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Analysis of a linear elastic model relative to a small pressurized cavity embedded in the half-space

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Table of Notations

Symbol	**Description**
$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$	Points in $\mathbb{R}^d$, with $d \geqslant 3$.
$\boldsymbol{x}'$	First $d-1$ components of the point $\boldsymbol{x}$ that is $\boldsymbol{x}' = (x_1, \cdots, x_{d-1})$.
$\widetilde{\boldsymbol{x}}$	Given a point $x \in \mathbb{R}^d_-$, $\widetilde{x}$ represents the reflected point $(\boldsymbol{x}', -x_d)$.
$\mathbb{R}^d_-$	It denotes the half-space $\{\boldsymbol{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d \, : \, x_d < 0\}$.
$\mathbb{R}^{d-1}$	Boundary of the half-space $\mathbb{R}^d_-$.
$B_r(\boldsymbol{x})$	It denotes the d-dimensional ball with centre $\boldsymbol{x}$ and radius $r > 0$.
Ω	Bounded Lipschitz domain in $\mathbb{R}^d$.
ω_d	Area of the $(d-1)$-dimensional unit sphere.
$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \ldots$	Vectors in $\mathbb{R}^d$.
$\boldsymbol{n}$	Unit outer normal vector to a surface.
$\boldsymbol{u} \cdot \boldsymbol{v}$	Inner product between vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.
$\boldsymbol{u} \times \boldsymbol{v}$	Cross vector between $\boldsymbol{u}$ and $\boldsymbol{v}$.
$\boldsymbol{u} \otimes \boldsymbol{v}$	Tensor product between vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.
$\mathbf{A}, \mathbf{B}, \ldots$	Matrices and secon-order tensors.
$\mathbf{I}$	Identity matrix.
$\mathbf{A}^T$	Transpose of the matrix $\mathbf{A}$.
$\widehat{\mathbf{A}}$	Symmetric part of the matrix $\mathbf{A}$, that is $\widehat{\mathbf{A}} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right)$.
$\mathbf{A} : \mathbf{B}$	Inner product between the two matrices $\mathbf{A}$ and $\mathbf{B}$ that is $\mathbf{A} : \mathbf{B} = \sum_{i,j} a_{ij} b_{ij}$.
$\lvert\mathbf{A}\rvert$	Norm induced by the matrix inner product, that is $\lvert\mathbf{A}\rvert = \sqrt{\mathbf{A} : \mathbf{A}}$.

Symbol	Description
I	In Chapter 2 it represents the identity map.
$\Gamma(\boldsymbol{x})$	Fundamental solution of the Laplace operator.
$S_{\Omega}\varphi(\boldsymbol{x})$	Single layer potential for the Laplace operator relative to the function φ.
$D_{\Omega}\varphi(\boldsymbol{x})$	Double layer potential for the Laplace operator relative to the function φ.
$N(\boldsymbol{x},\boldsymbol{y})$	Neumann function of the half-space for the Laplace operator.
κ_d	Constant in the definition of Γ function, $\kappa_d := 1/\omega_d(2-d)$.
$\mathbb{A},\mathbb{B},\ldots$	Fourth-order tensors.
$\mathbb{C}$	Fourth-order elasticity tensor.
$\mathbb{I}$	Fourth-order identity tensor such that $\mathbb{I}\mathbf{A} = \widehat{\mathbf{A}}$.
μ, λ	Lamé parameters of the linear elasticy theory.
ν	Poisson ratio. The identity $\nu = \lambda/2(\lambda+\mu)$ holds.
$\boldsymbol{\mathcal{L}}$	Elastostatic Lamé operator, that is $\boldsymbol{\mathcal{L}}\boldsymbol{u} := \mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\mathrm{div}\,\boldsymbol{u}$.
$\dfrac{\partial\boldsymbol{u}}{\partial\boldsymbol{\nu}}$	Conormal derivative, that is $\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\nu}} := (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n} = \lambda(\mathrm{div}\,\boldsymbol{u})\boldsymbol{n} + 2\mu(\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}$.
$\mathbf{\Gamma}(\boldsymbol{x})$	Fundamental solution of the lamé operator (Kelvin-Somigliana matrix).
$\mathbf{N}(\boldsymbol{x},\boldsymbol{y})$	Neumann function of the half-space related to the Lamé operator. $\mathbf{N}(\boldsymbol{x},\boldsymbol{y}) = \mathbf{\Gamma}(\boldsymbol{x},\boldsymbol{y}) + \mathbf{R}(\boldsymbol{x},\boldsymbol{y})$, with $\mathbf{R}$ regular part.
$\boldsymbol{N}^{(k)}(\boldsymbol{x},\boldsymbol{y})$	k-th column vector of the Neumann function $\mathbf{N}$.
$\mathbf{S}^{\Gamma}\boldsymbol{\varphi}(\boldsymbol{x})$	Single layer potential related to the Lamé operator with kernel $\mathbf{\Gamma}$.
$\mathbf{D}^{\Gamma}\boldsymbol{\varphi}(\boldsymbol{x})$	Double layer potential related to the Lamé operator with kernel $\mathbf{\Gamma}$.
$\mathbf{S}^{R}\boldsymbol{\varphi}(\boldsymbol{x})$	Single layer potential with kernel $\mathbf{R}$.
$\mathbf{D}^{R}\boldsymbol{\varphi}(\boldsymbol{x})$	Double layer potential with kernel $\mathbf{R}$.
c_ν	Constant $c_\nu := 4(1-\nu)(1-2\nu)$.
c'_ν	Constant $c'_\nu := (1-2\nu)/(8\pi(1-\nu))$.
$C_{\mu,\nu}$	Constant $C_{\mu,\nu} := 1/(16\pi\mu(1-\nu))$.

CHAPTER 1

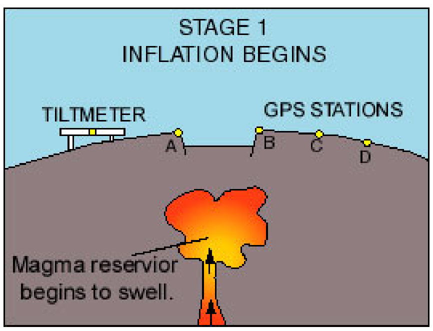
Introduction: from the physical problem to the mathematical model

This thesis is devoted to the mathematical study of a model arising from the volcanology. More precisely we establish a mathematical approach for surface deformation effects generated by a magma chamber embedded deep into the earth and exerting on it a uniform hydrostatic pressure. In the first part of this introduction, we will describe the underlying geophysical problem in order to better understand and appreciate the mathematical model under investigation. In the second part we will explain the tools developed for the mathematical analysis of the model and the results obtained.

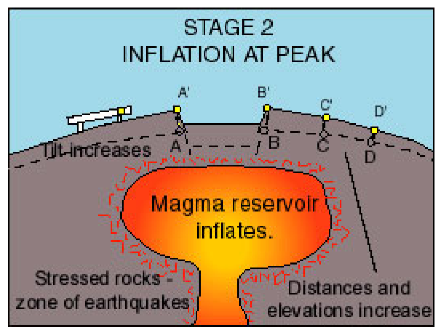
1.1 Volcano deformation

Monitoring of volcanoes activity is usually performed by combining different types of geophysical measurements. Ground deformations, seismic swarms and gravity changes are the principal means used to assess the risks of a possible imminent eruptive activity.

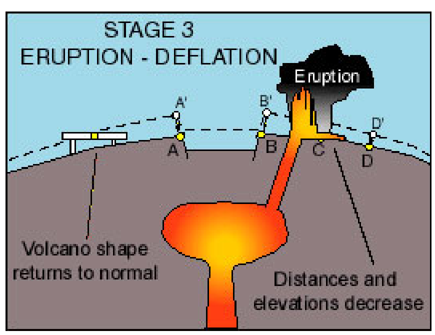
Ground deformations are among the most significant data being directly available. In fact, modern techniques of space geodesy, such as the Global Positioning System (GPS) and satellite radar interferometry (InSAR), now provide a large number of data of high quality both from temporal and spatial point of view [12, 14, 15, 56]. Modeling of the pattern and rate of displacement before and during eruptions can reveal much about the physics of active volcanoes [27]. This is especially true when studying stratovolcanoes or basaltic shield volcanoes, since their fast, short-term deformation is well associated with magma accumulation and eruptions, see [12, 15] and references therein. Specifically, the monitoring of ground deformation has showed a cyclical volcanic activity of inflation and deflation period [56],[57]. When magma accumulates in crustal reservoirs, volcanoes inflate (see Figure 1.1(a) and Figure 1.1(b)). The observations indicate relatively long period of volcanic uplift. After that, rapid periods of subsidence follow. These deflation episodes are accompanied either by eruptions or by dike intrusion into the flanks of the volcano (see Figure 1.1(c)).



(a) Magma comes from the mantle into the magma reservoir



(b) The inflation produces deformations



(c) Deflation period after an eruption

Figure 1.1. Inflation-deflation cycle. Courtesy Hawaiian Volcano Observatory website http://hvo.wr.usgs.gov/howwork/subsidence/inflate_deflate.html

Without being exaustive, we can briefly explain and simplify the physical phenomenon in this way: as magma migrates toward the earth's surface, it forces aside and exerts stresses on the surrounding crust causing ground deformations and in some cases, since the crust is brittle, earthquakes. Consequently, the redistribution of the mass at depth generates changes in the material density producing as an effect small anomalies in the gravity field. All these signals can be measured. However, since the subsurface structures beneath active volcanoes are extremely complex, the identification of the source of unrest is not straightforward. In fact, caldera unrest may be also caused by aqueous fluid intrusions, or interaction between the hydrothermal system and magma intrusions [17, 28, 59]. We highlight that the deformation measurements are sensitive only to changes in volume and pressure of the source hence they cannot give information on the material density. Gravity measurements, however, can constrain the mass of the intrusion. Given the significant density difference between silicate melts ($2500\ \mathrm{kg/m^3}$) and hydrothermal fluids ($1000\ \mathrm{kg/m^3}$), it is reasonable to use density estimates from gravity to distinguish between these two possible sources of caldera unrest.

In light of this, the main challenge is to interpret geodesy and gravity measurements jointly (see [12, 16, 53]) with the following goals

1. determine the geometry of subsurface magma bodies i.e., whether the source of deformation is a dike, a roughly equidimensional chamber, or a hybrid source (mixture of different mantle sources);

2. to quantify parameters of the source, for example its depth, dimensions, volume, density and internal magma pressure [56].

To achieve these objectives a simplified/conceptual model has been conceived with a central magma chamber that is supplied with melt from the mantle. The pressure increases, hence the ground is deformed producing gravity anomalies and changes in volcano shape. After some time, the increasing pressure causes the fracture of the walls and a dike propagates carrying magma either to the surface or into the volcano flanks [56].

From a modeling point of view, based on the elastic behaviour of the Earth's crust, the ground deformations are interpreted in the framework of the linear elasticity theory, see [13, 27, 57]. The gravity anomalies using the potential theory, see [12] and reference therein.

In this thesis we will focus the attention to the mathematical analysis of the most common elastic model which we now turn to present.

1.2 Towards the mathematical model

A well-established model is the one proposed by Mogi, [50], following previous results (see description in [26, 45, 56]). Mogi's model is based on the assumption that ground deformation effects are primarily generated by the presence of an underground magma chamber exerting a uniform pressure on the surrounding medium. Precisely, the model relies on three key founding schematisations:

1. ***Geometry of the model.*** The earth's crust is an infinite half-space (with free air/crust surface located on the plane $x_3 = 0$) and the magma chamber, buried in the half-space, is assumed to be a spherical cavity with radius r_0 and depth d_0 such that $r_0 \ll d_0$.
2. ***Geophysics of the crust.*** The crust is a perfectly elastic body, isotropic and homogeneous, whose deformations are described by the linearized elastostatic equations, hence are completely characterized by the Lamé parameters μ, λ (or, equivalently, Poisson ratio ν and shear modulus μ). The free air/crust boundary is assumed to be a traction-free surface.
3. ***Crust-chamber interaction.*** The cavity describing the magma chamber is assumed to be filled with an ideal incompressible fluid at equilibrium, so that the pressure p exterted on its boundary on the external elastic medium is hydrostatic and uniform.

In detail, assuming that the center of the sphere is located at $\boldsymbol{z} = (z_1, z_2, z_3)$ with $z_3 < 0$, the displacement $\boldsymbol{u} = (u_1, u_2, u_3)$ at a surface point $\boldsymbol{y} = (y_1, y_2, 0)$ is given by

$$u^\alpha(\boldsymbol{y}) = \frac{(1-\nu)}{\mu}\frac{\varepsilon^3 p(z_\alpha - y_\alpha)}{|\boldsymbol{z}-\boldsymbol{y}|^3}, \qquad u^3(\boldsymbol{y}) = -\frac{(1-\nu)}{\mu}\frac{\varepsilon^3 p\, d_0}{|\boldsymbol{z}-\boldsymbol{y}|^3} \tag{1.1}$$

in the limit $\varepsilon := r_0/d_0 \to 0$ (see Figure 1.2), for $\alpha = 1, 2$, where $d_0 = -z_3$. A second-order approximation has been proposed by McTigue [47] with the intent of providing a formal expansion able to cover the case of a spherical body with finite (but small) positive radius.

Being based on the assumption that the ratio radius/depth $\varepsilon := r_0/d_0$ is small, the Mogi model corresponds to the assumption that the magma chamber is well-approximated by a single point producing a uniform pressure in the radial direction; as such, it is sometimes referred to as a *point source model*. However, even if the source is reduced to a single point, the model still

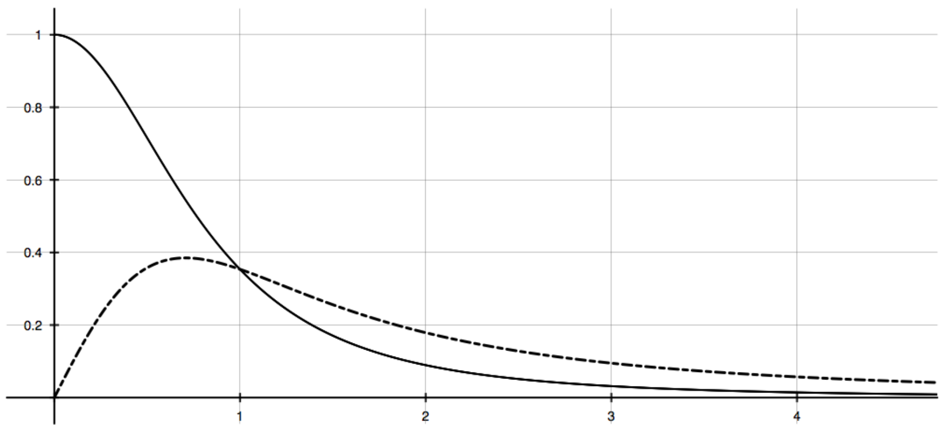


Figure 1.2. Normalized Mogi displacement profiles given in (1.1): horizontal components u_α, $\alpha = 1, 2$, dashed line; vertical component u_3, continuous line.

records the spherical form of the cavity. Different geometrical form may lead to different deformation effects (as will be clear in the subsequent analysis).

The Mogi model has been widely applied to real deformation data of different volcanoes to infer approximate location and strength of the magma chamber. The main benefit of such strategy lies in the fact that it provides a simple formula of the ground deformation expressed in terms of the basic physical parameters depth and total work (combining pressure and volume) and, thus, that it can be readily compared with real deformation data to provide explicit forecasts.

The simplicity of Mogi's formulas (1.1) makes the application model viable, but it compensates only partially the intrinsic reductions of the approach. As a consequence, variations of the basic assumptions have been proposed to provide more realistic frameworks. With no claim of completeness, we list here some generalizations of the Mogi model available in the literature.

Since real data sets often exhibit deviations from radially symmetric deformations, different shapes for the point source have been proposed. Guided by the request of furnishing explicit formulas, such attempt has primarily focused on ellipsoidal geometries and, in particular, on oblate and prolate ellipsoids, see [25, 61]. With respect to the spherical case, these new configurations are able to indicate the presence of some elongations of the chamber and possible tilt with respect to the surface. As a drawback, formulas for ellipsoid cavities turn to be rather complicated, involving, in the general case, elliptic integrals.

Different configurations, such as rectangular dislocations (see [52]) and horizontal penny-shaped cracks (see [33]), have also been considered still with the target of furnishing an explicit formula for ground deformation to be compared with real data by means of appropriate inversion algorithms. Still relative to the geometry of the model, efforts have also been directed to the case of a non-flat crust surface, with the intent of taking into account the specific topography, as given by the local elevation above mean sea level of the region under observation [60, 23].

Studies have been addressed to a finer description of the geophysical properties of the crust, with special attention to the case of heterogeneous rheologies. Indeed, different parts of the crust may exhibit different mechanical properties due to the presence of stiff (lava flows, welded pyroclasts, intrusions) and soft layers (non-welded pyroclasts, sediments), see [46] and references therein. Variations of elastic parameters may also arise as a consequence of the thermic properties of the magma inside the chamber, which determines different local behaviours in the neighborhood of the cavity, so that the presence of an additional layer surrounding the chamber could be appropriate. Additionally, nonuniform pressure distribution on the boundary of the chamber may arise, as an example, from a nonuniform nature of the material filling the cavity (see discussion in [26]). Incidentally, we stress that the use of nonlinear elastic models in this area is still in a germinal phase and it would require a more circumstantial analysis.

For completeness, we mention also the attempts of combining elastic properties with gravitational effects and time-dependent processes modeling of the crust by means of elasto-dynamic equations or viscoelastic rheologies (among others, see [12] and [22]).

In all cases, refined descriptions have the inherent drawback of requiring a detailed knowledge of the crust elastic properties. In absence of reliable complete data and measurements, the risk of introducing an additional degree of freedom in the parameter choice is substantial. This observation partly supports the approach of the Mogi model which consists in keeping as far as possible the parameters choice limited and, consequently, the model simple.

As the above overview shows, the geological literature on the topic is extensive. On the contrary, the mathematical contributions seem to be still lacking. In the following section we summarize the principal aim of this thesis, showing the mathematical generalization of the Mogi model referred to the shape of the cavity which will be not forced to be neither a sphere nor an ellipsoid, but an arbitrary bounded domain of the half-space.

1.3 The mathematical model

Let us introduce in detail the boundary value problem which arises from the previous assumptions on the geometry of the model, geophysics of the crust and crust-chamber interaction (see previous section) in the case of a generic shape for the magma chamber. Denoting by $\mathbb{R}^3_-$ the (open) half-space described by the condition $x_3 < 0$, the domain occupied by the Earth's crust is $\mathbb{R}^3_- \backslash \overline{C}$, where $C \subset \mathbb{R}^3_-$, describing the magma chamber, is assumed to be an open set with connected and bounded Lipschitz boundary ∂C. Hence, the boundary of $\mathbb{R}^3_- \backslash \overline{C}$ is composed by two disconnected components: the two-dimensional plane $\mathbb{R}^2 := \{\boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3 \,:\, y_3 = 0\}$, which constitutes the free air/crust border, and the set ∂C, corresponding to the crust/chamber edge.

Given $\mathbf{A} \in \mathbb{R}^{3\times 3}$ we denote by $\widehat{\mathbf{A}}$ its symmetric part, that is $\widehat{\mathbf{A}} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$. We introduce the elastic description of the medium filling $\mathbb{R}^3_- \backslash \overline{C}$. Assuming that the medium is homogeneous and isotropic and subjected to small elastic deformations, we derive the following boundary value problem for the displacement field $\boldsymbol{u}$, that is

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \mathbf{0} & \text{in } \mathbb{R}^3_- \backslash C \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n} = p\,\boldsymbol{n} & \text{on } \partial C \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{e}_3 = \mathbf{0} & \text{on } \mathbb{R}^2 \\ \boldsymbol{u} = o(\mathbf{1}), \quad \nabla \boldsymbol{u} = o(|\boldsymbol{x}|^{-1}) & |\boldsymbol{x}| \to \infty, \end{cases} \tag{1.2}$$

where $\mathbb{C} := \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu\mathbb{I}$ is the fourth-order isotropic elasticity tensor with $\mathbf{I}$ the 3×3 identity matrix and $\mathbb{I}$ the fourth-order tensor defined by $\mathbb{I}\mathbf{A} := \widehat{\mathbf{A}}$, C is the cavity modelling the magma chamber, p is a constant representing the pressure and $\widehat{\nabla}\boldsymbol{u}$ the strain tensor.

At this point, the model provides the displacement $\boldsymbol{u}$ of a generic <u>finite</u> cavity C. The next step is to deduce a corresponding point source model, in the spirit of the Mogi spherical one. To this aim, we assume the cavity C of the form

$$C = d_0\boldsymbol{z} + r_0\Omega$$

where $d_0, r_0 > 0$ are charateristic length-scales for depth and diameter of the cavity, its center $d_0\boldsymbol{z}$ belongs to $\mathbb{R}^3_-$ and its shape Ω is a bounded Lipschitz domain containing the origin. The Mogi model (1.1) corresponds to Ω given by a sphere of radius 1.
Introducing the rescaling $(\boldsymbol{x}, \boldsymbol{u}) \mapsto (\boldsymbol{x}/d_0, \boldsymbol{u}/r_0)$ and denoting the new variables again by $\boldsymbol{x}$ and $\boldsymbol{u}$, the above problem takes the form

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \boldsymbol{0} & \text{in } \mathbb{R}^3_-\setminus\overline{C}_\varepsilon \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n} = p\,\boldsymbol{n} & \text{on } \partial C_\varepsilon \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{e}_3 = \boldsymbol{0} & \text{in } \mathbb{R}^2 \\ \boldsymbol{u} = o(\boldsymbol{1}), \quad \nabla\boldsymbol{u} = o(|\boldsymbol{x}|^{-1}) & |\boldsymbol{x}| \to \infty, \end{cases} \tag{1.3}$$

where $\varepsilon = r_0/d_0$, $C_\varepsilon := \boldsymbol{z} + \varepsilon\Omega$ and p is a "rescaled" pressure, ratio of the original pressure p and ε. Denoting by $\boldsymbol{u}_\varepsilon$ the solution to the boundary value problem (1.3), the point reduction consists in considering the limiting behaviour as $\varepsilon \to 0$ of $\boldsymbol{u}_\varepsilon$ and, precisely, in determining an asymptotic expansion valid for $\boldsymbol{y} \in \mathbb{R}^2$ of the form

$$\boldsymbol{u}_\varepsilon(\boldsymbol{y}) = \varepsilon^\alpha p\,\boldsymbol{U}(\boldsymbol{z}, \boldsymbol{y}) + o(\varepsilon^\alpha) \qquad \text{as } \varepsilon \to 0^+ \qquad (\boldsymbol{y} \in \mathbb{R}^2)$$

for some appropriate exponent $\alpha > 0$ and principal term $\boldsymbol{U}$.

The well-posedness of (1.2) and the asymptotic analysis of (1.3) are the main subjects of this thesis.

1.4 An overview of the mathematical literature

The derivation of asymptotic expansions, in the presence of small inclusions or cavities, has been very successful in the field of the inverse problems.

A pioneering work is due to Friedman and Vogelius during the 80s, see [36], where the authors analyse the electrostatic problem for a conductor consisting of finitely many small inhomogeneities of extreme conductivity (infinite or zero conductivity) represented by regular domains. They first establish an asymptotic formula of first order for the perturbed potential. Secondly, from that, they prove that locations and relative sizes of the inhomogeneities depend Lipschitz-continuously on the potential measurement on any open subset of the boundary.

After this work, much effort has been made to improve and generalize the results, treating also the elasticity case [4], for its potential application in medical diagnosis or nondestructive evaluation of materials, see for example [5, 7]. In particular, the extensions to generic inhomogeneities, not necessarily regular and not perfectly conducting or insulating, and the implementation of reconstruction algorithms have been addressed, see [5, 7, 19, 38] and the reference therein for a vast bibliography. Specifically, starting from boundary measurements given by the couple potentials/currents or deformations/tractions, in the case, respectively, of electrical impedence tomography and linear elasticity, information about the conductivity profile and the elastic parameters of the medium have been inferred. We recall that without any a priori assumptions on the unknown inhomogeneities/cavities (for example without the smallness assumption), the reconstruction procedures give poor quality results. This is due to the severe ill-posedness of the inverse boundary value problem modelling both the electrical impedance tomography [2] and the elasticity problems [3, 51]. However, in certain situations one has some a priori information about the structure of the medium to be reconstructed. These additional details allow to restore the well-posedness of the problem and, in particular, to gain uniqueness and Lipschitz continuous dependence of inclusions or cavities from the boundary measurements. The smallness of the inhomogeneities, embedded in a medium with a smooth background conductivity or with smooth elastic parameters, is one of the way to obtain the well-posedness of the inverse problem as pointed out by Friedman and Vogelius in [36]. Therefore, by means of partial or complete asymptotic formulas of solutions to the conductivity/elastic problems and some efficient algorithms, information about the location and size of the inclusions can be reconstructed, see [4, 7, 38].

It is essential to highlight that the approach introduced by Ammari and Kang, see for example [4, 7], based on layer potentials techniques has been a powerful method to obtain asymptotic expansion of any order for solutions to the transmission problems and, as a particular case, to cavities and perfectly conducting inclusions' problem. For this reason, we decide to follow the same approach in this thesis. Despite the extensive literature in this field [4, 6, 7, 8, 19, 36, 38, 40], we remark that the mathematical problem of this work represents an intriguing novelty because we have to deal with a pressurized cavity, that is a hole with nonzero tractions on its boundary, buried in a half-space. These two peculiarities do not allow to reduce the boundary value problem to a classical one based on cavities (see, for example, problems in [4, 7] and reference therein). In fact, we can not create a background displacement vector field, which is both indipendent from the geometry of the hole and satisfies the decay conditions at infinity, in order to nullify the traction datum on the boundary of the pressurized cavity.

1.5 Organization of the thesis and main results

Guided by the historical approach summarized in Section 1.4, for which the electrostatic problem was the first one considered in the field of the asymptotic analysis in the presence of small inclusions, in Chapter 2 we analyse a simplified scalar version of the elastic model presented in Section 1.3. Specifically, denoting by $\mathbb{R}^d_-$ the half-space and $\mathbb{R}^{d-1}$ its boundary, with $d \geqslant 3$, we consider the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^d_- \backslash C \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g & \text{on } \partial C, \\ \dfrac{\partial u}{\partial x_d} = 0 & \text{on } \mathbb{R}^{d-1}, \\ u(\boldsymbol{x}) \to 0 & \text{as } |\boldsymbol{x}| \to +\infty \end{cases} \tag{1.4}$$

where C is the analogous of the pressurized cavity in the elastic case, $g \in L^2(\partial C)$ and $\boldsymbol{n}$ is the unit outer normal vector. Obviously, the choice to focus the attention on dimensions greater than two comes from the limitation imposed by the elastic physical problem.

As far as we know, this boundary value problem does not have a real physical meaning. On the other hand, being mathematically more manageable than the system of elasticity, it is useful to mark and shed light on the path to solve the elastic problem.

To prove the well-posedness of this boundary value problem we use the method of reflection. Let $\boldsymbol{x}' = (x_1, \cdots, x_{d-1})$, we consider the cavity image $\widetilde{C} = \{(\boldsymbol{x}', x_d),\ (\boldsymbol{x}', -x_d) \in C\}$ and the function G defined as

$$G(x) := \begin{cases} g(\boldsymbol{x}', x_d) & \text{if } x_d \leqslant 0 \\ g(\boldsymbol{x}', -x_d) & \text{if } x_d > 0 \end{cases}$$

Then we look at the extended problem

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}^d \backslash \left(C \cup \widetilde{C}\right) \\ \dfrac{\partial U}{\partial \boldsymbol{n}} = G & \text{on } \partial C \cup \partial \widetilde{C} \\ U \to 0 & \text{as } |\boldsymbol{x}| \to +\infty \end{cases} \tag{1.5}$$

where the condition $U \to 0$ is equivalent to $|U| = O(|\boldsymbol{x}|^{2-d})$. Classic theory on the exterior problems for harmonic functions guarantees existence and uniqueness of the solution U. Moreover, symmetry ensures the equivalence between (1.4) and the extended problem (1.5) in the half-space. In fact, the unique solution $U(\boldsymbol{x})$ of the extended problem (1.5), for $\boldsymbol{x} \in \mathbb{R}^d \backslash \left(C \cup \widetilde{C}\right)$, satisfies the scalar problem (1.4) when $x_d \leqslant 0$.

To apply the integral approach, we first look for an integral representation formula of the solution. To do this, we take advantage of the explicit expression of the Neumann function for the Laplace operator in the half-space

$$N(\boldsymbol{x}, \boldsymbol{y}) = \Gamma(\boldsymbol{x} - \boldsymbol{y}) + \Gamma(\widetilde{\boldsymbol{x}} - \boldsymbol{y}),$$

where Γ is the fundamental solution of the Laplacian, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d_-$ and $\widetilde{\boldsymbol{x}}$ is the symmetric point of $\boldsymbol{x}$ with respect to the x_d-plane, in order to get a representation formula containing only integral contributions on the boundary of C. In detail, we find that

$$u(\boldsymbol{x}) = \int_{\partial C} \left[N(\boldsymbol{x}, \boldsymbol{y}) g(\boldsymbol{y}) - \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{y}}} N(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y})\right] d\sigma(\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^d_- \backslash C,$$

where f is the trace of the solution u on ∂C. From the point of view of the inverse problem we are interested in evaluating the solution u on the boundary of the half-space; since $\Gamma(\boldsymbol{x} - \boldsymbol{y}) = \Gamma(\widetilde{\boldsymbol{x}} - \boldsymbol{y})$, for $\boldsymbol{x} \in \mathbb{R}^{d-1}$, the integral formula becomes

$$u(\boldsymbol{x}) = 2 \int_{\partial C} \left[\Gamma(\boldsymbol{x} - \boldsymbol{y}) g(\boldsymbol{y}) - \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{y}}} \Gamma(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y})\right] d\sigma(\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^{d-1}.$$

Taking Ω a bounded Lipschitz domain containing the origin and $\boldsymbol{z} \in \mathbb{R}^d_-$ we consider $C_\varepsilon := C = \boldsymbol{z} + \varepsilon\Omega$ with the assumption that $\operatorname{dist}(\boldsymbol{z}, \mathbb{R}^{d-1}) \geqslant \delta_0 > 0$.

We also define $g^\sharp(\boldsymbol{\zeta};\varepsilon) = g(\boldsymbol{z} + \varepsilon\boldsymbol{\zeta})$, with $\boldsymbol{\zeta} \in \Omega$, and $S_\Omega\varphi$ the single layer potential

$$S_\Omega\varphi(\boldsymbol{x}) := \int_{\partial\Omega} \Gamma(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})d\sigma(\boldsymbol{y}), \qquad \boldsymbol{x} \in \mathbb{R}^d.$$

Then, denoting with u_ε the dependence of u from ε and taking $g \in L^2(\partial C_\varepsilon)$ such that $g^\sharp$ is independent on ε, at any $\boldsymbol{x} \in \mathbb{R}^{d-1}$, the following asymptotic expansion holds

$$\begin{aligned} u_\varepsilon(\boldsymbol{x}) &= 2\varepsilon^{d-1}\Gamma(\boldsymbol{x}-\boldsymbol{z})\int_{\partial\Omega} g^\sharp(\boldsymbol{\zeta})d\sigma(\boldsymbol{\zeta}) \\ &+ 2\varepsilon^d\nabla\Gamma(\boldsymbol{x}-\boldsymbol{z}) \cdot \int_{\partial\Omega}\left\{\boldsymbol{n}_{\boldsymbol{\zeta}}\left(\frac{1}{2}\mathbf{I}+K_\Omega\right)^{-1} Sg^\sharp(\boldsymbol{\zeta}) - \boldsymbol{\zeta}g^\sharp(\boldsymbol{\zeta})\right\}d\sigma(\boldsymbol{\zeta}) + O(\varepsilon^{d+1}), \end{aligned}$$

as $\varepsilon \to 0$, where $O(\varepsilon^{d+1})$ denotes a quantity uniformly bounded by $C\varepsilon^{d+1}$ with $C = C(\delta_0)$ which tends to infinity when δ_0 goes to zero. The singular integral operator K_Ω is defined by

$$K_\Omega\varphi(\boldsymbol{x}) = \frac{1}{\omega_d}p.v.\int_{\partial\Omega}\frac{(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}_{\boldsymbol{y}}}{|\boldsymbol{x}-\boldsymbol{y}|^d}\varphi(\boldsymbol{y})d\sigma(\boldsymbol{y}), \qquad \boldsymbol{x} \in \partial\Omega,$$

where ω_d is the area of the $(d-1)$-dimensional unit sphere.

Finally, with the asymptotic expansion at hand, we consider the specific Neumann boundary datum $g = -\boldsymbol{p}\cdot\boldsymbol{n}$ where $\boldsymbol{p}$ is a constant vector in $\mathbb{R}^d$. This particular choice has a double purpose: to imitate the constant boundary conditions of the elastic model and to make more explicit the integrals in the asymptotic formula. The result gives the same polarization tensor obtained by Friedman and Vogelius in [36] for cavities in a bounded domain. Specifically, it holds

$$u_\varepsilon(\boldsymbol{x}) = 2\,\varepsilon^d|\Omega|\nabla\Gamma(\boldsymbol{x}-\boldsymbol{z})\cdot\mathbf{M}\boldsymbol{p} + O(\varepsilon^{d+1}), \qquad \boldsymbol{x} \in \mathbb{R}^{d-1},$$

where $\mathbf{M}$ is the symmetric positive definite tensor given by

$$\mathbf{M} := \mathbf{I} + \frac{1}{|\Omega|}\int_{\partial\Omega}\big(\boldsymbol{n}_{\boldsymbol{\zeta}} \otimes \boldsymbol{\Psi}(\boldsymbol{\zeta})\big)\,d\sigma(\boldsymbol{\zeta})$$

and the auxiliary function $\boldsymbol{\Psi}$ has components Ψ_i, $i = 1,\ldots,d$, solving

$$\begin{cases} \Delta\Psi_i = 0 & \text{in } \mathbb{R}^d\backslash\Omega \\ \dfrac{\partial\Psi_i}{\partial\boldsymbol{n}} = -n_i & \text{on } \partial\Omega \\ \Psi_i \to 0 & \text{as } |\boldsymbol{x}| \to +\infty. \end{cases}$$

In Chapter 3 we finally analyse the elastic problem (1.2) presented in Section 1.3. For the well-posedness of (1.2) we cannot use the same approach employed in the scalar case because the extention of the problem by symmetry in $\mathbb{R}^3$ doesn't work. In fact, it is impossible to build an extended problem to the whole space such that the third component u_3 of the displacement vector field $\boldsymbol{u}$ is continuous across the boundary of the half-space. One way to overcome this obstacle is to prove directly the invertibility of the boundary operators that come out from the integral representation formula of the solution $\boldsymbol{u}$. To do that, we need the Neumann function $\mathbf{N}$ of the Lamé operator, solution to

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\mathbf{N}(\cdot,\boldsymbol{y})) = \delta_{\boldsymbol{y}}\mathbf{I} & \text{in } \mathbb{R}^3_-, \\ (\mathbb{C}\widehat{\nabla}\mathbf{N}(\cdot,\boldsymbol{y}))\boldsymbol{n} = \mathbf{0} & \text{on } \mathbb{R}^2 \end{cases}$$

with the decay conditions at infinity

$$\mathbf{N} = O(|\boldsymbol{x}|^{-1}), \qquad |\nabla\mathbf{N}| = O(|\boldsymbol{x}|^{-2}).$$

$\mathbf{N}$ has an explicit expression and can be decomposed as $\mathbf{N} = \mathbf{\Gamma} + \mathbf{R}$, where $\mathbf{\Gamma}$ is the fundamental solution of the Lamé operator and $\mathbf{R}$ is the regular part (see Chapter 3 for details). Given $\mathbf{A} \in \mathbb{R}^{3\times3}$ we represent the transpose as $\mathbf{A}^T$. From that, we find the following representation formula for the solution $\boldsymbol{u}$ to (1.2)

$$\boldsymbol{u}(\boldsymbol{y}) = \int_{\partial C} \left[p\mathbf{N}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{n}(\boldsymbol{y}) - (\mathbb{C}\widehat{\nabla}\mathbf{N}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{n}(\boldsymbol{y}))^T\boldsymbol{f}(\boldsymbol{y})\right] d\sigma(\boldsymbol{x}), \qquad \boldsymbol{y} \in \mathbb{R}^3_-\backslash\overline{C}$$

where $\boldsymbol{f}$ is the trace of $\boldsymbol{u}$ on ∂C. In particular, $\boldsymbol{f}$ solves the integral equation

$$\left(\tfrac{1}{2}\mathbf{I} + \mathbf{K} + \mathbf{D}^R\right)\boldsymbol{f} = p\left(\mathbf{S}^\Gamma\boldsymbol{n} + \mathbf{S}^R\boldsymbol{n}\right), \qquad \text{on } \partial C$$

where $\mathbf{D}^R$, $\mathbf{S}^R$ are, respectively, the double and single layer potentials related to $\mathbf{R}$ while $\mathbf{S}^\Gamma$ is the single layer potential relative to $\mathbf{\Gamma}$ (see Chapter 3). In this framework, the well-posedness of the problem (1.2) follows showing the invertibility of the operator $\frac{1}{2}\mathbf{I} + \mathbf{K} + \mathbf{D}^R$ in $\mathbf{L}^2(\partial C)$. In order to prove the injectivity of this operator, we show the uniqueness of $\boldsymbol{u}$ following the classical approach based on the application of the Green's formula and the energy method. In particular, we consider two solutions, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, of the problem (1.2) and their difference $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$. Then, we cut the half-space with a hemisphere of radius r containing the cavity and we represent $\boldsymbol{u}$ in integral formulation by means of $\mathbf{N}$. The uniqueness result follows using the decay conditions at infinity of $\mathbf{N}$ and $\boldsymbol{u}$ as $r \to +\infty$. From the injectivity, it follows the existence of $\boldsymbol{u}$ proving the surjectivity of $\frac{1}{2}\mathbf{I} + \mathbf{K} + \mathbf{D}^R$ in $\boldsymbol{L}^2(\partial C)$ which is obtained by the application of the index theory regarding bounded and linear operators.

