Università degli Studi di Roma "La Sapienza"
Dipartimento di Matematica "G. Castelnuovo"

Ph. D. Thesis

Solenoidal Differential Inclusions and H-measures

Mariapia Palombaro

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Contents

1. Introduction

In recent years a lot of attention has been devoted to the understanding of phenomena arising from mechanics and from the composites literature. The main theme is to understand the appearance of "microstructures". These are extremely fine oscillations in the physical properties of materials, some of which seem to arise spontaneously. One of the models which has been mostly studied in the mathematical communities interested in the calculus of variations and in systems of PDEs attempts to explain the phenomenon via energy-minimization. Unfortunately, most often, the actual form of the putative energy density is unknown. However, the crucial observation is that one can establish (via measurements) the exact form of set K on which the density of energy is minimal. Typically one has a set of admissible maps $U : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ subject to belong to some functional spaces and one has an unknown function $f : \mathbb{M}^{m\times n} \to \mathbb{R}$ which is continuous and bounded from below. The stored energy has the form

$$DU \to G(DU) := \int_\Omega f(DU)\,dx\,.$$

Without loss of generality we may assume $f \geq 0$ and one sets

$$K = \{A \in \mathbb{M}^{m\times n} \,:\, f(A) = 0\}.$$

The set K may be of different types depending on the problem under consideration. Some relevant examples are the following

$$K = \{A_1, \dots, A_N\}\,, \quad K = SO(n)\,, \quad K = \bigcup_{i=1}^{N} SO(n)D_i\,,$$

where D_i are given diagonal matrices. In this contexts one has to content herself with the task of establishing when

$$\inf_{U\in W} \int_\Omega f(DU)\,dx = 0 \tag{0.1}$$

for a given (*i.e.* known) set K. We will come back later in the introduction to the issue of which space W would be desirable to use.
Problem (0.1) splits into two different subjects

(*i*) *exact solutions* ,

(*ii*) *approximate solutions* .

The first one is just a way of saying that if there exists $U \in W$ such that $DU \in K$ a.e. then, obviously (0.1) is satisfied and indeed the minimum is achieved. We call these mappings exact solutions of the differential inclusion $DU \in K$ a.e., and we say that K is rigid for exact solutions if the only such functions are globally affine. Clearly, due to the need to satisfy boundary conditions, one is interested in how large the class of exact solutions is and in particular in the question of whether it contains non-affine maps. The second one is slightly more sophisticated. When the minimum is not attained one can still find minimizing sequences, that is sequences $\{U_h\}$ such that $G(DU_h) \to 0$. Any such sequence is said to be an approximate solution and it satisfies $\mathrm{dist}(DU_h, K) \to 0$ in

measure. Therefore we say that K is rigid for approximate solutions if $\operatorname{dist}(DU_h, K) \to 0$ in measure and the following condition holds: if $\{U_h\}$ is weakly convergent to an affine map then $DU_h \to A \in K$ in measure.
A relatively simple instance of (0.1) arises when considering densities which are of the following form

$$f(A) = \min_{i=1,\dots,N} \{f_i(A)\}\,, \quad \text{where } f_i(A) = \alpha_i |A - A_i|^2 + \lambda_i\,, \quad \alpha_i, \lambda_i > 0\,,$$

where A_i are given matrices. We will return to this issue later in the introduction.

Many interesting results have been proved in this field (see for instance Ball & James [**7**], Šverák [**30, 31**], Tartar [**40**], Zhang [**41**], Chlebík & B. Kirchheim [**13**], Kirchheim & Preiss [**23**]). We refer to Müller [**25**] and Kirchheim [**22**] for a broader introduction into this subject.

In this thesis we address a different but very closely related problem. Typically one wants to minimize integrals of the form

$$V \to G(V) = \int_\Omega f(V(x))\,dx \tag{0.2}$$

where f satisfies some regularity assumptions and V is subject to the constraint $\mathcal{A}V = 0$ for some partial differential operator $\mathcal{A}$. We replace in (0.1) the constraint of being a gradient, *i.e.* of satisfying Curl $U = 0$ in the sense of distributions, with that of being solenoidal, *i.e.* satisfying

$$\operatorname{Div} V = 0$$

(where Div is the operator which acts as the divergence in the sense of distributions on each row). The latter still falls in the framework named $\mathcal{A}$-quasiconvexity which finds its roots in the work of Tartar on compensated compactness ([**33**], [**34**], [**35**], [**38**], [**39**]). For a comprehensive treatment of this part of the literature see Fonseca & Müller [**17**].

The notion of $\mathcal{A}$-quasiconvexity is introduced as a necessary and sufficient condition ensuring sequential weak lower semicontinuity of the functional (0.2) under appropriate growth conditions on f. The function f is said to be $\mathcal{A}$-quasiconvex (simply quasiconvex if $\mathcal{A}$= Curl) if

$$f(A) \le \fint_Q f(A + V(x))\,dx \tag{0.3}$$

for all $A \in \mathbb{M}^{m\times n}$ and all $V \in C^\infty(\mathbb{R}^n, \mathbb{M}^{m\times n})$ such that V is Q-periodic, with $Q = (0, 2\pi)^n$, $\mathcal{A}V = 0$ and $\fint_Q V(x)\,dx = A$.
In particular we will say that a function f is S-quasiconvex if f satisfies the inequality (0.3) for $\mathcal{A}$ =Div. With these motivations in mind we address the following kind of problems.

Problem 1. Given two integers $m, n \ge 2$, a set of real $m \times n$ matrices $K \subset \mathbb{M}^{m\times n}$ and a bounded open set Ω in $\mathbb{R}^n$, find $B \in L^\infty(\Omega, \mathbb{M}^{m\times n})$ such that

$$(0.4)\qquad \begin{cases} \mathrm{Div} B = 0 & \text{in } \mathcal{D}'(\Omega, \mathbb{R}^m)\,, \\ B(x) \in K & \text{a.e. in } \Omega\,, \\ B \text{ is non-constant}\,. \end{cases}$$

Problem 2. Find uniformly bounded sequences $\{B_h\} \subset L^\infty(\Omega, \mathbb{M}^{m\times n})$ such that

$$(0.5)\qquad \begin{cases} \mathrm{Div} B_h = 0 & \text{in } \mathcal{D}'(\Omega, \mathbb{R}^m)\,, \\ \mathrm{dist}(B_h, K) \to 0 & \text{in measure,} \\ \forall\, \{B_{h_j}\} \text{ subsequence of } \{B_h\}, \text{ if } B_{h_j} \to A \text{ in measure, then } A \notin K\,. \end{cases}$$

By analogy with the Curl-free case, we give the following definitions.

Definition 1. Any $B \in L^\infty(\Omega, \mathbb{M}^{m\times n})$ satisfying (0.4) is called an exact solution to *Problem 1.*

Definition 2. Any sequence $\{B_h\} \in L^\infty(\Omega, \mathbb{M}^{m\times n})$ satisfying (0.5) is called an approximate solution to *Problem 1.*

Definition 3. We say that K is rigid for exact or approximate solutions of Solenoidal type if there is no solution to *Problem 1* or *Problem 2* respectively.

Besides the interest this issue deserves on its own, our research is also motivated by the applications to the homogenization theory. We now explain briefly why. Indeed, in the context of bounding effective moduli for composites one is often required to understand the range of a functional of the form

$$(0.6)\qquad G(A) = \inf_U \int_\Omega \sum_{i=1}^N \chi_i(x) g(W_i(DU(x)))dx$$

where U belongs for instance to $W_0^p(\Omega, \mathbb{R}^n) + Ax$, the χ_i's are characteristic functions of disjoint measurable subsets of Ω and g is a scalar function on the space of matrices. We assume that the positive numbers $m_i = \fint_\Omega \chi_i(x)dx$ are given. The elementary upper bound is

$$(0.7)\qquad \sup_{\chi_i} G(A) \le \sum_{i=1}^N m_i(x) g(W_i(A))\,.$$

One may wonder whether this bound could be "attainable" or at least "optimal" for some matrix A. We will say that the bound is attainable if there exist $\chi_1, \dots, \chi_N$ for which the equality holds in (0.7). We say that the bound (0.7) is optimal if the inequality (0.7) cannot be improved. One can check that a sufficient condition for the attainability of the bound is the existence of characteristic functions $\chi_1, \dots, \chi_N$ such that

$$\mathrm{Div}\left(\sum_{i=1}^N \chi_i(x) W_i(A)\right) = 0\,.$$

The choice $W_i(A) \in K$ and $B = \sum_{i=1}^{N} \chi_i(x) W_i(A)$, gives a problem of the kind described in *Problem 1*. See also Garroni *et al.* [**19**] for further examples and Garroni & Nesi [**18**] for some discussion of the connection between approximate solutions to (0.4) and "optimality" of the bounds.

We now return to *Problem 1* and *Problem 2*.

The case when K consists of two matrices, say A_1 and A_2, has been studied in detail by Garroni & Nesi [**18**]. They proved that a necessary and sufficient condition for the existence of both exact and approximate solutions is the so-called rank-$(n-1)$ connectedness, that is the condition that $\operatorname{rank}(A_1 - A_2) \leq n - 1$.
In Chapter 1 we prove one of the main results of this thesis, namely that the latter condition is still necessary for the existence of non-constant divergence free matrix fields taking at most *three* values. In Chapter 1 we will prove the following theorem.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and let $K = \{A_1, A_2, A_3\} \subset \mathbb{M}^{m\times n}$, with $m \geq n$ and* $\operatorname{rank}(A_i - A_j) = n$ *for $i \neq j$. If $B : \Omega \to K$ is a measurable function satisfying* $\operatorname{Div} B = 0$ *in $\mathcal{D}'(\Omega, \mathbb{R}^m)$, then B is constant.*

(Here $\mathcal{D}'(\Omega, \mathbb{R}^m)$ denotes the space of distributions). In addition to the previous theorem, we also describe a different class of sets that does not support non-constant divergence free fields (Theorem 1.8). In this case the assumption on the cardinality of the set K is replaced by a certain algebraic condition which is weaker than simultaneous diagonalizability of the matrices in K.
Besides the question of non-existence of exact solutions, we consider the problem of finding approximate solutions to the "three-divergence problem". Garroni & Nesi [**18**] exhibited an example of set $K \subset \mathbb{M}^{3\times 3}$ of three pairwise rank-2 disconnected matrices supporting approximate solutions. The construction they used is based on an infinite-rank sequential lamination and is akin to that employed by Nesi & Milton [**26**], Scheffer [**28**], Talbot & Willis [**32**], Aumann & Hart [**6**], Tartar [**40**], Casadio Tarabusi [**12**], Bhattacharya *et al.* [**10**], Smyshlyaev & Willis [**29**]. In particular in [**40**] this construction is used to show that in the "four-gradient problem" the absence of rank-1 connections does not guarantee absence of microstructure (*i.e.* strong convergence of approximating sequences). This is the well-known example sometimes called "Tartar's square": four incompatible matrices, *i.e.* matrices A_i such that $\operatorname{rank}(A_i - A_j) > 1$ if $i \neq j$, that support a non-constant minimizing sequence.
Turning back to the Div-free setting, in Section 4 of Chapter 1 we show that the method implemented in [**18**] actually applies to a larger class of sets K. Theorem 1.10 gives a characterization of all such K's, which turn out to be non-rigid for approximate solutions.

Theorem 1.10. *For every q_1, q_2, $q_3 \in (0,1)$, let $A \in \mathbb{M}^{3\times 3}$ be defined as follows*

$$A = \frac{1}{q_3}\left[\left(1-\prod_{i=1}^{3}(1-q_i)\right)G^{-1}\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}G - q_2(1-q_3)I\right],$$

where $\lambda_1 = 0\,, \lambda_2 = 1/(1-q_1)\,, \lambda_3 = q_2/(q_1+q_2-q_1q_2)$, and G is an arbitrary matrix in $GL(3)$. Then, for every $M \in \mathbb{M}^{3\times 3}$ and $N \in GL(3)$, the set

$$K = \{M\,, N+M\,, NA+M\}$$

is non-rigid for approximate solutions.

The latter result will play an important role in the other main results of this work. In Chapter 2 we address the problem of characterizing the "relaxation" of the sets K described in Theorem 1.10. The "relaxation" of K, denoted by K^{qc}_S, is the S-quasiconvex hull of K and it is defined as follows.

Definition 0.1. For any $K \subset \mathbb{M}^{3\times 3}$, the set K^{qc}_S consists of all matrices $B_0 \in \mathbb{M}^{3\times 3}$ such that there exists a bounded sequence $\{B_h\} \subset L^2_{loc}(\mathbb{R}^3, \mathbb{M}^{3\times 3})$ which is Q-periodic and satisfies

$$\begin{cases}\mathrm{Div} B_h = 0 & \text{in } \mathcal{D}'(\mathbb{R}^3,\mathbb{R}^3)\,,\\ \mathrm{dist}(B_h, K) \to 0 & \text{in measure,}\\ \fint_Q B_h = B_0\,. & \end{cases}$$

Remark 0.2. The reason for choosing, in Definition 0.1, bounded sequences in L^2 will be clear in Chapter 2.

The characterization of K^{qc}_S is in general an open problem since it is intimately connected to the notion of S-quasiconvexity of functions, and the understanding of S-quasiconvexity turns out to be a very hard task.

A possible way of finding inner bounds for K^{qc}_S is to understand which is the set of points that are "realizable" by laminate structures. In many cases this set is the right candidate to be K^{qc}_S. Whenever we establish the existence of positive S-quasiconvex functions for which the intersections of their zero level sets coincides exactly with the candidate one, then the inner bound is indeed optimal. This strategy can be successfully applied, within the framework of the Curl-free problems, to the example of Tartar's, the so-called "Tartar square", that we briefly illustrate. The set K consists in this case of four 2×2 matrices A_1, A_2, A_3, A_4, that can thus be identified with points in $\mathbb{R}^2$ (see Figure 1):

$$K = \{\mathrm{diag}(-1,-3),\ \mathrm{diag}(-1,-3),\ \mathrm{diag}(-1,-3),\ \mathrm{diag}(-1,-3)\}\,.$$

The "relaxation" of K, denoted by K^{qc}, is defined in a way similar to (0.1). Roughly speaking, it consists in this case of all affine boundary conditions that can be satisfied by approximate solutions for the set K. As already remarked, the set K is non-rigid for approximate solutions that can be constructed using infinite-rank laminations. For such sequences, all admissible affine boundary conditions are those points which lie inside the

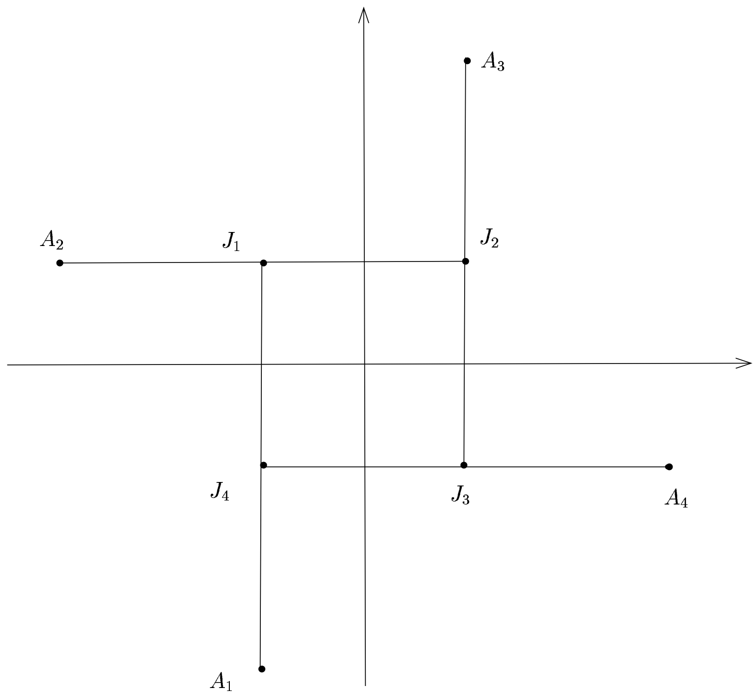

FIGURE 1. The “Tartar’s square”.

square $J_1J_2J_3J_4$ or on the rank-1 segments $[A_1J_1]$, $[A_2J_2]$, $[A_3J_3]$, $[A_4J_4]$. In order to prove that these are the only points in K^{qc}, one uses the fact that the function $f_i : \mathbb{M}^{2\times 2}_{sym} \to \mathbb{R}$ defined by

$$f_i(A) = \begin{cases} |\det(A)| & \text{if index}(A) = i \\ 0 & \text{otherwise} \end{cases}$$

is quasiconvex on its domain for every $i = 0, 1, 2$ (for the proof of this result see Šverák [**31**]), where index(A) is the number of negative eigenvalues counting multiplicities. The sought quasiconvex functions are obtained composing f_i with suitable translations:

$$\tilde{f}_i(A) = f_i(A - J_r) \quad \text{for } r = 1, 2, 3, 4\,.$$

The intersection of the zero level sets of the above functions is exactly the union of the square and the four arms.

Our strategy for characterizing K^{qc}_S will be similar in spirit. The key idea of this argument is presented in Lemma 2.13: starting from a S-quasiconvex function suggested by Tartar [**36**], we construct three functions living on 2-dimensional subspaces of $\mathbb{M}^{3\times 3}$. Each of them is S-quasiconvex on its domain and, in some way, plays the role of one of the functions $\tilde{f}_i$. However, since our functions are only defined on subspaces of $\mathbb{M}^{3\times 3}$, in order to prove the optimality of the bounds we have to use some more sophisticated tools. Indeed in Chapter 2 we introduce the notion of H-measures. In our toolkit we only require the representation of the H-measures in the special case of the periodic setting (see Kohn [**24**] for a self-contained introduction).

DEFINITION 0.3. Let $N, d \in \mathbb{N}$, $N \geq 2$, $d \geq 2$, $Q = (0, 2\pi)^d$, $\theta = (\theta_1, \dots, \theta_N) \in [0,1]^N$, with $\sum_{i=1}^{N} \theta_i = 1$ and let

$$I(\theta) = \left\{ \chi : \mathbb{R}^d \to \{0,1\}^N, Q\text{–periodic and measurable } : \sum_{i=1}^{N} \chi_i = 1 \text{ a.e.}, \fint_Q \chi = \theta \right\}.$$

For every $\chi \in I(\theta)$, we call H-measures generated by χ the matrix-valued measure $\mu = (\mu_{ij})_{i,j}$ defined as follows:

$$\mu_{ij} = \mathrm{Re} \sum_{k \neq 0} \hat{\chi}_i(k) \overline{\hat{\chi}_j(k)} \delta_{k/|k|}, \quad 1 \leq i, j \leq N.$$

Our definition is in fact a restrictive one, since the general construction of the H-measures may involve functions that need not be periodic (see Tartar [**38, 39**] and Gerard [**20**]). Roughly speaking, an H-measure is defined whenever a partition of Q into N phases is given. The H-measures associated to N-component microstructures depend in particular on the volume fraction θ_i of each phase, $i = 1, \dots, N$. More specifically, they are matrix-valued measures on the unit sphere S^{d-1} satisfying the following algebraic restrictions:

$$\mu_{ij} = \mu_{ji} \text{ and } \sum_{i=1}^{N} \mu_{ij} = 0, \quad 1 \leq j \leq N, \tag{0.8}$$

$$\int_{S^{d-1}} \mu_{ij} ds(\xi) = \delta_{ij}\theta_i - \theta_i\theta_j, \tag{0.9}$$

$$\mu_{ij}(\xi) = \mu_{ij}(-\xi), \tag{0.10}$$

$$\sum_{i,j=1}^{N} \int_{S^{d-1}} \varphi_i(\xi)\varphi_j(\xi)\mu_{ij} ds(\xi) \geq 0 \text{ for any continuous function } \varphi. \tag{0.11}$$

The following theorem, which is due to Kohn [**24**], states that for $N = 2$ the restrictions (0.8)-(0.11) are necessary and sufficient to characterize the set of all H-measures.

Theorem. (Kohn, [**24**]). *For any $0 < \theta < 1$, the set of H-measures associated to microstructures with average value θ consists precisely of the nonnegative measures $\mu(\xi)$ on S^{d-1} which have total mass $\theta(1-\theta)$ and satisfy*

$$\int_{S^{d-1}} f(\xi)\, d\mu(\xi) = \int_{S^{d-1}} f(-\xi)\, d\mu(\xi)$$

for every continuous function f.

When $N > 2$ the known restrictions (0.8)-(0.11) on the H-measures define a convex superset of Borel measures on the unit sphere, which contains all the H-measures, *i.e.* those measures realizable by a microstructure, but may also contain measures that are not realizable. Therefore it would be extremely interesting to find additional properties characterizing the H-measures. This is exactly what will be achieved in this thesis as a by-product of our main results. In Section 6 of Chapter 2 we address the problem of understanding under which conditions a three-point measure on the unit sphere S^2 (*i.e.* a sum of three Dirac masses supported on S^2) satisfying the known restrictions is an H-measure. Theorem 2.18 gives the necessary and sufficient conditions under an additional assumption on the mass of each of the three Dirac's (see formulae (2.56)-(2.57) in Theorem 2.15). On the other hand, Theorem 2.15 establishes the equivalence between the problem of characterizing the three-point H-measures and that of characterizing the "relaxation" of the sets K presented in Theorem 1.10. The two results put together amount to providing the optimal bounds for K_S^{qc} (see Corollary 2.19). A more detailed description of these results is outlined in the first section of Chapter 2.

