SAPIENZA
UNIVERSITÀ DI ROMA

SCUOLA DI DOTTORATO IN ECONOMIA

DOTTORATO DI RICERCA IN MATEMATICA PER LE APPLICAZIONI

ECONOMICO-FINANZIARIE XXIV CICLO

Pricing American Bond Options under HJM: an Infinite Dimensional Variational Inequality

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY IN

MATHEMATICS FOR ECONOMIC-FINANCIAL APPLICATIONS

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And then the harder they come

the harder they'll fall,

one and all.

Jimmy Cliff

The ice age is coming, the sun's zooming in

Engines stop running, the wheat is growing thin

A nuclear error but I have no fear,

'Cause London is drowning and I live by the river

The Clash

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Introduction

A major challenge in mathematical finance is pricing derivatives with an increasing degree of complexity. A huge theoretical effort has been made in the last forty years to provide suitable tools for this purpose. The volume of traded options and the wide variety of their structures require a deep analysis of both theoretical and numerical methods. Since the seminal paper by F. Black and M.S. Scholes [5] the connection between the pricing procedure and the solution of Partial Differential Equations (PDEs) has become more and more appearent.

An important class of traded options is that of *American options*. The basic idea behind these options is the holder's the right to buy/sell a security at any time prior to a given maturity at a fixed price which is specified in the contract. The mathematical formulation of this problem was given in the eighties by A. Bensoussan [3] and I. Karatzas [39] among others. In mathematical terms the pricing problem for an American option corresponds to an *optimal stopping* problem (for a good survey cf. [50]); that is, a problem in which a finite dimensional stochastic process is optimally stopped in order to maximize/minimize a given reward function. This problem, as many others in Mathematical Finance, is a stochastic control problem in which the controls are stopping times. When the underlying process is a diffusion one may find a variational formulation for the value function of the optimal stopping problem which corresponds to a free-boundary problem, in the language of PDE. To study this problems one must analyze the properties of the value function, characterize the optimal stopping time, i.e. the optimal exercise time, and the free-boundary. For a survey on variational inequalities see [30], [40]; for applications of variational inequalities to stochastic control problems see [4]. In the context of American options the security underlying the contract is usually a diffusion process. For example Jacka [36] studies the case of a geometric Brownian motion; more general diffusions are considered in [37]).

In this Thesis we study the optimal stopping problem corresponding to the American option on a Bond having the forward interest rate process as underlying. The forward rate is the instantaneous interest rate agreed at time t for a loan which will take place at a future time $T \geq t$. It is often denoted by $f(t,T)$ and taking $T = t$ one recovers the "so called" spot rate $R(t) = f(t,t)$. The price of the Bond, $B(t,T)$, is linked to the forward rate by the ordinary differential equation

$$f(t,T) = -\frac{\partial}{\partial T} \ln \left(B(t,T)\right). \tag{1}$$

There exists a vast literature on interest rate models concerning both theoretical and numerical aspects (for good surveys cf. for instance [6], [12], [48]). One of the most reliable models of forward rates is the famous HJM model, introduced in 1992 by D. Heath, R. Jarrow and A. Morton, [35]. We choose this framework for the present work. The peculiarity of the stochastic process representing the forward interest rate is its infinite dimensional character. In essence the HJM model describes the stochastic dynamics of the whole term structure of forward rates; in fact, at each time t the model's output is the family of rates $f(t,T)$, with $T \in [t, T_{max}]$. It follows that a suitable parametrization of $f(t,T)$ may be modeled by an infinite dimensional stochastic differential equation, as pointed out by M. Musiela [47] in 1993. A complete description of the HJM model in the context of infinite dimensional diffusion processes can be found in [26] and [27]. Notice however that a connection between PDE's in Hilbert spaces (cf. [17], [22]) and American options with infinite dimensional underlying is not straightforward. Such connection is known in the case of *European options* and forward rates. In fact, in this case the value function may be characterized as the unique smooth solution of the Kolmogorov equation (cf. [33]). In some sense that is the natural extension of the Black and Scholes pricing formula to the infinite dimensional setting.

Here we consider an *American Bond option*; specifically an American *Put* option on a stochastic Zero Coupon Bond (ZCB). This option gives the holder the right to sell the ZCB for a fixed price K at any time prior to the maturity T. The payoff at time t is given by $(K - B(t, \hat{T}))^+$ for $T \le \hat{T} \le T_{\max}$, where $(\,\cdot\,)^+$ denotes the positive part. In terms of the forward rate (1) the optimal stopping problem associated to the pricing of the American Bond option may be written as

$$\sup_{t \le \tau \le T} \mathbb{E}\left[e^{-\int_t^\tau f(u,u)du} \left(K - e^{-\int_\tau^{\hat{T}} f(\tau,u)du} \right)^+ \right]. \tag{2}$$

The theory of infinite dimensional stochastic analysis (cf. [19]) guarantees that (2) is Markovian in the sense that it only depends on the starting time t and on the initial data $\{f(t,u),\ u \in [t, \hat{T}]\}$. The dependence on the entire forward curve is typical of infinite dimensional processes, however for such processes the theory of optimal stopping is not as developed as its finite dimensional counterpart. A first attempt in this direction was made by J. Zabczyk [55] in 1997 from a purely probabilistic point of view. The variational approach was introduced by Zabczyk himself in 2001 (cf. [56]) and further contributions by Barbu and Marinelli [2] followed several years later. These Authors considered a diffusion process on functional space $\mathcal{H}$ (usually a Hilbert space) and solved the variational problem in a suitable L^2-space with respect to a measure on $\mathcal{H}$. The solution was characterized in a mild sense, adopting the general theory of monotone operators and the associated semigroups, cf. [13]. Optimal stopping and variational inequalities in infinite dimensions were also considered by Chow and Menaldi [15] in a particular case. In principle the results by Zabczyk and Barbu-Marinelli might be applied to the pricing of American Bond options on forward rates but these Authors outline the arguments. On the other hand D. Gątarek and A. Święch [31] study the problem by means of viscosity theory in infinite dimensions. They characterize the value function and the optimal stopping time under suitable assumptions on the dynamics of the forward rate. In particular they make use of the Goldys-Musiela-Sondermann parametrization (cf. [34]) and that completely determines the volatility structure of the dynamics. The adoption of this scheme simplifies the stochastic differential equation (SDE) in infinite dimensions removing an unbounded term from the drift. A possible drawback of this model is the lack of consistency with the observations on the market. This fact is discussed in details by D. Filipovic [26].

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a SDE on a Hilbert space $\mathcal{H}$ is formally written as

$$\begin{cases} dX_t = (AX_t + F(X_t))dt + \sigma(X_t)dW_t, & t \in [0,T] \\ X_0 = x. \end{cases} \tag{3}$$

Usually W represents a $\mathcal{H}$-valued Brownian motion but full generality may be obtained by introducing the concept of cylindrical Brownian motion (cf. [19]). The coefficients F and σ are suitable functions on $\mathcal{H}$, taking values respectively on $\mathcal{H}$ and on the space of bounded linear operators from $\mathcal{H}$ to itself; A is an unbounded linear operator with domain $D(A)$ dense in $\mathcal{H}$. There are three different notions of solution: *strong*, *weak* and *mild*. The last one relies on the theory of semigroups (cf. [49]) and it is the most widely adopted. Notice that the forward rate in the Musiela's parametrization is indeed a mild solution of (3) with suitable coefficients.

In this Thesis we show that the price of an American Bond option in the HJM framework is a solution (in some sense) of a suitable infinite dimensional variational inequality. Our results generalize Bensoussan and Lions' theory [4] of optimal stopping in finite dimensions to the infinite dimensional case by exploiting both probabilistic and analytical results. We also obtain a higher degree of regularity as compared to the results in [2], [31], [55], [56]. Our infinite dimensional variational inequality is highly degenerate, parabolic, on an unbounded domain. This kind of problems is non standard in the context of PDE's theory (cf. [53]) even at the finite dimensional level. Optimal stopping of degenerate diffusions on $\mathbb{R}^n$ was mostly studied by J.L. Menaldi in the early eighties (cf. [43], [44]). We solve the variational inequality in a suitable Banach space; the regularity of the solution turns out to be the infinite dimensional counterpart of the finite dimensional results. We also characterize the optimal stopping time and give some intuitive results about the shape of the continuation and stopping regions.

The whole analysis is based on a preliminary finite dimensional reduction of the diffusion process, which we associate a suitable optimal stopping problem to. Then we localize this problem to a bounded regular subset of $[0,T] \times \mathbb{R}^n$ so to exploit standard results on variational inequalities and optimal stopping. We prove that the value function of the localized finite dimensional optimal stopping problem solves a specific variational inequality and we characterize its optimal stopping time. Next, we obtain a number of a priori estimates which allow us to move to the infinite dimensional case and we manage to keep the connection between the value function of the optimal stopping problem and the variational inequality by exploiting the probabilistic representation of the solution. Similarly we characterize the optimal stopping time for the infinite dimensional case.

It is worth mentioning that there exists a large literature about finite dimensional realizations of the forward rate curve's dynamics, see for instance [7], [8], [9], [10], [28], [29]. These Authors show that, in some cases, for each initial condition there exists a suitable random time interval on which the solution of (3) has a representation in terms of a finite dimensional diffusion, whose coefficients depend on the initial data x.

This Thesis is organized as follows. In Chapter 1 we set the problem and we describe the connection between the dynamics of the forward interest rate and the theory of stochastic differential equations in infinite dimensions. In particular, we recall the Musiela's parametrization and the Filipovic's space of functions. In fact we consider the forward curve as an element of the Hilbert space $\mathcal{H}_w$ introduced in [26]. We characterize the value function of the option by writing explicitly the optimal stopping problem in $\mathcal{H}_w$ and afterwards we analyze some of its regularity properties. This is done by means of purely probabilistic considerations which however rely on the particular choice of the space $\mathcal{H}_w$.

In Chapter 2 we embed our problem in a more general class of optimal stopping problems for Hilbert space-valued diffusion processes. In particular we consider a diffusion X with features analogous to those of the forward rate and define the optimal stopping problem

$$V(t,x) := \sup_{t \le \tau \le T} \mathbb{E}\left[\Psi(\tau, X_\tau^{t,x})\right], \tag{4}$$

where the gain function Ψ has the same regularity of the discounted payoff of the original Put option. The regularity of X allows us to obtain the Lipschitz property of the value function V in the initial data x.

In Chapter 3 we introduce a finite dimensional reduction of the diffusion process X. First we take the Yosida approximation of the unbounded operator A (cf. eq. (3)). This leads to an approximating diffusion $X^{(\alpha)}$. Next, we adopt a Galerkin scheme for the finite dimensional reduction. The approximating diffusion is denoted by $X^{(\alpha,n)}$. To each approximation we associate an optimal stopping problem. In particular V_α is the value function in (4) corresponding to $X^{(\alpha)}$ and similarly for $V_\alpha^{(n)}$. Then we prove convergences of different kinds for $n \to \infty$ and for $\alpha \to \infty$.

In Chapter 4 we shortly recall some results about variational inequalities in $\mathbb{R}^n$. Hence we localize the value function $V_\alpha^{(n)}$ and we show that it is the unique solution of a suitable variational problem. Then we characterize the optimal stopping time of $V_\alpha^{(n)}$ as the first time the value function equals the gain function.

Chapter 5 is mostly devoted to a priori estimates needed to take limits as n and α go to infinity. In particular, as a first step, we extend the variational inequality from the bounded domain to the whole $\mathbb{R}^n$. In order to do so we introduce a Gaussian measure on $\mathbb{R}^n$. Then we restrict the variational inequality to a closed convex subset of a suitable Banach space, denoted by $\mathcal{V}_n$. We provide a number of a priori estimates both on the solution of the localized variational inequality and on the bilinear form appearing in the variational inequality itself. The relevance of these estimates is in their independence on the order of the approximation procedure. In fact the estimates turn out to be *universal*, i.e. they are uniform with respect to all the indexes characterizing the finite dimensional approximation and the localization. Finally, we prove that $V_\alpha^{(n)}$ is a solution in a weak sense of a suitable variational inequality on $\mathbb{R}^n$.

In Chapter 6 we extend our results to infinite dimensions; that is we analyze what happens to the sequence of variational inequalities in the limit as $n \to \infty$. First, we provide a natural extension of the Gaussian measure to infinite dimensions (cf. [11]) that allows us to extend the Banach space $\mathcal{V}_n$ to its infinite dimensional counterpart, denoted by $\mathcal{V}$. We show that the finite dimensional optimal stopping problems and their related variational inequalities are special cases of a general infinite dimensional theory.

We start by providing some preliminary results about coefficients' convergence in the bilinear forms found in the variational inequalities. The PDE approach relies on a weak formulation which makes use of test functions from a convex set at each finite dimensional step. We then select a test function from each convex set so to obtain a sequence converging to a *good* test function in the corresponding infinite dimensional convex subset. We are then able to take the limit and prove that the value function V_α is a solution of an infinite dimensional variational inequality. Again the optimal stopping time is characterized as the first time the value function equals the gain function. The proof is based on both the dynamic programming principle at the finite dimensional level and probabilistic arguments. By passing to the limit in the Yosida approximation, i.e. as $\alpha \to \infty$, we prove that the original value function V is a solution of a variational inequality which no longer depends on the parameter α. The arguments of the proof are similar to those adopted in the limit as $n \to \infty$. However, when we try to explicitly characterize the bilinear form in the infinite dimensional variational inequality, we end up dealing with the unbounded operator A. The universal estimate found in Chapter 5 allows us to control the rate of growth of the Yosida approximation A_α and hence A itself. The bilinear form, well defined on a domain contained in $\mathcal{V}$, is then extended to the whole $\mathcal{V}$ in a suitable way.

In Chapter 7 we consider a simpler case that does not require the Yosida approximation. In such case we are able to characterize the asymptotic properties of the sequence of finite dimensional continuation regions and show how that sequence relates to the continuation region in infinite dimensions.

The Thesis is completed by several technical appendixes. In particular Appendix A contains a smoothing method used to take limits from the finite dimensional setting to the infinite dimensional one.

In conclusion, we study the problem of pricing an American Bond option when the underlying process is the whole forward rate curve under the HJM model. From the mathematical point of view, this problem gives rise to a parabolic degenerate variational inequality in a Hilbert space which, formally, is the analogue of what one would expect in $\mathbb{R}^n$. We provide a solution of the infinite dimensional variational inequality. Our existence result is new in the literature and contributes an original method of solution. Finally we characterize the solution as the value function of a suitable optimal stopping problem for a Hilbert space-valued diffusion process.

Chapter 1

The HJM model for forward interest rates.

In this chapter we describe the financial setting of our problem. In particular we introduce the mathematical tools needed to catch some fundamental features of the forward rate dynamics. We describe the Heath-Jarrow-Morton model in both its original formulation and in the Musiela's once. Afterwards we recall the well known connection of this model with the theory of SDE's in infinite dimensions. We introduce the suitable space of functions where the whole forward curve can be considered. Once the mathematical setting is determined we present the pricing problem as an optimal stopping problem. Some regularity properties of the gain function are discussed in the concluding section.

1.1 Standard formulation

The forward rate at time t for a loan taking place at a future time $v \geq t$ and returned one instant later, i.e. at $v + dv$, is commonly denoted by $f(t, v)$. When we set $v = t$ we recover the instantaneous interest rate (spot rate) and we denote it by $R(t) = f(t, t)$. For any fixed maturity v, the time evolution of the forward rate is described by a map $t \mapsto f(t, v)$, $t \leq v$. We are particularly interested in a stochastic model for this dynamics and hence the natural framework is the Heath-Jarrow-Morton model (HJM), [35].

Let us introduce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ generated by a d-dimensional Brownian motion $\{W_t\}_{0 \leq t \leq T}$. For simplicity we assume $d = 1$ and the filtration is taken continuous and augmented with sets of null measure. Let $\tilde{\sigma} : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a non negative, bounded and continuous function. Let us assume $\tilde{\sigma}$ is Lipschitz continuous w.r.t. the last variable. For any $\hat{T} > 0$ and any maturity $v \in [0, \hat{T}]$, the dynamic of the forward rate under the risk-neutral probability measure $\mathbb{P}$ is described, according to HJM, by the SDE

$$\begin{aligned} f(t,v) = f(0,v) + \int_0^t \tilde{\sigma}(u,v,f(u,v)) \int_u^v \tilde{\sigma}(u,s,f(u,s))ds\,du \\ + \int_0^t \tilde{\sigma}(u,v,f(u,v))dW_u\,. \end{aligned} \tag{1.1}$$

Here $f(0,v)$ is deterministic and denotes the initial data at time 0. This representation can be understood as an infinite family of SDEs depending on the continuous time parameter v. When $\tilde{\sigma}(u,v,f(u,v))$ is bounded, non negative and Lipschitz there exists a unique strong solution $f(\cdot,\cdot)$ continuous in both variables [46]. It was shown in [35] that the boundedness of the volatility cannot be substantially weakened. Doing so would produce the unpleasant fact that the forward rate process explodes in a finite time $\mathbb{P}$-a.s.

The particular form of the drift coefficient in the SDE is the key feature of the HJM model. This expression is in fact necessary and sufficient condition for the existence of a risk neutral probability measure and hence for the absence of arbitrage condition. One of the most striking features is that for any $v \leq \hat{T}$ fixed, the process $\{f(t,v), 0 \leq t \leq v\}$ is not Markovian. If we look at the drift term in the SDE, we see that the dynamics depends on the evolution of the whole curve. This fact remains true even though we consider a deterministic coefficient $\tilde{\sigma}$, as long as it explicitly depends on the process. Nevertheless the Markovian feature is recovered if we consider the infinite dimensional process $t \mapsto \{f(t,v), t \leq v \leq \hat{T}\}$ and it substantially changes our perspective. From now on, for any given $\omega \in \Omega$, we consider the map that associates at each time t the whole forward curve with maturities between time t and $\hat{T}$, i.e. $t \mapsto \{f(t,v,\omega), t \leq v \leq \hat{T}\}$. The price of a stochastic Zero Coupon Bond with maturity $\hat{T}$ is expressed in terms of the forward rate curve as

$$B(t,\hat{T},\omega) = \exp\left(-\int_t^{\hat{T}} f(t,v,\omega)dv\right). \tag{1.2}$$

It follows the Itô dynamics

$$dB(t,\hat{T}) = B(t,\hat{T})R(t)dt + B(t,\hat{T})a(t,\hat{T})dW_t, \tag{1.3}$$

where

$$a(t,\hat{T}) = -\int_t^{\hat{T}} \tilde{\sigma}(t,v,f(t,v))dv. \tag{1.4}$$

The stochastic discount factor process D is obtained as usual as the exponential of minus the cumulated spot rate up to time t, i.e.

$$D(t,\omega) = \exp\left(-\int_0^t f(s,s,\omega)ds\right). \tag{1.5}$$

Our aim is to characterize the price function of an American Put Option on a Zero Coupon Bond. We consider the option with a fixed maturity $T \leq \hat{T}$ and a strike price $K < 1$. The discounted payoff clearly holds in the form

$$D(t,\omega)\left(K - B(t,\hat{T},\omega)\right)^+ = e^{-\int_0^t f(u,u,\omega)du}\left(K - e^{-\int_t^{\hat{T}} f(t,v,\omega)dv}\right)^+.$$

When we evaluate the price of the option at a generic time t, we denote the forward rate curve at that time as $f_t := \{f(t,v),\ t \leq v \leq \hat{T}\}$. The no-arbitrage condition enables us to express the price of the option under the risk neutral probability measure, $\mathbb{P}$, as

$$V(t,f_t) = \sup_{t\leq\tau\leq T} \mathbb{E}_{t,f_t}\left[e^{-\int_t^\tau f(u,u)du}\left(K - e^{-\int_\tau^{\hat{T}} f(\tau,v)dv}\right)^+\right]. \tag{1.6}$$

The main feature of this valuation formula is that the price of the option depends on the whole structure of the forward rates between time t and $\hat{T}$. As we mentioned above, for $t \in [0,\hat{T}]$ fixed, the map $v \mapsto f(t,v)$ is a real valued continuous function on $[t,\hat{T}]$. This fact can be expressed as $f(t,\cdot) \in C([t,\hat{T}];\mathbb{R})$. It then turns out that at any given time $t \in [0,T]$ the value function can be understood as a map $V(t,\cdot) : C([t,\hat{T}];\mathbb{R}) \to \mathbb{R}$. As we see there is a quite complicated functional dependence which connects the time of evaluation to the set where the value function is defined. In particular the value function is defined on a time dependent infinite dimensional space.

