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Orbits in symmetric varieties

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In dubious battle on the plains of heaven, what though the field be lost?

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Introduction

In [DC-P], De Concini and Procesi defined, for any connected semisimple adjoint algebraic group G, a smooth projective variety X endowed with a $G \times G$-action, containing G as a dense open subset, the complement of which being a normal crossings divisor. Together with these compactifications, they defined and studied more generally compactifications of symmetric varieties G/H. The interest in this class of varieties came originally from the problem of giving rigourous foundation to Schubert calculus. Recently however, the compactifications X have found applications of other types; for example they are an ingredient in Lafforgue's proof of Langlands' correspondence over function fields of positive characteristic. Yet another link is with Lusztig's theory of character sheaves. In [L], Lusztig defines a generalization of the notion of character sheaf and is lead to study the geometry of G-orbits in the compactification X of G.

Unlike the case of the $G \times G$-orbits, the geometry of G-orbits of X is not at all well understood, even for low rank groups. The object of this thesis is the study of G-orbits inside group compactifications and the study of the relations of closure between such orbits. We consider De Concini-Procesi compactifications and canonical projections of such varieties. The main question we address is whether, for a given compactification, the closure of any G-orbit is a finite union of G-orbits. In the open part of any compactification this property holds (it is a consequence of the finiteness of the number of unipotent conjugacy classes in a reductive group, see [Spa]).

The question is, does such property continue to hold at the boundary? For De Concini-Procesi compactifications this was conjecturally believed to be the case.

We will check that it holds in the simplest example of a variety which is a projection of a De Concini-Procesi compactification, namely the case of $X = \mathbb{P}(\mathrm{End}V)$ and $G = PGL(V)$, for V a vector space. This extremely simplified version of the problem may be approached by techniques from linear algebra and basic algebraic geometry.

For $G = PGL(2)$, the De Concini-Procesi compactification is the projectivization of the space of two-by-two matrices, so it falls under this first typology. But already for $G = PGL(3)$, one has to develop new techniques to give a classification of G-orbits.

There are two different possible ways to tackle such problem. One may use the geometric interpretation of $G \times G$-orbits in the De Concini-Procesi compactification and undertake a geometric classification of these objects. This has the advantage of being geometric, hence might be the appropriate setting to study the question of the closure of G-orbits. The drawback is that, such geometric classification may be very hard for general G. The second appoach available is due to Lusztig. It is algebraic in nature, one completely loses track of the geometry in this approach. But it has the advantage of giving a classification in terms of basic classifying data, moreover representatives and dimensions of orbits are easily determined through an inductive procedure. Unfortunately, it says very little on the closures of G-orbits. Nonetheless, this is the approach we choose to follow.

This forces us to develop adequate techniques for the study of the closures of G-orbits. A first general technique consists in a modification of Lusztig's procedure for the classification of G-orbits in X. But, losing the possibility of being iterated, it gives little more information than the normal method. For the smaller orbits one may argue by explicit degenerations and the modification of Lusztig's algorithm to obtain orbits in the closure, and by dimension arguments and geometric considerations to rule out orbits from the closure.

For regular orbits, i.e. those of maximum dimension, the principal method is the determination of invariants. The conjecture is that the closure of a regular orbit is the union of all orbits having the same invariants plus the null-cone, i.e. the union of those orbits on which all invariants vanish. We do not prove in general the conjecture, but we check its validity in the cases $G = PGL(2)$, $G = PGL(3)$, $G = PSp(4)$; the method employed works also for the case $G = PGL(4)$.

For the first three cases we give a complete list of G-orbits in the compactification X together with their closure relations. To determine the closure of intermediate orbits in the canonical compactification of $PSp(4)$, the modified algorithm, the computation of invariants and dimension considerations are usually not sufficient. One must also use considerations from representation theory.

The determination of the G-orbits in X and their closures for $G = PGL(2)$, $G = PGL(3)$ and $G = PSp(4)$, allows for various considerations. For one, it disproves the conjecture that for any group G, the De Concini-Procesi compactification X of G has the property that the closure of any G-orbit in X is the union of a finite number of G-orbits. In fact it is not so in the case $G = PSp(4)$, while the conjecture is true in the cases $G = PGL(2)$ and $G = PGL(3)$. Another interesting consideration is that, since the classification of G-orbits in X as afforded by Lusztig's algorithm clearly depends only on the combinatorics of the Weyl group, one could be lead to ask whether also the closures of the orbits depend only on the combinatorics of the Weyl group. This is the case for $PGL(3)$, but in such case there is an automorphism exchanging the two boundary divisors, i.e. there is an automorphism of the root system A_2 exchanging the two simple roots. The case of $G = PSp(4)$ shows in fact that the relations of closure between G-orbits do depend on the geometry of the boundary divisors and not just on the combinatorics.

Regarding the case $G = G_2$, we investigate it only as far as to be able to conclude that its canonical compactification does contain G-orbits which close to infinitely many G-orbits. We base our analysis on the study of G-orbits contained in a minimal nonclosed $G \times G$-orbit. In such case the modification of Lusztig's algorithm permits a complete understanding and reduces the question to combinatorics in the Weyl group. The method of study of G-orbits in minimal nonclosed $G \times G$-orbits works in general; we apply it also in the case of $G = PGL(4)$ to deduce that in its canonical compactification there are G-orbits closing to infinitely many G-orbits.

Having proved that the conjecture about closure of G-orbits in X does not hold for the cases $G = PSp(4)$, $G = G_2$ and $G = PGL(4)$, we deduce, by Dynkin diagram considerations and application of the modified Lusztig algorithm, that if a connected semisimple adjoint algebraic group G is such that the closure of any G-orbit in its canonical compactification X is a finite union of G-orbits, then G is a product of simple factors each of which isomorphic either to $PGL(2)$ or $PGL(3)$.

Let us very briefly recall how this material is divided into chapters.

The first chapter deals with G-orbits and closure of G-orbits in the case $G = PGL(V)$ and the compactification is $\mathbb{P}(\mathrm{End}V)$. The results are easily stated in terms of partitions and a natural partial order on partitions.

The second chapter develops the techniques to tackle the study of G-orbits and their closures inside the canonical compactification X of G. Lusztig's iterative procedure for the determination of G-orbits is exposed. Various criteria to study the closure of G-orbits are developed: the modified Lusztig method, a geometric lemma which is the foundation for considerations based on invariants, and a basic but useful representation theoretic criterium.

In the third chapter, Lusztig's algorithm is employed to determine the G-orbits in X and their dimensions together with representatives, in the cases $G = PGL(2)$, $G = PGL(3)$ and $G = PSp(4)$; the closure relations between orbits are determined. The study of minimal nonclosed $G \times G$-orbits in X, for $G = G_2$ and $G = PGL(4)$, is expounded. The chapter ends with the result that the conjecture holds only for products of $PGL(2)$ and $PGL(3)$.

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Chapter 1

Classification and closure of $GL(V)$–orbits in $\mathbb{P}(\mathrm{End}(V))$

If G is a connected semisimple algebraic group of adjoint type and V_{ω_1}, ..., V_{ω_1} the irreducible representations of G associated to the fundamental weights ω_1, ..., ω_l, then the canonical compactification X of G may be defined as the closure of G inside $\mathbb{P}(\mathrm{End}V_{\omega_1}) \times \ldots \times \mathbb{P}(\mathrm{End}V_{\omega_l})$, (see [DC-P]). By its very definition, X comes equipped with $G \times G$-equivariant morphism to the projective spaces $\mathbb{P}(\mathrm{End}V_{\omega_1})$, ..., $\mathbb{P}(\mathrm{End}V_{\omega_l})$. The images of X in these projective spaces are also compactifications of G, in general not smooth. One may approximate the problem of studying the G-orbits in X with the problem of studying the G-orbits in the image of X inside one of the above projective spaces. The simplest such case being $G = PGL(V)$ and the compactification being the whole $\mathbb{P}(\mathrm{End}V)$, where V is any vector space. The problem of determining the $PGL(V)$-orbits and their closures in $\mathbb{P}(\mathrm{End}V)$ turns out to be much simpler than the analogous problem for the canonical compactification of $PGL(V)$. The determination of $PGL(V)$-orbits is reduced to the problem of $GL(V)$-orbits in $\mathrm{End}V$, which is the theory of Jordan normal form of an endomorphism. The closure relations between $PGL(V)$-orbits in $\mathbb{P}(\mathrm{End}V)$ are not all consequences of closure relations between $GL(V)$-orbits in $\mathbb{P}(\mathrm{End}V)$.

A $PGL(V)$-orbits in $\mathbb{P}(\mathrm{End}V)$ may degenerate to nilpotent $PGL(V)$-orbits. To which nilpotent $PGL(V)$-orbits it may degenerate is well characterized in terms of a partial order on partitions.

In the course of the discussion, we recall the classification and closure relations of $GL(V)\times GL(V)$-orbits and of $GL(V)$-orbits in $\mathrm{End}V$; moreover we establish a link in general between the classification and closure of G-orbits in a representation V and the classification and closure of G-orbits in $\mathbb{P}(V)$, for a general connected (reductive) algebraic group G.

1.1 Classification of $\mathrm{GL(V)}-$orbits in $\mathbb{P}(\mathrm{End}(V))$

Let V be a vector space of dimension n over $\mathbb{C}$ and $G=\mathrm{GL(V)}$. Let S be the graded ring $\mathbb{C}[\mathrm{End(V)}]$ of polynomials on End(V). On End(V) acts the group $G\times G$ by left and right multiplication.This induces an action of $G\times G$ on S, restricted to the diagonal it gives an action of G on S which is the conjugacy action. Let X be the generic element of S. Putting $T_i(X):=Tr(\bigwedge^i X)$, we have $S^G=\mathbb{C}[T_1,\ldots,T_n]$, S^G being the subring of invariants of S under the action of G.

We recall the classification of $G\times G$-orbits and of $G-$orbits in $\mathbb{P}(\mathrm{End(V)})$.

Let

$$\mathcal{D}_i=\mathbb{P}(\{X\in\mathrm{End(V)}:rank(X)=i\}).$$

$\mathcal{D}_i$ is a locally closed subset of End(V), it is a $G\times G-$orbit, its closure $\overline{\mathcal{D}}_i$ is defined by the ideal

$$I_{\overline{\mathcal{D}}_i}=Span\big((i+1)\times(i+1)-\mathrm{minors}\big).$$

We have the following obvious inclusion relations

$$\overline{\mathcal{D}}_n \supset \overline{\mathcal{D}}_{n-1} \supset \ldots \supset \overline{\mathcal{D}}_1, \quad \text{with} \qquad \overline{\mathcal{D}}_1 = \mathcal{D}_1 \cong \left(\mathbb{P}^n\right)^* \times \mathbb{P}^n.$$

So the orbit decomposition of $\mathbb{P}(\text{End(V)})$ is

$$\mathbb{P}(\text{End(V)}) \;=\; \mathcal{D}_n \sqcup \mathcal{D}_{n-1} \sqcup \ldots \sqcup \mathcal{D}_1.$$

The dimension of the $\mathcal{D}_i$ is

$$\begin{aligned} \dim \mathcal{D}_i &= \dim Gr(i,n) \;+\; \dim Gr(n-i,n) \;+\; \dim Mat_{i\times i} \;-\; 1 \\ &= 2i(n-i) + i^2 - 1. \end{aligned}$$

A standard representative of the orbit $\mathcal{D}_l$ is

$$\begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix}.$$

While the classification of $G-$orbits in $\mathbb{P}(\text{End(V)})$.

Let l be an integer such that $1 \leq l \leq n$ and let $\lambda_1, \lambda_2, \ldots, \lambda_l \in \mathbb{C}$ be distinct complex numbers. Let $a_1, a_2, \ldots, a_l \in \mathbb{N}$ such that $a_1 + \ldots + a_l = n$ and $a_1 \geq a_2 \geq \ldots \geq a_l$.

Let $p^{(i)}$ be a partition of a_i, more explicitly $p^{(i)} = \left(p_1^{(i)} \geq \ldots \geq p_{r_i}^{(i)}\right)$.
Let us call $O^{\lambda_1,\lambda_2,\ldots,\lambda_l}_{p^{(1)},p^{(2)},\ldots,p^{(l)}}$ the $G-$orbit of the element

$$A^{\lambda_1,\lambda_2,\dots,\lambda_l}_{p^{(1)},p^{(2)},\dots,p^{(l)}} = \begin{pmatrix} \overbrace{\begin{matrix}\lambda_1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1\end{matrix}}^{p_1^{(1)}} & & & & \\ & \overbrace{\begin{matrix}\lambda_1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1\end{matrix}}^{p_2^{(1)}} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \overbrace{\begin{matrix}\lambda_l & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_l\end{matrix}}^{p_{r_l}^{(l)}} \end{pmatrix}$$

Every $G-$orbit in $\mathbb{P}(\mathrm{End(V)})$ contains elements of the type $A^{\lambda_1,\lambda_2,\dots,\lambda_l}_{p^{(1)},p^{(2)},\dots,p^{(l)}}$, moreover $A^{\lambda_1,\lambda_2,\dots,\lambda_l}_{p^{(1)},p^{(2)},\dots,p^{(l)}}$ is conjugate to $A^{\mu_1,\mu_2,\dots,\mu_l}_{q^{(1)},q^{(2)},\dots,q^{(l)}}$ if and only if there exists $\nu \in \mathbb{C}^*$ and a permutation $\sigma \in \mathcal{S}_l$ such that

$$\mu_i \;=\; \nu\lambda_{\sigma(i)} \qquad \text{and} \qquad q^{(i)} \;=\; p^{(\sigma(i))}.$$

Since $O^{\lambda_1,\lambda_2,\dots,\lambda_l}_{p^{(1)},p^{(2)},\dots,p^{(l)}} = G \cdot A^{\lambda_1,\lambda_2,\dots,\lambda_l}_{p^{(1)},p^{(2)},\dots,p^{(l)}}$, then the following holds

$$\dim O^{\lambda_1,\lambda_2,\dots,\lambda_l}_{p^{(1)},p^{(2)},\dots,p^{(l)}} \;=\; n^2 - \dim \mathrm{Stab}_G\, A^{\lambda_1,\lambda_2,\dots,\lambda_l}_{p^{(1)},p^{(2)},\dots,p^{(l)}}.$$

The calculation of $\dim \mathrm{Stab}_G\, A^{\lambda_1,\lambda_2,\dots,\lambda_l}_{p^{(1)},p^{(2)},\dots,p^{(l)}}$ is reduced to the case of a nilpotent endomorphism.

1.1.1 Classification and closure of $GL(V)$-orbits in End(V)

We briefly recall results about the classification and closure of $GL(V)$-orbits in End(V). These are consequences of the theory of the Jordan canonical form of a matrix.

Let V be a vector space of dimension n over an algebraically closed field k. Let l be an integer smaller than or equal to n, $\lambda_1, \ldots, \lambda_l$ distinct elements of k, $n_1, \ldots, n_l$ a partition of n, i.e. $n_1 + \cdots + n_l = n$, and for i from 1 to l a sequence of integers $a_1^{(i)} \leq a_2^{(i)} \leq \ldots \leq a_{s_i}^{(i)} = n_i$ with $a_j^{(i)} \leq 2a_{j-1}^{(i)}$; then to such a datum corresponds a $GL(V)$-orbit O in End(V), with representative A having $\lambda_1, \ldots, \lambda_l$ as eigenvalues with multiplicities $n_1, \ldots, n_l$ and such that $\dim(\ker(A - \lambda_1 I)^j) = a_j^{(i)}$. The closure of such orbit are the orbits with $\lambda_1, \ldots, \lambda_l$ as eigenvalues of multiplicities $n_1, \ldots, n_l$, but satisfying the equations $\dim(\ker(A - \lambda_i I)^j) \geq a_j^{(i)}$.

Defining $q_j^{(i)} = a_{j-1}^{(i)} - a_j^{(i)}$, we get $q_1^{(i)} \geq q_2^{(i)} \geq \ldots \geq q_{s_i}^{(i)} > 0$ and $q_1^{(i)} + \ldots + q_{s_i}^{(i)} = n_i$. Let us make a diagram with $q_1^{(i)}$ boxes in the first column, $q_2^{(i)}$ boxes in the second column, and so on. The number of boxes in each row give the integers of another partition of n_i, $p_1^{(i)} + \ldots + p_{r_i}^{(i)} = n_i$, this partition is by definition the transposed partition of $q_1^{(i)} + \ldots + q_{s_i}^{(i)} = n_i$, and of course the transposed partition of $p_1^{(i)} + \ldots + p_{r_i}^{(i)} = n_i$ is again $q_1^{(i)} + \ldots + q_{s_i}^{(i)} = n_i$.

So to there is a bijection between the set of sequences of integers $a_1^{(i)} \leq a_2^{(i)} \leq \ldots \leq a_{s_i}^{(i)} = n_i$ with $a_j^{(i)} \leq 2a_{j-1}^{(i)}$ and the set of partitions $p_1^{(i)} + \ldots + p_{r_i}^{(i)} = n_i$ of n_i. Then we can restate the parametrization of $GL(V)$-orbits in End(V) as follows:

Let l be an integer smaller than or equal to n, $\lambda_1, \ldots, \lambda_l$ distinct elements of k, $n_1, \ldots, n_l$ a partition of n, and $p^{(i)} = \{p_1^{(i)} \geq p_2^{(i)} \geq \ldots \geq p_{r_i}^{(i)}\}$ a partition of n_i. To this datum we associate a Jordan matrix with, for each $p_j^{(i)}$, a Jordan block of eigenvalue λ_i and size $p_j^{(i)}$. The endomorphisms having

Jordan canonical form of this type make up a $GL(V)$-orbit in End(V).

Let p and $p^{'}$ be two partitions of the same integer, q and $q^{'}$ the transposed partitions respectively of p and $p^{'}$, then we denote $p \leq p^{'}$ the relation $\forall j,\ q_1 + \ldots + q_1 \geq q_1^{'} + \ldots + q_1^{'}$. The relation of closure between orbits reads in the following way on the parameters:

An orbit $O^{'}$ is in the closure of another orbit O if and only if they correspond to the same $\lambda_1, \ldots, \lambda_l$ and $n_1, \ldots, n_l$, and moreover, for each i the inequality $p^i \leq p^{'i}$ holds.

1.2 Closure of GL(V)−orbits in $\mathbb{P}$(End(V))

Definition 1.2.1. *If $p^{(1)}$, $p^{(2)}$, $\ldots$, $p^{(l)}$ are partitions of respectively n_1, n_2, $\ldots$, n_l, then we define the partition $p = p^{(1)} + p^{(2)} + \ldots + p^{(l)}$ of $n = n_1 + n_2 + \ldots + n_l$ as follows:*

$$p_1 = p_1^{(1)} + p_1^{(2)} + \ldots + p^{(l)}{}_1, \quad \ldots \quad , \; p_l = p_l^{(1)} + p_l^{(2)} + \ldots + p^{(l)}{}_l.$$

We now determine the closure of a GL(V)−orbit O in $\mathbb{P}$(End(V)). If $O^{'}$ is a GL(V)−orbit in (End(V)) projecting to O, then naturally the projections of orbits in the closure of $O^{'}$ are in the closure of O, but these do not make up the whole closure. The remaining orbits are the nilpotent orbits associated to partitions smaller than or equal to the sum of the partitions associated to O.

Theorem 1.2.2. $\overline{O^{\lambda_1,\lambda_2,\ldots,\lambda_l}_{p^{(1)},p^{(2)},\ldots,p^{(l)}}} \supset O^{\mu_1,\mu_2,\ldots,\mu_{l'}}_{q^{(1)},q^{(2)},\ldots,q^{(l')}}$ *if and only if*

either *$l = l^{'}$ and there exists a permutation $\sigma \in \mathcal{S}_l$ and a complex number $\lambda \in \mathbb{C}^*$ such that*

$$\begin{cases} l\mu_{\sigma(i)} &= \lambda\lambda_i \ , \\ |q^{(\sigma(i))}| &= |p^{(i)}| \ , \\ p^{(i)} &\geq q^{(\sigma(i))} \ , \end{cases}$$

or *$l' = 1$, $\mu_1 = 0$ and $p \geq q^{(1)}$, where $p = p^{(1)} + p^{(2)} + \cdots + p^{(l)}$.*

Proof. Let us prove $\Leftarrow$. The first case reduces to $\mathrm{End}(V)$. Let us consider the second case. We observe that, for $\epsilon \neq 0$,

$$A^{\lambda_1,\lambda_2,\ldots,\lambda_l}_{p^{(1)},p^{(2)},\ldots,p^{(l)}}(\epsilon) =$$

$$\left(\begin{array}{cccccccc}
\overbrace{\epsilon\lambda_1 \quad 1 \qquad\qquad}^{p_1^{(1)}} & & & & & & & \\
\quad\ddots \quad \ddots & & 1 & & & & & \\
\qquad\quad \ddots \; \begin{smallmatrix}1\\ \epsilon\lambda_1\end{smallmatrix} & & & & & & & \\
 & \ddots & & & & & & \\
 & & \ddots & & & & & \\
 & & & \overbrace{\epsilon\lambda_2 \quad 1 \qquad\qquad}^{p_1^{(2)}} & & & & \\
 & & & \quad\ddots \quad \ddots & & 1 & & \\
 & & & \qquad\quad \ddots \; \begin{smallmatrix}1\\ \epsilon\lambda_2\end{smallmatrix} & & & & \\
 & & & & \ddots & & & \\
 & & & & & \ddots & & \\
 & & & & & & \overbrace{\epsilon\lambda_3 \quad 1 \qquad\qquad}^{p_1^{(3)}} & \\
 & & & & & & \quad\ddots \quad \ddots & \\
 & & & & & & \qquad\quad \ddots \; \begin{smallmatrix}1\\ \epsilon\lambda_3\end{smallmatrix} & \\
 & & & & & & & \overbrace{\epsilon\lambda_l \quad 1 \qquad\qquad}^{p_{r_l}^{(l)}} \\
 & & & & & & & \quad\ddots \quad \ddots \\
 & & & & & & & \qquad\quad \ddots \; \begin{smallmatrix}1\\ \epsilon\lambda_l\end{smallmatrix}
\end{array}\right).$$

(where the 1's above the block-diagonal are in the row corresponding to the last row of the k-th block associated to λ_i and the column corresponding to the first column of the k-th block associated to λ_{i+1}), belongs to $O^{\lambda_1,\lambda_2,\ldots,\lambda_l}_{p^{(1)},p^{(2)},\ldots,p^{(l)}}$. And

$$\lim_{\epsilon\to 0} A^{\lambda_1,\lambda_2,\ldots,\lambda_l}_{p^{(1)},p^{(2)},\ldots,p^{(l)}}(\epsilon) \;\in\; O^0_p.$$

Let's prove $\Rightarrow$. But first we state some lemmas and propositions. Let as usual $\lambda_1, \lambda_2, \ldots, \lambda_l$ be complex numbers (not necessarily distinct), and $a_1, a_2, \ldots, a_l$ be natural numbers such that $\sum a_i = n$. Let's define

$$\mathcal{E}^{\lambda_1,\lambda_2,\ldots,\lambda_l}_{a_1,a_2,\ldots,a_l} =$$

$$= \mathbb{P}\big(\{X \in \mathrm{End(V)} \mid \exists\lambda \in \mathbb{C}, \text{ such that } \det(tI - X) = (t-\lambda\lambda_1)^{a_1}\cdots(t-\lambda\lambda_l)^{a_l}\}\big).$$

Then we have the following lemmas:

Lemma 1.2.3. *Denoting with $\mathcal{N}$ the cone of nilpotent endomorphisms, then for any $(\{\lambda\}, \mathbf{a})$,*

$$\mathcal{N} \subset \mathcal{E}^{\lambda_1,\lambda_2,\ldots,\lambda_l}_{a_1,a_2,\ldots,a_l}.$$

Proof. Obvious. □

Lemma 1.2.4. *Supposing that $\exists i \; : \; \lambda_i \neq 0$, and denoting $\Lambda_s = \sum a_i \lambda_i^s$, we have that $\mathcal{E}^{\lambda_1,\lambda_2,\ldots,\lambda_l}_{a_1,a_2,\ldots,a_l}$ is a Zariski-closed subset of $\mathbb{P}(\mathrm{End(V)})$ cut out set-theoretically by the following equations:*

for any $((s_i, b_i))$, $((r_i, c_i)$ such that $\sum s_i b_i = \sum r_i c_i$,

$$Tr(X^{s_1})^{b_1} \cdots Tr(X^{s_u})^{b_u} \cdot \Lambda^{c_1}_{r_1} \cdots \Lambda^{c_v}_{r_v} = Tr(X^{r_1})^{c_1} \cdots Tr(X^{r_v})^{c_v} \cdot \Lambda^{b_1}_{s_1} \cdots \Lambda^{b_u}_{s_u}.$$

Proof. One implication is straightforward. We check now that if the given equations hold for X, then X belongs to $\mathcal{E}^{\lambda_1,\lambda_2,\ldots,\lambda_l}_{a_1,a_2,\ldots,a_l}$. We make some remarks.

Remark 1.2.5. *Since there is a* $\lambda_i \neq 0$*, then there is a* $\Lambda_{s_0} \neq 0$.

Remark 1.2.6. *If* X *is such that, for all* r, $Tr(X^r) = 0$*, then* $X \in \mathcal{N}$ *and a fortiori* $X \in \mathcal{E}^{\lambda_1,\lambda_2,\ldots,\lambda_l}_{a_1,a_2,\ldots,a_l}$*, for any* $(\{\lambda\}, \mathbf{a})$.

If X is nilpotent, this last remark concludes the lemma. So we may suppose from now on that X is not nilpotent.

Remark 1.2.7. *If for a specific* r *we have* $Tr(X^r) \neq 0$ *then we will have also* $\Lambda_r \neq 0$*. This follows from the following identity for* X*:*

$$Tr(X^{s_0})^r \Lambda_r^{s_0} \;=\; Tr(X^r)^{s_0} \Lambda_{s_0}^r \;\neq\; 0.$$

Remark 1.2.8. *In a similar way we see that* $Tr(X^r) = 0 \iff \Lambda_r = 0$.

Now, to prove the lemma, it is sufficient to check that there is a $\lambda \in \mathbb{C}^*$ such that $Tr((\lambda X)^r) = \Lambda_r$, for every r. We prove this in three steps.
step one. If $Tr((X)^d) = \Lambda_d \neq 0$, then it holds also for any multiple of d. This follows from

$$Tr((X)^{di}) = \frac{\Lambda_{di}}{(\Lambda_d)^i}(Tr(X^d))^i = \Lambda_{di}.$$

step two. If $Tr((X)^r) = \Lambda_r \neq 0$ and $Tr((X)^s) = \Lambda_s \neq 0$, then the same will hold also for any natural number in the monoid generated by r and s. Indeed, if t is one such natural number, then $\forall u \gg 0$ there will be $a \geq 0$ and $b \geq 0$ such that $tu = ar + bs$, and so the following equation holds

$$(Tr((X)^t))^u = \frac{Tr(X^r)^a}{(\Lambda_r)^a}\frac{Tr(X^s)^b}{(\Lambda_s)^b}(\Lambda_t)^u = (\Lambda_t)^u \ , \ \forall u \gg 0 \ .$$

And so

$$Tr((X)^t) = \Lambda_t \ .$$

step three. Let p_1, p_2, ..., p_n be natural numbers. Making the assumption that $Tr((X)^{p_i}) \neq 0$, then we can find $\lambda \in \mathbb{C}^*$ such that $Tr((\lambda X)^{p_i}) = \Lambda_{p_i} \neq 0$. Let us do explicitly only the case of two natural numbers, p and q, the general case follows the same lines. It is clear that one can choose a particular λ such that $Tr((X)^p) = \Lambda_p$. And, from step one, one has also $(Tr((X)^q))^{\frac{p}{\gcd(p,q)}} = (\Lambda_q)^{\frac{p}{\gcd(p,q)}}$. There exists then a root of unity $\zeta^{\frac{p}{\gcd(p,q)}} = 1$, such that $Tr((X)^q) = \zeta\Lambda_q$. Letting $\zeta = \omega^q$, then clearly $\omega^{\frac{p}{\gcd(p,q)}q} = 1$, and

$$Tr((\omega^{-1-a\frac{p}{\gcd(p,q)}}X)^q) = \Lambda_q \ ,$$

$$Tr((\omega^{-1-a\frac{p}{\gcd(p,q)}}X)^p) = \omega^{-p-\frac{ap^2}{\gcd(p,q)}}\Lambda_p \ .$$

It is possible to find a natural number a satisfying the congruence $-p - \frac{ap^2}{\gcd(p,q)} \equiv 0 \bmod(\mathrm{mcm}(p,q))$, i.e. $\frac{ap}{\gcd(p,q)} \equiv\ -1 \bmod(\frac{q}{\gcd(p,q)})$. This proves the third step.