Besides the link to the problem discussed so far, the results presented in Theorem 2.18 seem to be of independent interest. The resulting information on the H-measures also contributes to the understanding of many other questions. Interesting applications can be found for example in the relaxation of three-well energies in the context of infinitesimal elasticity. In this situation we are given a density of energy of the form

$$W(\eta) = \min\{W_1(\eta), W_2(\eta), W_3(\eta)\}$$

associated to a system of three linearly elastic phases with different stress-free strains but the same elastic moduli. The energy of each phase is a quadratic function of the linear strain η:

$$W_i(\eta) = \tfrac{1}{2}\langle \alpha(\eta - \eta_i), \eta - \eta_i\rangle + w_i\,, \quad i = 1, 2, 3\,.$$

Here η_i is the stress-free strain of the ith phase, and w_i the associated minimum energy; α is the tensor of elastic moduli, viewed as a symmetric linear map on the space of symmetric tensors, and assumed to be the same for the three phases. The relaxation of W is defined in the following way:

$$QW(\eta) = \inf_{\phi} \fint_Q W(\eta + e(\phi))\, dx \tag{0.12}$$

where ϕ represents the elastic displacement and ranges over all Q-periodic maps in $H^1_{loc}(\mathbb{R}^3, \mathbb{R}^3)$, and $e(\phi) = \frac{1}{2}(D\phi + D\phi^T)$ is its linear strain. Using Fourier analysis one can rewrite (0.12) as a minimization problem over the set of all H-measures associated to three-component microstructures (see [**24**], [**29**], [**2**] and Chapter 2 for further details). Minimization with respect to the superset of the matrix measures on the unit sphere S^2 subject to the known restrictions, leads in general to a lower bound for the relaxed energy. Although there could in principle be more than one minimizing measure on the superset, it has been shown (Smyshlyaev & Willis [**29**]) that it is always possible to construct a minimizing measure that can be expressed as a sum of at most three Dirac masses (identifying the point ξ on the sphere and its antipode $-\xi$). In [**29**] a sufficient condition for the realizability of a subclass of such three-point measures is also given. We will call it the Smyshlyaev-Willis' algorithm and it is valid when a certain condition is satisfied. In this case we say that we are working in the Smyshlyaev-Willis' regime. Our result (Theorem 2.18) says that this condition is also necessary, thus showing that a three-point measure belonging to this subclass is an H-measure if and only if it is realizable by a three-phase sequential lamination of infinite rank. Therefore, whenever the minimizing measure do fall within this subclass, we are able to establish the optimality of the delivered bound. We remark in conclusion that this argument extends to other cases. For instance to the case when $\alpha = I$ and we have gradients instead of linear strains in (0.12).

Applications of the H-measures techniques in homogenization can also be found in the work of Allaire & Maillot [**4**]. In the latter work the analysis is restricted to the case when the H-measures are supported on two Dirac masses. Our work may shed more light on the case when three Dirac masses are present.

CHAPTER 1

The three divergence free matrix fields problem

1. Mathematical setting and brief history

The problem of characterizing solenoidal matrix fields which take values in a finite set of matrices, has been recently considered by A. Garroni and V. Nesi [**18**]. This kind of problem is analogous to that on curl free matrix fields in which one asks whether a Lipschitz mapping using a finite number of gradients exists. Here the differential constraint of being the gradient of a mapping, and hence a curl free matrix field, is replaced by that of being a divergence free matrix field (*i.e.* a matrix valued function whose rows are divergence free in the distributional sense). To describe the problem we recall some definitions from the introduction.

Definition 1. Given two integers $m, n \geq 2$, a set of real $m \times n$ matrices $K \subset \mathbb{M}^{m\times n}$ and a bounded open set Ω in $\mathbb{R}^n$, we say that any $B \in L^\infty(\Omega, \mathbb{M}^{m\times n})$ satisfying

$$
\begin{cases}
\mathrm{Div} B = 0 & \text{in } \mathcal{D}'(\Omega, \mathbb{R}^m)\,, \\
B(x) \in K & \text{a.e. in } \Omega\,, \\
B \text{ is non-constant}\,,
\end{cases}
\tag{1.1}
$$

is an exact solution of (1.1). We say that K is rigid for exact solutions if there is no solution to (1.1).

Definition 2. We say that Problem (1.1) admits an approximate solution if there exists a uniformly bounded sequence $\{B_h\} \subset L^\infty(\Omega, \mathbb{M}^{m\times n})$ such that

$$
\begin{cases}
\mathrm{Div} B_h = 0 & \text{in } \mathcal{D}'(\Omega, \mathbb{R}^m)\,, \\
\mathrm{dist}(B_h, K) \to 0 & \text{in measure,} \\
\forall\, \{B_{h_j}\} \text{ subsequence of } \{B_h\}, \text{ if } B_{h_j} \to A \text{ in measure, then } A \notin K\,.
\end{cases}
\tag{1.2}
$$

We say that K is rigid for approximate solutions of (1.1) if there is no solution to (1.2).

We remark that if K is rigid for approximate solutions and there exists a sequence $\{B_h\}$ satisfying the first two conditions of (1.2), then any accumulation point of the sequence $\{B_h\}$ has to be a constant matrix belonging to K.

Let us briefly digress now to describe the situation in the context of the "gradient problem", that is: find $f \in W^{1,\infty}(\Omega, \mathbb{R}^m)$ such that $Df \in K$ a.e. in Ω and f is not affine. In this setting, let us consider the case $K = \{A_1, \ldots, A_N\} \subset \mathbb{M}^{m\times n}$. It is well-known that if $N = 3$ or $N = 2$, then a sufficient condition for the set K to be rigid is that $\mathrm{rank}(A_i - A_j) > 1$ for $i \neq j$. The condition $\mathrm{rank}(A_i - A_j) = 1$ is called rank-1 connectedness. J. M. Ball and R. D. James studied in detail the case $N = 2$ and proved, under the latter assumption, a rigidity result both for exact and for approximate solutions (see [**7**]). For $N = 3$ the following rigidity result is due to Šverák and will be used later.

THEOREM 1.1. (V. Šverák, [**31**]). *Let $\Omega \subset \mathbb{R}^n$ be an open connected set and let $K = \{A_1, A_2, A_3\} \subset \mathbb{M}^{m\times n}$, with $\mathrm{rank}(A_i - A_j) > 1$ for $i \neq j$. If $f \in W^{1,\infty}(\Omega, \mathbb{R}^m)$ satisfies $Df \in K$ a.e., then Df is constant.*
Let $p > 2$ and let $f_h \rightharpoonup f$ in $W^{1,p}(\Omega, \mathbb{R}^m)$. If $\mathrm{dist}(Df_h, K) \to 0$ in measure, then $Df_h \to A_i$ in measure, for some $i \in \{1, 2, 3\}$.

The previous result, specialized to the case $m = n = 2$, will be a crucial tool in the proof of Theorem 1.7.
For completeness let us also recall that for $N = 4$ rigidity still holds for exact solutions (see [**13**]) but it can fail for approximate ones and a suitable choice of $\{A_1, A_2, A_3, A_4\}$ (see [**37**], [**40**]). The case $N = 5$ is nicely illustrated in [**23**] by a non-rigid five point configuration without any rank-1 connection. For a comprehensive treatment of this part of the literature including the result by B. Kirchheim and D. Preiss [**23**] we refer to [**22**].

REMARK 1.2. All the previous works provide results also for the "divergence problem" when the working space is $\mathbb{M}^{m\times 2}$, since any set of solenoidal matrix fields defines a set of gradients via right-multiplication by $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since the converse is also true, when $n = 2$ the "Curl-free" and the "Div-free" problems are equivalent.

The problems are no longer equivalent if $m \geq n > 2$. The right notion of connectedness which comes into play, in this case, is that of the so called rank-$(n-1)$ connectedness. Given $A_1, A_2 \in \mathbb{M}^{m\times n}$, with $m \geq n$ and $\mathrm{rank}(A_1 - A_2) \leq n - 1$, one can construct solenoidal matrix fields which take both the values A_1 and A_2 on a set of positive measure (indeed one can check that simple laminates work). In contrast, if $\mathrm{rank}(A_1 - A_2) = n$, the following rigidity result holds.

PROPOSITION 1.3. (A. Garroni, V. Nesi, [**18**]). *Let Ω be an open and connected set in $\mathbb{R}^n$. Let $A_1, A_2 \in \mathbb{M}^{m\times n}$, with $m \geq n$ and $\mathrm{rank}(A_1 - A_2) = n$. Let $B : \Omega \to \{A_1, A_2\}$ be a measurable function satisfying $\mathrm{Div}B = 0$ in $\mathcal{D}'(\Omega, \mathbb{R}^m)$. Then B is constant.*

The case when K is made of two matrices has been completely solved and given a negative answer also for what concerns approximate solutions. The following proposition establishes rigidity for approximate solutions under the hypothesis of rank-$(n-1)$ disconnectedness.

PROPOSITION 1.4. (A. Garroni, V. Nesi, [**18**]). *Let Ω be a bounded open and connected set in $\mathbb{R}^n$, and let $K = \{A_1, A_2\} \subset \mathbb{M}^{m\times n}$, $m \geq n \geq 1$, be such that $\mathrm{rank}(A_1 - A_2) = n$. Let B_h be a sequence weakly convergent to B in $L^p(\Omega, \mathbb{M}^{m\times n})$, with $p > 1$, such that*

$$\mathrm{Div}B_h \to 0 \quad \text{strongly in } W^{-1,p}(\Omega, \mathbb{R}^m)$$

and

$$\mathrm{dist}(B_h, K) \to 0 \quad \text{in measure}.$$

Then

$$B_h \to A_1 \quad or \quad B_h \to A_2 \quad in\ measure.$$

So far everything seems to be parallel to the "gradient problem", but the case when K consists of three matrices turns out to be different. Indeed one can construct a sequence of matrix fields which are divergence free and whose distance from the set K approaches zero. In other words, approximate solutions exist for a suitable choice of $\{A_1, A_2, A_3\}$. The following result clarifies the situation.

LEMMA 1.5. (A. Garroni, V. Nesi, [**18**]). *Given $m \geq n \geq 3$, there exist three pairwise rank-n connected $m \times n$ matrices A_1, A_2, A_3, and there exists a sequence $B_h \in L^\infty(\Omega, \mathbb{M}^{m\times n})$ such that setting $K = \{A_1, A_2, A_3\}$, one has*

$$\text{dist}(B_h, K) \to 0 \qquad \textit{strongly in } L^p(\Omega),\ \forall p \geq 1, \tag{1.3}$$

$$\text{Div} B_h \to 0 \qquad \textit{strongly in } W^{-1,p}(\Omega; \mathbb{R}^m),\ \forall p \geq 1, \tag{1.4}$$

and $B_h \overset{*}{\rightharpoonup} B$ in L^∞, with $B \neq A_i$ for any $i = 1, 2, 3$.

REMARK 1.6. (A. Garroni, V. Nesi, [**18**]). In Lemma 1.5 one can achieve the stronger requirement $\text{Div}B_h = 0$ rather then (1.4), by suitably projecting the fields B_h onto Divergence-free matrix fields. Moreover, using results from [**17**] (see Lemma 2.17 therein), one can achieve the condition that $\{B_h\}$ is also bounded in L^∞.

The explicit formula for A_1, A_2, A_3, can be found in Section 4 (see Remark 1.21).

Next, we state the main theorem of the this chapter, namely a rigidity result for three-valued matrix fields under the assumption of rank-$(n-1)$ disconnectedness.

THEOREM 1.7. *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and let $K = \{A_1, A_2, A_3\} \subset \mathbb{M}^{m\times n}$, with $m \geq n$ and $\text{rank}(A_i - A_j) = n$ for $i \neq j$. If $B : \Omega \to K$ is a measurable function satisfying $\text{Div}B = 0$ in $\mathcal{D}'(\Omega, \mathbb{R}^m)$, then B is constant.*

In this chapter we will mainly deal with the problem of non-existence of exact solutions. In addition to the previous theorem, we prove a rigidity result for a particular class of matrix fields taking an arbitrary number of values. This is the precise statement.

THEOREM 1.8. *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and let $K \subset \mathbb{M}^{m\times n}$ be bounded, with $m \geq n$ and rank$(A_i - A_j) = n$ for every $A_i, A_j \in K$, with $i \neq j$. Suppose that K satisfies the following condition:*
there exist $n-1$ independent vector subspaces $\pi_1, \ldots, \pi_{n-1}$ of co-dimension 1 in $\mathbb{R}^n$, and $n-1$ vector subspaces $\tau_1, \ldots, \tau_{n-1}$ of dimension $n-1$ in $\mathbb{R}^m$, such that

$$A_i : \pi_r \to A_i(\pi_r) \subseteq \tau_r\,, \ \textit{for every } A_i \in K \textit{ and } r = 1, \ldots n-1\,. \tag{1.5}$$

Then every measurable matrix field $B : \Omega \to K$ satisfying $\text{Div}B = 0$ in $\mathcal{D}'(\Omega, \mathbb{R}^m)$ is constant.

An easy corollary of this theorem is that a set of simultaneously diagonalizable rank-n connected matrices is rigid for exact solutions.

We remark that Theorem 1.8, specialized to the case $m = n = 2$, reduces itself to a result which is well-known in the setting of the gradient problem. For the reader's convenience we state it explicitly.

PROPOSITION 1.9. (M. Chlebík, B. Kirchheim, [**13**]). *Assume $K \subset \mathbb{M}^{2\times 2}$ does not contain any rank-one connection and that there are two unit vectors $v, w \in \mathbb{R}^2$ with $w || Av$ for all $A \in K$. If $f \in W^{1,\infty}([0,1]^2, \mathbb{R}^2)$ satisfies $Df \in K$ a.e., then Df is constant.*

In some applications to composite materials, it makes sense to study more general linear differential constraints acting on the matrix field B. From the mathematical point of view, partial results in this direction are contained in [**9**].

The plan of the present chapter is as follows.

In Section 2, we present an algebraic argument which implies that the right dimension to study the problem is $n \times n$ (see Lemma 1.14). Next we remark that the condition of being divergence-free is invariant under any orthogonal change of variables (Remark 1.12). Using this invariance, in order to prove Theorem 1.7 it is enough to consider a very special situation. This kind of argument does not work for an arbitrary number of matrices and one does in fact expect that rigidity fails for a sufficiently large number of them. Yet, under the assumptions of Theorem 1.8, one can prove that rigidity still holds for an arbitrary number of values and actually even for a continuum of them. For the reader's convenience we give, in Lemma 1.11, the Gauss-Green formula for L^∞ fields, which will be the main ingredient in the proof of this result.

In Section 3, we give the proofs of Theorem 1.7 and Theorem 1.8.

The final section departs from the main focus of the present chapter. Indeed, in the spirit of Lemma 1.5, we address the problem of finding approximate solutions to the "three divergence problem". More precisely, we show that the construction used by Garroni and Nesi actually applies to a larger class of sets K. Theorem 1.10 gives a characterization of all such K's, which turn out to be non-rigid for approximate solutions.

THEOREM 1.10. *For every q_1, q_2, $q_3 \in (0,1)$, let $A \in \mathbb{M}^{3\times 3}$ be defined as follows*

$$A = \frac{1}{q_3}\left[\left(1 - \prod_{i=1}^{3}(1-q_i)\right) G^{-1}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} G - q_2(1-q_3)I\right],$$

where $\lambda_1 = 0\,, \lambda_2 = 1/(1-q_1)\,, \lambda_3 = q_2/(q_1+q_2-q_1q_2)$, and G is an arbitrary matrix in $GL(3)$. Then, for every $M \in \mathbb{M}^{3\times 3}$ and $N \in GL(3)$, the set

$$K = \{M\,, N + M\,, NA + M\} \tag{1.6}$$

is non-rigid for approximate solutions.

2. Mathematical background

In this Section we set some notations and a few preliminary results needed in the proof of the main results of Section 3.

Throughout this chapter Ω is an open connected subset of $\mathbb{R}^n$. We denote by $\mathbb{M}^{m\times n}$ the set of the real $m \times n$ matrices; 0 and I will indicate the zero matrix and the identity matrix in $\mathbb{M}^{n\times n}$ respectively. Left-multiplication of A times a vector $v \in \mathbb{R}^n$ is denoted by $A \cdot v$. The symbol $\langle v, w\rangle$ denotes the standard inner product in $\mathbb{R}^n$.

For every measurable subset E of $\mathbb{R}^n$, $|E|$ is the n-dimensional Lebesgue measure of E while, for $s \in \mathbb{R}^+$, $\mathcal{H}^s(E)$ is its s-dimensional Hausdorff measure.
Given a function $f \in L^1(\Omega, \mathbb{R})$, we say that $x \in \Omega$ is a Lebesgue point for f, and that $\lambda(x) \in \mathbb{R}$ is the Lebesgue value of f at x, if

$$\lim_{r\to 0} \fint_{B_r(x)} |f(y) - \lambda(x)|\, dy = 0,$$

where $B_r(x)$ is the open ball of radius r and center x and the symbol $\fint_{B_r(x)}$ stands for $\frac{1}{|B_r(x)|}\int_{B_r(x)}$. This definition extends in the obvious way to vector valued functions.
It is well-known that the set of Lebesgue points for f, which from now on we will denote by $\mathbb{L}(f)$, has full measure in Ω. For every $k \in \mathbb{N}$ and $f \in L^1(\Omega, \mathbb{R}^k)$, we will denote by $\tilde{f}$ a Lebesgue representative of f (*i.e.* $\tilde{f}(x) = \lambda(x)$ for every $x \in \mathbb{L}(f)$), so that $\tilde{f}$ coincides with f a.e. in Ω. For more details we refer the reader to [**14**].

Recall that a vector field $f \in L^\infty(\Omega, \mathbb{R}^n)$ is said to be divergence free if for every $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega \langle f(x), \nabla\varphi(x)\rangle\, dx = 0.$$

For the reader's convenience, we prove a Gauss-Green formula in the particular setting of our problem (much more general results can be found in [**5**]). In the sequel the symbol $\nu(x)$ will denote the outward normal to a given surface at the point x.

LEMMA 1.11. *Let $f \in L^\infty(\Omega, \mathbb{R}^n)$ be a divergence free vector field, and let $U \subset \overline{U} \subset \Omega$ be open with Lipschitz boundary. Suppose that $\mathcal{H}^{n-1}(\mathbb{L}(f) \cap \partial U) = \mathcal{H}^{n-1}(\partial U)$. Then*

$$\int_{\partial U} \langle \tilde{f}(s), \nu(s)\rangle\, d\mathcal{H}^{n-1}(s) = 0\,.$$

PROOF. Consider a sequence $\{\rho_n\}$ of mollifiers and set $f_n := f * \rho_n$. We have

$$\mathrm{div} f_n = \mathrm{div} f * \rho_n = 0.$$

The standard Gauss-Green formula for smooth functions on a Lipschitz domain yields

$$\int_{\partial U} \langle f_n(s), \nu(s)\rangle\, d\mathcal{H}^{n-1}(s) = 0. \tag{1.7}$$

It is easy to see that $f_n(x) \to \tilde{f}(x)$ for all $x \in \partial U \cap \mathbb{L}(f)$. Passing to the limit in (1.7) and using the dominated convergence Theorem, we get

$$\int_{\partial U} \langle \tilde{f}(s), \nu(s)\rangle\, d\mathcal{H}^{n-1}(s) = 0.$$

□

REMARK 1.12. Let $B \in L^\infty(\Omega, \mathbb{M}^{m\times n})$ be a divergence free matrix field and let R be an orthogonal matrix in $\mathbb{M}^{n\times n}$. Using a convolution argument as in Lemma 1.11 and the classical chain rule formula for smooth functions, one can check that for every $C \in \mathbb{M}^{m\times n}$ and $F \in \mathbb{M}^{n\times m}$, the matrix field $\widehat{B} : \{y \in \mathbb{R}^n \mid y = R^T x,\, x \in \Omega\} \to \mathbb{M}^{n\times n}$ defined by

$$\widehat{B}(y) := R^T F\big(B(Ry) + C\big)R \tag{1.8}$$

is divergence free.

The next lemma shows that, given a set $K \subset \mathbb{M}^{m\times n}$, the property of rank-$(n-1)$ disconnectedness is preserved under left multiplication by suitable matrices in $\mathbb{M}^{n\times m}$.

LEMMA 1.13. *Let $K \subset \mathbb{M}^{m\times n}$ be at most countable, with $m > n$ and $\operatorname{rank}(A_i - A_j) = n$ for every $A_i,\ A_j \in K$, with $i \neq j$. Then there exists $F \in \mathbb{M}^{n\times m}$ such that $\operatorname{rank}(FA_i - FA_j) = n$ for every $A_i,\ A_j \in K$ with $i \neq j$.*

PROOF. Since the image of $A_i - A_j$ is a n-dimensional subspace of $\mathbb{R}^m$, we can always find a linear operator $F : \mathbb{R}^m \to \mathbb{R}^n$ such that $\operatorname{Ker}(F(A_i - A_j)) = \operatorname{Ker}(F) \cap \operatorname{Im}(A_i - A_j) = \{0\}$ for every $i \neq j$. Then $\operatorname{rank}(FA_i - FA_j) = n$ for every $A_i,\ A_j \in K$ with $i \neq j$. □

The previous results show that, as long as we consider discrete valued matrix fields, we can always set the problem in the space of square matrices. This is the claim of the next Lemma.

LEMMA 1.14. *Let $K \subset \mathbb{M}^{m\times n}$ be at most countable, with $m > n$ and $\operatorname{rank}(A_i - A_j) = n$ for every $A_i, A_j \in K$, with $i \neq j$. Suppose we are given a divergence free matrix field $B : \Omega \to K$. Then there exists a divergence free matrix field $B' : \Omega \to K'$, where $K' \subset \mathbb{M}^{n\times n}$, $\operatorname{card}(K') = \operatorname{card}(K)$, and $\operatorname{rank}(A'_i - A'_j) = n$ for every $A'_i, A'_j \in K'$, with $i \neq j$.*

PROOF. Apply Lemma 1.13 and use Remark 1.12 with $R = I$ and $C = 0$. □

3. Proofs of the main results

In this section we give the proofs of Theorem 1.7 and Theorem 1.8.

Proof of Theorem 1.7. By Lemma (1.14) it is enough to consider the case $m = n > 2$. Moreover, due to the local character of our problem, we can make any convenient change of variables. Hence, as it is customary in this kind of problems, we begin with some reductions to special cases. We use Remark 1.12 choosing $F = (A_2 - A_1)^{-1}$ and $C = -A_1$. In this way we can assume that $A_1 = 0$ and $A_2 = I$. Moreover, since for any linear operator in $\mathbb{R}^n$ there exists a two-dimensional invariant subspace, we can choose the orthogonal matrix R in (1.8) so that A_3 is of the form

$$A_3 := A = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}. \tag{1.9}$$

and $\operatorname{rank}(A) = \operatorname{rank}(A - I) = n$. By assumptions we can write

$$B = \chi_{E_1} 0 + \chi_{E_2} I + \chi_{E_3} A\,,$$

where E_i are disjoint measurable sets and $E_1 \cup E_2 \cup E_3 = \Omega$. Next remark that, due to (1.9), the first two equations of $\text{Div}B = 0$ only involve derivatives with respect to directions e_1 and e_2. Roughly speaking, the idea is to use this information to conclude that B does not depend on the variables x_1 and x_2. This allows us to make a "projection" of the original problem into a lower dimensional space (actually a section of Ω) and to proceed by induction on the dimension n, using, at the final step, that by Theorem 1.1 and Remark 1.2, rigidity for exact solutions holds for $n = 2$.