For the problem being meaningful we require $K < 1$, otherwise the payoff would be always strictly positive because $B(t, \hat{T}) \leq 1$ for all $t \leq \hat{T}$. The basic condition $T \leq \hat{T}$ cannot be substantially weakened. In principle one could study the infinite horizon problem but, in order for the problem to be well posed from the financial angle, we must remember that $B(\hat{T}, \hat{T}) = 1$. Then, if we try to extend the ZCB dynamics over an infinite time interval, we must consider the original dynamics up to the maturity and then paste the ZCB process with a process greater or equal to one. It then implies that the optimal stopping time can only be before the ZCB's maturity $\hat{T}$.

1.2 The Musiela parametrization.

In order to overcome the formal difficulties arising in the definition of the value function, it is useful to rely on the theory of Infinite Dimensional Stochastic Differential Equations, cf. [19]. The first one to adopt this perspective was Marek Musiela, [47]. The so called Musiela's parametrization describes the forward rate curve $f(t, v)$ in terms of the time to maturity $x = v - t$ instead of the maturity v. This is simply done defining the forward rate curve by means of a new function $(t, x) \mapsto r_t(x)$. In terms of the original forward curve we have $f(t, v) = f(t, t + x) =: r_t(x)$. At any given time t the model input is the forward rate curve as a function $x \mapsto r_t(x)$. One can hence assume that the curve $r_t(\cdot)$ belongs to a suitable space of functions. The main features of this space will be explained in the next section. For the short rate we have the notation $R(t) = f(t, t) = r_t(0)$, which explicitly denotes the fact that it is the instantaneous rate, i.e. $x = 0$.

In order to rewrite the forward rate dynamics in the new parametrization we introduce the semigroup of bounded linear operators $\{S(t) \,|\, t \in \mathbb{R}_+\}$. It denotes the semigroup of right shifts which is defined as $S(t)h(x) = h(t + x)$ for any function $h : \mathbb{R}_+ \to \mathbb{R}$. Starting from equation (1.1) we can write

$$\begin{aligned} r_t(x) &= f(t, t + x) = f(0, t + x) \\ &\quad + \int_0^t \tilde{\sigma}(u, u + x + t - u, f(u, u + x + t - u)) \int_u^v \tilde{\sigma}(u, u + s - u, f(u, u + s - u)) ds\, du \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \tilde\sigma(u,u+t-u+x,f(u,u+t-u+x))dW_u\\
&= S(t)r_0(x)+\int_0^t \tilde\sigma(u,u+x+t-u,r_u(x+t-u))\int_0^{t-u+x}\tilde\sigma(u,u+y,r_u(y))dy\,du\\
&+\int_0^t \tilde\sigma(u,u+t-u+x,r_u(t-u+x))dW_u\\
&= S(t)r_0(x)+\int_0^t S(t-u)\sigma(u,x,r_u(x))\int_0^x \sigma(u,y,r_u(y))dy\,du\\
&+\int_0^t S(t-u)\sigma(u,x,r_u(x))dW_u,
\end{aligned}
$$

where $\sigma(t,x,r_t(x)) = \tilde\sigma(t,t+x,f(t,t+x))$. There is no substantial loss in generality if we assume time homogeneous volatility, i.e. if we set $\sigma(t,x,r_t(x)) = \sigma(r_t)(x)$.

For simplicity we denote

$$
F_\sigma(r_t)(x) := \sigma(r_t)(x)\int_0^x \sigma(r_t)(y)dy, \qquad x\in\mathbb{R}_+. \tag{1.7}
$$

The dynamic of the forward rate curve is completely characterized by the integral equation

$$
r_t(x) = S(t)r_0(x)+\int_0^t S(t-u)F_\sigma(r_u)(x)du+\int_0^t S(t-u)\sigma(r_u)(x)dW_u. \tag{1.8}
$$

The connection with the theory of infinite dimensional SDEs is then rather natural (cf. [19]). Set $S>\hat T$, then under appropriate conditions (1.8) represents the so called *mild* solution to the SDE

$$
\begin{cases}
dr_t = [Ar_t + F_\sigma(r_t)]\,dt + \sigma(r_t)dW_t, & t\in[0,S],\\
r_0 = r\in\mathcal{H}.
\end{cases} \tag{1.9}
$$

Here $\mathcal{H}$ is a suitable function space wherein the semigroup $S(t)$ is strongly continuous and A represents its infinitesimal generator (cf. [49]). From now on we assume $\mathcal{H}$ to be a Hilbert space, with norm $\|\cdot\|_{\mathcal{H}}$ and scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. The infinitesimal generator of the semigroup is an unbounded linear operator $A : D(A)\subset\mathcal{H}\to\mathcal{H}$. The choice of the Hilbert space has to be coherent with this formulation. The time horizon $[0,S]$ is arbitrary but it has to be large enough in order to satisfy $T\le\hat T\le S$. An exhaustive study in this sense has been carried out by Filipovic [26]. Here we follow his approach and provide a short summary of the essential facts.

Since (1.8) holds pointwisely the Hilbert space $\mathcal{H}$ has to be such that the pointwise evaluation is well defined on it. In order for the ZCB price to be meaningful it must also be $\mathcal{H} \subset L^1_{loc}(\mathbb{R}_+)$. The semigroup $\{S(t) \mid t \in \mathbb{R}_+\}$ has to be strongly continuous on $\mathcal{H}$ and the drift term has to fulfill some integrability conditions. Filipovic summarizes these hypothesis as follows

(H1) Each function $h \in \mathcal{H}$ has a continuous representative and the pointwise evaluation $\mathcal{J}_x(h) := h(x)$ is a continuous linear functional on $\mathcal{H}$, for all $x \in \mathbb{R}_+$.

(H2) The semigroup $\{S(t) \mid t \in \mathbb{R}_+\}$ is strongly continuous in $\mathcal{H}$ with infinitesimal generator denoted by A.

(H3) There exists a constant K such that

$$\|F_\sigma(h)\|_{\mathcal{H}} \leq K\|h\|^2_{\mathcal{H}},$$

for all $h \in \mathcal{H}$ with $F_\sigma(h) \in \mathcal{H}$.

In the next section we will shortly introduce the Hilbert space proposed by Filipovic and we will see how it satisfies all these hypotheses.

Before characterizing the Hilbert space it is worth mentioning the standard measurability conditions on the coefficients of (1.8) (cf. [19], Chapter 7). In particular it is required that $\sigma : \mathcal{H} \to L^0_2(\mathcal{H})$ is measurable as a map from $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ to $(L^0_2(\mathcal{H}), \mathcal{B}(L^0_2(\mathcal{H})))$. Here $L^0_2(\mathcal{H})$ represents the space of Hilbert-Schmidt operators in $\mathcal{H}$. Since we deal with a one dimensional Brownian motion, $\sigma(h) \in L(\mathbb{R}, \mathcal{H})$ for $h \in \mathcal{H}$. For $\sigma(h)$ to be Hilbert-Schmidt it is enough $\sigma(h) \in \mathcal{H}$.
We recall now the precise notions of *mild* and *weak* solution for (1.9), cf. [19].

Definition 1.2.1 *Let us consider a predictable $\mathcal{H}$-valued process $\{r_t,\, t \in [0, S]\}$ such that*

$$\mathbb{P}\left(\int_0^S \|r_t\|^2_{\mathcal{H}} dt < \infty\right) = 1,$$

then

i) r_t *is said to be a mild solution to (1.9) if for arbitrary* $t \in [0, S]$ *we have*

$$r_t = S(t)r_0 + \int_0^t S(t-u)F_\sigma(r_u)du + \int_0^t S(t-u)\sigma(r_u)dW_u.$$

ii) r_t *is said to be a weak solution to (1.9) if*

$$\mathbb{P}\left(\int_0^T \|\sigma(r_s)\|^2_{L_2^0} ds < \infty\right) = 1$$

and, for arbitrary $t \in [0, S]$ *and* $\zeta \in D(A^*)$ *we have*

$$\langle r_t, \zeta\rangle = \langle r_0, \zeta\rangle + \int_0^t \Big(\langle r_s, A^*\zeta\rangle_{\mathcal{H}} + \langle F_\sigma(r_s), \zeta\rangle_{\mathcal{H}}\Big)ds + \int_0^t \langle \zeta, \sigma(r_s)dW_s\rangle_{\mathcal{H}}.$$

We can finally introduce the Hilbert space where it is natural to set the whole problem.

1.3 Hilbert space characterization

The results contained here represent a short survey of [26], Chapter 5. It is reasonable to assume the integrability of the forward curve in the following sense

$$\int_{\mathbb{R}_+} |r_t(x)|^2 dx < \infty.$$

This is coherent with the bootstrapping and smoothing algorithms adopted by the practitioners when estimating the forward curve by data points. We also expect the forward curve to flatten for large maturities since it seems reasonable that the prices of loans with large maturities cannot substantially differ one from another. This can be modeled by adopting some increasing weighting function $w(x)$ in order to get

$$\int_{\mathbb{R}_+} |r_t(x)|^2 w(x) dx < \infty.$$

It does not provide a norm yet. In fact all the flat curves are indistinguishable. To avoid this unpleasant feature we add the square of the short rate $|r_t(0)|^2$. We then define the space $\mathcal{H}_w$ as follows:

Definition 1.3.1 *Let $w : \mathbb{R}_+ \to [1, +\infty)$ be a non decreasing C^1-function such that*

$$w^{-\frac{1}{3}} \in L^1(\mathbb{R}_+).$$

We write

$$\|h\|_w := |h(0)|^2 + \int_{\mathbb{R}_+} |h'(x)|^2 w(x) dx$$

and define

$$\mathcal{H}_w := \{h \in L^1_{loc}(\mathbb{R}_+) \,|\, \exists h' \in L^1_{loc}(\mathbb{R}_+) \text{ and } \|h\|_w < \infty\}.$$

The derivative is understood in the weak sense. It is known from real analysis that if h has a weak derivative h' then there exists an absolutely continuous representative of h, which we still denote by h, such that

$$h(x) = h(0) + \int_0^x h'(y) dy.$$

The choice of the space $\mathcal{H}_w$ is the right one according to the hypotheses (**H1**)-(**H2**) as stated in the following theorem:

Theorem 1.3.1 *The set $\mathcal{H}_w$ equipped with the norm $\|\cdot\|_w$ forms a separable Hilbert space meeting **(H1)-(H2)**.*

The proof of the theorem is given in [26], Theorem 5.1.1. It is anyway interesting to state some intermediate results which better clarify the role of the Hilbert space. An important property for our purposes is the following continuous embedding (cf. [26], Chapter 5, equation 5.4):

$$\sup_{x \in \mathbb{R}_+} |h(x)| \leq C \|h\|_w, \qquad h \in \mathcal{H}_w. \tag{1.10}$$

The choice of the Hilbert space sets some constraints on the possible candidate functions describing the volatility structure. We summarize it in the next proposition.

Proposition 1.3.1 *Let us denote by $\mathcal{H}^0_w$ the set*

$$\mathcal{H}^0_w := \{h \in \mathcal{H}_w \,|\, h(\infty) = 0\}.$$

Then $\mathcal{H}^0_w$ is a closed subspace of $\mathcal{H}_w$ and F_σ takes values in $\mathcal{H}_w$ if and only if σ takes values in $\mathcal{H}^0_w$.

The proof of this fact is in [26], Chapter 5, equation (5.13). This result basically gives a condition on the allowed volatility structures. We are supposed to chose volatility structures such that $\sigma(h)(x) \to 0$ when $x \to \infty$ for any $h \in \mathcal{H}_w$. A simple extension of Corollary 5.1.2 in [26], Chapter 5, guarantees that the following proposition holds.

Proposition 1.3.2 *Let $\sigma : \mathcal{H}_w \to \mathcal{H}_w^0$ be bounded and uniformly Lipschitz. Then*

$$\|F_\sigma(f) - F_\sigma(h)\|_w \leq L\|f - h\|_w, \qquad \forall f, h \in \mathcal{H}.$$

The next result identifies the infinitesimal generator A of $S(t)$ on $\mathcal{H}_w$.

Proposition 1.3.3 *We have $D(A) = \{h \in \mathcal{H}_w \,|\, h' \in \mathcal{H}_w\}$ and $Ah = h'$.*

Let us now state the existence and uniqueness theorem for the solution of the $\infty-$dimensional SDE (1.9).

Theorem 1.3.2 *Let $\sigma : \mathcal{H}_w \to \mathcal{H}_w^0$ be bounded and uniformly Lipschitz. Then there exists a unique mild solution to the stochastic differential equation (1.9). Moreover for any $T \in \mathbb{R}_+$ and $p \geq 2$ there exists a constant C_T such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|r_t\|_w^p\right] \leq C_T\left(1 + \|r_0\|_w^p\right). \tag{1.11}$$

The unique mild solution also coincides with the unique weak solution.

PROOF: We know from Proposition (1.3.2) that under the hypotheses of the theorem, F_σ is bounded and uniformly Lipschitz. Existence and uniqueness for a mild solution and the coincidence of mild and weak solutions hold as a consequence of [19], Theorem 7.4 and Theorem 6.5, or [26], Theorem 2.4.1. ■

Remark 1.3.1 *It is worth clarifying that [19], Theorem 7.4 provides the estimate*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|r_t\|_w^p\right] \leq C_{T,p}\left(1 + \|r_0\|_w^p\right), \tag{1.12}$$

only for all $p > 2$ and [26], Theorem 2.4.1 provides an estimate for $p = 2$ but with an arbitrary positive constant C_T independent of p. This two notations can be connected adopting a simple argument relying on Jensen's inequality. For any $p > 2$

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|r_t\|_w^2\right] \le \left(\mathbb{E}\left[\sup_{t\in[0,T]} \|r_t\|_w^p\right]\right)^{\frac{2}{p}} \le C_T\left(1 + \|r_0\|_w^2\right),$$

and the dependency on p can be suppressed because of the arbitrariness of p itself. For instance by taking the infimum over all $p > 2$ in the family $(C_{T,p})_{p>2}$ of expression (1.12)*, provided that it is strictly positive.*

A very important feature of the HJM model is that it can describe positive rates. More precisely in [35], Proposition 5, it is shown that the HJM model is compatible with non-negative forward curves. In particular, for $\hat{T} > 0$, $\lambda > 0$, $C > 0$, the choice $\tilde{\sigma}(t, \hat{T}, f(t, \hat{T})) = C \min\{\lambda, f(t, \hat{T})\}$ guarantees existence and uniqueness of a non-negative solution to the SDE (1.1) for any non-negative initial data $f(0, \hat{T})$. In our framework the equivalent choice is $\sigma(r)(x) = \gamma(x)\min\{\lambda, r(x)\}$, $r \in \mathcal{H}_w$ and $\gamma \in \mathcal{H}_w^0$. In this case the non negativity is granted for any $t \in [0, T]$ in the portion of curve $\{r_t(x),\, x \in [0, T - t]\}$. Therefore this is enough to guarantee positivity of the rates in the pricing formula that we will introduce below.

1.4 The pricing problem in Musiela's parametrization

When adopting the Musiela's parametrization one has to rewrite the price function according to the new notation. In particular the ZCB price is described in terms of the process r, as

$$B(t, \hat{T}) = \exp\left(-\int_0^{\hat{T}-t} r_t(x)dx\right). \tag{1.13}$$

It easily follows, for the stochastic discount factor D, the notation

$$D(t) = \exp\left(-\int_0^t r_s(0)ds\right). \tag{1.14}$$

The discounted payoff is then written as

$$e^{-\int_0^t r_u(0)du}\left(K - e^{-\int_0^{\hat{T}-t} r_t(x)dx}\right)^+.$$

As we mentioned above, it is not restrictive to assume a non negative rate in the HJM model. Thanks to this fact there is no loss of generality in considering a different definition of the discounted payoff, namely

$$e^{-\int_0^t (r_u(0))^+ du} \left(K - e^{-\int_0^{\hat{T}-t} r_t(x)dx} \right)^+ .$$

It does not introduce any modification of the original problem but it will turn out to be a useful trick in the finite dimensional analysis that we are going to perform in the next sections of this thesis. In practice we just strengthen the uniform boundedness of the discount factor. It is now natural to write the price function as

$$V(t, r_t) = \sup_{t \le \tau \le T} \mathbb{E}_{t,r_t} \left[e^{-\int_t^\tau (r_u(0))^+ du} \left(K - e^{-\int_0^{\hat{T}-\tau} r_\tau(x)dx} \right)^+ \right] . \tag{1.15}$$

It is worth noticing that as long as we pick initial datas from the set of functions in $r \in \mathcal{H}_w$ such that $r(x) \ge 0$, $x \in [0, \hat{T} - t]$, the value function defined above coincides with the original one. Roughly speaking, if $V(t, r_t) \in C([0, T] \times \mathcal{H})$, we can consider this modification as a continuous extension of the original value function outside the set of positive forward curves.