To resume the proof, we consider the monoid generated by the r's for which $Tr((\lambda X)^r) \neq 0$. It will be generated by a finitely many natural numbers p_1, p_2, ..., p_n. Thus, by step three we can choose $\lambda \in \mathbb{C}^*$ such that $Tr((\lambda X)^r) = \Lambda_r\ ,\ \forall r \in (p_1, p_2, \ldots, p_n)$. For the r's outside of the monoid $(p_1, p_2, \ldots, p_n)$, the equality is trivially satisfied, both sides of which being zero by the last remark. So the statement holds for all $r \in \mathbb{N}$. It is then clear that the characteristic polynomial of λX is determined and $\lambda X \in \mathcal{E}^{\lambda_1,\lambda_2,\ldots,\lambda_l}_{a_1,a_2,\ldots,a_l}$. This concludes the proof. □

Corollary 1.2.9. *Let* $A = A^{\lambda_1,\lambda_2,\ldots,\lambda_l}_{p^{(1)},p^{(2)},\ldots,p^{(l)}}$ *and* $O = O^{\lambda_1,\lambda_2,\ldots,\lambda_l}_{p^{(1)},p^{(2)},\ldots,p^{(l)}}$ *its orbit under* G*. Let* $B \in \overline{O}$*, then*

either B *is nilpotent,*

or $B \in O^{\lambda_1,\lambda_2,\ldots,\lambda_l}_{q^{(1)},q^{(2)},\ldots,q^{(l)}}$, *with* $|q^{(i)}| = |p^{(i)}|$.

Now we resume proving the theorem, which will follow at once from this proposition and the later general digression.

Proposition 1.2.10. *Let* $B \in \overline{O}$*,* B *not nilpotent. Then* $\exists \lambda \in \mathbb{C}^*$ *such that* $B \in \overline{\mathrm{GL(V)} \cdot \lambda A} \subset \mathrm{End(V)}$*. In particular, the first instance of the theorem holds.*

Proof. $B \in \overline{O}$ implies $B \in \mathcal{E}^{\lambda_1,\lambda_2,\ldots,\lambda_l}_{a_1,a_2,\ldots,a_l}$. Moreover, $\exists \lambda(\varepsilon) \in \mathbb{C}^*$ and $g(\varepsilon) \in$ GL(V) such that

$$\lim_{\varepsilon \to 0} \lambda(\varepsilon) g(\varepsilon) A g(\varepsilon)^{-1} = B \ .$$

Being as usual $\Lambda_j = \sum a_i \lambda_i^j$, and s be such that $\Lambda_s \neq 0$ (B is not nilpotent, and thus also A is not). We have then $A,\ B \in \mathcal{E}^{\lambda_1,\lambda_2,\ldots,\lambda_l}_{a_1,a_2,\ldots,a_l}$, $A,\ B$ not nilpotent, and

$$Tr(A^s) \neq 0 \quad , \quad Tr(B^s) \neq 0 \ .$$

Now, we have

$$\lambda(\varepsilon)^s g(\varepsilon) A^s g(\varepsilon)^{-1} \longrightarrow B^s \ ,$$

so

$$\lambda(\varepsilon)^s Tr(A^s) \longrightarrow Tr(B^s) \ ,$$

from which derives $\lambda(\varepsilon)^s \to \mu = \frac{Tr(B^s)}{Tr(A^s)} \in \mathbb{C}^*$. It follows then that $\lambda(\varepsilon) \to \lambda \in \mathbb{C}^*$, and thus

$$g(\varepsilon) A g(\varepsilon)^{-1} \longrightarrow \lambda^{-1} B \in \mathrm{End(V)} \ .$$

□

We could go on and prove the second instance of the theorem along the same line but we prefer to obtain it as a byproduct of a more general discussion. So we now make a digression and later come back to finish off the proof. The forthcoming discussion will not depend on the theorem. □

1.3 Classification and closure of orbits in $\mathbb{P}(V)$

Let V be a vector space of dimension n over the algebraically closed field k on which acts linearly a connected reductive algebraic group G, i.e. V is the underlying vector space of a linear representation of the group G. Then the action of G on V induces an action of G on $\mathbb{P}(V)$. We will discuss the classification of the orbits of G in $\mathbb{P}(V)$ and the closures of such orbits, ultimately reducing the problem to the classification and closure of G-orbits in V, and to the determination of the ideal of polynomial functions on V vanishing on an orbit. It is a generalization of the previous section to an arbitrary connected reductive group G and to an arbitrary linear representation V of G. For such purpose we introduce the completion $\overline{V}$ of V with a hyperplane at infinity and study the extended action of G on $\overline{V}$.

1.3.1 The completion $\overline{V}$ of V

If $x_1, \ldots, x_n$ are coordinates in V corresponding to a basis $(e_1, \ldots, e_n)$, then $\overline{V}$ is identified with $\mathbb{P}(V')$, where V' denotes a vector space of dimension $n+1$ with basis $(e_0,\ e_1, \ldots, e_n)$. Thus $\overline{V}$ is a projective space of dimension n, if $x_0,\ x_1, \ldots, x_n$ are the coordinates of V' corresponding to the basis $(e_0,\ e_1, \ldots, e_n)$, then one can introduce the homogeneous coordinates $[x_0 : x_1 : \ldots : x_n]$ in $\overline{V}$. In such coordinates, the inclusion $V \hookrightarrow \overline{V}$, is given by $(x_1, \ldots, x_n) \mapsto [1 : x_1 : \ldots : x_n]$, hence the complement, i.e. the part at infinity, is the subspace with homogeneous coordinates $[0 : x_1 : \ldots : x_n]$, and this is identified with $\mathbb{P}(V)$ by the the map $[0 : x_1 : \ldots : x_n] \mapsto [x_1 : \ldots : x_n]$.

G acts on $\overline{V}$ by its action on the coordinates $x_1, \dots, x_n$. In more intrinsic terms, $V' = k \oplus V$, where k is the trivial representation of G, $\overline{V} = \mathbb{P}(V')$ and the action of G is the one induced by the action on V'. We may restate this as follows. We have a G-equivariant commutative diagram

$$\begin{array}{ccccc} V & \longrightarrow & \overline{V} & \longleftarrow & \mathbb{P}(V) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{A}^n & \longrightarrow & \mathbb{P}^n & \longleftarrow & \mathbb{P}^{n-1} \end{array},$$

and the decomposition $\overline{V} = V \coprod \mathbb{P}(V)$ is G-stable. Furthermore, the bundle map

$$\begin{array}{c} V - \{0\} \\ {\scriptstyle\pi}\big\downarrow \\ \mathbb{P}(V) \end{array}$$

is G-equivariant. Hence the image under π of a G-orbit in $V - \{0\}$ is a G-orbit in $\mathbb{P}(V)$ and, conversely the inverse image of a G-orbit in $\mathbb{P}(V)$ is a union of G-orbits in $V - \{0\}$. One can extend the map π to all $\overline{V} - \{0\}$, it is just the projection in the projective space $\mathbb{P}^n$ from the point $0 = [1 : 0 : \dots : 0]$ to the hyperplane $\mathbb{P}^{n-1} = \{[0 : x_1 : \dots : x_n] \in \mathbb{P}^n$, i.e. the projection on the last n coordinates. We have seen that every G-orbit O in $\mathbb{P}(V)$ is the projection of a G-orbit O' in $V - \{0\}$. Now we relate the closure $\overline{O}$ of the orbit O in $\mathbb{P}(V)$ with the closure of the orbit O' in $\overline{V}$, in the case that the closure of the orbit O' in V does not contain the origin.

Proposition 1.3.1. *Assuming that the closure of the G-orbit O' in V does not contain the origin* 0 *of V, we have the equality*

$$\overline{\pi(O')} = \pi(\overline{O'}).$$

Proof. This is a special instance of a basic and general fact, which we state here in the form of a lemma:

Lemma 1.3.2. *Let $Y \subset \mathbb{P}^n$ be a closed subvariety of projective n-space, o a point in $\mathbb{P}^n$ not contained in Y, H a hyperplane of $\mathbb{P}^n$ not containing o,*

$\mathrm{pr}_o : \mathbb{P}^n - \{o\} \to H$ *the projection from the point* o *to the hyperplane* H. *Then we have that the image* $\mathrm{pr}_o(Y)$ *of the subvariety* Y *under the projection* pr_o *is a closed subvariety of the projective space* H.

Proof. Y is projective, hence complete. $\mathrm{pr}_o(Y)$ is the image of a complete variety under a regular map, so it is a closed subvariety of H. □

a variant of this is:

Lemma 1.3.3. *Let* $A \subset Y$ *be an open subset of* Y, *then* $\pi(\overline{A}) = \overline{\pi(A)}$.

Proof. Since π is regular on Y, we have $\pi(\overline{A}) \subset \overline{\pi(A)}$, i.e. π maps $\overline{A}$ into $\overline{\pi(A)}$, this map is dominant since A maps onto $\pi(A)$, but by the previous lemma the image of $\overline{A}$ must be closed. Thus $\pi(\overline{A}) = \overline{\pi(A)}$. □

Now the proposition follows by taking the projective space to be $\overline{V}$, the hyperplane to be $\mathbb{P}(V)$, the point o to be the origin of V, and the locally closed subset A to be the G-orbit O' in V. □

In this way the problem of determining the closure of a G-orbit in $\mathbb{P}(V)$ is translated into the problem of determining the closure inside the projective space $\overline{V}$ of a G-orbit O' which projects onto O.

Now we relate the closure of O' inside $\overline{V}$ with the ideal $I(O') \subset k[V]$ of polynomial functions on V restricting to zero on O', thus achieving the aim of reducing the study of orbits and orbit closures in $\mathbb{P}(V)$ to the study of orbits and orbit closures in V. This second reduction also is consequence of a basic and general fact.

Let us digress a bit at this point. We consider the following situation, $Z \subset \mathbb{P}^n$ is an irriducible projective subvariety of projective n-space, not contained in the hyperplane H of $\mathbb{P}^n$. Let us denote by $I(Z)$ the graded ideal of the graded algebra $k[x_0, \ldots, x_n]$ of those polynomials which are zero on Z. Consider the affine part of Z, i.e. the intersection $Z \cap \mathbb{A}^n$ of the variety

Z with the affine space $\mathbb{A}^n = \mathbb{P}^n - H$. We may summarize the inclusion relations between these varieties in the following commutative diagram

$$\begin{array}{ccc} Z \cap \mathbb{A}^n & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathbb{P}^n. \end{array}$$

We may choose as hyperplane H the one given by the equation $x_0 = 0$; in this case $\mathbb{A}^n$ has coordinates $x_1, \dots, x_n$, and the algebra $k[\mathbb{A}^n]$ of regular functions on $\mathbb{A}^n$ is identified with the algebra of polynomials $k[x_1, \dots, x_n]$ in the indeterminates $x_1, \dots, x_n$. Let $I(Z \cap \mathbb{A}^n)$ be the ideal in $k[x_1, \dots, x_n]$ of the polynomials in $x_1, \dots, x_n$ restricting to zero on $Z \cap \mathbb{A}^n$. There is a map α of dehomogeneization, which is just a restriction, from the algebra $k[x_0, x_1, \dots, x_n]$ to the algebra $k[x_1, \dots, x_n]$ given by $f(x_0, x_1, \dots, x_n) \mapsto f(1, x_1, \dots, x_n)$. This map sends the graded ideal $I(Z)$ to the ideal $I(Z \cap \mathbb{A}^n)$:

$$\begin{array}{ccc} k[x_0, x_1, \dots, x_n] & \xrightarrow{\alpha} & k[x_1, \dots, x_n] \\ \downarrow & & \downarrow \\ I(Z) & \longrightarrow & I(Z \cap \mathbb{A}^n). \end{array}$$

There is also a map β of homogeneization:

$$k[x_1, \dots, x_n] \xrightarrow{\beta} k[x_0, x_1, \dots, x_n],$$

which sends a degree r polynomial in $k[x_1, \dots, x_n]$ to a homogeneous polynomial of the same degree in $k[x_0, x_1, \dots, x_n]$. Observe that the map β does not preserve any of the algebraic structures present on $k[x_1, \dots, x_n]$ and $k[x_0, x_1, \dots, x_n]$. The image of the map β is the set of homogeneous polynomials in $k[x_0, x_1, \dots, x_n]$. So, if J is the ideal in $k[x_0, x_1, \dots, x_n]$ generated by $\beta(I(Z \cap \mathbb{A}^n))$, J is a graded ideal of $k[x_0, x_1, \dots, x_n]$.

Now the basic lemma:

Lemma 1.3.4. $\operatorname{rad}(J) = I(Z)$.

Proof. If $f \in k[x_0, x_1, \ldots, x_n]$ is homogeneous, irreducible and $f \neq x_0$, then $\beta(\alpha(f)) = f$. Since Z is irreducible, there is a set of generators $\{f_i\}$, with $f_i \in k[x_0, x_1, \ldots, x_n]$ homogeneous and irreducible. The fact that Z is not conatined in H implies that $f_i \neq x_0$ for all i. Now, it is easy to see that β sends $I(Z \cap \mathbb{A}^n)$ to $I(Z)$ and that $\alpha(\beta(f)) = f$ for any f in $k[x_1, \ldots, x_n]$, so $\alpha(I(Z)) = I(Z \cap \mathbb{A}^n)$; thus

$$J = \beta(I(Z \cap \mathbb{A}^n)) \cdot k[x_0, x_1, \ldots, x_n] = \beta(\alpha(I(Z))) \cdot k[x_0, x_1, \ldots, x_n] \supset$$

$$\supset \beta(\alpha(\{f_i\}) \cdot k[x_0, x_1, \ldots, x_n] = (f_i) \cdot k[x_0, x_1, \ldots, x_n] = I(Z).$$

So J is a graded ideal containing $I(Z)$. The closed subset Z' which it defines must satisfy $Z \cap \mathbb{A}^n) \subset Z' \subset Z$. This implies $Z' = Z$, since $Z \cap \mathbb{A}^n)$ is dense in Z. And so, by the nullstellensatz, $\mathrm{rad}(J) = I(Z)$. □

We may now return to the discussion of the main problem which we may restate as follows: given a linear representation of the connected reductive group G in the vector space V, supposing known the classification of G-orbits in V and the relations of closure between these G-orbits, to determine the G-orbits in $\mathbb{P}(V)$ and the orbit closures.

Remark 1.3.5. *The map $\pi : V - \{0\} \to \mathbb{P}(V)$ is G-equivariant, thus G-orbits in $\mathbb{P}(V)$ correspond to unions of G-orbits in $V - \{0\}$. Another way to state this is that the torus k^* acting by homotheties on V preserves G-orbits, thus one wants to understand the equivalence relation on the set of G-orbits in $V - \{0\}$ afforded by the action of the torus k^*.*

A first step in the study of an action of a group is the determination of the invariants for that action. The algebra of invariants for the action of G on V is the graded subalgebra $k[V]^G \subset k[V]$. A fundamental result states that $k[V]^G$ is finitely generated as an algebra over k (G is reductive). Let $\{f_1, \ldots, f_r\}$ be a minimal set of homogeneous generators of $k[V]^G$ as k-algebra, the affine closed subset of V associated to the ideal $(f_1, \ldots, f_r)k[V]$

is a cone in V, the null-cone of V. If we define the map $\Phi : V \to \mathbb{A}^r$, by $v \mapsto (f_1(v), \ldots, f_r(v))$, i.e. Φ associates to v its invariants, then the fibers of Φ are stable under G, and the null-cone N is the fiber over the origin $(0, \ldots, 0) \in \mathbb{A}^r$. The null-cone consists of those elements having the same invariants as $0 \in V$; the closure in V of a G-orbit outside of the null-cone does not contain 0 since it must be contained in a fiber of the map Φ different from the null-cone. N determines a closed subvariety of the projective space $\mathbb{P}(V)$ which we will also call N. The main result for G-orbits in $\mathbb{P}(V) - N$ is:

Proposition 1.3.6. *Let O be a G-orbit in $\mathbb{P}(V) - N$, O' a G-orbit in $V - N$ projecting to O through the map π, let $\overline{O}$ be the closure of O in $\mathbb{P}(V)$, $\overline{O'}$ that of O' in $\overline{V}$. If $Q \subset (\mathbb{P}(V) - N) \cap \overline{O}$ is a G-orbit, then there exists a G-orbit $Q' \subset (V - N) \cap \overline{O'}$. The G orbits of $\mathbb{P}(V)$ which are in the null-cone and in $\overline{O}$ are the G-orbits in $\overline{O'} \cap \mathbb{P}(V)$, where $\mathbb{P}(V)$is identified with the hyperplane at infinity of $\overline{V}$. Such intersection is cut out set-theoretically inside $\mathbb{P}(V)$ by the ideal obtained homogenizing $I(O')$ and then replacing x_0 with 0.*

Proof. Follows from the above discussion. □

With the adjunction of a restrictive hypothesis on the action of G, we may complete this with a statement about nilpotent orbits.

Proposition 1.3.7. *If the action of G on V is such that the fibers of the map Φ are finite unions of G-orbits, then the orbits in the nullcone are invariant with respect to homotheties and the closure relations are consequences of the affine closure relations. Moreover the closure of a G-orbit in $\mathbb{P}(V)$ is a finite union of G-orbits.*

Proof. Since k^* is connected, a homothety class of an orbit consists of either just one orbit or of an infinite number of orbit. Since an orbit homothetic to an orbit in the null-cone is again an orbit of the null-cone then all orbits in the null-cone must be invariant with respect to homotheties. This implies that the closure relations in the null-cone as subvariety of $\mathbb{P}(V)$ are just the closure relations in the null-cone as subvariety of V. □

1.4 Conclusion of the proof of theorem 1.2.2

We now finish off the proof of theorem 1.2.2. The strategy is to use proposition 1.3.6 to find the equations of the closure of a $GL(V)$-orbit in $\mathbb{P}(V)$ and to translate such equations in terms of partitions. For that, we use an elementary lemma.

Lemma 1.4.1. *Let $p^{(1)}, \ldots, p^{(l)}$ be partitions of $n_1, \ldots, n_l$ respectively. We denote by $q^{(1)}, \ldots, q^{(l)}$ the transposed partitions of the partitions $p^{(1)}, \ldots, p^{(l)}$ respectively. For $1 \leq i \leq l$, $j \geq 1$ we define $a_j^{(i)} = q_1^i + \ldots + q_j^i$. Let $p = p^{(1)} + \ldots + p^{(l)}$ be the partition of $n = n_1 + \ldots + n_l$, sum of the partitions $p^{(1)}, \ldots, p^{(l)}$, and q the transposed partition of p. Defining, for $j \geq 1$, $a_j = p_1 + \ldots + p_j$, we have the identities*

$$a_j = a_j^{(1)} + \ldots + a_j^{(l)}.$$

Proof. Follows from the definitions by elementary arithmetic. □

Let now O be a $GL(V)$-orbit in $\mathbb{P}(V)$, O' a $GL(V)$-orbit in $\mathrm{End}(V)$ projecting onto O. Let $\lambda_1, \ldots, \lambda_l$ the eigenvalues associated to O' with the multiplicities $n_1, \ldots, n_l$, and $p^{(1)}, \ldots, p^{(l)}$ the partitions specifying the Jordan canonical form of endomorphisms in O'. Define $a_j^{(i)}$, p and a_j as in lemma 1.4.1. By proposition 1.3.6, one gets equations satisfied by O by taking equations satisfied by O' and taking the homogeneous part of maximum total degree if there is one. The equations

$$\dim(\ker(\prod_i (X - \lambda_i I))^j) \geq a_j^{(1)} + \ldots + a_j^{(l)}$$

are satisfied by O'. These, by lemma 1.4.1 become

$$\dim(\ker(\prod_i (X - \lambda_i I))^j) \geq a_j.$$

It is readily seen that the homogeneous parts of maximum total degree of these polynomial equations are

$$\dim(\ker(X^j) \geq a_j.$$

This implies that a nilpotent orbit in the closure of O must have the corresponding partition smaller than or equal the partition p. This concludes the proof of theorem 1.2.2.

Chapter 2

Lusztig's method

This chapter is divided in two parts. In the first we develop an inductive procedure for finding the G-orbits for a class of homogeneous spaces under the group $G \times G$, for any G connected reductive. The class of homogeneous spaces under consideration contains the homogeneous spaces which arise as $G \times G$-orbits in the canonical compactification X of G. In some sense, the class considered is the smallest class of homogeneous spaces under $G \times G$ which is closed under the inductive procedure exposed. The procedure is due to Lusztig in the main lines. In [L], Lusztig applied it to the study of parabolic character sheaves. This presentation is a refinement of Lusztig's procedure as exposed in unpuplished notes [S] by Springer. While the second part of the chapter deals with the problem of the closure of G-orbits in an ambient space. Again, the aim is to develop a bit of general machinery to be used in finding which G-orbits of the canonical compactification X of G are in the closure of a given G-orbit. There isn't a simple general way to determine the closure of a G-orbit in X, so one is forced to an almost case by case analysis. One can divide the G-orbits of X roughly in three classes, those of high dimension i.e. the regular orbits and some subregular orbits, the orbits of intermediate dimension, and finally the smaller orbits. For regular orbits, the method of computing its invariants usually allows the determination of its closure. For G classical group of rank smaller than or equal to two, i.e. for the cases in which the computations were carried out, the result is that the closure of a regular G-orbit in X is made up of the regular orbit in question plus all nilpotent G-orbits in X, i.e. those orbits every component of which is nilpotent. Such result should be true for any G, although it does not follow immediately from the methods developed here. The intermediate orbits are the most problematic since computing invariants is not very useful in such case, and also dimension arguments are usually not sufficient. So one must resort to explicit degenerations and to a basic but handy negative criterion which will be exposed in this chapter. While for low dimensional orbits one may usually just argue by dimension considerations, thus no new method is needed. The techniques developed in this chapter will be put to use in the next.

2.1 Basic definitions

Let G be a connected reductive algebraic group over an algebraically closed field k, $\mathfrak{g}$ its Lie algebra. Let us fix a maximal torus T of G; $\mathfrak{t} \subset \mathfrak{g}$ be the Lie algebra of T. Denote by R the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$, i.e. $R \subset \mathfrak{t}^*$ and there exists a root decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is defined by the property $[t, g] = \alpha(t)g$, $\forall t \in \mathfrak{t}$, $\forall g \in \mathfrak{g}_\alpha$.
The Weyl group W of (G, T) is defined as the quotient $N_G(T)/T$. W acts on T, through the action induced by the action of $N_G(T)$ on T by conjugation. In the same way, the adjoint action of $N_G(T)$ on $\mathfrak{t}$ induces an action of W on $\mathfrak{t}$.
Let us fix a Borel subgroup B of G. This yields a choice of simple roots $\Delta \subset R$, i.e. a subset Δ of the set of roots R of G such that any root $\alpha \in R$ can be uniquely expressed as a linear combination $\sum n_i\alpha_i$ $\alpha_i \in \Delta$, with integer coefficients, either all nonnegative or all nonpositive. The subset $R^+ \subset R$ of positive roots, relative to the choice of simple roots Δ, is defined as the

set of roots which are linear combination of simple roots with nonnegative integral coefficients. By what has been said and properties of root systems $R = R^+ \coprod -R^+$.

A parabolic subgroup of G is defined as a connected subgroup containing a Borel subgroup of G. A standard parabolic subgroup P of G (with respect to the choice of B) is a parabolic subgroup P containing the Borel subgroup B; since all Borel subgroups of G are conjugate to B, then any parabolic subgroup of G is conjugate to a standard parabolic subgroup of G.

The Lie algebra $\mathfrak{p}$ of a standard parabolic P has a decomposition in root spaces

$$\mathfrak{p} = \mathfrak{t} \oplus \mathfrak{b} \oplus \bigoplus_{\alpha \in R_I^+} \mathfrak{g}_{-\alpha},$$

where for I a subset of the set of simple roots Δ, R_I denotes the subroot system generated by I, and R_I^+ the set of positive roots for such system.

A parabolic subgroup P of G containing T is conjugate to a standard parabolic by an element of $N_G(T)$; since conjugation by an element of T stabilizes Q, then the action of $N_G(T)$ on parabolics containing T passes to the Weyl group W. Let us, for example, suppose that $Q = wPw^{-1}$, $w \in W$, then its Lie algebra $\mathfrak{q}$ has root space decomposition

$$\mathfrak{q} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_{w(\alpha)} \oplus \bigoplus_{\alpha \in R_I^+} \mathfrak{g}_{w(-\alpha)}.$$

If $P = L{\cdot}U$ is a Levi decomposition of P, then obviously $Q = wLw^{-1}{\cdot}wUw^{-1}$ is a Levi decomposition of Q. The corresponding decomposition of the Lie algebra $\mathfrak{q}$ is

$$\mathfrak{q} = \bigoplus_{\alpha \in w(R_{\Delta - I})} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in w(R_I)} \mathfrak{g}_\alpha.$$

Remark 2.1.1. *If P, Q are two parabolic subgroups of G containing T, with Lie algebra, respectively, $\mathfrak{p}$, $\mathfrak{q}$, then $P \cap Q$ has Lie algebra $\mathfrak{p} \cap \mathfrak{q}$; moreover if*

$$\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha, \qquad \mathfrak{q} = \mathfrak{t} \oplus \bigoplus_{\alpha \in B} \mathfrak{g}_\alpha,$$

with $A \subset R$ and $B \subset R$, are the root space decompositions of $\mathfrak{p}$ and $\mathfrak{q}$, then $\mathfrak{p} \cap \mathfrak{q}$ has root decomposition

$$\mathfrak{p} \cap \mathfrak{q} = \mathfrak{t} \oplus \bigoplus_{\alpha \in A \cap B} \mathfrak{g}_\alpha .$$

Thus $\mathfrak{p} \cap \mathfrak{q}$ is completely determined by $A \cap B$; and, more important, $A \cap B$ determines $P \cap Q$. If $I(P)$, $I(Q)$, $I(P \cap Q)$ are respectively the ideals in $\mathbb{C}[G]$ corresponding to P, Q and $P \cap Q$, then of course $I(P \cap Q) = I(P) + I(Q)$ and the union of a basis of $I(P)$ and a basis of $I(Q)$ is a basis of $I(P) \cap I(Q)$. Now there is a family $\{g_\alpha\}_{\alpha \in R}$, $g_\alpha \in \mathbb{C}[G]$, such that, if P is a parabolic subgroup of G containing T, with root space decomposition of the Lie algebra

$$\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha ,$$

then $\{g_\alpha\}_{\alpha \in R - A}$ is a basis of the ideal $I(P)$. This implies that $\{g_\alpha\}_{\alpha \in R - A \cap B}$ is a basis for the ideal $I(P \cap Q)$ of $P \cap Q$. Thus one can determine the conjugate of a parabolic containing T by an element of W by looking at the action on roots and determine the intersection of two parabolics containing T just by looking at roots.

Lemma 2.1.2. *Let P and Q be two parabolic subgroups containing the maximal torus T; $P = L \cdot U$, $Q = M \cdot V$ Levi decomposition respectively of P and Q, compatible with the root space decomposition, i.e. $T \subset L$ and $T \subset M$. Let $\pi_P : P \longrightarrow L$ be the canonical projection of P to L, and $\pi_Q : Q \longrightarrow M$ be the canonical projection of Q to M. Then the following decomposition holds:*

$$P \cap Q = (L \cap M) \cdot (L \cap V) \cdot (U \cap M) \cdot (U \cap V).$$

moreover

$$\pi_P(P \cap Q) = (L \cap M) \cdot (L \cap V),$$

and

$$\pi_Q(P \cap Q) = (L \cap M) \cdot (U \cap M).$$

Proof. Let us prove the decomposition $P \cap Q = (L \cap M) \cdot (L \cap V) \cdot (U \cap M) \cdot (U \cap V)$. Since $L \cap U = \{1\}$ and $M \cap V = \{1\}$, the four subgroups $L \cap M$, $L \cap V$, $U \cap M$, $U \cap V$ are disjoint. Let g be an element of $P \cap Q$. It decomposes in a unique way in a product $hk = g$, $h \in L$, $k \in U$. $h = \pi_P(g)$, so $h \in \pi_P(P \cap Q) = L \cap Q$. While $k = h^{-1}g \in U \cap Q$. Now h, as an element of Q, decomposes uniquely as $h = h_1 h_2$, $h_1 \in M$, $h_2 \in V$. $h_1 = \pi_P(h) \in \pi_P(L \cap Q) = L \cap M$, while $h_2 = h_1^{-1}h \in L \cap V$. Analogously for k, $k = k_1 k_2$, with $k_1 = \pi_Q(k) \in \pi_Q(U \cap Q)$ and $\pi_Q(U \cap Q) = U \cap M$, while $k_2 = k_1^{-1}k \in U \cap V$. The proof of this decomposition yelds also the decompositions $\pi_P(P \cap Q) = (L \cap M) \cdot (L \cap V)$ and $\pi_Q(P \cap Q) = (L \cap M) \cdot (U \cap M)$ □

Corollary 2.1.3. *$\pi_P(P \cap Q)$ is a parabolic subgroup of L, its unipotent radical being $L \cap V$, and $L \cap M$ being a Levi factor. $\pi_Q(P \cap Q)$ is a parabolic subgroup of M with unipotent radical $U \cap M$, and $L \cap M$ a Levi factor.*

Corollary 2.1.4. *If A is the set of roots of P and B the set of roots of Q, $A = A_1 \coprod A_2$ the partition of A afforded by the Levi decomposition $P = L \cdot U$, $B = B_1 \coprod B_2$ the partition afforded by the Levi decomposition $Q = M \cdot V$, then $P \cap Q$ is the subgroup corresponding to $A \cap B$, $\pi_P(P \cap Q) = L \cap Q$ the subgroup corresponding to $A_1 \cap B$, its Levi decomposition corresponds to the partition $A_1 \cap B = (A_1 \cap B_1) \coprod (A_1 \cap B_2)$. Analogously, $\pi_Q(P \cap Q) = P \cap M$ is the subgroup corresponding to $A \cap B_1$ and its Levi decomposition corresponds to the partition $A \cap B_1 = (A_1 \cap B_1) \coprod (A_2 \cap B_1)$.*

2.2 The group $\Delta_{P,Q,\sigma}$ and its orbits in G

Let P, Q, L, M, U, V, π_P, π_Q be defined as above, and let $\sigma : L \longrightarrow M$ be an isomorphism such that $\sigma(T) = T$. Then the following lemma is easily checked:

Lemma 2.2.1. *If $A_1 \subset \mathfrak{t}^*$ is the set of roots of L, and $B_1 \subset \mathfrak{t}^*$ that of M,* *i.e.*

$$\mathfrak{l} = \mathfrak{t} \oplus \bigoplus_{\alpha \in A_1} \mathfrak{g}_\alpha,$$

$$\mathfrak{m} = \mathfrak{t} \oplus \bigoplus_{\alpha \in B_1} \mathfrak{g}_\alpha$$

where $\mathfrak{l}$ *and* $\mathfrak{m}$ *denote respectively the Lie algebra of* L *and* M*, then* $\sigma(A_1) = B_1$ *and* $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma(\alpha)}$ *for* $\alpha \in A_1$*.*

We now define the subgroup $\Delta_{P,Q,\sigma}$ of $G \times G$.