Afterwards, taking again a cavity of the form $C_\varepsilon = \boldsymbol{z} + \varepsilon\Omega$, with $\boldsymbol{z} \in \mathbb{R}^3_-$ and Ω is a bounded Lipschitz domain containing the origin, we find the asymptotic expansion of the solution to problem (1.3) for $\boldsymbol{y} \in \mathbb{R}^2$,

$$u^k_\varepsilon(\boldsymbol{y}) = p\varepsilon^3|\Omega|\widehat{\nabla}_{\boldsymbol{z}}\boldsymbol{N}^{(k)}(\boldsymbol{z}, \boldsymbol{y}) : \mathbb{M}\mathbf{I} + O(\varepsilon^4),$$

for $k = 1, 2, 3$, as $\varepsilon \to 0$, where u^k_ε stands for the k-th component of the displacement vector and $\boldsymbol{N}^{(k)}$ for the k-th column vector of the matrix $\mathbf{N}$. Here $p\,\varepsilon^3$ represents the total work exerted by the cavity on the half-space. $\mathbb{M}$ is the fourth-order moment elastic tensor defined by

$$\mathbb{M} := \mathbb{I} + \frac{1}{|\Omega|}\int_{\partial\Omega} \mathbb{C}(\boldsymbol{\theta}^{qr}(\boldsymbol{\zeta}) \otimes \boldsymbol{n}(\boldsymbol{\zeta}))\, d\sigma(\boldsymbol{\zeta}),$$

for $q, r = 1, 2, 3$, where $\mathbb{I}$ is the symmetric identity tensor, $\mathbb{C}$ is the isotropic elasticity tensor and $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega$.
Finally, the functions $\boldsymbol{\theta}^{qr}$, with $q, r = 1, 2, 3$, are solutions to

$$\operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\theta}^{qr}) = 0 \quad \text{in}\,\mathbb{R}^3\backslash\Omega, \qquad \frac{\partial\boldsymbol{\theta}^{qr}}{\partial\boldsymbol{\nu}} = -\frac{1}{3\lambda + 2\mu}\mathbb{C}\boldsymbol{n} \quad \text{on}\,\partial\Omega,$$

with the decay conditions at infinity

$$|\boldsymbol{\theta}^{qr}| = O(|\boldsymbol{x}|^{-1}), \qquad |\nabla\boldsymbol{\theta}^{qr}| = O(|\boldsymbol{x}|^{-2}), \qquad \text{as}\,|\boldsymbol{x}| \to \infty.$$

CHAPTER 2

The scalar model

The aim of this chapter is to provide a detailed mathematical study of the simplified scalar version of the elastic problem presented in the introduction. In particular, recalling that $\mathbb{R}^d_-$ is the half-space and $\mathbb{R}^{d-1}$ its boundary, for $d \geqslant 3$, we consider the Laplace equation

$$\Delta u = 0 \quad \text{in } \mathbb{R}^d_- \backslash C \tag{2.1}$$

with boundary conditions

$$\frac{\partial u}{\partial \boldsymbol{n}} = g \quad \text{on } \partial C, \qquad \frac{\partial u}{\partial x_d} = 0 \quad \text{on } \mathbb{R}^{d-1}, \qquad u(\boldsymbol{x}) \to 0 \quad \text{as } |\boldsymbol{x}| \to +\infty \tag{2.2}$$

where C is a cavity (analogous to the pressurized one for the elastic case), g is a function defined on ∂C. After proving the well-posedness of this boundary value problem, we will consider the case of a cavity of the form $C = \boldsymbol{z} + \varepsilon\Omega$, where $\boldsymbol{z} \in \mathbb{R}^d_-$ and Ω is a Lipschitz bounded domain containing the origin. The aim is to establish an asymptotic expansion of the solution of the problem as $\varepsilon \to 0$.

As far as we know, this model does not have a real physical application, however the mathematical result has an interest on its own. In fact, as explained in the Introduction, it belongs to the same stream of the asymptotic analysis of the conductivity equation in bounded domains. With respect to the vast literature in this field, see for example [5, 6, 7, 8, 36] and the reference therein, the principal novelty of this chapter concern the asymptotic analysis in the case of unbounded domain with unbounded boundary and with non-homogeneous Neumann datum on the boundary of the hole. To tackle the issue of the well-posedness of this boundary value problem and the corresponding asymptotic analysis, we follow the approach of Ammari and Kang based on integral equations.

This chapter is organized as follows. In Section 2.1 we recall some well-known results about harmonic functions and layer potentials tecnhiques for the Laplace operator useful in the next. In Section 2.2 we examine the well-posedness of the scalar problem and we get an integral representation formula of the solution. In Section 2.3, we state and prove a spectral result crucial for the derivation of our main asymptotic expansion in the case of regular domains. In Section 2.4 we present and prove the theorem on the asymptotic expansion and finally we illustrate the result for the particular choice $g = -\boldsymbol{p} \cdot \boldsymbol{n}$.

2.1 Preliminaries

The main aim here is to collect together the various concepts, basic definitions and key theorems useful for the next sections. In detail, we recall some important properties about the decay rate of harmonic functions in unbounded domains and single and double layer potentials for the Laplace operator on Lipschitz domains. As already explained, we focus the attention only to dimension $d \geqslant 3$; however we remark that most of the results we recall are true also for $d = 2$.

We skip the proofs of the basic concepts while we give them for some theorems that may be unfamiliar. Results about harmonic functions in unbounded domains are contained, for example, in [30, 35, 55]; those on properties of single and double layer potentials can be found in [7, 29, 43, 58].

2.1.1 Harmonic functions in exterior domains

The specific symmetry of the half-space permits to show the well-posedness of the scalar model by extending the problem to an exterior domain, viz. the complementary set of a bounded set. Hence, it is useful to recall the classical results on the asymptotic behaviour of harmonic functions in exterior domains.

Theorem 2.1.1. *If v is harmonic in $\mathbb{R}^d \backslash \Omega$, with $d \geqslant 3$, the following statements are equivalent*

1. *v is harmonic at infinity.*

2. *$v(\boldsymbol{x}) \to 0$ as $|\boldsymbol{x}| \to \infty$.*

3. *$|v(\boldsymbol{x})| = O\left(|\boldsymbol{x}|^{2-d}\right)$ as $|\boldsymbol{x}| \to \infty$.*

In addition, from the behaviour of the gradient of harmonic functions on the boundary of the $d-$*dimensional* balls, that is if v is a harmonic function in $B_R(\boldsymbol{x})$, then

$$|\nabla v| \leqslant \frac{d}{R} \max_{\partial B_R(\boldsymbol{x})} |v| \tag{2.3}$$

we can deduce the behaviour of the gradient of harmonic functions at infinity. We summarize the results in the following theorem

Theorem 2.1.2. *If v is harmonic in $\mathbb{R}^d \backslash \Omega$, $d \geqslant 3$, and $v(\boldsymbol{x}) \to 0$ as $|\boldsymbol{x}| \to \infty$, then there exist r and a constant M, depending on r, such that if $|\boldsymbol{x}| \geqslant r$, we have*

$$|v| \leqslant \frac{M}{|\boldsymbol{x}|^{d-2}}, \qquad |\nabla v| \leqslant \frac{M}{|\boldsymbol{x}|^{d-1}} \tag{2.4}$$

In conclusion, we recall the Green's second identity which plays a crucial role to convert differential problems into integral ones.

Proposition 2.1.3. *Let Ω be a Lipschitz domain in $\mathbb{R}^d$. Given the pair of functions (u, v) defined in Ω it holds*

$$\int_\Omega (\Delta u(\boldsymbol{x}) v(\boldsymbol{x}) - u(\boldsymbol{x}) \Delta v(\boldsymbol{x}))\, d\boldsymbol{x} = \int_{\partial\Omega} \left(\frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}) v(\boldsymbol{x}) - u(\boldsymbol{x}) \frac{\partial v}{\partial \boldsymbol{n}}(\boldsymbol{x}) \right) d\sigma(\boldsymbol{x}) \tag{2.5}$$

2.1.2 Single and double layer potentials

Denoting with ω_d the area of the $(d-1)$-dimensional unit sphere, we recall the fundamental solution of the Laplace operator, that is the solution to

$$\Delta \Gamma(\boldsymbol{x}) = \delta_{\mathbf{0}}(\boldsymbol{x}),$$

where $\delta_{\mathbf{0}}(\boldsymbol{x})$ represents the delta function centered at $\mathbf{0}$. It is well known that Γ is radially symmetric and has this expression

$$\Gamma(\boldsymbol{x}) = \frac{1}{\omega_d (2-d) |\boldsymbol{x}|^{d-2}}. \tag{2.6}$$

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ and a function $\varphi(\boldsymbol{y}) \in L^2(\partial\Omega)$, we introduce the integral operators

$$\begin{aligned} S_\Omega\varphi(\boldsymbol{x}) &:= \int_{\partial\Omega} \Gamma(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\, d\sigma(\boldsymbol{y}), && \boldsymbol{x} \in \mathbb{R}^d \\ D_\Omega\varphi(\boldsymbol{x}) &:= \int_{\partial\Omega} \frac{\partial\Gamma(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\varphi(\boldsymbol{y})\, d\sigma(\boldsymbol{y}), && \boldsymbol{x} \in \mathbb{R}^d\backslash\partial\Omega \end{aligned} \tag{2.7}$$

which are called, respectively, single and double layer potentials relative to the set Ω.
We summarize some of their properties below

i. By definition, $S_\Omega\varphi(\boldsymbol{x})$ and $D_\Omega\varphi(\boldsymbol{x})$ are harmonic in $\mathbb{R}^d\backslash\partial\Omega$.

ii. $S_\Omega\varphi(\boldsymbol{x}) = O(|\boldsymbol{x}|^{2-d})$ as $|\boldsymbol{x}| \to +\infty$.

iii. If $\int_{\partial\Omega} \varphi(\boldsymbol{x})\, d\sigma(\boldsymbol{x}) = 0$ then $S_\Omega\varphi(\boldsymbol{x}) = O(|\boldsymbol{x}|^{1-d})$ as $|\boldsymbol{x}| \to +\infty$.

iv. $D_\Omega\varphi(\boldsymbol{x}) = O(|\boldsymbol{x}|^{1-d})$ as $|\boldsymbol{x}| \to +\infty$.

Next, we introduce the boundary operator $K_\Omega : L^2(\partial\Omega) \to L^2(\partial\Omega)$

$$K_\Omega\varphi(\boldsymbol{x}) = \frac{1}{\omega_d} p.v. \int_{\partial\Omega} \frac{(\boldsymbol{y}-\boldsymbol{x})\cdot \boldsymbol{n}_{\boldsymbol{y}}}{|\boldsymbol{x}-\boldsymbol{y}|^d}\varphi(\boldsymbol{y})\, d\sigma(\boldsymbol{y}) \tag{2.8}$$

and its L^2-adjoint

$$K_\Omega^*\varphi(\boldsymbol{x}) = \frac{1}{\omega_d} p.v. \int_{\partial\Omega} \frac{(\boldsymbol{x}-\boldsymbol{y})\cdot \boldsymbol{n}_{\boldsymbol{x}}}{|\boldsymbol{x}-\boldsymbol{y}|^d}\varphi(\boldsymbol{y})\, d\sigma(\boldsymbol{y}) \tag{2.9}$$

where $p.v.$ denotes the Cauchy principal value. The operators K_Ω and K_Ω^* are singular integral operators, bounded on $L^2(\partial\Omega)$.

Given a function v defined in a neighbourhood of $\partial\Omega$, we set

$$\begin{aligned} v(\boldsymbol{x})\Big|_\pm &:= \lim_{h\to 0^+} v(\boldsymbol{x} \pm h\boldsymbol{n}_{\boldsymbol{x}}), && \boldsymbol{x} \in \partial\Omega, \\ \frac{\partial v}{\partial \boldsymbol{n}_{\boldsymbol{x}}}(\boldsymbol{x})\Big|_\pm &:= \lim_{h\to 0^+} \nabla v(\boldsymbol{x} \pm h\boldsymbol{n}_{\boldsymbol{x}})\cdot \boldsymbol{n}_{\boldsymbol{x}}, && \boldsymbol{x} \in \partial\Omega. \end{aligned} \tag{2.10}$$

The following theorem about the jump relations of the single and double potentials for Lipschitz domains is a consequence of Coifman-McIntosh-Meyer results on the boundedness of the Cauchy integral on Lipschitz curves, see

[21], and the method of rotations of Calderón, see [18].
In the sequel, $t_1, \cdots, t_{d-1}$ represent an orthonormal basis for the tangent plane to $\partial\Omega$ at a point $\boldsymbol{x}$ and $\partial/\partial\boldsymbol{t} = \sum_{k=1}^{d-1} \partial/\partial t_k \, t_k$ the tangential derivative on $\partial\Omega$.

Theorem 2.1.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. For $\varphi \in L^2(\partial\Omega)$ the following relations hold, a.e. in $\partial\Omega$,*

$$\begin{aligned} S_\Omega\varphi(\boldsymbol{x})\Big|_+ &= S_\Omega\varphi(\boldsymbol{x})\Big|_- \\ \frac{\partial S_\Omega\varphi}{\partial\boldsymbol{t}}(\boldsymbol{x})\Big|_+ &= \frac{\partial S_\Omega\varphi}{\partial\boldsymbol{t}}(\boldsymbol{x})\Big|_- \\ \frac{\partial S_\Omega\varphi}{\partial\boldsymbol{n}_{\boldsymbol{x}}}(\boldsymbol{x})\Big|_\pm &= \left(\pm\frac{1}{2}I + K^*_\Omega\right)\varphi(\boldsymbol{x}) \\ D_\Omega\varphi(\boldsymbol{x})\Big|_\pm &= \left(\mp\frac{1}{2}I + K_\Omega\right)\varphi(\boldsymbol{x}) \end{aligned} \tag{2.11}$$

Using Green's identity it follows that $D_\Omega(1) = 1$ hence, by the jump relations for the double layer potential, we have $K_\Omega(1) = 1/2$.

In the sequel, to determine the well-posedness of the scalar model, presented at the beginning of this chapter, rewritten in terms of integral equations, we will need to generalize the result about the invertibility of the operators $1/2I + K^*_\Omega$ and $1/2I + K_\Omega$ when a regular compact operator is added. Therefore we recall here what is known about the eigenvalues of K^*_Ω and K_Ω in $L^2(\partial\Omega)$ and then the invertibility of the operators $\lambda I - K^*_\Omega$ and $\lambda I - K_\Omega$, for suitable $\lambda \in \mathbb{R}$. These results, for the case of Lipschitz domains, are contained in [29]. We define

$$L^2_0(\partial\Omega) := \left\{\varphi \in L^2(\partial\Omega), \int_{\partial\Omega} \varphi \, d\sigma = 0\right\}$$

Theorem 2.1.5. *Let λ be a real number. The operator $\lambda I - K^*_\Omega$ is injective on*

(a) $L^2_0(\partial\Omega)$ *if* $|\lambda| \geqslant 1/2$*;*

(b) $L^2(\partial\Omega)$ *if* $\lambda \in (-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, +\infty)$.

Proof. By contradiction, let $\lambda \in (-\infty, -1/2] \cup (1/2, +\infty)$ and suppose there exists $\varphi \in L^2(\partial\Omega)$, not identically zero, satisfying $(\lambda I - K^*_\Omega)\varphi = 0$. Since $K_\Omega(1) = 1/2$, it follows by duality that φ has mean value zero on $\partial\Omega$, in fact

$$0 = \langle 1, (\lambda I - K^*_\Omega)\varphi\rangle_{L^2(\partial\Omega)} = \langle \lambda - K_\Omega(1), \varphi\rangle_{L^2(\partial\Omega)} \\ = \langle \lambda - 1/2, \varphi\rangle_{L^2(\partial\Omega)}.$$

Thus, from the properties of single layer potential, $S_\Omega\varphi(\boldsymbol{x}) = O(|\boldsymbol{x}|^{1-d})$ and $\nabla S_\Omega\varphi(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-d})$ for $|\boldsymbol{x}| \to \infty$. Since φ is not identically zero, the two numbers

$$A = \int_\Omega |\nabla S_\Omega\varphi|^2 \, d\boldsymbol{x}, \qquad B = \int_{\mathbb{R}^d\setminus\overline{\Omega}} |\nabla S_\Omega\varphi|^2 \, d\boldsymbol{x}$$

cannot be zero. Applying the divergence theorem and the jump relations of the single layer potentials in Theorem 2.1.4 to A and B, we get

$$A = \int_{\partial\Omega} \big(-\frac{1}{2}I + K^*_\Omega\big)\varphi \, S_\Omega\varphi \, d\sigma(\boldsymbol{x}), \qquad B = -\int_{\partial\Omega} \big(\frac{1}{2}I + K^*_\Omega\big)\, \varphi S_\Omega\varphi \, d\sigma(\boldsymbol{x}).$$

Since $(\lambda I - K^*_\Omega)\varphi = 0$, it follows that

$$\lambda = \frac{1}{2}\frac{B - A}{B + A},$$

hence $|\lambda| < 1/2$, which is a contradiction. This implies that the operator $\lambda I - K^*_\Omega$ is injective in $L^2(\partial\Omega)$ for $\lambda \in (-\infty, -1/2] \cup (1/2, \infty)$.
Instead, in the case $\lambda = 1/2$, we suppose by contradiction there exists $\varphi \in L^2_0(\partial\Omega)$, not identically zero, such that $(1/2I - K^*_\Omega)\varphi = 0$. Then, we define A and B as before, but in this case we find

$$A = \int_{\partial\Omega} \big(-\frac{1}{2}I + K^*_\Omega\big)\varphi \, S_\Omega\varphi \, d\sigma(\boldsymbol{x}) = 0,$$

hence $S_\Omega\varphi = \mathit{cost}$ in Ω. By the continuity property of single layer potential on $\partial\Omega$ (see Theorem 2.1.4) we have that $S_\Omega\varphi$ is constant on $\partial\Omega$. Moreover, $S_\Omega\varphi$ is harmonic in $\mathbb{R}^d\setminus\partial\Omega$ and behaves like $O(|\boldsymbol{x}|^{1-d})$ as $|\boldsymbol{x}| \to \infty$ because $\varphi \in L^2_0(\partial\Omega)$. Therefore, by the decay rate at infinity we find that $S_\Omega\varphi = 0$ in $\mathbb{R}^d$, hence $\varphi = 0$ on $\partial\Omega$. This contradicts the hypothesis, hence $1/2I - K^*_\Omega$ is injective in $L^2_0(\partial\Omega)$. $\square$

The invertibility results of $\lambda I - K^*_\Omega$ and $\lambda I - K_\Omega$ are not straightforward. If the domain Ω is regular, at least C^1, it can be proven that the boundary operators K_Ω and K^*_Ω are compact, hence the invertibility of $\lambda I - K^*_\Omega$ and $\lambda I - K_\Omega$ can be obtained by the Fredholm theory. Instead, in the Lipschitz domains, K_Ω and K^*_Ω lose the compactness property, see the example proposed by Fabes, Jodeit and Lewis in [31], hence we cannot use the Fredholm theory to infer the invertibility. Verchota in [58] solved this problem showing the fundamental idea that the Rellich identities are the appropriate substitutes of compactness in the case of Lipschitz domains. Here, we recall the Rellich identity for the Laplace equation.

Proposition 2.1.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let u be a function such that either*

(i) u is Lipschitz in $\overline{\Omega}$ and $\Delta u = 0$ in Ω,

or

(ii) u is Lipschitz in $\mathbb{R}^d \backslash \Omega$ and $\Delta u = 0$ in $\mathbb{R}^d \backslash \overline{\Omega}$ with $|u| = O(|\boldsymbol{x}|^{2-d})$

Let $\boldsymbol{\alpha}$ be a C^1-vector field in $\mathbb{R}^d$ with compact support. Then

$$\int_{\partial\Omega} (\boldsymbol{\alpha}\cdot\boldsymbol{n})\Big|\frac{\partial u}{\partial \boldsymbol{n}}\Big|^2 = \int_{\partial\Omega} (\boldsymbol{\alpha}\cdot\boldsymbol{n})\Big|\frac{\partial u}{\partial \boldsymbol{t}}\Big|^2 - 2\int_{\partial\Omega} \Big(\boldsymbol{\alpha}\cdot\frac{\partial u}{\partial \boldsymbol{t}}\Big)\frac{\partial u}{\partial \boldsymbol{n}} + \begin{cases} \displaystyle\int_{\Omega} 2(\nabla\boldsymbol{\alpha}\nabla u\cdot\nabla u) - \operatorname{div}\boldsymbol{\alpha}|\nabla u|^2 & \text{if } u \text{ satisfies (i)} \\ \displaystyle\int_{\mathbb{R}^d\backslash\overline{\Omega}} 2(\nabla\boldsymbol{\alpha}\nabla u\cdot\nabla u) - \operatorname{div}\boldsymbol{\alpha}|\nabla u|^2 & \text{if } u \text{ satisfies (ii)} \end{cases} \tag{2.12}$$

As a consequence of the previous Rellich formula we have there exists a positive constant C depending only on the Lipschitz character of Ω such that

$$\frac{1}{C}\Big\|\frac{\partial u}{\partial \boldsymbol{t}}\Big\|_{L^2(\partial\Omega)} \leqslant \Big\|\frac{\partial u}{\partial \boldsymbol{n}}\Big\|_{L^2(\partial\Omega)} \leqslant C\Big\|\frac{\partial u}{\partial \boldsymbol{t}}\Big\|_{L^2(\partial\Omega)}. \tag{2.13}$$

In the proof of the invertibility of the operators $\lambda I - K^*_\Omega$ a crucial role is played by the following theorem.

Theorem 2.1.7. *For $0 \leqslant h \leqslant 1$ suppose that the family of operators $A_h : L^2(\mathbb{R}^{d-1}) \to L^2(\mathbb{R}^{d-1})$ satisfy*

(i) $\|A_h\varphi\|_{L^2(\mathbb{R}^{d-1})} \geqslant C\|\varphi\|_{L^2(\mathbb{R}^{d-1})}$, where C is independent of h;

(ii) $h \to A_h$ *is continuous in norm;*

(iii) $A_0 : L^2(\mathbb{R}^{d-1}) \to L^2(\mathbb{R}^{d-1})$ *is invertible.*

Then, $A_1 : L^2(\mathbb{R}^{d-1}) \to L^2(\mathbb{R}^{d-1})$ *is invertible.*

With all the ingredients at hand we can state and prove the invertibility theorem of the operators $\lambda I - K_\Omega^*$, for λ in the range expressed in Proposition 2.1.5. These results are due to Verchota [58] (for $\lambda = \pm 1/2$) and Escauriaza, Fabes and Verchota [29].

Theorem 2.1.8 ([29])**.** *Let Ω be a Lipschitz domain. The operator $\lambda I - K_\Omega^*$ is invertible on*

(i) $L_0^2(\partial\Omega)$ *if* $|\lambda| \geqslant \frac{1}{2}$;

(ii) $L^2(\partial\Omega)$ *if* $\lambda \in (-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, \infty)$.

Proof. We first prove the invertibility of the operators $\pm 1/2 I + K_\Omega^* : L_0^2(\partial\Omega) \to L_0^2(\partial\Omega)$. Since $K_\Omega(1) = 1/2$ we have that, for all $f \in L^2(\partial\Omega)$,

$$\int_{\partial\Omega} K_\Omega^* f(\boldsymbol{x})\, d\sigma(\boldsymbol{x}) = \frac{1}{2} \int_{\partial\Omega} f(\boldsymbol{x})\, d\sigma(\boldsymbol{x})$$

hence $\pm 1/2 I + K_\Omega^*$ maps $L_0^2(\partial\Omega)$ into $L_0^2(\partial\Omega)$. Let $u(\boldsymbol{x}) = S_\Omega f(\boldsymbol{x})$, where $f \in L_0^2(\partial\Omega)$. Then u satisfies conditions (i) and (ii) in Proposition 2.1.6. Moreover by the properties of single layer potentials on the boundary of Ω, we have that $\partial u/\partial \boldsymbol{t}$ is continuous across the boundary and the jump relation holds

$$\frac{\partial u}{\partial \boldsymbol{n}}\Big|_{\pm} = \Big(\pm \frac{1}{2} I + K_\Omega^* \Big) f.$$

Applying (2.13) in Ω and $\mathbb{R}^d \backslash \overline{\Omega}$ we obtain that

$$\begin{aligned} \frac{1}{C} \big\| \big(\frac{1}{2} I + K_\Omega^* \big) f \big\|_{L^2(\partial\Omega)} &\leqslant \big\| \big(\frac{1}{2} I - K_\Omega^* \big) f \big\|_{L^2(\partial\Omega)} \\ \big\| \big(\frac{1}{2} I - K_\Omega^* \big) f \big\|_{L^2(\partial\Omega)} &\leqslant C \big\| \big(\frac{1}{2} I + K_\Omega^* \big) f \big\|_{L^2(\partial\Omega)}. \end{aligned} \tag{2.14}$$

Since

$$f = \left(\frac{1}{2} I + K_\Omega^* \right) f + \left(\frac{1}{2} I - K_\Omega^* \right) f,$$

from (2.14) we have that

$$\left\| \left(\frac{1}{2}I + K_{\Omega}^*\right) f \right\|_{L^2(\partial\Omega)} \geqslant C\|f\|_{L^2(\partial\Omega)}. \tag{2.15}$$

Localizing the situation, we can assume that $\partial\Omega$ is the graph of a Lipschitz function in order to simplify as much as possible the proof. Therefore $\partial\Omega = \{(\boldsymbol{x}', x_d) : x_d = \varphi(\boldsymbol{x}')\}$ where $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipschitz function. To show that $A = (1/2)I + K_{\Omega}^*$ is invertible we consider the Lipschitz graph corresponding to $h\varphi$ that is

$$\partial\Omega_h = \big\{(\boldsymbol{x}', x_d) : x_d = h\varphi(\boldsymbol{x}')\big\}, \qquad 0 \leqslant h \leqslant 1$$

and the corresponding operators $K_{\Omega_h}^*$ and A_h. Then $A_0 = (1/2)I$ and $A_1 = A$. In addition, A_h are continuous in norm as a function of h. Hence, from the inequality in (2.15) we have that $\|A_h f\|_{L^2(\partial\Omega_h)} \geqslant C\|f\|_{L^2(\partial\Omega_h)}$, since the constant C is independent of h but depends only on the Lipschitz character of Ω. Applying the continuity method of Theorem 2.1.7 we find that $1/2I + K_{\Omega}^*$ is invertible on $L_0^2(\partial\Omega)$. Next, we prove that $1/2I+K_{\Omega}^*$ is invertible on $L^2(\partial\Omega)$ showing that the operator is onto on $L^2(\partial\Omega)$. By duality argument, since $K_{\Omega}(1) = 1/2$, for all $f \in L^2(\partial\Omega)$ we get

$$\int\limits_{\partial\Omega} \left(\frac{1}{2}I + K_{\Omega}^*\right) f \, d\sigma(\boldsymbol{x}) = \int\limits_{\partial\Omega} f \, d\sigma(\boldsymbol{x})$$

hence $1/2I + K_{\Omega}^*$ maps $L^2(\partial\Omega)$ into $L^2(\partial\Omega)$. For $g \in L^2(\partial\Omega)$ we consider

$$g = g - c\left(\frac{1}{2}I + K_{\Omega}^*\right)(1) + c\left(\frac{1}{2}I + K_{\Omega}^*\right)(1),$$

where

$$c = \frac{1}{|\partial\Omega|} \int\limits_{\partial\Omega} g \, d\sigma(\boldsymbol{x}).$$

Defining $g_0 := g - c(1/2I + K_{\Omega}^*)(1)$, since

$$\int\limits_{\partial\Omega} (\frac{1}{2}I + K_{\Omega}^*)(1) \, d\sigma(\boldsymbol{x}) = |\partial\Omega|,$$

we have that $g_0 \in L_0^2(\partial\Omega)$. Let $f_0 \in L_0^2(\partial\Omega)$ be such that

$$\left(\frac{1}{2}I + K_{\Omega}^*\right) f_0 = g_0.$$

Then, defining $f := f_0 + c$ we find that

$$\left(\frac{1}{2}I + K_\Omega^*\right) f = g_0 + c\left(\frac{1}{2}I + K_\Omega^*\right)(1) = g.$$

This means that $1/2I + K_\Omega^*$ is onto in $L^2(\partial\Omega)$.
For the operator $-1/2I + K_\Omega^*$ we can follow the same argument both for the case $L_0^2(\partial\Omega)$ and $L^2(\partial\Omega)$.

Next, suppose that $|\lambda| > 1/2$. To prove the invertibility of the operators in the general case we use the Rellich identity. Let $f \in L^2(\partial\Omega)$, c_0 a fixed positive number and set $u(\boldsymbol{x}) = S_\Omega f(\boldsymbol{x})$. Let $\boldsymbol{\alpha}$ be a vector field with support in the set dist($\boldsymbol{x}$,$\partial\Omega$)$< 2c_0$, $\forall \boldsymbol{x} \in \partial\Omega$, such that $\boldsymbol{\alpha} \cdot \boldsymbol{n} \geqslant \delta$, for some $\delta > 0$. Therefore, from the Rellich identity (2.1.6), we have

$$\begin{aligned}\int\limits_{\partial\Omega} (\boldsymbol{\alpha}\cdot\boldsymbol{n})\Big|\frac{\partial u}{\partial \boldsymbol{n}}\Big|^2 &= \int\limits_{\partial\Omega} (\boldsymbol{\alpha}\cdot\boldsymbol{n})\Big|\frac{\partial u}{\partial \boldsymbol{t}}\Big|^2 - 2\int\limits_{\partial\Omega}\Big(\boldsymbol{\alpha}\cdot\frac{\partial u}{\partial \boldsymbol{t}}\Big)\frac{\partial u}{\partial \boldsymbol{n}} \\ &\quad + \int\limits_{\Omega} 2(\nabla\boldsymbol{\alpha}\nabla u\cdot\nabla u) - \operatorname{div}\boldsymbol{\alpha}\,|\nabla u|^2.\end{aligned} \tag{2.16}$$

Observe that on $\partial\Omega$

$$\frac{\partial u}{\partial \boldsymbol{n}}\Big|_- = \left(-\frac{1}{2}I + K_\Omega^*\right) f = \left(\lambda - \frac{1}{2}\right) f - (\lambda I - K_\Omega^*) f.$$

Since $\boldsymbol{\alpha} = (\boldsymbol{\alpha}\cdot\boldsymbol{n})\boldsymbol{n} + \sum_{k=1}^{d-1}(\boldsymbol{\alpha}\cdot t_k)t_k$ and $\nabla Sf(\boldsymbol{x})|_+ = 1/2\boldsymbol{n}f + \mathcal{K}f$ where

$$\mathcal{K}f(\boldsymbol{x}) = \frac{1}{\omega_d}\, p.v. \int\limits_{\partial\Omega} \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^d} f(\boldsymbol{y})\, d\sigma(\boldsymbol{y}),$$

we find that

$$\begin{aligned}(\nabla u\cdot\boldsymbol{\alpha}) &= \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{\alpha}\cdot\boldsymbol{n}) + (\boldsymbol{\alpha}\cdot\frac{\partial u}{\partial \boldsymbol{t}}) \\ &= -\frac{1}{2}(\boldsymbol{\alpha}\cdot\boldsymbol{n})f + K_{\boldsymbol{\alpha}} f,\end{aligned} \tag{2.17}$$

where

$$K_{\boldsymbol{\alpha}} f(\boldsymbol{x}) = \frac{1}{\omega_d}\, p.v. \int\limits_{\partial\Omega} \frac{((\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{\alpha}(\boldsymbol{x}))}{|\boldsymbol{x}-\boldsymbol{y}|^d} f(\boldsymbol{y})\, d\sigma(\boldsymbol{y}).$$

We also have

$$\int_{\Omega} |\nabla u|^2 \, d\boldsymbol{x} = \int_{\partial\Omega} u \frac{\partial u}{\partial \boldsymbol{n}}\Big|_{-} \, d\sigma(\boldsymbol{x})$$
$$= \int_{\partial\Omega} S_{\Omega} f \left[\left(\lambda - \frac{1}{2} \right) f - (\lambda I - K_{\Omega}^*) f \right] d\sigma(\boldsymbol{x}).$$

By using the following integral identity obtained by multiplying (2.17) for $\partial u/\partial \boldsymbol{n}$, that is

$$-2 \int_{\partial\Omega} \left(\boldsymbol{\alpha} \cdot \frac{\partial u}{\partial \boldsymbol{t}} \right) \frac{\partial u}{\partial \boldsymbol{n}} \, d\sigma(\boldsymbol{x}) = -2 \int_{\partial\Omega} \frac{\partial u}{\partial \boldsymbol{n}} \left[-\frac{1}{2} (\boldsymbol{\alpha} \cdot \boldsymbol{n}) f + K_{\boldsymbol{\alpha}} f \right] d\sigma(\boldsymbol{x})$$
$$+ 2 \int_{\partial\Omega} (\boldsymbol{\alpha} \cdot \boldsymbol{n}) \Big| \frac{\partial u}{\partial \boldsymbol{n}} \Big|^2 \, d\sigma(\boldsymbol{x}),$$

we get from the Rellich formula (2.16) that

$$\frac{1}{2} \int_{\partial\Omega} (\boldsymbol{\alpha} \cdot \boldsymbol{n}) \Big| \frac{\partial u}{\partial \boldsymbol{n}} \Big|^2 \, d\sigma(\boldsymbol{x}) = -\frac{1}{2} \int_{\partial\Omega} (\boldsymbol{\alpha} \cdot \boldsymbol{n}) \Big| \frac{\partial u}{\partial \boldsymbol{t}} \Big|^2 \, d\sigma(\boldsymbol{x})$$
$$+ \int_{\partial\Omega} \frac{\partial u}{\partial \boldsymbol{n}} \left[-\frac{1}{2} (\boldsymbol{\alpha} \cdot \boldsymbol{n}) f + K_{\boldsymbol{\alpha}} f \right] d\sigma(\boldsymbol{x})$$
$$- \int_{\Omega} \left[(\nabla \boldsymbol{\alpha} \nabla u, \nabla u) + \frac{1}{2} \operatorname{div} \boldsymbol{\alpha} \, |\nabla u|^2 \right] d\sigma(\boldsymbol{x}).$$