We are mostly interested in the regularity properties of the discounted payoff. We first denote the undiscounted payoff by

$$\Psi(t, r_t) = \left(K - e^{-\int_0^{\hat{T}-t} r_t(x)dx} \right)^+ ,$$

and prove some regularity properties. We can state the following results:

Proposition 1.4.1 *There exist $C_1, C_2 > 0$ such that the payoff Ψ has the following properties*

$$\sup_{(t,h) \in [0,T] \times \mathcal{H}_w} |\Psi(t, h)| \le K,$$

$$\sup_{t \in [0,T]} |\Psi(t, h) - \Psi(t, g)| \le C_1 \|h - g\|_w, \qquad \forall g, h \in \mathcal{H}_w,$$

$$|\Psi(s, h) - \Psi(t, h)| \le C_2 \|h\|_w \, |t - s|, \qquad h \in \mathcal{H}_w.$$

PROOF: The first assertion is obvious. In order to prove the second one we rely on the continuous injection $\mathcal{H}_w \hookrightarrow L^\infty(\mathbb{R}_+)$. In particular we have

$$\big|\Psi(t,h) - \Psi(t,g)\big| = \Big|\,\Big(K - e^{-\int_0^{\hat{T}-t} h(x)dx}\Big)^+ - \Big(K - e^{-\int_0^{\hat{T}-t} g(x)dx}\Big)^+\,\Big|.$$

It is easy to verify that the weak derivative of the function $\zeta : \mathbb{R} \to \mathbb{R}$,

$$\zeta(x) := (K - e^x)^+$$

is

$$\zeta'(x) = \begin{cases} 0 & x \geq \ln K, \\ -e^x & x < \ln K. \end{cases}$$

It implies that $\|\zeta'\|_{L^\infty(\mathbb{R})} \leq K < 1$ and hence (cf. for instance [14], Ch. 8, Prop 8.4)

$$|\zeta(x) - \zeta(y)| \leq \|\zeta'\|_{L^\infty(\mathbb{R})}|x-y| \leq |x-y|. \tag{1.16}$$

Now, if we denote $X = -\int_0^{\hat{T}-t} h(x)dx$ and $Y = -\int_0^{\hat{T}-t} g(x)dx$, then clearly

$$\big|\,\big(K - e^X\big)^+ - \big(K - e^Y\big)^+\,\big| \leq |X - Y|.$$

Hence we can conclude

$$\begin{aligned}|\Psi(t,h) - \Psi(t,g)| &\leq \left|\int_0^{\hat{T}-t} h(x)dx - \int_0^{\hat{T}-t} g(x)dx\right| \leq \int_0^{\hat{T}-t} |h(x) - g(x)|dx \\ &\leq \hat{T} \sup_{x\in\mathbb{R}_+} |h(x) - g(x)| \leq C\,\hat{T}\,\|h-g\|_w.\end{aligned}$$

The second claim is proved. For the third one we proceed in the same way as above and we get

$$|\Psi(t,h) - \Psi(s,h)| \leq \Big|\int_0^{\hat{T}-t} h(x)dx - \int_0^{\hat{T}-s} h(x)dx\Big|.$$

With no loss in generality we assume $s \leq t$ and obtain

$$|\Psi(t,h) - \Psi(s,h)| \leq \int_{\hat{T}-t}^{\hat{T}-s} |h(x)|dx \leq \sup_{x\in\mathbb{R}_+} |h(x)|\,|t-s| \leq C\|h\|_w\,|t-s|.$$

The proof is now complete. ■

This regularity is substantially preserved when considering the discounted payoff. Nevertheless the regularity which will play a crucial role in the analysis of the next sections is the one w.r.t. the space variable.

Corollary 1.4.1 *Let X and Y be two $\mathcal{H}_w$-valued stochastic processes. Then*

$$\sup_{t\in[0,T]}\left|e^{-\int_0^t (X_u(0)(\omega))^+\,du}\Psi(t,X_t(\omega)) - e^{-\int_0^t (Y_u(0)(\omega))^+\,du}\Psi(t,Y_t(\omega))\right| \\ \leq (C_1 + K\,C\,T)\sup_{t\in[0,T]}\|X_t(\omega)-Y_t(\omega)\|_w, \quad \mathbb{P}\text{-a.e.}\,\omega\in\Omega,$$

$$\left|e^{-\int_0^s (X_u(0)(\omega))^+\,du}\Psi(s,X_s(\omega)) - e^{-\int_0^t (X_u(0)(\omega))^+\,du}\Psi(t,X_t(\omega))\right| \\ \leq (K + C_2)\sup_{t\in[0,T]}\|X_t\|_w\,|t-s| + C_1\|X_t(\omega)-X_s(\omega)\|_w, \quad \mathbb{P}\text{-a.e.}\,\omega\in\Omega.$$

PROOF: The proof follows the same lines as above, in fact

$$\begin{aligned}
&\sup_{t\in[0,T]}\left|e^{-\int_0^t (X_u(0)(\omega))^+\,du}\Psi(t,X_t(\omega)) - e^{-\int_0^t (Y_u(0)(\omega))^+\,du}\Psi(t,Y_t(\omega))\right| \\
&\leq K\sup_{t\in[0,T]}\left|e^{-\int_0^t (X_u(0)(\omega))^+\,du} - e^{-\int_0^t (Y_u(0)(\omega))^+\,du}\right| \\
&\quad + \sup_{t\in[0,T]}|\Psi(t,X_t(\omega)) - \Psi(t,Y_t(\omega))| \\
&\leq K\sup_{t\in[0,T]}\int_0^t \left|(X_u(0)(\omega))^+ - (Y_u(0)(\omega))^+\right| du \\
&\quad + C_1\sup_{t\in[0,T]}\|X_t(\omega)-Y_t(\omega)\|_w.
\end{aligned}$$

Here we use the fact that

$$\begin{aligned}
\left|(X_u(0)(\omega))^+ - (Y_u(0)(\omega))^+\right| &\leq |X_u(0)(\omega) - Y_u(0)(\omega)| \\
&\leq \|X_u(\omega)-Y_u(\omega)\|_{L^\infty(\mathbb{R}_+)} \leq C\|X_u(\omega)-Y_u(\omega)\|_w,
\end{aligned}$$

and conclude

$$\begin{aligned}
&\sup_{t\in[0,T]}\left|e^{-\int_0^t (X_u(0)(\omega))^+\,du}\Psi(t,X_t(\omega)) - e^{-\int_0^t (Y_u(0)(\omega))^+\,du}\Psi(t,Y_t(\omega))\right| \\
&\quad \leq (C_1 + K\,C\,T)\sup_{t\in[0,T]}\|X_t(\omega)-Y_t(\omega)\|_w.
\end{aligned}$$

The proof is the same for the second inequality. ∎

Chapter 2

The optimal stopping problem

In this chapter we simplify the notation of the original problem. In order to do so we consider a simplified SDE which is indeed equivalent to the original one. Moreover, we get rid of the stochastic discount factor in the optimal stopping problem. Since the discount rate is non negative, this simplification does not affect the rationale in the variational inequality approach. Some regularities of the value function are pointed out and they hold in the original problem as well.

2.1 Simplified setting

In order to study the variatonal inequality associated to the pricing problem we prefer to simplify the setting and without loss in generality we can consider a problem of the following form. Let $\mathcal{H}$ represent a separable Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and induced norm $\| \cdot \|_{\mathcal{H}}$. Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be the infinitesimal generator of a strongly continuous semigroup of operators $\{S(t), t \geq 0\}$ on $\mathcal{H}$, cf. [49]. Let us now consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $W := (W^0, W^1, W^2 \ldots)$ be an infinite dimensional standard Brownian motion on the probability space. The filtration generated by the Brownian motion is $\{\mathcal{F}_t, t \in [0, S]\}$ and is augmented with the sets of null measure. We consider a particular SDE in the Hilbert space and denote by X its unique mild solution, cf. [19]. As before we consider $S \geq \hat{T} \geq T$. Let the SDE hold in the form

$$\begin{cases} dX_t = AX_t dt + \sigma(X_t) dW^0_t, & t \in [0, S], \\ X_0 = x. \end{cases} \tag{2.1}$$

We are interested in the case of SDEs driven by a finite dimensional Brownian motion. There is no loss in generality assuming for simplicity a one dimensional case. The diffusion coefficient $\sigma : \mathcal{H} \to \mathcal{H}$ is a continuous map and satisfies Lipschitz condition and sublinear growth condition. Some other assumption on σ will be needed and we will discuss them later.

We introduce now the function $\Psi : [0, S] \times \mathcal{H} \to \mathbb{R}$ representing the gain function of the optimal stopping problem. Coherently with the observations of the previous section, we make the following assumptions

Assumption 2.1.1 *The function Ψ is uniformly bounded on $\mathcal{H}$ by a constant $\overline{\Psi} \in \mathbb{R}$, i.e.*

$$\sup_{(t,x)\in[0,S]\times\mathcal{H}} |\Psi(t, x)| \le \overline{\Psi}. \tag{2.2}$$

Moreover, the following regularities hold:

$$|\Psi(t, x) - \Psi(t, y)| \le L_1 \|x - y\|_{\mathcal{H}} \qquad \forall t \in [0, S],\ x, y \in \mathcal{H}, \tag{2.3}$$

and

$$|\Psi(t, x) - \Psi(s, x)| \le L_2 \eta(\|x\|_{\mathcal{H}})\, |t - s| \qquad \forall x \in \mathcal{H},\ 0 \le s \le t \le S. \tag{2.4}$$

Here $\eta : \mathcal{H} \to \mathbb{R}$ satisfies a polynomial growth condition $|\eta(x)| \le C(1 + \|x\|^p_{\mathcal{H}})$, for some $p \ge 1$.

The Optimal Stopping problem we are going to analyze has the following form: given $T \le S$ let the value function V be defined as

$$V(t, x) := \sup_{0\le\tau\le T} \mathbb{E}\left[\Psi(\tau, X^{t,x}_\tau)\right], \tag{2.5}$$

where the supremum is taken over the class of all stopping times of the filtration $\{\mathcal{F}_t, t \in [0, S]\}$. This class of problems was previously studied in [2, 55, 56] but we now propose a completely new algorithm for the characterization of the solution.

2.2 Regularity of the value function

In this section we will prove some regularity results about the value function. We first need an auxiliary result on our SDE.

Proposition 2.2.1 *Let X^x and X^y be two mild solutions of (2.1) with initial data respectively equal to x and y. Then for $p > 2$*

$$\mathbb{E}\left[\sup_{0\le t\le S}\|X_t^x - X_t^y\|_{\mathcal{H}}^p\right] \le C(p,S,M,L_\sigma)\|x-y\|_{\mathcal{H}}^p. \tag{2.6}$$

PROOF: From the definition of mild solution we have

$$X_t^x = S(t)x + \int_0^t S(t-u)\sigma(X_u^x)dW_u^0,$$

and

$$X_t^y = S(t)y + \int_0^t S(t-u)\sigma(X_u^y)dW_u^0.$$

Hence

$$\begin{aligned}\|X_t^x - X_t^y\|_{\mathcal{H}}^p &= \|S(t)(x-y) + \int_0^t S(t-u)\left[\sigma(X_u^x)-\sigma(X_u^y)\right]dW_u^0\|_{\mathcal{H}}^p \\ &\le 2^{p-1}\left(\|S(t)(x-y)\|_{\mathcal{H}}^p + \|\int_0^t S(t-u)\left[\sigma(X_u^x)-\sigma(X_u^y)\right]dW_u^0\|_{\mathcal{H}}^p\right).\end{aligned}$$

We now take the supremum over all times and the average value,

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le S}\|X_t^x - X_t^y\|_{\mathcal{H}}^p\right] \le 2^{p-1}\Bigg(&\sup_{0\le t\le S}\|S(t)(x-y)\|_{\mathcal{H}}^p \\ &+\mathbb{E}\left[\sup_{0\le t\le S}\|\int_0^t S(t-u)\left[\sigma(X_u^x)-\sigma(X_u^y)\right]dW_u^0\|_{\mathcal{H}}^p\right]\Bigg).\end{aligned}$$

We know from semigroup theory [49] that the C_0-semigroup is uniformly bounded on $[0,S]$, i.e. $\sup_{0\le t\le S}\|S(t)\|_{\mathcal{H}} \le M$. Moreover for the stochastic convolution we can rely on [20], which guarantees

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le S}\|\int_0^t S(t-u)\left[\sigma(X_u^x)-\sigma(X_u^y)\right]dW_u^0\|_{\mathcal{H}}^p\right] \\ &\quad\le c_p M^p S^{\frac{p}{2}-1}\mathbb{E}\left[\int_0^S \|\sigma(X_u^x)-\sigma(X_u^y)\|_{\mathcal{H}}^p du\right].\end{aligned} \tag{2.7}$$

Summarizing, for a suitable constant $\hat{c}(p, S, M) > 0$, we get

$$\mathbb{E}\left[\sup_{0\le t\le S}\|X_t^x - X_t^y\|_{\mathcal{H}}^p\right] \le \hat{c}(p,S,M)\Bigg(\|x-y\|_{\mathcal{H}}^p + \mathbb{E}\left[\int_0^S \|\sigma(X_u^x) - \sigma(X_u^y)\|_{\mathcal{H}}^p du\right]\Bigg).$$

From the Lipschitz property of σ with constant L_σ

$$\mathbb{E}\left[\sup_{0\le t\le S}\|X_t^x - X_t^y\|_{\mathcal{H}}^p\right] \le \hat{c}(p,S,M,L_\sigma)\Bigg(\|x-y\|_{\mathcal{H}}^p + \mathbb{E}\left[\int_0^S \sup_{0\le s\le u}\|X_s^x - X_s^y\|_{\mathcal{H}}^p du\right]\Bigg)$$

Now we apply Gronwall's lemma to $f(S) := \mathbb{E}\left[\sup_{0\le t\le S}\|X_t^x - X_t^y\|_{\mathcal{H}}^p\right]$ and then obtain the estimate

$$\mathbb{E}\left[\sup_{0\le t\le S}\|X_t^x - X_t^y\|_{\mathcal{H}}^p\right] \le \hat{c}(p,S,M,L_\sigma)\|x-y\|_{\mathcal{H}}^p \cdot e^{\hat{c}(p,S,M,L_\sigma)\cdot S}.$$

The proof is complete with $C(p,S,M,L_\sigma) = \hat{c}(p,S,M,L_\sigma)e^{\hat{c}(p,S,M,L_\sigma)\cdot S}$. ■

Even though the inequality (2.7) does not apply for $p = 1, 2$ (cf. [20]), if we set $\bar{p} > 2$ we can deduce another inequality.

Corollary 2.2.1 *Let X^x and X^y be respectively two mild solutions of (2.1) with initial data x and y. For $p > 2$, the following holds*

$$\mathbb{E}\left[\sup_{0\le t\le S}\|X_t^x - X_t^y\|_{\mathcal{H}}\right] \le \sqrt[\bar{p}]{C(p,S,M,L_\sigma)}\|x-y\|_{\mathcal{H}}. \tag{2.8}$$

PROOF: Since $p > 2$, the map $x \mapsto \sqrt[p]{x}$ is concave and monotone on $x \ge 0$. Then

$$\mathbb{E}\left[\sup_{0\le t\le S}\|X_t^x - X_t^y\|_{\mathcal{H}}\right] = \mathbb{E}\left[\sqrt[p]{\sup_{0\le t\le S}\|X_t^x - X_t^y\|_{\mathcal{H}}^p}\right],$$

and from Jensen's inequality

$$\mathbb{E}\left[\sup_{0\le t\le S}\|X_t^x - X_t^y\|_{\mathcal{H}}\right] \le \sqrt[p]{\mathbb{E}\left[\sup_{0\le t\le S}\|X_t^x - X_t^y\|_{\mathcal{H}}^p\right]} \le \sqrt[p]{C(p,S,M,L_\sigma)}\|x-y\|_{\mathcal{H}}.$$

■

In general we might set $\hat{C}(S, M, L_\sigma) := \inf_{p>2} \sqrt[p]{C(p, S, M, L_\sigma)}$ and obtain a universal constant.

We can now state the regularity properties of the value function.

Proposition 2.2.2 *The value function $V(t, x)$ is such that*

$$\sup_{(t,x)\in[0,T]\times\mathcal{H}} |V(t, x)| \leq \overline{\Psi}, \tag{2.9}$$

moreover there exists $L_V > 0$ such that

$$|V(t, x) - V(t, y)| \leq L_V \|x - y\|_{\mathcal{H}}, \qquad \forall t \in [0, S],\ x, y \in \mathcal{H}. \tag{2.10}$$

PROOF: The first claim is an obvious consequence of the uniform boundedness of the gain function Ψ. Let us verify the second claim. Let us consider

$$\begin{aligned}
V(t, x) - V(t, y) &= \sup_{t\leq\tau_1\leq T} \mathbb{E}\left[\Psi(\tau_1, X^{t,x}_{\tau_1})\right] - \sup_{t\leq\tau_2\leq T} \mathbb{E}\left[\Psi(\tau_2, X^{t,y}_{\tau_2})\right] \\
&= \sup_{t\leq\tau_1\leq T} \inf_{t\leq\tau_2\leq T} \mathbb{E}\left[\Psi(\tau_1, X^{t,x}_{\tau_1}) - \Psi(\tau_2, X^{t,y}_{\tau_2})\right] \\
&\leq \sup_{t\leq\tau_1\leq T} \mathbb{E}\left[\Psi(\tau_1, X^{t,x}_{\tau_1}) - \Psi(\tau_1, X^{t,y}_{\tau_1})\right] \\
&\leq \mathbb{E}\left[\sup_{t\leq s\leq T} |\Psi(s, X^{t,x}_s) - \Psi(s, X^{t,y}_s)|\right] \\
&\leq L_1 \mathbb{E}\left[\sup_{t\leq s\leq T} \|X^{t,x}_s - X^{t,y}_s\|_{\mathcal{H}}\right].
\end{aligned}$$

The same estimate holds for $V(t, y) - V(t, x)$ and hence

$$|V(t, x) - V(t, y)| \leq L_1 \mathbb{E}\left[\sup_{t\leq s\leq T} \|X^{t,x}_s - X^{t,y}_s\|_{\mathcal{H}}\right].$$

Notice that the coefficients in (2.1) are time homogeneous, hence

$$\mathbb{E}\left[\sup_{t\le s\le T}\|X_s^{t,x}-X_s^{t,y}\|_{\mathcal{H}}\right]=\mathbb{E}\left[\sup_{0\le s\le T-t}\|X_s^{0,x}-X_s^{0,y}\|_{\mathcal{H}}\right]$$
$$\le\mathbb{E}\left[\sup_{0\le s\le S}\|X_s^{0,x}-X_s^{0,y}\|_{\mathcal{H}}\right]\le\hat{C}(S,M,L_\sigma)\|x-y\|_{\mathcal{H}}.$$

We get the last inequality from Corollary 2.2.1. Hence, setting $L_V := L_1\,\hat{C}(S,M,L_\sigma)$, the proof is complete. ■

Chapter 3

Approximation scheme

In this chapter we reduce the infinite dimensional SDE to a sequence of suitable finite dimensional SDEs. In order to do so we rely first on the Yosida approximation of the unbounded operator appearing in the infinite dimensional SDE. Later on we perform a Galerkin scheme for the finite dimensional reduction. We prove numerous convergence results for the approximating processes and the respective approximating optimal stopping problems.