Definition 2.2.2. *G be a connected reductive group, T a maximal torus of G, P, Q parabolic subgroups of G containing T, with Levi decompositions $P = L{\cdot}U$, $Q = M{\cdot}V$, L, M reductive subgroups containing T, U, V unipotent radicals respectively of P and Q; then we define the subgroup $\Delta_{P,Q,\sigma} \subset G \times G$ as*

$$\Delta_{P,Q,\sigma} = \{(p,q) \in P \times Q \quad | \quad \sigma(\pi_P(p)) = \pi_Q(q)\}.$$

If Δ_σ is the graph of $\sigma : L \longrightarrow M$, i.e.

$$\{(l,m) \in L \times M \quad | \quad m = \sigma(l)\},$$

then $\Delta_{P,Q,\sigma}$ is the semidirect product of Δ_σ and $U \times V$:

$$\Delta_{P,Q,\sigma} = \Delta_\sigma \ltimes U \times V.$$

$\Delta_{P,Q,\sigma}$ as subgroup of $G \times G$ inherits the action on G by left and right multiplication, i.e. $\Delta_{P,Q,\sigma} \times G \longrightarrow G$ is defined by $((p,q),g) \mapsto pgq^{-1}$.
Let us study the orbits of $\Delta_{P,Q,\sigma}$ in G. For this purpose we first decompose G in $P \times Q-$orbits, and then study the $\Delta_{P,Q,\sigma}-$orbit decomposition of each $P \times Q-$orbit separetely.

2.3 $P \times Q-$orbits in G (Bruhat decomposition)

We recall some well-known facts (see [B]).

Let G be a connected reductive algebraic group, T a maximal torus and $W = N(G)/T$ the Weyl group of G relative to T. For two parabolic subgroups P, Q of G containing T, we define the subgroups W_P, W_Q of W as the subgroups of W stabilizing respectively P and Q.
$G \times G$ acts on G by left and right multiplication: $(g_1, g_2)g = g_1 g g_2^{-1}$. While $G \times G$ acts transitively on G, its subgroup $P \times Q$ does not. We have:

Proposition 2.3.1. *The action of $P \times Q$ partitions G into finitely many orbits; such orbits are locally closed subvarieties of G, they are indexed by the elements of $W_P \backslash W / W_Q$ in the following way*

$$[w] \mapsto P \times Q \cdot (1, w^{-1}) \cdot 1_G = \text{orbit of } w.$$

Here by w we denote both an element of W and a representative in $N_G(T)$.

The closure of a $P \times Q$-orbit is $P \times Q$-stable, so is the union of $P \times Q$-orbits.
Let $O_{[w]}$, $[w] \in W_P \backslash W / W_Q$ denote the $P \times Q$-orbit corresponding to the class $[w]$, then we may define an order relation on $W_P \backslash W / W_Q$ such that $[w] \geq [w']$ if and only if $O_{[w']} \subset \overline{O_{[w]}}$.
On the other hand, if we fix a Borel subgroup B of G, we have the Bruhat order on W (with respect to the choice of generetors yielded by B).
We state the relation between these two orders.

Lemma 2.3.2. *$w_1 \in W$ be such that $P' = w_1 P w_1^{-1} \supset B$, and $w_2 \in W$ such that $Q' = w_2 Q w_2^{-1} \supset B$. We may endow the set $W_{P'} \backslash W / W_{Q'}$ with the structure of a partially ordered set by embedding it in W in the following way:*

$$W_{P'} \backslash W / W_{Q'} \to W, \quad c \in W_{P'} \backslash W / W_{Q'} \mapsto w(c),$$

where with $w(c)$ we denote the minimum length representative of c. Then, with respect to these order relations, the map

$$W_P \backslash W / W_Q \to W_{P'} \backslash W / W_{Q'}$$

defined by

$$[w] \mapsto [w_1 w w_2^{-1}]$$

is an order preserving bijection.
Also the dimension of the orbit $O_{[w]}$ can be read off from the class $[w]$; in fact, the codimension of $O_{[w]}$ in G is equal to the length of the representative of minimum length in the class $[w_1 w w_2^{-1}]$.

2.4 $\Delta_{P,Q,\sigma}$-orbits in a $P \times Q$-orbit of G

Let $O_{[w]}$ be the $P \times Q$-orbit in G corresponding to $[w] \in W_P \backslash W / W_Q$, $\Delta_{P,Q,\sigma}$ acts on $O_{[w]}$ and we will study its orbit decomposition. Let $w \in W$ be the representative of $[w]$ of minimal length, we will also denote w a representative in $N_G(T)$. Let Z_w be the group $Z_w = \{(p, w^{-1}pw) \in P \times Q \mid p \in P \cap wQw^{-1}\}$, then it is easy to see that Z_w is the stabilizer of $w \in G$ in $P \times Q$. So we have an isomorphism

$$P \times Q / Z_w \to O_{[w]},$$
$$(p, q) \mapsto pwq^{-1}.$$

This isomorphism is $P \times Q$-equivariant, hence a fortiori $\Delta_{P,Q,\sigma}$-equivariant. Composing the natural projection from $P \times Q$ to $P \times Q / Z_w$ with the above isomorphism we get a map

$$P \times Q \to O_{[w]}$$

which is compatible with the following action of $\Delta_{P,Q,\sigma} \times Z_w$ on the source and on the target:

$$\Delta_{P,Q,\sigma} \times Z_w \times (P \times Q) \to P \times Q$$

defined by

$$(((p_1, q_1), (p_2, q_2)), (p, q)) \mapsto (p_1 p p_2^{-1}, q_1 q q_2^{-1}),$$

and

$$\Delta_{P,Q,\sigma} \times Z_w \times O_{[w]} \to O_{[w]}$$

defined by

$$((p_1, q_1), (p_2, q_2), x) \mapsto p_1 x q_1^{-1}.$$

This morphism induces a bijection between the set of $\Delta_{P,Q,\sigma} \times Z_w$-orbits in $P \times Q$ and the set of $\Delta_{P,Q,\sigma}$-orbits in $O_{[w]}$. Analogously, we can consider the coset space $\Delta_{P,Q,\sigma} \backslash P \times Q$ endowed with the natural action of Z_w:

$$Z_w \times (\Delta_{P,Q,\sigma} \backslash P \times Q) \to \Delta_{P,Q,\sigma} \backslash P \times Q,$$

$$((p_1, q_1), (\Delta_{P,Q,\sigma} \cdot (p, q))) \mapsto \Delta_{P,Q,\sigma} \cdot (p p_1^{-1}, q q_1^{-1});$$

and the natural projection

$$P \times Q \to \Delta_{P,Q,\sigma} \backslash P \times Q,$$

which will induce a bijection between the $\Delta_{P,Q,\sigma} \times Z_w$-orbits in $P \times Q$ and the Z_w-orbits in $\Delta_{P,Q,\sigma} \backslash P \times Q$.
Summing up, we have defined a bijective correspondence between the $\Delta_{P,Q,\sigma}$-orbits in $O_{[w]}$ and the Z_w-orbits in $\Delta_{P,Q,\sigma} \backslash P \times Q$.

Lemma 2.4.1. *The map $P \times Q \to L$ given by $(p, q) \mapsto (\sigma^{-1}(\pi_Q(q)))^{-1}\pi(p)$ induces an isomorphism $\alpha : \Delta_{P,Q,\sigma} \backslash P \times Q \to L$. Letting $G_1 = L$, the subgroups $P_1 = \sigma^{-1}(w^{-1}Pw \cap M)$ and $Q_1 = L \cap wQw^{-1}$ are two parabolic subgroups of the connected reductive group G_1 containing the maximal torus T, whose Levi subgroups containing T are respectively $L_1 = \sigma^{-1}(w^{-1}Lw \cap M)$ and $M_1 = L \cap wMw^{-1}$, and unipotent radicals are respectively $U_1 = \sigma^{-1}(w^{-1}Uw \cap M)$ and $V_1 = L \cap wVw^{-1}$. We may define between the subgroups L_1 and M_1 the isomorphism $\sigma_1 : L_1 \to M_1$ defined by $l \mapsto w\sigma(l)w^{-1}$; there is a morphism $Z_w \to \Delta_{P_1,Q_1,\sigma_1}$ defined by the commutativity of the diagram*

$$\begin{array}{ccccccc}
P \times Q & \xrightarrow{(\pi_P,\pi_Q)} & L \times M & \xrightarrow{(id,\sigma^{-1})} & L \times L & \xrightarrow{(l_1,l_2)\mapsto(l_2,l_1)} & L \times L \\
\uparrow & & & & & & \uparrow \\
Z_w & \xrightarrow{\beta} & & & & & \Delta_{P_1,Q_1,\sigma_1}
\end{array}$$

Considering the action of $\Delta_{P_1,Q_1,\sigma_1}$ on G_1 by $(p,q)\cdot g = pgq^{-1}$, $(p,q) \in \Delta_{P_1,Q_1,\sigma_1}$, $g \in G_1$; then the isomorphism α is compatible with the action of Z_w on $\Delta_{P,Q,\sigma}\backslash P \times Q$, the action of $\Delta_{P_1,Q_1,\sigma_1}$ on G_1 and the isomorphism $\beta : Z_w \to \Delta_{P_1,Q_1,\sigma_1}$.

Proof. The morphism $P \times Q \to L$ considered above is just the composition

$$P \times Q \xrightarrow{(\pi_P,\pi_Q)} L \times M \xrightarrow{(\mathrm{id},\sigma^{-1})} L \times L \xrightarrow{(x,y)\mapsto(y,x)} L \times L \xrightarrow{(x,y)\mapsto xy^{-1}} L.$$

It is surjective as composition of surjestive maps, and $(p_1,q_1),(p_2,q_2) \in P\times Q$ have the same image in L if and only if they are in the same right coset of the group $\{(p,q) \mid \sigma(\pi_P(p)) = \pi_Q(q)\}$, which is exactly $\Delta_{P,Q,\sigma}$. This proves that the morphism α is well-defined and that it is an isomorphism.

Z_w acts on $P\times Q$ on the right, andsuch action, through the morphism α is turned into the action $(p,q)\cdot l = \sigma^{-1}(\pi_Q(q)) l \pi_P(p)^{-1}$; so one must just check, to prove the second statement, that the image of Z_w by the homomorphism $Z_w \to P \times Q \to L \times M \to L \times L \to L \times L$ is the group $\Delta_{P_1,Q_1,\sigma_1}$. Let us check this. The first arrow above is the canonical inclusion, the second is the canonical projection, the third is to apply σ^{-1} to the second coordinate and the last is to interchange the first and second coordinates. Z_w is the image of the homomorphism $P\cap wQw^{-1} \to P\times Q$ defined by $p \mapsto (p, w^{-1}pw)$. We have seen that $P\cap wQw^{-1} = (L\cap wMw^{-1})(L\cap wVw^{-1})(U\cap wMw^{-1})(U\cap wVw^{-1})$, so the image of Z_w is the product of

$$H_1 = \{(\sigma^{-1}(\pi_Q(w^{-1}gw)), \pi_P(g)),\ g \in L \cap wMw^{-1}\} =$$

$$= \{(\sigma^{-1}(w^{-1}gw), g),\ g \in L \cap wMw^{-1}\},$$

$$H_2 = \{(\sigma^{-1}(\pi_Q(w^{-1}gw), \pi_P(g)),\ g \in U \cap wMw^{-1}\} =$$

$$= \{(\sigma^{-1}(w^{-1}gw), 1),\ g \in U \cap wMw^{-1}\},$$

$$H_3 = \{(\sigma^{-1}(\pi_Q(w^{-1}gw)), \pi_P(g)),\ g \in L \cap wVw^{-1}\} = \{(1, g),\ g \in L \cap wVw^{-1}\},$$

$$H_4 = \{(\sigma^{-1}(\pi_Q(w^{-1}gw)), \pi_P(g)),\ g \in U \cap wVw^{-1}\} = \{1\}.$$

So the image of Z_w is the product $H_1 H_2 H_3$; but we can write

$$H_1 = \{(g, w\sigma(g)w^{-1}),\ g \in \sigma^{-1}(w^{-1}Lw \cap M)\},$$

$$H_2 = \{(g, 1),\ g \in \sigma^{-1}(w^{-1}Uw \cap M)\}, \text{ and}$$

$$H_3 = \{(1, g),\ g \in L \cap wVw^{-1}\}.$$

This is exactly $\Delta_{P_1,Q_1,\sigma_1}$. Thus the lemma is proved. □

Summing up, to classify $\Delta_{P,Q,\sigma}$-orbits in G one partitions G into $P \times Q$-orbits; in each of the $P \times Q$-orbits the problem of classifying $\Delta_{P,Q,\sigma}$-orbits is reduced to the classification of $\Delta_{P_1,Q_1,\sigma_1}$-orbits in a smaller group G_1. So one can iterate the procedure, subdividing G_1 in $P_1 \times Q_1$-orbits and then analyzing each $P_1 \times Q_1$-orbit separately. Inside a $P_1 \times Q_1$-orbit one is reduced to the study of $\Delta_{P_2,Q_2,\sigma_2}$-orbits in a smaller group G_2, and so on. The procedure terminates when $P_n = Q_n = G_n$; in this case the $\Delta_{P_n,Q_n,\sigma_n}$-orbits in G_n are the twisted σ_n-conjugacy classes in G_n. Hence the classification of $\Delta_{P,Q,\sigma}$-orbits in G reads as follows:

Proposition 2.4.2. *There is a finite partition of G into $\Delta_{P,Q,\sigma}$-stable locally closed subsets indexed by sequences $([w_0], [w_1], \ldots, [w_n])$, where $[w_0] \in W_P \backslash W / W_Q$, $[w_i] \in W_{P_i} \backslash W_i / W_{Q_i}$, with W_i the Weyl group of G_i relative to T, and such that $P_{n+1} = Q_{n+1} = G_{n+1}$. Moreover, the $\Delta_{P,Q,\sigma}$-orbits inside the locally closed subset indexed by $([w_0], [w_1], \ldots, [w_n])$ correspond to the twisted conjugacy classes in G_{n+1} with respect to the automorphism σ_{n+1}.*

More important is the fact that this method of classification actually gives a set of representatives for the $\Delta_{P,Q,\sigma}$-orbits in G provided one can produce a set of representatives of the twisted conjugacy classes in the subgroups G_{n+1} appearing.

Proposition 2.4.3. *Let* $([w_0], [w_1], \ldots, [w_n])$ *be a sequence parametrizing a locally closed subset as above,* $w_i \in W_i$ *the representative of minimum length of the class* $[w_i]$*. We choose a lifting in* G_i *of* w_i *and we denote it* w_i *as well. Let* $\{s_\alpha\}$ *be a set of representatives of the* σ_{n+1}*-twisted conjugacy classes in* G_{n+1}*, then* $\{s_\alpha w_n w_{n-1} \ldots w_1 w_0 \in G\}$ *is a set of representatives of the* $\Delta_{P,Q,\sigma}$*-orbits in the locally closed subset of* G *indexed by* $([w_0], [w_1], \ldots, [w_n])$*.*

Proof. The way σ_{n+1}-twisted conjugacy classes in G_{n+1} correspond to $\Delta_{P,Q,\sigma}$-orbits in G is through the diagram

$$\begin{array}{ccccccccc}
G & & & & & & & & \\
\uparrow & & & & & & & & \\
O_{[w_0]} & \longleftarrow & P_0 \times Q_0 & \longrightarrow & G_1 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & O_{[w_1]} & \longleftarrow & \ldots & & \\
 & & & & & & & & \\
 & & & & & & \ldots & \longrightarrow & G_n \\
 & & & & & & & & \uparrow \\
 & & & & & & & & O_{[w_n]} \longleftarrow P_n \times Q_n \longrightarrow G_{n+1};
\end{array}$$

we just have to lift back the representatives s_α to G. A lift from G_{n+1} to $P_n \times Q_n$ of s_α is $(s_\alpha, 1)$. Pushing it down to $O_{[w_n]}$ one obtains $s_\alpha w_n$ in G_n. We lift it to $(s_\alpha w_n, 1)$ in $P_{n-1} \times Q_{n-1}$; then pushing it down to $O_{[w_{n-1}]}$ one gets $s_\alpha w_n w_{n-1} \in G_{n-1}$. Proceeding in such way up to G, one gets the element

$s_\alpha w_n w_{n-1} \dots w_0 \in G$. These are representatives of the $\Delta_{P,Q,\sigma}$-orbits in the subset in question. □

Proposition 2.4.4. *If O is a $\Delta_{P,Q,\sigma}$-orbit in the locally closed subset indexed by $([w_0],[w_1],\dots,[w_n])$ corresponding to a σ_{n+1}-twisted conjugacy class S in G_{n+1}, then* $\mathrm{codim}_G O = \sum \mathrm{codim}_{G_i} O_{[w_i]} + \mathrm{codim}_{G_{n+1}} S$.

Proof. Straightforward from the definition of O. □

This proposition permits to compute the dimension of $\Delta_{P,Q,\sigma}$-orbits by looking at their classifying data. The codimension of the $O_{[w_i]}$ is readily expressed in terms of the length function in W_i, while the codimension of S in G_{n+1} is equal to the dimension of the stabilizer $\mathrm{Stab}_{G_{n+1}}(s)$ of a representative s of S.

This is Lusztig's method for classifying $\Delta_{P,Q,\sigma}$-orbits in G, but of course it can also be seen as a classification of the G-orbits in $G \times G/\Delta_{P,Q,\sigma}$:

Proposition 2.4.5. *There is a bijection between G-orbits in $G \times G/\Delta_{P,Q,\sigma}$ and $\Delta_{P,Q,\sigma}$-orbits in G, moreover if $\{s_\alpha\}$ is a set of representatives of $\Delta_{P,Q,\sigma}$-orbits in G, then $\{((s_\alpha)^{-1},1)\}$ is a set of representatives of G-orbits in $G \times G/\Delta_{P,Q,\sigma}$.*

Proof. Indeed both sets of orbits correspond to the $G \times \Delta_{P,Q,\sigma}$-orbits in $G \times G$ under the maps $G \times G/\Delta_{P,Q,\sigma} \leftarrow G \times G \to G$, where the first is canonical projection and the second is $(g_1,g_2) \mapsto (g_1)^{-1}g_2$; so lifting a set of representatives to $G \times G$ one gets that $\{((s_\alpha)^{-1},1)\}$ is a set of representatives for G-orbits in $G \times G/\Delta_{P,Q,\sigma}$. □

We can also give a parallel, slightly more geometric presentation of Lusztig's method for classifying the G-orbits in $G \times G/\Delta_{P,Q,\sigma}$. Consider the projection $G \times G/\Delta_{P,Q,\sigma} \to G \times G/P \times Q$, it is a fibration with fiber isomorphic to $L\backslash L \times L$; over the point (P,Q) of $G \times G/P \times Q$ the fiber is $P \times Q/\Delta_{P,Q,\sigma}$ and is isomorphic to $L\backslash L \times L$ through the map $P \times Q/\Delta_{P,Q,\sigma} \to L\backslash L \times L$ given by $(p,q) \mapsto (\pi_P(p)^{-1}, (\sigma^{-1}(\pi_Q(q)))^{-1})$.

Lemma 2.4.6. *$G \times G/P \times Q$ decomposes into finitely many orbits under the action of G; a set of representatives is $\{(1,w)\cdot(P,Q)\}$, $w \in N_G(T)$ representative of minimum length of a double coset class in $W_P\backslash W/W_Q$. The stabilizer of the point $(1,w)\cdot(P,Q)$ in the group G is the group $S_w = \{(g,g),\ g \in P \cap wQw^{-1}\}$.*

Proof. The first part is just Bruhat decomposition. As for the second, the stabilizer of $(1,w)\cdot(P,Q)$ in G is the intersection of G with the stabilizer of $(1,w)\cdot(P,Q)$ in $G \times G$, which is $P \times wQw^{-1}$. □

Lemma 2.4.7. *The inverse images of the G-orbits in $G \times G/P \times Q$ are G-stable subsets of $G \times G/\Delta_{P,Q,\sigma}$. The G-orbits in $G \times G/\Delta_{P,Q,\sigma}$ over $G \cdot (1,w)\cdot(P,Q)$ are in bijection with the S_w-orbits in the fiber over $(1,w)\cdot(P,Q)$. Defining $Z_w = (1,w)^{-1}S_w(1,w) = \{(p, w^{-1}pw),\ p \in P \cap wQw^{-1}\}$, the map $x \mapsto (1,w)x$ takes the fiber over (P,Q) to the fiber over $(1,w)\cdot(P,Q)$ and defines a bijection between the Z_w-orbits in the first fiber and the S_w-orbits in the second fiber.*

Proof. All statements are straightforward. The first follows because the fibration is G-equivariant. The second follows by the basic general fact that in a G-equivariant fibration $X \to Y$, Y homogeneous under G, any point in X is G-conjugate to a point over any fixed $y \in Y$, and any two points over y are G-conjugate in X if and only if they are $\mathrm{Stab}_G(y)$-conjugate in the fiber X_y of X over y. That $x \mapsto (1,w)x$ maps the fiber over (P,Q) to the fiber over $(1,w)\cdot(P,Q)$ is obvious, but since through this map the Z_w-action becomes the S_w-action, the bijection on orbits follows. □

Lemma 2.4.8. *Let $P_1 = L \cap wQw^{-1}$, $Q_1 = \sigma^{-1}(w^{-1}Pw \cap M)$, $G_1 = L$. Then P_1, Q_1 are parabolic subgroups of G_1 containing T, with Levi factors respectively $L_1 = L \cap wMw^{-1}$ and $M_1 = \sigma^{-1}(w^{-1}Lw \cap M)$, unipotent radicals respectively $U_1 = L \cap wVw^{-1}$ and $V_1 = \sigma^{-1}(w^{-1}Uw \cap M)$. If $\sigma_1 : L_1 \to M_1$ is defined by $l \mapsto \sigma^{-1}(w^{-1}lw)$, then the Z_w-action on the fiber over (P,Q) fac-*

tors through the map $Z_w \to \Delta_{P_1,Q_1,\sigma_1}$ defined by $(p,q) \mapsto (\pi_P(p), \sigma^{-1}(\pi_Q(q))$ and the action of $\Delta_{P_1,Q_1,\sigma_1}$ on $G_1 \times G_1$ by right multiplication.

Proof. Since the fiber over (P,Q) is identified to $L\backslash L \times L$, one sees that the action of Z_w factors through $Z_w \to L \times L$, so one must check that the image of this map is $\Delta_{P_1,Q_1,\sigma_1}$. This follows readily from the decomposition $P \cap wQw^{-1} = (L \cap wMw^{-1})(L \cap wVw^{-1})(U \cap wMw^{-1})(U \cap wVw^{-1})$, and the definition of Z_w. □

Notice the slight change in the definition of P_1, Q_1, σ_1. This makes keeping track of representatives easier, at the expense of making the computation of the σ_i messier.

We state in this new guise Lusztig's method.

Lusztig's algorithm

- Set $G_0 = G$, $P_0 = P$, $Q_0 = Q$, $L_0 = L$, $M_0 = M$, $U_0 = U$, $V_0 = V$, $\sigma_0 = \sigma$, $W_0 = W$.

- Choose $[w_0] \in W_{P_0}\backslash W_0/W_{Q_0}$, and w_0 be a lifting to G_0 of the representative of minimum length of $[w_0]$.

- Set $G_1 = L_0$, $P_1 = L_0 \cap w_0Q_0w_0^{-1}$, $Q_1 = \sigma^{-1}(w_0^{-1}P_0w_0 \cap M_0$, $L_1 = L_0 \cap w_0M_0w_0^{-1}$, $M_1 = \sigma^{-1}(w_0^{-1}L_0w_0 \cap M_0$, $U_1 = L_0 \cap w_0V_0w_0^{-1}$, $V_1 = \sigma^{-1}(w_0^{-1}U_0w_0 \cap M_0$, $\sigma_1 : L_1 \to M_1$ such that $\sigma_1(l) = \sigma_0^{-1}(w_0^{-1}lw_0$, W_1 the Weyl group of G_1 with respect to the maximal torus T.

- Choose $[w_1] \in W_{P_1}\backslash W_1/W_{Q_1}$, and w_1 be a lifting to G_1 of the representative of minimum length of $[w_1]$.

- Iterate to get G_2, P_2, Q_2, L_2, M_2, U_2, V_2, σ_2, $\ldots$, and so on until $G_{k+1} = P_{k+1} = Q_{k+1}$.

- Classify the σ_{k+1}-twisted conjugacy classes in G_{k+1}, and choose representatives $\{s_\alpha^{([w_0],\ldots,[w_k])}\}$.

- Define
  $t(([w_0],\dots,[w_k]),s_\alpha) = (s_\alpha^{(-1)^{k+1}}, w_0w_2\dots w_{k-1}w_k^{-1}w_{k-2}^{-1}\dots w_1^{-1})$
  if k is odd,
  and
  $t(([w_0],\dots,[w_k]),s_\alpha) = (s_\alpha^{(-1)^{k+1}}, w_0w_2\dots w_kw_{k-1}^{-1}w_{k-3}^{-1}\dots w_1^{-1})$
  if k is even.

Proposition 2.4.9. *The family $\{t(([w_0],\dots,[w_k]),s_\alpha) \in G\times G\}$, as $(([w_0],\dots,[w_k]),s_\alpha)$ varies through all the different outputs of Lusztig's algorithm, is a set of representatives of G-orbits in $G\times G/\Delta_{P,Q,\sigma}$.*

Proof. We have already seen that the G-orbits in $G\times G/\Delta_{P,Q,\sigma}$ correspond to sequences $([w_0],\dots,[w_k])$ and to σ_{n+1}-twisted conjugacy classes in G_{n+1}; if $\{s_\alpha^{([w_0],\dots,[w_k])}\}$ is a set of representatives of such conjugacy classes, one must just lift these to $G\times G$. Then we get $s_\alpha \mapsto (s_\alpha,1)\mapsto(s_\alpha^{-1},w_k)\mapsto (s_\alpha,w_{k-1}w_k^{-1})\mapsto(s_\alpha^{-1},w_{k-2}w_kw_{k-1}^{-1})\mapsto(s_\alpha,w_{k-3}w_{k-1}w_k^{-1}w_{k-2}^{-1})\mapsto\dots$, and so on up to $G\times G$. In the end we get $(s_\alpha^{(-1)^{k+1}}, w_0w_2\dots w_{k-1}w_k^{-1}w_{k-2}^{-1}\dots w_1^{-1})$ if k is odd, otherwise we get $(s_\alpha^{(-1)^{k+1}}, w_0w_2\dots w_kw_{k-1}^{-1}w_{k-3}^{-1}\dots w_1^{-1})$. This proves the proposition. □

2.5 Lusztig's method: a generalization

We wish to apply Lusztig's method to study the G-orbits inside the canonical compactification X of G. We subdivide X into $G\times G$-orbits

$$X = \coprod_{I\subset\Delta} X_I$$

and study the G-orbits inside each X_I separately. If P_I denotes the standard parabolic subgroup of G associated to the subset I of the set of simple roots Δ, L_I its Levi factor containing T, $\tilde{L}_I = L_I/Z(L_I)$ its adjoint quotient, P_I^- the opposite parabolic to P_I; then (see [DC-P]), X_I fibers over $G/P_I\times G/P_I^-$

with fiber $\tilde{L}_I$, the stabilizer $P_I \times P_I^-$ acting on the fiber $\tilde{L}_I$ over (P_I, P_I^-) by

$$(p,q) \cdot l = \tilde{\pi}_{P_I}(p) l \tilde{\pi}_{P_I^-}(q)^{-1}.$$

This implies that the stabilizer of the point over (P_I, P_I^-) corresponding to $1_{\tilde{L}_I}$ is the group

$$\tilde{\Delta}_{P_I,P_I^-} = \{(p,q) \in P_I \times P_I^- \mid \tilde{\pi}_{P_I}(p) = \tilde{\pi}_{P_I^-}(q)\}.$$

So X_I is isomorphic to the coset space $G \times G/\tilde{\Delta}_{P_I,P_I^-}$, and not to the space $G \times G/\Delta_{P_I,P_I^-}$. Thus Lusztig's method is not applicable as it is, we need to generalize it to a class of groups including $\tilde{\Delta}_{P_I,P_I^-}$.