Thus it holds

$$\frac{1}{2} \left(\lambda - \frac{1}{2} \right)^2 \int_{\partial\Omega} (\boldsymbol{\alpha} \cdot \boldsymbol{n}) f^2 \, d\sigma(\boldsymbol{x})$$
$$\leqslant \int_{\partial\Omega} \left[-\frac{1}{2} (\boldsymbol{\alpha} \cdot \boldsymbol{n}) f + K_{\boldsymbol{\alpha}} f \right] \left[\left(\lambda - \frac{1}{2} \right) f - (\lambda I - K_{\Omega}^*) f \right] d\sigma(\boldsymbol{x})$$
$$+ C \|f\|_{L^2(\partial\Omega)} \left(\|S_{\Omega} f\|_{L^2(\partial\Omega)} + \|(\lambda I - K_{\Omega}^*) f\|_{L^2(\partial\Omega)} \right)$$
$$+ C \|S_{\Omega} f\|_{L^2(\partial\Omega)} \|(\lambda I - K_{\Omega}^*) f\|_{L^2(\partial\Omega)} + C \|(\lambda I - K_{\Omega}^*) f\|_{L^2(\partial\Omega)}^2,$$

where C depends on the Lipschitz character of Ω and λ. By multiplying the integrand of the right-hand side integral we get

$$\frac{1}{2}\left(\lambda^2-\frac{1}{4}\right)\int\limits_{\partial\Omega}(\boldsymbol{\alpha}\cdot\boldsymbol{n})f^2\,d\sigma(\boldsymbol{x}) \leqslant \left(\lambda-\frac{1}{2}\right)\int\limits_{\partial\Omega} fK_{\boldsymbol{\alpha}}f\,d\sigma(\boldsymbol{x})$$
$$+ C\|f\|_{L^2(\partial\Omega)}\left(\|S_\Omega f\|_{L^2(\partial\Omega)}+\|(\lambda I-K^*_\Omega)f\|_{L^2(\partial\Omega)}\right)$$
$$+ C\|S_\Omega f\|_{L^2(\partial\Omega)}\|(\lambda I-K^*_\Omega)f\|_{L^2(\partial\Omega)}+C\|(\lambda I-K^*_\Omega)f\|^2_{L^2(\partial\Omega)}.$$

Denoting with $K^*_{\boldsymbol{\alpha}}$ the adjoint operator in $L^2(\partial\Omega)$ of $K_{\boldsymbol{\alpha}}$ we find

$$K^*_{\boldsymbol{\alpha}}+K_{\boldsymbol{\alpha}}=R_{\boldsymbol{\alpha}}f=\frac{1}{\omega_d}p.v.\int\limits_{\partial\Omega}\frac{[(\boldsymbol{x}-\boldsymbol{y})\cdot(\boldsymbol{\alpha}(\boldsymbol{x})-\boldsymbol{\alpha}(\boldsymbol{y}))]}{|\boldsymbol{x}-\boldsymbol{y}|^d}f(\boldsymbol{y})\,d\sigma(\boldsymbol{y}).$$

By duality, we have

$$\int\limits_{\partial\Omega} fK_{\boldsymbol{\alpha}}f\,d\sigma(\boldsymbol{x})=\frac{1}{2}\int\limits_{\partial\Omega} fR_{\boldsymbol{\alpha}}f\,d\sigma(\boldsymbol{x}).$$

Since $|\lambda|>1/2$ and $\boldsymbol{\alpha}\cdot\boldsymbol{n}\geqslant\delta>0$, the norm $\|f\|_{L^2(\partial\Omega)}$ in the left-hand side can be hidden thus getting

$$\|f\|_{L^2(\partial\Omega)}\leqslant C\left(\|(\lambda I-K^*_\Omega)f\|_{L^2(\partial\Omega)}+\|S_\Omega f\|+\|R_\alpha f\|_{L^2(\partial\Omega)}\right). \tag{2.18}$$

Since S_Ω and $R_{\boldsymbol{\alpha}}$ are compact in $L^2(\partial\Omega)$, we conclude from the above estimate that $\lambda I-K^*_\Omega$ has a closed range.
Now, we prove the surjectivity of the operator $\lambda I-K^*_{\partial\Omega}$ in $L^2(\partial\Omega)$. From this result and the injectivity proved in Theorem 2.1.5 the invertibility follows.
Suppose on the contrary that for some λ real, $|\lambda|>1/2$, $\lambda I-K^*_\Omega$ is not invertible in $L^2(\partial\Omega)$. Then the intersection of the spectrum of K^*_Ω and the set $\{\lambda\in\mathbb{R}:|\lambda|>1/2\}$ is not empty and so there exists a real number λ_0 that belongs to this intersection and it is a boundary point of the set. To reach a contradiction it suffices to show that $\lambda_0 I-K^*_\Omega$ is invertible. We know that $\lambda_0 I-K^*_\Omega$ is injective and by (2.18) has a closed range. Therefore there exists a constant C such that for all $f\in L^2(\partial\Omega)$ the following estimate holds

$$\|f\|_{L^2(\partial\Omega)}\leqslant C\|(\lambda_0 I-K^*_\Omega)f\|_{L^2(\partial\Omega)}. \tag{2.19}$$

Since λ_0 is a boundary point of the intersection of the spectrum of K^*_Ω and the real line there exists a sequence of real numbers λ_k with $|\lambda_k|>1/2$,

$\lambda_k \to \lambda_0$, as $k \to \infty$, and $\lambda_k I - K^*_\Omega$ is invertible on $L^2(\partial\Omega)$. Therefore, given $g \in L^2(\partial\Omega)$ there exists a unique $f_k \in L^2(\partial\Omega)$ such that $(\lambda_k I - K^*_\Omega) f_k = g$. If $\{\|f_k\|_{L^2(\partial\Omega)}\}$ has a bounded subsequence then there exists another subsequence that converges weakly to some $f_0 \in L^2(\partial\Omega)$ and it holds

$$\begin{aligned}\int\limits_{\partial\Omega} h(\lambda_0 I - K^*_\Omega) f_0 \, d\sigma(\boldsymbol{x}) &= \lim_{k\to+\infty} \int\limits_{\partial\Omega} f_k(\lambda_0 I - K_\Omega) h \, d\sigma(\boldsymbol{x}) \\ &= \lim_{k\to+\infty} \int\limits_{\partial\Omega} h(\lambda_0 I - K^*_\Omega) f_k \, d\sigma(\boldsymbol{x}) = \int\limits_{\partial\Omega} gh \, d\sigma(\boldsymbol{x}).\end{aligned}$$

Therefore $(\lambda_0 I - K^*_\Omega) f_0 = g$. In the opposite case we may assume that $\|f_k\|_{L^2(\partial\Omega)} = 1$ and $(\lambda_0 I - K^*_\Omega) f_k$ converges to zero in $L^2(\partial\Omega)$. However from (2.19)

$$\begin{aligned}1 = \|f_k\|_{L^2(\partial\Omega)} &\leqslant C\|(\lambda_0 I - K^*_\Omega) f_k\|_{L^2(\partial\Omega)} \\ &\leqslant C\,|\lambda - \lambda_k| + C\|(\lambda_k I - K^*_\Omega) f_k\|_{L^2(\partial\Omega)}\end{aligned}$$

Since the right-hand side converges to zero as $k \to \infty$, we get a contradiction, hence for each λ real, $|\lambda| > 1/2$, $\lambda I - K^*_\Omega$ is invertible. □

Remark 2.1.9. *The invertibility of the operator $1/2I + K_\Omega$ follows exploiting the Banach's closed range theorem starting from the result for $1/2I + K^*_\Omega$. In particular, the result follows from the fact that $1/2I + K^*_\Omega$ has closed and dense range in $L^2(\partial\Omega)$. For more details see [58].*

2.2 The scalar problem

Thanks to the instruments of the preliminaries section, we are now ready to analyse the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^d_- \backslash C \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g & \text{on } \partial C \\ \dfrac{\partial u}{\partial x_d} = 0 & \text{on } \mathbb{R}^{d-1} \\ u \to 0 & \text{as } |\boldsymbol{x}| \to +\infty \end{cases} \tag{2.20}$$

where C is the cavity, g is a function defined on ∂C and $d \geqslant 3$.

In particular, we establish the well-posedness of the problem and provide an integral representation formula for any bounded Lipschitz domain C contained in the half-space.

Only in the next section, making the smallness assumption on the cavity, we find the asymptotic expansion.

2.2.1 Well-posedness

Proving existence and uniqueness results in the half-space and, in general, in unbounded domain with unbounded boundary, is much more difficult with respect to the case of bounded or exterior domains. The main obstacle is the control of both solution decay and integrability on the boundary. Indeed, it is typical to treat these problems by means of weighted Sobolev spaces, see for example [9]. Here, in order to mantain a simple mathematical interpretation of the results, we choose to use the particular symmetry of the half-space to prove the well-posedness of the problem (2.20). Therefore, we extend the problem to the whole space, specifically to an exterior domain, establishing the well-posedness in a standard Sobolev setting.

Given a bounded Lipschitz domain $C \subset \mathbb{R}^d_-$ and the function $g : \partial C \to \mathbb{R}$, we define

$$\widetilde{C} := \{(\boldsymbol{x}', x_d) : \ (\boldsymbol{x}', -x_d) \in C\},$$

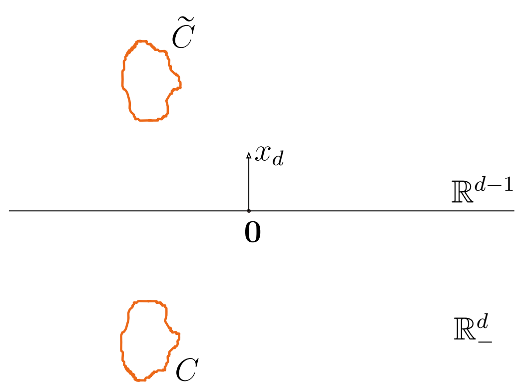


Figure 2.1. Reflection of the geometry

see Figure 2.1, and $G : \partial C \cup \partial \widetilde{C} \to \mathbb{R}$ as

$$G(\boldsymbol{x}) := \begin{cases} g(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \partial C \\ g(\widetilde{\boldsymbol{x}}) & \text{if } \boldsymbol{x} \in \partial \widetilde{C}. \end{cases} \tag{2.21}$$

Theorem 2.2.1. *The problem (2.20) has a unique solution. This solution coincides with the restriction to the half-space* $\mathbb{R}^d_-$ *of the solution to*

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}^d \backslash \left(C \cup \widetilde{C}\right) \\ \dfrac{\partial U}{\partial \boldsymbol{n}} = G & \text{on } \partial C \cup \partial \widetilde{C} \\ U \to 0 & \text{as } |\boldsymbol{x}| \to +\infty. \end{cases} \tag{2.22}$$

Proof. The proof is divided into three steps: uniqueness for (2.22), existence for (2.22), equivalence between (2.20) and (2.22).

1. For $\mathit{\Lambda} := C \cup \widetilde{C}$, let $R > 0$ be such that $\mathit{\Lambda} \subset B_R(\mathbf{0})$ and set $\mathit{\Omega}_R := B_R(\mathbf{0}) \backslash \mathit{\Lambda}$. Given two solutions, U_1 and U_2, to problem (2.22), the difference $W := U_1 - U_2$ solves the corresponding homogeneous problem. Multiplying equation $\Delta W = 0$ by W and integrating over the domain $\mathit{\Omega}_R = B_R(\mathbf{0}) \backslash \mathit{\Lambda}$, we infer

$$\begin{aligned} 0 &= \int_{\mathit{\Omega}_R} W(\boldsymbol{x}) \Delta W(\boldsymbol{x}) d\boldsymbol{x} \\ &= \int_{\partial \mathit{\Omega}_R} W(\boldsymbol{x}) \frac{\partial}{\partial \boldsymbol{n}} W(\boldsymbol{x}) d\sigma(\boldsymbol{x}) - \int_{\mathit{\Omega}_R} \left|\nabla W(\boldsymbol{x})\right|^2 d\boldsymbol{x} \\ &= \int_{\partial B_R(\mathbf{0})} W(\boldsymbol{x}) \frac{\partial}{\partial \boldsymbol{n}} W(\boldsymbol{x}) d\sigma(\boldsymbol{x}) - \int_{\mathit{\Omega}_R} \left|\nabla W(\boldsymbol{x})\right|^2 d\boldsymbol{x}, \end{aligned}$$

using integration by parts and boundary conditions. Exploiting the behaviour of the harmonic functions in exterior domains, as described in (2.1.2), we get

$$\left| \int_{\partial B_R(\mathbf{0})} W(\boldsymbol{x}) \frac{\partial}{\partial \boldsymbol{n}} W(\boldsymbol{x})\, d\sigma(\boldsymbol{x}) \right| \leqslant \frac{C}{R^{d-2}}.$$

Then, as $R \to \infty$, we find

$$\int_{\mathbb{R}^d \backslash \mathit{\Lambda}} \left|\nabla W(\boldsymbol{x})\right|^2 d\boldsymbol{x} = 0$$

which implies $W = 0$.

2. We represent the solution of (2.22) by means of single layer potential

$$S_{\mathit{\Lambda}} \psi(\boldsymbol{x}) = \int_{\partial \mathit{\Lambda}} \mathit{\Gamma}(\boldsymbol{x} - \boldsymbol{y}) \psi(\boldsymbol{y}) d\sigma(\boldsymbol{y}), \qquad \boldsymbol{x} \in \mathbb{R}^d \backslash \mathit{\Lambda}, \tag{2.23}$$

with function ψ to be determined. By the properties of single layer potential, $S_\Lambda\psi$ is harmonic in $\mathbb{R}^d \backslash \partial\Lambda$, $S_\Lambda\psi(\boldsymbol{x}){=}O(|\boldsymbol{x}|^{2-d})$ as $|\boldsymbol{x}| \to \infty$ and we have

$$\frac{\partial S_\Lambda \psi}{\partial \boldsymbol{n}}(\boldsymbol{x})\bigg|_+ = \frac{1}{2}\psi + K_\Lambda^* \psi, \qquad \boldsymbol{x} \in \partial\Lambda.$$

From the injectivity result on $L^2(\partial\Lambda)$ of the operator $1/2I + K_\Lambda^*$, in Theorem 2.1.5, there exists a function ψ such that

$$\left(\frac{1}{2}I + K_\Lambda^*\right)\psi(\boldsymbol{x}) = G(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial\Lambda, \tag{2.24}$$

observing that $G \in L^2(\partial\Lambda)$.

3. Let $u(\boldsymbol{x}', x_d) := U(\boldsymbol{x}', x_d)\big|_{x_d<0}$. From the boundary value problem (2.22) for U and the definition (2.21) of the function G, we have

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^d_- \backslash C \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g & \text{on } \partial C \\ u \to 0 & \text{as } |\boldsymbol{x}| \to +\infty \end{cases}$$

We have to verify that the normal derivative is null on the boundary of the half-space. For this purpose we first show that U is even with respect to the x_d-plane. We define

$$\bar{u}(\boldsymbol{x}', x_d) := U(\boldsymbol{x}', -x_d) \tag{2.25}$$

for $\boldsymbol{x} \in \mathbb{R}^d \backslash \left(C \cup \widetilde{C}\right)$; then $\bar{u}$ solves the following problem

$$\begin{cases} \Delta \bar{u} = 0 & \text{in } \mathbb{R}^d \backslash \left(C \cup \widetilde{C}\right) \\ \dfrac{\partial \bar{u}}{\partial \boldsymbol{n}} = G & \text{on } \partial C \cup \partial\widetilde{C} \\ \bar{u} \to 0 & \text{as } |\boldsymbol{x}| \to +\infty \end{cases} \tag{2.26}$$

since G is even with respect to x_d and on $\partial C \cap \partial\widetilde{C}$ we have

$$\frac{\partial \bar{u}}{\partial \boldsymbol{n}}(\boldsymbol{x}', x_d) = \frac{\partial U}{\partial \boldsymbol{n}}(\boldsymbol{x}', -x_d).$$

Problem (2.26) admits a unique solution $\bar{u}(\boldsymbol{x})$ as a consequence of the previous points, hence

$$U(\boldsymbol{x}', -x_d) = \bar{u}(\boldsymbol{x}', x_d) = U(\boldsymbol{x}', x_d).$$

From this last result, we obtain

$$\frac{\partial \bar{u}}{\partial x_d}(\boldsymbol{x}', x_d) = \frac{\partial U}{\partial x_d}(\boldsymbol{x}', x_d) = -\frac{\partial U}{\partial x_d}(\boldsymbol{x}', -x_d),$$

hence the derivative of U with respect to x_d computed at any point with $x_d = 0$ is zero. □

2.2.2 Representation formula

After proving the well-posedness of the boundary value problem (2.20), in this paragraph we derive an integral representation formula for the solution u to problem (2.20). This makes use of the single and double layer potentials defined in (2.7) and of contributions relative to the image cavity $\widetilde{C}$, given by

$$\begin{aligned} \widetilde{S}_C\varphi(\boldsymbol{x}) &:= \int_{\partial C} \Gamma(\widetilde{\boldsymbol{x}} - \boldsymbol{y})\varphi(\boldsymbol{y})d\sigma(\boldsymbol{y}), & \boldsymbol{x} \in \mathbb{R}^d, \\ \widetilde{D}_C\varphi(\boldsymbol{x}) &:= \int_{\partial C} \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\Gamma(\widetilde{\boldsymbol{x}} - \boldsymbol{y})\varphi(\boldsymbol{y})d\sigma(\boldsymbol{y}) & \boldsymbol{x} \in \mathbb{R}^d \backslash \partial\widetilde{C}. \end{aligned} \tag{2.27}$$

These operators, referred to as *image layer potentials*, can be read as single and double layer potentials on $\widetilde{C}$ applied to the reflection of the function φ with respect to x_d coordinate.

Theorem 2.2.2. *The solution u to problem* (2.20) *is such that*

$$u(\boldsymbol{x}) = S_C g(\boldsymbol{x}) - D_C f(\boldsymbol{x}) + \widetilde{S}_C g(\boldsymbol{x}) - \widetilde{D}_C f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^d_- \backslash C, \tag{2.28}$$

where S_C, D_C are defined in (2.7)*, $\widetilde{S}_C, \widetilde{D}_C$ in* (2.27)*, g is the Neumann boundary condition in* (2.20) *and f is the trace of u on ∂C.*

Using properties of layer potentials, from equation (2.28), we infer

$$f(\boldsymbol{x}) = S_C g(\boldsymbol{x}) - \left(-\tfrac{1}{2}I + K_C\right) f(\boldsymbol{x}) - \widetilde{D}_C f(\boldsymbol{x}) + \widetilde{S}_C g(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial C,$$

where K_C is defined in (2.8). Thus, the trace f satisfies the integral equation

$$\left(\frac{1}{2}I + K_C + \widetilde{D}_C\right) f = S_C g + \widetilde{S}_C g,$$

which will turn out to be useful in the sequel.

Before proving Theorem 2.2.2, we first recall the definition of the Neumann function for the Laplace operator, see for example [39], that is the solution $N = N(\boldsymbol{x}, \boldsymbol{y})$ to

$$\begin{cases} \Delta_{\boldsymbol{y}} N(\boldsymbol{x}, \boldsymbol{y}) = \delta_{\boldsymbol{x}}(\boldsymbol{y}) & \text{in } \mathbb{R}^d_- \\ \dfrac{\partial N}{\partial y_d}(\boldsymbol{x}, \boldsymbol{y}) = 0 & \text{on } \mathbb{R}^{d-1}, \end{cases}$$

where $\delta_{\boldsymbol{x}}(\boldsymbol{y})$ is the delta function with the center in a fixed point $\boldsymbol{x} \in \mathbb{R}^d$ and $\partial N/\partial y_d$ represents the normal derivative on the boundary of the half-space $\mathbb{R}^d_-$. The classical method of images provides the explicit expression

$$N(\boldsymbol{x}, \boldsymbol{y}) = \frac{\kappa_d}{|\boldsymbol{x} - \boldsymbol{y}|^{d-2}} + \frac{\kappa_d}{|\widetilde{\boldsymbol{x}} - \boldsymbol{y}|^{d-2}},$$

where $\kappa_d := 1/\omega_d(2-d)$. With the function N at hand, the representation formula (2.28) can be equivalently written as

$$\begin{aligned} u(\boldsymbol{x}) &= \mathcal{N}(f, g)(\boldsymbol{x}) \\ &:= \int_{\partial C} \left[N(\boldsymbol{x}, \boldsymbol{y}) g(\boldsymbol{y}) - \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{y}}} N(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y}) \right] d\sigma(\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^d_- \backslash C, \end{aligned} \tag{2.29}$$

which we now prove.

Proof of Theorem 2.2.2. Given $R, \varepsilon > 0$ such that $C \subset B_R(\mathbf{0})$ and $B_\varepsilon(\boldsymbol{x}) \subset \mathbb{R}^d_- \backslash C$, let

$$\Omega_{R,\varepsilon} := \left(\mathbb{R}^d_- \cap B_R(\mathbf{0})\right) \backslash \left(C \cup B_\varepsilon(\boldsymbol{x})\right).$$

We also define $\partial B^h_R(\mathbf{0})$ as the intersection of the hemisphere with the boundary of the half-space, and with $\partial B^b_R(\mathbf{0})$ the spherical cap (see Figure 2.2). Applying second Green's identity to $N(\boldsymbol{x}, \cdot)$ and u in $\Omega_{R,\varepsilon}$, we get

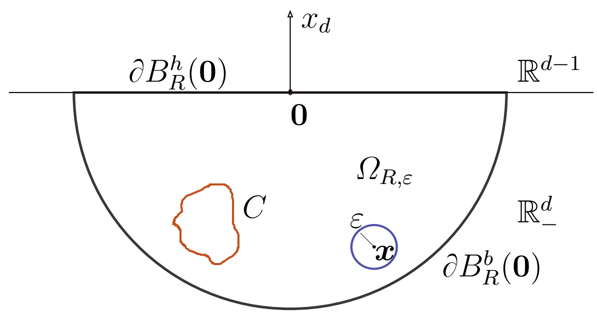


Figure 2.2. Domain $\Omega_{R,\varepsilon}$ used to obtain the integral representation formula (2.28).

$$\begin{aligned}
0 =& \int_{\Omega_{R,\varepsilon}} [N(\boldsymbol{x},\boldsymbol{y})\Delta u(\boldsymbol{y}) - u(\boldsymbol{y})\Delta_{\boldsymbol{y}} N(\boldsymbol{x},\boldsymbol{y})]\, d\boldsymbol{y} \\
=& \int_{\partial B_R^h(\mathbf{0})} \left[N(\boldsymbol{x},\boldsymbol{y})\frac{\partial u}{\partial y_d}(\boldsymbol{y}) - \frac{\partial}{\partial y_d}N(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})\right] d\sigma(\boldsymbol{y}) \\
&+ \int_{\partial B_R^b(\mathbf{0})} \left[N(\boldsymbol{x},\boldsymbol{y})\frac{\partial u}{\partial \boldsymbol{n}_{\boldsymbol{y}}}(\boldsymbol{y}) - \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{y}}}N(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})\right] d\sigma(\boldsymbol{y}) \\
&+ \int_{\partial B_\varepsilon(\boldsymbol{x})} \left[\frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{y}}}N(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y}) - N(\boldsymbol{x},\boldsymbol{y})\frac{\partial u}{\partial \boldsymbol{n}_{\boldsymbol{y}}}(\boldsymbol{y})\right] d\sigma(\boldsymbol{y}) \\
&- \int_{\partial C} \left[N(\boldsymbol{x},\boldsymbol{y})\frac{\partial u}{\partial \boldsymbol{n}_{\boldsymbol{y}}}(\boldsymbol{y}) - \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{y}}}N(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})\right] d\sigma(\boldsymbol{y}) \\
=& I_1 + I_2 + I_3 - \mathcal{N}(f,g)(\boldsymbol{x}).
\end{aligned}$$

The term I_1 is zero since both the normal derivative of the function N and u are zero above the boundary of the half-space.

Next, taking into account the behaviour of harmonic functions in exterior domains, formulas (2.4), we deduce

$$\begin{aligned}
\left|\int_{\partial B_R^b(\mathbf{0})} \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{y}}}N(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})\, d\sigma(\boldsymbol{y})\right| &\leqslant \frac{C}{R^{2d-3}}\int_{\partial B_R^b(\mathbf{0})} d\sigma(\boldsymbol{y}) = \frac{C}{R^{d-2}}, \\
\left|\int_{\partial B_R^b(\mathbf{0})} N(\boldsymbol{x},\boldsymbol{y})\frac{\partial u}{\partial \boldsymbol{n}_{\boldsymbol{y}}}(\boldsymbol{y})\, d\sigma(\boldsymbol{y})\right| &\leqslant \frac{C}{R^{2d-3}}\int_{\partial B_R^b(\mathbf{0})} d\sigma(\boldsymbol{y}) = \frac{C}{R^{d-2}},
\end{aligned}$$

where C denotes a generic positive constant. As $R \to +\infty$, I_2 tends to zero.

Finally, we decompose I_3 as

$$I_3 = I_{31} - I_{32} = \int_{\partial B_\varepsilon(\boldsymbol{x})} \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{y}}}N(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})\, d\sigma(\boldsymbol{y}) - \int_{\partial B_\varepsilon(\boldsymbol{x})} N(\boldsymbol{x},\boldsymbol{y})\frac{\partial u}{\partial \boldsymbol{n}_{\boldsymbol{y}}}(\boldsymbol{y}) d\sigma(\boldsymbol{y}).$$

Using the expression of N and the continuity of u, we derive

$$\begin{aligned}
I_{31} &= \int_{\partial B_\varepsilon(\boldsymbol{x})} \frac{\partial}{\partial \boldsymbol{n_y}} N(\boldsymbol{x},\boldsymbol{y}) u(\boldsymbol{y}) d\sigma(\boldsymbol{y}) \\
&= u(\boldsymbol{x}) \int_{\partial B_\varepsilon(\boldsymbol{x})} \frac{\partial}{\partial \boldsymbol{n_y}} N(\boldsymbol{x},\boldsymbol{y}) d\sigma(\boldsymbol{y}) \\
&\qquad\qquad + \int_{\partial B_\varepsilon(\boldsymbol{x})} \left[u(\boldsymbol{y}) - u(\boldsymbol{x})\right] \frac{\partial}{\partial \boldsymbol{n_y}} N(\boldsymbol{x},\boldsymbol{y}) d\sigma(\boldsymbol{y}),
\end{aligned}$$

which tends to $u(\boldsymbol{x})$ as $\varepsilon \to 0$. Moreover, we infer

$$\begin{aligned}
\left|I_{32}\right| &\leqslant C \sup_{\boldsymbol{y}\in\partial B_\varepsilon(\boldsymbol{x})} \left|\frac{\partial u}{\partial \boldsymbol{n_y}}\right| \int_{\partial B_\varepsilon(\boldsymbol{x})} \left|N(\boldsymbol{x},\boldsymbol{y})\right| d\sigma(\boldsymbol{y}) \\
&\leqslant C' \sup_{\boldsymbol{y}\in\partial B_\varepsilon(\boldsymbol{x})} \left|\frac{\partial u}{\partial \boldsymbol{n_y}}\right| \left[\int_{\partial B_\varepsilon(\boldsymbol{x})} \frac{1}{\varepsilon^{d-2}} d\sigma(\boldsymbol{y}) + \int_{\partial B_\varepsilon(\boldsymbol{x})} \frac{1}{|\widetilde{\boldsymbol{x}} - \boldsymbol{y}|^{d-2}}\, d\sigma(\boldsymbol{y})\right].
\end{aligned}$$

Observing that both the integrals tend to zero when ε goes to zero because the second one has a continuous kernel while the first one behaves as $O(\varepsilon)$, we infer that $I_{32} \to 0$ as $\varepsilon \to 0$. Putting together all the results, we obtain (2.29). $\square$

2.3 Spectral analysis

Following the approach of Ammari and Kang, see [7, 8], in this section, we prove the invertibility of the operator $\frac{1}{2}I + K_C + \widetilde{D}_C$ showing that, under suitable assumptions, the following inclusion holds

$$\sigma(K_C + \widetilde{D}_C) \subset (-1/2, 1/2]\,.$$

Such task is accomplished by determining the spectrum of the adjoint operator $K_C^* + \widetilde{D}_C^*$ in $L^2(\partial C)$, relying on the fact that the two spectra are conjugate.

The explicit expression of K_C^* is in (2.9). Computing the L^2-adjoint of $\widetilde{D}_C$ is straightforward: indeed, given $\psi \in L^2(\partial C)$, we have

$$\begin{aligned}
\int_{\partial C} \psi(\boldsymbol{x}) \widetilde{D}_C \varphi(\boldsymbol{x}) d\sigma(\boldsymbol{x}) &= \int_{\partial C} \psi(\boldsymbol{x}) \left(\frac{1}{\omega_d} \int_{\partial C} \frac{(\boldsymbol{y} - \widetilde{\boldsymbol{x}}) \cdot \boldsymbol{n_y}}{|\widetilde{\boldsymbol{x}} - \boldsymbol{y}|^d} \varphi(\boldsymbol{y}) d\sigma(\boldsymbol{y})\right) d\sigma(\boldsymbol{x}) \\
&= \int_{\partial C} \varphi(\boldsymbol{y}) \left(\frac{1}{\omega_d} \int_{\partial C} \frac{(\boldsymbol{y} - \widetilde{\boldsymbol{x}}) \cdot \boldsymbol{n_y}}{|\widetilde{\boldsymbol{x}} - \boldsymbol{y}|^d} \psi(\boldsymbol{x})\, d\sigma(\boldsymbol{x})\right) d\sigma(\boldsymbol{y})
\end{aligned}$$

and thus

$$\widetilde{D}_C^*\varphi(\boldsymbol{x}) = \frac{1}{\omega_d}\int_{\partial C} \frac{(\boldsymbol{x}-\widetilde{\boldsymbol{y}})\cdot \boldsymbol{n}_{\boldsymbol{x}}}{|\widetilde{\boldsymbol{y}}-\boldsymbol{x}|^d}\varphi(\boldsymbol{y})d\sigma(\boldsymbol{y}). \tag{2.30}$$

Note that the kernel of the integral operator $\widetilde{D}_C^*$ is smooth on ∂C.

As recalled in Theorem 2.1.5 and Theorem 2.1.8 the eigenvalues of K_C^* on $L^2(\partial C)$ lie in $(-1/2, 1/2]$. With the same approach, it can be shown that the same property holds true for $K_C^* + \widetilde{D}_C^*$.

Theorem 2.3.1. *Let C be an open bounded domain with Lipschitz boundary. Then*

$$\sigma(K_C^* + \widetilde{D}_C^*) \subset (-1/2, 1/2]\,.$$

For completeness, we provide here a complete proof of such fact.

Firstly, we observe that the regular operator $\widetilde{D}_C^*$ on the boundary of the cavity can be seen as the normal derivative of an appropriate single layer potential.

Lemma 2.3.2. *Given $\varphi \in L^2(\partial C)$ we have that*

$$\widetilde{D}_C^*\varphi(\boldsymbol{x}) = \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{x}}}\left(S_{\widetilde{C}}\widetilde{\varphi}(\boldsymbol{x})\right), \quad \boldsymbol{x} \in \partial C,$$

where $\widetilde{\varphi} \in L^2(\partial\widetilde{C})$ is defined by $\widetilde{\varphi}(\boldsymbol{x}) := \varphi(\widetilde{\boldsymbol{x}})$.

Proof. Using the expression (2.30) of $\widetilde{D}_C^*$ and the identity

$$\nabla_{\boldsymbol{x}}\left(\frac{1}{(2-d)|\boldsymbol{x}-\boldsymbol{y}|^{d-2}}\right) = \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^d},$$

we find that

$$\widetilde{D}_C^*\varphi(\boldsymbol{x}) = \nabla_{\boldsymbol{x}}\left(\int_{\partial C}\frac{\kappa_d\,\varphi(\boldsymbol{y})}{|\widetilde{\boldsymbol{y}}-\boldsymbol{x}|^{d-2}}\,d\sigma(\boldsymbol{y})\right)\cdot \boldsymbol{n}_{\boldsymbol{x}},$$

where $\kappa_d = 1/\omega_d(2-d)$. Given $\varphi \in L^2(\partial C)$ and $\widetilde{\varphi} \in L^2(\partial\widetilde{C})$ as previously defined, we have

$$\begin{aligned}\int_{\partial C}\frac{\varphi(\boldsymbol{y})}{|\widetilde{\boldsymbol{y}}-\boldsymbol{x}|^{d-2}}, d\sigma(\boldsymbol{y}) &= \int_{\partial\widetilde{C}}\frac{\varphi(\widetilde{\boldsymbol{z}})}{|\widetilde{\widetilde{\boldsymbol{z}}}-\boldsymbol{x}|^{d-2}}\,d\sigma(\boldsymbol{z})\\ &= \int_{\partial\widetilde{C}}\frac{\varphi(\widetilde{\boldsymbol{z}})}{|\boldsymbol{z}-\boldsymbol{x}|^{d-2}}\,d\sigma(\boldsymbol{z}) = \int_{\partial\widetilde{C}}\frac{\widetilde{\varphi}(\boldsymbol{z})}{|\boldsymbol{z}-\boldsymbol{x}|^{d-2}}\,d\sigma(\boldsymbol{z}),\end{aligned}$$

which gives the conclusion. $\square$

We are now ready to prove the main result of this section.

