suitable weak solutions are known to exist for all L^2 initial data, see [27] or the Appendix in [2]. Such solutions are also L^2 -weakly continuous as functions of time (see [35], pp. 281–282), namely

$$\int_{\mathbb{R}^3} u(t, x) w(x) dx \rightarrow \int_{\mathbb{R}^3} u(t', x) w(x) dx \quad (2.2)$$

for all $w \in L^2(\mathbb{R}^3)$ as $t \rightarrow t'$ ($t, t' \in [0, +\infty)$); thus it makes sense to impose the initial condition (2).

Next we define the *parabolic cylinder* of radius r and top point (t, x) as

$$Q_r(t, x) := \{(s, y) : |x - y| < r, t - r^2 < s < t\} \quad (2.3)$$

while the *shifted parabolic cylinder* is

$$Q_r^*(t, x) := Q_r(t + r^2/8, x) \equiv \{(s, y) : |x - y| < r, t - 7r^2/8 < s < t + r^2/8\} \quad (2.4)$$

The crucial regularity result in [2] ensures that:

Lemma 2.2 *There exists an absolute constant ε^* such that if (u, P) is a suitable weak solution of (1.1) and*

$$\limsup_{r \rightarrow 0} \frac{1}{r} \int \int_{Q_r^*(t, x)} |\nabla u|^2 \leq \varepsilon^*, \quad (2.5)$$

then (t, x) is a regular point.

We shall make frequent use of the following interpolation inequality from [1] (see also [9], [10] for extensions of the inequality):

Lemma 2.3 *Assume that*

1. $r \geq 0, 0 < a \leq 1, \gamma < 3/r, \alpha < 3/2, \beta < 3/2$;
2. $-\gamma + 3/r = a(-\alpha + 1/2) + (1 - a)(-\beta + 3/2)$;
3. $a\alpha + (1 - a)\beta \leq \gamma$;
4. *when $-\gamma + 3/r = -\alpha + 1/2$, assume also that $\gamma \leq a(\alpha + 1) + (1 - a)\beta$.*

Then

$$\|\sigma_\eta^\gamma u\|_{L^r(\mathbb{R}^3)} \leq C \|\sigma_\eta^\alpha \nabla u\|_{L^2(\mathbb{R}^3)}^a \|\sigma_\eta^\beta u\|_{L^2(\mathbb{R}^3)}^{1-a}, \quad (2.6)$$

where $\sigma_\eta := (\eta + |x|^2)^{-1/2}, \eta \geq 0$, with a constant C independent of η .

A key role in the following will be played by time-decay estimates for convolutions with the heat and Oseen kernels. It is convenient to introduce the notation

$$\Lambda(\alpha, p, \tilde{p}) := \alpha + \frac{2}{p} - \frac{2}{\tilde{p}}. \quad (2.7)$$

Proposition 2.4 ([22]) *Let $1 \leq p \leq q \leq \infty, 1 \leq \tilde{p} \leq \tilde{q} \leq \infty$ and*

$$\beta > -3/q, \quad \alpha < 3 - 3/p, \quad \Lambda(\alpha, p, \tilde{p}) \geq \Lambda(\beta, q, \tilde{q}). \quad (2.8)$$

For every (possibly zero) multi-index μ ,

1. if $|\mu| + \frac{3}{p} - \frac{3}{q} + \alpha - \beta \geq 0$, then

$$\| |x|^\beta \partial^\mu e^{t\Delta} u_0 \|_{L_{|x|}^q L_{d\theta}^{\tilde{q}}} \lesssim \frac{1}{t^{(|\mu| + \frac{3}{p} - \frac{3}{q} + \alpha - \beta)/2}} \| |x|^\alpha u_0 \|_{L_{|x|}^p L_{d\theta}^{\tilde{p}}}, \quad t > 0; \quad (2.9)$$

2. if $1 + |\mu| + \frac{3}{p} - \frac{3}{q} + \alpha - \beta > 0$, then

$$\| |x|^\beta \partial^\mu e^{t\Delta} \mathbb{P} \nabla \cdot F \|_{L_{|x|}^q L_{d\theta}^{\tilde{q}}} \lesssim \frac{1}{t^{(1 + |\mu| + \frac{3}{p} - \frac{3}{q} + \alpha - \beta)/2}} \| |x|^\alpha F \|_{L_{|x|}^p L_{d\theta}^{\tilde{p}}}, \quad t > 0. \quad (2.10)$$

An easy consequence of Proposition 2.4 is the embedding

$$L_{|x|}^p \alpha_p d_{|x|} L_{d\theta}^{\tilde{p}} \hookrightarrow B_{q,\infty}^{-1+3/q} \quad \text{if } \alpha = 1 - 3/p, \quad \tilde{p} \geq 2p/(p-1), \quad q \geq \max(p, \tilde{p}),$$

which is not needed in the following, but allows to compare Theorem 1.4 with earlier results; recall also that $B_{q,\infty}^{-1+3/q} \hookrightarrow BMO^{-1}$ for $q > 3$. Indeed, using estimate (2.9), we can write

$$\| e^{t\Delta} \phi \|_{L^q(\mathbb{R}^3)} \leq C t^{-(3/p-3/q+\alpha)/2} \| |x|^\alpha \phi \|_{L_{|x|}^p L_{d\theta}^{\tilde{p}}} \equiv C t^{-(1-3/q)/2} \| |x|^\alpha \phi \|_{L_{|x|}^p L_{d\theta}^{\tilde{p}}} \quad (2.11)$$

and then the embedding follows immediately from the following ‘caloric’ definition of Besov spaces; see e.g. [19].

Definition 2.5 A distribution $\phi \in \mathcal{S}'(\mathbb{R}^3)$ belongs to $B_{q,\infty}^{-1+3/q}(\mathbb{R}^3)$ ($q > 3$) if and only if

$$\| e^{t\Delta} \phi \|_{L^q(\mathbb{R}^3)} \leq C t^{-(1-3/q)/2} \quad \text{for } 0 < t \leq 1. \quad (2.12)$$

The best constant C in (2.12) is equivalent to the norm $\| \phi \|_{B_{q,\infty}^{-1+3/q}(\mathbb{R}^3)}$.

We conclude this section with an estimate for singular integrals in mixed radial-angular norms. Let $K \in C^1(\mathbb{S}^2)$ with zero mean value and

$$Tf(x) := \text{PV} \int_{\mathbb{R}^3} f(x-y) \frac{K(\hat{y})}{|y|^3} dy, \quad \hat{y} = \frac{y}{|y|}.$$

Theorem 2.6 Let $1 < p < \infty$, $1 < \tilde{p} < \infty$. Then

$$\| Tf \|_{L_{|x|}^p L_{d\theta}^{\tilde{p}}} \lesssim \| f \|_{L_{|x|}^p L_{d\theta}^{\tilde{p}}}. \quad (2.13)$$

The inequality (2.13) has been recently proved by A. Córdoba in the case $\tilde{p} = 2$ ([6], Theorem 2.1); essentially the same argument gives also the other cases.

Proof Consider first the case $p > \tilde{p}$. Let $1/q + \tilde{p}/p = 1$ and denote by X the set of all $g \in \mathcal{S}'(\mathbb{R})$ with $\int_0^{+\infty} g^q(\rho) \rho^2 d\rho = 1$. Then we can write

$$\begin{aligned} \| Tf \|_{L_{|x|}^p L_{d\theta}^{\tilde{p}}}^{\tilde{p}} &= \left(\int_0^{+\infty} \left(\int_{\mathbb{S}^2} |Tf(\rho, \theta)|^{\tilde{p}} dS_\theta \right)^{\frac{p}{\tilde{p}}} \rho^2 d\rho \right)^{\frac{\tilde{p}}{p}} \\ &= \sup_{g \in X} \int_0^{+\infty} \int_{\mathbb{S}^2} |Tf(\rho, \theta)|^{\tilde{p}} g(\rho) \rho^2 dS_\theta d\rho \\ &= \sup_{g \in X} \int_{\mathbb{R}^3} |Tf(x)|^{\tilde{p}} g(|x|) dx. \end{aligned}$$

Write $I(f, g) := \int_{\mathbb{R}^3} |Tf(x)|^{\tilde{p}} g(|x|) dx$. By Proposition 1 in [7] we have

$$I(f, g) \lesssim_s \int_{\mathbb{R}^3} |f(x)|^{\tilde{p}} (Mg^s(x))^{\frac{1}{s}} dx, \quad (2.14)$$

for all $1 < s < \infty$, where M is the Hardy–Littlewood maximal operator and $g^s(x) = (g(|x|))^s$. Since Mg^s is radially symmetric, this can be written

$$I(f, g) \lesssim_s \int_0^{+\infty} \int_{\mathbb{S}^2} |f(\rho, \theta)|^{\tilde{p}} (Mg^s(\rho))^{\frac{1}{s}} \rho^2 dS_\theta d\rho. \quad (2.15)$$

Now, for $s < q = \frac{p}{p-\tilde{p}}$, Hölder's inequality with exponents $p/\tilde{p}, q$ gives

$$\begin{aligned} I(f, g) &\lesssim \left(\int_0^{+\infty} \left(\int_{\mathbb{S}^2} |f(\rho, \theta)|^{\tilde{p}} dS_\theta \right)^{\frac{p}{\tilde{p}}} \rho^2 d\rho \right)^{\frac{\tilde{p}}{p}} \left(\int_0^{+\infty} (Mg^s(\rho))^{\frac{q}{s}} \rho^2 d\rho \right)^{\frac{1}{q}} \\ &\lesssim \|f\|_{L_{|x|}^{\tilde{p}} L_\theta^{\tilde{p}}}^{\tilde{p}} \|Mg^s\|_{L^{q/s}(\mathbb{R}^3)}^{1/s} \lesssim \|f\|_{L_{|x|}^{\tilde{p}} L_\theta^{\tilde{p}}}^{\tilde{p}} \|g^s\|_{L^{q/s}(\mathbb{R}^3)}^{1/s} \\ &\simeq \|f\|_{L_{|x|}^{\tilde{p}} L_\theta^{\tilde{p}}}^{\tilde{p}} \left(\int_0^{+\infty} g^q(\rho) \rho^2 d\rho \right)^{\frac{1}{q}} = \|f\|_{L_{|x|}^{\tilde{p}} L_\theta^{\tilde{p}}}^{\tilde{p}} \end{aligned}$$

and taking the supremum over all $g \in X$ we get the claim in the case $p > \tilde{p}$. The case $p = \tilde{p}$ is classical, and the case $p < \tilde{p}$ follows by duality.