2.5.1 The group $\tilde{\Delta}_{P,Q,\tilde{\sigma}}$ and the G-orbits in $G \times G/\tilde{\Delta}_{P,Q,\tilde{\sigma}}$

Let P and Q be as usual two parabolic subgroups of G containing T; L the Levi subgroup of P containing T, M the Levi subgroup of Q containing T, U the unipotent radical of P, V the unipotent radical of Q, $\pi_P : P \to L$ and $\pi_Q : Q \to M$ the canonical projection; furthermore let $Z_L \subset Z(L)$ be a subgroup of the center of L, $Z_M \subset Z(M)$ a subgroup of the center of M, $\tilde{L} = L/Z_L$, $\tilde{M} = M/Z_M$, $\tilde{\pi}_P : P \to \tilde{L}$, $\tilde{\pi}_Q : Q \to \tilde{M}$, $\tilde{\pi}_L : L \to \tilde{L}$, $\tilde{\pi}_M : M \to \tilde{M}$ the canonical projections. Let $\tilde{\sigma} : \tilde{L} \to \tilde{M}$ be an isomorphism of the groups $\tilde{L}$ and $\tilde{M}$, then we can define the group

$$\tilde{\Delta}_{P,Q,\tilde{\sigma}} = \{(p,q) \in P \times Q \mid \tilde{\sigma}(\tilde{\pi}_P(p)) = \tilde{\pi}_Q(q)\}.$$

We may proceed in the study of G-orbits in $G \times G/\tilde{\Delta}_{P,Q,\tilde{\sigma}}$. We consider the fibration

$$\begin{array}{c} G \times G/\tilde{\Delta}_{P,Q,\tilde{\sigma}} \\ \downarrow \\ G \times G/P \times Q; \end{array}$$

the fibers are isomorphic to $\tilde{L}\backslash\tilde{L}\times\tilde{L}$. The fibers over (P,Q) is $P\times Q/\tilde{\Delta}_{P,Q,\tilde{\sigma}}$, it is isomorphic to $\tilde{L}\backslash\tilde{L}\times\tilde{L}$ through the isomorphism

$$P\times Q/\tilde{\Delta}_{P,Q,\tilde{\sigma}}\to\tilde{L}\backslash\tilde{L}\times\tilde{L}$$

given by

$$(p,q)\mapsto(\tilde{\pi}_P(p)^{-1},\tilde{\pi}_Q(q)^{-1}).$$

By Bruhat decomposition, the base space decomposes into the disjoint union of a finite number of G-orbit

$$G\times G/P\times Q=\coprod_w G\cdot(1,w)\cdot(P,Q),$$

where w is the rapresentative of minimum length of a double coset class in $W_P\backslash W/W_Q$. As before, the G-orbits in $G\times G/\tilde{\Delta}_{P,Q,\tilde{\sigma}}$ above the G-orbit $G\cdot(1,w)\cdot(P,Q)$ correspond bijectively to the orbits in the fiber over $(1,w)\cdot(P,Q)$ under the group $\mathrm{Stab}_G((1,w)\cdot(P,Q))=S_w$. Moreover the map $x\mapsto(1,w)x$ maps the fiber over (P,Q) to the fiber over $(1,w)\cdot(P,Q)$, and transforms the orbits under the group $Z_w=(1,w)^{-1}S_w(1,w)$ into the orbits under S_w. Thus the problem is reduced to the study of Z_w-orbits in the fiber over (P,Q). If the fiber over (P,Q) is identified with $\tilde{L}\backslash\tilde{L}\times\tilde{L}$, then Z_w acts through its image $Z_w\hookrightarrow P\times Q\to\tilde{L}\times\tilde{M}\to\tilde{L}\times\tilde{L}$, and it acts by right multiplication.

Proposition 2.5.1. *Let* $G_1=\tilde{L}$, $P_1=\tilde{\pi}_P(L\cap wQw^{-1})$, $Q_1=\tilde{\sigma}^{-1}\tilde{\pi}_Q(w^{-1}Pw\cap M)$, $L_1=\tilde{\pi}_L(L\cap wMw^{-1})$, $M_1=\tilde{\sigma}^{-1}\tilde{\pi}_M(w^{-1}Lw\cap M)$, $Z_{L_1}=\tilde{\pi}_L(L\cap wZ_Mw^{-1})$, $Z_{M_1}=\tilde{\sigma}^{-1}\tilde{\pi}_M(w^{-1}Z_Lw\cap M)$, $\tilde{L}_1=L_1/Z_{L_1}$,$\tilde{M}_1=M_1/Z_{M_1}$, $\tilde{\sigma}_1:\tilde{L}_1\to\tilde{M}_1$ *such that* $\tilde{\sigma}_1(l_1)=\tilde{\sigma}^{-1}(w^{-1}l_1w)$. *Then the image of* Z_w *in* $\tilde{L}\times\tilde{L}$ *is* $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1}$.

Proof. We have $P\cap wQw^{-1}=(L\cap wMw^{-1})\cdot(L\cap wVw^{-1})\cdot(U\cap wMw^{-1})\cdot(U\cap wVw^{-1})$, and $Z_w=\{(p,w^{-1}pw),\ p\in P\cap wQw^{-1}\}$; so the image of Z_w in $\tilde{L}\times\tilde{L}$ is the product of the images of these four factors. The last one vanishes, the third has image of the type $\{(1,x)\}$, the second has images

$\{x,1\}$, the first goes injectively to $L \times M$ through the map $L \times M \to \tilde{L} \times \tilde{L}$ defined by $(l,m) \mapsto (\tilde{\pi}_L(l), \tilde{\sigma}^{-1}\tilde{\pi}_M(m))$. It is certainly a subgroup of

$$\tilde{\pi}_L(L \cap wMw^{-1}) \times \tilde{\sigma}^{-1}\tilde{\pi}_M(w^{-1}Lw \cap M) \cong$$

$$\cong (L \cap wMw^{-1}/Z_L \cap wMw^{-1}) \ \times \tilde{\sigma}^{-1}(w^{-1}Lw \cap M/w^{-1}Lw \cap Z_M),$$

so we want the image of a group in the direct product of two quotients by two different normal subgroups $H \to H/A \times H/B$; then, by the exact sequence

$$\{1\} \to H/A \cap B \to H/A \ \times H/B \to H/AB \to \{1\},$$

we may conclude that the image of Z_w in $\tilde{L} \times \tilde{L}$ is the subgroup defined as $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1}$. □

This propositon is the basic step in the classification of G-orbits in $G \times G/\tilde{\Delta}_{P,Q,\tilde{\sigma}}$. Indeed, the classification of the G-orbits over $(1,w) \cdot (P,Q)$ is reduced to the classification of G_1-orbits in $G_1 \times G_1/\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1}$. The group G_1 is smaller so, iterating, the process ends after a finite number of steps. For clarity, we state explicitely the algorithm for the classification.

Lusztig's algorithm (second version)

- Set $G_0 = G$, $T_0 = T$, $P_0 = P$, $Q_0 = Q$, $L_0 = L$, $M_0 = M$, $U_0 = U$, $V_0 = V$, $W_0 = W$, $Z_{L_0} = Z_L$, $Z_{M_0} = Z_M$, $\tilde{L}_0 = \tilde{L}$, $\tilde{M}_0 = \tilde{M}$, $\tilde{\sigma}_0 = \tilde{\sigma} : \tilde{L}_0 \to \tilde{M}_0$, $\tilde{\pi}_{L_0} = \tilde{\pi}_L : L_0 \to \tilde{L}_0$, $\tilde{\pi}_{M_0} = \tilde{\pi}_M : M_0 \to \tilde{M}_0$.

- Choose $[w_0] \in W_{P_0}\backslash W/W_{Q_0}$, and w_0 be a lifting to G_0 of the representative of minimum length of $[w_0]$.

- Set $G_1 = \tilde{L}_0$, $T_1 = \tilde{\pi}_{L_0}(T_0)$, $P_1 = \tilde{\pi}_{L_0}(L_0 \cap w_0Q_0w_0^{-1})$,
  $Q_1 = \tilde{\sigma}_0^{-1}(\tilde{\pi}_{M_0}(w_0^{-1}P_0w_0 \cap M_0))$,
  $L_1 = \tilde{\pi}_{L_0}(L_0 \cap w_0M_0w_0^{-1})$,
  $M_1 = \tilde{\sigma}_0^{-1}(\tilde{\pi}_{M_0}(w_0^{-1}L_0w_0 \cap M_0))$,
  $U_1 = \tilde{\pi}_{L_0}(L_0 \cap w_0V_0w_0^{-1})$,
  $V_1 = \tilde{\sigma}_0^{-1}(\tilde{\pi}_{M_0}(w_0^{-1}L_0w_0 \cap M_0))$,

W_1 the Weyl group of G_1 with respect to the maximal torus T_1, $Z_{L_1} = \tilde{\pi}_{L_0}(L_0 \cap w_0 Z_{M_0} w_0^{-1})$, $Z_{M_1} = \tilde{\sigma}_0^{-1}(\tilde{\pi}_{M_0}(w_0^{-1} Z_{M_0} w_0 \cap M_0))$, $\tilde{L}_1 = L_1/Z_{L_1}$, $\tilde{M}_1 = M_1/Z_{M_1}$, $\tilde{\sigma}_1 : \tilde{L}_1 \to \tilde{M}_1$ such that $\tilde{\sigma}_1(l_1) = \tilde{\sigma}^{-1}(w^{-1} l_1 w)$.

- Choose $[w_1] \in W_{P_1} \backslash W / W_{Q_1}$, and w_1 be a lifting to G_1 of the representative of minimum length of $[w_1]$.

- Iterate to get G_2, T_2, P_2, Q_2, L_2, M_2, U_2, V_2, W_2, Z_{L_2}, Z_{M_2}, $\tilde{L}_2$, $\tilde{M}_2$, $\tilde{\sigma}_2$, and so on until $G_{k+1} = P_{k+1} = Q_{k+1}$.

- Classify the $\tilde{\sigma}_{k+1}$-twisted conjugacy classes in G_{k+1} and choose representatives $\{s_\alpha, ([w_0], ..., [w_k])\}$.

- Choose liftings of w_i and s_α to G; define

  $$t(([w_0], ..., [w_k]), s_\alpha) = (s_\alpha, w_0 w_2 ... w_{k-1} w_k^{-1} w_{k-2}^{-1} ... w_1^{-1})$$

  if k is odd, and

  $$t(([w_0], ..., [w_k]), s_\alpha) = (s_\alpha^{-1}, w_0 w_2 ... w_k w_{k-1}^{-1} ... w_1^{-1})$$

  if k is even.

Now, as in the first version of Lusztig's algorithm, the following proposition holds.

Proposition 2.5.2. *The family* $\{t(([w_0], ..., [w_k]), s_\alpha)\}$*, as* $([w_0], ..., [w_k]), s_\alpha)$ *varies through all different outputs of Lusztig's algorithm, is a set of representatives of* G*-orbits in* $G \times G/\tilde{\Delta}_{P,Q,\tilde{\sigma}}$.

Proof. The only difference with the case of G-orbit in $G \times G/\Delta_{P,Q,\sigma}$ is that the G_i are not subgroups of G but subquotients, so one must each time lift representatives from G_i to G_{i-1}, all the way up to G; otherwise the proof runs parallel. □

For the codimensions of orbits we have:

Proposition 2.5.3.

$$\operatorname{codim}_{\tilde{\Delta}_{P,Q,\tilde{\sigma}}}(G \cdot t(([w_0], ..., [w_k]), s_\alpha) \cdot \tilde{\Delta}_{P,Q,\tilde{\sigma}}) =$$

$$\sum_{i=0}^{k} \operatorname{codim}_{G_i}(P_i w_i Q_i) \ + \ (\dim \, G_{k+1} \ - \ \dim \, \operatorname{Stab}_{G_{k+1}}(s_\alpha)).$$

Proof. Follows from the definition of the G-orbit $G \cdot t(([w_0], ..., [w_k]), s_\alpha) \cdot \tilde{\Delta}_{P,Q,\tilde{\sigma}}$. □

2.6 Closures of G-orbits

Let X be the canonical compactification of a connected semisimple algebraic group G of adjoint type. Our main aim is the study of G-orbits in X and their closures. The last two propositions give a satisfactory classification of G-orbits in X, unfortunately they say almost nothing on the relations of closures between orbits. Let us state in the form of a proposition what kind of information we can derive.

Proposition 2.6.1. *Let O_1, O_2 ($O_1 \neq O_2$) be two G-orbits in X. If $O_1 \subset \overline{O}_2$ then:*

(i) dim O_1 < dim O_2;

(ii) if O_2 is contained in the $G \times G$-orbit X_I, then O_1 is contained in its closure $\overline{X}_I = \coprod_{J \supset I} X_J$;

(iii) if O_1 and O_2 are contained in the same $G \times G$-orbit X_I, then the sequences $([w_1^{(1)}], ..., [w_{k_1}^{(1)}])$, $([w_1^{(2)}], ..., [w_{k_2}^{(2)}])$ associated respectively to O_1 and O_2, must satisfy the condition:
if $1 \leq i \leq$ min (k_1, k_2) is the integer such that, for $j < i$,

$$G_j = G_j^{(1)} = G_j^{(2)}, \;\; P_j = P_j^{(1)} = P_j^{(2)}, \;\; Q_j = Q_j^{(1)} = Q_j^{(2)}, \;\; [w_j^{(1)}] = [w_j^{(2)}]$$

and

$$[w_i^{(1)}] \neq [w_i^{(2)}],$$

then

$$P_{i-1}w_i^{(1)}Q_{i-1} \subset \overline{P_{i-1}w_{i-1}^{(2)}Q_{i-1}};$$

(iv) if the sequences associated to O_1 and O_2 are the same, then O_1 and O_2 correspond to σ_{k+1}-twisted conjugacy classes G_1 and G_2 in G_{k+1}; we must have $G_1 \subset \overline{G_2}$ in G_{k+1}; and vice versa if $G_1 \subset \overline{G_2}$ then $O_1 \subset \overline{O_2}$.

Proof. (i) and (ii) are obvious; (iii) and (iv) depend on the basic general fact that if $X \to Y$ is a G-equivariant fibration, for two G-orbits in X, O_1 and O_2, over the same G-orbit G_y in Y, we have $O_1 \subset \overline{O_2}$ if and only of the $\mathrm{Stab}_G(y)$-orbits in X_y, $O_1 \cap X_y$ and $O_2 \cap X_y$ satisfy $O_1 \cap X_y \subset \overline{O_2 \cap X_y}$. □

More loosely speaking, there is decomposition of X into G-stable locally closed pieces inside each of which the relations of closure between G-orbits are reduced to relations of closure between twisted conjugacy classes; i.e., the closure of a G-orbit inside the locally closed G-stable set which contains it is known, but outside such locally closed piece it is not clear which G-orbit it will contain. Of course it may contain only G-orbits from locally closed pieces in the closure of the locally closed piece which contains it, and of course it may contain only G-orbits of lesser dimension. The dimension of G-orbits is easy to compute and the relations of closure between the locally closed subsets of the decomposition is easy to understand; so this gives many candidates, but other criteria of closure are necessary to actually be able to determine wheter a candidate G-orbit is in the closure or not.

Let us discuss some results which in many cases enable us to gain more information on the closure of a G-orbit. One way to maintain control of closure relations is by considering the fibration $\overline{X_I} \to G/P_I \times G/P_I^-$ and not just its restriction to the open subset X_I. Such fibration has fiber isomorphic to $\overline{G}_I$, the canonical compactification of the adjoint quotient G_I of the Levi factor of P_I. One looks separately at G-orbits over each Bruhat cell in $G/P_I \times G/P_I^-$; for the orbits over the cell corresponding to w, the problem reduces to the classification and closure of Z_w-orbits in $\overline{G}_I$. Z_w acts on $\overline{G}_I$ through its image in $G_I \times G_I$, which we have seen to be $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1}$ for particular P_1, Q_1, $\tilde{\sigma}_1$. Unfortunately this method may not be iterated, and it is not clear how to solve the problem of closure of $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1}$-orbits in $\overline{G}_I$, except for few specific case. Summarizing, we have:

Lemma 2.6.2. *Consider the fibration*

$$\begin{array}{c} \overline{X}_I \\ \downarrow \\ G/P_I \times G/P_I^-, \end{array}$$

with fiber isomorphic to $\overline{G}_I$, then for the G-orbit in $\overline{X}_I$ over the cell $G \cdot (1,w) \cdot (P_I, P_I^-)$ of $G/P_I \times G/P_I^-$, the problem of understanding the closure of G-orbits is reduced to the problem of understanding the closure of $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1}$-orbits in $\overline{G}_I$.

Remark 2.6.3. *Even if one were to understand the closure relations of $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1}$-orbits in $\overline{G}_I$, one would still have no information on which orbits not over $G \cdot (1,w) \cdot (P_I, P_I^-)$ are in the closure of the orbit in consideration.*

Remark 2.6.4. *Over $G \cdot (P_I, P_I^-)$, the G-orbits correspond to G_I-orbits in $\overline{G}_I$, and as we have said, such correspondence is compatible with the relation of closure between orbits.*

Remark 2.6.5. *If $G_I \cong \mathrm{PGL}(2)$, it is easy to check that the only possible $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1}$ are either $P_1 = Q_1 = \mathrm{PGL}(2)$, $\tilde{\sigma}_1 : \mathrm{PGL}(2) \to \mathrm{PGL}(2)$, or $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1} = B \times B^-$, where B denotes a Borel subgroup of $\mathrm{PGL}(2)$.*

Remark 2.6.6. *Decomposing X into the G-stable locally closed subsets indexed by the sequences $([w_0], ..., [w_k])$,*

$$X = \coprod_{([w_0],\ldots,[w_k])} U_{([w_0],\ldots,[w_k])},$$

if a particular U decomposes into a finite disjoint union of G-orbits, then this decomposition corresponds to a Bruhat decomposition; the open orbit is dense in U, and thus closes to a union of the $U_{([w_0],\dots,[w_k])}$; which are in the closure is easily determined.

Others methods of determination of the closure of an orbit are be explicit degeneration, i.e. exhibiting a family or sequence of elements contained in the orbit and having limit inside another orbit; or, usually for big orbits, e.g. regular orbits, the lemma:

Lemma 2.6.7. *Let V be a smooth projective variety of dimension n, let G be a connected reductive group acting on V, let D_1, ..., D_k be G-stable divisors of V. Supposing that the intersection $Z = D_1 \cap \ldots \cap D_k$ contains just one G-orbit O of codimension not greater than k, and that there exist no i-dimensional families of G-orbits of codimension not greater than $k+i$, then $\overline{O} = Z$.*

Proof. The irreducible components of Z must all have codimension not greater than k. These irreducible components are G-stable, hence must contain either an open dense G-orbit or an i-dimensional family of G-orbits whose total space is dense. By the hypotheses, only O fulfills these requirements. Z is irreducible, containing O as an open subset. ☐

A useful negative criterion for an orbit not to be in the closure of another one is:

Lemma 2.6.8. *Let $X \hookrightarrow \mathbb{P}(\mathrm{End}V_{\omega_1}) \times \cdots \mathbb{P}(\mathrm{End}V_{\omega_l})$ be the canonical compactification of G with its canonical embedding in $\mathbb{P}(\mathrm{End}(V_{\omega_1}) \times \cdots \mathbb{P}(\mathrm{End}V_{\omega_l})$, where $\omega_1, \ldots \omega_l$ are the fundamental weights of G and V_{ω_i} is a representation of highest weight ω_i. Let $s^{(1)} = (s_1^{(1)}, \ldots s_l^{(1)}) \in \mathrm{End}V_{\omega_1} \times \cdots \mathrm{End}V_{\omega_l}$, $s^2 = (s_1^{(2)}, \ldots s_l^{(2)}) \in \mathrm{End}V_{\omega_1} \times \cdots \mathrm{End}V_{\omega_l}$ such that their image in $\mathbb{P}(\mathrm{End}V_{\omega_1}) \times \cdots \mathbb{P}(\mathrm{End}V_{\omega_l})$ belongs to I; supposing that $\mathrm{End}V_{\omega_i} = W' \oplus W''$ as G-module,* *and for such i, $s_i^{(1)}$ has nonzero component in W' while $s_i^{(2)}$ has zero component in W', then the G-orbit $G \cdot [s^{(1)}]$ of the image of $s^{(1)}$ in X is not in the closure of the orbit of $[s^{(2)}]$ under G.*

Proof. If one component is zero, it cannot degenerate to an element which has nonzero such component. $\square$

Chapter 3

G-orbits and closure of G-orbits in the canonical compactification X of G

In this chapter we apply the methods developed in chapter two to classify the G-orbits in the canonical compactification X of G for low rank G. More explicitly, we treat exhaustively the cases $G = PGL(2)$, $G = PGL(3)$, $G = PSp(4)$, in the sense that for each of these we give the list of G-orbits in X, for each G-orbit we produce a representative and moreover we give the closure relations between such orbits. For cases $G = PGL(4)$ and $G = G_2$ we do not provide such an exhaustive discussion of G-orbits; we limit ourselves to showing that in such cases there are G-orbits whose closure in X contains infinitely many G-orbits. That will be sufficient to give a negative answer to the question of whether, for such G, the closure of any G-orbit in X consists of a finite union of G-orbits. In fact, we are able to say much more; from these basic cases we deduce that if G semisimple adjoint is such that any G-orbit in its canonical compactification X closes to a finite number of G-orbits, then G is a product of simple factors, each of which being either $PGL(2)$ or $PGL(3)$.

3.1 $G = PGL(2)$

The case $G = PGL(2)$ is particularly simple, the analysis of G-orbits and of their closure relations can be carried out quite explicitly. Let $M_{2\times 2}$ denote the set/vector space/algebra of two-by-two matrices with coefficients in $\mathbb{C}$, then $\mathbb{P}(M_{2\times 2})$ carries a $G \times G$-action

$$G \times G \times \mathbb{P}(M_{2\times 2}) \to \mathbb{P}(M_{2\times 2}),$$

given by $(g_1, g_2, x) \mapsto g_1 x g_2^{-1}$.
Thus $X = \mathbb{P}(M_{2\times 2})$, and the $G \times G$-orbit decomposition of X is $X = X_\emptyset \coprod X_1$, with

$$X_\emptyset = G \times G \cdot \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix} \right) = PGL(2),$$

$$X_1 = G \times G \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right) = \mathbb{P}(\{\text{matrices of rank } 1\}) \cong G/B \times G/B \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

The G-orbits in $X_\emptyset$ are just the conjugacy classes in $G = PGL(2)$, these are classified by the Jordan form:

$$A_{[\lambda]} = G \cdot \left(\begin{smallmatrix} 1 & \\ & \lambda \end{smallmatrix} \right) \tag{3.1}$$

$$B = G \cdot \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right), \tag{3.2}$$

$A_{[\lambda]} = A_{[\lambda']} \Leftrightarrow \lambda' = \lambda$ or $\lambda' = \lambda^{-1}$, $\dim A_{[\lambda]} = 2$ for $\lambda \neq 1$, $\dim A_{[1]} = 0$, $\dim B = 2$.
The G-orbits in X_1 correspond to the Bruhat cells in G/B:

$$X_1 = G \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right) \coprod G \cdot \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right).$$

Denoting $C = G \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right)$ and $D = G \cdot \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right)$, we have $\dim C = 2$ and $\dim D = 1$.

3.1.1 Closure of G-orbits

By the results of chapter one, we have

Lemma 3.1.1. $\overline{A_\lambda} = A_\lambda \coprod D$, $\overline{B} = B \coprod D$, $\overline{C} = C \coprod D$.

3.2 $G = PGL(3)$

3.2.1 Basic definitions

The Lie algebra of G is $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. Let T be the torus of diagonal matrices inside G, $\mathfrak{t}$ its Lie algebra, $\alpha_1 : \mathfrak{t} \to \mathbb{C}$ be the root such that $\alpha_1((t_1, t_2, t_3)) = t_1 - t_2$, and $\alpha_2 : \mathfrak{t} \to \mathbb{C}$ be the root such that $\alpha_2((t_1, t_2, t_3)) = t_2 - t_3$. The roots of $\mathfrak{g}$ relative to $\mathfrak{t}$ are α_1, α_2, $\alpha_1 + \alpha_2$, $-\alpha_1$, $-\alpha_2$,$-\alpha_1 - \alpha_2$. Choosing the Borel subgroup B of upper triangular matrices in G, we get that the positive roots are α_1, α_2, $\alpha_1 + \alpha_2$, and the simple roots α_1, α_2. The Weyl group W of G relative to T is generated by the reflection s_1, for which $s_1(\alpha_1) = -\alpha_1$ and $s_1(\alpha_2) = \alpha_1 + \alpha_2$, and the reflection s_2, for which $s_2(\alpha_2) = -\alpha_2$ and $s_2(\alpha_1) = \alpha_1 + \alpha_2$. More precisely, we have $W = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1 = s_2 s_1 s_2\}$. The root spaces are the lines generated by the elementary matrices

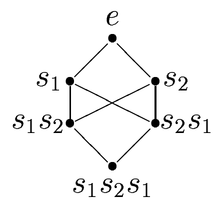
$$E_{\alpha_1} = \begin{pmatrix} 0&1&0\\0&0&0\\0&0&0 \end{pmatrix}, \quad E_{-\alpha_1} = \begin{pmatrix} 0&0&0\\1&0&0\\0&0&0 \end{pmatrix},$$

$$E_{\alpha_2} = \begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix}, \quad E_{-\alpha_2} = \begin{pmatrix} 0&0&0\\0&0&0\\0&1&0 \end{pmatrix},$$

$$E_{\alpha_1+\alpha_2} = \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix}, \quad E_{-\alpha_1-\alpha_2} = \begin{pmatrix} 0&0&0\\0&0&0\\1&0&0 \end{pmatrix}.$$

W is isomorphic to $\mathfrak{S}_3$, and can be identified with the group of permutation matrices inside G. So, for example, $s_1 = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}$, and $s_2 = \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}$. The

Bruhat order of W is as follows:



Let us now list the standard parabolic subgroups of G and make some remarks about the homogeneous spaces relative to those subgroups, which we will need later on. P be the standard parabolic subgroup of G thus defined: $P = \{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in PGL(3)\}$, for its Lie algebra $\mathfrak{p}$ we have $\mathfrak{p} = \mathbb{C}E_{\alpha_1} + \mathbb{C}E_{\alpha_2} + \mathbb{C}E_{\alpha_1+\alpha_2} + \mathbb{C}E_{-\alpha_2} + \mathfrak{t}$, and the subgroup W_P of the Weyl group W corresponding to P is $W_P = \{e, s_2\}$. Q be the standard parabolic subgroup of G thus defined: $Q = \{\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in PGL(3)\}$, for its Lie algebra $\mathfrak{q}$ we have $\mathfrak{q} = \mathbb{C}E_{\alpha_1} + \mathbb{C}E_{\alpha_2} + \mathbb{C}E_{\alpha_1+\alpha_2} + \mathbb{C}E_{-\alpha_1} + \mathfrak{t}$, and the subgroup W_Q of the Weyl group W corresponding to Q is $W_Q = \{e, s_1\}$.