Proof of Theorem (2.3.1). Given $\varphi \in L^2(\partial C)$, let ψ be defined by $\psi := S_C\varphi + S_{\widetilde{C}}\widetilde{\varphi}$. By the known properties of single layer potentials, we derive on ∂C

$$\frac{\partial\psi}{\partial\boldsymbol{n}}\bigg|_{\pm} = \left(\pm\tfrac{1}{2}I + K_C^* + \widetilde{D}_C^*\right)\varphi$$

and, as a consequence,

$$\frac{\partial\psi}{\partial\boldsymbol{n}}\bigg|_{+} + \frac{\partial\psi}{\partial\boldsymbol{n}}\bigg|_{-} = 2\left(K_C^* + \widetilde{D}_C^*\right)\varphi, \qquad \frac{\partial\psi}{\partial\boldsymbol{n}}\bigg|_{+} - \frac{\partial\psi}{\partial\boldsymbol{n}}\bigg|_{-} = \varphi. \tag{2.31}$$

Taking a linear combination of the two relations in (2.31), we deduce

$$\begin{aligned}\left(\lambda I - K_C^* - \widetilde{D}_C^*\right)\varphi &= \lambda\left(\frac{\partial\psi}{\partial\boldsymbol{n}}\bigg|_{+} - \frac{\partial\psi}{\partial\boldsymbol{n}}\bigg|_{-}\right) - \frac{1}{2}\left(\frac{\partial\psi}{\partial\boldsymbol{n}}\bigg|_{+} + \frac{\partial\psi}{\partial\boldsymbol{n}}\bigg|_{-}\right)\\ &= \left(\lambda - \frac{1}{2}\right)\frac{\partial\psi}{\partial\boldsymbol{n}}\bigg|_{+} - \left(\lambda + \frac{1}{2}\right)\frac{\partial\psi}{\partial\boldsymbol{n}}\bigg|_{-}.\end{aligned}$$

If λ is an eigenvalue of $K_C^* + \widetilde{D}_C^*$ with eigenfunction φ, then

$$\left(\lambda - \frac{1}{2}\right)\frac{\partial\psi}{\partial\boldsymbol{n}}\bigg|_{+} - \left(\lambda + \frac{1}{2}\right)\frac{\partial\psi}{\partial\boldsymbol{n}}\bigg|_{-} = 0, \qquad \text{on} \quad \partial C.$$

Multiplying such relation by the function ψ and integrating over ∂C, we get

$$\left(\lambda - \frac{1}{2}\right)\int_{\partial C}\psi(\boldsymbol{x})\frac{\partial\psi}{\partial\boldsymbol{n}}(\boldsymbol{x})\bigg|_{+} d\sigma(\boldsymbol{x}) - \left(\lambda + \frac{1}{2}\right)\int_{\partial C}\psi(\boldsymbol{x})\frac{\partial\psi}{\partial\boldsymbol{n}}(\boldsymbol{x})\bigg|_{-} d\sigma(\boldsymbol{x}) = 0. \tag{2.32}$$

Integrating by parts we have

$$\begin{aligned}\int_{\partial C}\psi(\boldsymbol{x})\frac{\partial\psi}{\partial\boldsymbol{n}}(\boldsymbol{x})\bigg|_{-} d\sigma(\boldsymbol{x}) &= \int_C \psi(\boldsymbol{x})\Delta\psi(\boldsymbol{x})\,d\boldsymbol{x} + \int_C \left|\nabla\psi(\boldsymbol{x})\right|^2 d\boldsymbol{x}\\ &= \int_C \left|\nabla\psi(\boldsymbol{x})\right|^2 d\boldsymbol{x}.\end{aligned} \tag{2.33}$$

The first integral in (2.32) can be dealt with as done in the proof of Theorem 2.2.2. Precisely, given large $R > 0$, applying the Green's formula in $\Omega_R :=$

$\left(\mathbb{R}^d_- \cap B_R(\mathbf{0})\right) \setminus C$, we get

$$\begin{aligned}\int_{\partial C} \psi(\boldsymbol{x}) \frac{\partial \psi}{\partial \boldsymbol{n}}(\boldsymbol{x})\bigg|_+ d\sigma(\boldsymbol{x})&\\ = \int_{\partial B^h_R(\mathbf{0})} \psi(\boldsymbol{x}) \frac{\partial \psi}{\partial x_d}(\boldsymbol{x})\, d\sigma(\boldsymbol{x}) &+ \int_{\partial B^b_R(\mathbf{0})} \psi(\boldsymbol{x}) \frac{\partial \psi}{\partial \boldsymbol{n}}(\boldsymbol{x})\bigg|_+ d\sigma(\boldsymbol{x})\\ &- \int_{\Omega_R} \psi(\boldsymbol{x}) \Delta \psi(\boldsymbol{x})\, d\boldsymbol{x} - \int_{\Omega_R} \left|\nabla \psi(\boldsymbol{x})\right|^2 d\boldsymbol{x},\end{aligned}$$

where $\partial B^h_R(\mathbf{0})$ is the intersection of the hemisphere with the half-space and $\partial B^b_R(\mathbf{0})$ is the spherical cap. The quantity $\partial\psi/\partial\boldsymbol{n}$ is identically zero on the boundary of the half-space since the kernel of the operator is the normal derivative of the Neumann function which, by hypothesis, is null on $\mathbb{R}^{d-1}$. Moreover, ψ is harmonic in Ω_R, so we infer

$$\int_{\partial C} \psi(\boldsymbol{x}) \frac{\partial \psi}{\partial \boldsymbol{n}}(\boldsymbol{x})\bigg|_+ d\sigma(\boldsymbol{x}) = \int_{\partial B^b_R(\mathbf{0})} \psi(\boldsymbol{x}) \frac{\partial \psi}{\partial \boldsymbol{n}}(\boldsymbol{x})\bigg|_+ d\sigma(\boldsymbol{x}) - \int_{\Omega_R} \left|\nabla \psi(\boldsymbol{x})\right|^2 d\boldsymbol{x}.$$

Recalling the asymptotic behaviour of simple layer potential,

$$\left|S_C\varphi\right| + \left|S_{\widetilde{C}}\varphi\right| = O(|\boldsymbol{x}|^{2-d}), \qquad \left|\nabla S_C\varphi\right| + \left|\nabla S_{\widetilde{C}}\varphi\right| = O(|\boldsymbol{x}|^{1-d}) \quad \text{as } |\boldsymbol{x}| \to \infty.$$

we obtain, for some $C > 0$,

$$\begin{aligned}\left| \int_{\partial B^b_R(\mathbf{0})} \psi(\boldsymbol{x}) \frac{\partial \psi}{\partial \boldsymbol{n}}(\boldsymbol{x})\bigg|_+ d\sigma(\boldsymbol{x}) \right| &\leqslant \int_{\partial B^b_R(\mathbf{0})} \left|\psi(\boldsymbol{x})\right| \left| \frac{\partial \psi}{\partial \boldsymbol{n}}(\boldsymbol{x})\bigg|_+ \right| d\sigma(\boldsymbol{x})\\ &\leqslant \frac{C}{R^{2d-3}} \int_{\partial B^b_R(\mathbf{0})} d\sigma(\boldsymbol{x}) = \frac{1}{R^{d-2}}.\end{aligned}$$

Passing to the limit $R \to +\infty$, we find

$$\int_{\partial C} \psi(\boldsymbol{x}) \frac{\partial \psi}{\partial \boldsymbol{n}}\bigg|_+ d\sigma(\boldsymbol{x}) = -\int_{\mathbb{R}^d_- \setminus \overline{C}} \left|\nabla \psi(\boldsymbol{x})\right|^2 d\boldsymbol{x}. \tag{2.34}$$

Plugging (2.33) and (2.34) into (2.32), we infer the identity

$$\left(\lambda - \frac{1}{2}\right) \int_{\mathbb{R}^d_- \setminus C} \left|\nabla \psi(\boldsymbol{x})\right|^2 d\boldsymbol{x} + \left(\lambda + \frac{1}{2}\right) \int_C \left|\nabla \psi(\boldsymbol{x})\right|^2 d\boldsymbol{x} = 0,$$

that is

$$(A+B)\lambda = \frac{1}{2}(A-B)$$

with

$$A := \int_{\mathbb{R}^d_-\setminus C} \left|\nabla\psi(\boldsymbol{x})\right|^2 d\boldsymbol{x} \qquad \text{and} \qquad B := \int_C \left|\nabla\psi(\boldsymbol{x})\right|^2 d\boldsymbol{x}.$$

The coefficient of λ is non-zero. On the contrary, if $A+B=0$ then $\nabla\psi = 0$ in $\mathbb{R}^d_-$ which means that $\psi \equiv 0$, hence, from the second equation in (2.31), we get $\varphi = 0$ in ∂C.

Therefore, solving with respect to λ, we finally get

$$\lambda = \frac{1}{2}\cdot\frac{A-B}{A+B} \in \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{2.35}$$

The value $\lambda = -1/2$ is not an eigenvalue for the operator $K_C^* + \widetilde{D}_C^*$. Indeed, in such a case, we would have

$$A = \int_{\mathbb{R}^d_-\setminus C} \left|\nabla\psi(\boldsymbol{x})\right|^2 d\boldsymbol{x} = 0,$$

and thus $\psi = 0$ in $\mathbb{R}^d_-\setminus C$. By definition of ψ, we deduce that $\psi = 0$ on ∂C and since ψ is harmonic in C, we get that $\psi = 0$ also in C. As before, by (2.31), this would imply that $\varphi = 0$ in ∂C. □

For completeness, let us observe that the value $\lambda = 1/2$ is an eigenvalue with geometric multiplicity equal to one. Indeed, identity (2.35) implies that, for such value of λ,

$$B = \int_C \left|\nabla\psi(\boldsymbol{x})\right|^2 d\boldsymbol{x} = 0,$$

hence ψ is constant in C. Normalizing $\psi = 1$ in C, the function ψ in $\mathbb{R}^d_-\setminus C$ is given by the restriction of the solution U to the Dirichlet problem in the exterior domain $\mathbb{R}^d\setminus\left(C \cup \widetilde{C}\right)$ with boundary data equal to 1. Then, by the second equation in (2.31), the function φ is the normal derivative of U at ∂C.

2.4 Asymptotic expansion

In this section, we derive an asymptotic formula for the solution to the problem (2.20) when the cavity C is small compared to the distance from the half-space $\mathbb{R}^{d-1}$. For the reader's convenience, we recall that the cavity C has the structure

$$C_\varepsilon := C = \boldsymbol{z} + \varepsilon\Omega$$

where Ω is a bounded Lipschitz set containing the origin. Moreover, we assume that

$$\operatorname{dist}(\boldsymbol{z}, \mathbb{R}^{d-1}) \geqslant \delta_0 > 0 \tag{2.36}$$

otherwise, for the application we have in mind, the problem does not have a real physical meaning. To emphasize the dependence of the solution to the direct problem by the parameter ε we denote it by u_ε. For brevity, we denote the layer potentials relative to C_ε by the index ε, *viz.*

$$S_\varepsilon = S_{C_\varepsilon}, \quad D_\varepsilon = D_{C_\varepsilon}, \quad \widetilde{S}_\varepsilon = \widetilde{S}_{C_\varepsilon}, \quad \widetilde{D}_\varepsilon = \widetilde{D}_{C_\varepsilon}, \quad K_\varepsilon = K_{C_\varepsilon}$$

and the trace of the solution u_ε on ∂C_ε by f_ε. In this way the representation formula (2.28) reads as

$$u_\varepsilon = S_\varepsilon g - D_\varepsilon f_\varepsilon - \widetilde{D}_\varepsilon f_\varepsilon + \widetilde{S}_\varepsilon g.$$

At $\boldsymbol{x} \in \mathbb{R}^{d-1}$, taking into account that $\boldsymbol{x} = \widetilde{\boldsymbol{x}}$, it follows that

$$S_\varepsilon g(\boldsymbol{x}) = \int_{\partial C_\varepsilon} \Gamma(\boldsymbol{x} - \boldsymbol{y}) g(\boldsymbol{y})\, d\sigma(\boldsymbol{y}) = \int_{\partial C_\varepsilon} \Gamma(\widetilde{\boldsymbol{x}} - \boldsymbol{y}) g(\boldsymbol{y})\, d\sigma(\boldsymbol{y}) = \widetilde{S}_\varepsilon g(\boldsymbol{x})$$

and

$$\begin{aligned} D_\varepsilon f_\varepsilon(\boldsymbol{x}) = \int_{\partial C_\varepsilon} \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{y}}} \Gamma(\boldsymbol{x} - \boldsymbol{y}) f_\varepsilon(\boldsymbol{y})\, d\sigma(\boldsymbol{y}) &= \int_{\partial C_\varepsilon} \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{y}}} \Gamma(\widetilde{\boldsymbol{x}} - \boldsymbol{y}) f_\varepsilon(\boldsymbol{y})\, d\sigma(\boldsymbol{y}) \\ &= \widetilde{D}_\varepsilon f_\varepsilon(\boldsymbol{x}) \end{aligned}$$

Hence, we obtain the equality

$$\tfrac{1}{2}\, u_\varepsilon(\boldsymbol{x}) = S_\varepsilon g(\boldsymbol{x}) - D_\varepsilon f_\varepsilon(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^{d-1}.$$

Associating with the relation at the boundary ∂C_ε

$$\left(\tfrac{1}{2} I + K_\varepsilon + \widetilde{D}_\varepsilon\right) f_\varepsilon(\boldsymbol{x}) = S_\varepsilon g(\boldsymbol{x}) + \widetilde{S}_\varepsilon g(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial C_\varepsilon, \tag{2.37}$$

we get the identity

$$\tfrac{1}{2}u_\varepsilon(\boldsymbol{x}) = J_1(\boldsymbol{x}) + J_2(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^{d-1}, \tag{2.38}$$

where

$$J_1(\boldsymbol{x}) := \int_{\partial C_\varepsilon} \Gamma(\boldsymbol{x}-\boldsymbol{y})g(\boldsymbol{y})\, d\sigma(\boldsymbol{y}),$$

$$J_2(\boldsymbol{x}) := -\int_{\partial C_\varepsilon} \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\Gamma(\boldsymbol{x}-\boldsymbol{y}) \left(\tfrac{1}{2}I + K_\varepsilon + \widetilde{D}_\varepsilon\right)^{-1} \left(S_\varepsilon g + \widetilde{S}_\varepsilon g\right)(\boldsymbol{y})\, d\sigma(\boldsymbol{y}).$$

Analyzing in details the dependence with respect to ε of such relation, we obtain an explicit expression for the first two terms in the asymptotic expansion of u_ε at $\mathbb{R}^{d-1}$.

In what follows, for any fixed value of $\varepsilon > 0$, given $h : \partial C_\varepsilon \to \mathbb{R}$, we introduce the function $h^\sharp : \partial\Omega \to \mathbb{R}$ defined by

$$h^\sharp(\boldsymbol{\zeta};\varepsilon) := h(\boldsymbol{z} + \varepsilon\,\boldsymbol{\zeta}), \qquad \boldsymbol{\zeta} \in \partial\Omega.$$

This definition is useful to consider integrals over a set that is independent on ε.

Theorem 2.4.1. *Let us assume (2.36). There exists ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ and $g \in L^2(\partial C_\varepsilon)$ such that $g^\sharp$ is independent on ε, at any $\boldsymbol{x} \in \mathbb{R}^{d-1}$ the following expansion holds*

$$u_\varepsilon(\boldsymbol{x}) = 2\varepsilon^{d-1}\Gamma(\boldsymbol{x}-\boldsymbol{z}) \int_{\partial\Omega} g^\sharp(\boldsymbol{\zeta})\, d\sigma(\boldsymbol{\zeta})$$
$$+ 2\varepsilon^d \nabla\Gamma(\boldsymbol{x}-\boldsymbol{z}) \cdot \int_{\partial\Omega} \left\{ \boldsymbol{n}_{\boldsymbol{\zeta}} \left(\tfrac{1}{2}I + K_\Omega\right)^{-1} S_\Omega g^\sharp(\boldsymbol{\zeta}) - \boldsymbol{\zeta} g^\sharp(\boldsymbol{\zeta}) \right\} d\sigma(\boldsymbol{\zeta}) + O(\varepsilon^{d+1}), \tag{2.39}$$

where $O(\varepsilon^{d+1})$ denotes a quantity uniformly bounded by $C\varepsilon^{d+1}$ with $C = C(\delta_0)$ which tends to infinity when δ_0 goes to zero.

To prove this theorem we first show the following expansion for the operator $\left(\tfrac{1}{2}I + K_\varepsilon + \widetilde{D}_\varepsilon\right)^{-1}$.

Lemma 2.4.2. *We have*

$$\left(\tfrac{1}{2}I + K_\varepsilon + \widetilde{D}_\varepsilon\right)^{-1} \left(S_\varepsilon g + \widetilde{S}_\varepsilon g\right)(\boldsymbol{z}+\varepsilon\boldsymbol{\zeta}) = \varepsilon \left(\tfrac{1}{2}I + K_\Omega\right)^{-1} S_\Omega g^\sharp(\boldsymbol{\zeta}) + O(\varepsilon^{d-1}) \tag{2.40}$$

for ε sufficiently small.

Proof. We analyse, separetely, the terms $\left(\frac{1}{2}I + K_\varepsilon + \widetilde{D}_\varepsilon\right)$ and $S_\varepsilon + \widetilde{S}_\varepsilon$, collecting, at the very end, the corresponding expansions.

At the point $\boldsymbol{z} + \varepsilon\boldsymbol{\zeta}$, where $\boldsymbol{\zeta} \in \partial\Omega$, we obtain

$$\begin{aligned} K_\varepsilon\varphi(\boldsymbol{z} + \varepsilon\boldsymbol{\zeta}) &= \frac{1}{\omega_d}\text{p.v.}\int_{\partial C_\varepsilon} \frac{(\boldsymbol{y} - \boldsymbol{z} - \varepsilon\boldsymbol{\zeta}) \cdot \boldsymbol{n}_{\boldsymbol{y}}}{|\boldsymbol{z} + \varepsilon\boldsymbol{\zeta} - \boldsymbol{y}|^d}\varphi(\boldsymbol{y})\, d\sigma(\boldsymbol{y}) \\ &= \frac{1}{\omega_d}\text{p.v.}\int_{\partial\Omega} \frac{(\boldsymbol{\eta} - \boldsymbol{\zeta}) \cdot \boldsymbol{n}_{\boldsymbol{\eta}}}{|\boldsymbol{\zeta} - \boldsymbol{\eta}|^d}\varphi^\sharp(\boldsymbol{\eta})\, d\sigma(\boldsymbol{\eta}) = K_\Omega\varphi^\sharp(\boldsymbol{\zeta}), \end{aligned}$$

and

$$\begin{aligned} \widetilde{D}_\varepsilon\varphi(\boldsymbol{z} + \varepsilon\boldsymbol{\zeta}) &= \int_{\partial C_\varepsilon} \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\Gamma(\widetilde{\boldsymbol{z}} + \varepsilon\widetilde{\boldsymbol{\zeta}} - \boldsymbol{y})\varphi(\boldsymbol{y})\, d\sigma(\boldsymbol{y}) \\ &= \varepsilon^{d-1}\int_{\partial\Omega} \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{\eta}}}\Gamma(\widetilde{\boldsymbol{z}} + \varepsilon\widetilde{\boldsymbol{\zeta}} - \boldsymbol{z} - \varepsilon\boldsymbol{\eta})\varphi^\sharp(\boldsymbol{\eta})\, d\sigma(\boldsymbol{\eta}) = \varepsilon^{d-1}R_\varepsilon\varphi^\sharp(\boldsymbol{\zeta}) \end{aligned}$$

where

$$R_\varepsilon\varphi^\sharp(\boldsymbol{\zeta}) := \int_{\partial\Omega} \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{\eta}}}\Gamma\left(\widetilde{\boldsymbol{z}} - \boldsymbol{z} + \varepsilon(\widetilde{\boldsymbol{\zeta}} - \boldsymbol{\eta})\right)\varphi^\sharp(\boldsymbol{\eta})\, d\sigma(\boldsymbol{\eta})$$

is uniformly bounded in ε.

Let us evaluate the term $S_\varepsilon + \widetilde{S}_\varepsilon$. We have

$$\begin{aligned} S_\varepsilon g(\boldsymbol{z} + \varepsilon\boldsymbol{\zeta}) &= \int_{\partial C_\varepsilon} \Gamma(\boldsymbol{z} + \varepsilon\boldsymbol{\zeta} - \boldsymbol{y})g(\boldsymbol{y})d\sigma(\boldsymbol{y}) \\ &= \varepsilon\int_{\partial\Omega} \Gamma(\boldsymbol{\zeta} - \boldsymbol{\theta})g^\sharp(\boldsymbol{\theta})d\sigma(\boldsymbol{\theta}) = \varepsilon S_\Omega g^\sharp(\boldsymbol{\zeta}) \end{aligned}$$

and

$$\begin{aligned} \widetilde{S}_\varepsilon g(\boldsymbol{z} + \varepsilon\boldsymbol{\zeta}) &= \int_{\partial C_\varepsilon} \Gamma\left(\widetilde{\boldsymbol{z}} + \varepsilon\widetilde{\boldsymbol{\zeta}} - \boldsymbol{y}\right)g(\boldsymbol{y})d\sigma(\boldsymbol{y}) \\ &= \varepsilon^{d-1}\int_{\partial\Omega} \Gamma\left(\widetilde{\boldsymbol{z}} - \boldsymbol{z} + \varepsilon(\widetilde{\boldsymbol{\zeta}} - \boldsymbol{\theta})\right)g^\sharp(\boldsymbol{\theta})d\sigma(\boldsymbol{\theta}) \\ &= \varepsilon^{d-1}\Gamma(\widetilde{\boldsymbol{z}} - \boldsymbol{z})\int_{\partial\Omega} g^\sharp(\boldsymbol{\theta})d\sigma(\boldsymbol{\theta}) + O(\varepsilon^d) \end{aligned}$$

where we have used the zero order expansion for Γ. Collecting we infer

$$\left(S_\varepsilon g + \widetilde{S}_\varepsilon g\right)(\boldsymbol{z} + \varepsilon\boldsymbol{\zeta}) = \varepsilon S_\Omega g^\sharp(\boldsymbol{\zeta}) + O(\varepsilon^{d-1})$$

To conclude, from (2.37) we have

$$\Big(\frac{1}{2}I + K_\Omega\Big)\Big(I + \varepsilon^{d-1}\Big(\frac{1}{2}I + K_\Omega\Big)^{-1} R_\varepsilon\Big) f^\sharp = \varepsilon S_\Omega g^\sharp(\boldsymbol{\zeta}) + O(\varepsilon^{d-1}).$$

From the continuous property of R_ε and the invertibility result of the operator $1/2I + K_\Omega$ as explained in Remark 2.1.9, we have

$$\Big\| \Big(\frac{1}{2}I + K_\Omega\Big)^{-1} R_\varepsilon \Big\| \leqslant C,$$

where $C > 0$ is independent from ε. On the other hand, choosing $\varepsilon_0^{d-1} = 1/2C$, it follows that for all $\varepsilon \in (0, \varepsilon_0)$ we have that $I + \varepsilon^{d-1}\Big(\frac{1}{2}I + K_\Omega\Big)^{-1} R_\varepsilon$ is invertible and

$$\Big(I + \varepsilon^{d-1}\Big(\frac{1}{2}I + K_\Omega\Big)^{-1} R_\varepsilon\Big)^{-1} = I + O(\varepsilon^{d-1}).$$

Therefore

$$f^\sharp = \varepsilon\Big(\frac{1}{2}I + K_\Omega\Big)^{-1} S_\Omega g^\sharp(\boldsymbol{\zeta}) + O(\varepsilon^{d-1}).$$

□

Remark 2.4.3. *If the domain C_ε is more regular, at least a C^1-domain, we have compactness of the operators K_ε and K_ε^*. Thefore we can prove the asymptotic expansion of the operator $\left(\frac{1}{2}I + K_\varepsilon + \widetilde{D}_\varepsilon\right)^{-1}$ in an alternative way. In fact, since $K_\varepsilon + \widetilde{D}_\varepsilon$ is compact and its spectrum is contained in $(-1/2, 1/2]$, there exists $\delta > 0$ such that*

$$\sigma\left(K_\varepsilon + \widetilde{D}_\varepsilon\right) \subset (-1/2 + \delta, 1/2].$$

Then, the operator

$$A_\varepsilon := \frac{1}{2}I - K_\varepsilon - \widetilde{D}_\varepsilon$$

is such that $\sigma(A_\varepsilon) \subset [0, 1-\delta)$ and thus has spectral radius strictly smaller than 1. As a consequence, taking the powers of the operator A_ε one finds

$$\|A_\varepsilon^h\| \leqslant 1 \quad \forall h \quad \textit{and} \quad \|A_\varepsilon^{h_0}\| < 1 \quad \textit{for some } h_0. \tag{2.41}$$

The inverse operator of $I - A_\varepsilon = \frac{1}{2}I + K_\varepsilon + \widetilde{D}_\varepsilon$ can be represented by the Neumann series that is

$$(I - A_\varepsilon)^{-1} = \sum_{h=0}^{+\infty} A_\varepsilon^h = \sum_{h=0}^{+\infty} \left(\frac{1}{2}I - K_\varepsilon - \widetilde{D}_\varepsilon\right)^h.$$

Taking R_ε of the proof of Lemma (2.4.2), we calculate A_ε^h highlighting the term that do not contain ε and the one of order $d-1$, that is

$$A_\varepsilon^h = \left(\frac{1}{2}I - K_\Omega\right)^h - \varepsilon^{d-1} E_{h,\varepsilon}$$

where

$$E_{h,\varepsilon} = \sum_{j=1}^{h} A_\varepsilon \cdots A_\varepsilon \underbrace{R_\varepsilon}_{j-th} A_\varepsilon \cdots A_\varepsilon.$$

For h_0 as in (2.41) and $h > h_0$ we have

$$\|E_{h,\varepsilon}\| \leqslant \|R_\varepsilon\| \|A_\varepsilon\|^{2h_0} \|A_\varepsilon^{h_0}\|^{[h/h_0]-1} \leqslant \|R_\varepsilon\| \|A_\varepsilon\|^{2h_0} \|A_\varepsilon^{h_0}\|^{h/h_0 - 1},$$

where $[\,\cdot\,]$ denotes the integer part, and thus

$$\sum_{h=0}^{+\infty} \|E_{h,\varepsilon}\| \leqslant C \sum_{h=0}^{+\infty} \|A_\varepsilon^{h_0}\|^{h/h_0}$$

giving the absolute convergence of $\sum E_{h,\varepsilon}$. Summarizing we conclude that

$$(I - A_\varepsilon)^{-1} = \left(\frac{1}{2}I + K_\Omega\right)^{-1} + O(\varepsilon^{d-1}). \tag{2.42}$$

Proof of Theorem 2.4.1. To prove (2.39), we analyse the two integrals J_1 and J_2 in (2.38).

For $\boldsymbol{x}, \boldsymbol{\zeta} \in \mathbb{R}^d$ with $\boldsymbol{x} \neq \boldsymbol{0}$ and ε sufficiently small, we have

$$\Gamma(\boldsymbol{x} - \varepsilon\boldsymbol{\zeta}) = \Gamma(\boldsymbol{x}) - \varepsilon \nabla\Gamma(\boldsymbol{x}) \cdot \boldsymbol{\zeta} + O(\varepsilon^2).$$

Hence, for $\boldsymbol{x} \in \mathbb{R}^{d-1}$, we get

$$\begin{aligned} J_1 &= \varepsilon^{d-1} \int_{\partial\Omega} \Gamma(\boldsymbol{x} - \boldsymbol{z} - \varepsilon\boldsymbol{\zeta})\, g^\sharp(\boldsymbol{\zeta})\, d\sigma(\boldsymbol{\zeta}) \\ &= \varepsilon^{d-1} \Gamma(\boldsymbol{x} - \boldsymbol{z}) \int_{\partial\Omega} g^\sharp(\boldsymbol{\zeta})\, d\sigma(\boldsymbol{\zeta}) \\ &\qquad - \varepsilon^d \nabla\Gamma(\boldsymbol{x} - \boldsymbol{z}) \cdot \int_{\partial\Omega} \boldsymbol{\zeta}\, g^\sharp(\boldsymbol{\zeta})\, d\sigma(\boldsymbol{\zeta}) + O(\varepsilon^{d+1}). \end{aligned} \tag{2.43}$$

Next we consider the second integral in (2.38), written as

$$J_2 = -\varepsilon^{d-1} \int_{\partial\Omega} \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{\zeta}}} \Gamma(\boldsymbol{x} - \boldsymbol{z} - \varepsilon\boldsymbol{\zeta}) h_\varepsilon^\sharp(\boldsymbol{\zeta})\, d\sigma(\boldsymbol{\zeta}),$$

where the function $h_\varepsilon^\sharp$ is given by

$$h_\varepsilon^\sharp(\boldsymbol{\zeta}) = \left(\frac{1}{2} I + K_\varepsilon + \widetilde{D}_\varepsilon\right)^{-1} \left(S_\varepsilon g + \widetilde{S}_\varepsilon g\right)(\boldsymbol{z} + \varepsilon\boldsymbol{\zeta}) \tag{2.44}$$

For $\boldsymbol{x}, \boldsymbol{\zeta} \in \mathbb{R}^d$ with $\boldsymbol{x} \neq \boldsymbol{0}$ and ε sufficiently small, it holds

$$\nabla_{\boldsymbol{x}} \Gamma(\boldsymbol{x} + \varepsilon\boldsymbol{\zeta}) = \nabla_{\boldsymbol{x}} \Gamma(\boldsymbol{x}) + O(\varepsilon), \tag{2.45}$$

therefore, taking advantage of the expansion (2.40),

$$\begin{aligned} J_2 &= \varepsilon^{d-1} \int_{\partial\Omega} \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{\zeta}}} \Gamma(\boldsymbol{x} - \boldsymbol{z}) h_\varepsilon^\sharp(\boldsymbol{\zeta})\, d\sigma(\boldsymbol{\zeta}) + O(\varepsilon^d) \\ &= \varepsilon^d \int_{\partial\Omega} \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{\zeta}}} \Gamma(\boldsymbol{x} - \boldsymbol{z}) \left(\tfrac{1}{2} I + K_\Omega\right)^{-1} S_\Omega g^\sharp(\boldsymbol{\zeta})\, d\sigma(\boldsymbol{\zeta}) + O(\varepsilon^{d+1}). \end{aligned}$$

Collecting the expansions for J_1 and J_2, we deduce (2.39). □

We show that the term $\left(\tfrac{1}{2} I + K_\Omega\right)^{-1} S_\Omega g(\boldsymbol{x})$, for $\boldsymbol{x} \in \partial\Omega$, represents the trace of the solution of the external domain related to the set Ω and with Neumann boundary condition given by g. To this aim, we consider the problem

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}^d \backslash \Omega \\ \dfrac{\partial U}{\partial \boldsymbol{n}} = g & \text{on } \partial\Omega \\ U \to 0 & \text{as } |\boldsymbol{x}| \to +\infty, \end{cases} \tag{2.46}$$

where the cavity Ω is such that $\boldsymbol{0} \in \Omega$.

Proposition 2.4.4. *Let us define $h(\boldsymbol{x}) := U(\boldsymbol{x})\big|_{\boldsymbol{x}\in\partial\Omega}$, then*

$$\left(\frac{1}{2} I + K_\Omega\right)^{-1} S_\Omega g(\boldsymbol{x}) = h(\boldsymbol{x}).$$

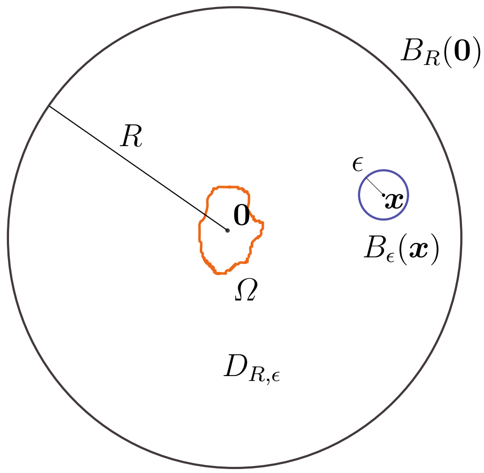


Figure 2.3. Domain used to get the representation formula for U.

Proof. The thesis comes from, as done in the proof of Theorem 2.2.2, by the application of the second Green's identity to the fundamental solution Γ and U in the domain $B_R(\mathbf{0})\backslash\Omega$, with R sufficiently large. We define $D_{R,\epsilon} := B_R(\mathbf{0})\backslash(\Omega \cup B_\epsilon(\boldsymbol{x}))$, with $\boldsymbol{x} \in (B_R(\mathbf{0})\backslash\Omega)$ (see Figure 2.3). By the second Green's identity, we get

$$\begin{aligned}
0 &= \int_{\partial D_{R,\epsilon}} \Big[U(\boldsymbol{y})\frac{\partial}{\partial \boldsymbol{n_y}}\Gamma(\boldsymbol{x}-\boldsymbol{y}) - \Gamma(\boldsymbol{x}-\boldsymbol{y})\frac{\partial}{\partial \boldsymbol{n_y}}U(\boldsymbol{y})\Big]d\sigma(\boldsymbol{y}) \\
&= \int_{\partial B_R(\mathbf{0})} \Big[U(\boldsymbol{y})\frac{\partial}{\partial \boldsymbol{n_y}}\Gamma(\boldsymbol{x}-\boldsymbol{y}) - \Gamma(\boldsymbol{x}-\boldsymbol{y})\frac{\partial}{\partial \boldsymbol{n_y}}U(\boldsymbol{y})\Big]d\sigma(\boldsymbol{y}) \\
&- \int_{\partial B_\epsilon(\boldsymbol{x})} \Big[U(\boldsymbol{y})\frac{\partial}{\partial \boldsymbol{n_y}}\Gamma(\boldsymbol{x}-\boldsymbol{y}) - \Gamma(\boldsymbol{x}-\boldsymbol{y})\frac{\partial}{\partial \boldsymbol{n_y}}U(\boldsymbol{y})\Big]d\sigma(\boldsymbol{y}) \\
&- \int_{\partial C} \Big[U(\boldsymbol{y})\frac{\partial}{\partial \boldsymbol{n_y}}\Gamma(\boldsymbol{x}-\boldsymbol{y}) - \Gamma(\boldsymbol{x}-\boldsymbol{y})\frac{\partial}{\partial \boldsymbol{n_y}}U(\boldsymbol{y})\Big]d\sigma(\boldsymbol{y}) \\
&:= I_1 - I_2 - \mathcal{I}(h,g)(\boldsymbol{x})
\end{aligned}$$

Using the decay rate of harmonic functions in unbounded domains, see Theorem 2.1.2, the integral I_1 gives

$$|I_1| \leqslant \left(\frac{C_1}{R^{2d-3}} + \frac{C_2}{R^{2d-3}}\right)\int_{\partial B_R(\mathbf{0})} d\sigma(\boldsymbol{y}) = \frac{C}{R^{d-2}}$$

where C is a positive constant. As $R \to +\infty$, I_1 tends to zero.

Finally, we decompose I_2 as

$$
\begin{aligned}
I_2 &= I_{21} - I_{22} \\
&= \int_{\partial B_\epsilon(\boldsymbol{x})} U(\boldsymbol{y}) \frac{\partial}{\partial \boldsymbol{n_y}} \Gamma(\boldsymbol{x}-\boldsymbol{y})\, d\sigma(\boldsymbol{y}) - \int\limits_{\partial B_\epsilon(\boldsymbol{x})} \Gamma(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial}{\partial \boldsymbol{n_y}} U(\boldsymbol{y})\, d\sigma(\boldsymbol{y}).
\end{aligned}
$$

Using the expression of Γ and the continuity of u, we derive

$$
\begin{aligned}
I_{21} = \int_{\partial B_\epsilon(\boldsymbol{x})} U(\boldsymbol{y}) \frac{\partial}{\partial \boldsymbol{n_y}} \Gamma(\boldsymbol{x}-\boldsymbol{y})\, d\sigma(\boldsymbol{y}) &= U(\boldsymbol{x}) \int_{\partial B_\epsilon(\boldsymbol{x})} \frac{\partial}{\partial \boldsymbol{n_y}} \Gamma(\boldsymbol{x}-\boldsymbol{y})\, d\sigma(\boldsymbol{y}) \\
&+ \int_{\partial B_\epsilon(\boldsymbol{x})} (U(\boldsymbol{y}) - U(\boldsymbol{x})) \frac{\partial}{\partial \boldsymbol{n_y}} \Gamma(\boldsymbol{x}-\boldsymbol{y}) d\sigma(\boldsymbol{y}),
\end{aligned}
$$

which tends to $U(\boldsymbol{x})$ as $\varepsilon \to 0$. Moreover, it holds

$$
|I_{22}| \leqslant C' \sup_{\boldsymbol{y} \in \partial B_\epsilon(\boldsymbol{x})} \left| \frac{\partial U(\boldsymbol{y})}{\partial \boldsymbol{n_y}} \right| \frac{1}{\epsilon^{d-2}} \int_{\partial B_\epsilon(\boldsymbol{x})} d\sigma(\boldsymbol{y}) = O(\epsilon)
$$

which goes to zero as ϵ goes to zero.
In conclusion we have the following integral representation formula

$$
\begin{aligned}
U(\boldsymbol{x}) &= -\mathcal{I}(h,g)(\boldsymbol{x}) \\
&= \int_{\partial \Omega} \left[\Gamma(\boldsymbol{x}-\boldsymbol{y}) g(\boldsymbol{y}) - h(\boldsymbol{y}) \frac{\partial}{\partial \boldsymbol{n_y}} \Gamma(\boldsymbol{x}-\boldsymbol{y}) \right] d\sigma(\boldsymbol{y}) \\
&= S_\Omega g(\boldsymbol{x}) - D_\Omega h(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^d \backslash \overline{\Omega}
\end{aligned}
\tag{2.47}
$$

where h is the trace of U on the boundary of the cavity Ω. Therefore, on $\partial\Omega$ from single and double layer potentials properties

$$
h(\boldsymbol{x}) = S_\Omega g(\boldsymbol{x}) - \left(-\frac{1}{2} I + K_\Omega \right) h(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega,
$$

hence

$$
h(\boldsymbol{x}) = \left(\frac{1}{2} I + K_\Omega \right)^{-1} S_\Omega g(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega,
$$

that is the assertion. □

2.4.1 A specific Neumann condition

Now, we want to consider a specific case of the Neumann condition on the boundary of the cavity C_ε so to get an explicit expression of the asymptotic expansion in terms of the polarization tensor and the fundamental solution.