Using the continuity of T in weighted Lebesgue spaces (see Stein [31])

$$\| |x|^\beta Tf \|_{L^p(\mathbb{R}^3)} \lesssim \| |x|^\beta f \|_{L^p(\mathbb{R}^3)} \quad \text{for } 1 < p < \infty, \quad -\frac{3}{p} < \beta < 3 - \frac{3}{p} \quad (2.16)$$

we can also obtain weighted versions of (2.13). In particular, by interpolation between (2.13) in the case $(\alpha_0, p_0, \tilde{p}_0) = (0, 2, 10)$ and (2.16) in the case $(\alpha_1, p_1, \tilde{p}_1) = (-4/3, 2, 2)$, with interpolation parameter $\theta = 3/8$ ($\Rightarrow (\alpha_\theta, p_\theta, \tilde{p}_\theta) = (-1/2, 2, 4)$), we get

$$\| |x|^{-1/2} Tf \|_{L_{|x|}^2 L_\theta^4} \lesssim \| |x|^{-1/2} f \|_{L_{|x|}^2 L_\theta^4}. \quad (2.17)$$

Remark 2.7 We denote with R_j the Riesz transform in the direction of the j -th coordinate and $R := (R_1, R_2, R_3)$. By (2.16, 2.17) the boundedness of R_j in $L^2(\mathbb{R}^3, |x|^{-1} dx)$ and $L_{|x|}^2 L_\theta^4(\mathbb{R}^3, |x|^{-1} dx)$ follows, and so that of $\mathbb{P} \equiv Id + R \otimes R$; see (1.4).

3 Proof of Theorem 1.4

We first need two technical lemmas. By standard machinery, integral estimates for the heat flow and for the bilinear operator appearing in the Duhamel representation (1.3) can be deduced by the time-decay estimates of Proposition 2.4.

Lemma 3.1 ([22]) *Let $\beta > -3/q$, $\alpha < 3 - 3/p$, $1 \leq \tilde{p} \leq \tilde{q} \leq \infty$, $1 < r < \infty$ and μ a (possibly zero) multi-index such that*

$$\begin{aligned} 1 \leq p \leq q \leq \infty & \quad \text{if } (|\mu| + \alpha - \beta)p + 1 < 0, \\ 1 \leq p \leq q < \frac{3p}{(|\mu| + \alpha - \beta)p + 1} & \quad \text{if } (|\mu| + \alpha - \beta)p + 1 \geq 0. \end{aligned} \quad (3.1)$$

Assume further that

$$|\mu| + \alpha + 3/p = \beta + 3/q + 2/r, \quad \Lambda(\alpha, p, \tilde{p}) \geq \Lambda(\beta, q, \tilde{q}). \quad (3.2)$$

Then

$$\| |x|^\beta \partial^\mu e^{t\Delta} u_0 \|_{L_t^q L_x^{\tilde{q}}} \lesssim \| |x|^\alpha u_0 \|_{L_t^p L_x^{\tilde{p}}}. \quad (3.3)$$

Remark 3.2 Once we have assumed the scaling relation in (3.2), it is straightforward to check that the assumption (3.1) is equivalent to $p < r$.

Proof The family of estimates (3.3) follows by the family of estimates (2.9) and by the Marcinkiewickz interpolation theorem. The condition $p < r$, which as remarked above turns out to be equivalent to (3.1), is necessary in order to apply the Marcinkiewickz theorem (see Proposition 3.4 in [22] for details).

Lemma 3.3 *Let $3 < q < \infty$, $2 < r < \infty$ satisfying $2/r + 3/q = 1$. Then*

$$\left\| \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes v)(s) ds \right\|_{L_t^q L_x^q} \lesssim \|u\|_{L_t^r L_x^q} \|v\|_{L_t^q L_x^q}. \quad (3.4)$$

The inequality (3.4) is well known, see for instance Theorem 3.1(i) in [12]. The $L_t^r L_x^q$ Lebesgue spaces have been extensively used in the context of Navier–Stokes equation since [12], [15].

Using the previous estimates, it is a simple matter to prove Theorem 1.4. We follow the scheme of the proof of Theorem 20.1(B) in [20] and we take advantage of the inequalities (2.9, 3.3).

Proof (Proof of Theorem 1.4)

Let $\tilde{p}_G := 2p/(p-1)$. We show that the space

$$X := \left\{ u : \|u\|_{L_t^q L_x^q} < +\infty, \sup_{t>0} t^{1/2} \|u\|_{L_x^\infty}(t) < +\infty \right\}, \quad (3.5)$$

equipped with the norm $\|\cdot\|_X := \|\cdot\|_{L_t^q L_x^q} + \sup_{t>0} t^{1/2} \|\cdot\|_{L_x^\infty}(t)$, is an admissible path space with adapted space $X_0 := L_{|x|^{\alpha p d}|x|}^p L_\theta^{\tilde{p}_G}$.

The estimate $\|e^{t\Delta} f\|_X \lesssim \|f\|_{X_0}$ follows indeed by the inequalities (2.9, 3.3); it is straightforward to check that (3.1) and $p, \tilde{p}_G \leq q$ are equivalent⁵ to (1.15) and that the last assumption in (3.2) and in (2.8) is satisfied because $\Lambda(\alpha, p, \tilde{p}_G) = \Lambda(0, q, q) = \Lambda(0, \infty, \infty) = 0$. Notice also that the set of q for which the third inequality in (1.15) is satisfied is not empty provided $p < 5$. It remains to show that

$$\|B(u, v)\|_X \lesssim \|u\|_X \|v\|_X.$$

The bound $\|B(u, v)\|_{L_t^q L_x^q} \lesssim \|u\|_{L_t^r L_x^q} \|v\|_{L_t^q L_x^q}$ follows by Lemma 3.3. In order to estimate $\sup_{t>0} t^{1/2} \|B(u, v)\|_{L_x^\infty}(t)$, we split this quantity as

$$\sup_{t>0} t^{1/2} \|B(u, v)\|_{L_x^\infty}(t) \leq I + II \quad (3.6)$$

⁵ Except that the value $q = p$ is not allowed in (1.15).

where

$$I = \sup_{t>0} t^{1/2} \left\| \int_0^{t/2} e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (u \otimes v)(s) ds \right\|_{L_x^\infty} \quad (3.7)$$

$$II = \sup_{t>0} t^{1/2} \left\| \int_{t/2}^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (u \otimes v)(s) ds \right\|_{L_x^\infty}, \quad (3.8)$$

and we use Minkowski inequality and the time-decay estimate (2.10). For I we have

$$\begin{aligned} I &\lesssim \sup_{t>0} t^{1/2} \int_0^{t/2} \frac{1}{(t-s)^{(1+3/(q/2))/2}} \|u\|_{L_x^q} \|v\|_{L_x^q}(s) ds \\ &\lesssim \sup_{t>0} t^{-3/q} \int_0^{t/2} \|u\|_{L_x^q} \|v\|_{L_x^q}(s) ds \\ &\lesssim \sup_{t>0} t^{-3/q} \|u\|_{L_t^r L_x^q} \|v\|_{L_t^r L_x^q} \left(\int \chi_{[0,t/2]}(s) ds \right)^{1-\frac{2}{r}} \\ &\lesssim \|u\|_{L_t^r L_x^q} \|v\|_{L_t^r L_x^q} t^{-3/q-2/r+1} = \|u\|_{L_t^r L_x^q} \|v\|_{L_t^r L_x^q} \end{aligned}$$

while for II we have

$$\begin{aligned} II &\lesssim \sup_{t>0} t^{1/2} \int_{t/2}^t \frac{1}{(t-s)^{1/2}} \frac{1}{s} \left(s^{1/2} \|u\|_{L_x^\infty}(s) \right) \left(s^{1/2} \|v\|_{L_x^\infty}(s) \right) ds \\ &\lesssim \left(\sup_{t>0} t^{1/2} \|u\|_{L_x^\infty} \right) \left(\sup_{t>0} t^{1/2} \|v\|_{L_x^\infty} \right) \sup_{t>0} t^{-1/2} \int_{t/2}^t \frac{1}{(t-s)^{1/2}} ds \\ &\lesssim \left(\sup_{t>0} t^{1/2} \|u\|_{L_x^\infty} \right) \left(\sup_{t>0} t^{1/2} \|v\|_{L_x^\infty} \right) \sup_{t>0} t^{-1/2} \left[(t-s)^{1/2} \right]_t^{t/2} \\ &\lesssim \left(\sup_{t>0} t^{1/2} \|u\|_{L_x^\infty} \right) \left(\sup_{t>0} t^{1/2} \|v\|_{L_x^\infty} \right). \end{aligned}$$

Summing up we obtain

$$\|B(u, v)\|_X \lesssim \|u\|_{L_t^r L_x^q} \|v\|_{L_t^r L_x^q} + \left(\sup_{t>0} t^{1/2} \|u\|_{L_x^\infty} \right) \left(\sup_{t>0} t^{1/2} \|v\|_{L_x^\infty} \right) \lesssim \|u\|_X \|v\|_X. \quad (3.9)$$

The existence of a unique solution u to Problem 1.3 satisfying

$$\|u\|_{L_t^r L_x^q} + \sup_{t>0} t^{1/2} \|u\|_{L_x^\infty}(t) \lesssim \| |x|^\alpha u_0 \|_{L_{|x|}^p L_\theta^{\bar{p}}} \quad (3.10)$$

follows by Proposition 1.1 and by the obvious inequality

$$\| |x|^\alpha u_0 \|_{L_{|x|}^p L_\theta^{\bar{p}}} \lesssim \| |x|^\alpha u_0 \|_{L_{|x|}^p L_\theta^{\bar{p}}}. \quad (3.11)$$

Finally, inequality (3.10) implies the boundedness of the solution u in $(\delta, +\infty) \times \mathbb{R}^3$ for all $\delta > 0$, and this implies smoothness of u (see Theorem 3.4 in [12] or [11], [15], [28], [30], [32], [36]).