3.1 Yosida approximation

It is often unpleasant to deal with unbounded linear operators. Our aim is to provide an approximation scheme as easy as possible. Then without further assumptions about the operator A a simple natural step is to introduce its Yosida approximation A_α (cf. Appendix B). For $\alpha > 0$ given, we introduce the diffusion process $X^{(\alpha)}$ as the unique solution of the SDE

$$\begin{cases} dX_t^{(\alpha)} = A_\alpha X_t^{(\alpha)} dt + \sigma(X_t^{(\alpha)}) dW_t^0, & t \in [0, S], \\ X_0^{(\alpha)} = x. \end{cases} \tag{3.1}$$

It is important now to stress that since A_α is a bounded linear operator on $\mathcal{H}$, then $X^{(\alpha)}$ is the unique *strong* solution for the SDE, i.e. the integral representation holds

$$X_t^{(\alpha)} = x + \int_0^t A_\alpha X_s^{(\alpha)} ds + \int_0^t \sigma(X_s^{(\alpha)}) dW_s^0, \qquad t \in [0, S], \ \mathbb{P}\text{-a.s.}$$

Every strong solution is also a mild solution and then $X^{(\alpha)}$ might be interpreted equivalently as

$$X_t^{(\alpha)} = e^{tA_\alpha}x + \int_0^t e^{(t-s)A_\alpha}\sigma(X_s^{(\alpha)})dW_s^0, \qquad t \in [0,S],\ \mathbb{P}\text{-a.s.}$$

The latter expression is useful to understand the following proposition which is proven in [19], Proposition 7.5, Chapter 7.

Proposition 3.1.1 *Let X^x be the unique mild solution of equation (2.1) and $X^{(\alpha)\,x}$ the unique strong solution of equation (3.1). For $p \geq 1$, the following convergence holds*

$$\lim_{\alpha\to\infty} \mathbb{E}\left[\sup_{0\leq t\leq S} \|X_t^{(\alpha)\,x} - X_t^x\|_{\mathcal{H}}^p\right] = 0, \qquad \forall x \in \mathcal{H}.$$

We can simply define an approximating optimal stopping problem substituting $X^{(\alpha)}$ to X, i.e. defining V_α as

$$V_\alpha(t,x) = \sup_{t\leq\tau\leq T} \mathbb{E}\left[\Psi(\tau, X_\tau^{(\alpha)\,t,x})\right].$$

The convergence shown above extends to the value function and we have the following theorem

Theorem 3.1.1 *For any $x \in \mathcal{H}$ given and fixed the pointwise convergence holds*

$$\lim_{\alpha\to\infty} \sup_{0\leq t\leq T} |V_\alpha(t,x) - V(t,x)| = 0. \tag{3.2}$$

PROOF: From the Lipschitz property of the gain function we simply argue as in Proposition 2.2.2 and obtain

$$\begin{aligned} |V_\alpha(t,x) - V(t,x)| &\leq \mathbb{E}\left[\sup_{t\leq s\leq T}\left|\Psi(s, X_s^{(\alpha)\,t,x}) - \Psi(s, X_s^{t,x})\right|\right] \\ &\leq \mathbb{E}\left[\sup_{t\leq s\leq T}\left\|X_s^{(\alpha)\,t,x} - X_s^{t,x}\right\|_{\mathcal{H}}\right]. \end{aligned}$$

Since the coefficients in (3.1) are time homogeneous we get

$$\begin{aligned} \mathbb{E}\left[\sup_{t\leq s\leq T}\left\|X_s^{(\alpha)\,t,x} - X_s^{t,x}\right\|_{\mathcal{H}}\right] &= \mathbb{E}\left[\sup_{0\leq s\leq T-t}\left\|X_s^{(\alpha)\,0,x} - X_s^{0,x}\right\|_{\mathcal{H}}\right] \\ &\leq \mathbb{E}\left[\sup_{0\leq s\leq S}\left\|X_s^{(\alpha)\,0,x} - X_s^{0,x}\right\|_{\mathcal{H}}\right]. \end{aligned}$$

The last expression is independent of time and hence the estimate on the value functions is uniform with respect to $t \in [0, T]$. Now we apply the convergence of Proposition 3.1.1 and get the result. A similar result holds in the case in which we only have local Lipschitz property of the gain function with respect to the space variable. Nevertheless the proof in that case is much more delicate and requires some localization arguments. ■

From dominated convergence theorem we can state the following corollary.

Corollary 3.1.1 *If μ is a finite measure on the Hilbert space then the following convergence result holds*

$$\lim_{\alpha\to\infty} \int_0^T \int_{\mathcal{H}} |V_\alpha(t,x) - V(t,x)|^p \mu(dx)dt = 0, \qquad \forall 1 \le p < \infty. \tag{3.3}$$

There are few crucial properties of the convergence which we obtain as a consequence of the Theorem C.0.5.

Theorem 3.1.2 *Let us assume that $V_\alpha \in C_b([0,T] \times \mathcal{H})$ for all $\alpha > 0$. Then $V_\alpha \to V$ uniformly on any compact subset $[0,T] \times \mathcal{K} \subset [0,T] \times \mathcal{H}$. Moreover the map $(t,x) \mapsto V(t,x)$ is continuous on the whole space $[0,T] \times \mathcal{H}$.*

PROOF: Since for $x \in \mathcal{H}$ fixed, $V_\alpha(\,\cdot\,, x) \to V(\,\cdot\,, x)$ uniformly on $t \in [0,T]$, then $t \mapsto V(t,x)$ has to be continuous on $[0,T]$ as well, i.e. $V(\,\cdot\,, x) \in C_b([0,T]; \mathbb{R})$. We can define the function

$$F_\alpha(x) := \sup_{t\in[0,T]} |V_\alpha(t,x) - V(t,x)|.$$

Clearly $\{F_\alpha\}_{\alpha\ge 0}$ is a equibounded family of real valued functions. Moreover, we have the following estimate for equi-Lipschitz property:

$$\begin{aligned}
\big|F_\alpha(x) - F_\alpha(y)\big| &= \Big| \sup_{t\in[0,T]} |V_\alpha(t,x) - V(t,x)| - \sup_{t\in[0,T]} |V_\alpha(t,y) - V(t,y)| \Big| \\
&\le \sup_{t\in[0,T]} \big|V_\alpha(t,x) - V_\alpha(t,y) + V(t,y) - V(t,x)\big| \\
&\le \sup_{t\in[0,T]} \big|V_\alpha(t,x) - V_\alpha(t,y)\big| + \sup_{t\in[0,T]} \big|V(t,y) - V(t,x)\big| \le 2L_V \|x-y\|_{\mathcal{H}}.
\end{aligned}$$

Then $\{F_\alpha\}_{\alpha\geq 0}$ is a family of equibounded and equicontinuous functions and $F_\alpha(x)\to 0$, as $\alpha\to\infty$ for all $x\in\mathcal{H}$. Then Theorem C.0.5 guarantees that $\{V_\alpha\}_{\alpha\geq 0}$ converges uniformly to zero on any compact subset $[0,T]\times\mathcal{K}$. This means that for arbitrary $\mathcal{K}$ it holds

$$\lim_{\alpha\to\infty}\sup_{(t,x)\in[0,T]\times\mathcal{K}}|V_\alpha(t,x)-V(t,x)|=0.$$

Then V is continuous on every compact subset $[0,T]\times\mathcal{K}$ as uniform limit of bounded continuous functions (cf. Theorem C.0.4).

We want to prove that this is enough for global continuity. Let (t_n,x_n) be a sequence in $[0,T]\times\mathcal{H}$ converging to a point $(t,x)\in[0,T]\times\mathcal{H}$. We now show the continuity in (t,x). Indeed we have

$$|V(t_n,x_n)-V(t,x)|\leq|V(t_n,x_n)-V(t_n,x)|+|V(t_n,x)-V(t,x)|.$$

We get the continuity of the first term from Proposition 2.2.2, indeed for a suitable constant $L_V>0$

$$|V(t_n,x_n)-V(t_n,x)|\leq L_V\|x_n-x\|_{\mathcal{H}}\to 0,\quad \text{as } n\to\infty.$$

For the second term we can always assume that the sequence $\{t_n\}_{n=1}^\infty$ belongs to the compact $[0,T]$. Hence clearly $[0,T]\times\{x\}$ is a compact subset of $[0,T]\times\mathcal{H}$. Then from the continuity on compact sets we have

$$|V(t_n,x)-V(t,x)|\to 0,\quad \text{as } n\to\infty.$$

Since (t,x) is an arbitrary point in $[0,T]\times\mathcal{H}$, the proof is complete. ■

We will see below that V_α is continuous on $[0,T]\times\mathcal{H}$ as a consequence of Theorem 3.2.2. The next step will be to reduce the Yosida approximating problem to a finite-dimensional Yosida approximating problem.

3.2 Finite dimensional reduction

Let $\{\varphi_1,\varphi_2,\dots\}$ be an orthonormal basis of $\mathcal{H}$ made of all elements in the domain of the unbounded operator A, i.e. $\varphi_i\in D(A)$, $i=1,2,\dots$. Such a set exists because $\mathcal{H}$ is separable and

$D(A)$ is dense in $\mathcal{H}$. We introduce now a trace class operator which will be crucial in the following analysis.

Definition 3.2.1 *Let $Q : \mathcal{H} \to \mathcal{H}$ be a positive, linear operator on $\mathcal{H}$ defined as*

$$Q\varphi_i = \lambda_i \varphi_i, \qquad \lambda_i > 0, \qquad i = 1, 2, \ldots,$$

and such that $\sum_{i=1}^{\infty} \lambda_i < \infty$, i.e. it is of trace class.

In the remainder of this thesis we will characterize a suitable connection between Q and A. We now state some technical assumptions on the diffusion coefficient.

Assumption 3.2.1 *The diffusion coefficient has the following properties*

(D1) *$\sigma(x) \in Q(\mathcal{H})$, $\forall x \in \mathcal{H}$, i.e. there exists $\gamma : \mathcal{H} \to \mathcal{H}$ such that $\sigma(x) = Q\gamma(x)$*

(D2) *$\gamma \in C_b^2(\mathcal{H}; \mathcal{H})$*

We are going now to perform a finite dimensional approximation of the diffusion process and of the associated optimal stopping problem. Let us consider the finite dimensional subset $\mathcal{H}^{(n)} := span\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ and the orthogonal projection operator $P_n : \mathcal{H} \to \mathcal{H}^{(n)}$. We approximate the diffusion coefficient by means of $\sigma^{(n)} := (P_n \sigma) \circ P_n$, more precisely $\sigma^{(n)}$ is given by the map $x \mapsto P_n \sigma(P_n x)$, $x \in \mathcal{H}$. Similarly we consider $A_{\alpha,n} := P_n A_\alpha P_n$, which represents a bounded linear operator on $\mathcal{H}^{(n)}$. Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a sequence such that $\epsilon_n > 0$ and $\epsilon_n \to 0$ as $n \to \infty$. We define the process $X_{\cdot}^{(\alpha,n)}(\omega) := \sum_{i=1}^{n} z_i^{(\alpha,n)}(\,\cdot\,, \omega)\varphi_i$ as the unique strong solution of the SDE on $\mathcal{H}^{(n)}$ given by

$$\begin{cases} dX_t^{(\alpha,n)\,x} = A_{\alpha,n} X_t^{(\alpha,n)\,x} dt + \sigma^{(n)}(X_t^{(\alpha,n)\,x}) dW_t^0 + \epsilon_n \sum_{i=1}^{n} \varphi_i \, dW_t^i, \qquad t \in [0, S], \\ X_0^{(\alpha,n)\,x} = P_n x. \end{cases} \tag{3.4}$$

Some comments are required in order to fully understand such a choice for the approximating SDE. The strong solution $X^{(\alpha,n)}$ can be interpreted as a solution of a SDE in $\mathcal{H}$ but living in the finite dimensional subspace of $\mathcal{H}^{(n)}$. It is worth noticing that at each time $t \in [0, S]$, $X_t^{(\alpha,n)}$ does not represent the projection of the process $X_t^{(\alpha)}$ on the finite dimensional subspace. Indeed a process with that property would not be markovian. Hence $X^{(\alpha,n)}$ has to be considered as an auxiliary diffusion process which turns out to be a good approximation of the original one. In particular the following convergence result holds

Proposition 3.2.1 *If the sequence $\{\epsilon_n\}_{n=1}^{\infty}$ is such that $n\,\epsilon_n \to 0$ as $n \to \infty$ then*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,S]} \left\| X_t^{(\alpha,n)\,x} - X_t^{(\alpha)\,x} \right\|^2\right] = 0. \tag{3.5}$$

Moreover, the convergence is uniform with respect to x varying on compact subsets of $\mathcal{H}$.

PROOF: Since we are dealing with the Yosida approximation of the original diffusion process, we have unique strong solutions of both the finite dimensional SDE and the infinite dimensional one. Then the solutions are

$$X^{(\alpha,n)\,x} = P_n x + \int_0^t A_{\alpha,n} X_s^{(\alpha,n)\,x} ds + \int_0^t P_n \sigma(X_s^{(\alpha,n)\,x}) dW_s^0 + \epsilon_n \sum_{i=1}^n \varphi_i W_t^i,$$

and

$$X^{(\alpha)\,x} = x + \int_0^t A_\alpha X_s^{(\alpha)\,x} ds + \int_0^t \sigma(X_s^{(\alpha)\,x}) dW_s^0.$$

From direct computations, using the fact that $A_{\alpha,n} X^{(\alpha,n)\,x} = P_n A_\alpha X^{(\alpha,n)\,x}$, we obtain

$$\begin{aligned} \|X_t^{(\alpha,n)\,x} - X_t^{(\alpha)\,x}\|_{\mathcal{H}}^2 \leq 6\Bigg[& \|P_n x - x\|_{\mathcal{H}}^2 + \Big\| \int_0^t P_n A_\alpha (X_s^{(\alpha,n)\,x} - X_s^{(\alpha)\,x}) ds \Big\|_{\mathcal{H}}^2 \\ & + \Big\| \int_0^t (I - P_n) A_\alpha X_s^{(\alpha)\,x} ds \Big\|_{\mathcal{H}}^2 \\ & + \Big\| \int_0^t P_n [\sigma(X_s^{(\alpha,n)\,x}) - \sigma(X_s^{(\alpha)\,x})] dW_s^0 \Big\|_{\mathcal{H}}^2 \\ & + \Big\| \int_0^t (I - P_n) \sigma(X_s^{(\alpha)\,x}) dW_s^0 \Big\|_{\mathcal{H}}^2 + \epsilon_n^2 \sum_{i=1}^n |W_t^i|^2 \Bigg]. \end{aligned}$$

We use Hölder inequality and take the supremum over $t \in [0, S]$ so to obtain

$$\begin{aligned} \sup_{0\leq t\leq S} \|X_t^{(\alpha,n)\,x} - X_t^{(\alpha)\,x}\|_{\mathcal{H}}^2 6\Bigg[& \|P_n x - x\|_{\mathcal{H}}^2 + S\|A_\alpha\|_{\mathcal{H}} \int_0^S \sup_{0\leq u\leq s} \|X_u^{(\alpha,n)\,x} - X_u^{(\alpha)\,x}\|_{\mathcal{H}}^2 ds \\ & + S \int_0^S \|(I - P_n) A_\alpha X_s^{(\alpha)\,x}\|_{\mathcal{H}}^2 ds \end{aligned}$$

$$+\sup_{0\le t\le S}\Big\|\int_0^t P_n[\sigma(X_s^{(\alpha,n)\,x})-\sigma(X_s^{(\alpha)\,x})]dW_s^0\Big\|_{\mathcal{H}}^2$$
$$+\sup_{0\le t\le S}\Big\|\int_0^t (I-P_n)\sigma(X_s^{(\alpha)\,x})dW_s^0\Big\|_{\mathcal{H}}^2+\epsilon_n^2\sum_{i=1}^n\sup_{0\le t\le S}|W_t^i|^2\Big].$$

We take the expectation and use the properties of the stochastic integral on Hilbert space, cf. [19], Chapter 4. We get

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le S}\|X_t^{(\alpha,n)\,x}-X_t^{(\alpha)\,x}\|_{\mathcal{H}}^2\right]\le 6\Bigg[&\|P_nx-x\|_{\mathcal{H}}^2\\&+S\|A_\alpha\|_{\mathcal{H}}\int_0^S\mathbb{E}\left[\sup_{0\le u\le s}\|X_u^{(\alpha,n)\,x}-X_u^{(\alpha)\,x}\|_{\mathcal{H}}^2\right]ds\\&+S\int_0^S\mathbb{E}\left[\|(I-P_n)A_\alpha X_s^{(\alpha)\,x}\|_{\mathcal{H}}^2\right]ds\\&+\int_0^S\mathbb{E}\left[\|\sigma(X_s^{(\alpha,n)\,x})-\sigma(X_s^{(\alpha)\,x})\|_{\mathcal{H}}^2\right]ds\\&+\int_0^S\mathbb{E}\left[\|(I-P_n)\sigma(X_s^{(\alpha)\,x})\|_{\mathcal{H}}^2\right]ds\\&+\epsilon_n^2\sum_{i=1}^n\mathbb{E}[\sup_{0\le t\le S}|W_t^i|^2]\Bigg].\end{aligned}$$

We exploit the Lipschitz property of the diffusion coefficient and finally get

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le S}\|X_t^{(\alpha,n)\,x}-X_t^{(\alpha)\,x}\|_{\mathcal{H}}^2\right]\le 6\Bigg[&\|P_nx-x\|_{\mathcal{H}}^2\\&+S\int_0^S\mathbb{E}\left[\|(I-P_n)A_\alpha X_s^{(\alpha)\,x}\|_{\mathcal{H}}^2\right]ds\\&+\int_0^S\mathbb{E}\left[\|(I-P_n)\sigma(X_s^{(\alpha)\,x})\|_{\mathcal{H}}^2\right]ds+\epsilon_n^2\,n\,S\\&+S\,\|A_\alpha\|_{\mathcal{H}}\int_0^S\mathbb{E}\left[\sup_{0\le u\le s}\|X_u^{(\alpha,n)\,x}-X_u^{(\alpha)\,x}\|_{\mathcal{H}}^2\right]ds\\&+L_\sigma^2\int_0^S\mathbb{E}\left[\sup_{0\le u\le s}\|X_u^{(\alpha,n)\,x}-X_u^{(\alpha)\,x}\|_{\mathcal{H}}^2\right]ds\Bigg].\end{aligned}$$