Corollary 2.4.5. *Given $\boldsymbol{p} \in \mathbb{R}^d$, let the boundary datum given by*

$$g = -\boldsymbol{p} \cdot \boldsymbol{n}.$$

Then, the following expansion holds

$$u_\varepsilon(\boldsymbol{x}) = 2\,\varepsilon^d |\Omega| \nabla \Gamma(\boldsymbol{x} - \boldsymbol{z}) \cdot \mathbf{M}\boldsymbol{p} + O(\varepsilon^{d+1}), \qquad \boldsymbol{x} \in \mathbb{R}^{d-1}, \tag{2.48}$$

where $\mathbf{M}$ is the symmetric positive definite tensor given by

$$\mathbf{M} := \mathbf{I} + \frac{1}{|\Omega|} \int_{\partial\Omega} \big(\boldsymbol{n}_{\boldsymbol{\zeta}} \otimes \boldsymbol{\Psi}(\boldsymbol{\zeta})\big)\, d\sigma(\boldsymbol{\zeta}) \tag{2.49}$$

and the auxiliary function $\boldsymbol{\Psi}$ has components Ψ_i, $i = 1, \ldots, d$, solving

$$\begin{cases} \Delta \Psi_i = 0 & \text{in } \mathbb{R}^d \backslash \Omega \\ \dfrac{\partial \Psi_i}{\partial \boldsymbol{n}} = -n_i & \text{on } \partial\Omega \\ \Psi_i \to 0 & \text{as } |\boldsymbol{x}| \to +\infty. \end{cases}$$

Proof. Let us set

$$\begin{aligned} J_1 &:= \nabla \Gamma(\boldsymbol{x} - \boldsymbol{z}) \cdot \int_{\partial\Omega} \boldsymbol{n}_{\boldsymbol{\zeta}} \left(\frac{1}{2} I + K_\Omega\right)^{-1} S_\Omega[-\boldsymbol{p} \cdot \boldsymbol{n}](\boldsymbol{\zeta})\, d\sigma(\boldsymbol{\zeta}), \\ J_2 &:= \nabla \Gamma(\boldsymbol{x} - \boldsymbol{z}) \cdot \int_{\partial\Omega} \boldsymbol{\zeta}\, \boldsymbol{p} \cdot \boldsymbol{n}_{\boldsymbol{\zeta}}\, d\sigma(\boldsymbol{\zeta}). \end{aligned}$$

Then, expansion (2.39) with $g = -\boldsymbol{p} \cdot \boldsymbol{n}$ gives

$$\begin{aligned} \frac{1}{2} u_\varepsilon(\boldsymbol{x}) &= -\varepsilon^{d-1} \Gamma(\boldsymbol{x} - \boldsymbol{z}) \int_{\partial\Omega} \boldsymbol{p} \cdot \boldsymbol{n}_{\boldsymbol{\zeta}}\, d\sigma(\boldsymbol{\zeta}) + J_1 + J_2 + O(\varepsilon^{d+1}) \\ &= J_1 + J_2 + O(\varepsilon^{d+1}) \end{aligned} \tag{2.50}$$

since divergence theorem guarantees that the first term in the expansion for u_ε is null.

From the equation (2.46), with $g = -\boldsymbol{p} \cdot \boldsymbol{n}$, since the problem for U is linear, we can decompose U as $U = \sum_i U_i$ where the functions U_i, for $i = 1, \cdots, d$, solve

$$\begin{cases} \Delta U_i = 0 & \text{in}\, \mathbb{R}^d \backslash \Omega \\ \dfrac{\partial U_i}{\partial \boldsymbol{n}} = -p_i n_i & \text{on } \partial \Omega \\ U_i \to 0 & \text{as}\, |\boldsymbol{x}| \to +\infty. \end{cases}$$

From the definition of the functions Ψ_i, we deduce $U = \boldsymbol{p} \cdot \boldsymbol{\Psi}$. Using Proposition 2.4.4, the term J_1 can be rewritten as

$$\begin{aligned} J_1 &= \nabla \Gamma(\boldsymbol{x} - \boldsymbol{z}) \cdot \int_{\partial \Omega} (\boldsymbol{\Psi}(\boldsymbol{\zeta}) \cdot \boldsymbol{p})\, \boldsymbol{n}_{\boldsymbol{\zeta}}\, d\sigma(\boldsymbol{\zeta}) \\ &= \nabla \Gamma(\boldsymbol{x} - \boldsymbol{z}) \cdot \int_{\partial \Omega} (\boldsymbol{n}_{\boldsymbol{\zeta}} \otimes \boldsymbol{\Psi}(\boldsymbol{\zeta}))\, \boldsymbol{p}\, d\sigma(\boldsymbol{\zeta}). \end{aligned}$$

To deal with the term J_2, we observe that

$$\int_{\partial \Omega} (\boldsymbol{n}_{\boldsymbol{\zeta}} \otimes \boldsymbol{\zeta})\ d\sigma(\boldsymbol{\zeta}) = |\Omega| \mathbf{I}.$$

Indeed, for $\boldsymbol{n}_{\boldsymbol{\zeta}} = (n_{\boldsymbol{\zeta},1}, \ldots, n_{\boldsymbol{\zeta},d})$, for any $i, j \in \{1, \ldots, d\}$, it follows

$$\begin{aligned} \int_{\partial \Omega} \zeta_i\, n_{\zeta,j}\, d\sigma(\boldsymbol{\zeta}) &= \int_{\partial \Omega} \boldsymbol{n}_{\boldsymbol{\zeta}} \cdot \zeta_i \boldsymbol{e}_j\, d\sigma(\boldsymbol{\zeta}) \\ &= \int_{\Omega} \operatorname{div}(\zeta_i \boldsymbol{e}_j)\ d\boldsymbol{\zeta} = \int_{\Omega} \boldsymbol{e}_j \cdot \boldsymbol{e}_i\, d\boldsymbol{\zeta} = |\Omega| \delta_{ij} \end{aligned}$$

where $\boldsymbol{e}_j$ is the j-th unit vector of $\mathbb{R}^d$. Thus, we get

$$J_2 = \nabla \Gamma(\boldsymbol{x} - \boldsymbol{z}) \cdot \int_{\partial \Omega} (\boldsymbol{\zeta} \otimes \boldsymbol{n}_{\zeta})\, \boldsymbol{p}\, d\sigma(\boldsymbol{\zeta}) = |\Omega| \nabla \Gamma(\boldsymbol{x} - \boldsymbol{z}) \cdot \boldsymbol{p}.$$

Collecting the expressions for J_1 and J_2, we obtain formula (2.48).

Symmetry of the tensor $\mathbf{M}$, defined in (2.49), follows from

$$\begin{aligned} \int_{\partial \Omega} \Psi_i(\boldsymbol{\zeta})\, n_{\boldsymbol{\zeta},j}\, d\sigma(\boldsymbol{\zeta}) &= -\int_{\partial \Omega} \Psi_i(\boldsymbol{\zeta}) \frac{\partial \Psi_j}{\partial \boldsymbol{n}}(\boldsymbol{\zeta})\, d\sigma(\boldsymbol{\zeta}) \\ &= \int_{\mathbb{R}^d \backslash \Omega} \operatorname{div}(\Psi_i(\boldsymbol{\zeta}) \nabla \Psi_j(\boldsymbol{\zeta}))\ d\boldsymbol{\zeta} \\ &= \int_{\mathbb{R}^d \backslash \Omega} \nabla \Psi_i(\boldsymbol{\zeta}) \cdot \nabla \Psi_j(\boldsymbol{\zeta})\, d\boldsymbol{\zeta} \end{aligned}$$

where the last term is obviously symmetric. Taking $\boldsymbol{\eta} \in \mathbb{R}^d$, we consider

$$\boldsymbol{\eta} \cdot \mathbf{M}\boldsymbol{\eta} = |\boldsymbol{\eta}|^2 + \frac{1}{|\Omega|} \int_{\partial\Omega} (\boldsymbol{n}_{\boldsymbol{\zeta}} \cdot \boldsymbol{\eta})(\Psi(\boldsymbol{\zeta}) \cdot \boldsymbol{\eta})\, d\sigma(\boldsymbol{\zeta}).$$

The positivity of the tensor follows from the divergence theorem, integration by parts and the definition of the function $\boldsymbol{\Psi}$, in fact

$$\begin{aligned}\int_{\partial\Omega} (\boldsymbol{n}_{\boldsymbol{\zeta}} \cdot \boldsymbol{\eta})(\boldsymbol{\Psi}(\boldsymbol{\zeta}) \cdot \boldsymbol{\eta})\, d\sigma(\boldsymbol{\zeta}) &= -\int_{\partial\Omega} \frac{\partial}{\partial \boldsymbol{n}} (\boldsymbol{\Psi}(\boldsymbol{\zeta}) \cdot \boldsymbol{\eta})(\boldsymbol{\Psi}(\boldsymbol{\zeta}) \cdot \boldsymbol{\eta})\, d\sigma(\boldsymbol{\zeta}) \\ &= \int_{\mathbb{R}^d \setminus \Omega} \operatorname{div}\left((\boldsymbol{\Psi}(\boldsymbol{\zeta}) \cdot \boldsymbol{\eta}) \nabla (\boldsymbol{\Psi}(\boldsymbol{\zeta}) \cdot \boldsymbol{\eta})\right)\, d\sigma(\boldsymbol{\zeta}) \\ &= \int_{\mathbb{R}^d \setminus \Omega} \left|\nabla(\boldsymbol{\Psi}(\boldsymbol{\zeta}) \cdot \boldsymbol{\eta})\right|^2 d\boldsymbol{\zeta},\end{aligned}$$

hence $\boldsymbol{\eta} \cdot \mathbf{M}\boldsymbol{\eta} > 0$. □

For specific forms of the cavity Ω, the auxiliary function $\boldsymbol{\Psi}$ can be determined explicitly, providing a corresponding explicit formula for the polarization tensor $\mathbf{M}$. The basic case is the one of a spherical cavity (see [36]). If $\Omega = \{\boldsymbol{x} \in \mathbb{R}^3 \,:\, |\boldsymbol{x}| < 1\}$, then a direct calculation shows that, for $i = 1, 2, 3$, it holds $\Psi_i(\boldsymbol{x}) = x_i/(2|\boldsymbol{x}|^3)$, and thus

$$\Psi_i(\boldsymbol{\zeta}) = \frac{1}{2}\, \zeta_i, \qquad \boldsymbol{\zeta} \in \partial\Omega.$$

As a consequence, the polarization tensor is a multiple of the identity and, precisely,

$$\mathbf{M} = \frac{3}{2}|\Omega|\mathbf{I} = 2\pi\mathbf{I}.$$

Then, the asymptotic expansion (2.48) becomes

$$u_\varepsilon(\boldsymbol{x}) = 4\pi\varepsilon^3 \nabla\Gamma(\boldsymbol{x} - \boldsymbol{z}) \cdot \boldsymbol{p} + O(\varepsilon^4), \qquad \boldsymbol{x} \in \mathbb{R}^2.$$

Explicit formulas can be provided also in the case of ellipsoidal cavities (see [5, 7, 8]).

In general, for given shapes of the cavity Ω, such auxiliary function can be numerically approximated and, thus, the first term in the expansion (2.48) can be considered as known in practical cases.

CHAPTER 3

The Elastic model

In this chapter we establish a sound mathematical approach for surface deformation effects generated by a magma chamber embedded into Earth's interior and exerting on it a uniform hydrostatic pressure. Modeling assumptions translate the problem into classical elasto-static system (homogeneous and isotropic) in an half-space with an embedded pressurized cavity. The boundary conditions are traction-free for the air/crust boundary and uniformly hydrostatic for the chamber boundary. These are complemented with zero-displacement condition at infinity (with decay rate). Therefore, representing the displacement vector field with $\boldsymbol{u}$ we get, from the mathematical point of view, the linear elasto-static boundary value problem

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \boldsymbol{0} & \text{in } \mathbb{R}^3_-\backslash C \\ \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} = p\,\boldsymbol{n} & \text{on } \partial C \\ \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} = \boldsymbol{0} & \text{on } \mathbb{R}^2 \\ \boldsymbol{u} = o(\boldsymbol{1}), \quad \nabla \boldsymbol{u} = o(|\boldsymbol{x}|^{-1}) & |\boldsymbol{x}| \to \infty, \end{cases}$$

where $\mathbb{C}$ is the elasticity tensor, C is the cavity, p is a constant representing the pressure and $\widehat{\nabla}\boldsymbol{u} = \frac{1}{2}\big(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\big)$ the strain tensor. With $\partial\boldsymbol{u}/\partial\boldsymbol{\nu}$ we depict the conormal derivative on the boundary of a domain, that is the traction vector $\partial\boldsymbol{u}/\partial\boldsymbol{\nu} := (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}$.

As done in the previous chapter for the scalar model, here we first establish the well-posedness of the problem and provide an appropriate integral formulation for its solution for cavities with general shape. Based on that, assuming that the chamber is centred at some fixed point $\boldsymbol{z}$ and has diameter $r > 0$, small with respect to the depth d, we derive rigorously the principal term in the asymptotic expansion for the surface deformation as $\varepsilon = r/d \to 0^+$. Such formula provides a rigorous proof of the Mogi point source model in the case of spherical cavities, presented in the Introduction, generalizing it to the case of cavities of arbitrary shape. For the application we have in mind, we focus the attention only to the dimensional case $d = 3$.

The chapter is organized as follows. In Section 3.1 we recall some arguments about linear elasticity and layer potentials techniques. In Section 3.2 we present the Neumann function for the Lamé operator in the half-space, then we analyze the well-posedness of the direct problem via an integral representation formula for the displacement field. Section 3.3 is devoted to the proof of the main result regarding the derivation of the asymptotic formula for the boundary diplacement field. In addition we analyse the properties of the moment elastic tensor and, as a consequence of the asymptotic expansion in the case of spherical cavity, we obtain the classical Mogi's formula.

3.1 Preliminaries

Let C be a bounded Lipschitz domain in $\mathbb{R}^3$ representing the region occupied by a homogeneous and isotropic elastic medium. Let λ and μ be the Lamé constants, i.e. the compression modulus and the shear modulus, we define the fourth-order elasticity tensor

$$\mathbb{C} := \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbb{I}$$

which satisfies the minor and major simmetry conditions, that is

$$\mathbb{C}_{ijkh} = \mathbb{C}_{khij} = \mathbb{C}_{jikh},$$

for all $i, j, k, h = 1, 2, 3$. If λ and μ satisfies the physical range $3\lambda + 2\mu > 0$ and $\mu > 0$, the elasticity tensor $\mathbb{C}$ is positive definite.
It is also common to use the Poisson ratio ν which is related to λ and μ by the identity $\nu = \lambda/2(\lambda + \mu)$.

In a homogeneous and isotropic elastic medium, the elastostatic Lamé operator $\mathcal{L}$ is defined by

$$\mathcal{L}\boldsymbol{u} := \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \mu\Delta\boldsymbol{u} + (\lambda + \mu)\nabla \operatorname{div} \boldsymbol{u},$$

where $\boldsymbol{u}$ represents the vector of the displacements. In terms of the Poisson ratio it becomes $\boldsymbol{\mathcal{L}u} = \mu(\Delta \boldsymbol{u} + 1/(1-2\nu)\nabla \mathrm{div}\, \boldsymbol{u})$.

The explicit expression of the conormal derivative is given by

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} := (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n} = \lambda(\mathrm{div}\, \boldsymbol{u})\boldsymbol{n} + 2\mu(\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}$$

or, equivalently,

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} = 2\mu\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + \lambda(\mathrm{div}\, \boldsymbol{u})\boldsymbol{n} + \mu(\boldsymbol{n} \times \mathrm{rot}\, \boldsymbol{u}).$$

For the sequel, we recall that the positive definiteness of the tensor $\mathbb{C}$ implies the strong ellipticity of the Lamé operator which corresponds to the request $\mu > 0$ and $\lambda + 2\mu > 0$, see [37].

We recall Betti's formulas for the Lamé system which can be obtained by integration by parts, see for example [4, 44]. Given a bounded Lipschitz domain $C \subset \mathbb{R}^3$ and two vectors $\boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^3$, the *first Betti formula* is

$$\int_{\partial C} \boldsymbol{u} \cdot \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}}\, d\sigma(\boldsymbol{x}) = \int_C \boldsymbol{u} \cdot \boldsymbol{\mathcal{L}v}\, d\boldsymbol{x} + \int_C Q(\boldsymbol{u}, \boldsymbol{v})\, d\boldsymbol{x}, \tag{3.1}$$

where the quadratic form Q associated to the Lamé system is

$$Q(\boldsymbol{u}, \boldsymbol{v}) := \lambda(\mathrm{div}\, \boldsymbol{u})(\mathrm{div}\, \boldsymbol{v}) + 2\mu\widehat{\nabla}\boldsymbol{u} : \widehat{\nabla}\boldsymbol{v}.$$

From (3.1) it is straightforward to find the *second Betti formula*

$$\int_C (\boldsymbol{u} \cdot \boldsymbol{\mathcal{L}v} - \boldsymbol{v} \cdot \boldsymbol{\mathcal{L}u})\, d\boldsymbol{x} = \int_{\partial C} \left(\boldsymbol{u} \cdot \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}} - \boldsymbol{v} \cdot \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}}\right) d\sigma(\boldsymbol{x}). \tag{3.2}$$

Formula (3.1) will be used to prove that the solution of the elastic problem proposed in this thesis is unique, and the equality (3.2) to get an integral representation formula for it. To accomplish this second goal, a leading role is played by the fundamental solution of the Lamé system: the *Kelvin matrix* $\boldsymbol{\Gamma}$ (or *Kelvin-Somigliana matrix*) solution to the equation

$$\mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\Gamma}) = \delta_{\boldsymbol{0}}\mathbf{I}, \qquad \boldsymbol{x} \in \mathbb{R}^3 \backslash \{\boldsymbol{0}\},$$

where $\delta_{\boldsymbol{0}}$ is the Dirac function centred at $\boldsymbol{0}$. Setting $C_{\mu,\nu} := 1/\{16\pi\mu(1-\nu)\}$, the explicit expression of $\boldsymbol{\Gamma} = (\Gamma_{ij})$ is

$$\Gamma_{ij}(\boldsymbol{x}) = -C_{\mu,\nu}\left\{\frac{(3-4\nu)\delta_{ij}}{|\boldsymbol{x}|} + \frac{x_i x_j}{|\boldsymbol{x}|^3}\right\}, \qquad i, j = 1, 2, 3, \tag{3.3}$$

where δ_{ij} is the Kronecker symbol and Γ_{ij} stands for the i-th component of the displacement when a force is applied in the j-th direction at the point $\mathbf{0}$. For reader convenience, we write also the gradient of $\mathbf{\Gamma}$ to highlight its behaviour at infinity

$$\frac{\partial \Gamma_{ij}}{\partial x_k}(\boldsymbol{x}) = C_{\mu,\nu}\left\{\frac{(3-4\nu)\delta_{ij}x_k - \delta_{ik}x_j - \delta_{jk}x_i}{|\boldsymbol{x}|^3} + \frac{3x_ix_jx_k}{|\boldsymbol{x}|^5}\right\}, \qquad i,j,k = 1,2,3. \tag{3.4}$$

Therefore from (3.3) and (3.4) it is straightforward to see that

$$|\mathbf{\Gamma}(\boldsymbol{x})| = O(|\boldsymbol{x}|^{-1}) \quad \text{and} \quad |\nabla\mathbf{\Gamma}(\boldsymbol{x})| = O(|\boldsymbol{x}|^{-2}) \qquad \text{as} \quad |\boldsymbol{x}| \to \infty. \tag{3.5}$$

3.1.1 Layer potentials for the Lamé operator

With the Kelvin matrix $\mathbf{\Gamma}$ at hand, we recall the definition of single and double layer potentials corresponding to the operator $\mathcal{L}$. Given $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\partial C)$ (see [4, 7, 44])

$$\begin{aligned} \mathbf{S}^{\Gamma}\boldsymbol{\varphi}(\boldsymbol{x}) &:= \int_{\partial C} \mathbf{\Gamma}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\, d\sigma(\boldsymbol{y}), & \boldsymbol{x} \in \mathbb{R}^3, \\ \mathbf{D}^{\Gamma}\boldsymbol{\varphi}(\boldsymbol{x}) &:= \int_{\partial C} \frac{\partial \mathbf{\Gamma}}{\partial \boldsymbol{\nu}(\boldsymbol{y})}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\, d\sigma(\boldsymbol{y}), & \boldsymbol{x} \in \mathbb{R}^3 \backslash \partial C, \end{aligned} \tag{3.6}$$

where $\partial\mathbf{\Gamma}/\partial\boldsymbol{\nu}$ denotes the conormal derivative applied to each column of the matrix $\mathbf{\Gamma}$.

We summarize here some properties of these operators

i. By definition, $\mathbf{S}^{\Gamma}\boldsymbol{\varphi}(\boldsymbol{x})$ and $\mathbf{D}^{\Gamma}\boldsymbol{\varphi}(\boldsymbol{x})$ satisfy the Lamé system in $\mathbb{R}^3\backslash\partial C$.

ii. $\mathbf{S}^{\Gamma}\boldsymbol{\varphi}(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1})$ and $\mathbf{D}^{\Gamma}\boldsymbol{\varphi}(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-2})$ as $|\boldsymbol{x}| \to +\infty$.

Next, we introduce $\mathbf{K}$ and $\mathbf{K}^*$ that is the $\boldsymbol{L}^2$-adjoint Neumann-Poincaré boundary integral operators defined, in the sense of Cauchy principal value, by

$$\begin{aligned} \mathbf{K}\boldsymbol{\varphi}(\boldsymbol{x}) &:= \text{p.v.}\int_{\partial C} \frac{\partial \mathbf{\Gamma}}{\partial \boldsymbol{\nu}(\boldsymbol{y})}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\, d\sigma(\boldsymbol{y}), \\ \mathbf{K}^*\boldsymbol{\varphi}(\boldsymbol{x}) &:= \text{p.v.}\int_{\partial C} \frac{\partial \mathbf{\Gamma}}{\partial \boldsymbol{\nu}(\boldsymbol{x})}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\, d\sigma(\boldsymbol{y}). \end{aligned}$$

As in the previous chapter, in the sequel the subscripts $+$ and $-$ indicate the limits from outside and inside of the set C, respectively (see (2.10) for the definition). We recall that $t_1, \cdots, t_{d-1}$ represent an orthonormal basis for the tangent plane to $\partial\Omega$ and $\partial/\partial \boldsymbol{t} = \sum_{k=1}^{d-1} \partial/\partial t_k \, t_k$ is the tangential derivative on $\partial\Omega$.

The following theorem about the jump relations of single and double potentials for Lipschitz domains is due to Dahlberg, Kenig and Verchota [24].

Theorem 3.1.1 ([24]). *Let C be a bounded Lipschitz domain in $\mathbb{R}^3$. For $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\partial C)$, the following relations hold, a.e on ∂C,*

$$\begin{aligned} \mathbf{D}^\Gamma \boldsymbol{\varphi}\Big|_\pm(\boldsymbol{x}) &= \left(\mp\tfrac{1}{2}\mathbf{I} + \mathbf{K}\right)\boldsymbol{\varphi}(\boldsymbol{x}), \\ \frac{\partial \mathbf{S}^\Gamma \boldsymbol{\varphi}}{\partial \boldsymbol{\nu}}\Big|_\pm(\boldsymbol{x}) &= \left(\pm\tfrac{1}{2}\mathbf{I} + \mathbf{K}^*\right)\boldsymbol{\varphi}(\boldsymbol{x}), \\ \frac{\partial \mathbf{S}^\Gamma \boldsymbol{\varphi}}{\partial \boldsymbol{t}}\Big|_-(\boldsymbol{x}) &= \frac{\partial \mathbf{S}^\Gamma \boldsymbol{\varphi}}{\partial \boldsymbol{t}}\Big|_+(\boldsymbol{x}) \end{aligned} \tag{3.7}$$

It is worth noticing that the two operators $\mathbf{K}$ and $\mathbf{K}^*$ are not compact even on smooth domains, in contrast with the analogous operators for the Laplace equation (see [7] and the considerations in the previous chapter), due to the presence in their kernels of the terms

$$\frac{n_i(x_j - y_j)}{|\boldsymbol{x} - \boldsymbol{y}|^3} - \frac{n_j(x_i - y_i)}{|\boldsymbol{x} - \boldsymbol{y}|^3}, \qquad i \neq j, \tag{3.8}$$

which make the kernel not integrable. Indeed, even in the case of smooth domains, we cannot approximate locally the terms $\boldsymbol{n} \times (\boldsymbol{x} - \boldsymbol{y})$ with a smooth function, that is in terms of powers of $|\boldsymbol{x} - \boldsymbol{y}|$ via Taylor expansion, in order to obtain an integrable kernel on ∂C. Therefore, the analysis to prove invertibility of the operators in (3.7) is complicated and usually based on a regularization procedure (see [44]) in the case of smooth domains. For Lipschitz domains the analysis is much more involved and, as for the Laplace operator, based on Rellich formulas. These results are contained in [24] and its companion article [32]. We recall here only the main aspects for the three-dimensional case.

Let Ψ be the vector space of all linear solutions of the equations

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{w}) = 0, & \text{in } C \\ \dfrac{\partial \boldsymbol{w}}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial C \end{cases}$$

or, alternatively
$$\Psi := \{\boldsymbol{w} : \nabla \boldsymbol{w} + (\nabla \boldsymbol{w})^T = 0\}.$$
The space Ψ has dimension 6. Such a function $\boldsymbol{w}$ is called infinitesimal rigid motion. We recall that $\boldsymbol{w}$ can be expressed as

$$\boldsymbol{w} = \boldsymbol{a} + \mathbf{A}\boldsymbol{x}, \tag{3.9}$$

where $\mathbf{A}$ is a skew-symmetric matrix and $\boldsymbol{a} \in \mathbb{R}^3$. We define

$$\boldsymbol{L}^2_\Psi(\partial C) := \left\{ \boldsymbol{f} \in \boldsymbol{L}^2(\partial C) : \int_{\partial C} \boldsymbol{f} \cdot \boldsymbol{w} \, d\sigma = 0, \forall \boldsymbol{w} \in \Psi \right\}$$

We have

Proposition 3.1.2 ([24]). *The operators*

$$-\frac{1}{2}\mathbf{I} + \mathbf{K}^* : \boldsymbol{L}^2_\Psi(\partial C) \to \boldsymbol{L}^2_\Psi(\partial C)$$
$$\frac{1}{2}\mathbf{I} + \mathbf{K}^* : \boldsymbol{L}^2(\partial C) \to \boldsymbol{L}^2(\partial C)$$

are injective.

We omit the proof since is similar to that one of the scalar case.
The range of $-1/2\mathbf{I}+\mathbf{K}^*$ as an operator on all $\boldsymbol{L}^2(\partial C)$ is contained in $\boldsymbol{L}^2_\Psi(\partial C)$ since
$$\int_{\partial C} \frac{\partial \mathbf{S}^\Gamma \boldsymbol{\varphi}}{\partial \boldsymbol{\nu}}\Big|_+ (\boldsymbol{x}) \boldsymbol{w}(\boldsymbol{x}) \, d\sigma = \int_{\partial C} \mathbf{S}^\Gamma \boldsymbol{\varphi} \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{\nu}} \, d\sigma(\boldsymbol{x}) = 0$$
for all $\boldsymbol{w} \in \Psi$. This is because $\boldsymbol{w}$ is a solution to the elastostatic systems satisfying $\partial \boldsymbol{w}/\partial \boldsymbol{\nu} = 0$.

In addition, it holds

Proposition 3.1.3 ([24]). *The operators*

$$-\frac{1}{2}\mathbf{I} + \mathbf{K}^* : \boldsymbol{L}^2_\Psi(\partial C) \to \boldsymbol{L}^2_\Psi(\partial C)$$
$$\frac{1}{2}\mathbf{I} + \mathbf{K}^* : \boldsymbol{L}^2(\partial C) \to \boldsymbol{L}^2(\partial C)$$

have closed range.

The key point to show that these two boundary operators have closed range, as in the case of the Laplace operator, is the following inequality

$$C^{-1}\Big\|\Big(-\frac{1}{2}\mathbf{I}+\mathbf{K}^*\Big)\boldsymbol{\varphi}\Big\|_{\boldsymbol{L}^2(\partial C)} \leqslant \Big\|\Big(\frac{1}{2}\mathbf{I}+\mathbf{K}^*\Big)\boldsymbol{\varphi}\Big\|_{\boldsymbol{L}^2(\partial C)} \leqslant C\Big\|\Big(-\frac{1}{2}\mathbf{I}+\mathbf{K}^*\Big)\boldsymbol{\varphi}\Big\|_{\boldsymbol{L}^2(\partial C)}$$

where C is a constant independent of $\boldsymbol{\varphi} \in \boldsymbol{L^2}(\partial C)$. However, we stress that the analysis to get the equivalence of the norms in the elastic case is very complicated since it is based on the twine of Rellich formulas for the Lamé operators, estimates derived from them, Korn's inequalities and results on the biharmonic equations.

In order to prove the invertibility of the operators, it remains to show dense range. To do that one can make use of the result on the invertibility for the same operators in the case of smooth domains. The minimum regularity we request on the domain is, at least, C^1 but here, without loss of generality, we consider C^∞ domains. As stated before, even if we use smooth domains we cannot apply the Fredholm's theory because $\mathbf{K}$ and $\mathbf{K}^*$ are not compact operators. However, the difference $\mathbf{K} - \mathbf{K}^*$ yields a compact operator, see [24] for details.

The following proposition is needed

Proposition 3.1.4. *Let H be a Hilbert space. If $T : H \to H$ is a bounded linear operator with closed range, with null space of dimension $l < \infty$, and such that $T - T^*$ is compact, then the range of T has codimension l also.*

Now, we state the invertibility result for smooth domains.

Lemma 3.1.5 ([24])**.** *Let C be a bounded smooth domain with connected boundary in $\mathbb{R}^3$. Let us consider the operators $\pm 1/2\mathbf{I} + \mathbf{K}^*$ on ∂C. Then*

(i) $-\frac{1}{2}\mathbf{I} + \mathbf{K}^* :: \boldsymbol{L}^2_\Psi(\partial C) \to \boldsymbol{L}^2_\Psi(\partial C)$

(ii) $\frac{1}{2}\mathbf{I} + \mathbf{K}^* : \boldsymbol{L}^2(\partial C) \to \boldsymbol{L}^2(\partial C)$

are invertible operators.

Proof. Let us prove (i) (the same argument yield for (ii) also). From the previous two propositions we know that the operator $-1/2\mathbf{I} + \mathbf{K}^*$ is one-to-one and has closed range. Moreover, the dimension of the null space is less than or equal to 6 and the codimension is greater than or equal to 6. Since $(-1/2\mathbf{I} + \mathbf{K}^*) - (-1/2\mathbf{I} + \mathbf{K})$ is compact, applying the Proposition 3.1.4, we have the assertion. $\square$

Now, we briefly explain the sequence of steps to deduce the invertibility of the operators $\pm 1/2\mathbf{I} + \mathbf{K}^*$ in the case of Lipschitz domains, giving only the main ideas.

The starting point is to consider a sequence of C^∞ domains, which we call C_j, that converge to the Lipschitz domain C (for all the details see Theorem 1.12 in [58]). In such a scheme the ∂C_j can be projected homeomorphically to ∂C so that the boundaries converge uniformly and so that the Lipschitz characters of the C_j are controlled by that of C. In fact, the unit normal vectors to the C_j will converge pointwise a.e. to those for C and in $L^p(\partial C)$ for all $1 \leqslant p \leqslant \infty$. If $\mathbf{K}_j$ denotes the singular operatos defined on ∂C_j we may project it onto ∂C and prove that

$$\lim_{j \to +\infty} \left\| \mathbf{K}_j^* \boldsymbol{f} - \mathbf{K}^* \boldsymbol{f} \right\|_{L^2(\partial \Omega_j)} = 0$$

and a result analogous for the adjoint operator $\mathbf{K}$. Then, since

$$\begin{aligned} \dim\Big(\mathrm{Ker}\Big(\pm\frac{1}{2}\mathbf{I} + \mathbf{K}^*\Big)\Big) &\leqslant \dim\Big(\mathrm{Ker}\Big(\pm\frac{1}{2}\mathbf{I} + \mathbf{K}_j^*\Big)\Big) \\ &= \dim\Big(\mathrm{Coker}\Big(\pm\frac{1}{2}\mathbf{I} + \mathbf{K}_j^*\Big)\Big) = l \\ &\leqslant \dim\Big(\mathrm{Coker}\Big(\pm\frac{1}{2}\mathbf{I} + \mathbf{K}^*\Big)\Big) \end{aligned}$$

where $l < \infty$ is independent of j, under other suitable assumptions, it can be proven that

$$\dim\Big(\mathrm{Coker}\Big(\pm\frac{1}{2}\mathbf{I} + \mathbf{K}^*\Big)\Big) = \dim\Big(\mathrm{Ker}\Big(\pm\frac{1}{2}\mathbf{I} + \mathbf{K}^*\Big)\Big) = l.$$

Finally, using the invertibility Lemma 3.1.5 about smooth domains we find

Theorem 3.1.6. *Let C be a bounded Lipschitz domain with connected boundary in $\mathbb{R}^3$. Then*

(i) $-\frac{1}{2}\mathbf{I} + \mathbf{K}^* :: \boldsymbol{L}^2_\Psi(\partial C) \to \boldsymbol{L}^2_\Psi(\partial C)$

(ii) $\frac{1}{2}\mathbf{I} + \mathbf{K}^* : \boldsymbol{L}^2(\partial C) \to \boldsymbol{L}^2(\partial C)$

are invertible operators.

Now, we have all the instruments to analyse the elastic boundary value problem.

3.2 The elastic problem

In this section we analyse the bounday value problem presented at the beginning of this chapter, that is

$$\begin{cases} \mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \boldsymbol{0} & \text{in } \mathbb{R}^3_-\backslash C \\ \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} = p\,\boldsymbol{n} & \text{on } \partial C \\ \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} = \boldsymbol{0} & \text{on } \mathbb{R}^2 \\ \boldsymbol{u} = o(\boldsymbol{1}), \quad \nabla \boldsymbol{u} = o(|\boldsymbol{x}|^{-1}) & |\boldsymbol{x}| \to \infty, \end{cases} \tag{3.10}$$

where C is the cavity and p the pressure.

In particular we provide an integral representation formula and establish the well-posedness of this problem. To do that, we give the expression of the Neumann function $\mathbf{N}$ of the half-space with null traction on the boundary, found by Mindlin in [48, 49]. Then we represent the solution to (3.10) by an integral formula through the Neumann function. Finally, all these objects will be used to prove the well-posedness of the problem (3.10).

3.2.1 Fundamental solution of the half-space

In this subsection we show the explicit expression of the Neumann function for the half-space presented for the first time in [48] by means of Galerkin vector and nuclei of strain of the theory of linear elasticity, and secondly in [49] using the Papkovich-Neuber representation of the displacement vector field and the potential theory. Here, we follow the second approach.

We consider the boundary value problem

$$\begin{cases} \mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{v}) = \boldsymbol{b} & \text{in } \mathbb{R}^3_- \\ \dfrac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}} = \boldsymbol{0} & \text{on } \mathbb{R}^2 \\ \boldsymbol{v} = o(\boldsymbol{1}), \quad \nabla \boldsymbol{v} = o(|\boldsymbol{x}|^{-1}) & \text{as } |\boldsymbol{x}| \to +\infty. \end{cases} \tag{3.11}$$

The Neumann function of (3.11) is the kernel $\mathbf{N}$ of the integral operator

$$\boldsymbol{v}(\boldsymbol{x}) = \int_{\mathbb{R}^3_-} \mathbf{N}(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{b}(\boldsymbol{y})\, d\boldsymbol{y}, \tag{3.12}$$

giving the solution to the problem.

Given $\boldsymbol{y} = (y_1, y_2, y_3)$, we set $\tilde{\boldsymbol{y}} := (y_1, y_2, -y_3)$.

Theorem 3.2.1. *The Neumann function* $\mathbf{N}$ *of problem* (3.11) *can be decomposed as*

$$\mathbf{N}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{\Gamma}(\boldsymbol{x} - \boldsymbol{y}) + \mathbf{R}^1(\boldsymbol{x} - \tilde{\boldsymbol{y}}) + y_3 \mathbf{R}^2(\boldsymbol{x} - \tilde{\boldsymbol{y}}) + y_3^2\, \mathbf{R}^3(\boldsymbol{x} - \tilde{\boldsymbol{y}}),$$

where $\boldsymbol{\Gamma}$ *is the Kelvin matrix, see* (3.3)*, and* $\mathbf{R}^k$*,* $k = 1, 2, 3$*, have components* R^k_{ij} *given by*

$$\begin{aligned} R^1_{ij}(\boldsymbol{\eta}) &:= C_{\mu,\nu}\big\{-(\tilde{f} + c_\nu \tilde{g})\delta_{ij} - (3-4\nu)\eta_i\eta_j \tilde{f}^3 + c_\nu\big[\delta_{i3}\eta_j - \delta_{j3}(1-\delta_{i3})\eta_i\big]\tilde{f}\tilde{g} \\ &\qquad\qquad + c_\nu(1-\delta_{i3})(1-\delta_{j3})\eta_i\eta_j\tilde{f}\tilde{g}^2\big\} \\ R^2_{ij}(\boldsymbol{\eta}) &:= 2C_{\mu,\nu}\big\{(3-4\nu)\big[\delta_{i3}(1-\delta_{j3})\eta_j + \delta_{j3}(1-\delta_{i3})\eta_i\big]\tilde{f}^3 \\ &\qquad\qquad - (1-2\delta_{3j})\delta_{ij}\eta_3\tilde{f}^3 + 3(1-2\delta_{3j})\eta_i\eta_j\eta_3\tilde{f}^5\big\} \\ R^3_{ij}(\boldsymbol{\eta}) &:= 2C_{\mu,\nu}(1-2\delta_{j3})\big\{\delta_{ij}\tilde{f}^3 - 3\eta_i\eta_j\tilde{f}^5\big\}. \end{aligned}$$

for $i, j = 1, 2, 3$*, where* $c_\nu := 4(1-\nu)(1-2\nu)$*,* $C_{\mu,\nu} = \frac{1}{16\pi\mu(1-\nu)}$ *and*

$$\tilde{f}(\boldsymbol{\eta}) := \frac{1}{|\boldsymbol{\eta}|}, \qquad \tilde{g}(\boldsymbol{\eta}) := \frac{1}{|\boldsymbol{\eta}| - \eta_3}.$$

The matrix $\mathbf{R}$, defined by

$$\mathbf{R}(\boldsymbol{\eta}, y_3) := \mathbf{R}^1(\boldsymbol{\eta}) + y_3\,\mathbf{R}^2(\boldsymbol{\eta}) + y_3^2\,\mathbf{R}^3(\boldsymbol{\eta}), \tag{3.13}$$

gives the regular part of the Neumann function since the singular point $\boldsymbol{\eta} = \mathbf{0}$ corresponds to $\boldsymbol{y} = (y_1, y_2, -y_3)$ with $y_3 < 0$, which belongs to $\mathbb{R}^3_+$.