We denote with $BC([0, \infty); L^2)$ the Banach space of bounded continuous functions $u : [0, \infty) \rightarrow L^2$ equipped with the norm $\sup_{t \geq 0} \|u(t)\|_{L^2}$.

Corollary 3.4 *Assume $u_0 \in L^2(\mathbb{R}^3)$ and that all the hypotheses of Theorem 1.4 are satisfied. Then the solution u belongs to $BC([0, \infty); L_x^2) \cap L_t^2 \dot{H}_x^1$. In particular u is a strong solution of (1.1), $u(t) \rightarrow u_0$ strongly in $L^2(\mathbb{R}^3)$ as $t \rightarrow 0$, and the energy identity $\|u(t)\|_{L_x^2}^2 + 2 \int_0^t \|\nabla u\|_{L_x^2}^2 = \|u_0\|_{L_x^2}^2$ holds for all $t \geq 0$.*

Proof Let X, X_0 be the same admissible and adapted spaces used in the proof of Theorem 1.4. We shall show that the space $X \cap BC([0, \infty); L_x^2) \cap L_t^2 \dot{H}_x^1$ equipped with the norm $\|\cdot\|_X + \|\cdot\|_{L_t^\infty L_x^2} + \|\cdot\|_{L_t^2 \dot{H}_x^1}$ is an admissible path space with adapted space $X_0 \cap L_x^2$ equipped with the norm $\|\cdot\|_{X_0} + \|\cdot\|_{L_x^2}$.

The estimate $\|e^{t\Delta} f\|_{X \cap BC([0, \infty); L_x^2)} \lesssim \|f\|_{X_0 \cap L_x^2}$ again follows by (2.9, 3.3), while the bound $\|e^{t\Delta} f\|_{L_t^2 \dot{H}_x^1} \lesssim \|f\|_{L_x^2}$, even if not covered by (3.3), is a well known property of the heat kernel. Since we have already proved $\|B(u, v)\|_X \lesssim \|u\|_X \|v\|_X$, it suffices to show that

$$\|B(u, v)\|_{L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1} \lesssim \|u\|_{X \cap BC([0, \infty); L_x^2) \cap L_t^2 \dot{H}_x^1} \|v\|_{X \cap BC([0, \infty); L_x^2) \cap L_t^2 \dot{H}_x^1}.$$

By the Minkowski and Hölder inequalities and (2.10)

$$\begin{aligned} \|B(u, v)\|_{L_t^\infty L_x^2} &\lesssim \sup_{t>0} \int_0^t \frac{1}{(t-s)^{1/2}} \frac{1}{s^{1/2}} \left(s^{1/2} \|u\|_{L_x^\infty}(s) \right) \|v\|_{L_x^2}(s) ds \\ &\leq \left(\sup_{t>0} t^{1/2} \|u\|_{L_x^\infty}(t) \right) \|v\|_{L_t^\infty L_x^2} \sup_{t>0} \int_0^t (t-s)^{-1/2} s^{-1/2} ds. \end{aligned} \quad (3.12)$$

Since $\int_0^t (t-s)^{-1/2} s^{-1/2} ds \leq C$ with C independent of t , (3.12) implies

$$\|B(u, v)\|_{L_t^\infty L_x^2} \lesssim \left(\sup_{t>0} t^{1/2} \|u\|_{L_x^\infty}(t) \right) \|v\|_{L_t^\infty L_x^2} \lesssim \|u\|_{X \cap BC([0, \infty); L_x^2)} \|v\|_{X \cap BC([0, \infty); L_x^2)}. \quad (3.13)$$

Similarly, using Minkowski's inequality, the $L^{p>1}$ boundedness of the Riesz transform and (2.9)

$$\|\nabla B(u, v)\|_{L_t^2 L_x^2} \lesssim \left\| \int_0^t \frac{1}{(t-s)^{(1+3/q)/2}} \|(u \cdot \nabla)v\|_{L_x^{2q/(q+2)}}(s) ds \right\|_{L_t^2}, \quad (3.14)$$

where $q > 3$. Then by Hölder's inequality and by the weak Young inequality for convolutions

$$\begin{aligned} &\lesssim \left\| \int_0^\infty \frac{1}{(t-s)^{(1+3/q)/2}} \|u\|_{L_x^q} \|\nabla v\|_{L_x^2}(s) ds \right\|_{L_t^2} \\ &\lesssim \left\| \frac{1}{(t-s)^{(1+3/q)/2}} \right\|_{L^{2q/(q+3), \infty}} \left\| \|u\|_{L_x^q} \|\nabla v\|_{L_x^2} \right\|_{L_t^{2q/(2q-3)}} \lesssim \|u\|_{L_t^q L_x^q} \|\nabla v\|_{L_t^2 L_x^2}, \end{aligned}$$

provided that $2/r + 3/q = 1$. These inequalities allow us to apply Proposition 1.1. Thus if

$$\|u_0\|_{X_0} = \| |x|^\alpha u_0 \|_{L_{|x|}^p L_\theta^{\tilde{p}}} + \|u_0\|_{L_x^2} < \bar{\varepsilon}, \quad (3.15)$$

with an $\bar{\varepsilon}$ possibly smaller than in Theorem 1.4, then $u \in X \cap BC([0, \infty); L_x^2) \cap L_t^2 \dot{H}_x^1$. On the other hand, rescaling the initial data and the solution as

$$u_0^\lambda = \lambda u_0(\lambda x), \quad u^\lambda = \lambda u(\lambda^2 t, \lambda x), \quad \lambda > 0; \quad (3.16)$$

we see that all norms remain fixed with the exception of $\|u_0^\lambda\|_{L_x^2}$, $\|u^\lambda\|_{L_t^\infty L_x^2}$, $\|u^\lambda\|_{L_t^2 \dot{H}_x^1}$ which goes to zero as $\lambda \rightarrow +\infty$. Thus the (3.15) is satisfied by u_0^λ , provided that $\| |x|^\alpha u_0 \|_{L_{|x|}^p L_\theta^{\tilde{p}}} =: \rho < \bar{\varepsilon}$ and $\lambda = \lambda(\rho)$ is large enough. In this way we find that $\| |x|^\alpha u_0 \|_{L_{|x|}^p L_\theta^{\tilde{p}}} < \bar{\varepsilon}$ implies that u^λ and hence u belongs to $BC([0, \infty); L_x^2) \cap L_t^2 \dot{H}_x^1$.

In particular we have $u(t) \rightarrow u_0$ strongly in $L^2(\mathbb{R}^3)$ as $t \rightarrow 0^+$. By this, and by the smoothness of u , it follows that u is a strong solution of (1.1) which satisfies the energy identity $\|u(t)\|_{L_x^2}^2 + 2 \int_0^t \|\nabla u\|_{L_x^2}^2 = \|u_0\|_{L_x^2}^2$, $t \geq 0$.

Remark 3.5 It is straightforward to check that the solution constructed in Corollary 3.4 is unique in the class of the weak solutions satisfying the energy inequality. More precisely, if u' is another weak solution of (1.1) satisfying $\|u'(t)\|_{L_x^2}^2 + 2 \int_0^t \|\nabla u'\|_{L_x^2}^2 \leq \|u_0\|_{L_x^2}^2$, $t \geq 0$, then the boundedness of $\|u\|_{L_t^q L_x^q} < \infty$ allows to apply the well known Prodi–Serrin uniqueness criterion [25], [29] to conclude $u = u'$.

4 Proof of Theorem 1.5

We note that the statement of Theorem 1.5 is invariant with respect to the natural scaling of the equation

$$u_0(x) \rightarrow u_0^\lambda(x) := \lambda u_0(\lambda x), \quad u(t, x) \rightarrow u^\lambda(t, x) := \lambda u(\lambda^2 t, \lambda x). \quad (4.1)$$

Thus it is sufficient to prove the result for $u_0^\lambda(x)$, $u^\lambda(t, x)$ instead of $u_0(x)$, $u(t, x)$, for an appropriate choice of the parameter λ . We choose $\lambda = \bar{\lambda}$ such that the following two quantities are equal:

$$\Gamma_1(\lambda, u_0, \tilde{p}) := \left(\int_0^{+\infty} \|u_0^\lambda(\rho \cdot)\|_{L_\theta^{\tilde{p}/2}}^{\tilde{p}/2} \rho \, d\rho \right)^{\frac{1}{2}} \equiv \lambda^{\frac{\tilde{p}}{4}-1} \| |x|^{-\frac{2}{\tilde{p}}} u_0 \|_{L_{|x|}^{\tilde{p}/2} L_\theta^{\tilde{p}}}^{\frac{\tilde{p}}{4}} \quad (4.2)$$

$$\Gamma_2(\lambda, u_0, \tilde{p}) := \left(\int_0^{+\infty} \|u_0^\lambda(\rho \cdot)\|_{L_\theta^{\tilde{p}}}^{\tilde{p}} \rho \, d\rho \right)^{\frac{1}{2}} \equiv \lambda^{\frac{\tilde{p}}{2}-1} \| |x|^{-\frac{1}{\tilde{p}}} u_0 \|_{L_x^{\tilde{p}}}^{\frac{\tilde{p}}{2}}. \quad (4.3)$$

It obvious that such a $\bar{\lambda}$ exists and one can easily calculate

$$\Gamma_1(\bar{\lambda}, u_0, \tilde{p}) = \Gamma_2(\bar{\lambda}, u_0, \tilde{p}) = [u_0]_{\tilde{p}} \equiv \varepsilon. \quad (4.4)$$

In the rest of the proof we shall drop the index $\bar{\lambda}$ and write simply $u_0 := u_0^{\bar{\lambda}}$, $u := u^{\bar{\lambda}}$.