Let us proceed with the formal proof. Set $n > 3$ and suppose that rigidity holds in $\mathbb{M}^{(n-1)\times(n-1)}$. We want to prove that B is constant. For every $x \in \Omega$, let Q be a open coordinate cube centered in x and such that $\overline{Q} \subset \Omega$. Without loss of generality we assume that $x = 0$, so that $Q = (-l, l)^n$ for some positive l.

Let $\{\rho_i\}$ be a sequence of mollifiers and set $B_i := \rho_i * B$. For i large enough we have that B_i is well defined in Q, $B_i \in C^\infty(Q, \mathbb{M}^{n\times n})$ and $\text{Div}B_i = 0$ in Q in the classical sense. Then fix $\overline{x} := (\overline{x}_3, \cdots, \overline{x}_n) \in (-l, l)^{n-2}$ such that for $\mathcal{H}^2$-a.e. $(x_1, x_2) \in (-l, l)^2$, $(x_1, x_2, \overline{x}) \in \mathbb{L}(B)$. By Fubini's Theorem, this is possible for $\mathcal{H}^{n-2}$-a.e. $\overline{x} \in (-l, l)^{n-2}$, since, as already remarked, $\mathbb{L}(B)$ has full measure in Ω. Now consider the field $\underline{B} : (-l, l)^2 \to \mathbb{M}^{2\times 2}$ defined by

$$\underline{B}(x_1, x_2) := \begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} \\ \tilde{b}_{21} & \tilde{b}_{22} \end{pmatrix} (x_1, x_2, \overline{x}), \tag{1.10}$$

where $\tilde{b}_{kl}$ is a Lebesgue representative of the kl-entry of the matrix B. Then let b^i_{kl} be the kl-entry of B_i and set

$$\underline{B_i}(x_1, x_2) := \begin{pmatrix} b^i_{11} & b^i_{12} \\ b^i_{21} & b^i_{22} \end{pmatrix} (x_1, x_2, \overline{x}). \tag{1.11}$$

Since $\underline{B}_i \in C^\infty((-l, l)^2, \mathbb{M}^{2\times 2})$ and Div $\underline{B}_i = 0$ in $(-l, l)^2$, we have

$$\int_{(-l,l)^2} \underline{B}_i \cdot \nabla\varphi \, dx_1 dx_2 = 0 \qquad \forall \varphi \in C_0^\infty(-l, l)^2. \tag{1.12}$$

Moreover $\underline{B}_i(x_1, x_2)$ converges to $\underline{B}(x_1, x_2)$ at every $(x_1, x_2) \in (-l, l)^2$ such that $(x_1, x_2, \overline{x}) \in \mathbb{L}(B)$. Then passing to the limit for $i \to \infty$ in (1.12) and using the dominated convergence Theorem we get

$$\int_{(-l,l)^2} \underline{B}(x_1, x_2) \cdot \nabla\varphi(x_1, x_2) \, dx_1 dx_2 = 0 \qquad \forall \varphi \in C_0^\infty(-l, l)^2. \tag{1.13}$$

By Theorem 1.1 we have that $\underline{B}$ is constant and hence $\widetilde{B}$ is constant on the section $(x_1, x_2, \overline{x})$. Since this is true for $\mathcal{H}^{n-2}$-a.e. $\overline{x} \in (-l, l)^{n-2}$, we deduce that $\widetilde{B}$ does not depend on (x_1, x_2) in Q. In particular we have

$$\frac{\partial \chi_{E_i}}{\partial x_1} = 0 \qquad \text{in } \mathcal{D}'(Q). \tag{1.14}$$

From (1.14), it is easy to see that there exist three measurable sets E_1', E_2', E_3' in $(-l,l)^{n-1}$ such that

$$E_i \cap Q = (-l,l) \times E_i' \quad \text{a.e.} \tag{1.15}$$

Now call $0'$, I', A' the $n \times (n-1)$-minors of the matrices 0, I, A respectively, obtained by eliminating the first column of each matrix. Notice that $\mathrm{rank}(A') = \mathrm{rank}(A' - I') = n-1$. Then set

$$B' := 0'\chi_{E_1'} + I'\chi_{E_2'} + A'\chi_{E_3'} \,.$$

Let us emphasize that B' lives in a space of dimension $n-1$. Combining (1.14) with the equation $\mathrm{Div}B = 0$ in $\mathcal{D}'(Q,\mathbb{R}^n)$, one concludes that B' satisfies

$$\mathrm{Div}B' = 0 \quad \text{in } \mathcal{D}'((-l,l)^{n-1},\mathbb{R}^n)\,. \tag{1.16}$$

By Lemma 1.13, there exists $F \in \mathbb{M}^{(n-1)\times n}$ such that

$$\mathrm{rank}(FI') = \mathrm{rank}(FA' - FI') = n-1\,. \tag{1.17}$$

Then set

$$\begin{aligned} A_2' &:= FI'\,, \\ A_3' &:= FA'\,, \\ B_{(n-1)} &:= 0\chi_{E_1'} + A_2'\chi_{E_2'} + A_3'\chi_{E_3'}\,. \end{aligned}$$

By (1.16) and (1.17), it follows that $B_{(n-1)}$ is an exact solution of the problem

$$\mathrm{Div}B_{(n-1)} = 0 \quad \text{in } \mathcal{D}'((-l,l)^{n-1},\mathbb{R}^{n-1})\,, \quad B_{(n-1)} \in \{0, A_2', A_3'\} \subset \mathbb{M}^{(n-1)\times(n-1)}\,. \tag{1.18}$$

By the inductive assumption, problem (1.18) is rigid. Then $|(-l,l)^{n-1}| = |E_i'|$ for some i and hence, by (1.15), $|Q| = |E_i \cap Q|$. By the arbitrariness of Q we conclude that B is constant. □

REMARK 1.15. Before giving the proof of Theorem 1.8, we make some considerations about its assumption (1.5). We want to show that, even in this case, we can reduce to a very special situation. Indeed, condition (1.5) implies that $\mathrm{Im}(A_i - A_j) = \mathrm{Im}(A_r - A_s)$ for every $A_i, A_j, A_r, A_s \in K$ with $i \neq j$ and $r \neq s$. We can thus apply the same argument as in Lemma 1.14, although no requirement is made on the cardinality of the set K. More precisely, we fix any two of the matrices in K, say A_1 and A_2, and choose a matrix $F \in \mathbb{M}^{n\times m}$ such that $F(A_1 - A_2) = I$. For such an F, we have that $FA_i \in \mathbb{M}^{n\times n}$ and $\mathrm{rank}(FA_i - FA_j) = n$ for every $A_i, A_j \in K$. Moreover, the subspaces $\pi_1, \ldots, \pi_{n-1}$ are stable under the action of every FA_i or, equivalently, the matrices FA_i have $n-1$ common eigenvectors.

We are now ready to prove Theorem 1.8.

Proof of Theorem 1.8. By Remark 1.12 and Remark 1.15, we can assume that $K \subset \mathbb{M}^{n\times n}$ and $A_i(\pi_r) \subseteq \pi_r$ for every $A_i \in K$ and $r = 1, \ldots, n-1$. For every r, let v_r be the unit vector orthogonal to π_r. We want to prove that B does not depend on any of the directions v_r, which are independent by assumptions. Then B would only depend on one direction, but the condition of being divergence free will imply that B is constant. We will only check the statement for one vector v_r. Choose any of the vectors v_r and a real number q, and by contradiction assume that there exist two points P, $Q \in \mathbb{L}(B)$, with $P - Q = qv_r$ and such that $\widetilde{B}(P) \neq \widetilde{B}(Q)$; for instance suppose that $\widetilde{B}(P) = A_1$ and $\widetilde{B}(Q) = A_2$.

Let us briefly digress to explain the idea of the proof in an informal way. We want to apply the Gauss-Green formula in a small cylinder with axis parallel to v_r and bases centered in P and Q respectively. Computing the flux of B through the boundary of the cylinder we will check that its contribution through the bases can never compensate the contribution through the lateral boundary, the former being a vector parallel to $(A_1 - A_2)v_r$, the latter belonging to the subspace π_r. This idea, however, will require some technical efforts, as B may be non-constant on the bases. What we do, in fact, is to consider a sequence of cylinders with vanishing radii. The contradiction will arise for a sufficiently small radius.

Now we continue the formal proof. To simplify the notations we will assume that $P = (0, \ldots, 0)$ and $v_r = e_n$, so that $Q = (0, \ldots, 0, q)$.
Let $\rho(x) := (x_1^2 + \cdots + x_{n-1}^2)^{1/2}$ and, for every $r \in \mathbb{R}^+$, set

$$C(r) := \{x \in \mathbb{R}^n : \quad 0 \leq \rho(x) \leq r,\ 0 \leq x_n \leq q\}.$$

Since $\widetilde{B}(P) = A_1$ and $\widetilde{B}(Q) = A_2$, we can find a cylinder on whose bases the mean value of B is A_1' and A_2' respectively, with $|A_1 - A_1'|$ and $|A_2 - A_2'|$ arbitrarily small (this can be checked by using Fubini's theorem). More precisely, for every given $\delta > 0$, we can find a radius $r_\delta \in \mathbb{R}^+$ and a vector $w_\delta \in \mathbb{R}^n$ such that, setting

$$C_\delta := C(r_\delta) + w_\delta$$

and denoting by D_δ^1 and D_δ^2 the bases of C_δ and by L_δ its lateral boundary (see Figure 1), the following hold:

$$\begin{aligned} & C_\delta \subset \Omega\,, \\ & \mathcal{H}^{n-1}(\partial C_\delta \cap \mathbb{L}(B)) = \mathcal{H}^{n-1}(\partial C_\delta)\,, \\ & \fint_{D_\delta^1} |\widetilde{B}(s) - A_1| d\mathcal{H}^{n-1}(s) + \fint_{D_\delta^2} |\widetilde{B}(s) - A_2| d\mathcal{H}^{n-1}(s) < \delta\,. \end{aligned} \tag{1.19}$$

By Lemma 1.11 we get

$$\int_{\partial C_\delta} \widetilde{B}(s) \cdot \nu(s)\, d\mathcal{H}^{n-1}(s) = 0\,. \tag{1.20}$$

We write (1.20) as the sum of three contributions as follows

$$\begin{aligned} & \int_{\partial C_\delta} \widetilde{B}(s) \cdot \nu(s)\, d\mathcal{H}^{n-1}(s) = \int_{D_\delta^1} \widetilde{B}(s) \cdot (-e_n)\, d\mathcal{H}^{n-1}(s) + \\ & \int_{D_\delta^2} \widetilde{B}(s) \cdot e_n\, d\mathcal{H}^{n-1}(s) + \int_{L_\delta} \widetilde{B}(s) \cdot \nu(s)\, d\mathcal{H}^{n-1}(s)\,, \end{aligned} \tag{1.21}$$

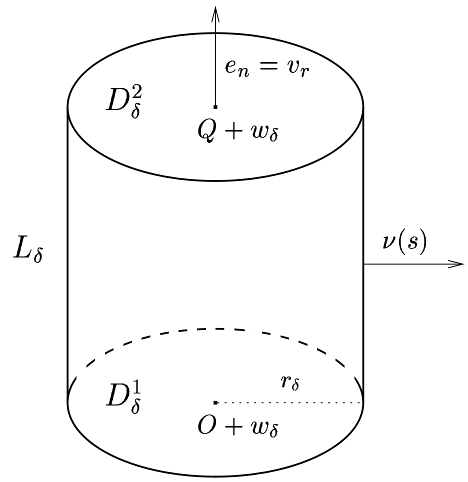

FIGURE 1. The cylinder C_δ

where $\nu(s) = \frac{1}{\rho(s)}(s_1, \ldots, s_{n-1}, 0)$ on L_δ. On the other hand we have

$$\int_{D_\delta^1} \widetilde{B}(s) \cdot (-e_n) \, d\mathcal{H}^{n-1}(s) = \mathcal{H}^{n-1}(D_\delta^1) \, A_1^\delta \cdot (-e_n) \, ,$$

$$\int_{D_\delta^2} \widetilde{B}(s) \cdot e_n \, d\mathcal{H}^{n-1}(s) = \mathcal{H}^{n-1}(D_\delta^1) \, A_2^\delta \cdot e_n \, , \tag{1.22}$$

where A_1^δ, $A_2^\delta \in \mathbb{M}^{n \times n}$ and, by (1.19), are such that $|A_1^\delta - A_1| + |A_2^\delta - A_2| < \delta$. Then we set

$$u_\delta := \int_{L_\delta} \widetilde{B}(s) \cdot \nu(s) \, d\mathcal{H}^{n-1}(s) \, .$$

Dividing the right hand side in (1.21) by $\mathcal{H}^{n-1}(D_\delta^1)$ and using (1.20) and (1.22), we obtain

$$(A_2^\delta - A_1^\delta) e_n + \frac{u_\delta}{\mathcal{H}^{n-1}(D_\delta^1)} = 0 \, . \tag{1.23}$$

Now recall that e_n is orthogonal to π_r, $\operatorname{rank}(A_2 - A_1) = n$ and $(A_2 - A_1)(\pi_r) = \pi_r$. It follows that $(A_2 - A_1) \cdot e_n \notin \pi_r$, and hence $(A_2^\delta - A_1^\delta) \cdot e_n \notin \pi_r$ for δ small enough. On the other hand we have that $u_\delta \in \pi_r$, the subspace π_r being preserved under the action of every $A_i \in K$. Then, for sufficiently small δ, (1.23) gives a contradiction. $\square$

REMARK 1.16. The assumption of boundedness required for the set K, in Theorem 1.8, can be actually removed. The previous proof, indeed, can be adapted to L^1_{loc} matrix fields by suitable modifications.

REMARK 1.17. It is worth noticing here that Theorem 1.7 is much stronger than Theorem 1.8 when applied to a three-element set K. Indeed, if we consider the case when $K = \{0, I, A\}$, in order to apply Theorem 1.8 we need the matrix A to have at least $n - 1$ eigenvectors in common with I.

4. Approximate solutions

In this section we give a refinement of Lemma 1.5 given in [**18**], where the authors exhibit an example of a set which is non-rigid for approximate solutions. Their construction is set in $\mathbb{M}^{3\times 3}$, but it can be extended to the case $m, n \geq 3$ by slight modifications. It actually provides an algorithm (similar to that of Tartar's for the gradients, [**40**]), which allows us to find approximate solutions for a large class of sets K. Similar constructions can be found in the works of several authors, see [**6**], [**12**], [**26**], [**28**], [**32**]. The particular case here resembles the construction in [**26**]. The key point is the following lemma.

LEMMA 1.18. (A. Garroni, V. Nesi, [**18**]). *Let $K = \{A_1, A_2, A_3\} \subset \mathbb{M}^{3\times 3}$ be a set of pairwise rank-3 disconnected matrices. If there exist three matrices $S_1, S_2, S_3 \in \mathbb{M}^{3\times 3}$ which satisfy the conditions*

$$\det(A_i - S_i) = 0, \quad \text{for } i = 1,2,3\,, \tag{1.24}$$

$$S_i = q_{i-1}A_{i-1} + (1-q_{i-1})S_{i-1} \qquad \bmod 3\,, \quad \text{for } i = 1,2,3\,, \tag{1.25}$$

for some $q_i \in (0,1)$, then the set K is non-rigid for approximate solutions.

In the next lemma, which will be used in the proof of Lemma 1.18, we denote by $Y_1, \dots, Y_n$ n copies of the unit cell $Y = (0,1)^3$, and by $C^\infty_\sharp(Y_1 \times \cdots \times Y_n)$ all smooth functions $\varphi(y_1, \dots, y_n)$ that are periodic with respect to each variable y_k, $k = 1, \dots, n$. Moreover, given $\varepsilon_1, \dots \varepsilon_n$, n positive functions of $\varepsilon > 0$ which converge to 0 as ε does, we will say that the scales $\varepsilon_1, \dots \varepsilon_n$ are well-separated if they satisfy

$$\lim_{\varepsilon\to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0 \quad \forall\, k \in \{1, \dots, n-1\}\,.$$

For any $\varphi \in \mathcal{D}(\Omega; C^\infty_\sharp(Y_1 \times \cdots \times Y_n))$, the oscillating function $[\varphi]_\varepsilon$ is defined by

$$[\varphi]_\varepsilon(x) = \varphi\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}\right)\,.$$

LEMMA 1.19. (G. Allaire, M. Briane, [**1**]). *Let $\varphi \in \mathcal{D}(\Omega; C^\infty_\sharp(Y_1 \times \cdots \times Y_n))$ be a function such that for $k \in \{1, \dots, n\}$ one has*

$$\int_{Y_k} \cdots \int_{Y_n} \varphi \, dy_k \dots dy_n = 0$$

Assume that the scales $\varepsilon_1, \dots \varepsilon_n$ are well separated; then $\dfrac{1}{\varepsilon_k}[\varphi]_\varepsilon$ is bounded in $H^{-1}(\Omega)$; in particular there exists $C > 0$ such that

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon_k} ||[\varphi]_\varepsilon||_{H^{-1}} \leq C \sup_{\Omega\times Y_1\times\cdots\times Y_n} |\varphi|\,.$$

Proof of Lemma 1.18. We will show that there exists a sequence $\{B_h\} \in L^\infty(Y, \mathbb{M}^{3\times 3})$ which satisfies

$$\operatorname{dist}(B_h, K) \to 0 \qquad \text{in measure,} \tag{1.26}$$

$$\operatorname{Div} B_h \to 0 \qquad \text{strongly in } H^{-1}((0,1)^3; \mathbb{R}^3), \text{ and} \tag{1.27}$$

$$B_h \overset{*}{\rightharpoonup} S_2 \quad \text{in } L^\infty. \tag{1.28}$$

Roughly speaking the idea is to construct, for any given ε and $N \in \mathbb{N}$, a matrix-valued function B_ε^N making N laminations with well-separated scales of oscillation $\varepsilon_j = \alpha_j(\varepsilon)$, with $\alpha_j(\varepsilon) \to 0$ as $\varepsilon \to 0$. In order to use Lemma 1.19, we have to care about some regularity conditions and therefore we will define smooth fields B_ε^N which coincide with piecewise-constant functions up to a set of small measure.

Let us proceed with the formal proof. By the assumption (1.24) there exists $v_i \in \mathbb{R}^3$ such that $v_i \in \operatorname{Ker}(A_i - S_i)$ for every $i = 1,2,3$. Let us assume for simplicity that $v_i = e_i$ for every $i = 1,2,3$. For $\delta < \min\{q_1, q_2, q_3\}$ we define smooth functions $\eta_1^\delta, \eta_2^\delta, \eta_3^\delta$ in the interval $[0,1]$ such that

$$\begin{aligned}
&0 \le \eta_i^\delta(t) \le 1 \quad && \forall\, t \in [0,1]\,,\\
&\eta_i^\delta(t) = 1 && \text{if } t \in [0, q_i - \delta]\,,\\
&\eta_i^\delta(t) = 0 && \text{if } t \in [q_i + \delta, 1]\,,\\
&|\eta_i^\delta{}'(t)| \le \frac{c}{\delta} && \forall\, t \in [0,1]\,,\\
&\int_0^1 \eta_i^\delta(t)\, dt = q_i\,.
\end{aligned}$$

Here the superscript $'$ denotes differentiation w.r.t. the variable t. For every $x \in (0,1)^3$, let $\alpha_i(x) = \eta_i^\delta(x_i)$ for $i = 1,2,3$. We begin our construction by laminating A_1 and S_1 "using the law α_1" with oscillations of size ε_1:

$$B_\varepsilon^1 = A_1 \alpha_1\left(\frac{x}{\varepsilon_1}\right) + S_1\left(1 - \alpha_1\left(\frac{x}{\varepsilon_1}\right)\right).$$

Clearly $|\{B_\varepsilon^1 \notin K\}| < q_1 + c_0\delta$, for some constant $c_0 > 0$. Next we replace S_1 by a laminate of A_3 and S_3 in direction e_3 with oscillations at the smaller scale ε_2:

$$B_\varepsilon^2 = A_1\alpha_1\left(\frac{x}{\varepsilon_1}\right) + \left(1 - \alpha_1\left(\frac{x}{\varepsilon_1}\right)\right)\left[A_3\alpha_3\left(\frac{x}{\varepsilon_2}\right) + S_3\left(1 - \alpha_3\left(\frac{x}{\varepsilon_2}\right)\right)\right],$$

and we have that $|\{B_\varepsilon^2 \notin K\}| < q_1 q_3 + c_0\delta$. We continue replacing S_3 by a laminate of A_2 and S_2 in direction e_2 with oscillations at the smaller scale ε_3 obtaining

$$\begin{aligned}
B_\varepsilon^3 = {}& A_1\alpha_1\left(\frac{x}{\varepsilon_1}\right) + \left(1 - \alpha_1\left(\frac{x}{\varepsilon_1}\right)\right) \times \\
&\times \left\{A_3\alpha_3\left(\frac{x}{\varepsilon_2}\right) + \left(1 - \alpha_3\left(\frac{x}{\varepsilon_2}\right)\right)\left[A_2\alpha_2\left(\frac{x}{\varepsilon_3}\right) + S_2\left(1 - \alpha_2\left(\frac{x}{\varepsilon_3}\right)\right)\right]\right\}.
\end{aligned}$$