Remark 3.2.1. $\dim G/P = 2$ *and* $\dim G/Q = 2$*, thus,* $\dim G/P \times G/P = 4$ *and* $\dim G/Q \times G/Q = 4$*.*

Remark 3.2.2. *[B, Bruhat decomposition]* $W_P \backslash W / W_P = \{e, s_1\}$*, and* $G/P \times G/P = G \cdot (P, P) \coprod G \cdot (P, s_1 P)$*. Analogously for* Q*,* $W_Q \backslash W / W_Q = \{e, s_2\}$*, and* $G/Q \times G/Q = G \cdot (Q, Q) \coprod G \cdot (Q, s_2 Q)$*. And for* B*,* $G/B \times G/B = \coprod_{s \in W} G \cdot (B, sB)$*,* $\dim G/B \times G/B = 6$*.*

Definition 3.2.3. *Let* Δ_P *be the subgroup of* $G \times G$ *defined by* $\Delta_P = \{(g_1, g_2) \in P \times P^- \mid \pi_P(g_1) = \pi_{P^-}(g_2)\}$*, and analogously* $\Delta_Q = \{(g_1, g_2) \in Q \times Q^- \mid \pi_Q(g_1) = \pi_{Q^-}(g_2)\}$*.*

3.2.2 The canonical compactification of $G = PGL(3)$

Let $M_{3\times 3}$ denote the set/vector space/algebra of three-by-three matrices with coefficients in $\mathbb{C}$. Let us consider the inclusion

$$G \hookrightarrow \mathbb{P}(M_{3\times 3}) \times \mathbb{P}(M_{3\times 3}),$$

given by $g \mapsto (g,\, {}^t g^{-1})$. This inclusion is $G \times G$-equivariant with respect to the $G \times G$-action on G defined by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$, and the $G \times G$-action on $\mathbb{P}(M_{3\times 3}) \times \mathbb{P}(M_{3\times 3})$ defined by $(g_1, g_2) \cdot (x, y) = (g_1 x g_2^{-1}, ({}^t g_1^{-1}) y ({}^t g_2))$. The closure of G in $\mathbb{P}(M_{3\times 3}) \times \mathbb{P}(M_{3\times 3})$ is a smooth projective variety with a $G \times G$-action which extends that on G (see [DC-P]). We denote this variety by X, it is the DeConcini-Procesi compactification of G. The $G \times G$-orbit decomposition is as follows,

$$X = X_\emptyset \coprod X_1 \coprod X_2 \coprod X_{12},$$

with

$$X_\emptyset = G \times G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \cong PGL(3),$$

$$X_1 = G \times G \cdot \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \cong G \times G / \Delta_P,$$

$$X_2 = G \times G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right) \cong G \times G / \Delta_Q,$$

$$X_{12} = G \times G \cdot \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right) \cong G/B \times G/B^-.$$

The $G \times G$-orbits are locally closed subsets of $\mathbb{P}(M_{3\times 3}) \times \mathbb{P}(M_{3\times 3})$, the closure relations are

$$\overline{G} = \overline{X_\emptyset} = X = X_\emptyset \coprod X_1 \coprod X_2 \coprod X_{12},$$

$$\overline{X_1} = X_1 \coprod X_{12},$$

$$\overline{X_2} = X_2 \coprod X_{12},$$

$$\overline{X_{12}} = X_{12}.$$

Let us give the equations defining these closed sets inside $\mathbb{P}(M_{3\times3})\times\mathbb{P}(M_{3\times3})$.

$$X = X_\emptyset \coprod X_1 \coprod X_2 \coprod X_{12} = \{(g,h) \in \mathbb{P}(M_{3\times3})\times\mathbb{P}(M_{3\times3}) \mid g\,{}^th = {}^thg = I_{3\times3}\},$$

$$X_1 \coprod X_2 \coprod X_{12} = \{(g,h) \in \mathbb{P}(M_{3\times3})\times\mathbb{P}(M_{3\times3}) \mid g\,{}^th = {}^thg = 0\},$$

$$X_1 \coprod X_{12} = \{(g,h) \in \mathbb{P}(M_{3\times3})\times\mathbb{P}(M_{3\times3}) \mid g\,{}^th = {}^thg = 0\ ,\ \mathrm{rk}\ g = 1\},$$

$$X_2 \coprod X_{12} = \{(g,h) \in \mathbb{P}(M_{3\times3})\times\mathbb{P}(M_{3\times3}) \mid g\,{}^th = {}^thg = 0\ ,\ \mathrm{rk}\ h = 1\},$$

$$X_{12} = \{(g,h) \in \mathbb{P}(M_{3\times3})\times\mathbb{P}(M_{3\times3}) \mid g\,{}^th = {}^thg = 0\ ,\ \mathrm{rk}\ g = \mathrm{rk}\ h = 1\}.$$

3.2.3 G-orbits in $X_\emptyset$

We begin the determination of G-orbits in X. The G-orbits in the open part $X_\emptyset$ are just conjugacy classes of $PGL(3)$. These are:

- $\mathcal{A} = \{I\}$, dim$\mathcal{A}$=0;
- $\mathcal{B} = G \cdot \begin{pmatrix} 1&1&0\\0&1&0\\0&0&1 \end{pmatrix}$, dim$\mathcal{B}$=4;
- $\mathcal{C} = G \cdot \begin{pmatrix} 1&1&0\\0&1&1\\0&0&1 \end{pmatrix}$, dim$\mathcal{C}$=6;
- $\mathcal{D}_\lambda = G \cdot \begin{pmatrix} 1&0&0\\0&1&0\\0&0&\lambda \end{pmatrix}$, with $\lambda \neq 0,1$, dim$\mathcal{D}_\lambda$=4;
- $\mathcal{E}_\lambda = G \cdot \begin{pmatrix} 1&1&0\\0&1&0\\0&0&\lambda \end{pmatrix}$, with $\lambda \neq 0,1$, dim$\mathcal{E}_\lambda$=6;
- $\mathcal{F}_{[\lambda,\mu]} = G \cdot \begin{pmatrix} 1&0&0\\0&\lambda&0\\0&0&\mu \end{pmatrix}$, with $\lambda \neq 0,1$, $\mu \neq 0,1$, $\lambda \neq \mu$, dim$\mathcal{F}$=6.

3.2.4 G-orbits in X_1

We now pass on to the determination of the G-orbits in the boundary piece of the compactification X of G. The boundary $X - X_\emptyset$ subdivides in the disjoint union of X_1, X_2 and X_{12}. We must determine the G-orbits inside each of these three pieces. The way we do it is by applying the results of 2.5, in particular Lusztig's algorithm and proposition 2.5.2.

We have the following G-equivariant fibration involving X_1:

$$\begin{array}{ccc} X_1 & \longleftarrow & G \times G/\Delta_P \\ \big\downarrow & & \big\downarrow \\ G/P \times G/P^- & \longleftarrow & G \times G/P \times P^- \end{array},$$

And the decomposition $G/P \times G/P^- = G \cdot (P, P^-) \coprod G \cdot (P, s_1 P^-)$.Thus the G-orbits in X_1 are partitioned in two classes, the G-orbits over $G \cdot (P, P^-)$ and the G-orbits over $G \cdot (P, s_1 P^-)$. Let us list these two types and gives their dimensions. Following the steps of Lusztig's algorithm as exposed in 2.5 we get the following.

G-orbits over $G \cdot (P, P^-)$

$\mathcal{G} = G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) = G \cdot \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right)$,
dim $\mathcal{G} = 4$;

$\mathcal{H}_{[\lambda]} = G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & \lambda \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) = G \cdot \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 1 & \\ & & \lambda \end{pmatrix} \right)$,
dim $\mathcal{H}_{[\lambda]} = 6$;

$\mathcal{I} = G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \substack{1 \\ 1} \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) = G \cdot \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 1 & \substack{1 \\ 1} \end{pmatrix} \right)$,
dim $\mathcal{I} = 6$.

G-orbits over $G \cdot (P, s_1 P^-)$

$\mathcal{J} = G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) = G \cdot \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$, dim $\mathcal{J} = 6$;

$\mathcal{K} = G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) = G \cdot \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$, $\dim \mathcal{K} = 5$.

3.2.5 G-orbits in X_2

As in the case of X_1, we apply the algorithm explained in 2.5.

We have the following G-equivariant fibration involving X_2:

$$\begin{array}{ccc} X_2 & \longleftarrow & G \times G/\Delta_Q \\ \downarrow & & \downarrow \\ G/Q \times G/Q^- & \longleftarrow & G \times G/Q \times Q^- \end{array},$$

And the decomposition $G/Q \times G/Q^- = G \cdot (Q, Q^-) \coprod G \cdot (Q, s_2 Q^-)$.Thus the G-orbits in X_2 are partitioned in two classes, the G-orbits over $G \cdot (Q, Q^-)$ and the G-orbits over $G \cdot (Q, s_2 Q^-)$. Let us list these two types and gives their dimensions. Going through Lusztig's algorithm we get the following.

G-orbits over $G \cdot (Q, Q^-)$

$\mathcal{G}' = G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right) = G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right)$,
$\dim \mathcal{G}' = 4$;

$\mathcal{H}_{[\lambda]}{}' = G \cdot \left(\begin{pmatrix} 1 & & \\ & \lambda & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right) = G \cdot \left(\begin{pmatrix} 1 & & \\ & \lambda & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right)$,
$\dim \mathcal{H}_{[\lambda]}{}' = 6$;

$\mathcal{I}' = G \cdot \left(\begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right) = G \cdot \left(\begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right)$,
$\dim \mathcal{I}' = 6$.

G-orbits over $G \cdot (Q, s_2 Q^-)$

$\mathcal{J}' = G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right) = G \cdot \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right)$, $\dim \mathcal{J}' = 6$;

$\mathcal{K}' = G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right) = G \cdot \left(\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & \end{pmatrix} \right)$, $\dim \mathcal{K}' = 5$.

3.2.6 G-orbits in X_{12}

The closed $G \times G$-orbit X_{12} is isomorphic to $G/B \times G/B^-$, so the G-orbits in X_{12} are just the Bruhat cells of the Bruhat decomposition. Hence we get:

$$\mathcal{L}_e = G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right) = G \cdot \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right),$$

dim $\mathcal{L}_e = 6$;

$$\mathcal{L}_{s_1} = G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right) = G \cdot \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right),$$

dim $\mathcal{L}_{s_1} = 5$;

$$\mathcal{L}_{s_2} = G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right) = G \cdot \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right),$$

dim $\mathcal{L}_{s_2} = 5$;

$$\mathcal{L}_{s_1 s_2} = G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right) = G \cdot \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right),$$

dim $\mathcal{L}_{s_1 s_2} = 4$;

$$\mathcal{L}_{s_2 s_1} = G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right) = G \cdot \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right),$$

dim $\mathcal{L}_{s_2 s_1} = 4$;

$$\mathcal{L}_{s_1 s_2 s_1} = G \cdot \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right) = G \cdot \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right),$$

dim $\mathcal{L}_{s_1 s_2 s_1} = 3$.

3.2.7 Closure of G-orbits in X

Let us give the closures of the regular orbits, i.e. those of maximal dimension.

Proposition 3.2.4. *The closure of a regular orbit contained in $X_\emptyset$ is that same orbit plus the set of nilpotent orbits of X.*

Proof. Let us consider the following G-invariant meromorphic functions on X: $f_1 : X \to \mathbb{P}^1$ and $f_2 : X \to \mathbb{P}^1$, defined respectively by $(g, h) \mapsto \frac{\text{Tr}(g)^3}{\text{Det}(g)}$ and $(g, h) \mapsto \frac{\text{Tr}(h)^3}{\text{Det}(h)}$. It is well known that these separate the regular G-orbits in $X_\emptyset$, i.e. if O_1, O_2 are two regular orbits in $X_\emptyset$ with $f_1(O_1) = f_1(O_2)$ and $f_2(O_1) = f_2(O_2)$, then $O_1 = O_2$. Thus, if O is a regular orbit in $X_\emptyset$ with $f_1(O) = \alpha$ and $f_2(O) = \beta$, we may consider the equations $\text{Tr}(g)^3 = \alpha \text{Det}(g)$ and $\text{Tr}(h)^3 = \alpha \text{Det}(h)$. The hypotheses of lemma 2.6.7 are satisfied with respect to these two equations. Moreover, by direct verification, we check that in $X - X_\emptyset$ only the nilpotent orbits satisfy these equations. ☐

Let us look at the regular orbits in $\overline{X_1}$.

Proposition 3.2.5. *Let us consider the meromorphic function $f : X \to \mathbb{P}^1$ defined by $(g, h) \mapsto \frac{\mathrm{Tr}(h)^2}{\mathrm{Tr}(h^2)}$. Then f separates all the regular orbits of $\overline{X_1}$ except $\mathcal{J}$ and $\mathcal{L}_e$, for which $f(\mathcal{J}) = f(\mathcal{L}_e) = 1$.*

Proof. Direct verification on representatives. □

Corollary 3.2.6. *If O is a regular G-orbit in X_1, different from $\mathcal{J}$, then its closure $\overline{O}$ is the union of all G-orbits in $\overline{X_1}$ projecting in the nilpotent variety by the second projection.*

Proof. Apply lemma 2.6.7. □

Proposition 3.2.7. *$\mathcal{L}_{s_1}$ is in the closure of $\mathcal{J}$, $\mathcal{L}_{s_2}$ is not.*

Proof. $\mathcal{L}_{s_2}$ cannot be in the closure of $\mathcal{J}$ since its first component is not nilpotent while that of $\mathcal{J}$ is. To prove that $\mathcal{L}_{s_1}$ is in the closure of $\mathcal{J}$ one can either produce an explicit degeneration of elements of $\mathcal{J}$ that tends to an element in $\mathcal{L}_{s_1}$, or apply lemma 2.6.2: both G-orbits intersect the fiber of $\overline{X_1}$ over $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ by the projection on the second component, in such way the problem is reduced to the closure of $B \times B^-$-orbits in the canonical compactification of $PGL(2)$, the orbits are given by Bruhat decomposition, $\mathcal{J}$ corresponding to the open orbit. □

Of course analogous arguments hold in the case of $\overline{X_2}$. Summing up:

Corollary 3.2.8. *In X, for regular orbits, we have:*

- $\overline{\mathcal{C}} = \mathcal{C} \cup \mathcal{B} \cup \mathcal{A} \cup \mathcal{K} \cup \mathcal{K}' \cup \mathcal{L}_{s_1 s_2} \cup \mathcal{L}_{s_2 s_1} \cup \mathcal{L}_{s_1 s_2 s_1}$;
- $\overline{\mathcal{E}_\lambda} = \mathcal{E}_\lambda \cup \mathcal{D}_\lambda \cup \mathcal{K} \cup \mathcal{K}' \cup \mathcal{L}_{s_1 s_2} \cup \mathcal{L}_{s_2 s_1} \cup \mathcal{L}_{s_1 s_2 s_1}$;
- $\overline{\mathcal{F}_{[\lambda,\mu]}} = \mathcal{F}_{[\lambda,\mu]} \cup \mathcal{K} \cup \mathcal{K}' \cup \mathcal{L}_{s_1 s_2} \cup \mathcal{L}_{s_2 s_1} \cup \mathcal{L}_{s_1 s_2 s_1}$;
- $\overline{\mathcal{I}} = \mathcal{I} \cup \mathcal{G} \cup \mathcal{L}_{s_2} \cup \mathcal{L}_{s_1 s_2} \cup \mathcal{L}_{s_2 s_1} \cup \mathcal{L}_{s_1 s_2 s_1}$;

- $\overline{\mathcal{H}_{[\lambda]}} = \mathcal{H}_{[\lambda]} \cup \mathcal{L}_{s_2} \cup \mathcal{L}_{s_1 s_2} \cup \mathcal{L}_{s_2 s_1} \cup \mathcal{L}_{s_1 s_2 s_1}$;
- $\overline{\mathcal{J}} = \mathcal{J} \cup \mathcal{K} \cup \mathcal{L}_{s_1} \cup \mathcal{L}_{s_1 s_2} \cup \mathcal{L}_{s_2 s_1} \cup \mathcal{L}_{s_1 s_2 s_1}$;
- $\overline{\mathcal{I}'} = \mathcal{I}' \cup \mathcal{G}' \cup \mathcal{L}_{s_1} \cup \mathcal{L}_{s_1 s_2} \cup \mathcal{L}_{s_2 s_1} \cup \mathcal{L}_{s_1 s_2 s_1}$;
- $\overline{\mathcal{H}'_{[\lambda]}} = \mathcal{H}'_{[\lambda]} \cup \mathcal{L}_{s_1} \cup \mathcal{L}_{s_1 s_2} \cup \mathcal{L}_{s_2 s_1} \cup \mathcal{L}_{s_1 s_2 s_1}$;
- $\overline{\mathcal{J}'} = \mathcal{J}' \cup \mathcal{K}' \cup \mathcal{L}_{s_2} \cup \mathcal{L}_{s_1 s_2} \cup \mathcal{L}_{s_2 s_1} \cup \mathcal{L}_{s_1 s_2 s_1}$;
- $\overline{\mathcal{L}_e} = \mathcal{L}_e \cup \mathcal{L}_{s_1} \cup \mathcal{L}_{s_2} \cup \mathcal{L}_{s_1 s_2} \cup \mathcal{L}_{s_2 s_1} \cup \mathcal{L}_{s_1 s_2 s_1}$.

Now we go on to the closures of the smaller orbits.

Proposition 3.2.9. *The closure $\overline{\mathcal{K}}$ of $\mathcal{K}$ is the union of the nilpotent G-orbits of X contained in $\overline{X_1}$, i.e.* $\overline{\mathcal{K}} = \mathcal{K} \cup \mathcal{L}_{s_1 s_2} \cup \mathcal{L}_{s_2 s_1} \cup \mathcal{L}_{s_1 s_2 s_1}$.

Proof. $\mathcal{K}$ is in the intersection of the two G-invariant divisors D_1 and D_2 of $\overline{X_1}$ defined by $D_1 = \{(g,h) \in \overline{X_1} \mid \mathrm{Tr}(g) = 0\}$ and $D_2 = \{(g,h) \in \overline{X_1} \mid \mathrm{Tr}(h) = 0\}$. We verify directly that these two conditions are equivalent respectively to g being nilpotent, h being nilpotent. Moreover the hypotheses of the lemma 2.6.7 are satisfied, thus $\overline{\mathcal{K}}$ is the intersection of the nilpotent variety of X with $\overline{X_1}$. □

An analogous argument holds for $\mathcal{K}'$.

Proposition 3.2.10. *The closure $\overline{\mathcal{K}'}$ of $\mathcal{K}'$ is the union of the nilpotent G-orbits of X contained in $\overline{X_2}$, i.e.* $\overline{\mathcal{K}'} = \mathcal{K}' \cup \mathcal{L}_{s_1 s_2} \cup \mathcal{L}_{s_2 s_1} \cup \mathcal{L}_{s_1 s_2 s_1}$.

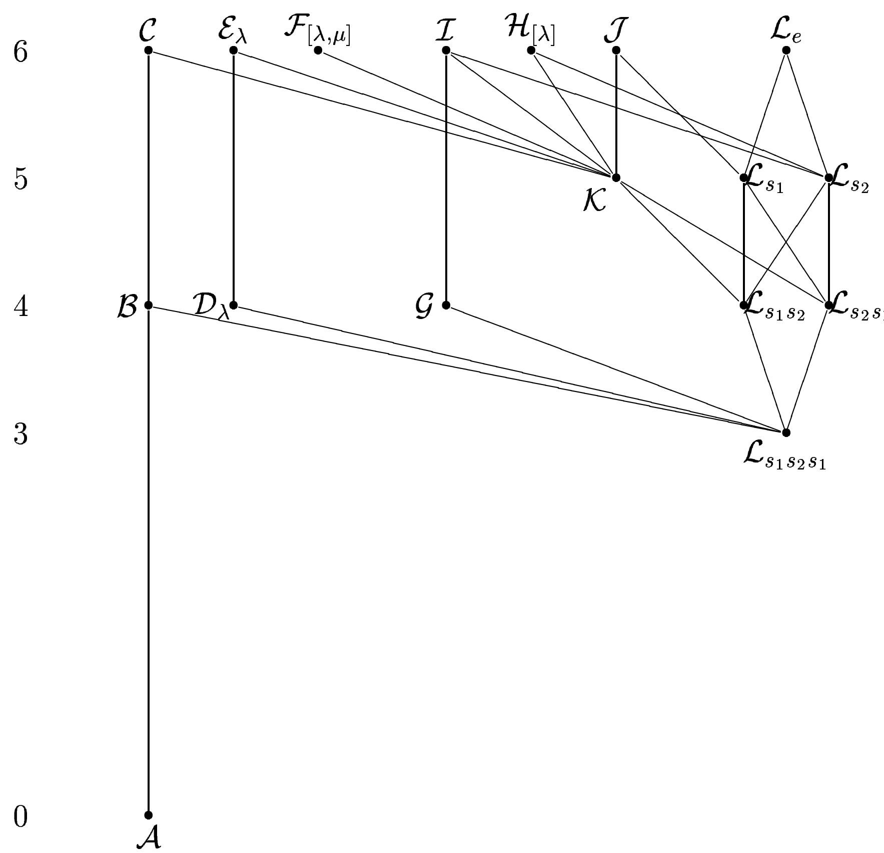
Proposition 3.2.11. $\overline{\mathcal{B}} = \mathcal{B} \cup \mathcal{A} \cup \mathcal{L}_{s_1 s_2 s_1}$, $\overline{\mathcal{D}_\lambda} = \mathcal{D}_\lambda \cup \mathcal{L}_{s_1 s_2 s_1}$.

Proof. the dimension of the G-orbits $\mathcal{B}$ and $\mathcal{D}_\lambda$ is four, hence the only G-orbits to which they can degenerate are $\mathcal{A}$ and $\mathcal{L}_{s_1 s_2 s_1}$. It is clear that $\mathcal{B}$ degenerates to $\mathcal{A}$; now $\mathcal{B} \cup \mathcal{A}$ cannot be closed since it has dimension four and is contained in the affine variety $X_\emptyset$. The same argument holds for $\mathcal{D}_\lambda$. □

Proposition 3.2.12. $\overline{\mathcal{G}} = \mathcal{G} \cup \mathcal{L}_{s_1s_2s_1}$, *and* $\overline{\mathcal{G}'} = \mathcal{G}' \cup \mathcal{L}_{s_1s_2s_1}$.

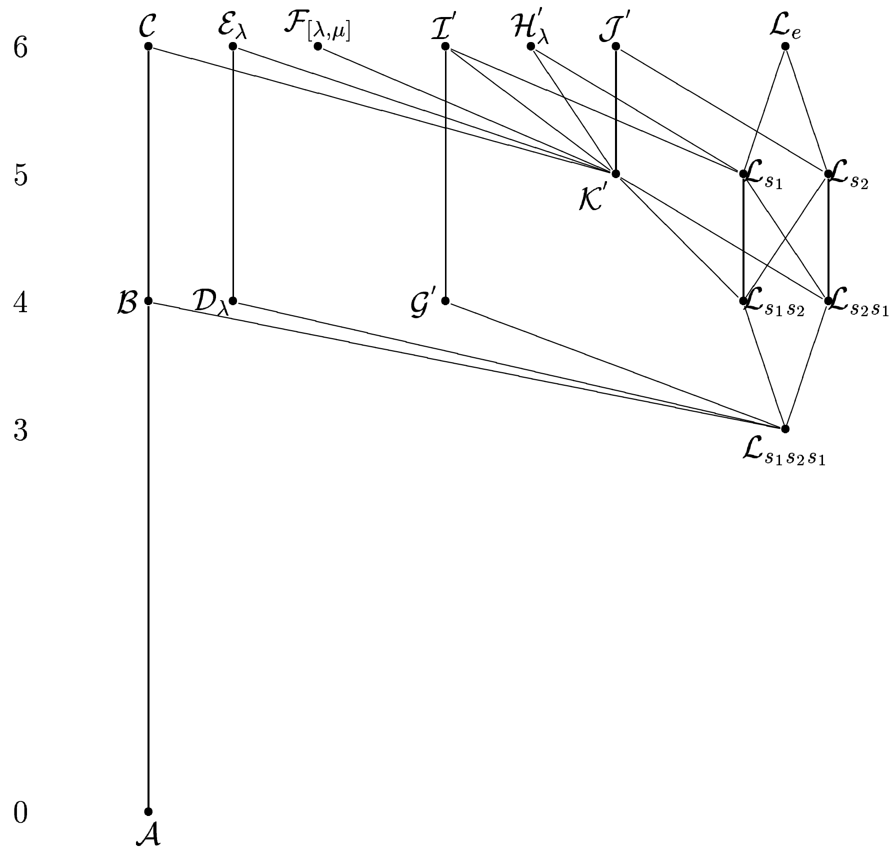
Proof. The G-orbit $\mathcal{G}$ is not closed since its projection on the first component is not. Again, by dimension reasons, the only G-orbit to which it can degenerate is $\mathcal{L}_{s_1s_2s_1}$. The analogous argument holds for $\mathcal{G}'$. □

We summarize the complete list of G-orbits in X together with their closure relations, for $G = PGL(3)$, in the following diagrams.



In the first three columns of this diagram we find the G-orbits in $X_\emptyset$, in the second three the G-orbits in X_1, and in the last three the G-orbits in X_{12}.

We reproduce also the analogous diagram with in the second three columns the G-orbits in X_2. There is a symmetry which reflects the symmetry between the roots α_1 and α_2 of $PGL(3)$.



For $G = PSp(4)$ there will be an analogous symmetry only as regards the classification of orbits, i.e. the graphs of depicting orbit closures for $PSp(4)$ will have the same vertices but different edges.

3.3 $G = PSp(4)$

Let J be the block $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, I being the two by two identity matrix, then the symplectic group of rank two is defined in the following way: $Sp(4) = \{g \in$

$GL(4) \mid gJ\,{}^t g = J\}$. The center of $Sp(4)$ is $\{\pm 1\}$, the group $PSp(4)$ is defined as the quotient $Sp(4)/\{\pm 1\}$, it is the adjoint group associated to the simply connected group $Sp(4)$.

3.3.1 Basic definitions

The Lie algebra of G is $\mathfrak{g} = \mathfrak{sp}(4, \mathbb{C})$. Let T be the torus of diagonal matrices inside G,i.e. $T = \{\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_1^{-1} & \\ & & & t_2^{-1} \end{pmatrix}\}$, $\mathfrak{t}$ its Lie algebra,i.e.$\mathfrak{t} = \{\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & -t_1 & \\ & & & -t_2 \end{pmatrix}\}$. $\alpha_1 : \mathfrak{t} \to \mathbb{C}$ be the root such that $\alpha_1(\text{diag }(t_1, t_2, -t_1, -t_2)) = t_1 - t_2$, and $\alpha_2 : \mathfrak{t} \to \mathbb{C}$ be the root such that $\alpha_2(\text{diag }(t_1, t_2, -t_1, -t_2)) = 2t_2$. The roots of $\mathfrak{g}$ relative to $\mathfrak{t}$ are α_1, α_2, $\alpha_1 + \alpha_2$, $2\alpha_1 + \alpha_2$,$-\alpha_1$, $-\alpha_2$,$-\alpha_1 - \alpha_2$,$-2\alpha_1 - \alpha_2$. Choosing the Borel subgroup B as the group $\{\begin{pmatrix} t_1 & * & * & * \\ 0 & t_2 & * & * \\ 0 & 0 & t_1^{-1} & 0 \\ 0 & 0 & * & t_2^{-1} \end{pmatrix} \in PSp(4)\}$, we get that the positive roots are α_1, α_2, $\alpha_1 + \alpha_2$, $2\alpha_1 + \alpha_2$, and the simple roots α_1, α_2. The Weyl group W of G relative to T is generated by the reflection s_1, for which $s_1(\alpha_1) = -\alpha_1$ and $s_1(\alpha_2) = 2\alpha_1 + \alpha_2$, and the reflection s_2, for which $s_2(\alpha_2) = -\alpha_2$ and $s_2(\alpha_1) = \alpha_1 + \alpha_2$. More precisely, we have $W = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, (s_1s_2)^2 = (s_2s_1)^2\}$. The root spaces are the lines generated by the matrices

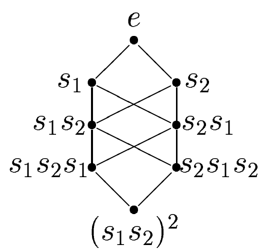
$$E_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \;,\; E_{-\alpha_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_{2\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;,\; E_{-2\alpha_1-\alpha_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_{\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;,\; E_{-\alpha_1-\alpha_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$E_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;,\; E_{-\alpha_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

W is isomorphic to the dihedral group $\mathfrak{D}_4$, and can be identified with the group of matrices inside G generated by $s_1 = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$, and $s_2 = \begin{pmatrix} 1 & & & \\ & 0 & & -1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix}$, the Bruhat order of W is as follows:



Let us now list the standard parabolic subgroups of G and make some remarks about the homogeneous spaces relative to those subgroups, which we will need later on. P be the standard parabolic subgroup of G thus defined: $P = \{\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \in PSp(4)\}$, for its Lie algebra $\mathfrak{p}$ we have $\mathfrak{p} = \mathbb{C}E_{\alpha_1} + \mathbb{C}E_{\alpha_2} + \mathbb{C}E_{\alpha_1+\alpha_2} + \mathbb{C}E_{2\alpha_1+\alpha_2} + \mathbb{C}E_{-\alpha_1} + \mathfrak{t}$, and the subgroup W_P of the Weyl group W corresponding to P is $W_P = \{e, s_1\}$. Q be the standard parabolic subgroup of G thus defined: $Q = \{\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & * & * & * \end{pmatrix} \in PSp(4)\}$, for its Lie algebra $\mathfrak{q}$ we have $\mathfrak{q} = \mathbb{C}E_{\alpha_1} + \mathbb{C}E_{\alpha_2} + \mathbb{C}E_{\alpha_1+\alpha_2} + \mathbb{C}E_{2\alpha_1+\alpha_2} + \mathbb{C}E_{-\alpha_2} + \mathfrak{t}$, and the subgroup W_Q of the Weyl group W corresponding to Q is $W_Q = \{e, s_2\}$.