We divide the proof into several steps. Note that in the course of the proof we shall reserve the symbol $Z \geq 1$ to denote several universal constants, which do not depend on u_0, u and $\tilde{p} \in [2, 4)$, and which may be different from line to line (and of course the final meaning of Z will be the maximum of all such constants).

4.1 Decomposition of the data

For $s > 0$ to be chosen, we write

$$u_{0,<s}(x) := u_0(x) \text{ if } |u_0(x)| < s, \quad u_{0,<s}(x) := 0 \text{ elsewhere} \quad (4.5)$$

and we decompose the initial data as

$$u_0 = v_0 + w_0, \quad w_0 := \mathbb{P}u_{0,<s}, \quad v_0 := \mathbb{P}(u_0 - u_{0,<s}) \quad (4.6)$$

which is possible since $u_0 = \mathbb{P}u_0$. We also write $u_{0,\geq s} := u_0 - u_{0,<s}$. It is clear that v_0, w_0 are divergence free. Moreover one has

$$\begin{aligned} \||x|^{-1/2}w_0\|_{L^2_{|x|}L^4_{\theta}} &\leq Zs^{1-\frac{\tilde{p}}{4}}(f\|u_0(\rho\cdot)\|_{L^{\tilde{p}}_{\theta}}^{\tilde{p}/2}\rho d\rho)^{\frac{1}{2}} = Zs^{1-\frac{\tilde{p}}{4}}\varepsilon \\ \||x|^{-1/2}v_0\|_{L^2_x} &\leq Zs^{1-\frac{\tilde{p}}{2}}(f\|u_0(\rho\cdot)\|_{L^{\tilde{p}}_{\theta}}^{\tilde{p}}\rho d\rho)^{\frac{1}{2}} = Zs^{1-\frac{\tilde{p}}{2}}\varepsilon \end{aligned} \quad (4.7)$$

for some universal constant $Z \geq 1$.

To prove (4.7), we use first the fact that the Leray projection \mathbb{P} is bounded on the weighted spaces $L^2(\mathbb{R}^3, |x|^{-1}dx)$ and $L^2_{|x|}L^4_{\theta}(\mathbb{R}^3, |x|^{-1}dx)$ (see Remark 2.7), then the elementary inequalities

$$\||x|^{-1/2}u_{0,<s}\|_{L^2_{|x|}L^4_{\theta}} \leq s^{1-\frac{\tilde{p}}{4}}(f\|u_0(\rho\cdot)\|_{L^{\tilde{p}}_{\theta}}^{\tilde{p}/2}\rho d\rho)^{\frac{1}{2}}, \quad (4.8)$$

$$\||x|^{-1/2}u_{0,\geq s}\|_{L^2_x} \leq s^{1-\frac{\tilde{p}}{2}}(f\|u_0(\rho\cdot)\|_{L^{\tilde{p}}_{\theta}}^{\tilde{p}}\rho d\rho)^{\frac{1}{2}}, \quad (4.9)$$

and finally property (4.4). Now we choose

$$s = \frac{2\tilde{p}-4}{4-\tilde{p}} \quad (4.10)$$

and this gives, with $\theta_1 = \theta_1(\tilde{p})$ and $\theta_2 = \theta_2(\tilde{p})$ defined as above (see (1.23)),

$$\||x|^{-1/2}w_0\|_{L^2_{|x|}L^4_{\theta}} \leq Z\theta_1\varepsilon, \quad \||x|^{-1/2}v_0\|_{L^2_x} \leq Z\theta_2\varepsilon. \quad (4.11)$$

Since the first norm in (4.11) is one of that we have considered in the well-posedness Theorem 1.4, when $Z\theta_1\varepsilon$ is small enough, we are allowed to look at the (unique/smooth) solution w of the the Navier–Stokes equation with data w_0 . This solution has good regularity properties and satisfies the powerful space-time integral estimate (1.22).

This suggests to decompose the weak solution $u = w + v$, so that we reduce to investigate the regularity properties of v instead of that of u . Looking again at the (4.11) and recalling $\theta_2(\tilde{p}) \rightarrow 0$ as $\tilde{p} \rightarrow 4^-$, this decomposition turns out to be convenient when \tilde{p} is close to 4, since, in this case, a substantially better smallness assumption on the data v_0 is available.

4.2 Decomposition of the weak solution

Consider the Cauchy problems

$$\begin{cases} \partial_t w + (w \cdot \nabla)w + \nabla P_w - \Delta w = 0 \\ \nabla \cdot w = 0 \\ w(0) = w_0 \\ P_w = R \otimes R (w \otimes w), \end{cases} \quad (4.12)$$

and

$$\begin{cases} \partial_t v + (v \cdot \nabla)v + (v \cdot \nabla)w + (w \cdot \nabla)v + \nabla P_v - \Delta v = 0 \\ \nabla \cdot v = 0 \\ v(0) = v_0 \\ P_v = R \otimes R (v \otimes v) + 2R \otimes R (v \otimes w). \end{cases} \quad (4.13)$$

Applying Theorem 1.4 (as in (1.21)) and Corollary 3.4, and recalling the first inequality in (4.11), we see that there exist two absolute constants ε_1, C_1 such that Problem 1.1 has a unique global smooth solution w provided the data satisfy

$$Z\theta_1\varepsilon < \varepsilon_1, \quad (4.14)$$

and in addition the solution w satisfies the estimate

$$\|w\|_{L_t^8 L_x^4} \leq C_1 \| |x|^{-1/2} w_0 \|_{L_{|x|}^2 L_\theta^4} \leq C_1 Z\theta_1\varepsilon \quad \Rightarrow \quad \|w\|_{L_t^8 L_x^4}^8 \leq C_1^8 (Z\theta_1\varepsilon)^7 \cdot Z\theta_1\varepsilon. \quad (4.15)$$

By possibly increasing Z by a factor bigger than both ε_1^{-1} and C_1^8 , this implies the following: if ε satisfies

$$Z\theta_1\varepsilon \leq 1 \quad (4.16)$$

then Problem 4.12 has a unique global smooth solution $w \in BC([0, \infty); L_x^2) \cap L_t^2 \dot{H}_x^1$ such that

$$\|w\|_{L_t^8 L_x^4}^8 \leq Z\theta_1\varepsilon. \quad (4.17)$$

As a consequence, the function $v = u - w$ is a weak solution of the second Cauchy Problem 4.13. Moreover, since u is a suitable weak solution, the function v inherits similar properties (we shall say for short that v is a suitable weak solution of the modified Problem 4.13).

4.3 A change of variables

Let $\xi \in \mathbb{R}^3, T > 0$ and consider the segment

$$L(T, \xi) := \{(s, \xi s) : s \in (0, T)\}.$$

We ask for which (T, ξ) the set $L(T, \xi)$ is a regular set. To this purpose we introduce the transformation

$$(t, y) = (t, x - \xi t), \quad v_\xi(t, y) = v(t, x), \quad w_\xi(t, y) = w(t, x), \quad (4.18)$$

which takes (4.12) into the system

$$\begin{cases} \partial_t w_\xi + ((w_\xi - \xi) \cdot \nabla) w_\xi + \nabla P_{w_\xi} - \Delta w_\xi = 0 \\ \nabla \cdot w_\xi = 0 \\ w_\xi(0) = w_0 \\ P_{w_\xi} = R \otimes R (w_\xi \otimes w_\xi), \end{cases} \quad (4.19)$$

and (4.13) into the system

$$\begin{cases} \partial_t v_\xi + ((v_\xi - \xi) \cdot \nabla) v_\xi + (v_\xi \cdot \nabla) w_\xi + (w_\xi \cdot \nabla) v_\xi + \nabla P_{v_\xi} - \Delta v_\xi = 0 \\ \nabla \cdot v_\xi = 0 \\ v_\xi(0) = v_0 \\ P_{v_\xi} = R \otimes R (v_\xi \otimes v_\xi) + 2R \otimes R (v_\xi \otimes w_\xi). \end{cases} \quad (4.20)$$

Note that this change of coordinates maps $L(T, \xi)$ in $(0, T) \times \{0\}$. Now we fix an arbitrary $M \geq 1$ and we define the set

$$S(M, T, \xi) := \left\{ s \in [0, T] : \int_s^{s+T/M} \int_{\mathbb{R}^3} |y|^{-1} |\nabla v_\xi(\tau, y)|^2 d\tau dy > M \right\} \quad (4.21)$$

and the number $\bar{s} \geq 0$

$$\bar{s} := \begin{cases} \inf \{s \in S(M, T, \xi)\} & \text{if } S(M, T, \xi) \neq \emptyset \\ T & \text{otherwise.} \end{cases} \quad (4.22)$$

From the definition of \bar{s} one has immediately

$$\int_0^{\bar{s}} \int_{\mathbb{R}^3} |y|^{-1} |\nabla v_\xi(\tau, y)|^2 d\tau dy \leq M(M+1) \leq 2M^2. \quad (4.23)$$

We next distinguish two cases.