At this point the only set where $B^3_\varepsilon \notin K$ is the region occupied by S_2 and thin layers of order δ. In particular $|\{B^3_\varepsilon \notin K\}| < q_1 q_3 q_2 + c_0\delta$.
Let us compute now the divergence of B^3_ε:

$$\begin{aligned}\operatorname{Div} B^3_\varepsilon = & \frac{1}{\varepsilon_1}\eta_1^{\delta\prime}\left(\frac{x_1}{\varepsilon_1}\right) \times \\ & \times \left\{ A_1 - A_3\alpha_3\left(\frac{x}{\varepsilon_2}\right) - \left(1-\alpha_3\left(\frac{x}{\varepsilon_2}\right)\right)\left[A_2\alpha_2\left(\frac{x}{\varepsilon_3}\right) + S_2\left(1-\alpha_2\left(\frac{x}{\varepsilon_3}\right)\right)\right]\right\} e_1 + \\ & + \left(1-\alpha_1\left(\frac{x}{\varepsilon_1}\right)\right)\left\{\frac{1}{\varepsilon_2}\eta_3^{\delta\prime}\left(\frac{x_3}{\varepsilon_2}\right)\left[A_3 - A_2\alpha_2\left(\frac{x}{\varepsilon_3}\right) - S_2\left(1-\alpha_2\left(\frac{x}{\varepsilon_3}\right)\right)\right]\right\} e_3\,,\end{aligned}$$

where we have used the fact that $\operatorname{Div}\left[A_2\alpha_2\left(\frac{x}{\varepsilon_3}\right) + S_2\left(1-\alpha_2\left(\frac{x}{\varepsilon_3}\right)\right)\right] = 0$.
We want to apply Lemma 1.19 to each of the two terms of $\operatorname{Div} B^3_\varepsilon$. To this end we set

$$\begin{aligned}&\phi^{3,1}(y_1,y_2,y_3) := \\ &\qquad \eta_1^{\delta\prime}(y_1) \times \{A_1 - A_3\alpha_3(y_2) - (1-\alpha_3(y_2))\left[A_2\alpha_2(y_3) + S_2(1-\alpha_2(y_3))\right]\} e_1, \\ &\phi^{3,2}(y_1,y_2,y_3) := (1-\alpha_1(y_1))\left\{\eta_3^{\delta\prime}(y_2)\left[A_3 - A_2\alpha_2(y_3) - S_2(1-\alpha_2(y_3))\right]\right\} e_3,\end{aligned}$$

so that

$$\operatorname{Div} B^3_\varepsilon = \frac{1}{\varepsilon_1}\phi^{3,1}\left(\frac{x}{\varepsilon_1},\frac{x}{\varepsilon_2},\frac{x}{\varepsilon_3}\right) + \frac{1}{\varepsilon_2}\phi^{3,2}\left(\frac{x}{\varepsilon_1},\frac{x}{\varepsilon_2},\frac{x}{\varepsilon_3}\right).$$

We observe that

$$\int_{Y_2}\int_{Y_3}\phi^{3,1}\,dy_3dy_2 = 0\,, \qquad \int_{Y_3}\phi^{3,2}\,dy_3 = 0\,.$$

Then by Lemma 1.19 there exists a positive constant C_1 such that

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon_2}||[\phi^{3,1}]||_{H^{-1}} \le C_1 \sup_{Y_1\times Y_3\times Y_3}|\phi^{3,1}|\,,$$

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon_3}||[\phi^{3,2}]||_{H^{-1}} \le C_1 \sup_{Y_1\times Y_3\times Y_3}|\phi^{3,2}|\,.$$

Therefore, choosing $\varepsilon_{k+1}/\varepsilon_k < \varepsilon$, we get

$$||\operatorname{Div} B^3_\varepsilon||_{H^{-1}} \le C_1 M\varepsilon\frac{c}{\delta} + c_1(\varepsilon)\,,$$

where $c_1(\varepsilon)\to 0$ as $\varepsilon\to 0$ and M is a positive constant which only depends on the norm of the matrices A_i and S_i, $i=1,2,3$. If we did not care about the condition (1.26), then it would be enough at this point to let ε tend to 0. In order to obtain a sequence that satisfies the condition (1.26), we have to use an infinite-rank sequential lamination. This can be done iterating the procedure described above. After N iterations the field B^{3N}_ε satisfies

$$|\{B^{3N}_\varepsilon \notin K\}| < (q_1q_2q_3)^N + c_0\delta$$

and

$$||\operatorname{Div} B^{3N}_\varepsilon||_{H^{-1}} \le C_N MN\varepsilon\frac{c}{\delta} + c_N(\varepsilon)\,.$$

where $c_N(\varepsilon) \to 0$ as $\varepsilon \to 0$. In order to obtain a sequence satisfying simultaneously (1.26) and (1.27) one needs a fine tuning of the parameter ε in terms of the number of the iterations N: we choose $\delta = \sqrt{\varepsilon}$ and ε such that $C_N N\sqrt{\varepsilon} + c_N(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $N \to +\infty$.

Finally, it easily seen that the sequence constructed in this way converges weakly to S_2. □

Theorem 1.10, given in Section 1, characterizes all possible triples $\{A_1, A_2, A_3\}$ satisfying the assumptions of Lemma 1.18, and it is a corollary of the following proposition.

PROPOSITION 1.20. *Let q_1, q_2, $q_3 \in (0,1)$ be given. Let $A_1 = 0$ and $A_2 = I$ in $\mathbb{M}^{3\times3}$. Then there exist $S_1, S_2, S_3, A_3 \in \mathbb{M}^{3\times3}$ satisfying conditions* (1.24) *and* (1.25) *of* Lemma 1.18, *if and only if A_3 is of the form*

$$A_3 = \frac{1}{q_3}\left[\left(1 - \prod_{i=1}^{3}(1-q_i)\right) G^{-1}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} G - q_2(1-q_3)I\right], \tag{1.29}$$

where G is an arbitrary matrix in $GL(3)$ and the λ_i's are defined as follows

$$\lambda_1 = 0\,, \quad \lambda_2 = \frac{1}{1-q_1}\,, \quad \lambda_3 = \frac{q_2}{q_1 + q_2 - q_1 q_2}\,.$$

PROOF. We rewrite (1.25) more explicitly:

$$\begin{aligned} &S_2 = q_1 A_1 + (1-q_1)S_1 = (1-q_1)S_1, \\ &S_3 = q_2 A_2 + (1-q_2)S_2 = q_2 I + (1-q_2)(1-q_1)S_1, \\ &S_1 = q_3 A_3 + (1-q_3)S_3 = q_3 A_3 + q_2(1-q_3)I + (1-q_1)(1-q_2)(1-q_3)S_1\,. \end{aligned} \tag{1.30}$$

Now let λ_i be the eigenvalues of S_1, then by (1.24) and (1.30) we get

$$\det(A_1 - S_1) = 0 \Longleftrightarrow \det(-S_1) = 0 \Longleftrightarrow \lambda_1 = 0\,,$$
$$\det(A_2 - S_2) = 0 \Longleftrightarrow \det(I - (1-q_1)S_1) = 0 \Longleftrightarrow \lambda_2 = \frac{1}{1-q_1}\,.$$

Moreover one can check that

$$\det(A_3 - S_3) = 0 \Longleftrightarrow \det[(q_1 + q_2 - q_1 q_2)S_1 - q_2 I] = 0 \Longleftrightarrow \lambda_3 = \frac{q_2}{q_1 + q_2 - q_1 q_2}\,.$$

Note that the λ_i's are all distinct, since $q_1, q_2, q_3 \in (0,1)$. Therefore the matrix S_1 is diagonalizable. Hence for any $S_1 \in \{G^{-1}\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)G,\ G \in GL(3)\}$, the matrices A_3, S_2, S_3 are uniquely determined by (1.30). In particular A_3 is of the form (1.29). Conversely, for any A_3 of the form (1.29), the matrices S_1, S_2, S_3 are uniquely determined and conditions (1.24) and (1.25) are satisfied. □

Proof of Theorem 1.10. Let $A, M, N \in \mathbb{M}^{3\times3}$ satisfy the assumptions of Theorem 1.10. Without loss of generality we can assume that $M = 0$ and $N = I$. Notice that A is of the form (1.29). Then, by Proposition 1.20 and Lemma 1.18, the set $K = \{0, I, A\}$ is non-rigid for approximate solutions. □

REMARK 1.21. If we choose $G = I$ in (2.36) the set K reduces itself to that given in [**18**], that is

$$A_3 = \operatorname{diag}\left(\frac{-q_2(1-q_3)}{q_3}, \frac{q_1+q_3-q_1q_3}{q_3(1-q_1)}, \frac{q_2}{q_1+q_2-q_1q_2}\right),$$
$$S_1 = \operatorname{diag}\left(0, \frac{1}{1-q_1}, \frac{q_2}{q_1+q_2-q_1q_2}\right),$$
$$S_2 = \operatorname{diag}\left(0, 1, \frac{q_2(1-q_1)}{q_1+q_2-q_1q_2}\right),$$
$$S_3 = \operatorname{diag}\left(q_2, 1, \frac{q_2}{q_1+q_2-q_1q_2}\right).$$

5. Link with bounds on effective moduli: a brief discussion

In this section we will give some examples which show the strict connection between the issues discussed so far and the problem of bounding the effective moduli for composites. The idea is that many problems of bounds can be reduced to the solvability of differential inclusions of solenoidal type.

Let us start with an example of G-closure problem in conductivity. Let $\Omega \subset \mathbb{R}^3$ be an open bounded and simply connected set and let $A_1, A_2, A_3 \in \mathbb{M}^{3\times 3}$ be symmetric positive definite matrices. Assume we are given three conductors with conductivities A_1, A_2, A_3 and consider a composite, filling the region Ω, made of these three materials mixed in "volume fractions" $\theta_1, \theta_2, \theta_3 \in (0,1)$, with $\theta_1 + \theta_2 + \theta_3 = 1$. By composite we mean a conducting material with conductivity at the point $x \in \Omega$ given by

$$\sigma(x) = A_1\chi_1(x) + A_2\chi_2(x) + A_3\chi_3(x);$$

here χ_i is the characteristic function of the set occupied by the material A_i, for $i = 1,2,3$. The function χ_i is assumed to be measurable and the numbers

$$\fint_\Omega \chi_i = \theta_i, \quad \forall\, i = 1,2,3,$$

are given. One of the basic task is to find, for any given "volume fractions" θ_1, θ_2, θ_3, the highest "overall conductivity"

$$G(F) := \sup_{\fint \chi_i = \theta_i} \left(\inf_{U - Fx \in W_0^{1,2}} \fint_\Omega \operatorname{tr}\left(DU(x)\sigma(x)DU^T(x)\right) dx \right). \tag{1.31}$$

The restriction on the "volume fractions" may be interpreted as the "cost" of using, say, the "most expensive phase" in the composite. For problem (1.31) one has the elementary bound

$$G(F) \le \sum_{i=1}^{3} \theta_i \operatorname{tr}(FA_iF^T), \tag{1.32}$$

which is obtained taking $U = Fx$ as a test field. The basic question is when this bound is attainable or at least optimal. We will say that the bound is attainable if there exist

χ_1, χ_2, χ_3 for which the equality holds in (1.32) and the supremum in (1.31) is achieved. We say that the bound is optimal if the inequality (1.32) cannot be improved. Now let us assume that the bound (1.32) is attainable. Then, using the quadratic character of the energy, one can show that there exist three characteristic functions $\bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3$ such that the corresponding minimum problem in (1.31) admits the affine function $U = Fx$ as minimizer. Using the Euler-Lagrange equation associated to the functional one gets

$$\operatorname{Div}\left(\sum_{i=1}^{3} A_i F^T \chi_i\right) = 0\,.$$

If the set $K = \{A_1F^T, A_2F^T, A_3F^T\}$ does not contain any rank-2 connection, then applying Theorem 1.7 yields a contradiction. It follows that in this case the bound (1.32) cannot be attainable. Similarly, assuming that (1.32) is optimal rather than attainable would imply the existence of approximate solutions for the set $K = \{A_1F^T, A_2F^T, A_3F^T\}$. In contrast, if the set K is of the type (1.6), then Theorem 1.10 implies the optimality of the bound (1.32) at least for certain values of the volume fractions θ_i. Assume for example that $F = I$ and choose any set $K = \{E_1, E_2, E_3\}$ of symmetric matrices in the class defined by (1.6). Then let $C \in \mathbb{M}^{3\times 3}$ be symmetric and such that $E_i + C$ is positive definite for every $i = 1, 2, 3$ and let A_i be defined as follows

$$A_i = E_i + C \quad \forall\, i = 1, 2, 3\,.$$

For such a choice of the matrices A_i the bound (1.32) is optimal at the point $F = I$. The characterization of the admissible volume fractions, *i.e.* those for which the trivial upper bound is optimal, will be studied in Chapter 2.

Other examples of less elementary nature are contained in [**18**].

CHAPTER 2

Relaxation and H-measures

1. An informal presentation of the problem

In the present chapter we address two apparently rather different issues which, however, will turn out to be very closely related. First we consider the problem of characterizing quasiconvex hulls for three "solenoidal wells" in dimension three when the wells are pairwise "incompatible". Second that of characterizing extremal three-point H-measures for three-phase mixtures of characteristic functions in dimension three. Let us explain this in more detail. The task is to characterize

$$\forall \eta \in \mathbb{M}^{3\times 3} \quad \mathcal{B}(\eta) = \inf_{\chi_i} \inf_{⨍_Q B=0} \frac{1}{2} ⨍_Q \left| B(x) + \eta - \sum_{i=1}^{3} \chi_i A_i \right|^2 dx \,. \tag{2.1}$$

In the context of gradient fields, $B = DU$ or $B = e(U)$ and the original motivations (and the most relevant known results) for our study have been recalled in the introduction. The most interesting case is when the A_i's are incompatible, *i.e.* $\mathrm{rank}(A_i - A_j) = 3$ for $i \neq j$. It is clear from the form of (2.1) that Div-free matrix fields taking values in the set $K = \{A_1, A_2, A_3\}$ will be important in the understanding of whether the infimum in (2.1) may be zero or not. We use the result of the first part of the thesis to establish the optimality of the lower bound $\mathcal{B}(\eta) = 0$ under appropriate circumstances on the set K. On the other hand, due to the work of Kohn [**24**], Allaire & Kohn [**3**] and Smyshlyaev & Willis [**29**] it is well-known, in the gradient context at least, that the calculation of $\mathcal{B}(\eta)$ can be done via Fourier analysis and it reduces itself to a minimization in the space of the so-called H-measures. This introduces the second topic of this part of the thesis, namely the characterization of a special class of H-measures. Indeed, we establish new sharp bounds on the set of possible H-measures arising as sums of three Dirac masses. Establishing bounds for H-measures requires two distinct steps. First, one needs to prove that a certain set of Borel measures can be realized by H-measures (in this case H-measures supported on three Dirac masses). This part is achieved via a relatively standard procedure namely "infinite-rank" lamination. All the essential ideas, in the Div-free context, have been introduced in the Garroni-Nesi construction [**18**] which in turn relies on the reiterated homogenization methods and ultimately on results by Allaire & Briane [**1**]. This part could also be realized using the construction of Smyshlyaev & Willis [**29**]. The second part of the program addresses the question of whether the set of measures constructed with the previous algorithm is the best possible. In other words the issue is whether a bound from the "outside" can be constructed for such the set under study. As already remarked in the introduction, when $N = 2$ Kohn solved the problem completely showing that the only restrictions to be taken into account are those listed in

(0.8)-(0.9). When $N = 3$ the situation is drastically different. Additional information is required. In dimension three we find that there are two regimes; the first, that we call Smyshlyaev-Willis regime and the second, that we call the complementary regime. In both cases we find bounds which were not previously known and that are sharp in the sense that they are achieved by some H-measures. The main difference is that, in the first regime, we solve completely the problem finding *all* possible restrictions and hence characterizing the H-measures. In the second regime our results provide a progress in bounding the set of H-measures but do not solve completely the problem. The technique we use has an independent interest. It is similar in spirit to ideas of Tartar [**40**], Sverak [**31**] and Faraco & Zhong [**15**]. However our approach is yet different. We find a suitable modification of a function introduced by Tartar [**36**]. The latter was originally introduced in the study of composites in homogenization. It is a rank-2 convex function which is quadratic and therefore quasiconvex on the space of Div-free fields. Our modification resembles Sverak's function because it behaves like his det^+ function on a certain two-dimensional set. However it also resembles the functions introduced by Faraco & Zhong because it has at most *quadratic* growth in dimension three. This new function will effectively act as a new S-quasiconvex function and provide the desired bound.

It is worth remarking that attempts to find non trivial information on H-measures, based on more standard bounding techniques arising in the context of calculus of variations and composites and exploiting continuity of minors of Jacobians or similar arguments, are bound to fail. Our work seems to really introduce information which is independent on the previous techniques.

The structure of the chapter is as follows. Section 2 reviews the definition and the basic properties of "multiwell" H-measures. The reformulation of the relaxation problem in the language of minimization with respect to H-measures is discussed in Section 3 and follows [**24**] and [**29**]. Section 4 specializes the latter to the three-divergence problem and Section 5 reviews the results from [**18**] and [**27**] and establishes some other technical results. The main results (Theorems 2.15 and 2.18 and the corollaries following them) which provide the characterization of the H-measures in the Smyshlyaev-Willis regime, are stated and proved in Section 6. We conclude the thesis with Section 7. The latter presents results which are not conclusive, but it explains how to extend the results of Section 6 to the complementary regime and with which limitations this can be done.

2. Definition and basic properties of H-measures

In the present section we recall the definition and some basic properties of the H-measures associated with periodic geometries. First we set some notation.

Let $N, d \in \mathbb{N}$, $N \geq 2$, $d \geq 2$, and let $\theta = (\theta_1, \ldots, \theta_N) \in [0,1]^N$, with $\sum_{i=1}^{N} \theta_i = 1$,

(d is the dimension, N is the number of "wells" and θ_i, $i = 1, ..., N$ are "volume fractions"). Let further the set $Q = (0, 2\pi)^d$ be "periodicity cell". We define $I(\theta)$ as the set of all characteristic functions $\chi(x) = (\chi_1(x), \ldots, \chi_N(x))$ of measurable subsets of Q with fixed volume fractions θ,

$$I(\theta)=\left\{\chi:\mathbb{R}^d\to\{0,1\}^N, Q\text{–periodic and measurable } : \sum_{i=1}^{N}\chi_i=1 \text{ a.e.}, \fint_Q \chi=\theta\right\},$$

where $\fint_Q \chi$ stands for $\frac{1}{|Q|}\int_Q \chi$.

We denote by $\hat{\chi}_j(k)\,,k\in\mathbb{Z}^d$, the Fourier coefficients for the Q-periodic functions χ_j:

$$\hat{\chi}_j(k)=\fint_Q \chi_j(x)e^{-ikx}dx\,.$$

For every $\chi\in I(\theta)$, we call H-measures generated by χ the matrix-valued measure $\mu=(\mu_{ij})_{i,j}$ defined as follows:

$$\mu_{ij}=\mathrm{Re}\sum_{k\neq 0}\hat{\chi}_i(k)\overline{\hat{\chi}_j(k)}\delta_{k/|k|}\,,\quad 1\le i,j\le N\,, \tag{2.2}$$

where $\delta_{k/|k|}$ denotes a Dirac mass at the point $\xi=k/|k|$ on the unit sphere S^{d-1} and k has integer components. In fact, our definition is a restrictive one. For the general construction, involving functions that need not be periodic, see [**20**] and [**38**].

We introduce the notation

$$\int_{S^{d-1}}\mu_{ij}(\xi)\varphi(\xi)ds(\xi)$$

to denote integration of the function φ with respect to the measure μ_{ij}.

The set of all possible H-measures, which we will denote by $Y^H(\theta)$, is characterized by including all weak limits of (2.2), *i.e.* all Borel measures $\mu_{ij}(\xi)$ such that there exists a sequence of measures $\mu_{ij}^m(\xi)$ of the form (2.2) for each $m=1,2,\dots$ and

$$\int_{S^{d-1}}\mu_{ij}^m(\xi)\varphi_{ij}(\xi)ds(\xi)\longrightarrow\int_{S^{d-1}}\mu_{ij}(\xi)\varphi_{ij}(\xi)ds(\xi)$$

as $m\to\infty$ for any continuous functions φ_{ij} on the unit sphere S^{d-1}. Conventional notation for the above convergence of measures is $\mu_{ij}^m\rightharpoonup\mu_{ij}$. So

$$Y^H(\theta)=\{\mu_{ij}:\exists\,\mu_{ij}^m \text{ of the form (2.2) and } \mu_{ij}^m\rightharpoonup\mu_{ij} \text{ as } m\to\infty\}\,.$$

Notice that $Y^H(\theta)$ is an infinite-dimensional convex set in the space of matrix measures, closed in the sense of the above convergence. It can be easily checked that the H-measures satisfy the following properties:

$$\mu_{ij}=\mu_{ji} \text{ and } \sum_{i=1}^{N}\mu_{ij}=0\,,\quad 1\le j\le N\,, \tag{2.3}$$

$$\int_{S^{d-1}}\mu_{ij}ds(\xi)=\delta_{ij}\theta_i-\theta_i\theta_j\,, \tag{2.4}$$

$$\mu_{ij}(\xi)=\mu_{ij}(-\xi)\,, \tag{2.5}$$

$$\sum_{i,j=1}^{N}\int_{S^{d-1}}\varphi_i(\xi)\varphi_j(\xi)\mu_{ij}ds(\xi)\ge 0 \text{ for any continuous function } \varphi\,. \tag{2.6}$$

We denote the set of all Borel measures on S^{d-1} subject to restrictions (2.3)-(2.6) by $Y(\theta)$:

$$Y(\theta) = \{\mu = (\mu_{ij})_{i,j} : (2.3) - (2.6) \text{ hold } \}.$$

The set $Y(\theta)$ is also convex and closed. Kohn (see [**24**], Theorem 6.4) has shown that for $N = 2$ the conditions (2.3)-(2.6) are necessary and sufficient to characterize the whole set $Y^H(\theta)$, *i.e.* then the sets $Y^H(\theta)$ and $Y(\theta)$ coincide. In contrast, for $N > 2$, the above restrictions are generally insufficient (Kohn, personal communications; see also discussion in [**29**]). This is a non-trivial fact, and one of the main results of the present work (Theorem 2.18 with corollaries) substantially clarifies it further, providing a criterion on whether or not the extremal points of the "bigger" (convex) set $Y(\theta)$ actually belong to the "true" set $Y^H(\theta)$, in effect introducing additional restrictions. Therefore the set $Y^H(\theta)$ is strictly contained, at least in some cases, in $Y(\theta)$: $Y^H(\theta) \subset Y(\theta)$.

3. N-well Relaxation vs H-measures

The aim of this section is to show how the H-measures arise in the relaxation of a "multi-well energy" of the form

$$F(\eta) = \tfrac{1}{2}\min\{|\eta - A_i|^2 \,, i = 1, \dots, N\}\,, \quad \eta \in \mathbb{M}^{d\times d}\,, \tag{2.7}$$

where $A_1, \dots, A_N$ are given matrices in $\mathbb{M}^{d\times d}$. We will also give an explicit expression for the relaxation in the case $N = 2$. Here the word "relaxation" is to be intended in the way that is pertinent to Solenoidal fields. Actually we will deal with the S-quasiconvexification at fixed "volume fractions" (see Definition 2.1).