A straightforward application of Gronwall's lemma and the choice of a suitable constant give us the final expression

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le S}\|X_t^{(\alpha,n)\,x}-X_t^{(\alpha)\,x}\|_{\mathcal{H}}^2\right]\\
&\quad\le C_S\Bigg[\|P_nx-x\|_{\mathcal{H}}^2+\int_0^S\mathbb{E}\left[\|(I-P_n)A_\alpha X_s^{(\alpha)\,x}\|_{\mathcal{H}}^2+\|(I-P_n)\sigma(X_s^{(\alpha)\,x})\|_{\mathcal{H}}^2\right]ds\\
&\qquad\qquad+\epsilon_n^2nS\Bigg]\exp\left(S^2\|A_\alpha\|_{\mathcal{L}(\mathcal{H})}+SL_\sigma^2\right).
\end{aligned}$$

We can use dominated convergence and the hypothesis about ϵ_n to show that the limit as $n\to\infty$ tends to zero. We want to prove that the limit is uniform on compact subsets of $\mathcal{H}$. For each n we define the real function $M_n(x)$ as

$$M_n(x):=\|P_nx-x\|_{\mathcal{H}}^2+\int_0^S\mathbb{E}\left[\|(I-P_n)A_\alpha X_s^{(\alpha)\,x}\|_{\mathcal{H}}^2+\|(I-P_n)\sigma(X_s^{(\alpha)\,x})\|_{\mathcal{H}}^2\right]ds+\epsilon_n^2nS.$$

Notice that

$$\begin{aligned}
&\left|\int_0^S\mathbb{E}\left[\|(I-P_n)A_\alpha X_s^{(\alpha)\,x}\|_{\mathcal{H}}^2-\|(I-P_n)A_\alpha X_s^{y\,(\alpha)}\|_{\mathcal{H}}^2\right]ds\right|\\
&\quad+\left|\int_0^S\mathbb{E}\left[\|(I-P_n)\sigma(X_s^{(\alpha)\,x})\|_{\mathcal{H}}^2-\|(I-P_n)\sigma(X_s^{y\,(\alpha)})\|_{\mathcal{H}}^2\right]ds\right|\\
&\quad\le(\|A_\alpha\|_{\mathcal{L}(\mathcal{H})}^2+L_\sigma^2)S\,\mathbb{E}\left[\sup_{0\le u\le S}\|X_u^{(\alpha)\,x}-X_u^{y\,(\alpha)}\|_{\mathcal{H}}^2\right].
\end{aligned}$$

From Corollary 2.2.1 we obtain the continuity of this term with respect to $x\in\mathcal{H}$. We might have obtained this result from [19], Chapter 9, Theorem 9.1, as well. This implies that $x\mapsto M_n(x)$ is continuous for all $n\ge 1$. Moreover $M_n(x)\to 0$, as $n\to\infty$ for all $x\in\mathcal{H}$ and the convergence is monotone, i.e. $M_n(x)\ge M_{n+1}(x)\ge\dots$ for all $x\in\mathcal{H}$. The Dini's theorem guarantees that the convergence is uniform on any compact subset $\mathcal{K}\subset\mathcal{H}$, cf. Appendix C. ■

From a simple application of Jensen's inequality one obtain the following corollary

Corollary 3.2.1 *If the sequence $\{\epsilon_n\}_{n=1}^\infty$ is such that $n\,\epsilon_n\to 0$ as $n\to\infty$ then*

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{t\in[0,S]}\left\|X_t^{(\alpha,n)\,x}-X_t^{(\alpha)\,x}\right\|\right]=0. \tag{3.6}$$

The convergence is uniform with respect to x varying on compact subsets of $\mathcal{H}$.

The following remark clarifies a useful extension.

Remark 3.2.1 *When considering diffusions starting at arbitrary initial time* $t \in [0, S]$ *the previous estimate holds in the same form thanks to the time homogeneous coefficients of the processes. In fact*

$$\mathbb{E}\left[\sup_{u\in[t,S]} \left\|X_u^{(\alpha,n)\,t,x} - X_u^{(\alpha)\,t,x}\right\|\right] = \mathbb{E}\left[\sup_{u\in[0,S-t]} \left\|X_u^{(\alpha,n)\,0,x} - X_u^{(\alpha)\,0,x}\right\|\right] \\ \leq M_n(x)\exp\left(S^2\|A_\alpha\|_{\mathcal{L}(\mathcal{H})} + SL_\sigma^2\right).$$

This implies that the convergence is uniform with respect to the initial time $t \in [0, S]$.

The interest toward this finite dimensional reduction arises from the fact that now we can write a SDE in $\mathbb{R}^n$ for the vector process $Z_\cdot^{(\alpha,n)}(\omega) := (z_1^{(\alpha,n)}(\,\cdot\,,\omega), \ldots, z_n^{(\alpha,n)}(\,\cdot\,,\omega))$. We recall that $X_\cdot^{(\alpha,n)}(\omega) = \sum_{i=1}^n z_i^{(\alpha,n)}(\,\cdot\,,\omega)\varphi_i$ and the SDE holds

$$\begin{cases} dZ_t^{(\alpha,n)} = b^{(\alpha,n)}(Z_t^{(\alpha,n)})dt + g^{(\alpha,n)}(Z_t^{(\alpha,n)})dW_t^{(n)}, & t \in [0,S], \\ Z_0^{(\alpha,n)} = (\langle x, \varphi_1\rangle_{\mathcal{H}}, \ldots, \langle x, \varphi_n\rangle_{\mathcal{H}}). \end{cases} \tag{3.7}$$

Here the Brownian motion is a $n+1$-dimensional projection of the infinite dimensional one defined above, i.e. $W^{(n)} = (W^0, W^1, \ldots, W^n)$. Adopting now the notation of the space $\mathbb{R}^n$ the drift coefficient is a vector $b^{(\alpha,n)}(z) = (b_1^{(\alpha,n)}(z), \ldots, b_n^{(\alpha,n)}(z))$ for $z \in \mathbb{R}^n$ and each component is given by

$$b_i^{(\alpha,n)}(z) = \sum_{j=1}^n z_j \langle A_\alpha\varphi_j, \varphi_i\rangle_{\mathcal{H}}.$$

The diffusion matrix turns out to be a $n \times (n+1)$-rectangualr matrix

$$g^{(\alpha,n)}(z) = \begin{pmatrix} \langle\sigma^{(n)}(z), \varphi_1\rangle_{\mathcal{H}} & \epsilon_n & 0 & 0 & \ldots & 0 & 0 \\ \langle\sigma^{(n)}(z), \varphi_2\rangle_{\mathcal{H}} & 0 & \epsilon_n & 0 & \ldots & 0 & 0 \\ \vdots & & & \ddots & & & \\ \langle\sigma^{(n)}(z), \varphi_n\rangle_{\mathcal{H}} & 0 & 0 & 0 & \ldots & 0 & \epsilon_n \end{pmatrix}.$$

What is relevant for the approximation scheme is that the diffusion $Z^{(\alpha,n)}$ is non-degenerate, i.e. $(g^{(\alpha,n)}(z)g^{(\alpha,n)*}(z)y,y)_{\mathbb{R}^n} \geq \epsilon_n^2|y|^2_{\mathbb{R}^n}$ for all $y,z \in \mathbb{R}^n$. It is also worth recalling that there is an isometry between the finite dimensional subspace $\left(\mathcal{H}^{(n)}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}, \|\cdot\|_{\mathcal{H}}\right)$ and the n-dimensional Euclidean space $(\mathbb{R}^n, (\,\cdot\,,\,\cdot\,)_{\mathbb{R}^n}, |\cdot|_{\mathbb{R}^n})$. It is now possible to define an approximated version of the optimal stopping problem. Let $\Psi^{(n)}(t,\cdot) := \Psi(t,\cdot)\circ P_n$ be the approximating gain function, then let $x^{(n)} \in \mathcal{H}^{(n)}$ represent the finite dimensional projection of the initial data $x \in \mathcal{H}$. We define the value function, $V_\alpha^{(n)}$, as:

$$V_\alpha^{(n)}(t,x^{(n)}) := \sup_{t\leq\tau\leq T} \mathbb{E}\left[\Psi^{(n)}(\tau, X_\tau^{(\alpha,n)\,t,x})\right] \tag{3.8}$$

Notice that $V_\alpha^{(n)}$ can equivalently be understood as a function defined on the subspace $[0,T]\times\mathcal{H}^{(n)}$ or by isometry as a function defined on $[0,T]\times\mathbb{R}^n$. In order for this fact to be more explicit we simply set $\hat{\Psi}^{(n)}(t,z_1^{(n)},\dots,z_n^{(n)}) := \Psi^{(n)}(t,x^{(n)})$ and define

$$U_\alpha^{(n)}(t,z^{(n)}) = \sup_{t\leq\tau\leq T} \mathbb{E}\left[\hat{\Psi}^{(n)}(\tau, Z_\tau^{t,z\,(\alpha,n)})\right], \tag{3.9}$$

where $z^{(n)} \in \mathbb{R}^n$. An important convergence result is summarized in the following theorem

Theorem 3.2.1 *For any $x \in \mathcal{H}$ given and fixed the pointwise convergence holds*

$$\lim_{n\to\infty} |V_\alpha^{(n)}(t,x^{(n)}) - V_\alpha(t,x)| = 0. \tag{3.10}$$

Moreover, this convergence is uniform on $[0,T]\times\mathcal{K}$, for any $\mathcal{K}$ compact subset of $\mathcal{H}$.

PROOF: We observe that $P_n X^{(\alpha,n)\,t,x} = X^{(\alpha,n)\,t,x}$. From the Lipschitz property of the gain function we simply get

$$|V_\alpha^{(n)}(t,x^{(n)}) - V_\alpha(t,x)| \leq \mathbb{E}\left[\sup_{t\leq s\leq T}\left|\Psi^{(n)}(s,X_s^{(\alpha,n)\,t,x}) - \Psi(s,X_s^{(\alpha)\,t,x})\right|\right]$$

$$= \mathbb{E}\left[\sup_{t\leq s\leq T}\left|\Psi(s,X_s^{(\alpha,n)\,t,x}) - \Psi(s,X_s^{(\alpha)\,t,x})\right|\right] \leq L_1\mathbb{E}\left[\sup_{t\leq s\leq T}\left\|X_s^{(\alpha,n)\,t,x} - X_s^{(\alpha)\,t,x}\right\|_{\mathcal{H}}\right].$$

We know from Remark 3.2.1 and Corollary 3.2.1 that the right hand side converges uniformly on compact sets of the form $[0,T] \times \mathcal{K}$. Then

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathcal{K}} |V_\alpha^{(n)}(t,x^{(n)}) - V_\alpha(t,x)| = 0.$$

A similar result holds in the case in which we only have local Lipschitz property of the gain function with respect to the space variable. Though the proof in that case is much more delicate and requires some localization arguments. ■

Dominated convergence theorem or monotone convergence imply the following corollary.

Corollary 3.2.2 *If μ is a finite measure on the Hilbert space then the following convergence result holds*

$$\lim_{n\to\infty} \int_0^T \int_{\mathcal{H}} |V_\alpha^{(n)}(t,x^{(n)}) - V_\alpha(t,x)|^p \mu(dx)dt = 0, \qquad \forall 1 \le p < \infty. \tag{3.11}$$

A useful continuity result is now derived from the previous considerations.

Theorem 3.2.2 *Let us assume $V_\alpha^{(n)} \in C([0,T] \times \mathcal{H}^{(n)})$ for all $n \geq 1$. Then the Yosida approximated value function V_α is continuous on the whole space $[0,T] \times \mathcal{H}$.*

PROOF: In the first place we notice that $\{V_\alpha^{(n)}(t,x^{(n)})\}_{n=1}^\infty$ is a uniformly bounded sequence, i.e.

$$\sup_{n\geq 1} \sup_{(t,x)\in[0,T]\times\mathcal{H}} |V_\alpha^{(n)}(t,x^{(n)})| \le \overline{\Psi}.$$

Moreover, we know that $V_\alpha^{(n)}(t,x^{(n)}) \to V_\alpha(t,x)$, as $n \to \infty$, uniformly on any compact subset $[0,T] \times \mathcal{K}$. These facts together with the continuity of $V_\alpha^{(n)}$ and Theorem C.0.4 guarantee that $V_\alpha(t,x)$ is continuous on $[0,T] \times \mathcal{K}$. Let now (t_n, x_n) be a sequence in $[0,T] \times \mathcal{H}$ converging to a point (t,x). It is easy to show the continuity from the same arguments as in Theorem 3.1.2. ■

The continuity of $V_\alpha^{(n)}$ is proved in Proposition 5.2.2 below and in the following remark. It is worth noticing that the uniform Lipschitz condition with respect to the space variable is crucial. In fact in Hilbert spaces one cannot assume that the sequence $\{x_n\}_{n=1}^\infty$ always belongs to a compact subset of $\mathcal{H}$.

We are now interested in giving an analytical characterization of the approximating value function in terms of a suitable variational inequality.

3.3 A short remark on the approximating scheme

From the results above it turns out that one can approximate the original value function $V(t,x)$ through a two indexes sequence of functions $\{V_\alpha^{(n)}\}_{\alpha\in\mathbb{R}_+, n\in\mathbb{N}}$. The slightly unpleasant fact is that when taking the limit one has to take care of the order of these limits. Indeed we first take the limit with respect to the finite dimensional scheme and only afterwards we can perform the limit with respect to the Yosida approximation. In formulae it means that for any $t \in [0,T]$

$$\lim_{\alpha\to\infty}\left[\lim_{n\to\infty}|V_\alpha^{(n)}(t,x^{(n)}) - V(t,x)|\right] = 0, \qquad x \in \mathcal{H}.$$

Nevertheless this fact does not really matter because our final aim would be to characterize the original value function in terms of a suitable Evolutionary Variational Inequality (EVI) on $\mathcal{H}$. Hence we will be mostly interested in showing existence of a solution to such a problem and the connection with the optimal stopping problem. Nevertheless in some cases the Yosida approximation can be avoided. Let us assume the following:

Assumption 3.3.1 *Let $\{S(t), t\geq 0\}$ be the C_0-semigroup associated to the infinitesimal generator A. Let then $\{e^{tA_n}, t \geq 0\}$ be the uniformly continuous semigroup associated to the bounded operator $A_n = P_n A P_n$. Then the following convergence holds*

$$\lim_{n\to\infty} e^{tA_n}x = S(t)x, \qquad x \in \mathcal{H}. \tag{3.12}$$

Under this assumption one can consider the Galerkin approximation directly on the original SDE, i.e. $X^{(n)}$ will be the unique strong solution (and hence mild solution) to the following SDE

$$\begin{cases} dX_t^{(n)} = A_n X_t^{(n)} dt + \sigma(X_t^{(n)}) dW_t^0 + \epsilon_n \sum_{i=1}^n \varphi_i \, dW_t^i, \qquad t\in[0,S], \\ X_0^{(n)} = x. \end{cases} \tag{3.13}$$

It is then easy to show that

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{t\in[0,S]}\left\|X_t^{(n)\,x} - X_t^x\right\|^p\right] = 0, \qquad 1\leq p<\infty. \tag{3.14}$$

This would not hold without the assumption about the semigroup. All the analysis about the approximating value function can be derived as above. It then turns out that this assumption simplifies the whole algorithm to a single approximation scheme. Yet for the application that we have in mind it is not clear whether the assumption holds or not. For this reason we wish to study the problem in full generality.

Another detail which deserves some attentions is the one concerning the universal Lipschitz regularity with respect to the space variable.

Remark 3.3.1 *It is easy to check that the Lipschitz condition in Proposition 2.2.2 holds for* V_α *and* $V_\alpha^{(n)}$ *as well. Moreover from Appendix B, Remark B.0.2, one derives that the Lipschitz constant* L_V *can be taken to be independent of* n *and* α*.*

Chapter 4

The finite dimensional variational inequality on bounded domains

Throughout this chapter we deal with a localized version of the finite dimensional optimal stopping problem. We fix the order α of the Yosida approximation and the order n of the Galerkin scheme; then we introduce the obstacle problem associated with the optimal stopping one. We provide a short survey about variational inequalities and their connection with obstacle problems and we focus on two different concepts of solution: *strong* solutions and *weak* solutions. We prove existence and uniqueness of a weak solution to our variational problem and its connection with the value function of the localized optimal stopping problem. We also prove the existence of an optimal stopping time and give its formal characterization.

4.1 The optimal stopping problem in $\mathbb{R}^n$

We will analyze the sequence of finite dimensional problems when the order of the Yosida approximation $\alpha > 0$ and the dimension $n > 0$ are fixed. In order to simplify the notation we skip the superscript and from now on we denote $U(t,z) := U_\alpha^{(n)}(t, z^{(n)})$, where it should be clear that $U : [0,T] \times \mathbb{R}^n \to \mathbb{R}$. Similarly the diffusion will be denoted by $Z^{t,z}_\cdot := Z^{(\alpha,n)\,t,z}_\cdot$ and will be the unique strong solution of the SDE in $\mathbb{R}^n$

$$\begin{cases} dZ_t = b(Z_t)dt + g(Z_t)dW_t, \qquad t \in [0,S], \\ Z_0 = z. \end{cases} \tag{4.1}$$

All the coefficients are the same as in Section 3.2 and the Brownian motion is in $\mathbb{R}^{n+1}$. Now it is useful to notice that for $x \in \mathcal{H}^{(n)}$ the gain function has the same form of his approximated version, i.e. $\Psi^{(n)}(t,x) = \Psi(t,x)$. As a consequence, since we are now dealing only with vectors in $\mathbb{R}^n$ for the value function we have the expression

$$U(t,z) = \sup_{t \leq \tau \leq T} \mathbb{E}\left[\hat{\Psi}(\tau, Z_\tau^{t,z})\right]. \tag{4.2}$$

There are many well known results about the connection between variational inequalities and optimal stopping problems in finite dimensional spaces, cf. [4]. We are going to rely on some of them but the main aim of this paper is to characterize the infinite dimensional variational inequality. It is then clear that at some point we will be expected to carry a limit on a sequence of finite dimensional variational inequalities in order to get an analogue in infinite dimensions. The main problem in doing so is that one cannot extend the Lebesgue measure to a Hilbert space. It is then clear that we have to adopt a suitable sequence of measures in order to give a meaning to this limit. For this reason we will carry out explicitly some crucial steps in the variational analysis. Not all of them represent a novelty but they are strictly necessary in order to produce a rigorous result at the infinite dimensional level.