4.4 First case: $\bar{s} = T$

In this case the entire segment $L(T, \xi)$ is a regular set. To prove this, we note first that by (4.23)

$$\int_0^T \int_{\mathbb{R}^3} \frac{|\nabla v(\tau, x)|^2}{|x - \xi \tau|} d\tau dx < +\infty. \quad (4.24)$$

Suppose we can also prove that

$$\int_0^T \int_{\mathbb{R}^3} \frac{|\nabla w(\tau, x)|^2}{|x - \xi \tau|} d\tau dx < +\infty. \quad (4.25)$$

Then summing the two we obtain

$$\int_0^T \int_{\mathbb{R}^3} \frac{|\nabla u(\tau, x)|^2}{|x - \xi \tau|} d\tau dx < +\infty. \quad (4.26)$$

Now let $0 < s < T$, and let $r > 0$ be so small that $0 < s - 7r^2/8 < s + r^2/8 < T$ and $|\xi|r \leq 1$. For each $(\tau, x) \in Q_r^*(s, \xi s)$ we have

$$|x - \xi \tau| \leq |x - \xi s| + |\xi||s - \tau| \leq r + r^2|\xi| \leq 2r \quad (4.27)$$

which implies

$$\frac{1}{r} \int \int_{Q_r^*(s, \xi s)} |\nabla u(\tau, x)|^2 d\tau dx \leq 2 \int_{s-7r^2/8}^{s+r^2/8} \int_{\mathbb{R}^3} \frac{|\nabla u(\tau, x)|^2}{|x - \xi \tau|} d\tau dx. \quad (4.28)$$

By continuity of the integral function, we obtain that the regularity condition (2.5) is satisfied at all $(s, \xi s) \in L(T, \xi)$, i.e. $L(T, \xi)$ is a regular set as claimed.

It remains to prove (4.25). By (4.17, 4.11) we know that

$$\|w_\xi\|_{L_t^8 L_x^4} = \|w\|_{L_t^8 L_x^4} < +\infty, \quad \| |x|^{-1/2} w_0 \|_{L^2(\mathbb{R}^3)} < +\infty \quad (4.29)$$

and that w , hence w_ξ , is a smooth solution. Thus we can write the energy inequality

$$\begin{aligned} \int_{\mathbb{R}^3} \phi |w_\xi|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \phi |\nabla w_\xi|^2 \leq \int_{\mathbb{R}^3} \phi |w_0|^2 dx \\ + \int_0^t \int_{\mathbb{R}^3} |w_\xi|^2 (\phi_t - \xi \cdot \nabla \phi + \Delta \phi) + \int_0^t \int_{\mathbb{R}^3} (|w_\xi|^2 + 2P_{w_\xi}) w_\xi \cdot \nabla \phi \end{aligned} \quad (4.30)$$

where $P_{w_\xi} = R \otimes R (w_\xi \otimes w_\xi)$ and $\phi \in C_c^\infty(\mathbb{R}^3)$ is any test function $\phi \geq 0$. We choose

$$\phi(y) := \sigma_\eta(y) \chi(\delta|y|), \quad \sigma_\eta(y) := (\eta + |y|^2)^{-\frac{1}{2}} \quad \eta, \delta > 0 \quad (4.31)$$

where $\chi \geq 0$ is a cut-off function supported in $[-1, 1]$ and equal to 1 near 0 (compare with the proof of Lemma 8.3 in [2]). Letting $\delta \rightarrow 0$ and using the inequalities

$$|\nabla \sigma_\eta| \leq (\eta + |y|^2)^{-1} = \sigma_\eta^2, \quad \Delta \sigma_\eta < 0, \quad (4.32)$$

we obtain

$$\begin{aligned} [\int_{\mathbb{R}^3} \sigma_\eta |w_\xi|^2]_0^t + 2 \int_0^t \int_{\mathbb{R}^3} \sigma_\eta |\nabla w_\xi|^2 \leq \\ \leq |\xi| \int_0^t \int_{\mathbb{R}^3} \sigma_\eta^2 |w_\xi|^2 + \int_0^t \int_{\mathbb{R}^3} \sigma_\eta^2 (|w_\xi|^3 + 2|P_{w_\xi}| |w_\xi|). \end{aligned}$$

Our goal is to prove an integral inequality for the quantities

$$a_\eta(t) = \int_{\mathbb{R}^3} \sigma_\eta(y) |w_\xi(t, y)|^2 dy, \quad B_\eta(t) = \int_0^t \int_{\mathbb{R}^3} \sigma_\eta(y) |\nabla w_\xi(\tau, y)|^2 dy d\tau. \quad (4.33)$$

To proceed, we use the weighted L^p inequality for the Riesz transform ([31]), uniform in $\eta \geq 0$

$$\|\sigma_\eta^m R\phi\|_{L^s} \leq Z \|\sigma_\eta^m \phi\|_{L^s}, \quad 1 < s < \infty, \quad m \in \left(-\frac{3(s-1)}{s}, \frac{3}{s}\right). \quad (4.34)$$

For the pressure term we have, using (2.6) and (4.34),

$$\begin{aligned} 2 \int_{\mathbb{R}^3} \sigma_\eta^2 |P_{w_\xi}| |w_\xi| &= 2 \int_{\mathbb{R}^3} \sigma_\eta^2 |w_\xi| |R \otimes R (w_\xi \otimes w_\xi)| \\ &\leq \|\sigma_\eta R \otimes R (w_\xi \otimes w_\xi)\|_{L^{8/5}} \|\sigma_\eta w_\xi\|_{L^{8/3}} \lesssim \|\sigma_\eta |w_\xi|^2\|_{L^{8/5}} \|\sigma_\eta w_\xi\|_{L^{8/3}} \\ &\leq \|w_\xi\|_{L^4} \|\sigma_\eta w_\xi\|_{L^{8/3}}^2 \lesssim \|w_\xi\|_{L^4} \|\sigma_\eta^{1/2} \nabla w_\xi\|_{L^2}^{7/4} \|\sigma_\eta^{1/2} w_\xi\|_{L^2}^{1/4} \\ &= \|w_\xi\|_{L^4} \dot{B}_\eta^{7/8} a_\eta^{1/8} \leq \frac{\dot{B}_\eta}{6} + C \|w_\xi\|_{L^4}^8 \cdot a_\eta \end{aligned} \quad (4.35)$$

for some universal constant C . In a similar way,

$$\begin{aligned} \int_{\mathbb{R}^3} \sigma_\eta^2 |w_\xi|^3 &\leq \|w_\xi\|_{L^4} \|\sigma_\eta^2 |w_\xi|^2\|_{L^{4/3}} = \|w_\xi\|_{L^4} \|\sigma_\eta w_\xi\|_{L^{8/3}}^2 \\ &\lesssim \|w_\xi\|_{L^4} \|\sigma_\eta^{1/2} \nabla w_\xi\|_{L^2}^{7/4} \|\sigma_\eta^{1/2} w_\xi\|_{L^2}^{1/4} \leq \frac{\dot{B}_\eta}{6} + C \|w_\xi\|_{L^4}^8 a_\eta \end{aligned} \quad (4.36)$$

and

$$|\xi| \int_{\mathbb{R}^3} \sigma_\eta^2 |w_\xi|^2 \lesssim |\xi| \cdot \|\sigma_\eta^{1/2} \nabla w_\xi\|_{L^2} \|\sigma_\eta^{1/2} w_\xi\|_{L^2} = |\xi| (\dot{B}_\eta a_\eta)^{1/2} \leq \frac{\dot{B}_\eta}{6} + C |\xi|^2 a_\eta. \quad (4.37)$$

Plugging these inequalities in the energy estimate we get

$$a_\eta(t) + B_\eta(t) \leq a_\eta(0) + \int_0^t (C|\xi|^2 + 3C \|w_\xi(s, \cdot)\|_{L^4}^8) a(s) ds, \quad (4.38)$$

and passing to the limit $\eta \rightarrow 0$ we obtain, for some larger universal constant C (note that $\|w_\xi(s)\|_{L^4} = \|w(s)\|_{L^4}$ for all s)

$$a(t) + B(t) \leq a(0) + C \int_0^t (|\xi|^2 + \|w(s, \cdot)\|_{L^4}^8) a(s) ds, \quad (4.39)$$

where

$$a(t) = \int_{\mathbb{R}^3} |y|^{-1} |w_\xi(t, y)|^2 dy, \quad B(t) = \int_0^t \int_{\mathbb{R}^3} |y|^{-1} |\nabla w_\xi(\tau, y)|^2 dy d\tau. \quad (4.40)$$

By a standard application of Gronwall's inequality, we obtain $a(t) < +\infty$ for all $t \geq 0$ which implies also $B(t) < +\infty$ for all $t \geq 0$ and thus the (4.25), as claimed.