Definition 2.1. For any $\theta = (\theta_1, \dots, \theta_N) \in [0,1]^N$, with $\sum_{i=1}^{N} \theta_i = 1$, we define the S-quasiconvexification of F at fixed "volume fractions" θ, and denote it by $Q_S^\theta F$, in the following way:

$$\forall \eta \in \mathbb{M}^{d\times d} \quad Q_S^\theta F(\eta) = \inf_{\chi \in I(\theta)} \inf_{B \in V} \tfrac{1}{2} \fint_Q \left| B(x) + \eta - \sum_{i=1}^{N} \chi_i A_i \right|^2 dx\,, \tag{2.8}$$

where V is the space of Q-periodic divergence free matrix fields with zero average on Q, that is

(2.9)

$$V = \left\{ B \in L^2_{loc}(\mathbb{R}^d, \mathbb{M}^{d\times d}),\ Q\text{−periodic},\ \fint_Q B(x)dx = 0\,, \mathrm{Div} B = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d, \mathbb{R}^d) \right\}.$$

Definition 2.1 is in fact a particular case of a more general definition which falls in the framework of $\mathcal{A}$-quasiconvexity (see, for instance, [**17**]). Indeed formula (2.8) involves matrix fields subject to differential constraints of Solenoidal-type (explaining the index S), rather then fields which satisfy more general differential constraints.

We will use the Fourier analysis to execute the so-called "internal" minimization in (2.8), *i.e.* minimization w.r.t. the variable B, for an arbitrary number N of "wells". Further minimization over χ will lead to the exact computation of $Q_S^\theta F$ only in the case $N = 2$ (see Proposition 2.2). We will basically follow the same method used by Kohn [**24**] (see also Allaire & Francfort [**2**]).

Let us fix $\chi \in I(\theta)$ and compute the minimum over B in (2.8). Elementary manipulation using the quadratic nature of F and the periodicity of B transforms the integral in (2.8) into

$$\tfrac{1}{2} \fint_Q \left| B(x) + \eta - \sum_{i=1}^{N} \chi_i A_i \right|^2 dx =$$

$$\tfrac{1}{2} \left\{ |\eta - \sum_{i=1}^{N} \theta_i A_i|^2 + \sum_{i=1}^{N} \theta_i (1-\theta_i) |A_i|^2 - \sum_{i \neq j = 1}^{N} \theta_i \theta_j \langle A_i, A_j \rangle + \right.$$

$$\left. \fint_Q \left(|B(x)|^2 - 2 \langle B(x), \sum_{i=1}^{N} \chi_i A_i \rangle \right) dx \right\}. \tag{2.10}$$

Rewriting (2.10) in Fourier space and using Plancherel's formula, the last term of (2.10) can be rewritten in the form

$$\tfrac{1}{2} \sum_{k \in \mathbb{Z}^n} \left(|\hat{B}(k)|^2 - 2 \langle \hat{B}(k), \sum_{i=1}^{N} \overline{\hat{\chi}_i(k)} A_i \rangle \right) \tag{2.11}$$

where $\hat{B}(k)$ and $\hat{\chi}_i(k)$ are Fourier coefficients for the Q-periodic functions B and χ_i respectively, and $\langle \cdot\,, \cdot \rangle$ is the symmetric inner product on complex matrices. Minimization of (2.11) can be done separately at each k. The frequency $k = 0$ contributes nothing to (2.11), since $\hat{B}(0) = 0$. For $k \neq 0$, the optimal value of $\hat{B}(k)$ turns out to be

$$\hat{B}(k) = \sum_{i=1}^{N} \hat{\chi}_i(k) \Pi_{V(k)} A_i \tag{2.12}$$

(see [**24**] for the necessary linear algebra), where $\Pi_{V(k)} A_i$ denotes the orthogonal projection of A_i onto the space

$$V(k) = \{ \zeta \in \mathbb{M}^{d \times d} \, : \, \zeta k = 0 \} \, .$$

Notice that $V(k)$ is the space of Fourier transforms of divergence free fields at frequency k and it really depends only on $k/|k|$. Moreover, the orthogonal space to $V(k)$ is given by the space $V(k)^{\perp}$ of Fourier transforms of gradient fields:

$$V(k)^{\perp} = \{ \zeta \in \mathbb{M}^{d \times d} : \zeta = v \otimes k \text{ for some } v \in \mathbb{R}^d \} \, .$$

Therefore, for every $\zeta \in \mathbb{M}^{d \times d}$ we have that

$$\Pi_{V(k)} \zeta = \zeta - (\zeta k) \otimes k \, . \tag{2.13}$$

Plugging (2.12) into (2.11) we find that the minimum value of (2.11) is given by

$$-\tfrac{1}{2} \sum_{k \neq 0} \left| \sum_{i=1}^{N} \hat{\chi}_i(k) \Pi_{V(k)} A_i \right|^2 . \tag{2.14}$$

Now we rewrite (2.14) in terms of the H-measures as follows:

$$-\tfrac{1}{2}\sum_{k\neq 0}\left|\sum_{i=1}^{N}\hat{\chi}_i(k)\Pi_{V(k)}A_i\right|^2 = -\tfrac{1}{2}\sum_{i,j=1}^{N}\int_{S^{d-1}}\langle \Pi_{V(k)}A_i, \Pi_{V(k)}A_j\rangle\, d\mu_{ij}.$$

Next we set

$$\forall \xi \in \mathbb{M}^{d\times d} \quad f^{ij}(\xi) := -\tfrac{1}{2}\langle \Pi_{V(\xi)}A_i, \Pi_{V(\xi)}A_j\rangle + \tfrac{1}{2}\langle A_i, A_j\rangle \tag{2.15}$$

and by (2.13) we find that

$$f^{ij}(\xi) = \tfrac{1}{2}\langle A_i\xi, A_j\xi\rangle\,.$$

Then the minimization problem for $Q^\theta_S F$ becomes

$$Q^\theta_S F(\eta) = \tfrac{1}{2}\left|\eta - \sum_{i=1}^{N}\theta_i A_i\right|^2 + \inf_{\mu\in Y^H(\theta)}\sum_{i,j=1}^{N}\int_{S^{d-1}} f^{ij}(\xi)\, d\mu_{ij}(\xi)\,, \tag{2.16}$$

where the minimization is taken with respect to all H-measures which come from N characteristic functions in the sense explained in Section 2.
The following proposition gives the precise formula for $Q^\theta_S F$ when the number of wells is $N = 2$, following the recipe of [**24**]. We will use the symbol $\lambda_{\min}(A)$ to denote the smallest principal value of the matrix A, that is the smallest eigenvalue of the symmetric non-negative matrix $(A^TA)^{1/2}$.

PROPOSITION 2.2. *Let $N = 2$ and $\theta \in [0,1]^2$ be given with $\theta_1 + \theta_2 = 1$. Then*

$$\forall \eta \in \mathbb{M}^{d\times d} \quad QF^\theta_S(\eta) = \tfrac{1}{2}\left|\eta - A_2 + \theta_1(A_2 - A_1)\right|^2 + \tfrac{1}{2}\theta_1(1-\theta_1)\lambda^2_{\min}(A_1 - A_2)\,. \tag{2.17}$$

PROOF. The condition (2.3) specializes to this case as follows:

$$\mu_{11} = \mu_{22} = -\mu_{12} = -\mu_{21}\,.$$

Therefore the H-measures are in effect scalar-valued and the minimization in (2.16) actually depends only on θ. Indeed we have

$$QF^\theta_S(\eta) = \tfrac{1}{2}\left|\eta - A_2 + \theta_1(A_2 - A_1)\right|^2 + \tfrac{1}{2}\inf_{\mu\in Y^H(\theta)}\int_{S^{d-1}} |A_1\xi|^2 - 2\langle A_1\xi, A_2\xi\rangle + |A_2\xi|^2\, d\mu_{11}(\xi)\,.$$

Now using (2.4) and the fact that $|(A_1 - A_2)\xi|^2 \geq \lambda^2_{\min}(A_1 - A_2)$ for every $\xi \in S^{d-1}$, we obtain

$$QF^\theta_S(\eta) \geq \tfrac{1}{2}\left|\eta - A_2 + \theta_1(A_2 - A_1)\right|^2 + \tfrac{1}{2}\theta_1(1-\theta_1)\lambda^2_{\min}(A_1 - A_2)\,. \tag{2.18}$$

To complete the proof of (2.17), we must show that the equality can be achieved in (2.18). To this end, note that the H-measures associated to a layered microstructure are supported on the line parallel to the layer normal. More precisely, if $\chi(x) = g(x\cdot\xi)$ with g periodic, then $\hat{\chi}(k) \neq 0$ only for $k = \xi$. Therefore, if the eigenvector $\bar{\xi}$ of $(A_1 - A_2)^T(A_1 - A_2)$ corresponding to the eigenvalue $\lambda_{\min}$ had entire coordinates, that is $\bar{\xi} \in \mathbb{Z}^d$, then the layer with normal $\bar{\xi}$ would provide an optimal geometry. Otherwise, if $\bar{\xi} \notin \mathbb{Z}^d$, one can find a sequence of directions $\{\xi_h\} \subset \mathbb{Z}^d$ such that $\xi_h \to \bar{\xi}$. Then the sequence of layers with normal ξ_h would do the job. □

REMARK 2.3. It is worth noticing that $\lambda_{\min}(A_1 - A_2) = 0$ when $\operatorname{rank}(A_1 - A_2) \leq d - 1$. In this case the matrices A_1 and A_2 are rank-$(d-1)$ connected and by (2.17) it follows that $QF_S^\theta(\eta) = \frac{1}{2}\left|\eta - A_2 + \theta_1(A_2 - A_1)\right|^2$.

4. The case $N = 3$

We now focus on the case $N = 3$. As already remarked, the set $Y^H(\theta)$ is not fully characterized in this case by the conditions (2.3)-(2.6). The aim of this section is to give a description of the set $Y(\theta)$ determined by (2.3)-(2.6), following [**29**]. More precisely, we will focus on the extremal points of $Y(\theta)$ and their representation as given in [**29**]. Before going into further details, let us see how the restriction (2.3) specializes to the case $N = 3$. Since $\chi_1 = 1 - \chi_2 - \chi_3$, we have:

$$\mu_{12} = \mu_{21} = -\mu_{22} - \mu_{23}\,, \quad \mu_{13} = \mu_{31} = -\mu_{23} - \mu_{33}\,, \quad \mu_{11} = \mu_{22} + 2\mu_{32} + \mu_{33}\,.$$

We can thus restrict our analysis and consider only those measures generated by χ_2 and χ_3, for every $\chi \in I(\theta)$. We set

$$\begin{aligned} a(\xi) &= \mu_{22}(\xi) \\ b(\xi) &= \mu_{23}(\xi) = \mu_{32}(\xi) \\ c(\xi) &= \mu_{33}(\xi)\,. \end{aligned} \tag{2.19}$$

Then relations (2.4) and (2.5) reduce to

$$\begin{aligned} &\int_{S^{d-1}} a(\xi)\,ds(\xi) = \theta_2(1-\theta_2)\,, \\ &\int_{S^{d-1}} b(\xi)\,ds(\xi) = -\theta_2\theta_3\,, \\ &\int_{S^{d-1}} c(\xi)\,ds(\xi) = \theta_3(1-\theta_3)\,, \end{aligned} \tag{2.20}$$

$$a(\xi) = a(-\xi)\,,\ b(\xi) = b(-\xi)\,,\ c(\xi) = c(-\xi)\,. \tag{2.21}$$

The condition of non-negativeness (2.6) can be rewritten as

$$\int_{S^{d-1}} a(\xi)\varphi^2(\xi) + 2b(\xi)\varphi(\xi)\psi(\xi) + c\psi^2(\xi)\,ds(\xi) \geq 0 \tag{2.22}$$

for any continuous functions φ and ψ.
The restriction (2.21) requires the measures to be distributed over the sphere symmetrically. Therefore we will always identify all the opposite points $\pm\xi$ on the sphere. Now consider the set $Y(\theta_2, \theta_3)$ of all Borel 2×2 matrix measures μ on S^{d-1} which satisfy (2.3)-(2.6):

$$Y(\theta_2,\theta_3) = \left\{ \mu(\xi) = \begin{pmatrix} a(\xi) & b(\xi) \\ b(\xi) & c(\xi) \end{pmatrix} : \text{(2.3)-(2.6) hold} \right\}. \tag{2.23}$$

Notice that condition (2.22) requires the matrix measures $\begin{pmatrix} a(\xi) & b(\xi) \\ b(\xi) & c(\xi) \end{pmatrix}$ to be non-negative. Moreover for any measure $\mu \in Y(\theta_2, \theta_3)$ the total mass M is fixed and we have

$$M = \begin{pmatrix} \theta_2(1-\theta_2) & -\theta_2\theta_3 \\ -\theta_2\theta_3 & \theta_3(1-\theta_3) \end{pmatrix} .$$

Smyshlyaev and Willis [**29**] have shown that the extremal points of the set $Y(\theta_2, \theta_3)$ have the form of a weighted sum of at most three Dirac masses (counting the pair $\pm\xi$ as one point):

$$\mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{\xi_r} , \tag{2.24}$$

where $\mu^r = \begin{pmatrix} a_r & b_r \\ b_r & c_r \end{pmatrix}$ and $\xi_r \in S^{d-1}$ for $r = 1, 2, 3$ [1]. On use of conditions (2.20) and (2.22), it is easily checked that the numbers a_r , b_r , c_r satisfy the following inequalities:

$$a_r \geq 0 , \; c_r \geq 0 , \; a_r c_r - b_r^2 \geq 0 , \text{ for each } r = 1, 2, 3 , \tag{2.25}$$

$$\sum_{r=1}^{3} a_r = \theta_2(1-\theta_2) , \sum_{r=1}^{3} b_r = -\theta_2\theta_3 , \sum_{r=1}^{3} c_r = \theta_3(1-\theta_3) . \tag{2.26}$$

We denote by $Y_3(\theta_2, \theta_3)$ the set of all measures of the form (2.24) subject to restrictions (2.25)-(2.26):

$$Y_3(\theta_2, \theta_3) = \left\{ \mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{\xi_r} : \text{(2.25)-(2.26) hold} \right\} . \tag{2.27}$$

Every matrix $\mu = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ satisfying the condition (2.25) belongs to the convex cone $\mathcal{K}$ of non-negative symmetric matrices in the (a, b, c) space:

$$\mathcal{K} = \{(a, b, c) \in \mathbb{R}^3 : a \geq 0 , \; c \geq 0 , \; ac - b^2 \geq 0\} .$$

Every matrix μ belonging to the cone $\mathcal{K}$ is uniquely characterized by its trace $\operatorname{tr} \mu = a + c$ and its "projection" μ_{cs} on the cross-section $\mathcal{K}_{cs}$ of the unit trace:

$$\mu = (\operatorname{tr} \mu) \mu_{cs} .$$

The cross-section $\mathcal{K}_{cs}$ is defined by the relations $a + c = 1$, $b^2 + (c - 1/2)^2 \leq 1/4$ and so can be identified with a unit disc in the (c, b)-plane (see Figure 1).
The total mass M belongs to $\mathcal{K}$ and it can be easily checked that its projection M_{cs} on the cross-section $\mathcal{K}_{cs}$ always lies inside the triangle defined by the points

$$\nu_1 = (0, 0) , \nu_2 = (1, 0) , \nu_3 = (1/2, -1/2)$$

in the (c, b)-plane. Moreover, the condition (2.26) implies that, for any $\mu \in Y_3(\theta_2, \theta_3)$, M_{cs} lies inside the triangle defined by the points $\mu^1_{cs}, \mu^2_{cs}, \mu^3_{cs}$.
We now focus our attention on those measures of $Y_3(\theta_2, \theta_3)$ for which the associated points

[1] It can in fact be shown using methods of [**29**] that for the extremal points one can always choose μ^r so that $\det \mu^r = 0$, $r = 1, 2, 3$, *c.f.* (2.28).

on the (c,b)-plane belong to the circle $C := \{(b,c) : b^2 + (c-1/2)^2 = 1/4\}$, *i.e.* those that are extremal for the set $\mathcal{K}_{cs}$. We denote by $\widehat{Y}_3(\theta_2,\theta_3)$ the set of all such measures:

$$\widehat{Y}_3(\theta_2,\theta_3) := \{\mu \in Y_3(\theta_2,\theta_3) : \mu^r_{cs} \in C \quad \forall\, r = 1,2,3\}\,. \tag{2.28}$$

It will be convenient to parametrize C by the angle $\phi \in [0, 2\pi]$:

$$\begin{cases} a = (1-\cos\phi)/2 \\ b = \frac{1}{2}\sin\phi \\ c = (1+\cos\phi)/2\,. \end{cases} \tag{2.29}$$

In this way, for any $\mu \in \widehat{Y}_3(\theta_2,\theta_3)$, the points μ^r_{cs} can be identified by the angles ϕ_r, $r = 1,2,3$.

Let us now go back to the problem of computing $Q^\theta_S F$. Since for $N = 3$ we do not know the set $Y^H(\theta)$, we can take in (2.16) the minimization over the larger set $Y(\theta)$. This strategy of course does not lead to the precise evaluation of the relaxed energy $Q^\theta_S F$, unless the extremizing measure turns out to be an H-measure. If not, it will simply provide a lower bound on $Q^\theta_S F$. Then we set

$$L(\theta) := \inf_{\mu\in Y(\theta)} \sum_{i,j=1}^{3} \int_{S^2} f^{ij}(\xi)\, d\mu_{ij}(\xi)\,, \tag{2.30}$$

where f^{ij} is defined by (2.15), and

$$\forall\, \eta \in \mathbb{M}^{d\times d} \quad Q^\theta_S F^L(\eta) := \tfrac{1}{2}\left|\eta - \sum_{i=1}^{3}\theta_i A_i\right|^2 + L(\theta)\,. \tag{2.31}$$

The next lemma, where we give an explicit formula for $L(\theta)$, clarifies the role of the set $\widehat{Y}_3(\theta_2,\theta_3)$ in the minimization problem defined in (2.30). It will be assumed, without loss of generality, that $A_1 = 0$.

LEMMA 2.4. ([**29**]) *Let $\theta \in (0,1)^3$ be given with $\sum_{i=1}^{3}\theta_i = 1$ and $A_1, A_2, A_3 \in \mathbb{M}^{d\times d}$ with $A_1 = 0$. Then the infimum in (2.30) is attained and the minimizing measure can be chosen in $\widehat{Y}_3(\theta_2,\theta_3)$. Moreover we have*

$$L(\theta) = (\operatorname{tr} M)\psi_c(M_{cs})\,,$$

where ψ_c denotes the convexification of the function $\psi : \mathcal{K} \to \mathbb{R}$ defined by

$$\psi(a,b,c) = \inf_{\xi\in S^{d-1}} \left\{a f^{22}(\xi) + 2b f^{23}(\xi) + c f^{33}(\xi)\right\}\,. \tag{2.32}$$

PROOF. The complete proof of Lemma 2.4 can be found in [**29**] (in particular see Proposition 5.1 and Lemma 5.2 therein). We will only give a sketch of the minimization algorithm in order to highlight the role of the set $\widehat{Y}_3(\theta_2,\theta_3)$ defined by (2.28). First of all we remark that in the minimization problem (2.30) the infimum is attained and the minimizing measure can always be chosen to belong to $Y_3(\theta_2,\theta_3)$. Moreover, since $f^{ij}(\xi) = 0$ for either $i = 1$ or $j = 1$, we can write

$$L(\theta) = \inf_{\mu \in Y_3(\theta_2,\theta_3)} \sum_{r=1}^{3} \{a_r f^{22}(\xi_r) + 2b_r f^{23}(\xi_r) + c_r f^{33}(\xi_r)\}. \tag{2.33}$$

Problem (2.33) may be approached using the following strategy:
(i) consider all possible splits of the total mass M into the sum of at most three "matrices" μ^r subject to condition (2.25):

$$M = \sum_{r=1}^{3} \mu^r \,; \tag{2.34}$$

(ii) for any decomposition (2.34) choose ξ_r in order to generate

$$\psi(a_r, b_r, c_r) = \inf_{\xi \in S^{d-1}} \left\{a_r f^{22}(\xi) + 2b_r f^{23}(\xi) + c_r f^{33}(\xi)\right\}, \; r = 1, 2, 3\,;$$

(iii) finally minimize with respect to all admissible splits (2.34). As a result

$$L(\theta) = \inf_{\{a_r, b_r, c_r\}} \sum_{r=1}^{3} \psi(a_r, b_r, c_r)\,.$$

Now remember that the total mass M is decomposed as $M = (\operatorname{tr} M) M_{cs}$. Then, for any decomposition (2.34), we have

$$M_{cs} = \sum_{r=1}^{3} \alpha_r \mu^r_{cs}$$

where $\alpha_r = \operatorname{tr} \mu^r / \operatorname{tr} M$.
Next notice that the function $\psi(\mu)$ is homogeneous of degree one, *i.e.* $\psi(t\mu) = t\psi(\mu)$ for any $t \geq 0$. Therefore, the problem of computing $L(\theta)$ reduces to minimizing

$$(\operatorname{tr} M) \sum_{r=1}^{3} \alpha_r \psi(\mu^r_{cs})$$

over all possible decomposition of M_{cs} into the convex combination of $\{\mu^r_{cs}\}$ on the cross-section. Moreover, since the function $\psi(\mu)$ is concave, it is enough to consider only those points μ^r_{cs} which lie on the circle C, *i.e.* are extremal for the set $\mathcal{K}_{cs}$. This procedure leads to finding (no more than) three critical points M_1, M_2, M_3 on C (see Figure 1) such that the extremizing measure μ can be written as

$$\mu = (\operatorname{tr} M) \sum_{i=1}^{3} \alpha_r M^r \delta_{\xi_r}, \quad \alpha_r \geq 0\,, \quad \alpha_1 + \alpha_2 + \alpha_3 = 1\,.$$

Finally, it is easily seen that $L(\theta) = (\operatorname{tr} M)\psi_c(M_{cs})$. □

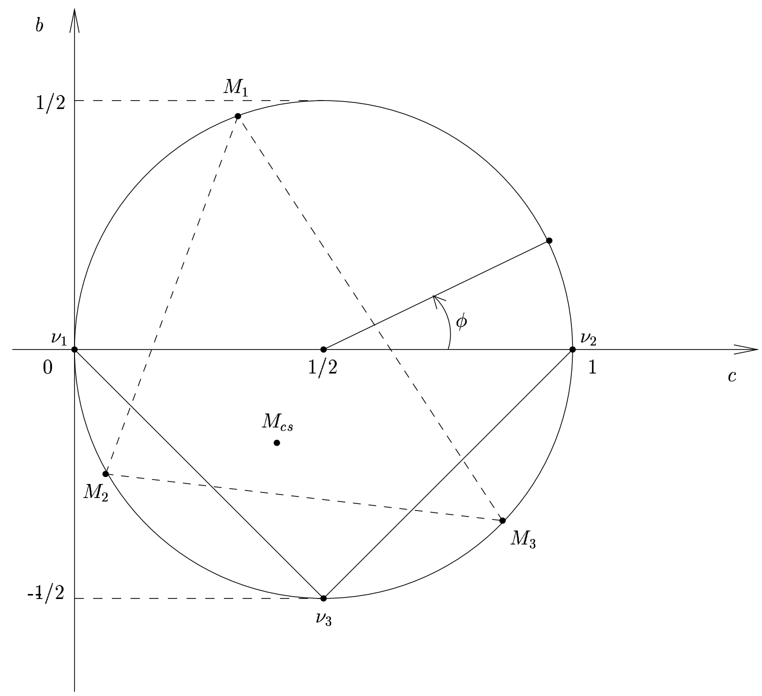

FIGURE 1. The cross-section $\mathcal{K}_{cs}$ of the cone $\mathcal{K}$ on the (c,b)-plane.