In order to prove this theorem, we recall the basic steps to deduce (3.12) using the potential approach in [49].

Papkovich-Neuber potentials

The starting point is the Helmholtz decomposition of the vector field $\boldsymbol{v}$ in (3.11) as

$$\boldsymbol{v} = \nabla\phi + \nabla \times \boldsymbol{\psi}, \tag{3.14}$$

where ϕ is a scalar potential and $\boldsymbol{\psi}$ a vector potential. Since the divergence of $\boldsymbol{\psi}$ is arbitrary, $\boldsymbol{\psi}$ can be chosen in such a way that $\operatorname{div}\boldsymbol{\psi} = 0$. From the Lamé operator with volume forces $\boldsymbol{b}$ and the Helmholtz representation (3.14), we find that

$$\Delta\Big[\boldsymbol{v} + \frac{1}{(1-2\nu)}\nabla\phi\Big] = \frac{\boldsymbol{b}}{\mu}.$$

We define

$$\boldsymbol{h} := 4\pi\mu\Big[\boldsymbol{v} + \frac{1}{(1-2\nu)}\nabla\phi\Big], \tag{3.15}$$

where the constant $4\pi\mu$ has been added to simplify the calculations in the sequel, hence

$$\Delta\boldsymbol{h} = 4\pi\boldsymbol{b}, \qquad\qquad \operatorname{div}\boldsymbol{h} = \frac{8\pi\mu(1-\nu)}{1-2\nu}\Delta\phi. \tag{3.16}$$

By the identity $\Delta(\boldsymbol{x}\cdot\boldsymbol{h}) = \boldsymbol{x}\cdot\Delta\boldsymbol{h} + 2\mathrm{div}\,\boldsymbol{h}$ and the relation $\Delta\boldsymbol{h} = 4\pi\boldsymbol{b}$, we find that

$$\operatorname{div}\boldsymbol{h} = \tfrac{1}{2}\big[\Delta(\boldsymbol{x}\cdot\boldsymbol{h}) - 4\pi\boldsymbol{x}\cdot\boldsymbol{b}\big]. \tag{3.17}$$

Combining this expression with the second one in (3.16) we get

$$\Delta\left[\frac{8\pi\mu(1-\nu)}{1-2\nu}\phi - \frac{\boldsymbol{x}\cdot\boldsymbol{h}}{2}\right] = -2\pi\boldsymbol{x}\cdot\boldsymbol{b}.$$

We define the scalar quantity β as

$$\beta := \frac{16\pi\mu(1-\nu)}{1-2\nu}\phi - \boldsymbol{x}\cdot\boldsymbol{h}, \tag{3.18}$$

hence

$$\Delta\beta = -4\pi\boldsymbol{x}\cdot\boldsymbol{b}.$$

Using the definition (3.18) of β, we can avoid the dependence from ϕ into the relation (3.15), that is

$$\boldsymbol{v} = C_{\mu,\nu}\{4(1-\nu)\boldsymbol{h} - \nabla\left(\beta + \boldsymbol{x}\cdot\boldsymbol{h}\right)\}, \tag{3.19}$$

where $\boldsymbol{h}$ and β are the Papkovich-Neuber potentials. Let us introduce the functions

$$\phi(\boldsymbol{x}) := \frac{1}{|\boldsymbol{x}|} \qquad \text{and} \qquad \psi(\boldsymbol{x}) := \frac{\phi(\boldsymbol{x})}{1 - x_3\phi(\boldsymbol{x})} = \frac{1}{|\boldsymbol{x}| - x_3},$$

observing that, apart from $\partial_i\phi = -x_i\phi^3$, $i = 1,2,3$, the following identities hold true for $\alpha = 1,2$,

$$\phi - \psi = -x_3\phi\,\psi, \qquad \partial_\alpha\psi = -x_\alpha\phi\,\psi^2, \qquad \partial_3\psi = \phi\,\psi, \qquad \partial_3(\phi\,\psi) = \phi^3.$$

We denote by ϕ and $\widetilde{\phi}$ the values $\phi(\boldsymbol{x}+\boldsymbol{e}_3)$ and $\phi(\boldsymbol{x}-\boldsymbol{e}_3)$, respectively, with analogous notation for ψ.

Proposition 3.2.2. *Let* $\mathbf{I}$ *be the identity matrix and* δ *the Dirac delta concentrated at* $-\boldsymbol{e}_3$*. Then, the matrix-valued function* $\mathcal{N} = \mathcal{N}(\boldsymbol{x})$ *solution to*

$$\boldsymbol{\mathcal{L}v} := \mathrm{div}\big(\mathbb{C}\widehat{\nabla}\boldsymbol{v}\big) = \delta\mathbf{I} \quad \text{in } \mathbb{R}^3_-, \qquad\qquad \big(\mathbb{C}\widehat{\nabla}\boldsymbol{v}\big)\boldsymbol{e}_3 = \mathbf{0} \quad \text{in } \mathbb{R}^2,$$

is given by

$$\begin{aligned}
\mathcal{N}_{\alpha\alpha} &= -C_{\mu,\nu}\big\{(3-4\nu)\phi + x_\alpha^2\phi^3 + \widetilde{\phi} + [(3-4\nu)x_\alpha^2 - 2x_3]\widetilde{\phi}^3 + 6x_\alpha^2 x_3\widetilde{\phi}^5 \\
&\qquad\qquad + c_\nu\big(\widetilde{\psi} - x_\alpha^2\widetilde{\phi}\,\widetilde{\psi}^2\big)\big\} \\
\mathcal{N}_{\alpha\beta} &= -C_{\mu,\nu}x_\alpha x_\beta\big\{\phi^3 + (3-4\nu)\widetilde{\phi}^3 + 6x_3\widetilde{\phi}^5 - c_\nu\widetilde{\phi}\,\widetilde{\psi}^2\big\} \\
\mathcal{N}_{3\alpha} &= -C_{\mu,\nu}x_\alpha\big\{(x_3+1)\phi^3 + (3-4\nu)(x_3+1)\widetilde{\phi}^3 + 6x_3(x_3-1)\widetilde{\phi}^5 - c_\nu\widetilde{\phi}\,\widetilde{\psi}\big\} \\
\mathcal{N}_{\alpha 3} &= -C_{\mu,\nu}\,x_\alpha\big\{(x_3+1)\phi^3 + (3-4\nu)(x_3+1)\widetilde{\phi}^3 - 6x_3(x_3-1)\widetilde{\phi}^5 + c_\nu\widetilde{\phi}\,\widetilde{\psi}\big\} \\
\mathcal{N}_{33} &= -C_{\mu,\nu}\big\{(3-4\nu)\phi + (x_3+1)^2\phi^3 + (1+c_\nu)\widetilde{\phi} \\
&\qquad\qquad + \big[(3-4\nu)(x_3-1)^2 + 2x_3\big]\widetilde{\phi}^3 - 6x_3(x_3-1)^2\widetilde{\phi}^5\big\}
\end{aligned} \tag{3.20}$$

where $C_{\mu,\nu} = 1/\{16\pi\mu(1-\nu)\}$, $c_\nu = 4(1-\nu)(1-2\nu)$ *and* $\alpha = 1,2$.

To establish (3.20), we observe that the columns $\mathcal{N}^{(i)}$ of $\mathcal{N}$ are determined by solving the equation $\boldsymbol{\mathcal{L}v} = \boldsymbol{e}_i\delta$ for $i = 1,2,3$ and using the Papkovich–Neuber representation

$$\boldsymbol{v} = C_{\mu,\nu}\big\{4(1-\nu)\boldsymbol{h} - \nabla\big(\boldsymbol{x}\cdot\boldsymbol{h} + \beta\big)\big\} \quad \text{with} \begin{cases} \Delta\boldsymbol{h} = 4\pi\boldsymbol{e}_i\delta \\ \Delta\beta = 4\pi\delta_{i3}\delta. \end{cases} \tag{3.21}$$

where δ_{ij} is the Kronecker symbol. The coupling between $\boldsymbol{h}$ and β is determined by the boundary conditions on the plane $\{x_3 = 0\}$, which are

$$\left.\begin{aligned}
&(1-2\nu)(\partial_3 h_\alpha + \partial_\alpha h_3) - \boldsymbol{x}\cdot\partial^2_{\alpha 3}\boldsymbol{h} - \partial^2_{\alpha 3}\beta = 0, \qquad (\alpha = 1,2), \\
&2\nu\,\mathrm{div}\,\boldsymbol{h} + 2(1-2\nu)\,\partial_3 h_3 - \boldsymbol{x}\cdot\partial^2_{33}\boldsymbol{h} - \partial^2_{33}\beta = 0,
\end{aligned}\right\} \quad \text{for } x_3 = 0. \tag{3.22}$$

Set

$$G(\boldsymbol{x},\boldsymbol{y}) := -\phi\big(\boldsymbol{x}-\boldsymbol{y}\big) + \phi\big(\boldsymbol{x}-\widetilde{\boldsymbol{y}}\big).$$

Denoting by $\langle f, g\rangle$ the action of the distribution f on the function g, we determine $\boldsymbol{h}$ and β taking advantage of the relation (which descends from the second Green identity)

$$F(\boldsymbol{x}) = \tfrac{1}{4\pi}\langle \Delta F, G(\boldsymbol{x},\cdot)\rangle, \tag{3.23}$$

applied to different choices of F.

Proof of Proposition 3.2.2. To determine $\mathcal{N}$, we consider separately the case of horizontal and vertical forcing. By symmetry, x_1 and x_2 can be interchanged.

Horizontal force: $\mathcal{L}\boldsymbol{v} = \boldsymbol{e}_1\delta$. We choose $h_2 = 0$, so that boundary conditions become

$$\begin{cases} (1-2\nu)(\partial_3 h_1 + \partial_1 h_3) - x_1\partial^2_{13}h_1 - \partial^2_{13}\beta = 0, \\ (1-2\nu)\partial_2 h_3 - x_1\partial^2_{23}h_1 - \partial^2_{23}\beta = 0, \\ 2\nu\,\partial_1 h_1 + 2(1-\nu)\,\partial_3 h_3 - x_1\partial^2_{33}h_1 - \partial^2_{33}\beta = 0, \end{cases} \qquad \text{for } x_3 = 0,$$

Differentiating the first equation with respect to x_1, the second with respect to x_2 and taking the difference, we obtain

$$0 = (1-2\nu)\partial^2_{23}h_1 + \partial^2_{23}h_1 = 2(1-\nu)\partial^2_{23}h_1 \qquad \text{for } x_3 = 0,$$

which suggests, after integration with respect to x_2, the choice $F := \partial_3 h_1$. Applying (3.23),

$$\partial_3 h_1 = -\partial_{y_3} G\big|_{\boldsymbol{y}=-\boldsymbol{e}_3} = -\partial_3(\phi + \widetilde{\phi}), \qquad \text{for } x_3 < 0,$$

and thus $h_1 = -(\phi + \widetilde{\phi})$.

Being $\partial_3 h_1$ null for $x_3 = 0$, integration of the second boundary condition encourages the choice $F := (1-2\nu)h_3 - \partial_3\beta$ which is zero for $x_3 = 0$. Hence, since $\Delta F = 0$, we deduce

$$(1-2\nu)h_3 - \partial_3\beta = 0, \qquad \text{for } x_3 < 0. \tag{3.24}$$

Concerning the third boundary condition, for $x_3 = 0$ we observe that

$$\begin{aligned} \partial_1 h_1 &= x_1(\phi^3 + \widetilde{\phi}^3) = 2x_1\widetilde{\phi}^3 = -2\partial_1\widetilde{\phi} \\ x_1\partial^2_{33}h_1 &= x_1(\phi^3 + \widetilde{\phi}^3 - 3\phi^5 - 3\widetilde{\phi}^5) = 2x_1(\widetilde{\phi}^3 - 3\widetilde{\phi}^5) = -2\big(\partial_1\widetilde{\phi} - \partial^2_{13}\widetilde{\phi}\big), \end{aligned}$$

since ϕ and $\widetilde{\phi}$ coincide when $x_3 = 0$. Substituting in the third boundary condition, we obtain

$$F := 2(1-\nu)\partial_3 h_3 - \partial^2_{33}\beta + 2(1-2\nu)\partial_1\widetilde{\phi} - 2\partial^2_{13}\widetilde{\phi} = 0 \quad \text{for } x_3 = 0.$$

Since $\Delta F = 0$, we infer

$$2(1-\nu)\partial_3 h_3 - \partial^2_{33}\beta + 2(1-2\nu)\partial_1\widetilde{\phi} - 2\partial^2_{13}\widetilde{\phi} = 0 \qquad \text{for } x_3 < 0,$$

and thus, being $\partial_1\widetilde{\phi} = -x_1\widetilde{\phi}^3 = -\partial_3(x_1\widetilde{\phi}\,\widetilde{\psi})$,

$$2(1-\nu)h_3 - \partial_3\beta = -2x_1\widetilde{\phi}^3 + 2(1-2\nu)x_1\widetilde{\phi}\,\widetilde{\psi} \qquad \text{for } x_3 < 0,$$

Coupling with (3.24), we deduce

$$\begin{cases} h_3 = -2x_1\widetilde{\phi}^3 + 2(1-2\nu)x_1\widetilde{\phi}\,\widetilde{\psi} \\ \partial_3\beta = -2(1-2\nu)x_1\widetilde{\phi}^3 + 2(1-2\nu)^2 x_1\widetilde{\phi}\,\widetilde{\psi} \end{cases} \qquad \text{for } x_3 < 0.$$

Recalling that $\widetilde{\phi}^3 = \partial_3(\widetilde{\phi}\,\widetilde{\psi})$ and $\widetilde{\phi}\,\widetilde{\psi} = \partial_3\widetilde{\psi}$, by integration,

$$\beta = -2(1-2\nu)x_1\widetilde{\phi}\,\widetilde{\psi} + 2(1-2\nu)^2 x_1\widetilde{\psi} \qquad \text{for } x_3 < 0.$$

Using the identity $(x_3-1)\widetilde{\phi}\,\widetilde{\psi} = \widetilde{\psi} - \widetilde{\phi}$, we infer

$$x_3 h_3 + \beta = x_1\big\{-2(1-2\nu)\widetilde{\phi} - 2x_3\widetilde{\phi}^3 + c_\nu\widetilde{\psi}\big\}.$$

Substituting in (3.21), we get the expressions for $\mathcal{N}_{i1}$ given in (3.20).

Vertical force: $\boldsymbol{\mathcal{L}v} = \boldsymbol{e}_3\delta$. Choosing $h_1 = h_2 = 0$, conditions (3.22) become

$$\begin{cases} (1-2\nu)\partial_\alpha h_3 - \partial^2_{\alpha 3}\beta = 0 \qquad (\alpha = 1,2), \\ 2(1-\nu)\partial_3 h_3 - \partial^2_{33}\beta = 0, \end{cases} \qquad \text{for } x_3 = 0.$$

Integrating the first relation with respect to x_α, we obtain

$$\begin{cases} (1-2\nu)h_3 - \partial_3\beta = 0, \\ 2(1-\nu)\partial_3 h_3 - \partial^2_{33}\beta = 0, \end{cases} \qquad \text{for } x_3 = 0.$$

Since $\Delta h_3 = \Delta\beta = \delta$, identity (3.23) with $F := (1-2\nu)h_3 - \partial_3\beta$ gives

$$\begin{aligned} (1-2\nu)h_3 - \partial_3\beta &= \big\{(1-2\nu)G + \partial_{y_3}G\big\}\big|_{\boldsymbol{y}=-\boldsymbol{e}_3} \\ &= (1-2\nu)(-\phi + \widetilde{\phi}) - (x_3+1)\phi^3 - (x_3-1)\widetilde{\phi}^3, \end{aligned} \tag{3.25}$$

for $x_3 < 0$. Applying (3.23) to $F := 2(1-\nu)\partial_3 h_3 - \partial^2_{33}\beta$, we deduce

$$\begin{aligned}2(1-\nu)\partial_3 h_3 - \partial^2_{33}\beta &= \left\{-2(1-\nu)\partial_{y_3}G - \partial^2_{y_3y_3}G\right\}\Big|_{\boldsymbol{y}=-\boldsymbol{e}_3}\\ &= \partial_3\left\{-2(1-\nu)(\phi+\widetilde{\phi}) + \partial_3(\phi-\widetilde{\phi})\right\}, \qquad \text{for } x_3<0.\end{aligned}$$

Integrating with respect to x_3, we infer

$$2(1-\nu)h_3 - \partial_3\beta = -2(1-\nu)(\phi+\widetilde{\phi}) - (x_3+1)\phi^3 + (x_3-1)\widetilde{\phi}^3, \quad \text{for } x_3<0.$$

Coupling with (3.25), we get explicit expressions for h_3 and $\partial_3\beta$, namely

$$\begin{cases} h_3 = -\phi - (3-4\nu)\widetilde{\phi} + 2(x_3-1)\widetilde{\phi}^3, \\ \partial_3\beta = (x_3+1)\phi^3 - c_\nu\widetilde{\phi} + (3-4\nu)(x_3-1)\widetilde{\phi}^3 \end{cases} \qquad \text{for } x_3<0.$$

Differentiation of $\partial_3\beta$ with respect to x_α gives

$$\begin{aligned}\partial^2_{3\alpha}\beta &= -3x_\alpha(x_3+1)\phi^5 + c_\nu x_\alpha\widetilde{\phi}^3 - 3(3-4\nu)x_\alpha(x_3-1)\widetilde{\phi}^5\\ &= \partial_3\left\{x_\alpha\phi^3 + c_\nu x_\alpha\widetilde{\phi}\,\widetilde{\psi} + (3-4\nu)x_\alpha\widetilde{\phi}^3\right\}\end{aligned}$$

and thus

$$\partial_\alpha\beta = x_\alpha\left\{\phi^3 + c_\nu\widetilde{\phi}\,\widetilde{\psi} + (3-4\nu)\widetilde{\phi}^3\right\}, \qquad \text{for } x_3<0.$$

Recalling identity (3.21), we deduce the corresponding expressions for $\mathcal{N}_{i3}$ in (3.20). □

With the explicit expression of function $\mathcal{N}(\boldsymbol{x})$ at hand we can now prove the Theorem 3.2.1.

Proof of Theorem 3.2.1. Uniqueness of the solution to (3.11) is similar to the one for problem (3.10) which we present in the following section.

The fundamental solution $\mathbf{N} = \mathbf{N}(\boldsymbol{x},\boldsymbol{y})$ in the half-space $\{x_3<0\}$ is such that its columns $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ solve $\boldsymbol{\mathcal{L}}\boldsymbol{v}_i = \delta_{\boldsymbol{y}}\boldsymbol{e}_i$ where $\delta_{\boldsymbol{y}}$ is the Dirac delta concentrated at $\boldsymbol{y}=(y_1,y_2,y_3)$ with $y_3<0$. Thus, the Neumann function $\mathbf{N}$ is given by

$$\mathbf{N}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{|y_3|}\mathcal{N}\left(\frac{x_1-y_1}{|y_3|}, \frac{x_2-y_2}{|y_3|}, \frac{x_3}{|y_3|}\right), \tag{3.26}$$

as a result of the homogeneity of δ and the second order degree of $\boldsymbol{\mathcal{L}}$.

Recalling the definitions of $\phi, \widetilde{\phi}, \widetilde{\psi}$ and computing at $(x_1 - y_1, x_2 - y_2, x_3)/|y_3|$, we obtain the identities

$$f := -\frac{\phi}{y_3} = \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}, \qquad \tilde{f} := -\frac{\widetilde{\phi}}{y_3} = \frac{1}{|\boldsymbol{x} - \tilde{\boldsymbol{y}}|},$$
$$\tilde{g} := -\frac{\widetilde{\psi}}{y_3} = \frac{1}{|\boldsymbol{x} - \tilde{\boldsymbol{y}}| - x_3 - y_3},$$

where $\tilde{\boldsymbol{y}} = (y_1, y_2, -y_3)$. Hence, the components of $C_{\mu,\nu}^{-1}\mathbf{N}$ are given by

$$\begin{aligned}
C_{\mu,\nu}^{-1} N_{\alpha\alpha} &= -(3-4\nu) f - (x_\alpha - y_\alpha)^2 f^3 - \tilde{f} - (3-4\nu)(x_\alpha - y_\alpha)^2 \tilde{f}^3 - c_\nu \tilde{g} \\
&\qquad + c_\nu (x_\alpha - y_\alpha)^2 \tilde{f}\tilde{g}^2 - 2x_3 y_3 \tilde{f}^3 + 6(x_\alpha - y_\alpha)^2 x_3 y_3 \tilde{f}^5 \\
C_{\mu,\nu}^{-1} N_{\alpha\beta} &= (x_\alpha - y_\alpha)(x_\beta - y_\beta)\big\{-f^3 - (3-4\nu)\tilde{f}^3 + c_\nu \tilde{f}\tilde{g}^2 + 6x_3 y_3 \tilde{f}^5\big\} \\
C_{\mu,\nu}^{-1} N_{3\alpha} &= (x_\alpha - y_\alpha)\big\{-(x_3 - y_3) f^3 - (3-4\nu)(x_3 - y_3)\tilde{f}^3 + c_\nu \tilde{f}\tilde{g} \\
&\qquad + 6x_3 y_3 (x_3 + y_3)\tilde{f}^5\big\} \\
C_{\mu,\nu}^{-1} N_{\alpha 3} &= (x_\alpha - y_\alpha)\Big\{-(x_3 - y_3) f^3 - (3-4\nu)(x_3 - y_3)\tilde{f}^3 - c_\nu \tilde{f}\tilde{g} \\
&\qquad - 6x_3 y_3 (x_3 + y_3)\tilde{f}^5\Big\} \\
C_{\mu,\nu}^{-1} N_{33} &= -(3-4\nu) f - (x_3 - y_3)^2 f^3 - (1 + c_\nu)\tilde{f} - (3-4\nu)(x_3 + y_3)^2 \tilde{f}^3 \\
&\qquad + 2x_3 y_3 \tilde{f}^3 - 6x_3 y_3 (x_3 + y_3)^2 \tilde{f}^5.
\end{aligned}$$

Recollecting the expression for fundamental solution $\mathbf{\Gamma}$ in the whole space and using the relation $\tilde{f} = \tilde{g} - (x_3 + y_3)\tilde{f}\tilde{g}$, the above formulas can be rewritten as $\mathbf{N} = \mathbf{\Gamma} + \mathbf{R}$ where $\mathbf{\Gamma}$ is computed at $\boldsymbol{x} - \boldsymbol{y}$ and the component R_{ij}, for $i, j = 1, 2, 3$, of $\mathbf{R}$ are given by

$$\begin{aligned}
R_{\alpha\alpha} &= C_{\mu,\nu}\big\{-(\tilde{f} + c_\nu \tilde{g}) - (3-4\nu)\eta_\alpha^2 \tilde{f}^3 + c_\nu \eta_\alpha^2 \tilde{f}\tilde{g}^2 - 2x_3 y_3 \big(\tilde{f}^3 - 3\eta_\alpha^2 \tilde{f}^5\big)\big\} \\
R_{\beta\alpha} &= C_{\mu,\nu}\eta_\alpha \eta_\beta \big\{-(3-4\nu)\tilde{f}^3 + c_\nu \tilde{f}\tilde{g}^2 + 6x_3 y_3 \tilde{f}^5\big\} \\
R_{3\alpha} &= C_{\mu,\nu}\eta_\alpha \big\{-(3-4\nu)(\eta_3 - 2y_3)\tilde{f}^3 + c_\nu \tilde{f}\tilde{g} + 6x_3 y_3 \eta_3 \tilde{f}^5\big\} \\
R_{\alpha 3} &= C_{\mu,\nu}\eta_\alpha \Big\{-(3-4\nu)(\eta_3 - 2y_3)\tilde{f}^3 - c_\nu \tilde{f}\tilde{g} - 6x_3 y_3 \eta_3 \tilde{f}^5\Big\} \\
R_{33} &= C_{\mu,\nu}\big\{-(\tilde{f} + c_\nu \tilde{g}) - (3-4\nu)\eta_3^2 \tilde{f}^3 + c_\nu \eta_3 \tilde{f}\tilde{g} + 2x_3 y_3 \big(\tilde{f}^3 - 3\eta_3^2 \tilde{f}^5\big)\big\},
\end{aligned}$$

where $\eta_\alpha = x_\alpha - y_\alpha$ for $\alpha = 1, 2$ and $\eta_3 = x_3 + y_3$, which can be recombined as

$$
\begin{aligned}
R_{ij} = C_{\mu,\nu}\big\{&-(\tilde f + c_\nu \tilde g)\delta_{ij} - (3-4\nu)\eta_i\eta_j \tilde f^3 \\
&+ 2(3-4\nu)y_3\big[\delta_{3i}(1-\delta_{3j})\eta_j + \delta_{3j}(1-\delta_{3i})\eta_i\big]\tilde f^3 \\
&+ c_\nu\big[\delta_{i3}\eta_j - \delta_{3j}(1-\delta_{3i})\eta_i\big]\tilde f\tilde g + c_\nu(1-\delta_{3j})(1-\delta_{3i})\eta_i\eta_j\tilde f\tilde g^2 \\
&- 2(1-2\delta_{3j})x_3y_3\big(\delta_{ij}\tilde f^3 - 3\eta_i\eta_j\tilde f^5\big)\big\}
\end{aligned}
$$

for $i,j = 1,2,3$. Since $x_3 = \eta_3 - y_3$, we obtain the decomposition $R_{ij} := R^1_{ij} + R^2_{ij} + R^3_{ij}$ where

$$
\begin{aligned}
R^1_{ij} &:= C_{\mu,\nu}\big\{-(\tilde f + c_\nu\tilde g)\delta_{ij} - (3-4\nu)\eta_i\eta_j\tilde f^3 + c_\nu\big[\delta_{3i}\eta_j - \delta_{3j}(1-\delta_{3i})\eta_i\big]\tilde f\tilde g \\
&\qquad + c_\nu(1-\delta_{3j})(1-\delta_{3i})\eta_i\eta_j\tilde f\tilde g^2\big\} \\
R^2_{ij} &:= 2C_{\mu,\nu}y_3\big\{(3-4\nu)\big[\delta_{3i}(1-\delta_{3j})\eta_j + \delta_{3j}(1-\delta_{3i})\eta_i\big]\tilde f^3 - (1-2\delta_{3j})\delta_{ij}\eta_3\tilde f^3 \\
&\qquad + 3(1-2\delta_{3j})\eta_i\eta_j\eta_3\tilde f^5\big\} \\
R^3_{ij} &:= 2C_{\mu,\nu}(1-2\delta_{3j})y_3^2\big\{\delta_{ij}\tilde f^3 - 3\eta_i\eta_j\tilde f^5\big\},
\end{aligned}
$$

that is the assertion.

□

To convert the problem (3.10) into an integral form, bounds on the decay at infinity of the Neumann function and its derivative at infinity are needed.

Proposition 3.2.3. *For any $M_x, M_y > 0$, there exists $C > 0$ such that*

$$
|\mathbf{N}(\boldsymbol{x},\boldsymbol{y})| \leqslant C\,|\boldsymbol{x}|^{-1} \quad \text{and} \quad |\nabla\mathbf{N}(\boldsymbol{x},\boldsymbol{y})| \leqslant C\,|\boldsymbol{x}|^{-2} \tag{3.27}
$$

for any $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^3_-$ with $|\boldsymbol{x}| \geqslant M_x$ and $|\boldsymbol{y}| \leqslant M_y$.

Proof. If ϕ is a homogeneous function of degree α defined and continuous in $\mathbb{R}^3_-\backslash\{\mathbf{0}\}$, then there exists a constant C such that

$$
|\phi(\boldsymbol{x})| \leqslant C|\boldsymbol{x}|^\alpha, \qquad \boldsymbol{x} \in \mathbb{R}^3_-\backslash\{\mathbf{0}\}.
$$

Thus, since $\mathbf{R}^k$ are homogeneous of degree $-k$, for $k = 1,2,3$, and

$$
|\boldsymbol{\eta}| - \eta_3 \geqslant |\boldsymbol{\eta}| = |\boldsymbol{x} - \tilde{\boldsymbol{y}}| \geqslant |\boldsymbol{x}| - M_y
$$

for $|\boldsymbol{x}|$ sufficiently large, the term $\mathbf{R}$ is bounded by

$$
|\mathbf{R}| \leqslant |\mathbf{R}^1| + |y_3||\mathbf{R}^2| + |y_3|^2|\mathbf{R}^3| \leqslant C\left(\frac{1}{|\boldsymbol{x}|} + \frac{|y_3|}{|\boldsymbol{x}|^2} + \frac{|y_3|^2}{|\boldsymbol{x}|^3}\right) \leqslant \frac{C}{|\boldsymbol{x}|}.
$$

Coupling with (3.5), we deduce the bound for $\mathbf{N}$.

The estimates on $|\nabla \mathbf{N}|$ is consequence of the homogeneity of derivatives of homogeneous functions together with the observation that $\tilde{f}$ and $\tilde{g}$ are C^1 in $\mathbb{R}^3_- \backslash \{\mathbf{0}\}$. □

3.2.2 Representation formula

Next, we derive an integral representation formula for $\boldsymbol{u}$ solution to the problem (3.10). For, we make use of single and double layer potentials defined in (3.6) and integral contributions relative to the regular part $\mathbf{R}$ of the Neumann function $\mathbf{N}$, defined by

$$\begin{aligned} \mathbf{S}^R \boldsymbol{\varphi}(\boldsymbol{x}) &:= \int_{\partial C} (\mathbf{R}(\boldsymbol{x}, \boldsymbol{y}))^T \boldsymbol{\varphi}(\boldsymbol{y})\, d\sigma(\boldsymbol{y}), && \boldsymbol{x} \in \mathbb{R}^3_-, \\ \mathbf{D}^R \boldsymbol{\varphi}(\boldsymbol{x}) &:= \int_{\partial C} \left(\frac{\partial \mathbf{R}}{\partial \boldsymbol{\nu}(\boldsymbol{y})}(\boldsymbol{x}, \boldsymbol{y}) \right)^T \boldsymbol{\varphi}(\boldsymbol{y})\, d\sigma(\boldsymbol{y}), && \boldsymbol{x} \in \mathbb{R}^3_-, \end{aligned} \tag{3.28}$$

where $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\partial C)$.

Theorem 3.2.4. *The solution $\boldsymbol{u}$ to (3.10) is such that*

$$\boldsymbol{u} = p\mathbf{S}^\Gamma \boldsymbol{n} - \mathbf{D}^\Gamma \boldsymbol{f} + p\mathbf{S}^R \boldsymbol{n} - \mathbf{D}^R \boldsymbol{f}, \qquad \text{in } \mathbb{R}^3_- \backslash \overline{C} \tag{3.29}$$

where $\mathbf{S}^\Gamma$, $\mathbf{D}^\Gamma$ are defined in (3.6), $\mathbf{S}^R$, $\mathbf{D}^R$ in (3.28), $p\boldsymbol{n}$ is the boundary condition in (3.10) and $\boldsymbol{f}$ is the trace of $\boldsymbol{u}$ on ∂C.

Before proving this theorem, we observe that $\boldsymbol{f}$ solves the integral equation

$$\left(\tfrac{1}{2}\mathbf{I} + \mathbf{K} + \mathbf{D}^R\right) \boldsymbol{f} = p\left(\mathbf{S}^\Gamma \boldsymbol{n} + \mathbf{S}^R \boldsymbol{n}\right), \qquad \text{on } \partial C, \tag{3.30}$$

obtained by the application of the trace properties of the double layer potential (3.7) in formula (3.29).