4.5 Second case: $0 \leq \bar{s} < T$

Since v_ξ is a suitable weak solution of Problem 4.20, the following modified energy inequality is valid (see e.g. [3] for details): for all $t \geq 0$ and $0 \leq \phi \in C_c^\infty([0, \infty) \times \mathbb{R}^3)$, we have

$$\begin{aligned} \int_{\mathbb{R}^3} \phi(t, y) |v_\xi(t, y)|^2 dy + 2 \int_0^t \int_{\mathbb{R}^3} \phi |\nabla v_\xi|^2 \\ \leq \int_{\mathbb{R}^3} \phi(0, y) |v_0(y)|^2 dy + \int_0^t \int_{\mathbb{R}^3} |v_\xi|^2 (\phi_t - \xi \cdot \nabla \phi + \Delta \phi) \\ + \int_0^t \int_{\mathbb{R}^3} (|v_\xi|^2 + 2P_{v_\xi}) v_\xi \cdot \nabla \phi + \int_0^t \int_{\mathbb{R}^3} |v_\xi|^2 (w_\xi \cdot \nabla \phi) \\ + 2 \int_0^t \int_{\mathbb{R}^3} (v_\xi \cdot w_\xi) (v_\xi \cdot \nabla \phi) + \phi (v_\xi \cdot \nabla) v_\xi \cdot w_\xi \end{aligned} \quad (4.41)$$

which implies

$$\begin{aligned} \int_{\mathbb{R}^3} \phi(t, y) |v_\xi(t, y)|^2 dy + 2 \int_0^t \int_{\mathbb{R}^3} \phi |\nabla v_\xi|^2 \leq \\ \leq \int_{\mathbb{R}^3} \phi(0, y) |v_0(y)|^2 dy + \int_0^t \int_{\mathbb{R}^3} |v_\xi|^2 (\phi_t - \xi \cdot \nabla \phi + \Delta \phi) \\ + \int_0^t \int_{\mathbb{R}^3} (|v_\xi|^2 + 2P_{v_\xi}) v_\xi \cdot \nabla \phi + \int_0^t \int_{\mathbb{R}^3} 3|v_\xi|^2 |w_\xi| |\nabla \phi| + 18|\phi| |v_\xi| |\nabla v_\xi| |w_\xi|. \end{aligned} \quad (4.42)$$

By a standard approximation procedure (see the proof of Lemma 8.3 in [2]) the estimate is valid for any test function of the form

$$\phi(t, y) := \psi(t)\phi_1(y) \quad (4.43)$$

with $\phi_1 \in C_c^\infty(\mathbb{R}^3)$, $\phi_1 \geq 0$, and

$$\psi : [0, \infty) \rightarrow [0, \infty) \quad \text{absolutely continuous with} \quad \dot{\psi} \in L^1([0, \infty)). \quad (4.44)$$

We shall choose here

$$\psi(t) \equiv 1, \quad \phi_1 = \sigma_\eta(y)\chi(\delta|y|), \quad (4.45)$$

where $\eta, \delta > 0$,

$$\sigma_\eta(y) = (\eta + |y|^2)^{-\frac{1}{2}}, \quad (4.46)$$

and $\chi : [0, \infty) \rightarrow [0, 1]$ is a smooth nonincreasing function such that

$$\chi = 1 \text{ on } [0, 1], \quad \chi = 0 \text{ on } [2, +\infty). \quad (4.47)$$

Passing to the limit $\delta \rightarrow 0$ in the energy inequality we obtain

$$\begin{aligned} & [\int_{\mathbb{R}^3} \sigma_\eta |v_\xi|^2]_0^t + 2 \int_0^t \int_{\mathbb{R}^3} \sigma_\eta |\nabla v_\xi|^2 \leq \\ & \leq \int_0^t \int_{\mathbb{R}^3} |v_\xi|^2 (-\xi \cdot \nabla \sigma_\eta + \Delta \sigma_\eta) + \int_0^t \int_{\mathbb{R}^3} (|v_\xi|^2 + 2P_{v_\xi}) v_\xi \cdot \nabla \sigma_\eta \\ & + 18 \int_0^t \int_{\mathbb{R}^3} \sigma_\eta |v_\xi| |\nabla v_\xi| |w_\xi| + 3 \int_0^t \int_{\mathbb{R}^3} |v_\xi|^2 |w_\xi| |\nabla \sigma_\eta|. \end{aligned} \quad (4.48)$$

Note that a similar argument is used in [2], one of the differences here being the presence of the last two perturbative terms, which we control using (4.17). Recalling (4.32), we deduce the estimate

$$\begin{aligned} & [\int_{\mathbb{R}^3} \sigma_\eta |v_\xi|^2]_0^t + 2 \int_0^t \int_{\mathbb{R}^3} \sigma_\eta |\nabla v_\xi|^2 \leq |\xi| \int_0^t \int_{\mathbb{R}^3} \sigma_\eta^2 |v_\xi|^2 \\ & + \int_0^t \int_{\mathbb{R}^3} \sigma_\eta^2 (|v_\xi|^3 + 2|P_{v_\xi}| |v_\xi| + 3|v_\xi|^2 |w_\xi|) + 18 \int_{\mathbb{R}^3} \sigma_\eta |v_\xi| |\nabla v_\xi| |w_\xi|. \end{aligned} \quad (4.49)$$

We can now proceed as in the first case, using (4.49) to obtain a Gronwall type inequality for the quantities

$$a_\eta(t) = \int_{\mathbb{R}^3} \sigma_\eta(y) |v_\xi(t, y)|^2 dy, \quad B_\eta(t) = \int_0^t \int_{\mathbb{R}^3} \sigma_\eta(y) |\nabla v_\xi(\tau, y)|^2 dy d\tau. \quad (4.50)$$

We first estimate the term in P_{v_ξ} ; recall that

$$P_{v_\xi} = R \otimes R (v_\xi \otimes v_\xi) + 2R \otimes R (v_\xi \otimes w_\xi). \quad (4.51)$$

We have

$$\begin{aligned} 2 \int_{\mathbb{R}^3} \sigma_\eta^2 |P_{v_\xi}| |v_\xi| & \leq 2 \int_{\mathbb{R}^3} \sigma_\eta^2 |v_\xi| |R \otimes R (v_\xi \otimes v_\xi)| \\ & + 4 \int_{\mathbb{R}^3} \sigma_\eta^2 |v_\xi| |R \otimes R (v_\xi \otimes w_\xi)| =: I + II. \end{aligned}$$

Here and in the following, as usual, Z denotes several universal constants, possibly different from line to line. By (4.34) we can write

$$\begin{aligned} I &\leq 2\|\sigma_\eta R \otimes R(v_\xi \otimes v_\xi)\|_{L^2} \|\sigma_\eta v_\xi\|_{L^2} \leq Z\|\sigma_\eta |v_\xi|^2\|_{L^2} \|\sigma_\eta v_\xi\|_{L^2} \\ &\leq Z\|\sigma_\eta^{1/2} v_\xi\|_{L^4}^2 \|\sigma_\eta v_\xi\|_{L^2} \end{aligned} \quad (4.52)$$

and then by the Caffarelli–Kohn–Nirenberg inequality we obtain

$$\begin{aligned} I &\leq Z\|\sigma_\eta^{1/2} \nabla v_\xi\|_{L^2}^{3/2} \|\sigma_\eta^{1/2} v_\xi\|_{L^2}^{1/2} \cdot \|\sigma_\eta^{1/2} \nabla v_\xi\|_{L^2}^{1/2} \|\sigma_\eta^{1/2} v_\xi\|_{L^2}^{1/2} \\ &= Z\dot{B}_\eta a_\eta^{1/2} \leq \frac{\dot{B}_\eta}{6} + Z\dot{B}_\eta a_\eta. \end{aligned} \quad (4.53)$$

In a similar way we have

$$\begin{aligned} II &\leq 4\|\sigma_\eta R \otimes R(v_\xi \otimes w_\xi)\|_{L^{\frac{8}{3}}} \|\sigma_\eta v_\xi\|_{L^{\frac{8}{3}}} \\ &\leq Z\|\sigma_\eta |v_\xi| |w_\xi|\|_{L^{\frac{8}{3}}} \|\sigma_\eta v_\xi\|_{L^{\frac{8}{3}}} \leq Z\|w_\xi\|_{L^4} \|\sigma_\eta v_\xi\|_{L^{\frac{8}{3}}}^2 \end{aligned}$$

and again by the CKN inequality

$$II \leq Z\|w_\xi\|_{L^4} \|\sigma_\eta^{1/2} v_\xi\|_{L^2}^{1/4} \|\sigma_\eta^{1/2} \nabla v_\xi\|_{L^2}^{7/4}, = Z\|w_\xi\|_{L^4} a_\eta^{\frac{1}{8}} \dot{B}_\eta^{\frac{7}{8}} \leq \frac{\dot{B}_\eta}{6} + Z\|w_\xi\|_{L^4}^8 a_\eta. \quad (4.54)$$

Consider now the other terms in (4.49). Proceeding as above, we have

$$|\xi| \int_{\mathbb{R}^3} \sigma_\eta^2 |v_\xi|^2 \leq Z|\xi| \|\sigma_\eta^{1/2} \nabla v_\xi\|_{L^2} \|\sigma_\eta^{1/2} v_\xi\|_{L^2} = Z|\xi| (\dot{B}_\eta a_\eta)^{1/2} \leq \frac{\dot{B}_\eta}{6} + Z|\xi|^2 a_\eta; \quad (4.55)$$

and

$$\int_{\mathbb{R}^3} \sigma_\eta^2 |v_\xi|^3 = \|\sigma_\eta^{2/3} v_\xi\|_{L^3}^3 \leq Z\|\sigma_\eta^{1/2} \nabla v_\xi\|_{L^2}^2 \|\sigma_\eta^{1/2} v_\xi\|_{L^2} = Z\dot{B}_\eta a_\eta^{1/2} \leq \frac{\dot{B}_\eta}{6} + Z\dot{B}_\eta a_\eta \quad (4.56)$$

while for the perturbative terms we can write

$$\begin{aligned} 3 \int_{\mathbb{R}^3} \sigma_\eta^2 |v_\xi|^2 |w_\xi| &\leq 3\|w_\xi\|_{L^4} \|\sigma_\eta v_\xi\|_{L^{8/3}}^2 \leq Z\|w_\xi\|_{L^4} \|\sigma_\eta^{1/2} v_\xi\|_{L^2}^{1/4} \|\sigma_\eta^{1/2} \nabla v_\xi\|_{L^2}^{7/4} \\ &= Z\|w_\xi\|_{L^4} a_\eta^{1/8} \dot{B}_\eta^{7/8} \leq \frac{\dot{B}_\eta}{6} + Z\|w_\xi\|_{L^4}^8 a_\eta \end{aligned} \quad (4.57)$$

and

$$\begin{aligned} 18 \int_{\mathbb{R}^3} \sigma_\eta |v_\xi| |\nabla v_\xi| |w_\xi| &\leq 18\|\sigma_\eta^{1/2} \nabla v_\xi\|_{L^2} \|w_\xi\|_{L^4} \|\sigma_\eta^{1/2} v_\xi\|_{L^4} \\ &\leq Z\|\sigma_\eta^{1/2} \nabla v_\xi\|_{L^2} \|w_\xi\|_{L^4} \|\sigma_\eta^{1/2} \nabla v_\xi\|_{L^2}^{3/4} \|\sigma_\eta^{1/2} v_\xi\|_{L^2}^{1/4} \\ &= Z\|w_\xi\|_{L^4} \dot{B}_\eta^{7/8} a_\eta^{1/8} \leq \frac{\dot{B}_\eta}{6} + Z\|w_\xi\|_{L^4}^8 a_\eta. \end{aligned} \quad (4.58)$$