Remark 3.3.1. $\dim G/P = 3$ *and* $\dim G/Q = 3$, *thus,* $\dim G/P \times G/P = 6$ *and* $\dim G/Q \times G/Q = 6$. *G/P parametrizes the set of isotropic planes in $\mathbb{C}^4$ with respect to the standard symplectic form J, while G/Q parametrizes the set of isotropic lines inside $\mathbb{C}^4$ with respect to the standard symplectic form J.*

Remark 3.3.2 (Bruhat decomposition). *For any connected reductive group G and two parabolic subgroups P_1, P_2 containing a maximal torus T of G, we have that the double coset classes in G with respect to P_1 and P_2 are indexed by elements of $W_{P_1}\backslash W/W_{P_2}$, where W_{P_1} is the subgroup of the Weyl group W of G associated to P_1, and likewise W_{P_2} for P_2. Moreover if $s \in N_G(T)$ is a representative of the double coset class $\mathbf{s} \in W_{P_1}\backslash W/W_{P_2}$, then the double coset class in G indexed by $\mathbf{s}$ is $P_1 s P_2$.*

We will need such decomposition in the case $G = PSp(4)$, $P_1 = P$, $P_2 = P^-$ the opposite parabolic of P and the case $G = PSp(4)$, $P_1 = Q$, $P_2 = Q^-$ the opposite parabolic of Q: $W_P\backslash W/W_P = \{e, s_2, s_2s_1s_2\}$, and $G/P\times G/P^- = G\cdot(P,P^-)\coprod G\cdot(P,s_2P^-)\coprod G\cdot(P,s_2s_1s_2P^-)$. Analogously for Q, $W_Q\backslash W/W_Q = \{e, s_1, s_1s_2s_1\}$, and $G/Q\times G/Q^- = G\cdot(Q,Q^-)\coprod G\cdot(Q,s_1Q^-)\coprod G\cdot(Q,s_1s_2s_1Q^-)$. And for B, $G/B\times G/B^- = \coprod_{s\in W} G\cdot(B,sB^-)$, $\dim G/B\times G/B^- = 8$.

Definition 3.3.3. *We define, following chapter two, L to be the unique Levi factor of P containing T, $\tilde{L}$ to be the semisimple adjoint quotient of L, and $\pi_P : P \to \tilde{L}$ the canonical surjection. We define M to be the unique Levi factor of Q containing T, $\tilde{M}$ to be the semisimple adjoint quotient of M, and $\pi_Q : Q \to \tilde{M}$ the canonical surjection. Now let Δ_P be the subgroup of $G\times G$ defined by $\Delta_P = \{(g_1,g_2)\in P\times P^- \mid \pi_P(g_1) = \pi_{P^-}(g_2)\}$, and analogously $\Delta_Q = \{(g_1,g_2)\in Q\times Q^- \mid \pi_Q(g_1) = \pi_{Q^-}(g_2)\}$.*

3.3.2 The canonical compactification of $G = PSp(4)$

Let us consider the standard representation $V = \mathbb{C}^4$ of $Sp(4)$, and its exterior square $\bigwedge^2(V)$, the representation V is irreducible with highest weight the fundamental weight ω_1, while $\bigwedge^2(V)$ is not irreducible but has a highest weight corresponding to the fundamental weight ω_1 and the corresponding weight space is one dimensional. Then by results of [DC-P], we have that the closure of the immersion

$$G \hookrightarrow \mathbb{P}(\mathrm{End}(V)) \times \mathbb{P}(\mathrm{End}(\bigwedge^2(V))$$

inside $\mathbb{P}(\mathrm{End}(V))\times\mathbb{P}(\mathrm{End}(\bigwedge^2(V))$ is a smooth projective variety endowed with a $G\times G$-action extending the one on G given by left and right multiplication. The $G\times G$-orbit decomposition is:

$$X = X_\emptyset \coprod X_1 \coprod X_2 \coprod X_{12},$$

with

$$X_\emptyset = G\times G\cdot x_0 \cong G\times G/\Delta(G) \cong G, \text{ with } x_0 = \left(\begin{pmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix}, \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}\right)$$

$$X_1 = G\times G\cdot x_1 \cong G\times G/\Delta_P, \text{ with } x_1 = \left(\begin{pmatrix} 1 &&& \\ & 1 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 1 &&&&& \\ & 0 &&&& \\ && 0 &&& \\ &&& 0 && \\ &&&& 0 & \\ &&&&& 0 \end{pmatrix}\right)$$

$$X_2 = G\times G\cdot x_2 \cong G\times G/\Delta_Q, \text{ with } x_2 = \left(\begin{pmatrix} 1 &&& \\ & 0 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 1 &&&&& \\ & 0 &&&& \\ && 1 &&& \\ &&& 0 && \\ &&&& 0 & \\ &&&&& 0 \end{pmatrix}\right)$$

$$X_{12} = G\times G\cdot x_{12} \cong G/B\times G/B^-, \text{ with } x_{12} = \left(\begin{pmatrix} 1 &&& \\ & 0 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 1 &&&&& \\ & 0 &&&& \\ && 0 &&& \\ &&& 0 && \\ &&&& 0 & \\ &&&&& 0 \end{pmatrix}\right).$$

It is also possible to give a geometrical interpretation of these subvarieties, and to base the classification and closure of G-orbits in X on such geometric interpretation. We follow the method of Lusztig for the classification of G-orbits which is more algebraic, hence we will not use such a geometric interpretation even though it would be of interest to understand the exact relations between these two methods.

3.3.3 G-orbits in $X_\emptyset$

The G-orbits in $X_\emptyset$ are the conjugacy classes in $PSp(4)$. The classification of conjugacy classes in $PSp(4)$ is well-known. If $\{g_\alpha \in PSp(4)\}$ is a family of representatives of conjugacy classes, then a family of representatives of G-orbits in $X_\emptyset$ is given by $\{(g_\alpha, \wedge^2 g_\alpha,\ {}^t g_\alpha^{-1} \in X_\emptyset\}$. Let us then list the conjugacy classes of $PSp(4)$:

- $\mathcal{A} = G\cdot \begin{pmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix}$, $\dim \mathcal{A} = 0$;

- $\mathcal{B} = G \cdot \begin{pmatrix} 1&0&1&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$, $\dim \mathcal{B} = 4$;
- $\mathcal{C} = G \cdot \begin{pmatrix} 1&0&0&1 \\ 0&1&1&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$, $\dim \mathcal{C} = 6$;
- $\mathcal{D} = G \cdot \begin{pmatrix} 1&2&2&2 \\ 0&1&-2&2 \\ 0&0&1&0 \\ 0&0&-2&1 \end{pmatrix}$, $\dim \mathcal{D} = 8$;
- $\mathcal{E}_{[\lambda]} = G \cdot \begin{pmatrix} \lambda&0&0&0 \\ 0&\lambda&0&0 \\ 0&0&\lambda^{-1}&0 \\ 0&0&0&\lambda^{-1} \end{pmatrix}$, $\dim \mathcal{E}_{[\lambda]} = 6$, $\lambda \neq 0, 1, -1$;
- $\mathcal{F}_{[\lambda]} = G \cdot \begin{pmatrix} \lambda&\lambda&0&0 \\ 0&\lambda&0&0 \\ 0&0&\lambda^{-1}&0 \\ 0&0&\lambda^{-1}&\lambda^{-1} \end{pmatrix}$, $\dim \mathcal{F}_{[\lambda]} = 8$, $\lambda \neq 0, 1, -1$;
- $\mathcal{G}_{[\lambda]} = G \cdot \begin{pmatrix} 1&0&0&0 \\ 0&\lambda&0&0 \\ 0&0&1&0 \\ 0&0&0&\lambda^{-1} \end{pmatrix}$, $\dim \mathcal{G}_{[\lambda]} = 6$, $\lambda \neq 0, 1$;
- $\mathcal{H}_{[\lambda]} = G \cdot \begin{pmatrix} 1&0&1&0 \\ 0&\lambda&0&0 \\ 0&0&1&0 \\ 0&0&0&\lambda^{-1} \end{pmatrix}$, $\dim \mathcal{H}_{[\lambda]} = 8$, $\lambda \neq 0$;
- $\mathcal{I}_{[\lambda_1,\lambda_2]} = G \cdot \begin{pmatrix} \lambda_1&0&0&0 \\ 0&\lambda_2&0&0 \\ 0&0&\lambda_1^{-1}&0 \\ 0&0&0&\lambda_2^{-1} \end{pmatrix}$, $\dim \mathcal{I}_{[\lambda_1,\lambda_2]} = 8$, $\lambda_1, \lambda_2 \neq 0, 1, -1$.

3.3.4 G-orbits in X_1

We now determine the G-orbits in the boundary of X. $X - X_\emptyset$ is the union of X_1, X_2 and X_{12}. As for the case of $PGL(3)$, we determine the G-orbits in each of these pieces by applying the results of 2.5 and in particular Lusztig's algorithm and proposition 2.5.2.

We have the following G-equivariant fibration involving X_1:

$$\begin{array}{ccccc} X_1 & \xleftarrow{\cong} & G \times G \cdot x_1 & \xleftarrow{\cong} & G \times G/\Delta_P \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \\ G/P \times G/P^- & \xleftarrow{\cong} & G \times G \cdot pr_2(x_1) & & \end{array} .$$

We have also the decomposition

$$G/P \times G/P^- = G \cdot pr_2(x_1) \coprod G \cdot (1, s_2) pr_2(x_1) \coprod G \cdot (1, s_2 s_1 s_2) pr_2(x_1).$$

Thus the G-orbits in X_1 are partitioned in three classes, the G-orbits over $G \cdot \begin{pmatrix} 1 &&&& \\ & 0 &&& \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{pmatrix}$, the G-orbits over $G \cdot (1, s_2) \begin{pmatrix} 1 &&&& \\ & 0 &&& \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{pmatrix}$, and the G-orbits over $G \cdot (1, s_2 s_1 s_2) \begin{pmatrix} 1 &&&& \\ & 0 &&& \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{pmatrix}$. Let us list these three types and gives their dimensions. Following the steps of Lusztig's algorithm as exposed in 2.5 we get the following.

G-orbits over $G \cdot pr_2 x_1$

The representatives of the orbits are:

- $\left(\begin{pmatrix} 1 &&& \\ & 1 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 1 &&&& \\ & 0 &&& \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{pmatrix} \right)$,
- $\left(\begin{pmatrix} 1 & 1 && \\ & 1 && \\ && 1 & \\ && -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 1 &&& \\ & 1 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 1 &&&& \\ & 0 &&& \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{pmatrix} \right) =$
  $= \left(\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 &&&& \\ & 0 &&& \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{pmatrix} \right)$,
- $\left(\begin{pmatrix} 1 &&& \\ & \lambda && \\ && 1 & \\ &&& \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 1 &&& \\ & 1 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 1 &&&& \\ & 0 &&& \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{pmatrix} \right) =$
  $= \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 &&&& \\ & 0 &&& \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{pmatrix} \right)$.

Thus the G-orbits are

- $\mathcal{J} = G \cdot \left(\begin{pmatrix} 1 &&& \\ & 1 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 1 &&&& \\ & 0 &&& \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{pmatrix} \right)$, $\dim \mathcal{J} = 6$;

- $\mathcal{K} = G \cdot \left(\begin{pmatrix} 1&1&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&&&&&\\ &0&&&&\\ &&0&&&\\ &&&0&&\\ &&&&0&\\ &&&&&0 \end{pmatrix} \right)$, $\dim \mathcal{K} = 8$;

- $\mathcal{L}_{[\lambda]} = G \cdot \left(\begin{pmatrix} 1&0&0&0\\ 0&\lambda&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&&&&&\\ &0&&&&\\ &&0&&&\\ &&&0&&\\ &&&&0&\\ &&&&&0 \end{pmatrix} \right)$, $\dim \mathcal{L}_{[\lambda]} = 8$, with $\lambda \neq 0, 1$.

G-orbits over $G \cdot (1, s_2)pr_2x_1$

The representatives of the orbits are:

- $$\left(\begin{pmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{pmatrix}, \begin{pmatrix} 1&&&\\ &0&&-1\\ &&1&\\ &1&&0 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 1&&&\\ &1&&\\ &&0&\\ &&&0 \end{pmatrix}, \begin{pmatrix} 1&&&&&\\ &0&&&&\\ &&0&&&\\ &&&0&&\\ &&&&0&\\ &&&&&0 \end{pmatrix} \right) =$$
$$= \left(\begin{pmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \right),$$

- $$\left(\begin{pmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{pmatrix}, \begin{pmatrix} 1&&&\\ &0&&-1\\ &&1&\\ &1&&0 \end{pmatrix} \right) \left(\begin{pmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{pmatrix}, \begin{pmatrix} 0&1&&\\ 1&0&&-1\\ &&0&\\ &-1&&0 \end{pmatrix} \right) \cdot$$
$$\cdot \left(\begin{pmatrix} 1&&&\\ &1&&\\ &&0&\\ &&&0 \end{pmatrix}, \begin{pmatrix} 1&&&&&\\ &0&&&&\\ &&0&&&\\ &&&0&&\\ &&&&0&\\ &&&&&0 \end{pmatrix} \right) = \left(\begin{pmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \right).$$

Thus the G-orbits are

- $\mathcal{M} = G \cdot \left(\begin{pmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \right)$, $\dim \mathcal{M} = 8$;

- $\mathcal{N} = G \cdot \left(\begin{pmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \right)$, $\dim \mathcal{N} = 7$.

G-orbits over $G \cdot s_2s_1s_2pr_2x_1$

The representatives are:

- $$\left(\begin{pmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{pmatrix}, \begin{pmatrix} &&&1\\ &&-1&\\ &-1&&\\ 1&&& \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 1&&&\\ &1&&\\ &&0&\\ &&&0 \end{pmatrix}, \begin{pmatrix} 1&&&&&\\ &0&&&&\\ &&0&&&\\ &&&0&&\\ &&&&0&\\ &&&&&0 \end{pmatrix} \right) =$$

$$= \left(\begin{pmatrix} 0&0&0&1\\ 0&0&-1&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \right),$$

- $$\left(\begin{pmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{pmatrix}, \begin{pmatrix} &&&1\\ &&-1&\\ &-1&&\\ 1&&& \end{pmatrix} \right) \left(\begin{pmatrix} 1&1&&\\ &1&&\\ &&1&\\ &&-1&1 \end{pmatrix}, \begin{pmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{pmatrix} \right) \cdot$$
$$\cdot \left(\begin{pmatrix} 1&&&\\ &1&&\\ &&0&\\ &&&0 \end{pmatrix}, \begin{pmatrix} 1&&&&&\\ &0&&&&\\ &&0&&&\\ &&&0&&\\ &&&&0&\\ &&&&&0 \end{pmatrix} \right) = \left(\begin{pmatrix} 0&0&1&-1\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \right),$$

- $$\left(\begin{pmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{pmatrix}, \begin{pmatrix} &&&1\\ &&-1&\\ &-1&&\\ 1&&& \end{pmatrix} \right) \left(\begin{pmatrix} 1&&&\\ &\lambda&&\\ &&1&\\ &&&\lambda^{-1} \end{pmatrix}, \begin{pmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{pmatrix} \right) \cdot$$
$$\cdot \left(\begin{pmatrix} 1&&&\\ &1&&\\ &&0&\\ &&&0 \end{pmatrix}, \begin{pmatrix} 1&&&&&\\ &0&&&&\\ &&0&&&\\ &&&0&&\\ &&&&0&\\ &&&&&0 \end{pmatrix} \right) = \left(\begin{pmatrix} 0&0&0&1\\ 0&0&-\lambda&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \right).$$

Thus the G-orbits are

- $\mathcal{O} = G \cdot \left(\begin{pmatrix} 0&0&0&1\\ 0&0&-1&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \right)$, $\dim \mathcal{O} = 3$;
- $\mathcal{P} = G \cdot \left(\begin{pmatrix} 0&0&1&-1\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \right)$, $\dim \mathcal{P} = 5$;
- $\mathcal{Q}_{[\lambda]} = G \cdot \left(\begin{pmatrix} 0&0&0&1\\ 0&0&\lambda&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \right)$, $\dim \mathcal{Q}_{[\lambda]} = 5$, with $\lambda \neq 0, 1$.

3.3.5 G-orbits in X_2

As for X_1, we apply the algorithm of 2.5.

We have the following G-equivariant fibration involving X_2:

$$\begin{array}{ccccc} X_2 & \xleftarrow{\cong} & G \times G \cdot x_2 & \xleftarrow{\cong} & G \times G/\Delta_Q \\ \downarrow & & \mathrm{pr}_1 \downarrow & & \\ G/Q \times G/Q^- & \xleftarrow{\cong} & G \times G \cdot pr_1(x_2) & & \end{array} .$$

We have also the decomposition

$$G/Q \times G/Q^{-} = G \cdot pr_1(x_2) \coprod G \cdot (1, s_1)pr_1(x_2) \coprod G \cdot (1, s_1 s_2 s_1)pr_2(x_1).$$

Thus the G-orbits in X_2 are partitioned in three classes, the G-orbits over $G \cdot \begin{pmatrix} 1 &&& \\ & 0 && \\ && 0 & \\ &&& 0 \end{pmatrix}$, the G-orbits over $G \cdot (1, s_1) \begin{pmatrix} 1 &&& \\ & 0 && \\ && 0 & \\ &&& 0 \end{pmatrix}$, and the G-orbits over $G \cdot (1, s_2 s_1 s_2) \begin{pmatrix} 1 &&& \\ & 0 && \\ && 0 & \\ &&& 0 \end{pmatrix}$. We list these three types and give their dimensions by Lusztig's algorithm.

G-orbits over $G \cdot pr_1 x_2$

The representatives of the orbits are:

- $\left(\begin{pmatrix} 1 &&& \\ & 0 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 1 &&&&& \\ & 0 &&&& \\ && 1 &&& \\ &&& 0 && \\ &&&& 0 & \\ &&&&& 0 \end{pmatrix} \right),$

- $\left(\begin{pmatrix} 1 &&& \\ & 1 && 1 \\ && 1 & \\ &&& 1 \end{pmatrix}, \begin{pmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 1 &&& \\ & 0 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 1 &&&&& \\ & 0 &&&& \\ && 1 &&& \\ &&& 0 && \\ &&&& 0 & \\ &&&&& 0 \end{pmatrix} \right) =$

  $= \left(\begin{pmatrix} 1 &&& \\ & 0 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right),$

- $\left(\begin{pmatrix} 1 &&& \\ & \lambda && \\ && 1 & \\ &&& \lambda_{-1} \end{pmatrix}, \begin{pmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 1 &&& \\ & 0 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 1 &&&&& \\ & 0 &&&& \\ && 1 &&& \\ &&& 0 && \\ &&&& 0 & \\ &&&&& 0 \end{pmatrix} \right) =$

  $= \left(\begin{pmatrix} 1 &&& \\ & 0 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} \lambda &&&&& \\ & 0 &&&& \\ && \lambda^{-1} &&& \\ &&& 0 && \\ &&&& 0 & \\ &&&&& 0 \end{pmatrix} \right).$

Thus the G-orbits are

- $\mathcal{J}' = G \cdot \left(\begin{pmatrix} 1 &&& \\ & 0 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 1 &&&&& \\ & 0 &&&& \\ && 1 &&& \\ &&& 0 && \\ &&&& 0 & \\ &&&&& 0 \end{pmatrix} \right)$, $\dim \mathcal{J}' = 6$;

- $\mathcal{K}' = G \cdot \left(\begin{pmatrix} 1 &&& \\ & 0 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right)$, $\dim \mathcal{K}' = 8$;

- $\mathcal{L}'_{[\lambda]} = G \cdot \left(\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} \lambda & & & & & \\ & 0 & & & & \\ & & \lambda^{-1} & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix} \right)$, $\dim \mathcal{L}'_{[\lambda]} = 8$, with $\lambda \neq 0, 1$.

G-orbits over $G \cdot (1, s_1) pr_1 x_2$

The representatives of the orbits are:

- $\left(\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix} \right) =$

  $= \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right)$,

- $\left(\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 0 & & -1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix} \right) \cdot$

  $\cdot \left(\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix} \right) = \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right)$.

Thus the G-orbits are

- $\mathcal{M}' = G \cdot \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right)$, $\dim \mathcal{M}' = 8$;

- $\mathcal{N}' = G \cdot \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right)$, $\dim \mathcal{N}' = 7$.

G-orbits over $G \cdot s_1 s_2 s_1 pr_1 x_2$

The representatives of the orbits are:

- $\left(\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} & & 1 & \\ & -1 & & \\ -1 & & & \\ & & & -1 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix} \right) =$

  $= \left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right)$,

- $\left(\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}, \begin{pmatrix}&&&1&&\\&&-1&&&\\-1&&&&&\\&&&&&-1\end{pmatrix}\right)\left(\begin{pmatrix}1&&&\\&1&&1\\&&1&\\&&&1\end{pmatrix}, \begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}\right)\cdot$
  $\cdot\left(\begin{pmatrix}1&&&\\&0&&\\&&0&\\&&&0\end{pmatrix}, \begin{pmatrix}1&&&&&\\&0&&&&\\&&1&&&\\&&&0&&\\&&&&0&\\&&&&&0\end{pmatrix}\right) = \left(\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&-1&0&1\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right),$

- $\left(\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}, \begin{pmatrix}&&&1&&\\&&-1&&&\\-1&&&&&\\&&&&&-1\end{pmatrix}\right)\left(\begin{pmatrix}1&&&\\&\lambda&&\\&&1&\\&&&\lambda^{-1}\end{pmatrix}, \begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}\right)\cdot$
  $\cdot\left(\begin{pmatrix}1&&&\\&0&&\\&&0&\\&&&0\end{pmatrix}, \begin{pmatrix}1&&&&&\\&0&&&&\\&&1&&&\\&&&0&&\\&&&&0&\\&&&&&0\end{pmatrix}\right) = \left(\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&\lambda\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right).$

Thus the G-orbits are

- $\mathcal{O}' = G\cdot\left(\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&-1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right)$, $\dim\mathcal{O}' = 3$;

- $\mathcal{P}' = G\cdot\left(\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&-1&0&1\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right)$, $\dim\mathcal{P}' = 5$;

- $\mathcal{Q}'_{[\lambda]} = G\cdot\left(\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&\lambda\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right)$, $\dim\mathcal{Q}'_{[\lambda]} = 5$, with $\lambda \neq 0, 1$.

3.3.6 G-orbits in X_{12}

The G-orbits in X_{12} are just the cells of the Bruhat decomposition. So we have:

$$\mathcal{V}_e = G\cdot\left(\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}, \begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}\right)\cdot\left(\begin{pmatrix}1&&&\\&0&&\\&&0&\\&&&0\end{pmatrix}, \begin{pmatrix}1&&&&&\\&0&&&&\\&&0&&&\\&&&0&&\\&&&&0&\\&&&&&0\end{pmatrix}\right) =$$

$$= G\cdot\left(\begin{pmatrix}1&&&\\&0&&\\&&0&\\&&&0\end{pmatrix}, \begin{pmatrix}1&&&&&\\&0&&&&\\&&0&&&\\&&&0&&\\&&&&0&\\&&&&&0\end{pmatrix}\right),$$

$\dim\mathcal{V}_e = 8$;

$$\mathcal{V}_{s_1} = G\cdot\left(\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}, \begin{pmatrix}0&-1&&\\1&0&&\\&&0&-1\\&&1&0\end{pmatrix}\right)\cdot\left(\begin{pmatrix}1&&&\\&0&&\\&&0&\\&&&0\end{pmatrix}, \begin{pmatrix}1&&&&&\\&0&&&&\\&&0&&&\\&&&0&&\\&&&&0&\\&&&&&0\end{pmatrix}\right) =$$

$$= G \cdot \left(\begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix} \right), \dim \mathcal{V}_{s_1} = 7;$$

$$\mathcal{V}_{s_2} = G \cdot \left(\begin{pmatrix} 1&&&\\&1&&\\&&1&\\&&&1 \end{pmatrix}, \begin{pmatrix} 1&&&\\&0&&-1\\&&1&\\&1&&0 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 1&&&\\&0&&\\&&0&\\&&&0 \end{pmatrix}, \begin{pmatrix} 1&&&&&\\&0&&&&\\&&0&&&\\&&&0&&\\&&&&0&\\&&&&&0 \end{pmatrix} \right) =$$

$$= G \cdot \left(\begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix} \right), \dim \mathcal{V}_{s_2} = 7;$$

$$\mathcal{V}_{s_1 s_2} = G \cdot \left(\begin{pmatrix} 1&&&\\&1&&\\&&1&\\&&&1 \end{pmatrix}, \begin{pmatrix} &&&1\\1&&0&\\&-1&&\\0&&1& \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 1&&&\\&0&&\\&&0&\\&&&0 \end{pmatrix}, \begin{pmatrix} 1&&&&&\\&0&&&&\\&&0&&&\\&&&0&&\\&&&&0&\\&&&&&0 \end{pmatrix} \right) =$$

$$= G \cdot \left(\begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix} \right), \dim \mathcal{V}_{s_1 s_2} = 6;$$

$$\mathcal{V}_{s_2 s_1} = G \cdot \left(\begin{pmatrix} 1&&&\\&1&&\\&&1&\\&&&1 \end{pmatrix}, \begin{pmatrix} &-1&&0\\&&-1&\\&0&&-1\\1&&& \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 1&&&\\&0&&\\&&0&\\&&&0 \end{pmatrix}, \begin{pmatrix} 1&&&&&\\&0&&&&\\&&0&&&\\&&&0&&\\&&&&0&\\&&&&&0 \end{pmatrix} \right) =$$

$$= G \cdot \left(\begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix} \right), \dim \mathcal{V}_{s_2 s_1} = 6;$$

$$\mathcal{V}_{s_1 s_2 s_1} = G \cdot \left(\begin{pmatrix} 1&&&\\&1&&\\&&1&\\&&&1 \end{pmatrix}, \begin{pmatrix} -1&&1&\\&-1&&\\-1&&0&\\&&&-1 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 1&&&\\&0&&\\&&0&\\&&&0 \end{pmatrix}, \begin{pmatrix} 1&&&&&\\&0&&&&\\&&0&&&\\&&&0&&\\&&&&0&\\&&&&&0 \end{pmatrix} \right) =$$

$$= G \cdot \left(\begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix} \right), \dim \mathcal{V}_{s_1 s_2 s_1} = 5,$$

$$\mathcal{V}_{s_2 s_1 s_2} = G \cdot \left(\begin{pmatrix} 1&&&\\&1&&\\&&1&\\&&&1 \end{pmatrix}, \begin{pmatrix} &&&1\\&&-1&\\&-1&&\\1&&& \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 1&&&\\&0&&\\&&0&\\&&&0 \end{pmatrix}, \begin{pmatrix} 1&&&&&\\&0&&&&\\&&0&&&\\&&&0&&\\&&&&0&\\&&&&&0 \end{pmatrix} \right) =$$

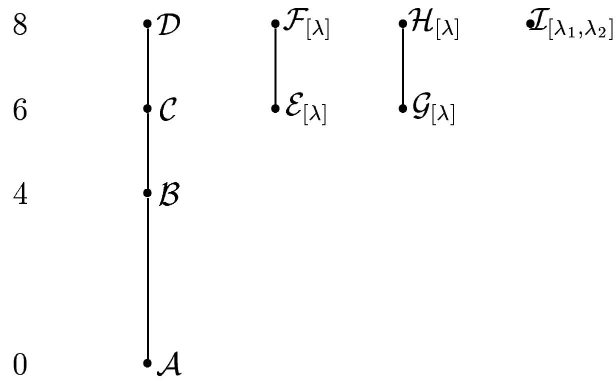
$$= G \cdot \left(\begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix} \right), \dim \mathcal{V}_{s_2 s_1 s_2} = 5,$$

$$\mathcal{V}_{(s_1 s_2)^2} = G \cdot \left(\begin{pmatrix} 1&&&\\&1&&\\&&1&\\&&&1 \end{pmatrix}, \begin{pmatrix} &&1&0\\&&0&1\\-1&0&&\\0&-1&& \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 1&&&\\&0&&\\&&0&\\&&&0 \end{pmatrix}, \begin{pmatrix} 1&&&&&\\&0&&&&\\&&0&&&\\&&&0&&\\&&&&0&\\&&&&&0 \end{pmatrix} \right) =$$

$$= G \cdot \left(\begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix} \right), \dim \mathcal{V}_{(s_1 s_2)^2} = 4.$$

3.3.7 Closure of G-orbits in X

In the open part $X_\emptyset$, the G-orbits correspond to the conjugacy classes in $PSp(4)$ and the relations of closure between G-orbits correspond to the relations of closure between the conjugacy classes in $PSp(4)$. These are well known and we recall them below in the form of a diagram:



The numbers to the left indicate the dimensions of the orbits, and the straight lines connect an orbit to another in its closure, going downwards of course. So the orbits on the top row are the regular orbits. In the first column there are the unipotent conjugacy classes, while the last one contains the semisimple regular conjugacy class.