Proof of Theorem 3.2.4. Given $r, \varepsilon > 0$ such that $C \subset B_r(\mathbf{0})$ and $B_\varepsilon(\boldsymbol{y}) \subset \mathbb{R}^3_- \backslash \overline{C}$, let

$$\Omega_{r,\varepsilon} = \left(\mathbb{R}^3_- \cap B_r(\mathbf{0})\right) \backslash (C \cup B_\varepsilon(\boldsymbol{y}))$$

with r sufficiently large such that to contain the cavity C; additionally, we define $\partial B^h_r(\mathbf{0})$ as the intersection of the hemisphere with the boundary of

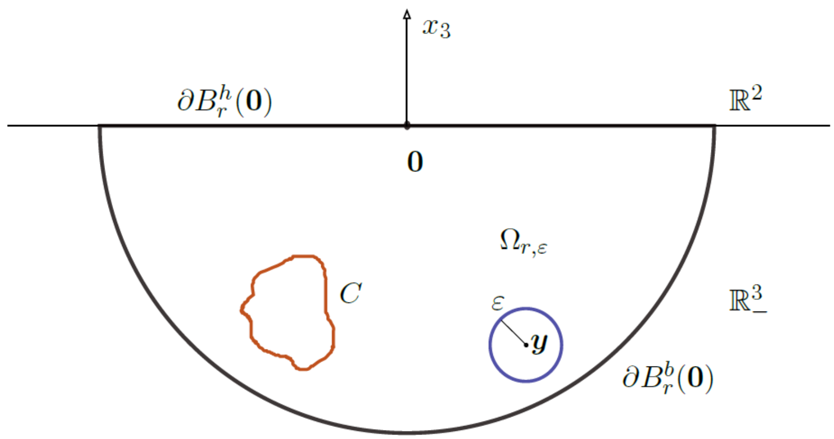


Figure 3.1. Domain $\Omega_{r,\varepsilon}$.

the half-space, and with $\partial B_r^b(\mathbf{0})$ the spherical cap (see Figure 3.1). Now, we apply Betti's formula (3.2) to $\boldsymbol{u}$ and the k-th column vector of $\mathbf{N}$, indicated by $\boldsymbol{N}^{(k)}$, for $k = 1, 2, 3$, in $\Omega_{r,\varepsilon}$, hence

$$
\begin{aligned}
0 &= \int_{\Omega_{r,\varepsilon}} \left[\boldsymbol{u}(\boldsymbol{x}) \cdot \mathcal{L}\boldsymbol{N}^{(k)}(\boldsymbol{x},\boldsymbol{y}) - \boldsymbol{N}^{(k)}(\boldsymbol{x},\boldsymbol{y}) \cdot \mathcal{L}\boldsymbol{u}(\boldsymbol{x})\right] d\boldsymbol{x} \\
&= \int_{\partial B_r^b(\mathbf{0})} \left[\frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{N}^{(k)}(\boldsymbol{x},\boldsymbol{y}) \cdot \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x})\right] d\sigma(\boldsymbol{x}) \\
&\quad - \int_{\partial B_\varepsilon(\boldsymbol{y})} \left[\frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{N}^{(k)}(\boldsymbol{x},\boldsymbol{y}) \cdot \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x})\right] d\sigma(\boldsymbol{x}) \\
&\quad - \int_{\partial C} \left[\frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{N}^{(k)}(\boldsymbol{x},\boldsymbol{y}) \cdot \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x})\right] d\sigma(\boldsymbol{x}) \\
&:= I_1 + I_2 + I_3,
\end{aligned}
$$

since, from (3.10) and the boundary condition in (3.11),

$$
\int_{\partial B_r^h(\mathbf{0})} \left[\frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{N}^{(k)}(\boldsymbol{x},\boldsymbol{y}) \cdot \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x})\right] d\sigma(\boldsymbol{x}) = 0.
$$

We show that the term I_1 goes to zero by using the behaviour at infinity of $\boldsymbol{u}$ given in (3.10) and of the Neumann function given in (3.27). Indeed, we have

$$\left|\int\limits_{\partial B_r^b(\mathbf{0})} \frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu_x}}(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{u}(\boldsymbol{x})\, d\sigma(\boldsymbol{x})\right| \leqslant \int\limits_{\partial B_r^b(\mathbf{0})} |\boldsymbol{u}| \left|\frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu_x}}\right| d\sigma(\boldsymbol{x})$$

$$\leqslant \frac{C}{r^2} \int\limits_{\partial B_r^b(\mathbf{0})} |\boldsymbol{u}(\boldsymbol{x})| d\sigma(\boldsymbol{x}).$$

This last integral can be estimated by means of the spherical coordinates $x_1 = r\sin\varphi\cos\theta$, $x_2 = r\sin\varphi\sin\theta$, $x_3 = r\cos\varphi$ where $\varphi \in [\pi/2, \pi]$, since $B_r^b(\mathbf{0})$ is a hemisphere in $\mathbb{R}^3_-$, and $\theta \in [0, 2\pi)$, indeed

$$\frac{C}{r^2} \int\limits_{\partial B_r^b(\mathbf{0})} |\boldsymbol{u}|\, d\sigma(\boldsymbol{x}) = C \int\limits_{\frac{\pi}{2}}^{\pi} \int\limits_{0}^{2\pi} |\boldsymbol{u}(r,\theta,\varphi)| \sin\varphi\, d\theta\, d\varphi$$

$$\leqslant C \sup_{\theta\in[0,2\pi),\varphi\in[\frac{\pi}{2},\pi]} |\boldsymbol{u}(r,\theta,\varphi)| \to 0,$$

as $r \to +\infty$, since $\boldsymbol{u} = o(\mathbf{1})$. Similarly

$$\left|\int\limits_{\partial B_r^b(\mathbf{0})} \boldsymbol{N}^{(k)}(\boldsymbol{x},\boldsymbol{y}) \cdot \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu_x}}(\boldsymbol{x})\, d\sigma(\boldsymbol{x})\right| \leqslant \int\limits_{\partial B_r^b(\mathbf{0})} |\boldsymbol{N}^{(k)}| \left|\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu_x}}\right| d\sigma(\boldsymbol{x})$$

$$\leqslant \frac{C}{r} \int\limits_{\partial B_r^b} \left|\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu_x}}\right| d\sigma(\boldsymbol{x}).$$

Again, passing through spherical coordinates, we get

$$\frac{C}{r} \int\limits_{\partial B_r^b(\mathbf{0})} \left|\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu_x}}\right| d\sigma(\boldsymbol{x}) \leqslant C \sup_{\theta\in[0,2\pi),\varphi\in[\frac{\pi}{2},\pi]} r\left|\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}}(r,\theta,\varphi)\right| \to 0, \tag{3.31}$$

as $r \to +\infty$, since $|\nabla \boldsymbol{u}| = o(r^{-1})$.
Integral I_2 gives the value of the function $\boldsymbol{u}$ in $\boldsymbol{y}$ as ε goes to zero. Indeed, we have

$$\left|\int\limits_{\partial B_\varepsilon(\boldsymbol{y})} \boldsymbol{N}^{(k)}(\boldsymbol{x},\boldsymbol{y}) \cdot \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu_x}}(\boldsymbol{x})\, d\sigma(\boldsymbol{x})\right| \leqslant \int\limits_{\partial B_\varepsilon(\boldsymbol{y})} |\boldsymbol{N}^{(k)}| \left|\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu_x}}\right| d\sigma(\boldsymbol{x})$$

$$\leqslant \sup_{\boldsymbol{x}\in\partial B_\varepsilon(\boldsymbol{y})} \left|\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu_x}}\right| \int\limits_{\partial B_\varepsilon(\boldsymbol{y})} \left[|\boldsymbol{\Gamma}^{(k)}| + |\boldsymbol{R}^{(k)}|\right] d\sigma(\boldsymbol{x}) = O(\varepsilon),$$

since the second integral has a continuous kernel. On the other hand

$$
\begin{aligned}
-\int\limits_{\partial B_\varepsilon(\boldsymbol{y})} \frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{u}(\boldsymbol{x})\, d\sigma(\boldsymbol{x}) &= -\boldsymbol{u}(\boldsymbol{y}) \cdot \int\limits_{\partial B_\varepsilon(\boldsymbol{y})} \frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y})\, d\sigma(\boldsymbol{x}) \\
&\quad + \int\limits_{\partial B_\varepsilon(\boldsymbol{y})} [\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}(\boldsymbol{x})] \cdot \frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y})\, d\sigma(\boldsymbol{x}).
\end{aligned}
$$

The latter integral tends to zero when ε goes to zero because

$$
\begin{aligned}
&\left| \int\limits_{\partial B_\varepsilon(\boldsymbol{y})} [\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}(\boldsymbol{x})] \cdot \frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y})\, d\sigma(\boldsymbol{x}) \right| \\
&\qquad \leqslant \sup_{\boldsymbol{x}\in\partial B_\varepsilon(\boldsymbol{y})} |\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}(\boldsymbol{x})| \int\limits_{\partial B_\varepsilon(\boldsymbol{y})} \left| \frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}} \right| d\sigma(\boldsymbol{x})
\end{aligned}
$$

and this last integral is bounded when ε goes to zero. Let finally observe that

$$
\begin{aligned}
-\boldsymbol{u}(\boldsymbol{y}) \cdot \int\limits_{\partial B_\varepsilon(\boldsymbol{y})} \frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y})\, d\sigma(\boldsymbol{x}) &= -\boldsymbol{u}(\boldsymbol{y}) \cdot \int\limits_{\partial B_\varepsilon(\boldsymbol{y})} \frac{\partial (\boldsymbol{\Gamma}^{(k)} + \boldsymbol{R}^{(k)})}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y})\, d\sigma(\boldsymbol{x}) \\
= -\boldsymbol{u}(\boldsymbol{y}) \cdot \int\limits_{\partial B_\varepsilon(\boldsymbol{y})} \frac{\partial \boldsymbol{\Gamma}^{(k)}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x}-\boldsymbol{y})\, d\sigma(\boldsymbol{x}) &- \boldsymbol{u}(\boldsymbol{y}) \cdot \int\limits_{\partial B_\varepsilon(\boldsymbol{y})} \frac{\partial \boldsymbol{R}^{(k)}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y})\, d\sigma(\boldsymbol{x}),
\end{aligned}
\tag{3.32}
$$

where the latter integral tends to zero as $\varepsilon \to 0$, since $\boldsymbol{R}^{(k)}$, for $k = 1, 2, 3$, represents the regular part of the Neumann function. To deal with the first integral, we preliminarily observe that direct differentiation gives

$$
\begin{aligned}
\left(\frac{\partial \boldsymbol{\Gamma}^{(k)}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}} \right)_h (\boldsymbol{x}-\boldsymbol{y}) = -c'_\nu \Bigg\{ & n_k(\boldsymbol{x}) \frac{\partial}{\partial x_h} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} - n_h(\boldsymbol{x}) \frac{\partial}{\partial x_k} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \\
& + \left[\delta_{hk} + \frac{3}{(1-2\nu)} \frac{\partial |\boldsymbol{x}-\boldsymbol{y}|}{\partial x_k} \frac{\partial |\boldsymbol{x}-\boldsymbol{y}|}{\partial x_h} \right] \frac{\partial}{\partial \boldsymbol{n}(\boldsymbol{x})} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \Bigg\},
\end{aligned}
\tag{3.33}
$$

where $c'_\nu := (1-2\nu)/(8\pi(1-\nu))$.
We substitute this expression into the integral (3.32) and we take into account

that

$$n_h(\boldsymbol{x}) = \frac{x_h - y_h}{|\boldsymbol{x}-\boldsymbol{y}|}, \qquad \frac{\partial}{\partial x_k}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} = -\frac{x_k - y_k}{|\boldsymbol{x}-\boldsymbol{y}|^3},$$

hence

$$\int\limits_{\partial B_\varepsilon(\boldsymbol{y})} n_h(\boldsymbol{x})\frac{\partial}{\partial x_k}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,d\sigma(\boldsymbol{x}) = -\int\limits_{\partial B_\varepsilon(\boldsymbol{y})} \frac{(x_h-y_h)(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^4}\,d\sigma(\boldsymbol{x}).$$

To solve this last integral we use spherical coordinates, that is

$$x_1 - y_1 = \varepsilon\sin\varphi\cos\theta, \qquad x_2 - y_2 = \varepsilon\sin\varphi\sin\theta, \qquad x_3 - y_3 = \varepsilon\cos\varphi,$$

where $\varphi \in [0,\pi]$ and $\theta \in [0, 2\pi)$. From a simple calculation it follows

$$-\int\limits_{\partial B_\varepsilon(\boldsymbol{y})} \frac{(x_h-y_h)(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^4}\,d\sigma(\boldsymbol{x}) = \begin{cases} 0 & \text{if } h \neq k \\ -\frac{4}{3}\pi & \text{if } h = k. \end{cases} \tag{3.34}$$

Therefore, from (3.33) and (3.34), we have

$$\int\limits_{\partial B_\varepsilon(\boldsymbol{y})} \left(n_k(\boldsymbol{x})\frac{\partial}{\partial x_h}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} - n_h(\boldsymbol{x})\frac{\partial}{\partial x_k}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right) d\sigma(\boldsymbol{x}) = 0, \tag{3.35}$$

for any h and k. Hence, (3.32) becomes

$$\begin{aligned} &-\boldsymbol{u}(\boldsymbol{y})\cdot \int\limits_{\partial B_\varepsilon(\boldsymbol{y})} \frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu}_{\boldsymbol{x}}}(\boldsymbol{x}-\boldsymbol{y})\,d\sigma(\boldsymbol{x}) \\ &= c'_\nu \sum_{h=1}^{3} u_h(\boldsymbol{y}) \int\limits_{\partial B_\varepsilon(\boldsymbol{y})} \left(\delta_{hk} + \frac{3}{(1-2\nu)}\frac{\partial|\boldsymbol{x}-\boldsymbol{y}|}{\partial x_k}\frac{\partial|\boldsymbol{x}-\boldsymbol{y}|}{\partial x_h}\right)\frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{x}}}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,d\sigma(\boldsymbol{x}) \\ &\qquad + O(\varepsilon). \end{aligned}$$

Employing again the spherical coordinates and the definition of c'_ν, we find that

$$\frac{1-2\nu}{8\pi(1-\nu)}\int\limits_{\partial B_\varepsilon(\boldsymbol{y})} \delta_{hk}\frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{x}}}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,d\sigma(\boldsymbol{x}) = \begin{cases} -\frac{1-2\nu}{2(1-\nu)} & \text{if } h = k \\ 0 & \text{if } h \neq k. \end{cases} \tag{3.36}$$

Similarly

$$\frac{3}{8\pi(1-\nu)}\int_{\partial B_\varepsilon(\boldsymbol{y})}\left(\frac{\partial|\boldsymbol{x}-\boldsymbol{y}|}{\partial x_k}\frac{\partial|\boldsymbol{x}-\boldsymbol{y}|}{\partial x_h}\right)\frac{\partial}{\partial \boldsymbol{n_x}}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,d\sigma(\boldsymbol{x}) = \begin{cases}-\frac{1}{2(1-\nu)} & \text{if}\, h=k\\ 0 & \text{if}\, h\neq k.\end{cases} \tag{3.37}$$

Putting together all the results in (3.36) and (3.37), we find that

$$\lim_{\varepsilon\to 0}\left(-\boldsymbol{u}(\boldsymbol{y})\cdot\int_{\partial B_\varepsilon(\boldsymbol{y})}\frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu_x}}(\boldsymbol{x}-\boldsymbol{y})\,d\sigma(\boldsymbol{x})\right) = -u_k(\boldsymbol{y}).$$

Using the definition of single and double layer potentials (3.6), (3.28) and splitting $\mathbf{N}$ as $\mathbf{\Gamma}+\mathbf{R}$ formula (3.29) holds. □

From the behaviour of the Neumann function given in (3.27) and the representation formula in (3.29), we immediately get

Corollary 3.2.5. *If $\boldsymbol{u}$ is a solution to* (3.10), *then*

$$\boldsymbol{u}(\boldsymbol{y}) = O(|\boldsymbol{y}|^{-1}) \qquad as \quad |\boldsymbol{y}|\to\infty. \tag{3.38}$$

3.2.3 Well-posedness

The well-posedness of the boundary value problem (3.10) reduces to show the invertibility of

$$\tfrac{1}{2}\mathbf{I}+\mathbf{K}+\mathbf{D}^R : \boldsymbol{L}^2(\partial C)\to \boldsymbol{L}^2(\partial C). \tag{3.39}$$

In particular, in order to prove the injectivity of the operator (3.39) we show the uniqueness of $\boldsymbol{u}$ following the classical approach based on the application of the Betti's formula (3.1) and the energy method, see [34, 44]. From the injectivity, it follows the existence of $\boldsymbol{u}$ proving the surjectivity of (3.39) which is obtained by the application of the index theory regarding bounded and linear operators.

First of all, let us recall the closed range theorem due to Banach (see [41, 62]).

Theorem 3.2.6 ([41, 62])**.** *Let X and Y be Banach spaces, and T a closed linear operator defined in X into Y such that $\overline{D(T)} = X$. Then the following propositions are all equivalent:*

a. $Im(T)$ is closed in Y; *b. $Im(T^*)$ is closed in X^*;*
c. $Im(T) = (Ker(T^*))^\perp$; *d. $Im(T^*) = (Ker(T))^\perp$.*

Through this theorem we can prove

Lemma 3.2.7. *The operator $\frac{1}{2}\mathbf{I} + \mathbf{K} : \boldsymbol{L}^2(\partial C) \to \boldsymbol{L}^2(\partial C)$ is invertible with bounded inverse.*

Proof. The assertion of this lemma is based on the invertibility of the operator $\frac{1}{2}\mathbf{I} + \mathbf{K}^*$ studied in [24]; it is known that

$$\tfrac{1}{2}\mathbf{I} + \mathbf{K}^* : \boldsymbol{L}^2(\partial C) \to \boldsymbol{L}^2(\partial C)$$

is a bounded linear operator, injective and with dense and closed range. Therefore, from Theorem 3.2.6 we have

$$\operatorname{Ker}\left(\tfrac{1}{2}\mathbf{I} + \mathbf{K}\right) = \{0\}, \qquad \operatorname{Im}\left(\tfrac{1}{2}\mathbf{I} + \mathbf{K}\right)^\perp = \{0\}$$

and $\operatorname{Im}(1/2\mathbf{I} + \mathbf{K})$ is closed. Then, it follows that the operator $\frac{1}{2}\mathbf{I} + \mathbf{K} : \boldsymbol{L}^2(\partial C) \to \boldsymbol{L}^2(\partial C)$ is bijective and the assertion follows exploiting the bounded inverse theorem. □

Since $\mathbf{D}^R$ has a continuous kernel we prove its compactness adapting the arguments contained in [43].

Lemma 3.2.8. *The operator $\mathbf{D}^R : \boldsymbol{L}^2(\partial C) \to \boldsymbol{L}^2(\partial C)$ is compact.*

Proof. For the sake of simplicity, we call

$$\mathbf{H}(\boldsymbol{x}, \boldsymbol{y}) := \frac{\partial \mathbf{R}}{\partial \boldsymbol{\nu}}(\boldsymbol{x}, \boldsymbol{y}), \qquad \boldsymbol{x}, \boldsymbol{y} \in \partial C$$

and we denote by $\boldsymbol{H}^{(k)}$, $k = 1, 2, 3$, the column vectors of the matrix $\mathbf{H}$.

Let $\boldsymbol{S}$ be a bounded set such that $\boldsymbol{S} \subset \boldsymbol{L}^2(\partial C)$, that is $\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\partial C)} \leqslant K$, for any $\boldsymbol{\varphi} \in \boldsymbol{S}$. Then, applying Cauchy-Schwarz inequality

$$|(\boldsymbol{D}^R \boldsymbol{\varphi}(\boldsymbol{y}))_k|^2 \leqslant \|\boldsymbol{H}^{(k)}(\cdot, \boldsymbol{y})\|^2_{\boldsymbol{L}^2(\partial C)} \|\boldsymbol{\varphi}\|^2_{\boldsymbol{L}^2(\partial C)} \leqslant K|\partial C| \max_{\boldsymbol{x}, \boldsymbol{y} \in \partial C} |\boldsymbol{H}^{(k)}|,$$

with $k = 1, 2, 3$, for all $\boldsymbol{y} \in \partial C$ and $\boldsymbol{\varphi} \in \boldsymbol{S}$. Hence $|\boldsymbol{D}^R(\boldsymbol{\varphi})| \leqslant K'$, with $K' > 0$, which implies that $\boldsymbol{D}^R(\boldsymbol{S})$ is bounded. Moreover, for all $\varepsilon > 0$ there exist $\boldsymbol{\varphi}, \boldsymbol{\varphi}' \in \boldsymbol{S}$ and $\delta > 0$ such that if $\|\boldsymbol{\varphi}(\boldsymbol{y}) - \boldsymbol{\varphi}'(\boldsymbol{y})\|_{\boldsymbol{L}^2(\partial C)} < \delta$ then, applying again the Cauchy-Schwarz inequality

$$|\boldsymbol{D}^R(\boldsymbol{\varphi} - \boldsymbol{\varphi}')(\boldsymbol{y})| < \varepsilon.$$

Thus $\mathbf{D}^R(\boldsymbol{S}) \subset \mathbf{C}(\partial C)$ where $\mathbf{C}(\partial C)$ indicates the space of continuous function on ∂C. Since each component of the matrix $\mathbf{H}$ is uniformly continuous on the compact set $\partial C \times \partial C$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\boldsymbol{H}^{(k)}(\boldsymbol{z}, \boldsymbol{x}) - \boldsymbol{H}^{(k)}(\boldsymbol{z}, \boldsymbol{y})| \leqslant \frac{\varepsilon}{\sqrt{3}K|\partial C|^{1/2}},$$

for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \partial C$ with $|\boldsymbol{x} - \boldsymbol{y}| < \delta$. Since

$$\begin{aligned} |(\boldsymbol{D}^R\boldsymbol{\varphi})_k(\boldsymbol{x}) - (\boldsymbol{D}^R\boldsymbol{\varphi})_k(\boldsymbol{y})| &\leqslant \int_{\partial C} |\boldsymbol{H}^{(k)}(\boldsymbol{z}, \boldsymbol{x}) - \boldsymbol{H}^{(k)}(\boldsymbol{z}, \boldsymbol{y})||\boldsymbol{\varphi}(\boldsymbol{z})| \, d\sigma(\boldsymbol{z}) \\ &\leqslant \|\boldsymbol{H}^{(k)}(\cdot, \boldsymbol{x}) - \boldsymbol{H}^{(k)}(\cdot, \boldsymbol{y})\|_{\boldsymbol{L}^2(\partial C)} \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\partial C)} \\ &\leqslant \frac{\varepsilon}{\sqrt{3}}, \end{aligned}$$

for $k = 1, 2, 3$, hence

$$|(\boldsymbol{D}^R\boldsymbol{\varphi})(\boldsymbol{x}) - (\boldsymbol{D}^R\boldsymbol{\varphi})(\boldsymbol{y})| \leqslant \varepsilon,$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \partial C$ and $\boldsymbol{\varphi} \in \boldsymbol{S}$, that is $\mathbf{D}^R(\boldsymbol{S})$ is equicontinuous. The assertion follows from Ascoli-Arzelà Theorem and noticing that $\boldsymbol{C}(\partial C)$ is dense in $\boldsymbol{L}^2(\partial C)$. $\square$

We now prove

Theorem 3.2.9 (uniqueness). *The boundary valure problem* (3.10) *admits a unique solution.*

Proof. Let $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$ be solutions to (3.10). Then the difference $\boldsymbol{v} := \boldsymbol{u}^1 - \boldsymbol{u}^2$ solves the homogeneous version of (3.10), that is

$$\operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{v}) = \mathbf{0} \qquad \text{in } \mathbb{R}^3_-\backslash C \tag{3.40}$$

with homogeneous boundary conditions

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}} = \mathbf{0} \qquad \text{on } \partial C, \qquad \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}} = \mathbf{0} \qquad \text{on } \mathbb{R}^2 \tag{3.41}$$

$$\boldsymbol{v} = O(|\boldsymbol{x}|^{-1}) \qquad \nabla \boldsymbol{v} = o(|\boldsymbol{x}|^{-1}) \qquad |\boldsymbol{x}| \to \infty,$$

where we make use of the decay condition at infinity comes from Corollary 3.2.5 for $\boldsymbol{v}$.
Applying Betti's formula (3.1) to $\boldsymbol{v}$ in $\Omega_r = (\mathbb{R}^3_- \cap B_r(\mathbf{0}))\backslash C$, we find

$$\int\limits_{\partial\Omega_r} \boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}}\, d\sigma(\boldsymbol{x}) = \int\limits_{\Omega_r} Q(\boldsymbol{v}, \boldsymbol{v})\, d\boldsymbol{x}$$

where Q is the quadratic form $Q(\boldsymbol{v}, \boldsymbol{v}) = \lambda(\operatorname{div} \boldsymbol{v})^2 + 2\mu|\widehat{\nabla} \boldsymbol{v}|^2$. From the behaviour of $\boldsymbol{v}$ and the boundary conditions (3.41), we estimate the previous integral defined on the surface of Ω_r; contributions over the surface of the cavity and the intersection of the hemisphere with the half-space are null by means of (3.41), whereas on the spherical cap

$$\left| \int\limits_{\partial B^b_r(\mathbf{0})} \boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}}\, d\sigma(\boldsymbol{x}) \right| \leqslant \int\limits_{\partial B^b_r(\mathbf{0})} |\boldsymbol{v}| \left| \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}} \right| d\sigma(\boldsymbol{x}) \leqslant \frac{C}{r} \int\limits_{\partial B^b_r(\mathbf{0})} \left| \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}} \right| d\sigma(\boldsymbol{x}).$$

As already done in (3.31) to obtain the representation formula, this integral can be evaluated by spherical coordinates; in particular, it tends to zero when $r \to +\infty$. Therefore

$$\int\limits_{\mathbb{R}^3_-\backslash C} \left\{ \lambda(\operatorname{div} \boldsymbol{v})^2 + 2\mu|\widehat{\nabla} \boldsymbol{v}|^2 \right\} d\boldsymbol{x} = 0.$$

Since the quadratic form is positive definite for the parameters range $3\lambda + 2\mu > 0$ and $\mu > 0$, we have that

$$\widehat{\nabla} \boldsymbol{v} = \mathbf{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \mathbb{R}^3_-\backslash C, \tag{3.42}$$

It follows that the rigid displacements $\boldsymbol{v} = \boldsymbol{a} + \mathbf{A}\boldsymbol{x}$, with $\boldsymbol{a} \in \mathbb{R}^3$ and $\mathbf{A}$ belonging to the space of the anti-symmetric matrices (see (3.9) and, for more details, [4, 20]), could be the only nonzero solutions which satisfy (3.40), the boundary conditions in (3.41) and (3.42). However, in this case they are excluded thanks to the behaviour of the function $\boldsymbol{v}$ at infinity. Hence, we obtain that $\boldsymbol{v} = \mathbf{0}$, that is $\boldsymbol{u}^1 = \boldsymbol{u}^2$ in $\mathbb{R}^3_-\backslash C$. $\square$

The uniqueness result for the problem (3.10) ensures the injectivity of the operator (3.39). In order to prove the surjectivity of the operator (3.39) we recall, for the reader convenience, the definition of the index of an operator (see [1, 41])

Definition 3.2.1 ([1, 41])**.** *Given a bounded operator $T : X \to Y$ between two Banach spaces, the index of the operator T is the extended real number defined as*

$$i(T) = dim(Ker(T)) - dim(Y/Im(T)),$$

where $dim(Ker(T))$ is called the ***nullity*** *and $dim(Y/Im(T))$ the* ***defect*** *of T. In particular, when the nullity and the defect are both finite the operator T is said to be* ***Fredholm****.*

We remember also an important theorem regarding the index of a bounded linear operator perturbed with a compact operator (see [1]).

Theorem 3.2.10 ([1])**.** *Let $T : X \to Y$ be a bounded linear operator and $K : X \to Y$ a compact operator from two Banach spaces. Then $T + K$ is Fredholm with index $i(T + K) = i(T)$.*

Now all the ingredients are supplied in order to prove the surjectivity of the operator.

Theorem 3.2.11. *The operator $\frac{1}{2}\mathbf{I} + \mathbf{K} + \mathbf{D}^R$ is onto in $\boldsymbol{L}^2(\partial C)$.*

Proof. From Lemma 3.2.7 we have that the operator $\frac{1}{2}\mathbf{I} + \mathbf{K} : \boldsymbol{L}^2(\partial C) \to \boldsymbol{L}^2(\partial C)$ is Fredholm with index $i\left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right) = 0$, because both the nullity and the defect of this operator are null. Moreover, since the operator $\mathbf{D}^R$ is compact from Lemma 3.2.8, it follows by means of Theorem 3.2.10 that

$$i\left(\frac{1}{2}\mathbf{I} + \mathbf{K} + \mathbf{D}^R\right) = 0.$$

Hence

$$\dim\left(\mathrm{Ker}\left(\frac{1}{2}\mathbf{I} + \mathbf{K} + \mathbf{D}^R\right)\right) = \dim\left(\mathbf{L}^2(\partial C)/\mathrm{Im}\left(\frac{1}{2}\mathbf{I} + \mathbf{K} + \mathbf{D}^R\right)\right).$$

Since the operator $\frac{1}{2}\mathbf{I} + \mathbf{K} + \mathbf{D}^R$ is injective it shows that $\dim(\mathrm{Ker}(\frac{1}{2}\mathbf{I} + \mathbf{K} + \mathbf{D}^R)) = 0$. Finally, $\dim(\mathbf{L}^2(\partial C)/\mathrm{Im}(\frac{1}{2}\mathbf{I} + \mathbf{K} + \mathbf{D}^R)) = 0$, that is

$$\mathrm{Im}\left(\frac{1}{2}\mathbf{I} + \mathbf{K} + \mathbf{D}^R\right) = \boldsymbol{L}^2(\partial C).$$

$\square$

Summing up, it follows

Corollary 3.2.12. *There exists a unique solution to* (3.10).

Proof. Uniqueness follows from Theorem 3.2.9 and the existence from Theorem 3.2.11. □

3.3 Rigorous derivation of the asymptotic expansion

In this section, with the integral representation formula (3.29) at hand, we consider the hypothesis that the cavity C is small compared to the distance from the boundary of the half-space. The aim is to derive an asymptotic expansion of the solution $\boldsymbol{u}$.

In particular, let us take the cavity, that from now on we denote by C_ε to highlight the dependence from ε, as

$$C_\varepsilon = \boldsymbol{z} + \varepsilon\Omega$$

where Ω is a bounded Lipschitz domain containing the origin. At the same time, we write the solution of the boundary value problem (3.10) as $\boldsymbol{u}_\varepsilon$. From (3.29), recalling that $\mathbf{N} = \boldsymbol{\Gamma} + \mathbf{R}$, we have

$$\begin{aligned} u_\varepsilon^k(\boldsymbol{y}) &= p \int\limits_{\partial C_\varepsilon} \boldsymbol{N}^{(k)}(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{x})\, d\sigma(\boldsymbol{x}) - \int\limits_{\partial C_\varepsilon} \frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu}}(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{f}(\boldsymbol{x})\, d\sigma(\boldsymbol{x}) \\ &:= I_1^{(k)}(\boldsymbol{y}) + I_2^{(k)}(\boldsymbol{y}), \quad \boldsymbol{y} \in \mathbb{R}^2 \end{aligned} \tag{3.43}$$

for $k = 1, 2, 3$, where u_ε^k indicates the k-th component of the displacement vector and $\boldsymbol{f}$ is the solution of (3.30), that is

$$\left(\tfrac{1}{2}\mathbf{I} + \mathbf{K}_\varepsilon + \mathbf{D}_\varepsilon^R\right) \boldsymbol{f}_\varepsilon(\boldsymbol{x}) = p\left(\mathbf{S}_\varepsilon^\Gamma(\boldsymbol{n})(\boldsymbol{x}) + \mathbf{S}_\varepsilon^R(\boldsymbol{n})(\boldsymbol{x})\right), \qquad \boldsymbol{x} \in \partial C_\varepsilon \tag{3.44}$$

where we add the dependence from ε to all the layer potentials to distinguish them, in the sequel, from the layer potential defined over a domain independent from ε. In what follows, with $\mathbb{I}$ we indicate the fourth-order symmetric tensor such that $\mathbb{I}\mathbf{A} = \widehat{\mathbf{A}}$ and for any fixed value of $\varepsilon > 0$, given $\boldsymbol{h} : \partial C_\varepsilon \to \mathbb{R}^3$, we introduce the function $\boldsymbol{h}^\sharp : \partial\Omega \to \mathbb{R}^3$ defined by

$$\boldsymbol{h}^\sharp(\boldsymbol{\zeta}) := \boldsymbol{h}(\boldsymbol{z} + \varepsilon\boldsymbol{\zeta}), \qquad \boldsymbol{\zeta} \in \partial\Omega.$$

Moreover, we consider the functions $\boldsymbol{\theta}^{qr}$, for $q, r = 1, 2, 3$, solutions to

$$\operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\theta}^{qr}) = 0 \quad \text{in } \mathbb{R}^3 \backslash \Omega, \qquad \frac{\partial \boldsymbol{\theta}^{qr}}{\partial \boldsymbol{\nu}} = -\frac{1}{3\lambda + 2\mu}\mathbb{C}\boldsymbol{n} \quad \text{on } \partial\Omega, \tag{3.45}$$

with the decay conditions at infinity

$$|\boldsymbol{\theta}^{qr}| = O(|\boldsymbol{x}|^{-1}), \qquad |\nabla\boldsymbol{\theta}^{qr}| = O(|\boldsymbol{x}|^{-2}), \qquad \text{as } |\boldsymbol{x}| \to \infty, \tag{3.46}$$

where the condition $\partial\boldsymbol{\theta}^{qr}/\partial\boldsymbol{\nu}$ has to be read as

$$\left(\frac{\partial \boldsymbol{\theta}^{qr}}{\partial \boldsymbol{\nu}}\right)_i = -\frac{1}{3\lambda + 2\mu}\mathbb{C}_{ijqr}n_j.$$

We now state our main result

Theorem 3.3.1 (asymptotic expansion)**.** *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ at any $\boldsymbol{y} \in \mathbb{R}^2$ the following expansion holds*

$$u_\varepsilon^k(\boldsymbol{y}) = \varepsilon^3|\Omega|p\widehat{\nabla}_{\boldsymbol{z}}\boldsymbol{N}^{(k)}(\boldsymbol{z}, \boldsymbol{y}) : \mathbb{M}\mathbf{I} + O(\varepsilon^4), \tag{3.47}$$

for $k = 1, 2, 3$, where $O(\varepsilon^4)$ denotes a quantity bounded by $C\varepsilon^4$ for some uniform constant $C > 0$, and $\mathbb{M}$ is the fourth-order elastic moment tensor defined by

$$\mathbb{M} := \mathbb{I} + \frac{1}{|\Omega|}\int_{\partial\Omega} \mathbb{C}(\boldsymbol{\theta}^{qr}(\boldsymbol{\zeta}) \otimes \boldsymbol{n}(\boldsymbol{\zeta}))\, d\sigma(\boldsymbol{\zeta}), \tag{3.48}$$

where $\boldsymbol{\theta}^{qr}$, for $q, r = 1, 2, 3$, solve the problem in (3.45) and (3.46).