5. More on divergence free matrix fields problems: S-quasiconvexity

The purpose of this section is the study of the differential inclusions problem related to the relaxation of the energy (2.7). Our strategy is to make a connection between this problem and that of characterizing the H-measures arising in (2.16) when $d = N = 3$. In order to proceed we need to give some definitions.

DEFINITION 2.5. Given a set of real 3×3 matrices $K \subset \mathbb{M}^{3\times 3}$ we say that K is non-rigid for periodic approximate solutions of Solenoidal-type, if there exists a bounded sequence $\{B_h\} \subset L^2_{loc}(\mathbb{R}^3, \mathbb{M}^{3\times 3})$, Q-periodic and such that

$$\begin{cases} \mathrm{Div} B_h = 0 & \text{in } \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)\,, \\ \mathrm{dist}(B_h, K) \to 0 & \text{in measure,} \end{cases} \tag{2.35}$$

and

$$\forall \{B_{h_j}\} \text{ subsequence of } \{B_h\}, \text{ if } B_{h_j} \to A \text{ in measure, then } A \notin K\,.$$

REMARK 2.6. Strictly speaking, we should call the above solutions, solutions of 2-Solenoidal-type by analogy with the corresponding notion arising in the context of $\mathcal{A}$-quasiconvexity. We will however keep the simpler notation. The reason for choosing L^2 will be transparent in the continuation of this section.

DEFINITION 2.7. We call S-quasiconvex hull of the set K, and denote it by K_S^{qc}, the set defined in the following way

$$K_S^{qc} = \left\{ B_0 \in \mathbb{M}^{3\times 3} : \exists \{B_h\} \text{ solution to (2.35) and } \fint_Q B_h = B_0 \, \forall \, h \right\} .$$

REMARK 2.8. Using results from [**17**] (in particular see Lemma 2.15 therein), one can prove that in the definition of the set K_S^{qc} one may assume w.l.o.g. that the sequence $\{B_h\}$ is L^2-equi-integrable.

We recall the following result from Chapter 1.

LEMMA 2.9. *For every q_1, q_2, $q_3 \in (0,1)$, we define $A \in \mathbb{M}^{3\times 3}$ as follows*

$$A = \frac{1}{q_3}\left[\left(1 - \prod_{i=1}^{3}(1-q_i)\right) G^{-1} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} G - q_2(1-q_3) I \right] , \tag{2.36}$$

where $\lambda_1 = 0\,, \lambda_2 = 1/(1-q_1)\,, \lambda_3 = q_2/(q_1+q_2-q_1q_2)$, and G is an arbitrary matrix in $GL(3,\mathbb{Q})$. Then, for every $M \in \mathbb{M}^{3\times 3}$ and $N \in GL(3,\mathbb{R})$, the set

$$K = \{M\,, N + M\,, NA + M\} \tag{2.37}$$

is non-rigid for periodic approximate solutions of Solenoidal-type.

Before stating the main results of this section we need to make a few comments about the results presented in Lemma (2.9) (see [**18**] and [**27**] for further details). A key property of every set $K = \{A_1, A_2, A_3\}$ of the type (2.37) is that one can find three matrices $S_1, S_2, S_3 \in \mathbb{M}^{3\times 3}$ such that

$$\begin{aligned} S_2 &= q_1 A_1 + (1-q_1) S_1\,, \\ S_3 &= q_2 A_2 + (1-q_2) S_2\,, \\ S_1 &= q_3 A_3 + (1-q_3) S_3\,, \end{aligned}$$

for some $q_1, q_2, q_3 \in (0,1)$ and

$$\det(A_i - S_i) = 0 \quad \forall\, i = 1,2,3\,.$$

Therefore the matrices A_i and S_i are rank-2 connected, while the matrices A_1, A_2, A_3 can be seen to be pairwise rank-2 disconnected since $\operatorname{rank}(A_i - A_j) = 3$, $i \neq j$. The rank-2 lines A_1S_1, A_2S_2, A_3S_3 are schematically represented in Figure 2. For every K we define the following sets:

$$\Gamma_1(K) = \{\xi \in \mathbb{M}^{3\times 3} : \xi = t_1 A_1 + t_2 S_1 + t_3 A_3\,,\ t_i \in [0,1)\,, t_1, t_3 \neq 0\,, \sum_{i=1}^{3} t_i = 1\}\,,$$

$$\Gamma_2(K) = \{\xi \in \mathbb{M}^{3\times 3} : \xi = t_1 A_1 + t_2 S_2 + t_3 A_2\,, t_i \in [0,1)\,, t_1, t_3 \neq 0\,, \sum_{i=1}^{3} t_i = 1\}\,,$$

$$\Gamma_3(K) = \{\xi \in \mathbb{M}^{3\times 3} : \xi = t_1 A_2 + t_2 S_3 + t_3 A_3\,, t_i \in [0,1)\,, t_1, t_3 \neq 0\,, \sum_{i=1}^{3} t_i = 1\}\,.$$

Notice that all the sets $\Gamma_i(K)$ are subsets of the convex hull of K, which we denote by K^c. Finally we denote by $T(K)$ the set of all points of K^c which are not in $\bigcup_{i=1}^{3} \Gamma_i(K)$:

$$T(K) := K^c - \bigcup_{i=1}^{3} \Gamma_i(K)\,. \tag{2.38}$$

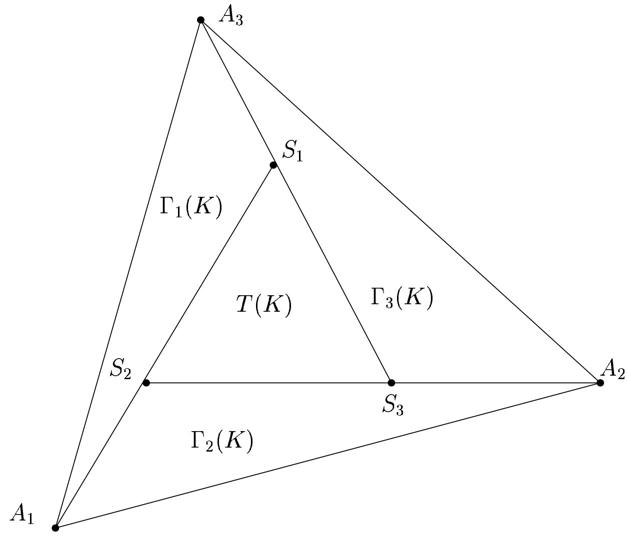

FIGURE 2. The sets $T(K)$, $\Gamma_1(K)$, $\Gamma_2(K)$, $\Gamma_3(K)$.

LEMMA 2.10. *$T(K) \subseteq K_S^{qc}$ for any K of the form* (2.37).

PROOF. The proof of Lemma 2.10 relies on a generalization of the construction presented in the proof of Lemma 1.18 (§4 in Chapter 1). Notice in passing that an alternative proof using the language of H-measures is to first notice that, for $\eta \in T(K)$ and associated θ, the lower bound $L(\theta)$ given by (2.30) vanishes and that the minimizer $\mu \in \widehat{Y}(\theta_2, \theta_3)$ satisfies in fact the H-measure realizability criterion given in [**29**], Proposition 6.1. □

In the sequel we will show that $K_S^{qc} = T(K)$. Since by Lemma 2.10 $K_S^{qc} \supseteq T(K)$, we only need to prove that $K_S^{qc} \subseteq T(K)$. The crucial step will be to prove the latter inclusion when K is of the type $K = \{0, I, D(q)\}$, where $D(q)$ is given by (2.36) with $G = I$ and $q = (q_1, q_2, q_3)$ any arbitrary point in $(0,1)^3$. Indeed we have the following lemma.

LEMMA 2.11. *Let* $K = \{A_1, A_2, A_3\} \subset \mathbb{M}^{3\times 3}$ *and* $\bar{K} = \{NGA_1G^{-1} + M, NGA_2G^{-1} + M, NGA_3G^{-1} + M\}$ *with* $G \in GL(3, \mathbb{Q})$, $N \in GL(3, \mathbb{R})$, $M \in \mathbb{M}^{3\times 3}$. *Then* $B_0 \in K_S^{qc}$ *if and only if* $NGB_0G^{-1} + M \in \bar{K}_S^{qc}$.

PROOF. Assume that $B_0 \in K_S^{qc}$. By definition there exists a sequence of Q-periodic divergence free matrix fields $\{B_h\}$ which satisfies

$$\operatorname{dist}(B_h, K) \to 0 \quad \text{in measure} \quad \text{and} \quad \fint_Q B_h = B_0.$$

Now we introduce the new variable y in $\mathbb{R}^3$ defined by

$$y = G^{-T} x$$

and define the sequence $\{\bar{B}_h\}$ in the following way:

$$\bar{B}_h(y) := NGB(G^T y)G^{-1} + M\,. \tag{2.39}$$

Then one can check that B_h is still divergence free in $\mathbb{R}^3$ and it satisfies

$$\operatorname{dist}(B_h, \bar{K}) \to 0 \quad \text{in measure.}$$

If the matrix G has rational entries, *i.e.* $G \in GL(3, \mathbb{Q})$, then there exists a positive integer l such that $\bar{B}_h$ is periodic with periodicity cube $(0, 2l\pi)^3$, and thus the proof is concluded. □

We recall from the introduction the notion of $\mathcal{A}$-quasiconvexity (see (0.3)). We specialize now to the context we will be dealing with in the present section.

DEFINITION 2.12. A continuous function $f : \mathbb{M}^{3\times 3} \to \mathbb{R}$ with quadratic growth is said to be S-quasiconvex if for every Q-periodic divergence free matrix field $B \in L^2_{loc}(\mathbb{R}^3, \mathbb{M}^{3\times 3})$ the following inequality holds:

$$\fint_Q f(B)\, dx \geq f\left(\fint_Q B\, dx\right)\,. \tag{2.40}$$

Such functions turn out to be the natural tool in bounding the S-quasiconvex hull of a given set K. Indeed if $\{B_h\}$ satisfies (2.35) for $K = \{A_1, A_2, A_3\}$ and $\fint_Q B_h = B_0$, then one has

$$B_0 = \sum_{i=1}^3 \theta_i A_i \quad \text{for some } \theta \in [0,1]^3\,, \text{ with } \sum_{i=1}^3 \theta_i = 1, \quad \text{and} \quad f(B_0) \leq \sum_{i=1}^3 \theta_i f(A_i)\,.$$

Unfortunately we do not know any explicit S-quasiconvex function which can provide the optimal bound on K_S^{qc} when the set K of the type (2.37). Therefore the characterization $K_S^{qc} = T(K)$ will be performed in several steps. The first one consists in the construction of a continuous function $\mathcal{T}^+$ defined on a two-dimensional subspace of $\mathbb{M}^{3\times 3}$ and for which the inequality (2.40) holds whenever B takes values in the domain of $\mathcal{T}^+$. (The function

$\mathcal{T}^+$ is loosely analogous to the function $\det^+$ of Sverak [**31**].) This is the topic of the next lemma. Before giving its statement we need to introduce some notation.
We set

$$V_1^1 = \mathrm{diag}(1,1,0), \quad V_2^1 = \mathrm{diag}(-1,0,-1)\,, \tag{2.41}$$

$$V_1^2 = \mathrm{diag}(-1,-1,0), \quad V_2^2 = \mathrm{diag}(0,1,1)\,, \tag{2.42}$$

$$V_1^3 = \mathrm{diag}(-1,0,-1), \quad V_2^3 = \mathrm{diag}(0,1,1)\,. \tag{2.43}$$

For every $i = 1, 2, 3$, we denote by π_i the plane generated by the pair V_1^i, V_2^i:

$$\pi_i := \{M \in \mathbb{M}^{3\times 3} \,:\, M = \eta_1 V_1^i + \eta_2 V_2^i \text{ for some } \eta_1, \eta_2 \in \mathbb{R}\}\,, \tag{2.44}$$

and by π_i^+ the subset of π_i defined as follows

$$\pi_i^+ := \{M \in \pi_i :\, M = \eta_1 V_1^i + \eta_2 V_2^i \text{ for some } \eta_1, \eta_2 > 0\}\,.$$

Finally we define V_{π_i} as the space of Q-periodic divergence free matrix fields which take values in π_i:

$$V_{\pi_i} = \left\{B \in L^2_{loc}(\mathbb{R}^3, \mathbb{M}^{3\times 3}),\ Q\text{-periodic},\ \mathrm{Div}B = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3,\mathbb{R}^3),\ B \in \pi_i \ a.e.\right\}. \tag{2.45}$$

Our next result is of crucial importance for establishing Theorem 2.18. It seems to have an interest on its own.

LEMMA 2.13. *For every $i = 1, 2, 3$, there exists a continuous function $\mathcal{T}_i^+ : \pi_i \to \mathbb{R}$ which satisfies the following properties:*

$$\mathcal{T}_i^+(M) > 0 \quad \text{if } M \in \pi_i^+, \tag{2.46}$$

$$\mathcal{T}_i^+(M) = 0 \quad \text{if } M \in \pi_i - \pi_i^+, \tag{2.47}$$

$$⨍_Q \mathcal{T}_i^+(B) \geq \mathcal{T}_i^+(⨍_Q B) \quad \forall\, B \in V_{\pi_i}\,. \tag{2.48}$$

PROOF. Let us consider the function $\mathcal{T} : \mathbb{M}^{3\times 3} \to \mathbb{R}$ given by

$$\mathcal{T}(M) = 2\mathrm{tr}\,(MM^T) - (\mathrm{tr}\, M)^2 \tag{2.49}$$

One can check that the above function, which is due to Tartar [**38**], satisfies for every $i = 1, 2, 3$ conditions (2.46) and (2.48), but not (2.47). The idea now is to modify the restriction of $\mathcal{T}$ to the space π_i in order to achieve condition (2.47). Let us first evaluate the restriction of $\mathcal{T}$ to π_i :

$$\forall\, \eta_1, \eta_2 \in \mathbb{R} \quad \mathcal{T}(\eta_1 V_1^i + \eta_2 V_2^i) = 2[(\eta_1 - \eta_2)^2 + \eta_1^2 + \eta_2^2] - 4(\eta_1 - \eta_2)^2 = 4\eta_1\eta_2.$$

Then, for every $i = 1, 2, 3$, we define $\mathcal{T}_i^+ : \pi_i \to \mathbb{R}$ in the following way:

$$\forall\, \eta_1, \eta_2 \in \mathbb{R} \qquad \mathcal{T}_i^+(\eta_1 V_1^i + \eta_2 V_2^i) = \eta_1^+ \eta_2^+,$$

where the symbol η_j^+ denotes the positive part of η_j: $\eta_j^+ := \max\{0, \eta_j\}$. The function $\mathcal{T}_i^+$ satisfies (2.46) and (2.47) by construction. It remains to check that the inequality (2.48) is also fulfilled. We will only prove that for $i = 1$, the other cases being analogous. To this end notice that if $B(x) = \eta_1(x) V_1^1 + \eta_2(x) V_2^1$, then the equation $\mathrm{Div}B = 0$ yields

$$\frac{\partial}{\partial x_1}(\eta_1 - \eta_2) = 0\,, \quad \frac{\partial \eta_1}{\partial x_2} = 0\,, \quad \frac{\partial \eta_2}{\partial x_3} = 0\,, \quad \text{in } \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)\,. \tag{2.50}$$

Using (2.50) one can show that for any $B \in V_{\pi_1}$ there exist three functions $u, v, z \in L^2_{loc}(\mathbb{R})$ such that B can be written in the form

$$B(x_1, x_2, x_3) = (u(x_3) - z(x_1))V_1^1 + (v(x_2) - z(x_1))V_2^1.$$

Hence we have

$$\fint_Q \mathcal{T}_1^+(B)\,dx = \fint_Q (u(x_3) - z(x_1))^+ (v(x_2) - z(x_1))^+ \, dx_1 dx_2 dx_3 \geq$$

$$\geq (\bar{u} - \bar{z})^+ (\bar{v} - \bar{z})^+ = \mathcal{T}_1^+ \left(\fint_Q B\,dx \right),$$

where the symbols $\bar{u}, \bar{v}, \bar{z}$ denote the average of the functions u, v, z over the interval $(0, 2\pi)$. [In the last inequality we have used the convexity of the integrand in $v(x_2)$, $u(x_3)$ and then in $z(x_1)$.] □

We can now focus our analysis on the case $K = \{0, I, D(q)\}$ and give the first partial result towards the characterization of K_S^{qc}. It will be convenient to give the explicit expression for the matrices $D(q), S_1(q), S_2(q), S_3(q)$ in this case:

$$D(q) = \operatorname{diag}\left(-\frac{q_2}{q_3}(1 - q_3),\, -\frac{(1 - q_1)(1 - q_3) - 1}{q_3(1 - q_1)},\, -\frac{q_2}{(1 - q_1)(1 - q_2) - 1} \right), \tag{2.51}$$

$$S_1(q) = \operatorname{diag}\left(0, \frac{1}{1 - q_1}, \frac{q_2}{q_1 + q_2 - q_1 q_2} \right), \tag{2.52}$$

$$S_2(q) = \operatorname{diag}\left(0, 1, \frac{q_2(1 - q_1)}{q_1 + q_2 - q_1 q_2} \right), \tag{2.53}$$

$$S_3(q) = \operatorname{diag}\left(q_2, 1, \frac{q_2}{q_1 + q_2 - q_1 q_2} \right). \tag{2.54}$$

In the next lemma we will use the symbol $p(q)$ to denote the two-dimensional subspace of $\mathbb{M}^{3\times 3}$ containing I and $D(q)$, for any fixed $q \in (0, 1)^3$.

LEMMA 2.14. *Let $q \in (0, 1)^3$ be given and let $\{B_h\}$ be a L^2-equi-integrable sequence which satisfies* (2.35) *with $K = \{0, I, D(q)\}$. If $\fint_Q B_h = B_0$ and $B_h \in p(q)$* a.e. *$\forall\, h$, then $B_0 \in T(K)$.*

PROOF. Since $B_0 \in K^c$, it is enough to prove that $B_0 \notin \bigcup_{i=1}^{3} \Gamma_i(K)$. For simplicity of notations we will omit the dependence on q of the matrices (2.51)-(2.54).
In order to prove that $B_0 \notin \Gamma_3(K)$, recall that the lines $S_3 S_1$ and $S_2 S_3$ are rank-2 lines. Therefore there exists a matrix N_1 such that

$$N_1(S_1 - S_3) = V_1^1 \quad \text{and} \quad N_1(S_3 - S_2) = V_2^1,$$

where the matrices V_1^1 and V_2^1 are defined in (2.41). It is easily seen that the matrix $N_1 = \operatorname{diag}\left(-\frac{1}{q_2}, \frac{1 - q_1}{q_1}, -\frac{q_1 + q_2 - q_1 q_2}{q_1 q_2} \right)$ does the job.