So the the relations of closure between orbits of the open part are not problematic. Let us determine the closure of a regular orbit of the open part inside all of X. These will follow by a computation of the invariants.

Proposition 3.3.4. *The closure of a regular orbit in $X_\emptyset$ is the set of nilpotent orbits of X.*

Proof. We consider the G-invariant meromorphic functions on X already considered in the case $G = PGL(3)$: $f_1 : X \to \mathbb{P}^1$ and $f_2 : X \to \mathbb{P}^1$, defined respectively by $(g,h) \mapsto \frac{\mathrm{Tr}(g)^3}{\mathrm{Det}(g)}$ and $(g,h) \mapsto \frac{\mathrm{Tr}(h)^3}{\mathrm{Det}(h)}$. These separate the regular G-orbits in $X_\emptyset$. Outside of $X_\emptyset$, having a complete list of representatives of orbits, we may check directly that the equations $\mathrm{Tr}(g) = 0$, $\mathrm{Tr}(g) = 0$, imply

g and h are nilpotent. We verify also that no regular G-orbit in $X - X_\emptyset$ is nilpotent. Lemma 2.6.7 thus concludes. □

A similar computation of invariants permits the determination of closures of regular orbits in the boundary. Let us start with the regular orbits in $\overline{X_1}$.

Proposition 3.3.5. *Again as in the case of $G = PGL(3)$, let us consider the meromorphic function $f : X \to \mathbb{P}^1$ defined by $(g,h) \mapsto \frac{\mathrm{Tr}(g)^2}{\mathrm{Tr}(g^2)}$. Then f separates all the regular orbits of $\overline{X_1}$ except $\mathcal{M}$ and $\mathcal{V}_e$, for which $f(\mathcal{M}) = f(\mathcal{V}_e) = 1$.*

Proof. Just compute the function on the representatives of the regular orbits in X_1. □

Corollary 3.3.6. *If O is a regular G-orbit in X_1, different from $\mathcal{M}$, then its closure $\overline{O}$ is the union of the G-orbits with the same value of f as O and of the G-orbits in $\overline{X_1}$ projecting in the nilpotent variety by the first projection.*

Proof. Follows by application of 2.6.7. □

For the closure of $\mathcal{M}$ one needs a further argument.

Proposition 3.3.7. *$\mathcal{V}_{s_2}$ is in the closure of $\mathcal{M}$, $\mathcal{L}_{s_1}$ is not.*

Proof. $\mathcal{L}_{s_1}$ cannot be in the closure of $\mathcal{M}$ since its second component is not nilpotent while that of $\mathcal{M}$ is. To prove that $\mathcal{L}_{s_2}$ is in the closure of $\mathcal{M}$ one goes about in exactly the same way as for $G = PGL(3)$:either by an easy explicit degeneration , or by 2.6.2, reducing the problem to the Bruhat decomposition of the canonical compactification of $PGL(2)$ by projecting on the second component. □

Now, in the case of $G = PSp(4)$ there is no symmetry between X_1 and X_2. Indeed, neither the homogeneous spaces nor X_1 and X_2 are isomorphic. Nonetheless, the arguments we used to find the closures of the regular G-orbits in X_1, work also in the case of regular G-orbits in X_2 by just interchanging 1 and 2 in their various occurences. We may conclude:

Corollary 3.3.8. *In X, for regular orbits, we have:*

- $\overline{\mathcal{K}} = \mathcal{K} \cup \mathcal{J} \cup \mathcal{N} \cup \mathcal{P} \cup \mathcal{Q}_{[\lambda]} \cup \mathcal{O} \cup \mathcal{V}_{s_1} \cup \mathcal{V}_{s_1 s_2} \cup \mathcal{V}_{s_2 s_1} \cup \mathcal{V}_{s_1 s_2 s_1} \cup \mathcal{V}_{s_2 s_1 s_2} \cup \mathcal{V}_{(s_1 s_2)^2}$;
- $\overline{\mathcal{L}_{[\lambda]}} = \mathcal{L}_{[\lambda]} \cup \mathcal{N} \cup \mathcal{P} \cup \mathcal{Q}_{[\lambda]} \cup \mathcal{O} \cup \mathcal{V}_{s_1} \cup \mathcal{V}_{s_1 s_2} \cup \mathcal{V}_{s_2 s_1} \cup \mathcal{V}_{s_1 s_2 s_1} \cup \mathcal{V}_{s_2 s_1 s_2} \cup \mathcal{V}_{(s_1 s_2)^2}$;
- $\overline{\mathcal{M}} = \mathcal{M} \cup \mathcal{N} \cup \mathcal{P} \cup \mathcal{Q}_{[\lambda]} \cup \mathcal{O} \cup \mathcal{V}_{s_2} \cup \mathcal{V}_{s_1 s_2} \cup \mathcal{V}_{s_2 s_1} \cup \mathcal{V}_{s_1 s_2 s_1} \cup \mathcal{V}_{s_2 s_1 s_2} \cup \mathcal{V}_{(s_1 s_2)^2}$;
- $\overline{\mathcal{K}'} = \mathcal{K}' \cup \mathcal{J}' \cup \mathcal{N}' \cup \mathcal{P}' \cup \mathcal{Q}'_{[\lambda]} \cup \mathcal{O}' \cup \mathcal{V}_{s_2} \cup \mathcal{V}_{s_1 s_2} \cup \mathcal{V}_{s_2 s_1} \cup \mathcal{V}_{s_1 s_2 s_1} \cup \mathcal{V}_{s_2 s_1 s_2} \cup \mathcal{V}_{(s_1 s_2)^2}$;
- $\overline{\mathcal{L}'_{[\lambda]}} = \mathcal{L}'_{[\lambda]} \cup \mathcal{N}' \cup \mathcal{P}' \cup \mathcal{Q}'_{[\lambda]} \cup \mathcal{O}' \cup \mathcal{V}_{s_2} \cup \mathcal{V}_{s_1 s_2} \cup \mathcal{V}_{s_2 s_1} \cup \mathcal{V}_{s_1 s_2 s_1} \cup \mathcal{V}_{s_2 s_1 s_2} \cup \mathcal{V}_{(s_1 s_2)^2}$;
- $\overline{\mathcal{M}'} = \mathcal{M}' \cup \mathcal{N}' \cup \mathcal{P}' \cup \mathcal{Q}'_{[\lambda]} \cup \mathcal{O}' \cup \mathcal{V}_{s_1} \cup \mathcal{V}_{s_1 s_2} \cup \mathcal{V}_{s_2 s_1} \cup \mathcal{V}_{s_1 s_2 s_1} \cup \mathcal{V}_{s_2 s_1 s_2} \cup \mathcal{V}_{(s_1 s_2)^2}$;
- $\overline{\mathcal{V}_e} = \mathcal{V}_e \cup \mathcal{V}_{s_1} \cup \mathcal{V}_{s_2} \cup \mathcal{V}_{s_1 s_2} \cup \mathcal{V}_{s_2 s_1} \cup \mathcal{V}_{s_1 s_2 s_1} \cup \mathcal{V}_{s_2 s_1 s_2} \cup \mathcal{V}_{(s_1 s_2)^2}$.

Now we go on to the closures of the smaller orbits. Only for $\mathcal{N}$ and $\mathcal{N}'$, considerations of invariants are sufficient for the determination of the closures.

Proposition 3.3.9. *The closure $\overline{\mathcal{N}}$ of $\mathcal{N}$ is the union of the nilpotent G-orbits of X contained in $\overline{X_1}$, i.e.* $\overline{\mathcal{N}} = \mathcal{N} \cup \mathcal{P} \cup \mathcal{Q}_{[\lambda]} \cup \mathcal{O} \cup \mathcal{V}_{s_1 s_2} \cup \mathcal{V}_{s_2 s_1} \cup \mathcal{V}_{s_1 s_2 s_1} \cup \mathcal{V}_{s_2 s_1 s_2} \cup \mathcal{V}_{(s_1 s_2)^2}$.

Proof. We use the same argument as in the case of $G = PGL(3)$. $\mathcal{N}$ is in the intersection of the two G-invariant divisors D_1 and D_2 of $\overline{X_1}$ defined by $D_1 = \{(g,h) \in \overline{X_1} \mid \mathrm{Tr}(g) = 0\}$ and $D_2 = \{(g,h) \in \overline{X_1} \mid \mathrm{Tr}(h) = 0\}$. We check directly that these two conditions are equivalent respectively to g being nilpotent, h being nilpotent. Moreover the hypotheses of lemma 2.6.7 are satisfied, thus $\overline{\mathcal{N}}$ is the intersection of the nilpotent variety of X with $\overline{X_1}$. □

Analogously for $\mathcal{N}'$.

Proposition 3.3.10. $\overline{\mathcal{N}'} = \mathcal{N}' \cup \mathcal{P}' \cup \mathcal{Q}'_{[\lambda]} \cup \mathcal{O}' \cup \mathcal{V}_{s_1 s_2} \cup \mathcal{V}_{s_2 s_1} \cup \mathcal{V}_{s_1 s_2 s_1} \cup \mathcal{V}_{s_2 s_1 s_2} \cup \mathcal{V}_{(s_1 s_2)^2}$.

This takes care of G-orbits of dimension greater than or equal to seven, for the orbits of dimension six this type of argument breaks down and one must resort to arguments based on lemma 2.6.8 instead.

Closure of the orbits of dimension six

In the open part the orbits of dimension six are $\mathcal{C}$, $\mathcal{E}_{[\lambda]}$, $\mathcal{G}_{[\lambda]}$; in X_1 there is $\mathcal{J}$, in X_2 $\mathcal{J}'$; and in X_{12} there are $\mathcal{L}_{s_1 s_2}$ and $\mathcal{L}_{s_2 s_1}$. The closure of these last two needs just Bruhat decomposition of X_{12}, for the rest of them we will apply lemma 2.6.8.

If V is the standard representation of $Sp(4)$, the canonical compactification X of $G = PSp(4)$ is the closure of the natural immersion of G in $\mathbb{P}(\mathrm{End}V) \times \mathbb{P}(\mathrm{End} \bigwedge^2 V)$, hence to apply lemma 2.6.8 we must decompose the representations $\mathrm{End}V$ and $\mathrm{End} \bigwedge^2 V$ of $Sp(4)$ into irreducible pieces and determine the components of representatives of the orbits in question with respect to this decomposition.

As usual, J denotes the standard symplectic form, which in block notation is written $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, the entries of the matrix being two-by-two matrices themselves. With such choice of J, the definition of $\mathfrak{sp}(4)$, in block notation is, $\mathfrak{sp}(4) = \{\left(\begin{smallmatrix} A & B \\ C & -{}^tA \end{smallmatrix}\right) \mid {}^tB = B, \; {}^tC = C\}$. The symplectic scalar product J on the vector space V defines an isomorphism $V \to V^*$; in the standard basis (e_1, e_2, e_3, e_4) for V, and the dual one $(e_1^*, e_2^*, e_3^*, e_4^*)$ for V^*, the matrix of such isomorphism is J.

J allows to identify $V \otimes V \cong \mathrm{Hom}(V^*, V)$ to $\mathrm{End}(V)$. In coordinates, the isomorphism from $V \otimes V$ to $\mathrm{End}(V)$, is defined by $B \to BJ$; the inverse isomorphism will of course be $M \to MJ^{-1}$. This identification is compatible with the action of $Sp(4)$ on both sides. Hence the decomposition of $\mathrm{End}(V)$ as a representation of $Sp(4)$ will follow from the decomposition of $V \otimes V$ as a representation of $Sp(4)$. This decomposition is easily obtained:

$V \otimes V = S^2V \oplus \bigwedge_0^2 V \oplus \mathbb{C}J^{-1}$,

with $S^2V = \{B \mid {}^tB = B\}$, $\bigwedge_0^2 V = \{B \mid {}^tB = -B \;,\; \mathrm{Tr}(BJ) = 0\}$, and of course $\mathbb{C}J^{-1}$ are the scalar multiples of J^{-1}. We list the projections from $V \otimes V \cong \mathrm{Hom}(V^*, V)$ to its components.

1. $\mathrm{Hom}(V^*, V) \to S^2V$,
   $B \mapsto \frac{B+\,{}^tB}{2}$;

2. $\mathrm{Hom}(V^*, V) \to \bigwedge^2 V$,
   $B \mapsto \frac{B-\,{}^tB}{2}$;

3. $\mathrm{Hom}(V^*, V) \to \bigwedge_0^2 V$,
   $B \mapsto \frac{B-\,{}^tB}{2} - \frac{\mathrm{Tr}(BJ)}{4}J^{-1}$;

4. $\mathrm{Hom}(V^*, V) \to \mathbb{C}J^{-1}$,
   $B \mapsto \frac{\mathrm{Tr}(BJ)}{4}J^{-1}$.

Note that this last projection, if $\mathrm{Hom}(V^*, V)$ is identified to $\mathrm{End}(V)$, is just projection on the line generated by the identity. To apply lemma 2.6.8, we must compute the projections of the orbits under consideration to these components. It is an easy computation, we display below, for each component whether the projection of the orbit under consideration is zero or not.

	proj. to $S^2(V \otimes V)$	proj. to $\bigwedge_0^2(V \otimes V)$	proj. to $\mathbb{C}J^{-1}$
$\mathcal{C}$	$\neq 0$	0	$\neq 0$
$\mathcal{E}_{[\lambda]}$, $\lambda \neq \pm 1$	$\neq 0$	0	$\neq 0$
$\mathcal{G}_{[\lambda]}$, $\lambda \neq 1$	$\neq 0$	$\neq 0$	$\neq 0$
$\mathcal{J}$	$\neq 0$	0	$\neq 0$
$\mathcal{O}$	0	$\neq 0$	0
$\mathcal{P}$	$\neq 0$	$\neq 0$	0
$\mathcal{Q}_{[\lambda]}$, $\lambda \neq 1$	$\neq 0$	$\neq 0$	0
$\mathcal{Q}_1$	$\neq 0$	0	0
$\mathcal{V}_{s_1 s_2 s_1}$	$\neq 0$	0	0
$\mathcal{V}_{s_2 s_1 s_2}$	$\neq 0$	$\neq 0$	0
$\mathcal{V}_{(s_1 s_2)^2}$	$\neq 0$	0	0

Thus we may deduce the following.

Proposition 3.3.11. *Neither $\mathcal{C}$, $\mathcal{E}_{[\lambda]}$ nor $\mathcal{J}$ close to $\mathcal{O}$, $\mathcal{P}$, $\mathcal{Q}_{[\lambda]}$ $(\lambda \neq 1)$, $\mathcal{V}_{s_2 s_1 s_2}$.*

Proof. $\mathcal{C}$, $\mathcal{E}_{[\lambda]}$and $\mathcal{J}$ have zero component in $\bigwedge_0^2(V \otimes V)$, while $\mathcal{O}$, $\mathcal{P}$, $\mathcal{Q}_{[\lambda]}$ $(\lambda \neq 1)$, $\mathcal{V}_{s_2 s_1 s_2}$ have nonzero such component. Apply lemma 2.6.8. □

Thus, for $\mathcal{C}$, $\mathcal{E}_{[\lambda]}$ and $\mathcal{J}$, the orbits we need to check are $\mathcal{Q}_1$, $\mathcal{V}_{s_1 s_2 s_1}$ and $\mathcal{V}_{(s_1 s_2)^2}$. For all these cases explicit degenerations are conclusive:

Proposition 3.3.12. *$\mathcal{J}$ closes to $\mathcal{Q}_1$, $\mathcal{V}_{s_1 s_2 s_1}$ and $\mathcal{V}_{(s_1 s_2)^2}$.*

Proof. The first component of the representative of $\mathcal{J}$ is $\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$, which is easily seen to be conjugate to $\begin{pmatrix} 1&0&0&1 \\ 0&1&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$ by an element of $Sp(4)$; as a matter of fact, the whole family $\begin{pmatrix} 1&0&0&t \\ 0&1&t&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$ is contained in $\mathcal{J}$. For $t \to \infty$ the limit of the family in $\mathbb{P}(\mathrm{End}V)$ is $\begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$ which is the first component of the representative of $\mathcal{Q}_1$; second components follow. $\mathcal{J}$ degenerates to $\mathcal{V}_{s_1 s_2 s_1}$ by lemma 2.6.2 and Bruhat decomposition. □

An analogous degeneration allows to conclude:

Proposition 3.3.13. *$\mathcal{E}_{[\lambda]}$ closes to $\mathcal{Q}_1$.*

Proof. The first component of the representative of $\mathcal{E}_{[\lambda]}$ is $\begin{pmatrix} \lambda&0&0&0 \\ 0&\lambda&0&0 \\ 0&0&\lambda^{-1}&0 \\ 0&0&0&\lambda^{-1} \end{pmatrix}$ which is conjugate to $\begin{pmatrix} \lambda&0&0&1 \\ 0&\lambda&1&0 \\ 0&0&\lambda^{-1}&0 \\ 0&0&0&\lambda^{-1} \end{pmatrix}$; so the family $\begin{pmatrix} \lambda&0&0&t \\ 0&\lambda&t&0 \\ 0&0&\lambda^{-1}&0 \\ 0&0&0&\lambda^{-1} \end{pmatrix}$ is contained in $\mathcal{E}_{[\lambda]}$ and tends to $\begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$ as t tends to infinity. The second components also follow. □

Similarly:

Proposition 3.3.14. *$\mathcal{C}$ closes to $\mathcal{Q}_1$.*

Proof. The first component of the representative of $\mathcal{C}$ is $\begin{pmatrix} 1&0&0&1\\ 0&1&1&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$; the family $\begin{pmatrix} 1&0&0&t\\ 0&1&t&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$, the elements of which are all conjugate to $\begin{pmatrix} 1&0&0&1\\ 0&1&1&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$, tends to $\begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$ as t tends to infinity. The second components also follow. □

This takes care of the closure of the orbits of dimension six in $\overline{X_1}$ and of the closure of part of the orbits in $X_\emptyset$ to orbits of $\overline{X_1}$. As regards the closure of $\mathcal{G}_{[\lambda]}$ such computations are not conclusive and we need to push the analysis to the second order. If we consider the locally closed subvarieties A and B of the affine space $\mathbb{A}^n$, then $A \subset \overline{B}$ if and only if $I(A) \supset I(B)$. So, a way to prove that A is not in the closure of B to look for a regular function on $\mathbb{A}^n$ which is zero on B but is nonzero on A. In our case the subsets in question are cones, so we only need to consider homogeneous polynomials. The homogeneous polynomials of degree d make up a finite dimensional vector space over the ground field k. Still, in general it is not easy to find the subspace of those homogeneous polynomials of degree d which are zero on B, and analogously to check that a homogeneous polynomial vanishes on A. Nonetheless, for the case in question, it is possible to reduce these problems to linear algebra by representation theoretic considerations. The point is that the subvarieties we consider are orbits containing explicitly given representatives, under the group G (or more exactly the simply connected covering of G). G acts linearly on the affine ambient space and thus on the regular functions on that affine space. The ideal of functions vanishing on a G-orbit constitutes a subrepresentation of the space of regular functions. The ambient space is $\mathrm{End}(V) \times \mathrm{End}(\bigwedge^2 V)$, but since to show that a subbariety in a product of two affine spaces does not degenerate to another subvariety in such product it is enough to prove that the projection on one of these actors of the first does not degenerate to the projection on the same factor of the second, we consider just $\mathrm{End}(V)$ as ambient space. We thus look at regular functions on End(V) and the action of G on these; the space of such functions is isomorphic to $S(\mathrm{End}(V))$, the symmetric algebra over End(V). An element A of End(V) is identified with a linear function on End(V) by considering the bilinear form $\mathrm{End}(V) \times \mathrm{End}(V) \to k$ which associates to the endomorphisms A and B the scalar $\mathrm{Tr}(AB)$. Extending the application $A \mapsto \mathrm{Tr}(AB)$ to the symmetric algebra of End(V), we obtain an identification of the symmetric algebra of End(V) with the algebra of regular functions on End(V); in this identification the r-th symmetric tensor power $S^r(\mathrm{End}(V))$ corresponds to the homogeneous regular functions of degree r.Hence, to check that the orbit A is not in the closure of the orbit B we proceed as follows:

- Consider the space of linear functions on End(V), which is identified with End(V) itself, decompose it into irreducible representations with respect to G, determine which of these representations are identically zero on B (one need only check that all elements of a representation are zero on a fixed representative of the orbit), check whether the irreducible components which annihilate B also annihilate A. If that is not the case, A is not in the closure of B. This is actually the content of lemma 2.6.8.

- If all linear functions on End(V) that are zero on B are also zero on A, one repeats the analysis for the degree two homogeneous regular functions on End(V). So, decompose the representation $S^2(\mathrm{End}(V))$ into irreducible components, determine which are in the ideal of B, then determine if these are also in the ideal of A. If not A is not in the closure of B.

- Else proceed to degree three and higher if necessary.

We will apply this procedure to prove:

Proposition 3.3.15. *$\mathcal{G}_{[\lambda]}$ closes to $\mathcal{Q}_{[\lambda]}$ and $\mathcal{V}_{(s_1 s_2)^2}$ inside X_1.*

Proof. Following the procedure in degree two, we consider $S^2(\mathrm{End}(V))$. If ω is an integral dominant weight of $\mathfrak{g}$, we will denote by L_ω the irreducible $\mathfrak{g}$-module (or G-module) of highest weight ω. The decomposition of $S^2(\mathrm{End}(V))$ is obtained by $S^2(\mathrm{End}(V)) \cong S^2(V \otimes V) \cong S^2(S^2(V) \oplus \bigwedge_0^2(V) \oplus \mathbb{C}J) = S^2(S^2(V)) \oplus S^2(\bigwedge_0^2(V)) \oplus (S^2(V) \otimes \bigwedge_0^2(V)) \oplus S^2(V) \oplus \bigwedge_0^2(V) \oplus \mathbb{C}$. Denoting ω_1, ω_2 the fundamental weights of $\mathfrak{g}$, we have $V = L_{\omega_1}$, $\bigwedge_0^2(V) = L_{\omega_2}$, $S^2(V) = L_{2\omega_1}$. To get the decomposition into irreducibles of $S^2(\mathrm{End}(V))$ it suffices to decompose into irreducibles the representations $S^2(S^2(V)) = S^2(L_{2\omega_1})$, $S^2(\bigwedge_0^2(V)) = S^2(L_{\omega_2})$ and $S^2(V) \otimes \bigwedge_0^2(V) = L_{2\omega_1} \otimes L_{\omega_2}$. It is easily checked that $S^2(L_{2\omega_1}) = L_{4\omega_1} \oplus L_{2\omega_2} \oplus L_{\omega_2} \oplus L_0$, $S^2(L_{\omega_2}) = L_{2\omega_2} \oplus L_0$ and $L_{2\omega_1} \otimes L_{\omega_2} = L_{\omega_1+\omega_2} \oplus L_0$. We thus observe that in the decomposition of $S^2(\mathrm{End}(V))$ there are two irreducible factors isomorphic to $L_{2\omega_2}$, one in $S^2(S^2(V))$ and another in $S^2(\bigwedge_0^2(V))$. If $\{e_1, e_2, e_3, e_4\}$ is the standard basis of V, then a basis of $S^2(V)$ is given by the products $\{e_i e_j\}$ and a basis of $S^2(S^2(V))$ is given by the products $\{(e_i e_j)(e_k e_l)\}$; where the factors inside different parentheses may not be mixed. With this choice of basis for the space $S^2(S^2(V))$, a highest weight vector for the irreducible G-module $L_{2\omega_2}$ is given by $X = (e_3 e_4)^2 - (e_3)^2(e_3)^2$. In an analogous manner we give a basis of $S^2(\bigwedge_0^2(V))$. A basis of $\bigwedge_0^2(V)$ may be given by $\{e_1 \wedge e_2,\ e_1 \wedge e_4,\ e_2 \wedge e_3,\ e_3 \wedge e_4,\ e_1 \wedge e_3 - e_2 \wedge e_4\}$, hence a basis for $S^2(\bigwedge_0^2(V))$ may be given by the products of these factors. With respect to such basis we find a highest weight vector inside $S^2(\bigwedge_0^2(V))$, spanning a G-module isomorphic to $L_{2\omega_2}$, having expression $Y = (e_3 \wedge e_4)^2$. The space of highest weight vectors inside $S^2(\mathrm{End}(V))$ of weight $2\omega_2$ and generating G-modules vanishing on $\mathcal{G}_{[\lambda]}$ is one dimensional spanned by the vector $Z_\lambda = (\lambda - 1)^2 X - (\lambda + 1)^2 Y$. One verifies that the module generated by Z_λ vanishes on $\mathcal{Q}_{[\mu]}$ if and only $\lambda = \mu$ or $\lambda = \mu^{-1}$. One verifies in the same manner that the module generated by Z_λ does not vanish on the orbit $\mathcal{P}$. This ends the proof. □

Now we turn to the closures of $\mathcal{C}$, $\mathcal{E}_{[\lambda]}$, $\mathcal{G}_{[\lambda]}$ and $\mathcal{J}'$ in $\overline{X_2}$. The strategy is the same as for the closures in $\overline{X_1}$, i.e. give explicit degenerations to show that certain orbits degenerate to certain other orbits and use the representation theoretic criterion to show that there are no other possible closure relations.