Before proving the theorem on the asymptotic expansion of $\boldsymbol{u}_\varepsilon$, we need to present some results

Lemma 3.3.2. *The integral equation (3.44), when $\boldsymbol{x} = \boldsymbol{z} + \varepsilon\boldsymbol{\zeta}$, with $\boldsymbol{\zeta} \in \partial\Omega$, is such that*

$$\left(\tfrac{1}{2}\mathbf{I} + \mathbf{K} + \varepsilon^2\boldsymbol{\Lambda}_{\Omega,\varepsilon}\right)\boldsymbol{f}^\sharp(\boldsymbol{\zeta}) = \varepsilon p\mathbf{S}^\Gamma(\boldsymbol{n})(\boldsymbol{\zeta}) + O(\varepsilon^2), \tag{3.49}$$

where

$$\boldsymbol{\Lambda}_{\Omega,\varepsilon}\boldsymbol{f}^\sharp(\boldsymbol{\eta}) := \int_{\partial\Omega} \frac{\partial\mathbf{R}}{\partial\boldsymbol{\nu}(\boldsymbol{\eta})}(\boldsymbol{z} + \varepsilon\boldsymbol{\eta}, \boldsymbol{z} + \varepsilon\boldsymbol{\zeta})\boldsymbol{f}^\sharp(\boldsymbol{\eta})\, d\sigma(\boldsymbol{\eta})$$

is uniformly bounded in ε. Moreover, when ε is sufficiently small, we have

$$\boldsymbol{f}^\sharp(\boldsymbol{\zeta}) = \varepsilon p\left(\tfrac{1}{2}\mathbf{I} + \mathbf{K}\right)^{-1}\mathbf{S}^\Gamma(\boldsymbol{n})(\boldsymbol{\zeta}) + O(\varepsilon^2), \qquad \boldsymbol{\zeta} \in \partial\Omega.$$

Proof. At the point $\boldsymbol{z}+\varepsilon\boldsymbol{\zeta}$, where $\boldsymbol{\zeta}\in\partial\Omega$, we obtain

$$\begin{aligned}\mathbf{D}^R_\varepsilon \boldsymbol{f}(\boldsymbol{z}+\varepsilon\boldsymbol{\zeta}) &= \int\limits_{\partial C_\varepsilon} \frac{\partial \mathbf{R}}{\partial \boldsymbol{\nu}(\boldsymbol{t})}(\boldsymbol{t},\boldsymbol{z}+\varepsilon\boldsymbol{\zeta})\boldsymbol{f}(\boldsymbol{t})\,d\sigma(\boldsymbol{t})\\ &= \varepsilon^2 \int\limits_{\partial\Omega} \frac{\partial \mathbf{R}}{\partial \boldsymbol{\nu}(\boldsymbol{\eta})}(\boldsymbol{z}+\varepsilon\boldsymbol{\eta},\boldsymbol{z}+\varepsilon\boldsymbol{\zeta})\boldsymbol{f}^\sharp(\boldsymbol{\eta})\,d\sigma(\boldsymbol{\eta}).\end{aligned}$$

Therefore, recalling that the kernel $\partial\mathbf{R}/\partial\boldsymbol{\nu}(\boldsymbol{\eta})$ is continuous we get

$$\mathbf{D}^R_\varepsilon = \varepsilon^2 \boldsymbol{\Lambda}_{\Omega,\varepsilon} \tag{3.50}$$

where $\|\boldsymbol{\Lambda}_{\Omega,\varepsilon}\| \leqslant C'$, with C' independent from ε.
For the integral

$$\mathbf{K}_\varepsilon \boldsymbol{f}(\boldsymbol{z}+\varepsilon\boldsymbol{\zeta}) = \text{p.v.} \int\limits_{\partial C_\varepsilon} \frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{\nu}(\boldsymbol{t})}(\boldsymbol{t}-\boldsymbol{z}-\varepsilon\boldsymbol{\zeta})\boldsymbol{f}(\boldsymbol{t})\,d\sigma(\boldsymbol{t})$$

we use the explicit expression of the conormal derivative of the fundamental solution of the Lamé operator given in (3.33). In particular, since (3.33) is a homogeneous function of degree -2, with the substitution $\boldsymbol{t}=\boldsymbol{z}+\varepsilon\boldsymbol{\eta}$, we find

$$\begin{aligned}&\left(\frac{\partial\boldsymbol{\Gamma}^{(k)}}{\partial\boldsymbol{\nu}}\right)_h (\varepsilon(\boldsymbol{\eta}-\boldsymbol{\zeta}))\\ &\quad= -\frac{1}{4\pi\varepsilon^2}\Bigg\{\left[\frac{1-2\nu}{2(1-\nu)}\delta_{hk} + \frac{3}{2(1-\nu)}\frac{\eta_k-\zeta_k}{|\boldsymbol{\eta}-\boldsymbol{\zeta}|}\frac{\eta_h-\zeta_h}{|\boldsymbol{\eta}-\boldsymbol{\zeta}|}\right]\frac{\partial}{\partial\boldsymbol{n}(\boldsymbol{\eta})}\frac{1}{|\boldsymbol{\eta}-\boldsymbol{\zeta}|}\\ &\qquad\qquad + \frac{1-2\nu}{2(1-\nu)}n_h(\boldsymbol{\eta})\frac{\eta_k-\zeta_k}{|\boldsymbol{\eta}-\boldsymbol{\zeta}|^3} - \frac{1-2\nu}{2(1-\nu)}n_k(\boldsymbol{\eta})\frac{\eta_h-\zeta_h}{|\boldsymbol{\eta}-\boldsymbol{\zeta}|^3}\Bigg\}\\ &\quad= \frac{1}{\varepsilon^2}\left(\frac{\partial\boldsymbol{\Gamma}^{(k)}}{\partial\boldsymbol{\nu}}\right)_h(\boldsymbol{\eta}-\boldsymbol{\zeta}),\end{aligned}$$

for $h,k=1,2,3$. Therefore, we immediately obtain that

$$\mathbf{K}_\varepsilon \boldsymbol{f}(\boldsymbol{z}+\varepsilon\boldsymbol{\zeta}) = \text{p.v.}\int\limits_{\partial\Omega} \frac{\partial\boldsymbol{\Gamma}}{\partial\boldsymbol{\nu}(\boldsymbol{\eta})}(\boldsymbol{\eta}-\boldsymbol{\zeta})\boldsymbol{f}^\sharp(\boldsymbol{\eta})\,d\sigma(\boldsymbol{\eta}) = \mathbf{K}\boldsymbol{f}^\sharp(\boldsymbol{\zeta}). \tag{3.51}$$

Evaluating the other integrals in (3.44) we obtain

$$\mathbf{S}_{\varepsilon}^{\Gamma}(\boldsymbol{n})(\boldsymbol{z}+\varepsilon\boldsymbol{\zeta}) = \int\limits_{\partial C_{\varepsilon}} \boldsymbol{\Gamma}(\boldsymbol{t}-\boldsymbol{z}-\varepsilon\boldsymbol{\zeta})\boldsymbol{n}(\boldsymbol{t})\, d\sigma(\boldsymbol{t})$$

hence, choosing $\boldsymbol{t} = \boldsymbol{z} + \varepsilon\boldsymbol{\eta}$, with $\boldsymbol{\eta} \in \partial\Omega$, we find

$$\mathbf{S}_{\varepsilon}^{\Gamma}(\boldsymbol{n})(\boldsymbol{z}+\varepsilon\boldsymbol{\zeta}) = \varepsilon^2 \int\limits_{\partial\Omega} \boldsymbol{\Gamma}(\varepsilon(\boldsymbol{\eta}-\boldsymbol{\zeta}))\boldsymbol{n}(\boldsymbol{\eta})\, d\sigma(\boldsymbol{\eta}) = \varepsilon\mathbf{S}^{\Gamma}(\boldsymbol{n})(\boldsymbol{\zeta}), \tag{3.52}$$

where the last equality follows noticing that the fundamental solution is homogeneous of degree -1. In a similar way

$$\mathbf{S}_{\varepsilon}^{R}(\boldsymbol{n})(\boldsymbol{z}+\varepsilon\boldsymbol{\zeta}) = \int\limits_{\partial C_{\varepsilon}} \mathbf{R}(\boldsymbol{t},\boldsymbol{z}+\varepsilon\boldsymbol{\zeta})\boldsymbol{n}(\boldsymbol{t})\, d\sigma(\boldsymbol{t}),$$

hence, taking again $\boldsymbol{t} = \boldsymbol{z} + \varepsilon\boldsymbol{\eta}$, we find

$$\mathbf{S}_{\varepsilon}^{R}(\boldsymbol{n})(\boldsymbol{z}+\varepsilon\boldsymbol{\zeta}) = \varepsilon^2 \int\limits_{\partial\Omega} \mathbf{R}(\boldsymbol{z}+\varepsilon\boldsymbol{\eta},\boldsymbol{z}+\varepsilon\boldsymbol{\zeta})\boldsymbol{n}(\boldsymbol{\eta})\, d\sigma(\boldsymbol{\eta})$$

and since $\mathbf{R}$ is regular it follows that

$$\mathbf{S}_{\varepsilon}^{R}(\boldsymbol{n})(\boldsymbol{z}+\varepsilon\boldsymbol{\zeta}) = O(\varepsilon^2). \tag{3.53}$$

Relation (3.49) follows putting together the result in (3.50), (3.51), (3.52) and (3.53).

To conclude, from (3.49) we have

$$\left(\tfrac{1}{2}\mathbf{I}+\mathbf{K}\right)\left(\mathbf{I}+\varepsilon^2\left(\tfrac{1}{2}\mathbf{I}+\mathbf{K}\right)^{-1}\boldsymbol{\Lambda}_{\varepsilon,\Omega}\right)\boldsymbol{f}^{\sharp} = \varepsilon p\mathbf{S}^{\Gamma}(\boldsymbol{n}) + O(\varepsilon^2), \qquad \text{on } \partial\Omega.$$

From Lemma 3.2.7 and the continuous property of $\boldsymbol{\Lambda}_{\varepsilon,\Omega}$ described before, we have

$$\left\|\left(\tfrac{1}{2}\mathbf{I}+\mathbf{K}\right)^{-1}\boldsymbol{\Lambda}_{\varepsilon,\Omega}\right\| \leqslant C$$

where $C > 0$ is independent from ε. On the other hand, choosing $\varepsilon_0^2 = 1/2C$, it follows that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\mathbf{I}+\varepsilon^2\left(\tfrac{1}{2}\mathbf{I}+\mathbf{K}\right)^{-1}\boldsymbol{\Lambda}_{\varepsilon,\Omega}$$

is invertible and

$$\left(\mathbf{I} + \varepsilon^2 \left(\tfrac{1}{2}\mathbf{I} + \mathbf{K}\right)^{-1} \boldsymbol{\Lambda}_{\varepsilon,\Omega}\right)^{-1} = \mathbf{I} + O(\varepsilon^2).$$

Therefore

$$\boldsymbol{f}^\sharp = \varepsilon p \left(\tfrac{1}{2}\mathbf{I} + \mathbf{K}\right)^{-1} \mathbf{S}^\Gamma(\boldsymbol{n}) + O(\varepsilon^2), \qquad \text{on } \partial\Omega,$$

that is the assertion. □

For ease of reading, we define the function $\boldsymbol{w} : \partial\Omega \to \partial\Omega$ as

$$\boldsymbol{w}(\boldsymbol{\zeta}) := -\left(\tfrac{1}{2}\mathbf{I} + \mathbf{K}\right)^{-1} \mathbf{S}^\Gamma(\boldsymbol{n})(\boldsymbol{\zeta}), \qquad \boldsymbol{\zeta} \in \partial\Omega. \tag{3.54}$$

Taking the problem

$$\operatorname{div}\left(\mathbb{C}\widehat{\nabla}\boldsymbol{v}\right) = 0 \qquad \text{in } \mathbb{R}^3\backslash\Omega, \qquad\qquad \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}} = -\boldsymbol{n} \qquad \text{on } \partial\Omega \tag{3.55}$$

with decay conditions at infinity

$$\boldsymbol{v} = O(|\boldsymbol{x}|^{-1}), \qquad\qquad |\nabla \boldsymbol{v}| = O(|\boldsymbol{x}|^{-2}) \qquad \text{as } |\boldsymbol{x}| \to +\infty, \tag{3.56}$$

we show that $\boldsymbol{w}(\boldsymbol{x})$, for $\boldsymbol{x} \in \partial\Omega$, is the trace of $\boldsymbol{v}$ on the boundary of Ω. The well-posedness of this problem is a classical result in the theory of linear elasticity so we remind the reader, for example, to [34, 37, 44].

Proposition 3.3.3. *The function $\boldsymbol{w}$, defined in (3.54), is such that $\boldsymbol{w} = \boldsymbol{v}\big|_{\boldsymbol{x}\in\partial\Omega}$ where $\boldsymbol{v}$ is the solution to (3.55) and (3.56).*

Proof. Applying second Betti's formula to the fundamental solution $\mathbf{\Gamma}$ and the function $\boldsymbol{v}$ into the domain $B_r(\mathbf{0})\backslash(\Omega \cup B_\varepsilon(\boldsymbol{x}))$, with $\varepsilon > 0$ and $r > 0$ sufficiently large such that to contain the cavity Ω, we obtain, as done in a similar way in the proof of the Theorem 3.2.4,

$$\boldsymbol{v}(\boldsymbol{x}) = -\mathbf{S}^\Gamma \boldsymbol{n}(\boldsymbol{x}) - \mathbf{D}^\Gamma \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^3\backslash\Omega$$

Therefore, from the single and double layer potential properties for the elastostatic equations, we find

$$\boldsymbol{v}(\boldsymbol{x}) = -\mathbf{S}^\Gamma \boldsymbol{n}(\boldsymbol{x}) - \left(-\tfrac{1}{2}\mathbf{I} + \mathbf{K}\right) \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial\Omega$$

hence

$$\boldsymbol{v}(\boldsymbol{x}) = -\left(\tfrac{1}{2}\mathbf{I} + \mathbf{K}\right)^{-1} \mathbf{S}^\Gamma(\boldsymbol{n})(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial\Omega$$

that is the assertion. □

We note that the function $\boldsymbol{v}$, as well as its trace $\boldsymbol{w}$ on $\partial\Omega$, can be written in terms of the functions $\boldsymbol{\theta}^{qr}$. Indeed, taking

$$\boldsymbol{v} = \boldsymbol{\theta}^{qr}\delta_{qr},$$

where we use the convention to sum up the repeated indices, and using (3.45) and (3.46), it is straightforward to see that the elastostatic equation and the boundary condition in (3.55) are satisfied.

Proof of Theorem 3.3.1. We study separately the two integrals $I_1^{(k)}, I_2^{(k)}$ defined in (3.43). Since $\boldsymbol{y} \in \mathbb{R}^2$ and $\boldsymbol{x} \in \partial C_\varepsilon = \boldsymbol{z} + \varepsilon\boldsymbol{\zeta}$, with $\boldsymbol{\zeta} \in \partial\Omega$, we consider the Taylor expansion for the Neumann function, that is

$$\boldsymbol{N}^{(k)}(\boldsymbol{z} + \varepsilon\boldsymbol{\zeta}, \boldsymbol{y}) = \boldsymbol{N}^{(k)}(\boldsymbol{z}, \boldsymbol{y}) + \varepsilon\nabla\boldsymbol{N}^{(k)}(\boldsymbol{z}, \boldsymbol{y})\boldsymbol{\zeta} + O(\varepsilon^2), \tag{3.57}$$

for $k = 1, 2, 3$. By the change of variable $\boldsymbol{x} = \boldsymbol{z} + \varepsilon\boldsymbol{\zeta}$ and substituting (3.57) in $I_1^{(k)}$, we find

$$\begin{aligned} I_1^{(k)} &= \varepsilon^2 p\boldsymbol{N}^{(k)}(\boldsymbol{z}, \boldsymbol{y}) \cdot \int_{\partial\Omega} \boldsymbol{n}\, d\sigma(\boldsymbol{\zeta}) + \varepsilon^3 p \int_{\partial\Omega} \boldsymbol{n}(\boldsymbol{\zeta}) \cdot \nabla\boldsymbol{N}^{(k)}(\boldsymbol{z}, \boldsymbol{y})\boldsymbol{\zeta}\, d\sigma(\boldsymbol{\zeta}) + O(\varepsilon^4) \\ &:= p\left(\varepsilon^2 I_{11}^{(k)} + \varepsilon^3 I_{12}^{(k)}\right) + O(\varepsilon^4). \end{aligned}$$

Integral $I_{11}^{(k)}$ is null, in fact, applying the divergence theorem

$$\int_{\partial\Omega} \boldsymbol{n}(\boldsymbol{\zeta})\, d\sigma(\boldsymbol{\zeta}) = 0.$$

For the integral $I_{12}^{(k)}$, we use the equality $\boldsymbol{n} \cdot \nabla\boldsymbol{N}^{(k)}\boldsymbol{\zeta} = \nabla\boldsymbol{N}^{(k)} : (\boldsymbol{n}(\boldsymbol{\zeta}) \otimes \boldsymbol{\zeta})$, therefore

$$I_1^{(k)} = \varepsilon^3 p\nabla\boldsymbol{N}^{(k)}(\boldsymbol{z}, \boldsymbol{y}) : \int_{\partial\Omega} \big(\boldsymbol{n}(\boldsymbol{\zeta}) \otimes \boldsymbol{\zeta}\big)\, d\sigma(\boldsymbol{\zeta}) + O(\varepsilon^4), \qquad k = 1, 2, 3. \tag{3.58}$$

For the term $I_2^{(k)}$ we use the result in Lemma 3.3.2 and the Taylor expansion of the conormal derivative of $\mathbf{N}^{(k)}(\boldsymbol{x}, \boldsymbol{y})$, for $k = 1, 2, 3$. In particular, for $\boldsymbol{x} = \boldsymbol{z} + \varepsilon\boldsymbol{\zeta}$, when $\boldsymbol{\zeta} \in \partial\Omega$ and $\boldsymbol{y} \in \mathbb{R}^2$, we consider only the first term of the asymptotic expansion, that is

$$\frac{\partial\boldsymbol{N}^{(k)}}{\partial\boldsymbol{\nu}(\boldsymbol{x})}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\partial\boldsymbol{N}^{(k)}}{\partial\boldsymbol{\nu}(\boldsymbol{\zeta})}(\boldsymbol{z}, \boldsymbol{y}) + O(\varepsilon), \qquad k = 1, 2, 3.$$

Therefore

$$
\begin{aligned}
I_2^{(k)} &= -\varepsilon^2 \int\limits_{\partial\Omega} \frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu}(\boldsymbol{x})}(\boldsymbol{z}+\boldsymbol{\varepsilon\zeta},\boldsymbol{y})\cdot \boldsymbol{f}^\sharp(\boldsymbol{\zeta})\,d\sigma(\boldsymbol{\zeta}) \\
&= \varepsilon^3 p \int\limits_{\partial\Omega} \frac{\partial \boldsymbol{N}^{(k)}}{\partial \boldsymbol{\nu}(\boldsymbol{\zeta})}(\boldsymbol{z},\boldsymbol{y})\cdot \boldsymbol{w}(\boldsymbol{\zeta})\,d\sigma(\boldsymbol{\zeta}) + O(\varepsilon^4),
\end{aligned}
$$

for any k, where $\boldsymbol{w}$ is defined in (3.54). Since $\partial \boldsymbol{N}^{(k)}/\partial \boldsymbol{\nu}(\boldsymbol{\zeta}) = \mathbb{C}\widehat{\nabla}\boldsymbol{N}^{(k)}\boldsymbol{n}(\boldsymbol{\zeta})$, we have

$$
\mathbb{C}\widehat{\nabla}\boldsymbol{N}^{(k)}\boldsymbol{n}(\boldsymbol{\zeta})\cdot \boldsymbol{w}(\boldsymbol{\zeta}) = \mathbb{C}\widehat{\nabla}\boldsymbol{N}^{(k)} : (\boldsymbol{w}(\boldsymbol{\zeta})\otimes \boldsymbol{n}(\boldsymbol{\zeta})).
$$

Therefore

$$
I_2^{(k)}(\boldsymbol{y}) = \varepsilon^3\, p\, \mathbb{C}\widehat{\nabla}\boldsymbol{N}^{(k)}(\boldsymbol{z},\boldsymbol{y}) : \int\limits_{\partial\Omega} (\boldsymbol{w}(\boldsymbol{\zeta})\otimes \boldsymbol{n}(\boldsymbol{\zeta}))\,d\sigma(\boldsymbol{\zeta}) + O(\varepsilon^4). \tag{3.59}
$$

Collecting the result in (3.58) and (3.59), equation (3.43) becomes

$$
\begin{aligned}
&u_\varepsilon^k(\boldsymbol{y}) = I_1^{(k)}(\boldsymbol{y}) + I_2^{(k)}(\boldsymbol{y}) \\
&= \varepsilon^3 p\left[\nabla \boldsymbol{N}^{(k)}(\boldsymbol{z},\boldsymbol{y}) : \int\limits_{\partial\Omega} (\boldsymbol{n}\otimes\boldsymbol{\zeta})\,d\sigma + \mathbb{C}\widehat{\nabla}\boldsymbol{N}^{(k)}(\boldsymbol{z},\boldsymbol{y}) : \int\limits_{\partial\Omega}(\boldsymbol{w}\otimes\boldsymbol{n})\,d\sigma\right] + O(\varepsilon^4).
\end{aligned}
$$

Now, handling this expression, we higlight the moment elastic tensor. We have

$$
\int_{\partial\Omega} (\boldsymbol{n}(\boldsymbol{\zeta})\otimes\boldsymbol{\zeta})\; d\sigma(\boldsymbol{\zeta}) = |\Omega|\mathbf{I}, \tag{3.60}
$$

indeed, for any $i,j = 1,2,3$, it follows

$$
\begin{aligned}
\int_{\partial\Omega} \zeta_i\, n_j\, d\sigma(\boldsymbol{\zeta}) &= \int_{\partial\Omega} \boldsymbol{n}\cdot \zeta_i \boldsymbol{e}_j\, d\sigma(\boldsymbol{\zeta}) \\
&= \int_{\Omega} \mathrm{div}\,(\zeta_i \boldsymbol{e}_j)\, d\boldsymbol{\zeta} = \int_{\Omega} \boldsymbol{e}_j\cdot\boldsymbol{e}_i\, d\boldsymbol{\zeta} = |\Omega|\delta_{ij},
\end{aligned}
$$

where $\boldsymbol{e}_j$ is the j-th unit vector of $\mathbb{R}^3$. Hence, by (3.60) and taking the symmetric part of $\nabla\boldsymbol{N}^{(k)}$, for any k, we find

$$
u_\varepsilon^k = \varepsilon^3 p\left[\widehat{\nabla}\boldsymbol{N}^{(k)} : \mathbf{I}|\Omega| + \mathbb{C}\widehat{\nabla}\boldsymbol{N}^{(k)} : \int\limits_{\partial\Omega} \boldsymbol{w}\otimes\boldsymbol{n}\, d\sigma(\boldsymbol{\zeta})\right] + O(\varepsilon^4).
$$

Using the symmetries of $\mathbb{C}$, we have

$$u_\varepsilon^k = \varepsilon^3 |\Omega| p \widehat{\nabla} \boldsymbol{N}^{(k)} : \left[\mathbf{I} + \frac{1}{|\Omega|} \int\limits_{\partial\Omega} \mathbb{C}(\boldsymbol{w} \otimes \boldsymbol{n})\, d\sigma(\boldsymbol{\zeta}) \right] + O(\varepsilon^4),$$

for $k = 1, 2, 3$. Now, taking into account that $\mathbf{I} = \mathbb{I}\mathbf{I}$ and using the equality $\boldsymbol{w} = \boldsymbol{\theta}^{qr} \delta_{qr}$, we have the assertion. □

3.3.1 Properties of the moment elastic tensor

In this section we analyse the symmetry and positivity properties of the moment elastic tensor $\mathbb{M}$. Starting from the problem (3.45) and passing through the weak formulation in $B_R(\mathbf{0}) \backslash \Omega$, we find

$$\begin{aligned}
\int\limits_{B_R(\mathbf{0})\backslash\Omega} & \mathbb{C} \widehat{\nabla} \boldsymbol{\theta}^{kh} : \widehat{\nabla} \boldsymbol{\varphi}\, d\boldsymbol{x} \\
&= \int\limits_{\partial B_R(\mathbf{0})} (\mathbb{C} \widehat{\nabla} \boldsymbol{\theta}^{kh} \boldsymbol{n}) \cdot \boldsymbol{\varphi}\, d\sigma(\boldsymbol{x}) - \int\limits_{\partial\Omega} (\mathbb{C} \widehat{\nabla} \boldsymbol{\theta}^{kh} \boldsymbol{n}) \cdot \boldsymbol{\varphi}\, d\sigma(\boldsymbol{x}) \\
&= \int\limits_{\partial B_R(\mathbf{0})} (\mathbb{C} \widehat{\nabla} \boldsymbol{\theta}^{kh} \boldsymbol{n}) \cdot \boldsymbol{\varphi}\, d\sigma(\boldsymbol{x}) + \frac{1}{3\lambda + 2\mu} \int\limits_{\partial\Omega} \mathbb{C}(\boldsymbol{n} \otimes \boldsymbol{\varphi})\, d\sigma(\boldsymbol{x})
\end{aligned}$$

Choosing $\boldsymbol{\varphi} = \boldsymbol{\theta}^{rs}$, with $r, s = 1, 2, 3$ we have

$$\begin{aligned}
\int\limits_{B_R(\mathbf{0})\backslash\Omega} & \mathbb{C} \widehat{\nabla} \boldsymbol{\theta}^{kh} : \widehat{\nabla} \boldsymbol{\theta}^{rs}\, d\boldsymbol{x} \\
&= \int\limits_{\partial B_R(\mathbf{0})} (\mathbb{C} \widehat{\nabla} \boldsymbol{\theta}^{kh} \boldsymbol{n}) \cdot \boldsymbol{\theta}^{rs}\, d\sigma(\boldsymbol{x}) + \frac{1}{3\lambda + 2\mu} \int\limits_{\partial\Omega} \mathbb{C}(\boldsymbol{n} \otimes \boldsymbol{\theta}^{rs})\, d\sigma(\boldsymbol{x})
\end{aligned}$$

Using the decay condition at infinity (3.46) of the functions $\boldsymbol{\theta}$, we get. as $R \to +\infty$,

$$\int\limits_{\partial B_R(\mathbf{0})} (\mathbb{C} \widehat{\nabla} \boldsymbol{\theta}^{kh} \boldsymbol{n}) \cdot \boldsymbol{\theta}^{rs}\, d\sigma(\boldsymbol{x}) \to 0,$$

hence

$$\int\limits_{\mathbb{R}^3\backslash\Omega} \mathbb{C} \widehat{\nabla} \boldsymbol{\theta}^{kh} : \widehat{\nabla} \boldsymbol{\theta}^{rs}\, d\boldsymbol{x} = \frac{1}{3\lambda + 2\mu} \int\limits_{\partial\Omega} \mathbb{C}(\boldsymbol{n} \otimes \boldsymbol{\theta}^{rs})\, d\sigma(\boldsymbol{x}) \tag{3.61}$$

or in components, summing up the repeated indices,

$$\int_{\mathbb{R}^3\setminus\Omega} \mathbb{C}_{ijlm}(\widehat{\nabla}\boldsymbol{\theta}^{kh})_{lm}(\widehat{\nabla}\boldsymbol{\theta}^{rs})_{ij}\, d\boldsymbol{x} = \frac{1}{3\lambda+2\mu}\int_{\partial\Omega} \mathbb{C}_{ijkh}\, n_i \theta_j^{rs}\, d\sigma(\boldsymbol{x}).$$

Positivity

Now, we prove the positivity of the tensor $\mathbb{M}$, i.e. $\mathbb{M}\mathbf{A} : \mathbf{A} > 0$, for all $\mathbf{A} \in \mathbb{R}^{3\times 3}$. By the definition (3.48) of $\mathbb{M}$ and applying (3.61) we have

$$\begin{aligned}\mathbb{M}_{khrs}A_{kh}A_{rs} &= |\mathbf{A}|^2 + (3\lambda+2\mu)\int_{\mathbb{R}^3\setminus\Omega} \mathbb{C}\widehat{\nabla}\boldsymbol{\theta}^{kh} : \widehat{\nabla}\boldsymbol{\theta}^{rs}A_{kh}A_{rs}\, d\boldsymbol{x}\\ &= |\mathbf{A}|^2 + (3\lambda+2\mu)\int_{\mathbb{R}^3\setminus\Omega} \mathbb{C}\widehat{\nabla}(\boldsymbol{\theta}^{kh}A_{kh}) : (\widehat{\nabla}\boldsymbol{\theta}^{rs}A_{rs})\, d\boldsymbol{x} > 0\end{aligned}$$

since $\mathbb{C}$ is positive definite.

Symmetries

First, we notice that from $\boldsymbol{w} = \boldsymbol{\theta}^{kh}\delta_{kh}$ we have $\boldsymbol{w} = \boldsymbol{\theta}^{hk}\delta_{hk}$, hence $\boldsymbol{\theta}^{hk}$ satisfy the same problem (3.45) and (3.46). Again, by the definition (3.48), the weak formulation (3.61) and the symmetries of the elastic tensor $\mathbb{C}$, it is straightforward to obtain the following symmetries for the moment elastic tensor

$$\mathbb{M}_{khrs} = \mathbb{M}_{hkrs} = \mathbb{M}_{khsr} = \mathbb{M}_{rskh},$$

where $k, h, r, s = 1, 2, 3$.

3.3.2 The Mogi model

In this subsection, starting from the asymptotic expansion (3.47), that is

$$u_\varepsilon^k(\boldsymbol{y}) = \varepsilon^3|\Omega|p\widehat{\nabla}_{\boldsymbol{z}}\boldsymbol{N}^{(k)}(\boldsymbol{z},\boldsymbol{y}) : \mathbb{M}\mathbf{I} + O(\varepsilon^4), \qquad k = 1, 2, 3,$$

where $\mathbb{M}$ is the tensor given in (3.48), we recover the Mogi model, presented within the Introduction (precisely in Section 1.2), related to a spherical cavity. We first recall that

$$\mathbb{M}\mathbf{I} = \left[\mathbb{I} + \frac{1}{|\Omega|}\int_{\partial\Omega} \mathbb{C}(\boldsymbol{\theta}^{qr}\otimes\boldsymbol{n})\, d\sigma(\boldsymbol{\zeta})\right]\mathbf{I} = \mathbf{I} + \frac{1}{|\Omega|}\int_{\partial\Omega} \mathbb{C}(\boldsymbol{w}\otimes\boldsymbol{n})\, d\sigma(\boldsymbol{\zeta}),$$

where, in the last equality, we use the link between the functions $\boldsymbol{w}$ and $\boldsymbol{\theta}^{qr}$ that is $\boldsymbol{w} = \boldsymbol{\theta}^{qr}\delta_{qr}$, $q, r = 1, 2, 3$. Therefore, to get the Mogi's formula, we first find the explicit expression of $\boldsymbol{w}$ when the cavity Ω is the unit sphere and then we calculate the gradient of the Neumann function $\mathbf{N}$.

We recall that $\boldsymbol{w}$ is the trace on the boundary of the cavity of the solution to the external problem

$$\operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{v}) = 0 \quad \text{in } \mathbb{R}^3 \backslash B_1(\mathbf{0}), \qquad \frac{\partial \boldsymbol{v}}{\partial \nu} = -\boldsymbol{n} \quad \text{on } \partial B_1(\mathbf{0}),$$

where $B_1(\mathbf{0}) = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| \leqslant 1\}$ with decay at infinity

$$\boldsymbol{v} = O(|\boldsymbol{x}|^{-1}), \qquad |\nabla \boldsymbol{v}| = O(|\boldsymbol{x}|^{-2}) \qquad \text{as } |\boldsymbol{x}| \to +\infty.$$

We look for a solution with the form

$$\boldsymbol{v}(\boldsymbol{x}) = \phi(r)\,\boldsymbol{x} \qquad \text{with } r := |\boldsymbol{x}|,$$

so that

$$\Delta v_i = \left\{\phi'' + \frac{4\phi'}{r}\right\} x_i, \qquad \operatorname{div} \boldsymbol{v} = r\phi' + 3\phi, \qquad \nabla \operatorname{div} \boldsymbol{v} = \left\{\phi'' + \frac{4\phi'}{r}\right\} \boldsymbol{x}.$$

By direct substitution, since $\boldsymbol{n} = \boldsymbol{x}$ on ∂B, we get

$$\operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{v}) = (\lambda + 2\mu)\left(\phi'' + \frac{4\phi'}{r}\right)\boldsymbol{x},$$
$$\frac{\partial \boldsymbol{v}}{\partial \nu} = \big\{(\lambda + 2\mu)r\phi' + (3\lambda + 2\mu)\phi\big\}\boldsymbol{x}$$

Thus, we need to find a function $\phi : [1, +\infty) \to \mathbb{R}$ such that

$$\phi'' + \frac{4\phi'}{r} = 0, \quad (\lambda + 2\mu)r\phi' + (3\lambda + 2\mu)\phi\Big|_{r=1} = -1, \quad \phi\Big|_{r=+\infty} = 0.$$

Condition at infinity implies that $B = 0$ and $A = 1/4\mu$. Therefore, the solution is $\mathbf{v}(\boldsymbol{x}) = \boldsymbol{x}/4\mu|\boldsymbol{x}|^3$, which implies that

$$\boldsymbol{w}(\boldsymbol{x}) := \boldsymbol{v}(\boldsymbol{x})\Big|_{|\boldsymbol{x}|=1} = \frac{\boldsymbol{x}}{4\mu}.$$

With the function $\boldsymbol{w}$ at hand, we have that

$$\mathbf{I} + \frac{1}{|B_1(\mathbf{0})|}\int_{\partial B_1(\mathbf{0})} \mathbb{C}(\boldsymbol{w}(\boldsymbol{\zeta}) \otimes \boldsymbol{n}(\boldsymbol{\zeta}))\, d\sigma(\zeta) = \mathbf{I} + \frac{3}{16\pi\mu}\int_{\partial B_1(\mathbf{0})} \frac{\mathbb{C}(\boldsymbol{\zeta} \otimes \boldsymbol{\zeta})}{|\boldsymbol{\zeta}|^3}\, d\sigma(\boldsymbol{\zeta}).$$

Through the use of spherical coordinates and ortogonality relations for the circular functions, it holds

$$\int_{\partial B_1(\mathbf{0})} \frac{\boldsymbol{\zeta}\otimes\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|^3}\, d\sigma(\boldsymbol{\zeta}) = \frac{4\pi}{3}\mathbf{I},$$

hence the second-order tensor $\mathbb{M}\mathbf{I}$ is given by

$$\mathbb{M}\mathbf{I} = \frac{3(\lambda+2\mu)}{4\mu}\mathbf{I}.$$

It implies

$$u_\varepsilon^k(\boldsymbol{y}) = \frac{\pi(\lambda+2\mu)}{\mu}\varepsilon^3 p\,\mathrm{Tr}(\widehat{\nabla}_{\boldsymbol{z}}\boldsymbol{N}^{(k)}(\boldsymbol{z},\boldsymbol{y})) + O(\varepsilon^4), \qquad k=1,2,3. \quad (3.62)$$

For the Neumann's function $\mathbf{N}$ (see the proof of the Theorem 3.2.1 for the explicit expression of the singular components of the matrix $\mathbf{N}$), we are interested only to the trace of $\nabla_{\boldsymbol{z}}\mathbf{N}(\boldsymbol{z},\boldsymbol{y})$ computed at $y_3=0$.

Evaluating $\mathbf{N}=\mathbf{N}(\boldsymbol{z},\boldsymbol{y})$ at $y_3=0$, we get

$$\begin{aligned}
\kappa_\mu^{-1}N_{\alpha\alpha} &= -f-(z_\alpha-y_\alpha)^2f^3-(1-2\nu)g+(1-2\nu)(z_\alpha-y_\alpha)^2fg^2\\
\kappa_\mu^{-1}N_{\beta\alpha} &= (z_\alpha-y_\alpha)(z_\beta-y_\beta)\big\{-f^3+(1-2\nu)fg\big\}\\
\kappa_\mu^{-1}N_{3\alpha} &= (z_\alpha-y_\alpha)\big\{-z_3f^3+(1-2\nu)fg\big\}\\
\kappa_\mu^{-1}N_{\alpha3} &= (z_\alpha-y_\alpha)\Big\{-z_3f^3-(1-2\nu)fg\Big\}\\
\kappa_\mu^{-1}N_{33} &= -2(1-\nu)f-z_3^2f^3
\end{aligned}$$

where $\alpha,\beta=1,2$ and $\kappa_\mu=1/(4\pi\mu)$, with

$$f=1/|\boldsymbol{z}-\boldsymbol{y}|, \qquad \text{and} \qquad g=1/(|\boldsymbol{z}-\boldsymbol{y}|-z_3).$$

Let $\rho^2:=(z_1-y_1)^2+(z_2-y_2)^2$. Using the identities

$$\rho^2f^2=1-z_3^2f^2, \qquad (1-z_3f)g=f$$

and the differentiation formulas

$$\begin{aligned}
&\partial_{\boldsymbol{z}_\alpha}f=-(z_\alpha-y_\alpha)f^3, &&\partial_{\boldsymbol{z}_3}f=-z_3f^3\\
&\partial_{\boldsymbol{z}_\alpha}g=-(z_\alpha-y_\alpha)fg, &&\partial_{\boldsymbol{z}_3}g=fg,\\
&\partial_{\boldsymbol{z}_\alpha}(fg)=-(z_\alpha-y_\alpha)(f+g)f^2g, &&\partial_{\boldsymbol{z}_3}(fg)=f^3,
\end{aligned}$$

we deduce the following formulas for some of the derivatives of $\kappa_\mu^{-1} N_{ij}$

$$\begin{aligned}
\kappa_\mu^{-1}\partial_{\boldsymbol{z}_\alpha} N_{\alpha\alpha} &= (z_\alpha - y_\alpha)\big\{-f^3 + 3(z_\alpha - y_\alpha)^2 f^5 \\
&\qquad + (1-2\nu)\big[3f - (z_\alpha - y_\alpha)^2 f^2 (f+2g)\big] g^2\big\} \\
\kappa_\mu^{-1}\partial_{\boldsymbol{z}_\beta} N_{\beta\alpha} &= (z_\alpha - y_\alpha)\big\{-f^3 + 3(z_\beta - y_\beta)^2 f^5 \\
&\qquad + (1-2\nu)\big[f - (z_\beta - y_\beta)^2 f^2 (f+2g)\big] g^2\big\} \\
\kappa_\mu^{-1}\partial_{\boldsymbol{z}_3} N_{3\alpha} &= (z_\alpha - y_\alpha)\big\{-2\nu f^3 + 3z_3^2 f^5\big\} \\
\kappa_\mu^{-1}\partial_{\boldsymbol{z}_\alpha} N_{\alpha 3} &= -z_3 f^3 + 3(z_\alpha - y_\alpha)^2 z_3 f^5 \\
&\qquad + (1-2\nu)\big[-1 + (z_\alpha - y_\alpha)^2 (f+g) f\big] f g \\
\kappa_\mu^{-1}\partial_{\boldsymbol{z}_3} N_{33} &= -2\nu z_3 f^3 + 3z_3^3 f^5.
\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}
\mathrm{Tr}\big(\hat{\nabla}\boldsymbol{N}^{(\alpha)}\big) &= 2\kappa_\mu(1-2\nu)(z_\alpha - y_\alpha) f^3, \qquad \text{for } \alpha = 1,2 \\
\mathrm{Tr}\big(\hat{\nabla}\boldsymbol{N}^{(3)}\big) &= 2\kappa_\mu(1-2\nu) z_3 f^3.
\end{aligned} \tag{3.63}$$

Combining (3.62), (3.63) and using the explicit expression for f, we find

$$\begin{aligned}
u_\varepsilon^\alpha(\boldsymbol{y}) &= \frac{1-\nu}{\mu}\frac{\varepsilon^3 p (z_\alpha - y_\alpha)}{|\boldsymbol{z}-\boldsymbol{y}|^3} + O(\varepsilon^4), \qquad \text{for } \alpha = 1,2 \\
u_\varepsilon^3(\boldsymbol{y}) &= \frac{1-\nu}{\mu}\frac{\varepsilon^3 p\, z_3}{|\boldsymbol{z}-\boldsymbol{y}|^3} + O(\varepsilon^4),
\end{aligned}$$

that are the components given in (1.1).

We highlight that, in general, for other shapes of the cavity Ω, the trace on $\partial\Omega$ of the auxiliary functions $\boldsymbol{\theta}^{qr}$, with $q, r = 1,2,3$, can be numerically approximated (if it can not be calculated explicitly) and, thus, the first term in the asymptotic expansion (3.47) can be considered as known in practical cases.

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