Next we define the sequence $\{B'_h\}$ and the set K' in the following way:

$$\forall\, h \quad B'_h := N_1(B_h - S_3)\,, \qquad K' = \{-N_1 S_3, N_1(I - S_3), N_1(D - S_3)\}\,.$$

Recalling the definition of the space π_1 given in (2.44), it is easily checked that $\{B'_h\}$ satisfies the following properties:

$$\begin{cases} B'_h \in \pi_1 \ \text{ a.e. } \forall\, h, \\ \text{Div} B'_h = 0 & \text{in } \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)\,, \\ \text{dist}(B_h, K') \to 0 & \text{in measure,} \\ \fint_Q B'_h = N_1(B_0 - S_3)\,. \end{cases} \tag{2.55}$$

Next we remark that the condition $B_0 \notin \Gamma_3(K)$ is equivalent to $N(B_0 - S_3) \in \pi_1 - \pi_1^+$. In order to prove the latter inclusion we use the function $\mathcal{T}_1^+$ of Lemma 2.13. Since $\mathcal{T}_1^+|_{K'} = 0$, using (2.48) and (2.55), one gets

$$\mathcal{T}_1^+(N_1(B_0 - S_3)) \leq 0.$$

In order to prove that $B_0 \notin \Gamma_1(K) \cup \Gamma_2(K)$, one finds two matrices N_2 and N_3 such that

$$N_2(S_1 - S_3) = V_1^2\,, \quad N_2(S_2 - S_1) = V_2^2, \quad N_3(S_3 - S_2) = V_1^3\,, \quad N_3(S_2 - S_1) = V_2^3\,,$$

and in a similar fashion one shows that

$$\mathcal{T}_2^+(N_2(B_0 - S_1)) \leq 0\,, \quad \mathcal{T}_3^+(N_3(B_0 - S_2)) \leq 0\,.$$

□

6. Main results: the Smyshlyaev-Willis regime

In the present section we go back to the problem of characterizing the H-measures arising in (2.16) for $d = N = 3$. We will show that the solution to the problem discussed in Section 5, that is finding the optimal bound on the set K_S^{qc}, also provides a further clarification for this problem. The main results of this section (Theorem 2.18 with Corollary 2.20 and Remark 2.23) establishes in effect that the sufficient conditions ([**29**], Proposition 6.1) for realizability of some extremal three-point measures of $Y(\theta)$ by the H-measures are also necessary. The result formulated in Theorem 2.18 is believed new and having potential for applications to other quasiconvexification and multi-well problems.
Let us introduce some notation needed in the presentation of the first result. For the three-points measure $\mu = \sum_{r=1}^{3} \mu^r \delta_{e_r} \in \widehat{Y}_3(\theta_2, \theta_3)$, let ϕ_r be the angle associated with the mass μ^r via (2.24)-(2.29), and let t_r be defined as follows:

$$t_r := \tan \frac{\phi_r}{2}\,, \quad r = 1, 2, 3\,.$$

We use the conventional notation (e_1, e_2, e_3) to indicate the canonical basis of $\mathbb{R}^3$ and denote by P the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

THEOREM 2.15. *Let $\theta \in (0,1)^3$, $d = 3$ and let $\bar\mu \in \widehat{Y}_3(\theta_2, \theta_3)$ be supported on (e_1, e_2, e_3):*

$$\bar\mu(\xi) = \sum_{r=1}^{3} \bar\mu^r \delta_{e_r} \,. \tag{2.56}$$

Suppose that

$$\bar\phi_1 \in (0,\pi)\,, \quad \bar\phi_2 \in (\pi, \tfrac{3}{2}\pi)\,, \quad \bar\phi_3 \in (\tfrac{3}{2}\pi, 2\pi)\,, \tag{2.57}$$

and $t_1(1+t_3) \neq t_3(1+t_2)$. Then we have:
(i) if $t_1(1+t_3) < t_3(1+t_2)$ then $\exists\ \bar q \in (0,1)^3$ such that $D(\bar q) = \mathrm{diag}(t_1,t_2,t_3)$ and

$$\bar\mu \in Y^H(\theta) \iff \theta_2 I + \theta_3 D(\bar q) \in \{0, I, D(\bar q)\}_S^{qc}\,; \tag{2.58}$$

(ii) if $t_1(1+t_3) > t_3(1+t_2)$ then $\exists\ \bar q \in (0,1)^3$ such that $PD(\bar q)P = \mathrm{diag}\left(\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}\right)$ and

$$\bar\mu \in Y^H(\theta) \iff \theta_2 PD(\bar q)P + \theta_3 I \in \{0, I, PD(\bar q)P\}_S^{qc}\,. \tag{2.59}$$

PROOF. *Case* (i). Assume that $t_1(1+t_3) < t_3(1+t_2)$. We choose $\bar q = (\bar q_1, \bar q_2, \bar q_3)$ in the following way:

$$\bar q_1 = \frac{t_1(1+t_3) - t_3(1+t_2)}{t_3(t_1 - t_2)}\,, \quad \bar q_2 = \frac{t_1(1+t_3) - t_3(1+t_2)}{t_2 - t_3}\,, \quad \bar q_3 = \frac{t_1(1+t_3) - t_3(1+t_2)}{(1+t_2)(t_1 - t_3)}\,. \tag{2.60}$$

By assumption $t_1 \in (0,+\infty)$, $t_2 \in (-\infty,-1)$, $t_3 \in (-1,0)$. It follows that $\bar q \in (0,1)^3$ and it can be easily checked that $D(\bar q) = \mathrm{diag}(t_1,t_2,t_3)$.
Then set

$$A_1 = 0\,, \quad A_2 = I\,, \quad A_3 = D(\bar q)\,, \quad K = \{A_1, A_2, A_3\}\,, \quad B_0 = \theta_1 A_1 + \theta_2 A_2 + \theta_3 A_3\,, \tag{2.61}$$

and define the problem (2.8) corresponding to the above choice of matrices:

$$\forall \eta \in \mathbb{M}^{3\times 3} \quad Q_S^\theta F(\eta) = \inf_{\chi \in I(\theta)} \inf_{B \in V} \tfrac{1}{2} \int_Q \left| B(x) + \eta - \sum_{i=1}^{3} \chi_i A_i \right|^2 dx\,, \tag{2.62}$$

where V is defined in (2.9). Using (2.15), (2.16), (2.19), (2.61) we specialize $Q_S^\theta F$ at the point $\eta = B_0$ in terms of the H-measures:

$$Q_S^\theta F(B_0) = \inf_{\mu \in Y^H(\theta)} \int_{S^2} \left(a(\xi) f^{22}(\xi) + 2b(\xi) f^{23}(\xi) + c(\xi) f^{33} \right) ds(\xi)\,. \tag{2.63}$$

Next remark that the condition $B_0 \in K_S^{qc}$ is equivalent to $Q_S^\theta F(B_0) = 0$. In order to prove (2.58) we will show that $Q_S^\theta F(B_0) = 0$ if and only if the extremizing measure in (2.63) is the given measure $\bar{\mu}$. To this end note that by condition (2.22) and by definition of f^{ij} it follows that

$$a(\xi)f^{22}(\xi) + 2b(\xi)f^{23}(\xi) + c(\xi)f^{33} \geq 0$$

anywhere on the sphere S^2. Therefore it will be enough to prove that the function $\psi(\mu)$ defined in (2.32) vanishes only at the points $\mu = \bar{\mu}^r$, $r = 1, 2, 3$. Then parametrize C by the angle ϕ as in (2.29) and set $e(\phi) = \sin(\phi/2)A_2 + \cos(\phi/2)A_3$. Evaluation of the function $\psi(\mu)$ for μ belonging to the circle C gives:

$$\psi(a,b,c) = \tfrac{1}{2}\inf_{|k|=1}\{a|A_2k|^2 + 2b\langle A_2k, A_3k\rangle + c|A_3k|^2\} = \tfrac{1}{2}\inf_{|k|=1}\langle e(\phi)^T e(\phi)k, k\rangle. \tag{2.64}$$

Therefore the value of $\psi(a,b,c)$ is the smallest eigenvalue of the symmetric non-negative matrix

$$e(\phi)^T e(\phi) = \left(\sin\frac{\phi}{2}A_2 + \cos\frac{\phi}{2}A_3\right)^T \left(\sin\frac{\phi}{2}A_2 + \cos\frac{\phi}{2}A_3\right).$$

Recalling the definition of the matrices A_2 and A_3 given in (2.61) we find that

$$e(\phi)^T e(\phi) = \left(\sin\frac{\phi}{2}I + \cos\frac{\phi}{2}D(q)\right)^2.$$

In other words ψ is the infimum of three linear functions whose graphs are planes intersecting the cylinder $\{(b,c,\psi) : (b,c) \in C\}$ over ellipses. It is easily seen that there are exactly three critical points ϕ_1, ϕ_2 and ϕ_3, that can be obtained by equating to zero the eigenvalues of $e(\phi)$:

$$\begin{aligned} &\tan\frac{\phi_1}{2} - \frac{\bar{q}_2}{\bar{q}_3}(1-\bar{q}_3) = 0\,, \\ &\tan\frac{\phi_2}{2} - \frac{1}{\bar{q}_3}\left[1 - \bar{q}_3 - \frac{1}{1-\bar{q}_1}\right] = 0\,, \\ &\tan\frac{\phi_3}{2} - \frac{\bar{q}_2}{(1-\bar{q}_1)(1-\bar{q}_2)-1} = 0. \end{aligned} \tag{2.65}$$

Using the definition of $\bar{q}$ given in (2.60), one can check that $\tan\dfrac{\phi_r}{2} = t_r$ in (2.65) and therefore $\phi_r = \bar{\phi}_r$ for every $r = 1, 2, 3$. Moreover one can see that for every $r = 1, 2, 3$, the extremizing point in (2.64) for $\phi = \bar{\phi}_r$ is the eigenvector of $e(\phi_r)^T e(\phi_r)$ corresponding to the eigenvalue zero, that is the vector e_r.

Case (ii). Now assume that $t_1(1+t_3) > t_3(1+t_2)$. Then choose $\bar{q} = (\bar{q}_1, \bar{q}_2, \bar{q}_3)$ in the following way:

$$\bar{q}_1 = \frac{t_1(1+t_3) - t_3(1+t_2)}{t_1 - t_3}\,, \quad \bar{q}_2 = \frac{t_1(1+t_3) - t_3(1+t_2)}{t_1(t_3 - t_2)}\,, \quad \bar{q}_3 = \frac{t_1(1+t_3) - t_3(1+t_2)}{(1+t_3)(t_1 - t_2)}\,, \tag{2.66}$$

and set

$$A_1 = 0\,, \quad A_2 = PD(\bar{q})P\,, \quad A_3 = I\,, \quad K = \{A_1, A_2, A_3\}\,, \quad B_0 = \theta_1 A_1 + \theta_2 A_2 + \theta_3 A_3\,. \tag{2.67}$$

Then one defines the problem (2.62) corresponding to the choice (2.66) and proceeds as in the previous case. □

REMARK 2.16. If in Theorem 2.15 the measure $\bar{\mu}$ is supported on any three independent vectors $\xi_1, \xi_2, \xi_3 \in \mathbb{Q}^3 \cap S^2$, the result still holds if in (2.58) and (2.59) we replace $D(q)$ by $G^{-1}D(q)G$, where G^{-1} is the matrix of columns ξ_1, ξ_2, ξ_3.

REMARK 2.17. Theorem 2.15 establishes the equivalence between two problems: the one of understanding whether a three-points measure in $\widehat{Y}_3(\theta_2, \theta_3)$ is an H-measure and the one of characterizing the S-quasiconvex hull of the set K defined via (2.61) or (2.67). The nature of this correspondence can be visualized as follows. On the cross-section $\mathcal{K}_{cs}$ in the (c, b)-plane consider the triangle specified by the points $\nu_1 = (0,0)$, $\nu_2 = (1,0)$, $\nu_3 = (\frac{1}{2}, -\frac{1}{2})$. Every point in the interior of K^c can be identified with a point inside the triangle $\nu_1\nu_2\nu_3$ via the correspondence

(2.68)
$$\theta_1 A_1 + \theta_2 A_2 + \theta_3 A_3 \in \mathrm{Int}(K^c) \xrightarrow{\rho} \left(\frac{\theta_3(1-\theta_3)}{\theta_2(1-\theta_2) + \theta_3(1-\theta_3)}, \frac{-\theta_2\theta_3}{\theta_2(1-\theta_2) + \theta_3(1-\theta_3)} \right) \in \mathcal{K}_{cs}\,.$$

Now let $\bar{\mu} = \sum_{r=1}^{3} \bar{\mu}^r \delta_{e_r} \in \widehat{Y}_3(\bar{\theta}_2, \bar{\theta}_3)$ be given and let K be the set associated with $\bar{\mu}$ via (2.60) or (2.66). Then set

$$K^{qc}_{cs} := \rho(K^{qc}_s \cap \mathrm{Int}(K^c))\,, \quad T_{cs}(K) := \rho(T(K) \cap \mathrm{Int}(K^c))\,,$$

where $T(K)$ is the set defined by (2.38). Theorem 2.15 can be rephrased by saying that $\bar{\mu}$ is an H-measure *if and only if* the projection M_{cs} on $\mathcal{K}_{cs}$ of the total mass of $\bar{\mu}$ belongs to K^{qc}_{cs}. Due to Lemma 2.10 we can say that the set K^{qc}_{cs} contains the set $T_{cs}(K)$. Therefore if $M_{cs} \in T_{cs}(K)$ then $\bar{\mu} \in Y^H(\theta)$. Moreover it can be checked that the set $T_{cs}(K)$ is the region delimited by the lines $\nu_1\bar{\mu}^3_{cs}$, $\nu_2\bar{\mu}^2_{cs}$ and $\nu_3\bar{\mu}^1_{cs}$ on the (c, b)-plane (see Figure 3). We briefly illustrate its construction. On the (c, b)-plane draw the three lines $\nu_3\bar{\mu}^1_{cs}$, $\nu_2\bar{\mu}^2_{cs}$, $\nu_1\bar{\mu}^3_{cs}$. Consider the intersections of each of the lines with the two others and with the lines $\nu_1\nu_2$, $\nu_1\nu_3$, $\nu_2\nu_3$:

$$\{R_1\} = \nu_3\bar{\mu}^1_{cs} \cap \nu_1\bar{\mu}^3_{cs}\,, \quad \{R_2\} = \nu_3\bar{\mu}^1_{cs} \cap \nu_2\bar{\mu}^2_{cs}\,, \quad \{R_3\} = \nu_2\bar{\mu}^2_{cs} \cap \nu_1\bar{\mu}^3_{cs}\,,$$
$$\{R_4\} = \nu_3\bar{\mu}^1_{cs} \cap \nu_1\nu_2\,, \quad \{R_5\} = \nu_2\bar{\mu}^2_{cs} \cap \nu_1\nu_3\,, \quad \{R_6\} = \nu_1\bar{\mu}^3_{cs} \cap \nu_3\nu_2\,.$$

Then the set $T_{cs}(K)$ is given by the union of the closed triangle $R_1R_2R_3$ and the segments $[R_1, R_6)$, $[R_2, R_4)$, $[R_3, R_5)$.

We are now ready to formulate the following theorem on the characterization of the extremal three-point H-measures for three phases.

THEOREM 2.18. *Let $\theta \in (0,1)^3$, $d = 3$ and let $\bar{\mu} \in \widehat{Y}_3(\theta_2, \theta_3)$ be supported on (e_1, e_2, e_3):*

$$\bar{\mu}(\xi) = \sum_{r=1}^{3} \bar{\mu}^r \delta_{e_r}\,. \tag{2.69}$$

Suppose that

$$\bar{\phi}_1 \in (0, \pi)\,, \quad \bar{\phi}_2 \in (\pi, \tfrac{3}{2}\pi)\,, \quad \bar{\phi}_3 \in (\tfrac{3}{2}\pi, 2\pi)\,, \quad t_1(1+t_3) \neq t_3(1+t_2)\,, \tag{2.70}$$

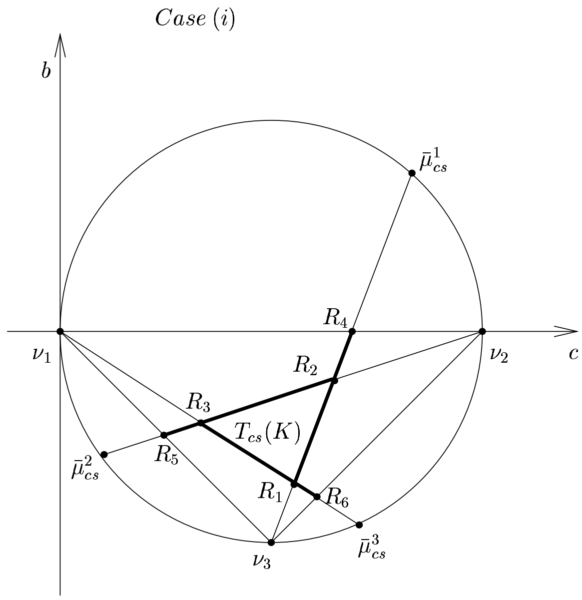

FIGURE 3. The set $T_{cs}(K)$ in the cases (i) and (ii) of Theorem 2.15.

and let K be the set associated with $\bar\mu$ via (2.61) *or* (2.67).
Then

$$\bar\mu \in Y^H(\theta) \Longleftrightarrow M_{cs} \in T_{cs}(K)\,. \tag{2.71}$$

PROOF. Let $\bar\mu$ be given and assume that $t_1(1+t_3) < t_3(1+t_2)$.
By Theorem 2.15 and Remark 2.16 it follows that if $M_{cs} \in T_{cs}(K)$ then $\bar\mu \in Y^H(\theta)$ (alternatively, it follows from [**29**] Proposition 6.1).
In order to prove the converse implication, we will argue that if $M_{cs} \notin T_{cs}(K)$ then $\bar\mu \notin Y^H(\theta)$. Choose $\bar q \in (0,1)^3$ as in (2.60) and set

$$N = \mathrm{diag}\left(-\frac{1}{\bar q_2}, \frac{1-\bar q_1}{\bar q_1}, -\frac{\bar q_1+\bar q_2-\bar q_1\bar q_2}{\bar q_1\bar q_2}\right)\,. \tag{2.72}$$

Then let A_1, A_2, A_3, K be given by (2.61) and define
(2.73)

$$E_1 = NA_1\,, \quad E_2 = NA_2\,, \quad E_3 = NA_3\,, \quad \widetilde K = \{E_1, E_2, E_3\}\,, \quad \widetilde B_0 = \theta_1 E_1 + \theta_2 E_2 + \theta_3 E_3\,.$$

By definition we have that $T_{cs}(\widetilde K) = T_{cs}(K)$. It is convenient to define the following matrices

$$V_1 = \mathrm{diag}\left(0, -\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)\,, \quad V_2 = \mathrm{diag}\left(-\frac{1}{\sqrt2}, 0, -\frac{1}{\sqrt2}\right)\,, \quad V_3 = \mathrm{diag}\left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}, 0\right)\,.$$

Now let us consider the space V_π defined as follows:

$$V_\pi = \left\{B \in L^2_{loc}(\mathbb{R}^3, \mathbb{M}^{3\times3}),\ Q\text{–periodic},\ \mathrm{Div}B = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3,\mathbb{R}^3),\ B \in \pi\ a.e.\right\}\,,$$

where π is the two-dimensional space generated by the matrices V_2 and V_3. Next notice that $E_1, E_2, E_3 \in \pi$ and consider the following problem

$$\forall \eta \in \pi \quad \mathcal{F}(\eta) = \inf_{\chi \in I(\theta)} \inf_{B \in V_\pi} \tfrac{1}{2} \fint_Q \left| B(x) + \eta - \sum_{i=1}^{3} \chi_i E_i \right|^2 dx \,. \tag{2.74}$$

By Lemma 2.14 it follows that if $\mathcal{F}(\widetilde{B}_0) = 0$ then $\widetilde{B}_0 \in T(\widetilde{K})$, that is $M_{cs} \in T_{cs}(K)$, hence if $M_{cs} \notin T_{cs}(K)$ then $\mathcal{F}(\widetilde{B}_0) > 0$. As in the proof of Theorem 2.15 we will express $\mathcal{F}(\widetilde{B}_0)$ in terms of the H-measures and show that the lower bound (2.30) is always zero and is delivered by the given measure $\bar{\mu}$. (This will imply that the minimizing three-point measure is *not* an H-measure if $M_{cs} \notin T_{cs}(K)$.) To this end note that the Fourier transform of any field B in V_π is only supported on the lines through e_r, $r = 1, 2, 3$. Indeed, as already remarked in the proof of Lemma 2.14, for any $B \in V_\pi$ there exist three functions $u, v, z \in L^2_{loc}(\mathbb{R})$ such that B can be written in the form

$$B(x_1, x_2, x_3) = \begin{pmatrix} u(x_3) - v(x_2) & 0 & 0 \\ 0 & u(x_3) - z(x_1) & 0 \\ 0 & 0 & z(x_1) - v(x_2) \end{pmatrix} .$$

Therefore we find that the non trivial Fourier coefficients of B are those in the coordinate directions:

$$\forall\, i \in \{1, 2, 3\}\,, \forall\, n \in \mathbb{Z} \quad \exists\ t \in \mathbb{R} \ \text{ such that } \hat{B}(ne_i) = tV_i\,, \ \text{ and}$$
$$\hat{B}(k) = 0 \ \text{ if } \forall\, n \in \mathbb{Z} \quad k \notin \{ne_1, ne_2, ne_3\}\,.$$

Formulae (2.16) and (2.30) specialize to this case as follows:

$$\mathcal{F}(\widetilde{B}_0) \geq L(\theta) = \inf_{\mu \in \widehat{Y}_3(\theta_2, \theta_3)} \sum_{r=1}^{3} a_r \gamma^{22}(e_r) + 2b_r \gamma^{23}(e_r) + c_r \gamma^{33}(e_r)\,, \tag{2.75}$$

where

$$\gamma^{ij}(e_r) = \langle E_i, E_j \rangle - \langle \Pi_{V_r} E_i, E_j \rangle$$

and $\Pi_{V_r} E_i$ denotes the orthogonal projection of the matrix E_i onto the space generated by V_r. Then set

$$\psi_r(a, b, c) = a\gamma^{22}(e_r) + 2b\gamma^{23}(e_r) + c\gamma^{33}(e_r)\,.$$

We are left to show that the function ψ_r vanishes at the point $\mu = \bar{\mu}_r$, $r = 1, 2, 3$. Indeed parametrize a, b, c by the angle ϕ and set

$$e(\phi) = \sin \frac{\phi}{2} E_2 + \cos \frac{\phi}{2} E_3\,.$$

Then

$$\psi_r(a, b, c) = 0 \Longleftrightarrow |e(\phi)|^2 = |\Pi_{V_r} e(\phi)|^2\,. \tag{2.76}$$

Condition (2.76) requires the matrix $e(\phi)$ to belong to the space $\mathrm{span}(V_r)$. Since E_2 and E_3 do not belong to $\mathrm{span}(V_r)$ for each $r = 1, 2, 3$, this condition is achieved at one and only one point ϕ_r for every $r = 1, 2, 3$. Then using the definition of the matrices E_r (2.73) it can be checked that $E_3 + t_r E_1 \,||\, V_r$ (see Figure 4) and therefore $\phi_r = \bar{\phi}_r$. This all implies

that in (2.75) $L(\theta) = 0$ with the infimum achieved *only* at the above constructed measure $\bar{\mu}$. Finally, for $\widetilde{B}_0 \notin T(\widetilde{K})$ we conclude that $\bar{\mu} \notin Y^H(\theta)$.
The case when $t_1(1+t_3) < t_3(1+t_2)$ is handled in a similar way: in (2.72) and (2.73) one chooses $\bar{q}$ as in (2.66) and the matrices A_r as in (2.67), and then proceeds as before. $\square$

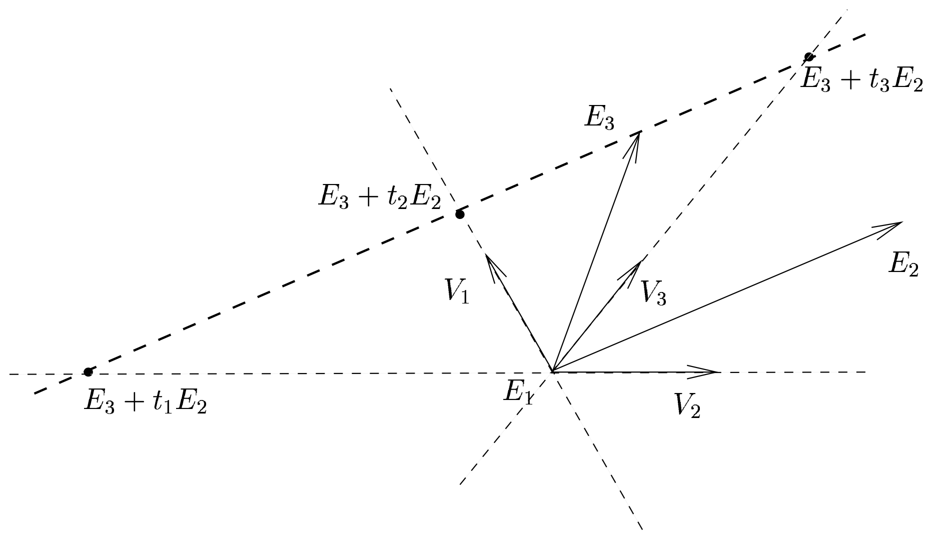

FIGURE 4. The intersections of the line $E_3 + \mathbb{R}E_2$ with directions V_1, V_2 and V_3 on the plane π.