Proposition 3.3.16. *$\mathcal{C}$ degenerates to $\mathcal{P}'$.*

Proof. The standard representative for $\mathcal{C}$ is $\begin{pmatrix} 1 & & & 1 \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, one checks easily that the matrix $\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is symplectic and is conjugate to the above representative. Conjugating by the torus $\begin{pmatrix} t & & & \\ & 1 & & \\ & & t^{-1} & \\ & & & 1 \end{pmatrix}$ and taking the limit as $t \to \infty$ one gets that $\mathcal{C}$ degenerates to $\mathcal{P}'$. □

Proposition 3.3.17. *$\mathcal{E}_{[\lambda]}$ degenerates to $\mathcal{Q}'_{[-\lambda^2]}$.*

Proof. The usual representative $\begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda^{-1} & \\ & & & \lambda^{-1} \end{pmatrix}$ is conjugate to $\begin{pmatrix} \lambda & 0 & 1 & 0 \\ 0 & \lambda & 0 & 1 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix}$. The exterior square of this matrix is of type $\begin{pmatrix} * & * & * & -\lambda & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & \lambda^{-1} \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}$, hence conjugating by the torus $\begin{pmatrix} t^{10} & & & \\ & t^{-1} & & \\ & & t^{-10} & \\ & & & t \end{pmatrix}$ and sending t to infinity, one shows that $\mathcal{E}_{[\lambda]}$ degenerates to $\mathcal{Q}'_{[-\lambda^2]}$. □

Proposition 3.3.18. *$\mathcal{G}_{[\lambda]}$ degenerates to $\mathcal{Q}'_{[-\lambda^2]}$.*

Proof. One checks that the matrix $\begin{pmatrix} 1 & \lambda-1 & \lambda+1 & 1 \\ 0 & \lambda & \lambda & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1-\lambda^{-1} & \lambda^{-1} \end{pmatrix}$ is symplectic and is conjugate to the usual representative $\begin{pmatrix} 1 & & & \\ & \lambda & & \\ & & 1 & \\ & & & \lambda^{-1} \end{pmatrix}$ of $\mathcal{G}_{[\lambda]}$. The exterior square of such matrix is of the type $\begin{pmatrix} * & * & * & -2\lambda & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & 2\lambda^{-1} \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}$, hence again conjugating by the torus $\begin{pmatrix} t^{10} & & & \\ & t^{-1} & & \\ & & t^{-10} & \\ & & & t \end{pmatrix}$ and sending t to infinity, one shows that $\mathcal{G}_{[\lambda]}$ degenerates to $\mathcal{Q}'_{[-\lambda^2]}$. □

Proposition 3.3.19. *$\mathcal{J}'$ degenerates to $\mathcal{P}'$.*

Proof. The standard representative of $\mathcal{J}'$ is

$$\left(\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}\right);$$

the conjugate of such representative by the symplectic matrix $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is the element

$$\left(\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right).$$

Conjugating by the torus $\begin{pmatrix} t & & & \\ & 1 & & \\ & & t^{-1} & \\ & & & 1 \end{pmatrix}$ and taking the limit as $t \to \infty$ one gets that $\mathcal{J}'$ degenerates to $\mathcal{P}'$. ☐

Proposition 3.3.20. *$\mathcal{J}'$ degenerates to $\mathcal{V}_{s_2 s_1 s_2}$.*

Proof. $\overline{X_2}$ fibers over $G/Q \times G/Q^-$ G-equivariantly, the fiber being isomorphic to the canonical compactification of $PGL(2)$. $G/Q \times G/Q^-$ has three G-orbits, $\mathcal{J}'$ lies over the open one. Since $\overline{X_2}$ is compact, $\mathcal{J}'$ must close to some G-orbit over the intermediate G-orbit of $G/Q \times G/Q^-$. For dimension reasons, $\mathcal{V}_{s_2 s_1 s_2}$ is the only possible candidate. ☐

Proposition 3.3.21. *The above stated degenerations, together with their consequences, are the only possible closure relations between orbits of $X - X_1$.*

Proof. As we said above, the strategy is to look for functions which are zero on the dimension six orbit under consideration but are nonzero on the smaller orbits to which we wish to show that the dimension six orbit does not degenerate. In this case, the linear functions are of no help since no nonzero linear function vanishes on the orbits in question. Going to the homogeneous functions of degree two, we are looking at $S^2(\mathrm{End}(\bigwedge_0^2 V))$. Since $\bigwedge_0^2 V = L_{\omega_2}$, $\mathrm{End}(\bigwedge_0^2 V) = L_{\omega_2} \otimes L_{\omega_2} = L_{2\omega_2} \oplus L_{2\omega_1} \oplus L_0$. Hence $S^2(\mathrm{End}(\bigwedge_0^2 V)) =$

$S^2(L_{2\omega_2} \oplus L_{2\omega_1} \oplus L_0) = S^2(L_{2\omega_2}) \oplus S^2(L_{2\omega_1}) \oplus (L_{2\omega_2} \otimes L_{2\omega_1}) \oplus L_{2\omega_2} \oplus L_{2\omega_1} \oplus L_0$. By straightforward calculations using Weyl's character formula, we get: $S^2(L_{2\omega_2}) = L_{4\omega_2} \oplus L_{4\omega_1} \oplus L_{2\omega_2} \oplus L_0$, $S^2(L_{2\omega_1}) = L_{4\omega_1} \oplus L_{2\omega_2} \oplus L_{\omega_2} \oplus L_0$, $L_{2\omega_2} \otimes L_{2\omega_1} = L_{2\omega_2+2\omega_1} \oplus L_{\omega_2+2\omega_1} \oplus L_{2\omega_2} \oplus L_{2\omega_1}$. Carrying out the computations for the submodule $L^2_{4\omega_1}$ we get that no other degeneration is possible. □

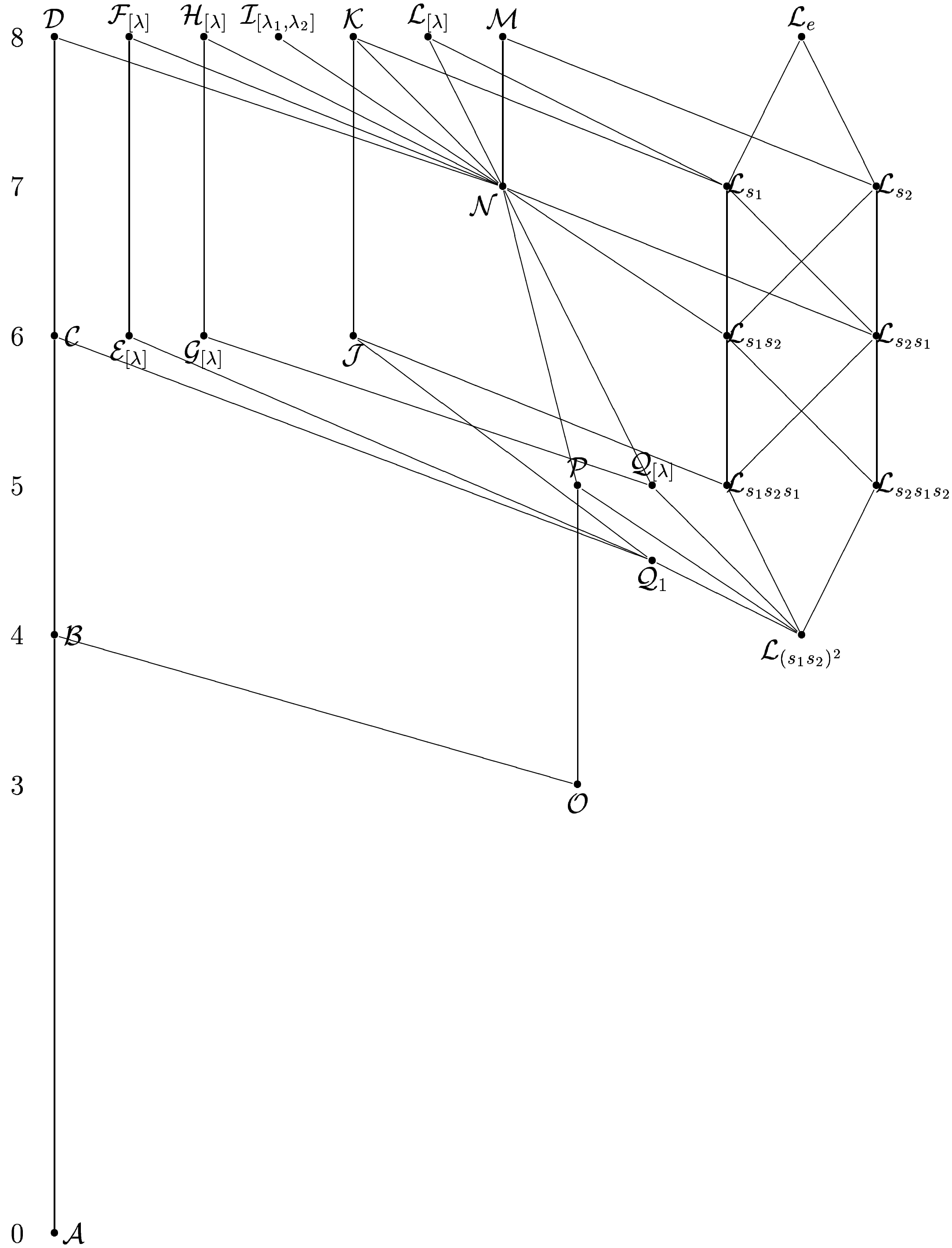
Closure of the orbits of smaller dimension

Proposition 3.3.22. *$\overline{\mathcal{P}} = \mathcal{P} \cup \mathcal{O} \cup \mathcal{V}_{(s_1s_2)^2}$, $\overline{\mathcal{Q}_{[\lambda]}} = \mathcal{Q}_{[\lambda]} \cup \mathcal{V}_{(s_1s_2)^2}$, $\overline{\mathcal{P}'} = \mathcal{P}' \cup \mathcal{O}' \cup \mathcal{V}_{(s_1s_2)^2}$, $\overline{\mathcal{Q}'_{[\lambda]}} = \mathcal{Q}'_{[\lambda]} \cup \mathcal{V}_{(s_1s_2)^2}$, $\mathcal{O}$ and $\mathcal{O}'$ are closed.*

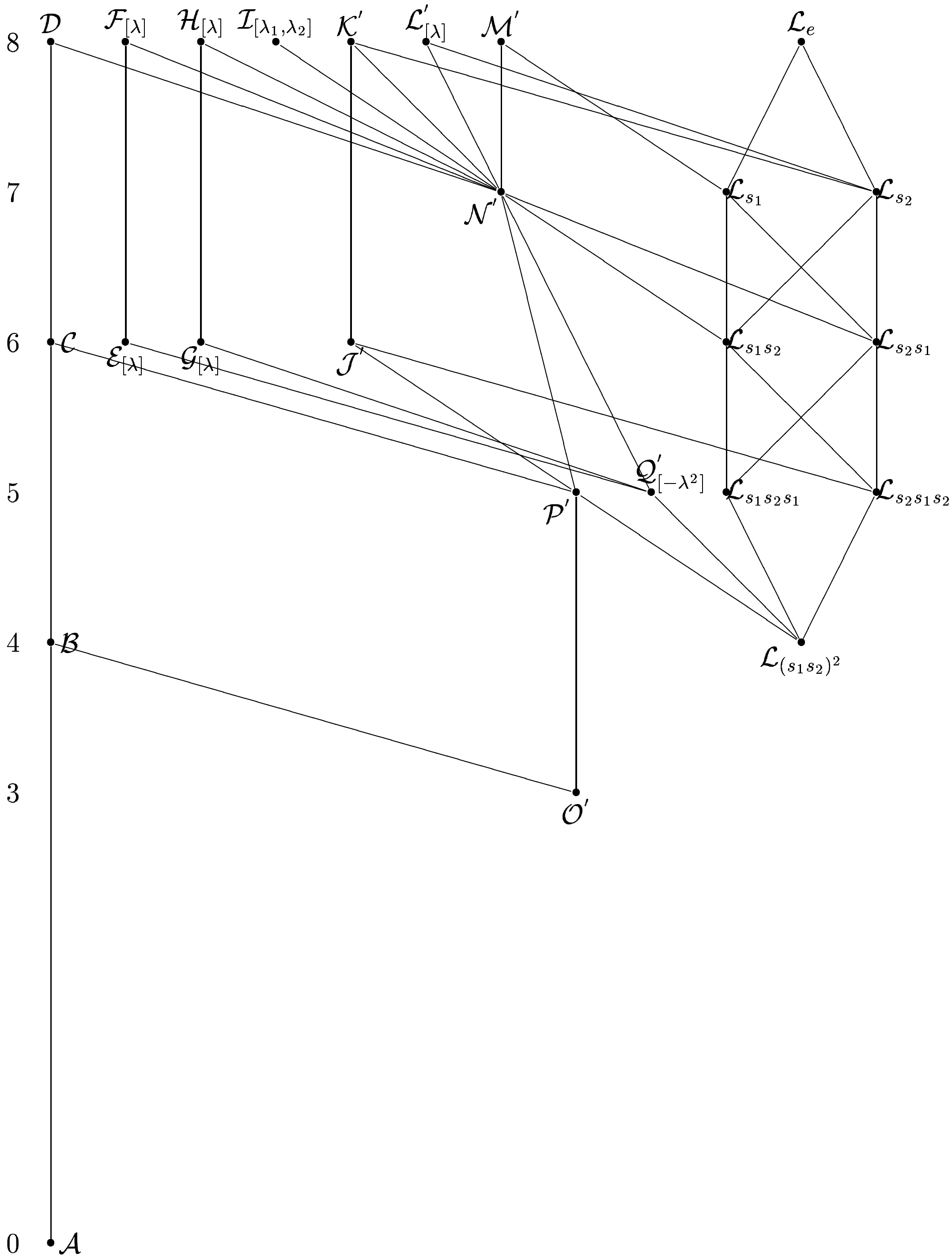
Proof. $\mathcal{P}$ degenerates to $\mathcal{O}$, but $\mathcal{P} \cup \mathcal{O}$ is not closed in X. Indeed, $\mathcal{P} \cup \mathcal{O} \subset X_1$. So it must degenerate to some G-orbit in X_{12}; for dimension reasons, the only possibility is $\overline{\mathcal{P}} = \mathcal{P} \cup \mathcal{O} \cup \mathcal{V}_{(s_1s_2)^2}$. For the same reason $\overline{\mathcal{P}'} = \mathcal{P}' \cup \mathcal{O}' \cup \mathcal{V}_{(s_1s_2)^2}$. $\mathcal{Q}_{[\lambda]}$ is closed in X_1, but not in X, so again, the only possibility is $\overline{\mathcal{Q}_{[\lambda]}} = \mathcal{Q}_{[\lambda]} \cup \mathcal{V}_{(s_1s_2)^2}$. Same for $\mathcal{Q}'_{[\lambda]}$. That $\mathcal{O}$ and $\mathcal{O}'$ are closed is easily seen by considering dimensions for example. □

Summary of orbit closures

We summarize the complete list of G-orbits in X together with their closure relations , for $G = PSp(4)$, in the following diagrams.



8
7
6
5
4
3
0
$\mathcal{D}$
$\mathcal{F}_{[\lambda]}$
$\mathcal{H}_{[\lambda]}$
$\mathcal{I}_{[\lambda_1,\lambda_2]}$
$\mathcal{K}$
$\mathcal{L}_{[\lambda]}$
$\mathcal{M}$
$\mathcal{L}_e$
$\mathcal{N}$
$\mathcal{L}_{s_1}$
$\mathcal{L}_{s_2}$
$\mathcal{C}$
$\mathcal{E}_{[\lambda]}$
$\mathcal{G}_{[\lambda]}$
$\mathcal{J}$
$\mathcal{L}_{s_1 s_2}$
$\mathcal{L}_{s_2 s_1}$
$\mathcal{P}$
$\mathcal{Q}_{[\lambda]}$
$\mathcal{L}_{s_1 s_2 s_1}$
$\mathcal{L}_{s_2 s_1 s_2}$
$\mathcal{Q}_1$
$\mathcal{B}$
$\mathcal{L}_{(s_1 s_2)^2}$
$\mathcal{O}$
$\mathcal{A}$



8
7
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5
4
3
0
$\mathcal{D}$
$\mathcal{F}_{[\lambda]}$
$\mathcal{H}_{[\lambda]}$
$\mathcal{I}_{[\lambda_1,\lambda_2]}$
$\mathcal{K}'$
$\mathcal{L}'_{[\lambda]}$
$\mathcal{M}'$
$\mathcal{L}_e$
$\mathcal{N}'$
$\mathcal{L}_{s_1}$
$\mathcal{L}_{s_2}$
$\mathcal{C}$
$\mathcal{E}_{[\lambda]}$
$\mathcal{G}_{[\lambda]}$
$\mathcal{J}'$
$\mathcal{L}_{s_1 s_2}$
$\mathcal{L}_{s_2 s_1}$
$\mathcal{Q}'_{[-\lambda^2]}$
$\mathcal{P}'$
$\mathcal{L}_{s_1 s_2 s_1}$
$\mathcal{L}_{s_2 s_1 s_2}$
$\mathcal{B}$
$\mathcal{L}_{(s_1 s_2)^2}$
$\mathcal{O}'$
$\mathcal{A}$

The first diagram depicts the closure relations between the G-orbits in $X_\emptyset$, X_1 and X_{12}; while the second depicts the closure relations between the G-orbits in $X_\emptyset$, X_2 and X_{12}. Together they give a complete picture of closure relations between G-orbits in X. This clearly shows that while the G-orbits depend only on the combinatorics of the Weyl group, the closure relations between such orbits in general depend on the geometry of the boundary divisors of the compactification.

3.4 The conjecture on the closure of G-orbits in the canonical compactification of G is false

The G-orbits in the open part of the compactification coincide with the conjugacy classes of G. It is known that the closure in G of a conjugacy class is the union of a finite number of conjugacy classes (see [Spa]). So the question whether in X the same result holds seems natural. By Lusztig's method, the variety X is subdivided in a finite union of locally closed subvarieties, stable with repsect to the G-action, and inside each of these, the G-orbits correspond to twisted conjugacy classes in smaller reductive groups in a way compatible with closures. So the result still holds for a G-orbit inside the locally closed subvariety which contains it, since it is known that the closure of a twisted conjugacy class inside a reductive group is the union of a finite number of twisted conjugacy classes ([Spa]). The problem is in passing from a locally closed piece to another. For a given G-orbit, determining explicitely its closure may be quite hard, but to show that the conjecture does not hold for a group G, one must just look at one particular G-orbit and prove that its closure in X contains infinitely many G-orbits, which is a quite easier task. We prove that the only groups for which the conjecture holds are PGL(2), PGL(3) and products of these. We do this by reducing the problem to the rank two case and to PGL(4). We have determined explicitely all the G-orbits in the compactification and the closure relations for the cases $G = \mathrm{PGL}(2)$, $G = \mathrm{PGL}(3)$, $G = \mathrm{PSp}(4)$; for the case $G = G_2$ we give only some of the closure relations, enough to show that the conjecture is false in such a case; for the case $G = \mathrm{PGL}(4)$ we give just a counterexample to the conjecture. These counterexamples are based on an analysis of small $G \times G$-orbit, i.e. $G \times G$-orbits whose canonical fibration has fiber PGL(2), in other words $G \times G$-orbits corresponding to parabolic subgroups minimal among the nonBorel parabolic subgroups. For the $G \times G$-orbits the finite locally closed decomposition is a Bruhat decomposition, and each of these decomposes either as twisted conjugacy classes of PGL(2), i.e. in two cells one of which is dense. Which of these two cases occurs may be read off from the G-orbit decomposition of the closed $G \times G$-orbits, i.e. from combinatorial data. Let state it precisely.

Lemma 3.4.1. *Let G be a connected reductive group, T a maximal torus, W the weyl group, P a parabolic subgroup containing T, L Levi factor of P which contains T. Suppose $L/Z(L) \cong \mathrm{PGL}(2)$. Let $w \in W$, and $Z_w = \{(p, w^{-1}pw),\ p \in P \cap wP^{-}w^{-1}\}$. Then the image of Z_w in* $\mathrm{PGL}(2) \times \mathrm{PGL}(2)$ *through the composition $Z_w \hookrightarrow P \times P^{-} \to \mathrm{PGL}(2) \times \mathrm{PGL}(2)$, is either the graph of an automorphism $\tilde{\sigma}_1 : \mathrm{PGL}(2) \to \mathrm{PGL}(2)$, or it is the product $B' \times B''$ of two Borel subgroups of* PGL(2).

Proof. We have seen that the image is a subgroup $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1}$, where P_1, Q_1 are parabolic subgroups of PGL(2), and $\tilde{\sigma}_1 : \tilde{L}_1 \to \tilde{M}_1$ is an isomorphism between two quotients of the Levi factor of P_1 and Q_1 by central subgroups. If $P_1 = Q_1 = \mathrm{PGL}(2)$, then $\tilde{\sigma}_1 : \mathrm{PGL}(2) \to \mathrm{PGL}(2)$ and the $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1}$ is the graph of $\tilde{\sigma}_1$. If P_1 is not all of PGL(2) then it is a Borel subgroup B' of PGL(2). In such case also Q_1 must be a Borel subgroup B'' of PGL(2), since PGL(2) has no center and a Levi factor of B' is a one-dimensional torus. Since $P_1 = B'$, $Q_1 = B''$, conjugation by w does not stabilize L; applying the second version of Lusztig's algorithm to compute Z_{L_1} and Z_{M_1} we have that both are nontrivial. Since L_1 and M_1 are one-dimensional tori, we have $Z_{L_1} = L_1$, $Z_{M_1} = M_1$. This implies $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1} = B' \times B''$ in this case. $\square$

Let G be a connected semisimple group of adjoint type, T a maximal torus, W the Weyl group of G with respect to T, B a Borel subgroup containing T, Δ the set of simple roots corresponding to the choice of B. Let $\alpha \in \Delta$, and $I = \Delta - \{\alpha\}$, P_I be the standard parabolic subgroup associated to I, $W_{P_I} = \{e, s_\alpha\} \subset W$, X_I be the $G \times G$-orbit in the canonical compactification X of G induced by I. Consider the fibration

$$\begin{array}{c} X_I \\ \downarrow \\ G/P \times G/P^- \end{array} ;$$

the G-orbits in $G/P \times G/P^-$ are indexed by elements of $W_{P_I}\backslash W/W_{P_I}$. Then the basic result is:

Lemma 3.4.2. *The G-orbit in X_I over the G-orbit $G \cdot (1, w_0) \cdot (P_I, P_I^-)$ are infinitely many if the double coset class $W_{P_I} w_0 W_{P_I}$ contains two elements. The G-orbits in X_I over $G \cdot (1, w_0) \cdot (P_I, P_I^-)$ are two if $W_{P_I} w_0 W_{P_I}$ contains four elements.*

Proof. The G-orbit in X_I over $G \cdot (1, w_0) \cdot (P_I, P_I^-)$ corresponds to $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1}$-orbits in PGL(2). Now, by the previous lemma, either $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1}$ is the graph of $\tilde{\sigma}_1 : \mathrm{PGL}(2) \to \mathrm{PGL}(2)$ or $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1} = B' \times B''$, with B', B'' Borel subgroups of PGL(2). In the first case the G-orbits in X_I over $G \cdot (1, w_0) \cdot (P_I, P_I^-)$ correspond to twisted conjugacy classes in PGL(2), so there are infinitely many. In the second case, they correspond to $B' \times B''$-orbits in PGL(2), so there are two. One must distinguish these two cases in terms of double coset class $W_{P_I} w_0 W_{P_I}$. Along the lines of lemma 2.6.2, the fibration $X_I \to G/P_I \times G/P_I^-$ may be extended to a fibration $\overline{X}_I \to G/P_I \times G/P_I^-$ with fiber the canonical compactification $\overline{\mathrm{PGL}(2)}$ of PGL(2), $X_\Delta \cong G/B \times G/B^-$ mapping to the closed $\mathrm{PGL}(2) \times \mathrm{PGL}(2)$-orbit $\mathrm{PGL}(2)/B_1 \times \mathrm{PGL}(2)/B_1^-$, where B_1 is the Borel subgroup of PGL(2) which is the image of B through the projection to PGL(2). The G-orbit in $X_\Delta \cong G/B \times G/B^-$ over $G \cdot (1, w_0) \cdot (P_I, P_I^-)$ are indexed by the elements of the double coset class $W_{P_I} w_0 W_{P_I}$. On the other hand they correspond to $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1}$-orbits in $\mathrm{PGL}(2)/B_1 \times \mathrm{PGL}(2)/B_1^-$. So when $|W_{P_I} w_0 W_{P_I}| = 4$, $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1} = B' \times B''$, and there are only two orbits in X_I above $G \cdot (1, w_0) \cdot (P_I, P_I^-)$. While, if $|W_{P_I} w_0 W_{P_I}| = 2$, then $\tilde{\Delta}_{P_1,Q_1,\tilde{\sigma}_1}$ is the graph of an automorphism of PGL(2), and there are infinitely many orbits in X_I above $G \cdot (1, w_0) \cdot (P_I, P_I^-)$ □

Corollary 3.4.3. *The conjecture does not hold for $G = G_2$, i.e. if X is the canonical compactification $G = G_2$, then there is at least one G-orbit in X whose closure contains infinitely many G-orbits.*

Proof. Using the preceding lemma it is easy to produce one such orbit. The Weyl group W of G_2 has the presentation

$$W =< \tau_1, \tau_2;\ (\tau_1\tau_2)^3 = (\tau_2\tau_1)^3 >,$$

so

$$W = \{e, \tau_1, \tau_2, \tau_1\tau_2, \tau_2\tau_1, \tau_1\tau_2\tau_1, \tau_2\tau_1\tau_1, \tau_1\tau_2\tau_1\tau_2, \tau_2\tau_1\tau_2\tau_1, \tau_1\tau_2\tau_1\tau_2\tau_1, \tau_2\tau_1\tau_2\tau_1\tau_2,$$

$$\tau_1\tau_2\tau_1\tau_2\tau_1\tau_2 = \tau_2\tau_1\tau_2\tau_1\tau_2\tau_1\}.$$

Let P the standard parabolic subgroup of G_2 such that $W_P = \{e, \tau_2\}$, then $W_P \backslash W / W_P = \{e, \tau_2, \tau_2\tau_1\tau_2, \tau_2\tau_1\tau_2\tau_1\tau_2\}$, the Bruhat order induced on these is $e < \tau_2 < \tau_2\tau_1\tau_2 < \tau_2\tau_1\tau_2\tau_1\tau_2\}$. Now, $|W_P e W_P| = |W_P| = 2$, $|W_P \tau_2 W_P| = 4$, $|W_P \tau_2\tau_1\tau_2 W_P| = 4$, $|W_P \tau_2\tau_1\tau_2\tau_1\tau_2 W_P| = 2$. So over $G \cdot (1, \tau_2) \cdot (P, P^-)$ there are only two G-orbits, one of which is open in the inverse image. The same holds for $G \cdot (1, \tau_2\tau_1\tau_2) \cdot (P, P^-)$, while over $G \cdot (1, \tau_2\tau_1\tau_2\tau_1\tau_2) \cdot (P, P^-)$ there are infinitely many. Taking for example the big orbit over $G \cdot (1, \tau_2) \cdot (P, P^-)$, its closure will contain infinitely many orbits, since it will contain all orbits over $G \cdot (1, \tau_2\tau_1\tau_2) \cdot (P, P^-)$ and over $G \cdot (1, \tau_2\tau_1\tau_2\tau_1\tau_2) \cdot (P, P^-)$ □

Corollary 3.4.4. *The conjecture does not hold for* $G = \mathrm{PGL}(4)$*, i.e. if* X *is the canonical compactification of* $G = \mathrm{PGL}(4)$*, then there is at least one* G*-orbit in* X *whose closure contains infinitely many orbits.*

Proof. $W \cong S_4$ is the Weyl group of PGL(4), it has the presentation

$$W =< s_1,\ s_2,\ s_3,\ s_1s_2s_1 = s_2s_1s_2,\ s_2s_3s_2 = s_3s_2s_3,\ s_1s_3 = s_3s_1 > .$$

Let P be the standard parabolic of PGL(4) such that $W_P = \{e, s_2\}$. We will look at G-orbits in the $G\times G$-orbit of X which fibers over $G/P\times G/P_-$. Using the isomorphism $W \to S_4$ given by $s_1 \mapsto (12)$, $s_2 \mapsto (23)$, $s_3 \mapsto (34)$, we have $W_P = \{e, (23)\}$, and $W_P\backslash W/W_P = \{e, (14), (12), (34), (12)(34), (1234), (1432)\}$. It is easily seen that $|W_PeW_P| = |W_P| = 2$, $|W_P(14)W_P| = 2$, $|W_P(12)W_P| = 4$, $|W_P(34)W_P| = 4$, $|W_P(1234)W_P| = 4$, $|W_P(1432)W_P| = 4$. To understand the Bruhat ordering between these elements one must write them in terms of the simple reflection (12), (23), (34):

$$(14) = (12)(23)(34)(23)(12),\ (12),\ (34),\ (12)(34),\ (1234) = (12)(23)(34),$$

$$(1432) = (34)(23)(12).$$

An element in a Coxeter group is smaller than another for the Bruhat ordering if a reduced expression of the first appears as a subword in a reduced expression of the second. This shows that (14) is bigger than all others. So the orbit of greater dimension over $G \cdot (1, s) \cdot (P, P^-)$, with $s = (12), (34), (12)(34), (1234), (1432)$ closes to infinitely many orbits, viz. those over $G \cdot (1, (14)) \cdot (P, P^-)$ □

We could apply the same strategy to $PSp(4)$, or just take into consideration the explicit computation carried out for that group, to argue that the conjecture does not hold in that case. Now, these three special cases suffice to prove that the conjecture is false for all simple groups of rank greater than or equal to three. The explicit computations carried out for the cases of $PGL(2)$ and $PGL(3)$ show that for such groups the conjecture is true. So we get:

Proposition 3.4.5. *If the connected semisimple group G of adjoint type is such that in its canonical compactification X the closures of G-orbits are unions of a finite number of G-orbits, then $G = PGL(2)^a \times PGL(3)^b$, with $a, b \in \mathbb{N}$.*

Proof. G is the product of simple groups of adjoint type G_i, and X is the product of the compactifications X_i of the factors of G. G-orbits in X are products of G_i-orbits in X_i for varying i, so the conjecture is false for G if and only if it is false for at least one factor G_i of G. We are thus reduced to the case of G a simple group. Simple groups are classified by irreducible Dynkin diagrams. These are of type A_l (for $l \geq 1$), B_l (for $l \geq 2$), C_l (for $l \geq 3$), D_l (for $l \geq 4$), E_6,E_7, E_8, F_4, G_2. $PGL(2)$ corresponds to A_1, $PGL(3)$ to A_2, $PGL(4)$ to A_3, $PSp(4)$ to B_2 and the group G_2 to the Dynkin diagram by the same name. Any Dynkin diagram different from A_1 or A_2 contains either A_3, B_2 or G_2 as subdiagrams; so if G is different from $PGL(2)$ or $PGL(3)$, it contains a parabolic subgroup P whose semisimple component of adjoint type G_I is either $PGL(4)$, $PSp(4)$ or G_2. If $\overline{X_I}$ is the closure of the $G \times G$-orbit X_I fibering over $G/P \times G/P^-$, then the G-orbits in $\overline{X_I}$ over the G-orbit $G\dot{(}P, P^-)$ in $G/P \times G/P^-$ correspond to the G_I-orbits in the canonical compactification of G_I, (lemma 2.6.2). Hence if the conjecture is false for G_I, it is false also for G. Applying such arguments to the simple factors of a semisimple group of adjoint type G, we get that the conjecture is true for G if and only if the simple factors of G are isomorphic either to $PGL(2)$ or to $PGL(3)$. □

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