In the first place we consider μ_n to be a finite measure on $\mathbb{R}^n$. As usual we introduce the L^2-norm with respect to such a measure

$$\|u\|_{L^2_\mu(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |u(z)|^2 \mu_n(dz)\right)^{\frac{1}{2}}.$$

This characterizes the Hilbert space $L^2_\mu(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R} \,|\, \|u\|_{L^2_\mu(\mathbb{R}^n)} < \infty\}$. We can now introduce a weighted Sobolev space and we denote it by $W^{1,2}(\mathbb{R}^n, \mu_n)$. In particular

$$W^{1,2}(\mathbb{R}^n, \mu_n) := \{u \in L^2_\mu(\mathbb{R}^n) \,:\, \|\nabla u\|_{L^2_\mu(\mathbb{R}^n)} < \infty\}, \tag{4.3}$$

where

$$\|\nabla u\|^2_{L^2_\mu(\mathbb{R}^n)} := \sum_{i=1}^n \int_{\mathbb{R}^n} \left|\frac{\partial u(z)}{\partial z_i}\right|^2 \mu_n(dz).$$

We want now to point out a regularity property for the gain function.

Proposition 4.1.1 *For the gain function $\hat{\Psi}$ there exists a constant $C > 0$ such that it holds*

$$\sup_{t\in[0,T]} \|\hat{\Psi}(t)\|_{W^{1,2}(\mathbb{R}^n,\mu_n)} < C. \tag{4.4}$$

Moreover, for $1 \le p < \infty$ as in Assumption 2.1.1, if the measure μ is such that

$$\int_{\mathbb{R}^n} |z|^{2p}\mu(dz) < \infty, \tag{4.5}$$

then there exists $C_p > 0$ such that

$$\int_0^T \left\| \frac{\partial\hat{\Psi}}{\partial t}(t)\right\|^2_{L^2_\mu(\mathbb{R}^n)} dt < C_p. \tag{4.6}$$

PROOF: Let us begin with the first claim (the infinite dimensional anologue of this result is in [18], Ch. 10). Let us recall that for $t \in [0,T]$, $\Psi(t,\cdot)$ is bounded and Lipschitz, uniformly with respect to t, on the whole space. Since $\sup_{(t,x)\in[0,T]\times\mathcal{H}} |\Psi(t,x)| \le \overline{\Psi}$, then clearly $\sup_{t\in[0,T]} \|\Psi^{(n)}(t)\|_{L^\infty(\mathbb{R}^n)} \le \overline{\Psi}$. In particular this bound does not depend on the dimension n of the space. We can conclude that $\sup_{t\in[0,T]} \|\hat{\Psi}(t)\|_{L^2_\mu(\mathbb{R}^n)} \le \overline{\Psi}\mu(\mathbb{R}^n)$. We have now to discuss the bound on the derivative.

We may mollify the gain function by means of the standard mollifiers $\{\rho_k\}_{k=1}^\infty$. For $t \in [0,T]$ given and fixed we define $\hat{\Psi}_k(t,\cdot) := \rho_k \star \hat{\Psi}(t,\cdot)$. Clearly the pointwise convergence holds $\hat{\Psi}_k(t,z) \to \hat{\Psi}(t,z)$, $z \in \mathbb{R}^n$, cf. [14], Chapter 4, Proposition 4.21. From uniform boundedness we obtain

$$|\hat{\Psi}_k(t,z)| = \Big|\int_{\mathbb{R}^n} \rho_k(y)\hat{\Psi}(t,z-y)dy\Big| \le \left\|\hat{\Psi}(t)\right\|_{L^\infty(\mathbb{R}^n)},$$

and hence clearly $\|\hat{\Psi}_k(t)\|_{L^\infty(\mathbb{R}^n)} \le \left\|\hat{\Psi}(t)\right\|_{L^\infty(\mathbb{R}^n)}$. We can then use dominated convergence to prove that $\hat{\Psi}_k(t,\cdot) \to \hat{\Psi}(t,\cdot)$ in $L^2_\mu(\mathbb{R}^n)$ as $k \to \infty$ (this is also a very well known fact, cf. [14], Chapter 4, Theorem 4.22). It is easy to verify that the mollified functions are equilipschitz and hence have uniformly bounded first derivatives. In fact for t given and fixed and every $z \in \mathbb{R}^n$ the gradient $\nabla\hat{\Psi}_k(t,z)$ is linear functional on $\mathbb{R}^n$. We have for the directional derivative in the generic direction y,

$$|(\nabla\hat{\Psi}_k(t,z),y)_{\mathbb{R}^n}| = \left|\lim_{\varepsilon\to 0}\frac{\hat{\Psi}_k(t,z+\varepsilon y)-\hat{\Psi}_k(t,z)}{\varepsilon}\right| \le L_1|y|_{\mathbb{R}^n},$$

where now $|\cdot|_{\mathbb{R}^n}$ is the Euclidean norm. This fact clearly implies $|\nabla\hat{\Psi}_k(t,z)|_{\mathbb{R}^n} \leq L_1$ uniformly with respect to to t and z. Hence $\|\nabla\hat{\Psi}_k(t)\|_{L^2_\mu(\mathbb{R}^n)} \leq L_1\mu(\mathbb{R}^n)$. This implies that

$$\|\hat{\Psi}_k(t)\|_{L^2_\mu(\mathbb{R}^n)} + \|\nabla\hat{\Psi}_k(t)\|_{L^2_\mu(\mathbb{R}^n)} \leq (\overline{\Psi} + L_1)\mu_n(\mathbb{R}^n). \tag{4.7}$$

Then there exists a function $\phi(t,\cdot) \in W^{1,2}(\mathbb{R}^n,\mu)$ and a subsequence such that $\hat{\Psi}_{k_j}(t) \rightharpoonup \phi(t)$ in $W^{1,2}(\mathbb{R}^n,\mu)$. From the uniqueness of the limit we have $\hat{\Psi}(t) = \phi(t)$ and hence $\hat{\Psi}(t) \in W^{1,2}(\mathbb{R}^n,\mu)$. This fact guarantees that $\nabla\hat{\Psi}$ is well defined in $L^2_\mu(\mathbb{R}^n)$ and enables us to strenghten the convergence. In fact from standard results about mollifiers we can now say that poinwise convergence $\nabla\hat{\Psi}_k(t,z) \to \nabla\hat{\Psi}(t,z)$ holds. From dominated convergence we obtain that $\hat{\Psi}_k(t) \to \hat{\Psi}(t)$ in $W^{1,2}(\mathbb{R}^n,\mu)$. This implies that

$$\|\nabla\hat{\Psi}_k(t)\|_{L^2_\mu(\mathbb{R}^n)} \to \|\nabla\hat{\Psi}(t)\|_{L^2_\mu(\mathbb{R}^n)} \leq L_1\mu(\mathbb{R}^n).$$

Putting these results together and setting $C_\Psi = \overline{\Psi}^2 + L_1^2$ we obtain the uniform bound

$$\sup_{0\leq t\leq T} \|\hat{\Psi}(t)\|^2_{W^{1,2}(\mathbb{R}^n,\mu_n)} \leq C_\Psi\, \mu_n(\mathbb{R}^n). \tag{4.8}$$

It is quite relevant that the bound (4.8) depends on the dimension of the space only through the measure $\mu_n(\mathbb{R}^n)$. We anticipate than when μ_n is a probability measure and $\mu_n(\mathbb{R}^n) = 1$, $n \in \mathbb{N}$, the estimate becomes independent of the dimension of the space. We will discuss it in further details later. For the second claim we can adapt the same rationale. For $z \in \mathbb{R}^n$ fixed, $\hat{\Psi}(\cdot,z)$ is a locally Lipschitz function. Then the same arguments as before hold except for the fact that now the bound will depend on the space variable and in particular from the hypothesis on $\hat{\Psi}$ we obtain

$$\|\hat{\Psi}_k(z)\|^2_{L^2([0,T])} + \|\frac{\partial\hat{\Psi}_k}{\partial t}(z)\|^2_{L^2([0,T])} \leq C(1+|z|^{2p}). \tag{4.9}$$

The idea is that the set $\chi := \{v \in L^2(0,T;L^2_\mu(\mathbb{R}^n)) : \frac{\partial v}{\partial t} \in L^2(0,T;L^2_\mu(\mathbb{R}^n))\}$ is a separable Hilbert space when equipped with the obvious scalar product. Then we take the integral in (4.9) with respect to the measure μ and apply the Fubini's theorem to exchange the order of the integrals. If the hypothesis holds, we have a uniform bound for $\hat{\Psi}_k$ in the Hilbert space χ. Hence as before there exists a subsequence weakly convergent to an element of χ and the whole sequence converges strongly in $L^2(0,T;L^2_\mu(\mathbb{R}^n))$. From the uniqueness of the limit we have the thesis. ∎

4.2 Regularity of the coefficients

It is worth now discussing in some details the properties of the coefficients of the SDE in $\mathbb{R}^n$. It is reasonable to expect that these coefficients, being a projected version of the original ones, inherit the same regularity properties as their infinite dimensional counterparts. For the drift term we have the expression $b(z) = b^{(\alpha,n)}(z) := (b_1^{(\alpha,n)}(z), \dots, b_n^{(\alpha,n)}(z))$ and for each component

$$b_i^{(\alpha,n)}(z) = \sum_{j=1}^{n} z_j \langle A\varphi_j, \varphi_i \rangle_{\mathcal{H}}.$$

It is clear that $|b^{(\alpha,n)}(z)|_{\mathbb{R}^n} \leq C_{\alpha,n}(1 + |z|_{\mathbb{R}^n})$. Then in particular $b^{(\alpha,n)}(z)$ is bounded on bounded subsets of $\mathbb{R}^n$ along with its derivatives of all orders . For our purposes it is fundamental to notice that all the bounds on the drift coefficient actually depend on α and n, i.e. on the order of the Yosida approximation and on the number of dimensions of the space.

We consider now the diffusion matrix and in particular we focus on the generic element $g_{i,1}^{(\alpha,n)}(z) = \langle \sigma^{(n)}(z), \varphi_i \rangle_{\mathcal{H}}$. From Assumption 3.2.1, $\sigma(x) \in C_b^2(\mathcal{H}; \mathcal{H})$. Then there exists a family $\{D\sigma(x)\}_{x \in \mathcal{H}}$ of bounded linear operators $D\sigma(x) : \mathcal{H} \to \mathcal{H}$ defined through

$$\lim_{\|h\|_{\mathcal{H}} \to 0} \frac{\|\sigma(x+h) - \sigma(x) - D\sigma(x)h\|_{\mathcal{H}}}{\|h\|_{\mathcal{H}}} = 0, \qquad \forall h \in \mathcal{H}. \tag{4.10}$$

This operator is unique and $D\sigma(x)h$ represents the Fréchet derivative of σ in the direction of h, cf. [24], Chapter 8. Notice that the Fréchet differentiability is the strongest concept of differentiability in general metric spaces.

We define the directional derivative along the direction φ_k as $D\sigma_k(x) := D\sigma(x)\varphi_k$. We want to show that

$$\frac{\partial g_{i,1}^{(\alpha,n)}}{\partial z_k}(z) = \langle P_n D\sigma_k(z), \varphi_i \rangle_{\mathcal{H}} = \langle D\sigma_k(z), \varphi_i \rangle_{\mathcal{H}}. \tag{4.11}$$

Clearly we can always think of $z \in \mathbb{R}^n$ as an element in $\mathcal{H}$ with only n components. Now let $z \in \mathbb{R}^n$ be given and fixed. If we keep in mind the isometry between $\mathcal{H}^{(n)}$ and $\mathbb{R}^n$, with a slight abuse in the notation we will say that $P_n z = z$. For any $\varphi_k \in \mathcal{H}^{(n)}$, we can evaluate the limit representing the Gateaux derivative along the direction φ_k,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\left|\frac{g_{i,1}^{(\alpha,n)}(z+\varepsilon\varphi_k)-g_{i,1}^{(\alpha,n)}(z)}{\varepsilon}-\langle P_n D\sigma_k(z),\varphi_i\rangle_{\mathcal{H}}\right| \\
&=\lim_{\varepsilon\to 0}\left|\frac{\langle\sigma^{(n)}(z+\varepsilon\varphi_k),\varphi_i\rangle_{\mathcal{H}}-\langle\sigma^{(n)}(z),\varphi_i\rangle_{\mathcal{H}}}{\varepsilon}-\langle P_n D\sigma_k(z),\varphi_i\rangle_{\mathcal{H}}\right| \\
&=\lim_{\varepsilon\to 0}\left|\frac{\langle P_n\sigma(z+\varepsilon\varphi_k)-P_n\sigma(z)-\varepsilon P_n D\sigma_k(z),\varphi_i\rangle_{\mathcal{H}}}{\varepsilon}\right| \\
&\le\lim_{\varepsilon\to 0}\left\|\frac{\sigma(z+\varepsilon\varphi_k)-\sigma(z)-\varepsilon D\sigma_k(z)}{\varepsilon}\right\|_{\mathcal{H}}\|\varphi_i\|_{\mathcal{H}} \\
&\le\lim_{\varepsilon\to 0}\left\|\frac{\sigma(z+\varepsilon\varphi_k)-\sigma(z)}{\varepsilon}-D\sigma(z)\varphi_k\right\|_{\mathcal{H}}=0.
\end{aligned}
$$

The last equality is due to the definition of the Frechét derivative and to the fact that along the direction φ_k it is equivalent to the Gateaux derivative. From the uniqueness of such a derivative we deduce (4.11). Given that $g_{i,j}^{(\alpha,n)}(z)=\delta_{i,j+1}\epsilon_n$ when $j>1$ then $g_{i,j}^{(\alpha,n)}(z)\in C_b^1(\mathbb{R}^n)$ and it is easy to prove that actually $g_{i,j}^{(\alpha,n)}(z)\in C_b^2(\mathbb{R}^n)$.

Even though the notation might be misleading to some extent, it is worth to remark that what the previous relation is telling to us in terms of the original Hilbert space $\mathcal{H}$, is the following

$$
D\sigma^{(n)}(x)\varphi_k=P_n D\sigma(P_n x)\varphi_k,\quad \text{for } x\in\mathcal{H} \text{ and } k=1,\dots,n. \tag{4.12}
$$

A matrix which will play a crucial role in what follows is $g^{(\alpha,n)}g^{(\alpha,n)*}(z)$ which for ease of notation we dnote by $g\,g^*(z)$ since α,n are fixed now. The generic element is of the form

$$
(g\,g^*)_{i,j}(z)=\langle\sigma^{(n)}(z),\varphi_i\rangle_{\mathcal{H}}\langle\sigma^{(n)}(z),\varphi_j\rangle_{\mathcal{H}}+\delta_{i,j}\epsilon_n^2. \tag{4.13}
$$

We can now evaluate the derivative

$$
\begin{aligned}
\frac{\partial}{\partial z_j}(g\,g^*)_{i,j}(z) &= \langle P_n D\sigma_j(z),\varphi_i\rangle_{\mathcal{H}}\langle\sigma^{(n)}(z),\varphi_j\rangle_{\mathcal{H}} \\
&+ \langle P_n D\sigma_j(z),\varphi_j\rangle_{\mathcal{H}}\langle\sigma^{(n)}(z),\varphi_i\rangle_{\mathcal{H}} \\
&= \langle P_n D\sigma(z)\varphi_j,\varphi_i\rangle_{\mathcal{H}}\langle\sigma^{(n)}(z),\varphi_j\rangle_{\mathcal{H}} \\
&+ \langle P_n D\sigma(z)\varphi_j,\varphi_j\rangle_{\mathcal{H}}\langle\sigma^{(n)}(z),\varphi_i\rangle_{\mathcal{H}}.
\end{aligned}
$$

This will be recalled when we analyze the limit as $n \to \infty$ in the sequence of variational inequalities. What is important here is that we have verified $(g\,g^*)_{i,j}(z) \in C^1_b(\mathbb{R}^n)$ for all $i, j = 1, 2, \ldots, n$. Actually we also have $C^2_b(\mathbb{R}^n)$-regularity.

4.3 Variational inequalities on bounded domains

Now we will analyze the optimal stopping on a bounded open regular domain $\mathcal{O} \subset \mathbb{R}^n$. Let us denote the first exit time of the diffusion $Z^{t,z}$ from $\mathcal{O}$ as

$$\tau^{t,z}_{\mathcal{O}} := \inf\{s \geq t \,:\, Z^{t,z}_s \notin \mathcal{O}\}. \tag{4.14}$$

We have a early stopped version of the original optimal stopping problem when we define

$$U^{\mathcal{O}}(t,z) := \sup_{t \leq \tau \leq T} \mathbb{E}\left[\hat{\Psi}(\tau \wedge \tau_{\mathcal{O}}, Z^{t,z}_{\tau\wedge\tau_{\mathcal{O}}})\right], \tag{4.15}$$

Since in this section the domain $\mathcal{O}$ is given and fixed we can simplify the notation defining $v := U^{\mathcal{O}}$. We can associate a second order differential operator, namely $\mathcal{L}$, to the diffusion $Z^{t,z}$. In particular for any $f \in C^2_c(\mathbb{R}^n)$, the infinitesimal generator of the diffusion reads

$$\mathcal{L}f(z) = \frac{1}{2}\sum_{i,j=1}^{n} (g\,g^*)_{i,j}(z)\frac{\partial^2 f}{\partial z_i \partial z_j}(z) + \sum_{i=1}^{n} b_i(z)\frac{\partial f}{\partial z_i}(z). \tag{4.16}$$

From standard heuristic arguments about dynamic programming one would expect the value function $v(t,z)$ to fulfill the following variational inequality

$$\begin{cases} \max\left\{\frac{\partial v}{\partial t} + \mathcal{L}v\,,\, \hat{\Psi} - v\right\} = 0, & (t,z) \in (0,T) \times \mathcal{O}, \\ v(T,z) = \hat{\Psi}(T,z), & z \in \mathcal{O}, \\ v(t,z) = \hat{\Psi}(t,z), & (t,z) \in (0,T) \times \partial\mathcal{O}, \\ v(t,z) \geq \hat{\Psi}(t,z), & (t,z) \in [0,T] \times \bar{\mathcal{O}}. \end{cases} \tag{4.17}$$

It is worth recalling that there are several concepts of solution to this formal equation and we are going to introduce some of them in what follows. Yet, before going on, it is fundamental to remark a critical issue.