COROLLARY 2.19. *For any K of the form* (2.37) *we have:*

$$K_S^{qc} = T(K)\,.$$

PROOF. Let $\tilde{q} \in (0,1)^3$ be given. Assume for the time being that $K = \{0, I, D(\tilde{q})\}$, where $D(\tilde{q})$ is defined by (2.51). Using the algorithm presented in Lemma 2.4 and formula (2.65) for $\bar{q} = \tilde{q}$, one verifies that the lower bound $L(\theta)$ is zero for every θ and that the measure μ that delivers the bound satisfies the assumptions (2.70) of Theorem 2.18. Applying Theorem 2.18 and Theorem 2.15 one concludes that $K_S^{qc} = T(K)$. Now let K be defined by (2.37) for some $G \in GL(3,\mathbb{Q})$, $N \in GL(3,\mathbb{R})$ and $M \in \mathbb{M}^{3\times 3}$. Then use Lemma 2.11 to conclude the proof. $\square$

COROLLARY 2.20. *Under the assumptions of Theorem 2.18,* (2.71) *still holds if $\bar{\mu}$ is supported on any three independent vectors $\xi_1, \xi_2, \xi_3 \in \mathbb{Q}^3 \cap S^2$.*

PROOF. We simply use Corollary 2.19, Theorem 2.15 and Remark 2.16. $\square$

We want to discuss now what happens if the triangle $R_1R_2R_3$ on the (c,b)-plane degenerates into one single point, which we denote by R_0 (see Figure 5). In this case the associated measure $\bar{\mu}$ satisfies

$$t_1(1+t_3) = t_3(1+t_2)$$

and therefore there is no set K of the type (2.37) for which (2.71) may hold. More precisely the problem associated to $\bar{\mu}$ in the sense of Theorems 2.15 and 2.18 may be regarded as a degenerate case among those discussed so far. The following proposition clarifies the situation.

PROPOSITION 2.21. *Let* $\theta \in (0,1)^3$, $d = 3$ *and let* $\mu \in \widehat{Y}_3(\theta_2, \theta_3)$ *be supported on* (e_1, e_2, e_3)*:*

$$\mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{e_r} .$$

Suppose that

$$\phi_1 \in (0, \pi)\,, \quad \phi_2 \in (\pi, \tfrac{3}{2}\pi)\,, \quad \phi_3 \in (\tfrac{3}{2}\pi, 2\pi)\,, \quad t_1(1 + t_3) = t_3(1 + t_2)\,.$$

Then

$$\mu \in Y^H(\theta) \Longleftrightarrow M_{cs} \in [R_0, R_4) \cup [R_0, R_5) \cup [R_0, R_6) \Longleftrightarrow \theta_2 I + \theta_3 A_0 \in \{0, I, A_0\}_S^{qc},$$

where the matrix A_0 *is defined as follows:*

$$A_0 = \mathrm{diag}(-t_1, -t_2, -t_3)\,.$$

PROOF. Set $K_0 = \{0, I, A_0\}$ and $S_0 = \mathrm{diag}(0, 1, -t_3)$. The first observation we make is that the matrix S_0 is rank-2 connected with each of the three matrices $0, I, A_0$ and that the set $T(K_0)$ is given in this case by the union of three segments:

$$T(K_0) = [0, S_0] \cup [I, S_0] \cup [A_0, S_0].$$

Moreover it is easily checked that $T_{cs}(K_0) = [R_0, R_4) \cup [R_0, R_5) \cup [R_0, R_6)$ and $\rho(S_0) = R_0$. Arguing as before one can show that every point outside $T(K_0)$ does not belong to K_{0S}^{qc} and therefore if $\mu \in Y^H(\theta)$ then $M_{cs} \in T_{cs}(K_0)$.
Now let $M_{cs} \in [R_0, R_5) \cup [R_0, R_6)$. A possible way to prove that $\mu \in Y^H(\theta)$ is to use an approximation argument. We consider a sequence of points M_m^1 on the circle C such that $M_m^1 \to \mu_{cs}^1$ as $m \to \infty$ (see Figure 5). By Theorem 2.18 it follows that for every m the measure μ^m corresponding to the split $M_m^1, \mu_{cs}^2, \mu_{cs}^3$ is an H-measure. By construction $\mu^m \rightharpoonup \mu$ and therefore $\mu \in Y^H(\theta)$. If $M_{cs} \in [R_0, R_4)$ then one introduces a perturbation around the point μ_{cs}^2 or μ_{cs}^3 and proceeds as before. □

REMARK 2.22. The result of Proposition 2.21 extends straightforwardly to the case when μ is supported on any basis of $\mathbb{R}^3$.

REMARK 2.23. Using the approximation argument introduced in Proposition 2.21, one can show that if a measure μ is such that the points μ_{cs}^r coincide with the basic points ν_r, then $\mu \in Y^H(\theta)$ for any value of the volume fractions θ. Another way to see this is to study the problem (2.8) when the matrices A_1, A_2, A_3 are pairwise rank-2 connected and therefore $\{A_1, A_2, A_3\}_S^{qc} = \{A_1, A_2, A_3\}^c$. Using the algorithm illustrated in Lemma 2.4, one finds that the extremizing measure is indeed of that kind.

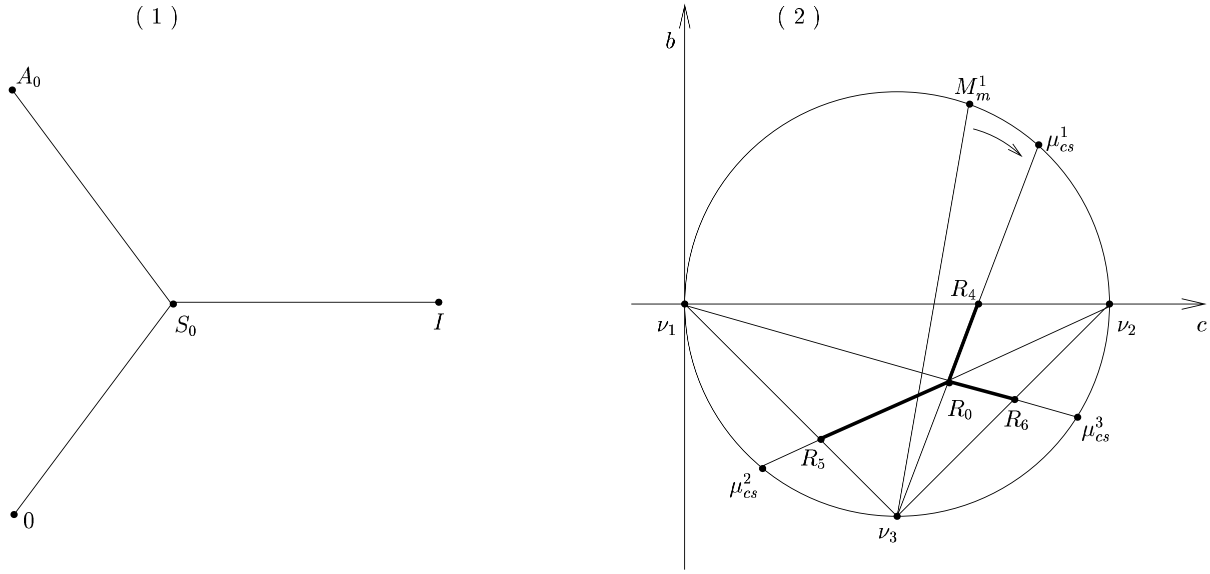

FIGURE 5. (1) The set K_0. (2) The set $T_{cs}(K_0)$ on the (c, b)-plane.

7. Partial results for the complementary regime

The results presented in Section 6 provide the characterization of all three-point H-measures of the form $\mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{\xi_r}$ when the associated points μ_{cs}^r on the (c, b)-plane lie on the circular segments $\nu_2\nu_1$, $\nu_1\nu_3$, $\nu_3\nu_2$, *one* on each segment including the endpoints. We have shown the equivalence between this problem and that of characterizing the S-quasiconvex hull of a set K when K is of the type (2.37) or when K contains rank-2 connections. In this section we will briefly discuss the case when two of the points μ_{cs}^r lie on the *same* circular segment. In this case we give new constraints on the H-measures which were not previously known. However our result is not conclusive. In certain regimes we were not able to fully characterize the set of H-measures. An analogous of Theorem 2.15 still holds (Theorem 2.24 below); indeed we can still define a set $K \subset \mathbb{M}^{3\times 3}$ and establish the equivalence between the problem of understanding whether such measures could be "realizable" by H-measures and that of characterizing the S-quasiconvex hull of the associated sets K.

In the next result we keep the notation introduced in Section 6 and we denote by G^{-1} the matrix of columns the vectors ξ_1, ξ_2, ξ_3.

THEOREM 2.24. *Let $\theta \in (0,1)^3$, $d = 3$ and let $\mu \in \widehat{Y}_3(\theta_2, \theta_3)$ be supported on three independent vectors $\xi_1, \xi_2, \xi_3 \in \mathbb{Q}^3 \cap S^2$:*

$$\mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{\xi_r} .$$

Suppose that two of the three points μ_{cs}^1, μ_{cs}^2, μ_{cs}^3, lie on the same circular segment. Then

$$\mu \in Y^H(\theta) \Longleftrightarrow \theta_2 I + \theta_3 A \in \{0, I, A\}_S^{qc},$$

where the matrix A is defined in the following way:

$$A = G^{-1} \begin{pmatrix} -t_1 & 0 & 0 \\ 0 & -t_2 & 0 \\ 0 & 0 & -t_3 \end{pmatrix} G. \tag{2.77}$$

PROOF. The proof is analogous to that of Theorem 2.15 and therefore it will be omitted. □

REMARK 2.25. It can be checked that for every A defined by (2.77), the set $K = \{0, I, A\}$ is *not* of the type (2.37). Indeed only two of the rank-2 lines through 0, I and A intersect at some point inside the convex hull K^c. Therefore the quasiconvexity-type argument used in Lemma 2.14 can only provide in this case a partial result.

We will illustrate the situation by giving an explicit example. A more systematic treatment will be the object of future work.

7.1. Example. Let $\mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{e_r} \in \widehat{Y}_3(\theta_2, \theta_3)$ and assume that

$$\mu_{cs}^1 = \left(\frac{4}{5}, \frac{2}{5}\right), \quad \mu_{cs}^2 = \left(\frac{1}{10}, -\frac{3}{10}\right), \quad \mu_{cs}^3 = \left(\frac{1}{5}, -\frac{2}{5}\right).$$

The matrix A defined via (2.77) is given by

$$A = \operatorname{diag}\left(-\frac{1}{2}, 3, 2\right).$$

It is readily seen that the matrix $S = \operatorname{diag}\left(0, 1, \frac{5}{7}\right)$ is rank-2 connected to the matrices 0 and I and that $S = \frac{1}{7}I + \frac{2}{7}A$. Moreover one can verify that the rank-2 lines $0S$ and IS intersect the rank-2 planes through A at points outside the convex hull of the set $K = \{0, I, A\}$. The rank-2 lines $0S$ and IS are represented on the (θ_2, θ_3)-plane by the lines $r_1 : \theta_3 = 2\theta_2$ and $r_2 : \theta_3 = -\frac{1}{3}\theta_2 + \frac{1}{3}$ respectively (see Figure 6). The set of the "volume fractions" is represented on the (θ_2, θ_3)-plane by the simplex $\Delta_2 := \{(\theta_2, \theta_3) : 0 \leq \theta_2 + \theta_3 \leq 1\}$. Using the same arguments as in Section 6 one can show that all points in the interior of the triangle specified by the points $(0,0)$, $(1,0)$, $\left(\frac{1}{7}, \frac{2}{7}\right)$ correspond to those values of the "volume fractions" θ for which $\mu \notin Y^H(\theta)$. On the (c, b)-plane these points correspond to those inside the polygon $P_0P_1P_3P_4P_5$. Indeed let $\sigma : \operatorname{Int}(\Delta_2) \to \mathcal{K}_{cs}$ be defined by

$$\sigma(\theta_2, \theta_3) = \left(\frac{\theta_3(1-\theta_3)}{\theta_2(1-\theta_2) + \theta_3(1-\theta_3)}, \frac{-\theta_2\theta_3}{\theta_2(1-\theta_2) + \theta_3(1-\theta_3)}\right).$$

Then we have

$$\sigma : \{(\theta_2, \theta_3) : \theta_2 > 0,\ \theta_3 - 2\theta_2 \leq 0,\ 3\theta_3 + \theta_2 - 1 \leq 0\} \to P_0P_1P_3P_4P_5.$$

Therefore we conclude that if the measure μ is such that $M_{cs} \in P_0P_1P_3P_4P_5$, then μ is not an H-measure. If instead M_{cs} lies inside the triangle $P_5P_6P_7$, then we are not able to establish whether or not μ is an H-measure. On the other hand the points inside the triangle $P_5P_6P_7$ correspond on the (θ_2,θ_3)-plane to the set U delimited by the line r_2 and the hyperbola $p_2(\theta_2,\theta_3) = 2\theta_2^2 - 2\theta_3^2 + 3\theta_2\theta_3 - 2\theta_2 + 2\theta_3 = 0$ (the shadowed region in Figure 6 (1)):

$$\sigma^{-1} : \operatorname{Int}(P_5P_6P_7) \to U := \{(\theta_2,\theta_3) : 3\theta_3 + \theta_2 - 1 > 0,\ p_2(\theta_2,\theta_3) < 0\}\,.$$

The set $\overline{U}$ defines an outer bound on K_S^{qc}.

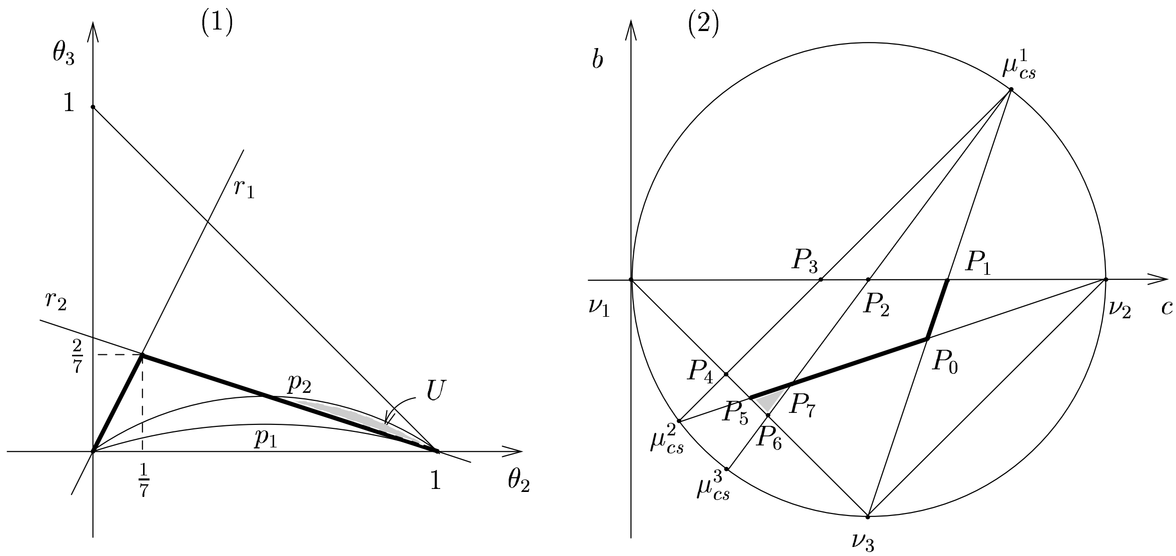

FIGURE 6. (1) The set U on the (θ_2,θ_3)-plane. (2) The set $\sigma(U)$ on the (c,b)-plane.

We briefly describe now an alternative way to derive the equation of the hyperbola $\{p_2 = 0\}$. For any $D \in \mathbb{M}^{3\times3}$ let $\mathcal{T}_D : \mathbb{M}^{3\times3} \to \mathbb{R}$ be defined as follows

$$\forall\, M \in \mathbb{M}^{3\times3} \quad \mathcal{T}_D(M) = \mathcal{T}(DM)\,,$$

where $\mathcal{T}$ is the function defined in (2.49). The S-quasiconvexity of $\mathcal{T}$, and hence of $\mathcal{T}_D$, yields the inequality

$$\theta_2\mathcal{T}_D(I) + \theta_3\mathcal{T}_D(A) - \mathcal{T}_D(\theta_2 I + \theta_3 A) \le 0\,.$$

The above inequality defines an outer bound on K_S^{qc} for every $D \in \mathbb{M}^{3\times3}$. In particular for any diagonal matrix D=diag(d_1,d_2,d_3), one gets

$$[d_1^2 + (d_2-d_3)^2 - 2d_1(d_2+d_3)]\theta_2 + \tfrac{1}{4}[d_1^2 + 4(3d_2-2d_3)^2 + 4d_1(3d_2+2d_3)]\theta_3 +$$
$$-\tfrac{1}{4}d_1^2(-2\theta_2+\theta_3)^2 - [d_3(\theta_2+2\theta_3) - d_2(\theta_2+3\theta_3)]^2 +$$
$$+d_1(2\theta_2-\theta_3)[d_3(\theta_2+2\theta_3) + d_2(\theta_2+3\theta_3)] \ge 0\,. \tag{2.78}$$

The "best" possible bound is obtained optimizing (2.78) over D. To this end we observe that formula (2.78) defines a quadratic form on $\mathbb{R}^3$ whose coefficients depend on the parameters θ_2 and θ_3:

$$\theta_2 \mathcal{T}_D(I) + \theta_3 \mathcal{T}_D(A) - \mathcal{T}_D(\theta_2 I + \theta_3 A) = \langle Z(\theta_2, \theta_3) D, D \rangle ,$$

where $Z(\theta_2, \theta_3)$ is the symmetric matrix of entries m_{ij} given by

$$\begin{aligned}
m_{11} &= -\theta_2^2 - \frac{1}{4}(-1+\theta_3)\theta_3 + \theta_2(1+\theta_3) , \\
m_{12} = m_{21} &= \frac{1}{2}[2(-1+\theta_2)\theta_2 + (3+5\theta_2)\theta_3 - 3\theta_3^2] , \\
m_{13} = m_{31} &= (-1+\theta_2)\theta_2 + \theta_3 + \frac{3}{2}\theta_2\theta_3 - \theta_3^2 , \\
m_{22} &= \theta_2 - \theta_2^2 - 6\theta_2\theta_3 - 9(-1+\theta_3)\theta_3 , \\
m_{23} = m_{32} &= \theta_2^2 + 6(-1+\theta_3)\theta_3 + \theta_2(-1+5\theta_3) , \\
m_{33} &= \theta_2 - \theta_2^2 - 4\theta_2\theta_3 - 4(-1+\theta_3)\theta_3 .
\end{aligned}$$

The determinant of $Z(\theta_2, \theta_3)$ factorizes as follows

$$\det\, Z(\theta_2, \theta_3) = p_1(\theta_2, \theta_3)\, p_2(\theta_2, \theta_3)\, p_3(\theta_2, \theta_3) ,$$

where

$$\begin{aligned}
p_1(\theta_2, \theta_3) &= 2\theta_2^2 - 3\theta_3^2 + 5\theta_2\theta_3 - 2\theta_2 + 3\theta_3 , \\
p_2(\theta_2, \theta_3) &= 2\theta_2^2 - 2\theta_3^2 + 3\theta_2\theta_3 - 2\theta_2 + 2\theta_3 , \\
p_3(\theta_2, \theta_3) &= \theta_2^2 + 6\theta_3^2 + 5\theta_2\theta_3 - \theta_2 - 6\theta_3 .
\end{aligned}$$

It can be checked that the matrix $Z(\theta_2, \theta_3)$ is positive definite in the set

$$P := \{(\theta_2, \theta_3) \in \Delta_2 : \, p_1(\theta_2, \theta_3) > 0, \, p_2(\theta_2, \theta_3) < 0\}$$

and negative definite elsewhere in Δ_2. It follows that for every $(\theta_2, \theta_3) \in \Delta_2 - \overline{P}$ the matrix $\theta_2 I + \theta_3 A$ does not belong to K_S^{qc}. The latter implication however cannot provide any further information on the H-measures, since $P \supset U$. More precisely one can check that

$$\sigma(p_1) = (P_4, P_3) , \quad \sigma(p_2) = (P_6, P_2) \quad \text{and} \quad \sigma(P) = \mathrm{Int}(P_2 P_3 P_4 P_6) ,$$

where

$$p_i := \{(\theta_2, \theta_3) \in \mathrm{Int}(\Delta_2) : p_i(\theta_2, \theta_3) = 0\} \quad \text{for } i = 1, 2 .$$

A systematic study of the complementary regime has not been completed yet. This is part of ongoing work. However this example shows that the approach developed in Section 6 can be used to obtain sharp bounds on the H-measures. It also shows that new ideas are needed to solve the problem completely.

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