Now recalling (4.49), summing all the inequalities and absorbing a term $\int_0^t \dot{B}_\eta(s) ds = B_\eta(t)$ from the left hand side, we obtain

$$a_\eta(t) + B_\eta(t) \leq a_\eta(0) + Z \int_0^t (|\xi|^2 + \dot{B}_\eta(s) + \|w_\xi(s, \cdot)\|_{L^4}^8) a(s) ds, \quad (4.59)$$

and passing to the limit $\eta \rightarrow 0$, we arrive at the estimate

$$a(t) + B(t) \leq a(0) + Z \int_0^t (|\xi|^2 + \dot{B}(s) + \|w_\xi(s, \cdot)\|_{L^4}^8) a(s) ds, \quad (4.60)$$

for some universal constant Z , where

$$a(t) = \int_{\mathbb{R}^3} |y|^{-1} |v_\xi(t, y)|^2 dy, \quad B(t) = \int_0^t \int_{\mathbb{R}^3} |y|^{-1} |\nabla v_\xi(\tau, y)|^2 dy d\tau. \quad (4.61)$$

Noting $\|w_\xi(s, \cdot)\|_{L^4} = \|w(s, \cdot)\|_{L^4}$, by a standard application of Gronwall's lemma we get for $0 \leq t \leq \bar{s}$

$$a(t) \leq a(0)(1 + ZAe^{ZA}), \quad A = B(\bar{s}) + \|w\|_{L_t^8 L_x^4}^8 + \bar{s}|\xi|^2. \quad (4.62)$$

By (4.23, 4.17) we have $A \leq 2M^2 + Z + \bar{s}|\xi|^2$, while by (4.11) we have $a(0) \leq (Z\theta_2\varepsilon)^2$. If we restrict to the vectors ξ such that⁶

$$|\xi|^2 \bar{s} \leq M^2 \quad (4.63)$$

the estimate becomes

$$a(\bar{s}) \leq (Z\theta_2\varepsilon)^2 (1 + (3M^2 + Z)e^{3M^2 + Z}) \quad (4.64)$$

and taking a possibly larger universal constant Z , this implies

$$a(\bar{s}) \leq Ze^{4M^2} (\theta_2\varepsilon)^2. \quad (4.65)$$

Notice that (4.63) is satisfied provided that

$$L(T, \xi) \subset \left\{ (\tau, z) : \tau \geq \frac{|z|^2}{M^2} \right\}. \quad (4.66)$$

We now repeat the argument, starting from the point $(\bar{s}, \bar{s}\xi)$. We write the analogous of the energy inequality (4.42) on the time interval $\bar{s} \leq s \leq t$ with $t \leq \bar{s} + T$, choosing as test function $\phi(t, y) := \psi_\eta(t) \sigma_\eta(y) \chi(\delta|y|)$ where χ and σ_η are as before, while

$$\psi_\eta(t) := e^{-kB_{\bar{s}, \eta}(t)}, \quad B_{\bar{s}, \eta}(t) := \int_{\bar{s}}^t \int_{\mathbb{R}^3} \sigma_\eta |\nabla v_\xi|^2 \quad (4.67)$$

with k a positive constant to be specified. Note that $B_{\bar{s}, \eta}$ is bounded if $\eta > 0$ by the properties of v . In this way we obtain, letting $\delta \rightarrow 0$,

$$\begin{aligned} & \left[\int_{\mathbb{R}^3} \psi_\eta \sigma_\eta |v_\xi|^2 \right]_{\bar{s}}^t + 2 \int_{\bar{s}}^t \int_{\mathbb{R}^3} \psi_\eta \sigma_\eta |\nabla v_\xi|^2 \leq \\ & \leq \int_{\bar{s}}^t \int_{\mathbb{R}^3} \psi_\eta |v_\xi|^2 (-kB_{\bar{s}, \eta} \sigma_\eta - \xi \cdot \nabla \sigma_\eta + \Delta \sigma_\eta) + \int_{\bar{s}}^t \int_{\mathbb{R}^3} \psi_\eta (|v_\xi|^2 + 2P_{v_\xi} v_\xi) \cdot \nabla \sigma_\eta \\ & + 18 \int_{\bar{s}}^t \int_{\mathbb{R}^3} \psi_\eta \sigma_\eta |v_\xi| |\nabla v_\xi| |w_\xi| + 3 \int_{\bar{s}}^t \int_{\mathbb{R}^3} \psi_\eta |v_\xi|^2 |w_\xi| |\nabla \sigma_\eta|, \end{aligned}$$

⁶ Remember that \bar{s} is a function of ξ .

for $\bar{s} \leq t \leq \bar{s} + T$, and this implies, recalling (4.32),

$$\begin{aligned} & [\int_{\mathbb{R}^3} \Psi_\eta \sigma_\eta |v_\xi|^2]_{\bar{s}}^t + 2 \int_{\bar{s}}^t \int_{\mathbb{R}^3} \Psi_\eta \sigma_\eta |\nabla v_\xi|^2 \leq \\ & \leq \int_{\bar{s}}^t \int_{\mathbb{R}^3} \Psi_\eta |v_\xi|^2 (|\xi| \sigma_\eta^2 - k \dot{B}_{\bar{s}, \eta} \sigma_\eta) \\ & + \int_{\bar{s}}^t \int_{\mathbb{R}^3} \Psi_\eta \sigma_\eta^2 (|v_\xi|^3 + 2|P_{v_\xi}| |v_\xi| + 3|v_\xi|^2 |w_\xi|) + 18 \sigma_\eta |v_\xi| |\nabla v_\xi| |w_\xi|. \end{aligned} \quad (4.68)$$

Our goal now is to prove an integral inequality involving the quantities

$$a_\eta(t) = \int_{\mathbb{R}^3} \sigma_\eta(y) |v_\xi(t, y)|^2 dy, \quad B_{\bar{s}, \eta}(t) = \int_{\bar{s}}^t \int_{\mathbb{R}^3} \sigma_\eta(y) |\nabla v_\xi(\tau, y)|^2 dy d\tau. \quad (4.69)$$

We estimate the terms at the right hand side of (4.68). First of all we have

$$\begin{aligned} 2 \int_{\mathbb{R}^3} \sigma_\eta^2 |P_{v_\xi}| |v_\xi| & \leq 2 \int_{\mathbb{R}^3} \sigma_\eta^2 |v_\xi| |R \otimes R (v_\xi \otimes v_\xi)| \\ & + 4 \int_{\mathbb{R}^3} \sigma_\eta^2 |v_\xi| |R \otimes R (v_\xi \otimes w_\xi)| =: I + II. \end{aligned}$$

With computations similar to those of the first step, using the boundedness of the Riesz transform and the CKN inequality, we obtain

$$I \leq \frac{\dot{B}_{\bar{s}, \eta}}{8} + Z \dot{B}_{\bar{s}, \eta} a_\eta, \quad (4.70)$$

and, by possibly increasing the value of Z at each step,

$$\begin{aligned} II & \leq Z \|w_\xi\|_{L^4} \|\sigma_\eta^{1/2} v_\xi\|_{L^2}^{1/4} \|\sigma_\eta^{1/2} \nabla v_\xi\|_{L^2}^{7/4} = Z \|w_\xi\|_{L^4} a_\eta^{1/8} \dot{B}_{\bar{s}, \eta}^{7/8} \\ & \leq \frac{\dot{B}_{\bar{s}, \eta}}{8} + \|w_\xi\|_{L^4}^8 + Z a_\eta \dot{B}_{\bar{s}, \eta}. \end{aligned} \quad (4.71)$$

Next we have

$$\begin{aligned} |\xi| \int_{\mathbb{R}^3} \sigma_\eta^2 |v_\xi|^2 & = |\xi| \|\sigma_\eta v_\xi\|_{L^2}^2 \leq Z |\xi| \|\sigma_\eta^{1/2} \nabla v_\xi\|_{L^2} \\ & = Z |\xi| (\dot{B}_{\bar{s}, \eta} a_\eta)^{1/2} \leq |\xi|^2 + Z \dot{B}_{\bar{s}, \eta} a_\eta; \end{aligned} \quad (4.72)$$

and

$$\int_{\mathbb{R}^3} \sigma_\eta^2 |v_\xi|^3 = \|\sigma_\eta^{2/3} v_\xi\|_{L^3}^3 \leq Z \|\sigma_\eta^{1/2} \nabla v_\xi\|_{L^2}^2 \|\sigma_\eta^{1/2} v_\xi\|_{L^2} = Z \dot{B}_{\bar{s}, \eta} a_\eta^{1/2} \leq \frac{\dot{B}_{\bar{s}, \eta}}{8} + Z \dot{B}_{\bar{s}, \eta} a_\eta. \quad (4.73)$$