Remark 4.3.1 *Let us assume that v is a solution "in some sense" of problem (4.17). Then for $\lambda > 0$ given and fixed this is equivalent to claim that $\tilde{v}(t,z) = e^{\lambda t} v(t,z)$ is a solution "in the same sense" of the variational inequality in which we replace $\mathcal{L}$ by $\mathcal{L} - \lambda$ and the boundary conditon Φ reads $\Phi = e^{\lambda t}\hat{\Psi}$.*

For this reason in the following we will be quite relaxed in switching back and forth between the equivalent problems. We adopt the approach of [4], taking into account the fact that the function $\hat{\Psi}$ represents at the same time the boundary condition and the obstacle in our variational formulation. It is remarkable that the lack of regularity of the payoff function $\hat{\Psi}$, and then of the obstacle, prevents us from adopting standard penalization techniques to prove existence and uniqueness of a *strong* solution to the variational problem (cf. [30], Chapter 1, Theorem 3.2). We will proceed to an approximation of the payoff function and then we will provide a characterization of the value function as an appropriate *weak* solution of the variational inequality. Before going into details it is worth reformulating the problem in terms of homogeneous boundary condition, i.e. we define $u = v - \hat{\Psi}$ and formally rewrite the problem as

$$\begin{cases} \max\left\{\frac{\partial u}{\partial t} + \mathcal{L}u + f\,,\, -u\right\} = 0, & (t,z) \in (0,T) \times \mathcal{O}, \\ u(T,z) = 0, & z \in \mathcal{O}, \\ u(t,z) = 0, & (t,z) \in (0,T) \times \partial\mathcal{O}, \\ u(t,z) \geq 0, & (t,z) \in [0,T] \times \bar{\mathcal{O}}. \end{cases} \tag{4.18}$$

We have formally set

$$f(t,z) = \frac{\partial \hat{\Psi}}{\partial t}(t,z) + \mathcal{L}\hat{\Psi}(t,z). \tag{4.19}$$

Now we have to tackle a variational inequality with homogeneous (zero) boundary condition on the parabolic boundary $\partial\mathcal{O} \times (0,T]$. It is worth noticing that the obstacle is now represented by the constant function $\varphi \equiv 0$. In order for the previous expression to be rigorous we introduce the following Hilbert spaces [14]:

- $L^2(\mathcal{O})$ as usual represents the set of the square integrable real functions defined on $\mathcal{O}$. Let $(\,\cdot\,,\,\cdot\,)$ be the scalar product in this space and $|\cdot|_{L^2}$ the induced norm.

- $L^{2*}(\mathcal{O})$ is the dual of $L^2(\mathcal{O})$.

- $H^1(\mathcal{O})$ represents the subset of $L^2(\mathcal{O})$ of functions having square integrable partial weak derivatives of first order, namely $H^1(\mathcal{O}) := \{w \in L^2(\mathcal{O}) : \frac{\partial w}{\partial z_j} \in L^2(\mathcal{O}),\, j = 1, \dots, n\}$. This space is a Banach space when equipped with the norm

$$\|w\|_{H^1} = |w|_{L^2} + |\nabla w|_{L^2}.$$

- $H^2(\mathcal{O})$ represents the subset of $H^1(\mathcal{O})$ of functions having square integrable partial weak derivatives of second order, namely $H^2(\mathcal{O}) := \{w \in H^1(\mathcal{O}) : \frac{\partial^2 w}{\partial z_i \partial z_j} \in L^2(\mathcal{O}),\, i, j = 1, \dots, n\}$. This space is a Banach space when equipped with the norm

$$\|w\|_{H^1} = |w|_{L^2} + |\nabla w|_{L^2} + |D^2 w|_{L^2}.$$

- $H^1_0(\mathcal{O})$ is the closure of $C^\infty_c(\mathcal{O})$ with respect to the norm of $H^1(\mathcal{O})$.

- $H^{-1}(\mathcal{O})$ is the dual of $H^1_0(\mathcal{O})$ and $\|\cdot\|_{H^{-1}}$ is its norm.

Identifying $L^2(\mathcal{O})$ with its dual we get the usual Gelfand triple

$$H^1_0(\mathcal{O}) \hookrightarrow L^2(\mathcal{O}) \simeq L^{2*}(\mathcal{O}) \hookrightarrow H^{-1}(\mathcal{O}). \tag{4.20}$$

The injections are all compact. It is worth noticing that since $\partial\mathcal{O}$ is a C^2-boundary the trace operator $w|_{\partial\mathcal{O}}$ is well defined (cf. [4], Chapter 2, Sec. 5) and $w \in H^1_0(\mathcal{O})$ if and only if $w|_{\partial\mathcal{O}} = 0$. Moreover the Poincare's inequality (cf. [14], Corollary 9.19, pag. 290)

$$|w|_{L^2} \le C|\nabla w|_{L^2}, \qquad \forall w \in H^1_0(\mathcal{O}),$$

holds and hence the norm $\|w\|_{H^1}$ and $|\nabla w|_{L^2}$ are equivalent. Before proceeding we introduce, for any $t \in [0, T]$ given, the bilinear form $a(t; \cdot, \cdot) : H^1(\mathcal{O}) \times H^1(\mathcal{O}) \to \mathbb{R}$ defined as

$$\begin{aligned} a(t; u, w) \; := \; & \frac{1}{2} \sum_{i,j=1}^n \int_{\mathcal{O}} (g\, g^*)_{i,j}(z) \frac{\partial u}{\partial z_i} \frac{\partial w}{\partial z_j}(t, z) dz \\ + \; & \sum_{i=1}^n \int_{\mathcal{O}} \left(\frac{1}{2} \sum_{j=1}^n \frac{\partial (g\, g^*)_{i,j}}{\partial z_j}(z) - b_i(z) \right) \frac{\partial u}{\partial z_i} w(t, z) dz, \end{aligned}$$

The derivatives $\frac{\partial(g\,g^*)_{i,j}}{\partial z_j}$, $i,j=1,\ldots,n$ make sense in light of our previous discussion about the regularity of coefficients.

We want now to clarify the role of the bilinear form in our variational inequality. Let us multiply the first expression in (4.18) by $w \in H_0^1(\mathcal{O})$, then integrate over $\mathcal{O}$. It holds

$$\int_{\mathcal{O}} \frac{\partial u}{\partial t}(t,x)w(x)dx + \int_{\mathcal{O}} \mathcal{L}u(t,x)w(x)dx \leq -\int_{\mathcal{O}} f(t,x)w(x)dx.$$

Thanks to the regularity of $\partial\mathcal{O}$ we can adopt the Green's formula, [4], Chapter 2, Sec. 5, to perform integration by parts. Taking into account that $w|_{\partial\mathcal{O}}=0$, some simple computations produce the final expression

$$-(\frac{\partial u}{\partial t}(t),w) + a(t;u(t),w) \geq (f(t),w).$$

The drift coefficient in the SDE (4.1) has sublinear growth and then is bounded on $\mathcal{O}$. The diffusion term is bounded by hypothesis and hence it is easy to verify that the bilinear form is bounded in $H^1(\mathcal{O})$ uniformly with respect to $t \in [0,T]$, i.e.

$$|a(t;u,w)| \leq C_{\mathcal{O}}\|u\|_{H^1}\,\|w\|_{H^1}, \qquad \forall u,w \in H^1(\mathcal{O}). \tag{4.21}$$

It is worth noticing that $C_{\mathcal{O}}>0$ does not depend on T because all the coefficients are time homogeneous, yet it depends on the size of the set $\mathcal{O}$. We still have to clarify the meaning of the term $(f(t),w)$, in fact for $f(t)$ to be an element of $L^2(\mathcal{O})$ we need $\hat{\Psi}(t,\cdot)$ to admit a second order weak derivative with respect to the space variables. As we pointed out in Proposition 4.1.1 the best we can hope for is $\hat{\Psi} \in W^{1,p}((0,T)\times\mathcal{O})$, $1 \leq p \leq \infty$. Given the expression

$$(f(t),w) = \int_{\mathcal{O}} \frac{\partial\hat{\Psi}}{\partial t}(t,z)w(z)dz + \int_{\mathcal{O}} \mathcal{L}\hat{\Psi}(t,z)w(z)dz,$$

we can adopt the Green's formula in order to give a meaning to the term involving $\mathcal{L}\hat{Psi}$. Indeed it holds

$$(f(t),w) = (\frac{\partial\hat{\Psi}}{\partial t}(t),w) - a(t;\hat{\Psi}(t),w). \tag{4.22}$$

For $\hat{\Psi}$ given, the right hand side of the last expression can be interpreted as a continuous linear functional acting on elements of $H_0^1(\mathcal{O})$. Then for $t \in [0,T]$ fixed we introduce $\mathcal{T}_\psi(t) \in H^{-1}(\mathcal{O})$ defined through the dual pairing $\langle \cdot, \cdot \rangle_{H_0^1, H^{-1}}$ between $H_0^1(\mathcal{O})$ and $H^{-1}(\mathcal{O})$ as

$$\langle \mathcal{T}_\psi(t), w \rangle_{H_0^1, H^{-1}} := (\frac{\partial \hat{\Psi}}{\partial t}(t), w) - a(t; \hat{\Psi}(t), w), \qquad \forall w \in H_0^1(\mathcal{O}). \tag{4.23}$$

From previous considerations one easily verifies that the norm $\|\mathcal{T}_\psi(t)\|_{H_{-1}}$ is well defined and finite for all $t \in [0,T]$. It is useful in what follows to explicitly evaluate such a norm. For any $w \in H_0^1(\mathcal{O})$

$$|\langle \mathcal{T}_\psi(t), w \rangle_{H_0^1, H^{-1}}| \leq \big|(\frac{\partial \hat{\Psi}}{\partial t}(t), w)\big| + |a(t; \hat{\Psi}(t), w)|,$$

then by continuity of the bilinear form one gets

$$|\langle \mathcal{T}_\psi(t), w \rangle_{H_0^1, H^{-1}}| \leq \left(\big|\frac{\partial \hat{\Psi}}{\partial t}(t)\big|_{L^2(\mathcal{O})} + C_{\mathcal{O}} \|\hat{\Psi}(t)\|_{H^1(\mathcal{O})} \right) \|w\|_{H_0^1(\mathcal{O})}.$$

It then implies that

$$\|\mathcal{T}_\psi(t)\|_{H^{-1}} = \left(\big|\frac{\partial \hat{\Psi}}{\partial t}(t)\big|_{L^2(\mathcal{O})} + C_{\mathcal{O}} \|\hat{\Psi}(t)\|_{H^1(\mathcal{O})} \right). \tag{4.24}$$

The VI is well defined in the form

$$-(\frac{\partial u}{\partial t}(t), w) + a(t; u(t), w) \geq \langle \mathcal{T}_\psi(t), w \rangle_{H_0^1, H^{-1}}. \tag{4.25}$$

4.4 On the concept of solution

We can now introduce the concept of *strong* and *weak* solution of the evolutionary variational inequality as in [4], Chapter 3, Sec. 2. From now on we refer to the problem (4.18), because it is equivalent to the original problem up to the transformation $u = v - \hat{\Psi}$. As mentioned above we are tackling a very particular case of *obstacle problem*. In our case the obstacle is the constant function $\varphi = 0$ and then things get to be relatively simplified.

4.4.1 Strong solutions

At first we analyze what we consider the “best case”, i.e. the case of strong solutions.

Definition 4.4.1 *We say $u(t,z)$ is a strong solution to the evolutionary variational inequality (4.18) if*

$$u \in L^2(0,T;H_0^1(\mathcal{O})),\ \frac{du}{dt} \in L^2(0,T;L^2(\mathcal{O})),\ u(T)=0,\ u \geq 0 \, \textit{a.e. in} \, (0,T)\times\mathcal{O},$$

and satisfies

$$-(\frac{\partial u}{\partial t}(t), w-u(t)) + a(t;u(t),w-u(t)) \geq \langle \mathcal{T}_\psi(t), w-u(t)\rangle_{H_0^1,H^{-1}}, \quad \textit{a.e.}\, t\in[0,T],$$

for all $w \in H_0^1(\mathcal{O})$ such that $w \geq 0$ a.e. in $\mathcal{O}$.

It is worth noticing that from [23], Theorem 1, Chapter VIII, we know that the continuous injection

$$W(0,T;H_0^1(\mathcal{O}),H^{-1}(\mathcal{O})) \hookrightarrow C(0,T;L^2(\mathcal{O})) \tag{4.26}$$

holds (cf. Appendix D). It is then clear that considering the continuous version of $u(t,z)$ it makes sense to interpret the boundary condition $u(T)=0$ pointwisely.

Remark 4.4.1 *When $\hat{\Psi}$ is regular enough we have $f := \partial_t\hat{\Psi} + \mathcal{L}\hat{\Psi} \in L^2(\mathcal{O})$. Hence on the right hand side of the inequality we find the scalar product $(f, w-u)$ instead of the dual pairing. In some cases there is a substantial difference in the arguments needed to prove existence and uniqueness of solutions to this kind of problems.*

It is now interesting to point out the connection between the *strong* solution and the heuristic variational inequality (4.17). This can be achieved if we manage somehow to add some regularity to the solution u. Let us assume for a moment that $v = u + \hat{\Psi}$ is in the class $W^{12,2}((0,T)\times\mathcal{O})$ so that the derivatives in (4.17) hold in the sense almost everywhere. We will show in the remainder that under suitable regularity conditions for the coefficients and for the boundary value, such a regularity can be obtained.

In terms of $v = u + \hat{\Psi}$ the strong solution of definition above solves the inequality

$$-(\frac{\partial v}{\partial t}(t), w(t) - v(t)) + a(t; v(t), w(t) - v(t)) \geq 0, \qquad \text{a.e. } t \in [0, T],$$

for all $w \in L^2(0, T; H^1(\mathcal{O}))$ such that $w(t) \in H^1(\mathcal{O})$, $w(t) - \hat{\Psi}(t) \in H_0^1(\mathcal{O})$ a.e. $t \in [0, T]$ and $w \geq \hat{\Psi}$ a.e. in $[0, T] \times \mathcal{O}$. It can be rewritten in terms of the infinitesimal generator $\mathcal{L}$ as

$$(\frac{\partial v}{\partial t}(t) + \mathcal{L}v(t), w(t) - v(t)) \leq 0, \qquad \text{a.e. } t \in [0, T].$$

If we choose $w(t) = v(t) + \zeta$, for $\zeta \in C_0^\infty(\mathcal{O})$ and $\zeta \geq 0$ we obtain

$$(\frac{\partial v}{\partial t}(t) + \mathcal{L}v(t), \zeta) \leq 0, \qquad \text{a.e. } t \in [0, T],$$

and it clearly implies

$$\frac{\partial v}{\partial t} + \mathcal{L}v \leq 0, \qquad \text{a.e. } \in (0, T) \times \mathcal{O}.$$

If we now choose $w = \hat{\Psi}(t)$ we get

$$(\frac{\partial v}{\partial t} + \mathcal{L}v, \hat{\Psi} - v) \leq 0, \qquad \text{a.e. } t \in [0, T],$$

so that since $\hat{\Psi} - v \leq 0$, we should have

$$\frac{\partial v}{\partial t} + \mathcal{L}v \geq 0, \qquad \text{a.e. } \in (0, T) \times \mathcal{O}.$$

This fact is in contradiction with the previous observation, hence we conclude that

$$\max\left\{ \left(\frac{\partial v}{\partial t} + \mathcal{L}v \right), (\hat{\Psi} - v) \right\} = 0, \qquad \text{a.e. } \in (0, T) \times \mathcal{O}.$$

We have then established the connection between the formal variational inequality and the concept of strong solution. Besides that the concept of strong solution is sometimes too tight in order to precisely characterize the solution of our variational inequality and hence we introduce a weaker concept of solution.

4.4.2 Weak solutions

The basic idea behind the concept of weak solution is to relax the regularity with respect to the time variable. First we introduce a convex set

$$\mathcal{K} := \{w : \ w \in L^2(0,T;H_0^1(\mathcal{O})), \frac{\partial w}{\partial t} \in L^2(0,T;L^2(\mathcal{O})), w(t,z) \geq 0 \text{ a.e. in } (0,T)\times\mathcal{O}\}. \quad (4.27)$$

This set is not empty. Let us assume for a moment that u is a strong solution as in Definition (4.4.1), then let us consider the expression

$$Y := \int_0^T \left[-(\frac{\partial w}{\partial t}(t), w(t)-u(t)) + a(t;u(t),w(t)-u(t)) - \langle \mathcal{T}_\psi(t), w(t)-u(t)\rangle_{H_0^1,H^{-1}} \right] dt.$$

Let us skip the explicit time dependence and the indexes in the dual pairing, since there is no ambiguity. We have

$$\begin{aligned} Y &= \int_0^T \left[-(\frac{\partial u}{\partial t}, w-u) + a(t;u,w-u) - \langle \mathcal{T}_\psi, w-u\rangle \right] dt \\ &+ \int_0^T \left[-(\frac{\partial}{\partial t} w-u, w-u) \right] dt \geq \int_0^T \left[-\frac{1}{2}\frac{d}{dt}|w-u|_{L^2}^2 \right] dt \end{aligned}$$

From the definition of u we get

$$Y \geq \int_0^T \left[-\frac{1}{2}\frac{d}{dt}|w-u|_{L^2}^2 \right] dt = \frac{1}{2}\left(|w(0)-u(0)|_{L^2}^2 - |w(T)-u(T)|_{L^2}^2 \right).$$

It is then clear that

$$Y + \frac{1}{2}|w(T)-u(T)|_{L^2}^2 \geq 0.$$

From this arguments we see that the following definition makes sense and in particular that any *strong* solution is also a *weak* solution.

Definition 4.4.2 *We say $u(t,z)$ is a weak solution to the evolutionary variational inequality (4.18) if*

$$u \in L^2(0,T;H_0^1(\mathcal{O})), \ \ u \geq 0 \textit{ a.e. in } (0,T)\times\mathcal{O},$$

and satisfies

$$\int_0^T \left[-(\frac{\partial w}{\partial t}, w-u) + a(t;u,w-u) - \langle \mathcal{T}_\psi, w-u \rangle \right] dt + \frac{1}{2}|w(T)|^2_{L^2} \geq 0, \tag{4.28}$$

for all $w \in \mathcal{K}$.

It is worth remarking that the boundary condition $u(T) = 0$ is embedded in the last term of expression (4.28). Indeed the full formal expression of the last term would be $|w(T) - u(T)|_{L^2}$. Then by replacing it with $|w(T)|_{L^2}$ necessarily implies $u(T) = 0$, a.e. $z \in \mathcal{O}$.

4.5 General results about strong solutions

We give now a short survey about results of existence and uniqueness of strong solutions to Evolutionary Variational Inequalities (EVI). The standing assumptions are

$$\begin{aligned}
&(g\,g^*)_{i,j}(z),\ b_i(z) \in L^\infty((0,T)\times\mathcal{O}),\ \ i,j = 1,\dots,n,\\
&\frac{\partial}{\partial z_k}(g\,g^*)_{i,j}(z) \in L^\infty((0,T)\times\mathcal{O}),\ \ i,j,k = 1,\dots,n,\\
&\frac{\partial}{\partial t}(g\,g^*)_{i,j}(z) \in L^\infty((0,T)\times\mathcal{O}),\ \ i,j = 1,\dots,n,\\
&(g\,g^*)_{i,j}(z) = (g\,g^*)_{j,i}(z).
\end{aligned} \tag{4.29}$$

Notice that from the discussion in Section 4.2 we know that these assumptions are fulfilled in our case.