Finally, for the perturbative terms we have

$$\begin{aligned} 3 \int_{\mathbb{R}^3} \sigma_\eta^2 |v_\xi|^2 |w_\xi| & \leq 3 \|w_\xi\|_{L^4} \|\sigma_\eta v_\xi\|_{L^{8/3}}^2 \\ & \leq Z \|w_\xi\|_{L^4} \|\sigma_\eta^{1/2} v_\xi\|_{L^2}^{1/4} \|\sigma_\eta^{1/2} \nabla v_\xi\|_{L^2}^{7/4} \\ & = Z \|w_\xi\|_{L^4} a_\eta^{1/8} \dot{B}_{\bar{s}, \eta}^{7/8} \leq \frac{\dot{B}_{\bar{s}, \eta}}{8} + \|w_\xi\|_{L^4}^8 + Z \dot{B}_{\bar{s}, \eta} a_\eta, \end{aligned} \quad (4.74)$$

and

$$\begin{aligned} 18 \int_{\mathbb{R}^3} \sigma_\eta |v_\xi| |\nabla v_\xi| |w_\xi| & \leq 18 \|\sigma_\eta^{1/2} \nabla v_\xi\|_{L^2} \|w_\xi\|_{L^4} \|\sigma_\eta^{1/2} v_\xi\|_{L^4} \\ & \leq Z \|\sigma_\eta^{1/2} \nabla v_\xi\|_{L^2} \|w_\xi\|_{L^4} \|\sigma_\eta^{1/2} \nabla v_\xi\|_{L^2}^{3/4} \|\sigma_\eta^{1/2} v_\xi\|_{L^2}^{1/4} \\ & = Z \|w_\xi\|_{L^4} a_\eta^{1/8} \dot{B}_{\bar{s}, \eta}^{7/8} \leq \frac{\dot{B}_{\bar{s}, \eta}}{8} + \|w_\xi\|_{L^4}^8 + Z \dot{B}_{\bar{s}, \eta} a_\eta, \end{aligned} \quad (4.75)$$

We now plug the previous inequalities in (4.68) and we obtain

$$\begin{aligned} a_\eta(t)\psi_\eta(t) - a_\eta(\bar{s}) + 2 \int_{\bar{s}}^t \dot{B}_{\bar{s},\eta}(s)\psi_\eta(s)ds &\leq \\ &\leq \int_{\bar{s}}^t \psi_\eta(s) \left[\frac{5}{8} \dot{B}_{\bar{s},\eta} + 6Z\dot{B}_{\bar{s},\eta}a_\eta + |\xi|^2 + 3\|w_\xi\|_{L^4}^8 - k\dot{B}_{\bar{s},\eta}a_\eta \right](s)ds. \end{aligned}$$

We subtract the first term at the right hand side from the left hand side; then we choose $k = 6Z$ and note that

$$\int_{\bar{s}}^t \dot{B}_{\bar{s},\eta}\psi_\eta \equiv -\frac{1}{6Z} \int_{\bar{s}}^t \dot{\psi}_\eta = \frac{\psi_\eta(\bar{s}) - \psi_\eta(t)}{6Z} = \frac{1 - \psi_\eta(t)}{6Z} \quad (4.76)$$

so that, for $\bar{s} \leq t \leq \bar{s} + T$, we obtain

$$a_\eta(t)\psi_\eta(t) - a_\eta(\bar{s}) + \frac{1 - \psi_\eta(t)}{6Z} \leq |\xi|^2 \int_{\bar{s}}^t \psi_\eta(s)ds + 3 \int_{\bar{s}}^t \|w_\xi(s, \cdot)\|_{L^4}^8 ds. \quad (4.77)$$

Consider now the increasing function, for $t \geq \bar{s}$,

$$B_{\bar{s}}(t) := \int_{\bar{s}}^t \int_{\mathbb{R}^3} |y|^{-1} |\nabla v_\xi(\tau, y)|^2 dy d\tau \quad (4.78)$$

which may become infinite at some point $t = t_0 > \bar{s}$. By the definition of \bar{s} , we know that $B_{\bar{s}}(t) \geq M$ for $t \geq \bar{s} + T/M$; since $B_{\bar{s},\eta} \rightarrow B_{\bar{s}}$ pointwise as $\eta \rightarrow 0$, we have also

$$B_{\bar{s},\eta}(s) \geq \frac{M}{2} \quad \text{for } s \geq \bar{s} + \frac{T}{M} \quad \text{and } \eta \text{ small enough.} \quad (4.79)$$

Using this estimate for $s \geq \bar{s} + T/M$ and the obvious one $B_{\bar{s},\eta} \geq 0$ for $s < \bar{s} + T/M$, we have easily

$$\int_{\bar{s}}^{\bar{s}+T} \psi_\eta(s) ds = \int_{\bar{s}}^{\bar{s}+T} e^{-6ZB_{\bar{s},\eta}(s)} ds \leq \frac{T}{M} + e^{-3ZM} \left(T - \frac{T}{M}\right) \leq \frac{2T}{M} \quad (4.80)$$

(remember $Z \geq 1$). We now use the estimate $a(\bar{s}) \leq Ze^{4M^2}(\theta_2\varepsilon)^2$ (proved in (4.65)) and note that we can assume

$$\theta_2\varepsilon \leq 1 \quad \Rightarrow \quad a(\bar{s}) \leq Ze^{4M^2}\theta_2\varepsilon. \quad (4.81)$$

Moreover by (4.17) we have also

$$\|w_\xi\|_{L_t^8 L_x^4}^8 = \|w\|_{L_t^8 L_x^4}^8 \leq Z\theta_1\varepsilon \quad (4.82)$$

so that inequality (4.77) implies

$$(a_\eta(t) - \frac{1}{6Z})\psi_\eta(t) + \frac{1}{6Z} - 3Z\theta_1\varepsilon - Ze^{4M^2}\theta_2\varepsilon - 2|\xi|^2 \frac{T}{M} \leq 0 \quad (4.83)$$

or equivalently

$$a_\eta(t) + (\frac{1}{6Z} - 3Z\theta_1\varepsilon - Ze^{4M^2}\theta_2\varepsilon - 2|\xi|^2 \frac{T}{M})e^{6ZB_{\bar{s},\eta}(t)} \leq \frac{1}{6Z}. \quad (4.84)$$

We now assume ε is so small that

$$3Z\theta_1\varepsilon \leq \frac{1}{30Z}, \quad Ze^{4M^2}\theta_2\varepsilon \leq \frac{1}{30Z}, \quad (4.85)$$

(this implies also (4.81) and (4.16)), so that (4.84) implies

$$a_\eta(t) + (\frac{1}{10Z} - 2|\xi|^2 \frac{T}{M})e^{6ZB_{\bar{s},\eta}(t)} \leq \frac{1}{6Z}. \quad (4.86)$$

Assume in addition that ξ satisfies

$$\left(\frac{1}{10Z} - 2|\xi|^2 \frac{T}{M}\right) > 0 \quad \text{i.e.} \quad |\xi|^2 T < \frac{M}{20Z}. \quad (4.87)$$

Note that this condition is stronger than the first condition (4.63) on ξ , i.e. $|\xi|^2 \bar{s} \leq M^2$, since $M, Z \geq 1$ and $\bar{s} \leq T$. Then, if we let $\eta \rightarrow 0$, we have

$$a_\eta(t) \rightarrow a(t) := \int_{\mathbb{R}^3} |y|^{-1} |v_\xi(t, y)|^2 dy, \quad (4.88)$$

$$B_{\bar{s}, \eta}(t) \rightarrow B_{\bar{s}}(t) := \int_{\bar{s}}^t \int_{\mathbb{R}^3} |y|^{-1} |\nabla v_\xi(s, y)|^2 dy ds \quad (4.89)$$

and (4.86) implies, for all $\bar{s} \leq t \leq \bar{s} + T$

$$a(t) + \left(\frac{1}{10Z} - 2|\xi|^2 \frac{T}{M}\right) e^{6ZB_{\bar{s}}(t)} \leq \frac{1}{6Z}. \quad (4.90)$$

Thus, using (4.87), we see that $a(t)$ and $B_{\bar{s}}(t)$ are finite for $\bar{s} \leq t \leq \bar{s} + T$. Since by the definition of \bar{s} we already know that $B(\bar{s}) \leq 2M^2 < +\infty$, we conclude that

$$B(s) < +\infty \quad \text{for all} \quad 0 \leq s \leq \bar{s} + T. \quad (4.91)$$

In particular we have

$$B(T) = \int_0^T \int |y|^{-1} |\nabla v_\xi(s, y)|^2 dy ds \equiv \int_0^T \int |x - s\xi|^{-1} |\nabla v(s, x)|^2 dy ds < +\infty \quad (4.92)$$

and then the same argument used to conclude the proof in the first case ($\bar{s} = T$) gives also in the second case ($\bar{s} < T$) that $L(T, \xi)$ is a regular set, provided (4.85, 4.87) are satisfied.

4.6 Conclusion of the proof

Summing up, we have proved that there exists a universal constant Z such that for any $\tilde{p} \in [2, 4)$, $M \geq 1$, $T > 0$ and $\xi \in \mathbb{R}^3$ the following holds: if $\varepsilon \equiv [u_0]_{\tilde{p}}$ is small enough to satisfy (4.85), and T, ξ are such that (4.87) holds, then the segment $L(T, \xi)$ is a regular set for the weak solution u .

Now define

$$\delta = \frac{1}{90Z^2}. \quad (4.93)$$

Then (4.85) is implied by

$$\theta_1 \varepsilon \leq \delta, \quad \theta_2 \varepsilon \leq \delta e^{-4M^2} \quad (4.94)$$

while (4.87) is implied by

$$|\xi|^2 T < M\delta \quad \iff \quad T > \frac{|T\xi|^2}{M\delta} \quad (4.95)$$

or equivalently

$$(T, T\xi) \in \Pi_{M\delta}, \quad \Pi_{M\delta} := \left\{ (t, x) : t > \frac{|x|^2}{M\delta} \right\}. \quad (4.96)$$

In other words, if ε satisfies (4.94) and $(T, T\xi)$ belongs to $\Pi_{M\delta}$, then $L(T, \xi)$ is a regular set. Since $\Pi_{M\delta}$ is the union of such segments for arbitrary $\xi \in \mathbb{R}^3$, $T > 0$, we conclude that $\Pi_{M\delta}$ is a regular set for the solution u , provided (4.94) holds.

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