The finite dimensional approximation algorithm is such that the diffusion (4.1) is non degenerate. This in turn guarantees the uniform ellipticity condition, indeed at the n-th step we have

$$\sum_{i,j=1}^n (g\,g^*)_{i,j}(z) y_i y_j \geq \epsilon_n |y|^2_{\mathbb{R}^n},\ \text{a.e. in } \mathcal{O},\ \ \forall y \in \mathbb{R}^n. \tag{4.30}$$

Unfortunately in our case $\epsilon_n \to 0$, then the uniform ellipticity tends to vanish when we pass to the limit and we definitely have to deal with a degenerate diffusion. This fact is rather clear if we remember that the infinite dimensional diffusion is driven by a one dimensional Brownian motion. Nevertheless, thanks to the standing assumptions we obtain the continuity and the coerciveness of the bilinear form $a(t;\cdot,\cdot)$, namely there exist $C_{\mathcal{O}},\ \lambda_{\mathcal{O}},\ \hat{\lambda}_{\mathcal{O}} > 0$ independent on time such that

$$|a(t;u,w)| \leq C_{\mathcal{O}} \|u\|_{H_0^1} \|w\|_{H_0^1}, \qquad \forall u, w \in H_0^1(\mathcal{O}), \tag{4.31}$$

and

$$a(t;w,w) + \lambda_{\mathcal{O}} \|w\|_{L^2}^2 \geq \hat{\lambda}_{\mathcal{O}} \|w\|_{H_0^1}^2, \qquad \forall w \in H_0^1(\mathcal{O}). \tag{4.32}$$

From Remark 4.3.1 we can swicth to the equivalent problem and we simply assume

$$a(t;w,w) \geq \hat{\lambda}_{\mathcal{O}} \|w\|_{H_0^1}^2, \qquad \forall w \in H_0^1(\mathcal{O}). \tag{4.33}$$

The obstacle in our problem is represented by the function $\varphi = 0$ and then is smooth in both time and space variables. The crucial fact is that the regularity of the payoff function affects the formulation of the whole problem. As stated earlier we have $\hat{\Psi} \in C([0,T] \times \bar{\mathcal{O}})$ and for all $1 \leq p < \infty$

$$\frac{\partial \hat{\Psi}(t,z)}{\partial t},\ \frac{\partial \hat{\Psi}(t,z)}{\partial z_j} \in L^p((0,T) \times \mathcal{O}),\ \ j = 1, \ldots, n.$$

As we will see this is not enough in order to guarantee the existence of a strong solution. We state now the existence and uniqueness theorem for strong solutions of variational inequalities. The statement is almost the same as in [4] and summarizes Theorem 2.2 and Corollary 2.2, Chapter 3, Section 2 therein. The proof here is omitted. The obstacle is denoted by φ and the boundary condition by $\bar{u}$.

Theorem 4.5.1 *Let us consider the following general variational problem in the strong formula-*

tion:

$$\begin{cases} \textit{Find } u(t,z) \textit{ such that:} \\ u \in L^2(0,T;H_0^1(\mathcal{O})),\ \frac{du}{dt} \in L^2(0,T;L^2(\mathcal{O})), \\ u(T) = \bar{u},\ u \geq \varphi \textit{ a.e. in } (0,T) \times \mathcal{O}, \\ -(\frac{\partial u}{\partial t}(t), w - u(t)) + a(t;u(t), w-u(t)) \geq (f, w-u(t)), \qquad \textit{a.e. } t \in [0,T], \\ \forall w \in H_0^1(\mathcal{O}) \textit{ such that } w \geq \varphi \textit{ a.e. in } \mathcal{O}. \end{cases} \tag{4.34}$$

Let conditions (4.29), (4.31) and (4.33) hold and let $f \in L^2(0,T;L^2(\mathcal{O}))$*. Let us also assume*

$$\varphi,\ \frac{\partial \varphi}{\partial t} \in L^2(0,T;H^1(\mathcal{O})), \tag{4.35}$$

$$\varphi \geq 0,\ \frac{\partial \varphi}{\partial t} = 0, \quad \textit{on } (0,T) \times \partial\mathcal{O}.$$

For the boundary condition let us assume $\bar{u} = u(T,z) \in H_0^1(\mathcal{O})$ *and* $\bar{u} \geq 0$*. Then there exists a unique solution,* $u(t,z)$*, to problem (4.34) and* $u \in L^2(0,T;H^2(\mathcal{O})) \cap L^\infty(0,T;H_0^1(\mathcal{O}))$*.*

We present now another result that refines the previous theorem. The statement we will make summarizes [4], Theorem 2.13 and Corollary 2.3, Chapter 3, Section 2. The basic idea is that, given the hypotheses of the previous theorem, the regularity of the solution only depends on the regularity of f and φ.

Theorem 4.5.2 *Let us assume the same hypotheses as in Theorem 4.5.1. Let also the following regularity conditions hold*

$$f \in L^p((0,T) \times \mathcal{O}),\ \varphi \in L^p((0,T) \times \mathcal{O}),$$

$$-\frac{\partial \varphi}{\partial t}(t,z) + \mathcal{L}\varphi(t,z) \in L^p((0,T) \times \mathcal{O}),$$

$$\bar{u} \in W^{2,p}(\mathcal{O}) \cap W_0^{1,p}(\mathcal{O}).$$

Then the regularity of the unique strong solution is

$$u \in L^p(0,T;W^{2,p}(\mathcal{O})), \ \frac{du}{dt} \in L^p((0,T) \times \mathcal{O}).$$

We can now discuss about the solution of our specific problem. In our case the hypothesis $f \in L^2(0,T;L^2(\mathcal{O}))$ fails to be true in the homogeneized formulation of the EVI. Moreover the payoff function $\hat{\Psi}$ plays, in the non-homogeneous formulation, the role of the obstacle in problem (4.5.1). It would be then desirable to have for $\hat{\Psi}$ the same regularity as in (4.35) but this fact fails to be true as well. These two difficulties represent two ways of analyzing the lack of regularity of our problem. It turns out to be impossible to carry out the arguments as in [4] to prove existence for the strong solution to our variational inequality.

4.6 Existence and uniqueness of a weak solution

The rationale adopted in [4], Chapter 3, Section 2, to find a unique strong solution for EVI cannot be applied straightforwardly in the present case. This fact is mainly due to the impossibility to rewrite rigorously the dual pairing as a scalar product in $L^2(\mathcal{O})$. Moreover, specific difficulties arise in proving the existence of the distributional time derivative of u. This happens because $\hat{\Psi}$ not being regular makes impossible to prove some estimates in the penalization procedure that usually one tries to carry out (cf. [4], eq. 255-258, pag. 244-245). On the other hand the regularity of the data in the variational problem is good enough to guarantee the existence of a maximum weak solution as in [4], Section 2, Theorem 2.6. Nevertheless it is worth to remark that the set of weak solutions does not in general reduce to a single element, (for examples of non uniqueness cf. [45]). Motivated by this remark we prefer to rely on the results of Appendix A. We then need the following additional assumption about the gain function.

Assumption 4.6.1 *Let* $\Psi : [0,T]\times\mathcal{H} \to \mathbb{R}$ *be such that it is possible to find a sequence* $\{\Psi_k\}_{k\geq 1} \subset C^\infty([0,T] \times \mathcal{H})$ *and for any finite measure* μ *it holds*

$$\lim_{k\to\infty} \int_0^T \int_{\mathcal{H}} |\Psi_k(t,x) - \Psi(t,x)|^2 \mu(dx)dt = 0,$$

and

$$\lim_{k\to\infty}\int_0^T\int_{\mathcal{H}}\|D\Psi_k(t,x)-D\Psi(t,x)\|^2_{\mathcal{H}}\mu(dx)dt=0.$$

Let moreover the convergence be uniform, i.e.

$$\lim_{k\to\infty}\sup_{(t,x)\in[0,T]\times\mathcal{H}}|\Psi_k(t,x)-\Psi(t,x)|=0.$$

It is clear that under this assumption the regularity of the gain function keep being true at the finite dimensional level and hence we shall replace $\hat{\Psi}$ by $\hat{\Psi}_k\in C^\infty([0,T]\times\mathbb{R}^n)$. Then we know that we have a unique strong solution for the variational inequality with such a regular obstacle. Nevertheless this regularity vanishes when the order of the approximation increases. For this reason we will set the whole problem into the framework of the weak formulation and try to characterize the solution in this context. This will turn out to be the natural setting of our variational problem at the infinite dimensional level. At this point it may seem that the adoption of the regularized gain function is useless, yet it will play a crucial role when taking the limit to infinite dimensions. This fact will be discussed in more details later on.

4.6.1 Regularized variational inequality

As a first step we turn our problem into the framework of Theorem 4.5.1. As a consequence of our regularization we can define a sequence $\{\mathcal{T}_{\psi,k}\}_{k=1}^\infty$ in $L^2(0,T;H^{-1}(\mathcal{O}))$, through the dual pairing

$$\langle\mathcal{T}_{\psi,k}(t),w\rangle:=(\frac{\partial\hat{\Psi}_k}{\partial t}(t),w)-a(t;\hat{\Psi}_k(t),w),\qquad\forall w\in H^1_0(\mathcal{O}).\tag{4.36}$$

It is rather simple to verify that $\mathcal{T}_{\psi,k}\to\mathcal{T}_\psi$ in $L^2(0,T;H^{-1}(\mathcal{O}))$ as $k\to\infty$. At the end of our analysis we will see how the same limit holds in a suitable infinite dimensional setting. For $k\in\mathbb{N}$ given and fixed, the regularity of $\hat{\Psi}_k$ enables us to define $f^{(k)}\in L^p((0,T)\times\mathcal{O})$, for all $1\le p<\infty$ as

$$f^{(k)}(t,z):=\frac{\partial\hat{\Psi}_k}{\partial t}(t,z)+\mathcal{L}\hat{\Psi}_k(t,z).$$

We can now write down a regularized version of our EVI in terms of $\hat{\Psi}_k$ and $f^{(k)}$, i.e.

$$\begin{cases} \text{Find } u^{(k)}(t,z) \text{ such that:} \\ u^{(k)} \in L^2(0,T;H_0^1(\mathcal{O})),\ \frac{du^{(k)}}{dt} \in L^2(0,T;L^2(\mathcal{O})), \\ u^{(k)}(T)=0,\ u^{(k)} \geq 0 \text{ a.e. in } (0,T)\times\mathcal{O}, \\ -(\frac{\partial u^{(k)}}{\partial t}(t), w-u^{(k)}(t)) + a(t;u^{(k)}(t), w-u^{(k)}(t)) \geq (f^{(k)}, w-u^{(k)}(t)),\ \text{a.e. } t\in[0,T], \\ \forall w \in H_0^1(\mathcal{O}) \text{ such that } w \geq 0 \text{ a.e. in } \mathcal{O}. \end{cases} \tag{4.37}$$

From Theorem 4.5.1 and Theorem 4.5.2 we know that there exists a unique strong solution $u^{(k)}$ such that

$$u^{(k)} \in L^p(0,T;W_0^{1,p}(\mathcal{O})) \cap L^p(0,T;W^{2,p}(\mathcal{O})),\ \ \frac{du^{(k)}}{dt} \in L^p(0,T;L^p(\mathcal{O})),$$

for all $1 \leq p < \infty$. Moreover, the regularity of this solution allows us to understand the EVI in the almost everywhere sense. In particular it holds

$$\begin{cases} \max\left\{\frac{\partial u^{(k)}}{\partial t} + \mathcal{L}u^{(k)} + f^{(k)}\,,\, -u^{(k)}\right\} = 0, & \text{a.e. } (t,z)\in(0,T)\times\mathcal{O}, \\ u^{(k)}(T,z)=0, & \text{a.e. } z\in\mathcal{O}, \\ u^{(k)}(t,z)=0, & \text{a.e. } (t,z)\in(0,T)\times\partial\mathcal{O}, \\ u^{(k)}(t,z)\geq 0, & \text{a.e. } (t,z)\in[0,T]\times\bar{\mathcal{O}}. \end{cases} \tag{4.38}$$

It is worth to stress the connection with the non homogeneous EVI. If we go back to the original problem, i.e. we consider $v^{(k)} = u^{(k)} + \hat{\Psi}_k$, we obtain

$$v^{(k)} \in L^p(0,T;W^{2,p}(\mathcal{O})),\ \ \frac{dv^{(k)}}{dt} \in L^p(0,T;L^p(\mathcal{O})),$$

and

$$v^{(k)} - \hat{\Psi}_k \in L^p(0,T;W_0^{1,p}(\mathcal{O})).$$

Moreover $v^{(k)}$ is the unique solution of

$$\begin{cases} \max\left\{\frac{\partial v^{(k)}}{\partial t} + \mathcal{L}v^{(k)}\,,\,\hat{\Psi}_k - v^{(k)}\right\} = 0, & \text{a.e. } (t,z) \in (0,T) \times \mathcal{O}, \\ v^{(k)}(T,z) = \hat{\Psi}_k(T,z), & \text{a.e. } z \in \mathcal{O}, \\ v^{(k)}(t,z) = \hat{\Psi}_k(t,z), & \text{a.e. } (t,z) \in (0,T) \times \partial\mathcal{O}, \\ v^{(k)}(t,z) \geq \hat{\Psi}_k(t,z), & \text{a.e. } (t,z) \in [0,T] \times \bar{\mathcal{O}}. \end{cases} \tag{4.39}$$

This formulation will be useful later to clarify the connection between the optimal stopping problem and the variational formulation. As we already mentioned we wish to set our variational problem in a weaker form. We give a simple result

Theorem 4.6.1 *The unique strong solution of (4.37), $u^{(k)}$, is also the unique solution of: find $u \in L^2(0,T;H_0^1(\mathcal{O}))$ such that*

$$u(T) = 0, \;\; u \geq 0 \textit{ a.e. in } (0,T) \times \mathcal{O},$$

and satisfies

$$\int_0^T \left[-(\frac{\partial w}{\partial t}, w - u) + a(t; u, w - u) - \langle \mathcal{T}_{\psi,k}, w - u\rangle\right] dt + \frac{1}{2}|w(T)|^2_{L^2} \geq 0, \tag{4.40}$$

for all $w \in \mathcal{K}$.

PROOF: The fact that the strong solution $u^{(k)}$ is also a weak solution was previously discussed. The only thing that still needs to be proven is the uniqueness. First of all notice the obvious fact that

$$\int_0^T \left[-(\frac{\partial u^{(k)}}{\partial t}, w - u^{(k)}) + a(t; u^{(k)}, w - u^{(k)}) - \langle \mathcal{T}_{\psi,k}, w - u^{(k)}\rangle \right] dt \geq 0, \tag{4.41}$$

for all $w \in L^2(0,T;H_0^1(\mathcal{O}))$ and $w \geq 0$. Let v be another solution of (4.40). Since $u^{(k)} \in \mathcal{K}$ we can set $w = u^{(k)}$ in expression (4.40) and obtain

$$\int_0^T \left[-(\frac{\partial u^{(k)}}{\partial t}, u^{(k)} - v) + a(t; v, u^{(k)} - v) - \langle \mathcal{T}_{\psi,k}, u^{(k)} - v\rangle \right] dt + \frac{1}{2}|u^{(k)}(T)|^2_{L^2} \geq 0,$$

and clearly, since $u^{(k)}(T) = 0$ a.e. on $\mathcal{O}$ the expression simplifies to

$$\int_0^T \left[-(\frac{\partial u^{(k)}}{\partial t}, u^{(k)} - v) + a(t; v, u^{(k)} - v) - \langle \mathcal{T}_{\psi,k}, u^{(k)} - v \rangle \right] dt \geq 0. \tag{4.42}$$

We substitute $w = v$ in the expression (4.41) and sum it with (4.42). We then obtain

$$\int_0^T \left[a(t; u^{(k)}, v - u^{(k)}) + a(t; v, u^{(k)} - v) \right] dt \geq 0,$$

and hence

$$\int_0^T a(t; u^{(k)} - v, v - u^{(k)}) dt \geq 0.$$

It implies

$$\int_0^T a(t; v - u^{(k)}, v - u^{(k)}) dt \leq 0,$$

and now by coerciveness (4.33) we obtain the uniqueness,

$$\hat{\lambda}_{\mathcal{O}} \int_0^T \|v - u^{(k)}\|^2_{H_0^1} dt \leq 0.$$

■

Hence we are now in the setting of weak solutions and our aim is to point out the connection between this variational problem and the optimal stopping problem restricted to a bounded domain in $\mathbb{R}^n$. Before doing so it is worth noticing that the same formulation holds for the non-homogenous problem. We need to define the following closed convex set

Definition 4.6.1 *Let $\mathcal{K}_\Psi^{(k)}$ be the closed convex set of functions $w \in L^2(0, T; H^1(\mathcal{O}))$ such that*

$$\begin{cases} \frac{\partial w}{\partial t} \in L^2(0, T; L^2(\mathcal{O})), \\ w(t) - \hat{\Psi}_k(t) \in H_0^1(\mathcal{O}) \text{ a.e. } t \in [0, T], \\ w \geq \hat{\Psi}_k \text{ a.e. } [0, T] \times \mathcal{O} \end{cases} \tag{4.43}$$

From the definition we have that if $w \in \mathcal{K}_\Psi^{(k)}$ then $w(t,z) \geq \hat{\Psi}_k(t,z)$ a.e. in $(0,T) \times \mathcal{O}$. Now we see that substituting $u^{(k)} = v^{(k)} - \hat{\Psi}_k$ in equation (4.40), the result in the previous theorem is equivalent to say that there exists a unique function v such that

$$v \in L^2(0,T; H^1(\mathcal{O})), \ \ v - \hat{\Psi}_k \in L^2(0,T; H_0^1(\mathcal{O})),$$

$$v \geq \hat{\Psi}_k \text{ a.e. in } (0,T) \times \mathcal{O},$$

and satisfies

$$\int_0^T \left[-(\frac{\partial w}{\partial t}, w - v) + a(t; v, w - v) \right] dt + \frac{1}{2} |w(T) - \hat{\Psi}_k(T)|^2_{L^2(\mathcal{O})} \geq 0, \tag{4.44}$$

for all $w \in \mathcal{K}_\Psi^{(k)}$.

We have proved that there exists a unique strong (weak) solution v to the regularized version of the problem (4.17). We want now to prove that such a solution is actually a suitable optimal stopping functional, namely a regularized version of (4.15).

4.7 Connection with the optimal stopping problem

Before starting our analysis we remind some results about the Sobolev spaces we are dealing with. In particular we recall the concept of *segment property* for a domain.

Definition 4.7.1 *A domain $\Omega \subset \mathbb{R}^n$ has the segment property if $\forall x \in \partial\Omega$ there exists an open set U_x and a non-zero vector $y_x \in \mathbb{R}^n$ such that $x \in U_x$ and if $z \in \bar{\Omega} \cap U_x$, then $z + ty_x \in \Omega$ for* $0 < t < 1$.

It is worth noticing that such a domain must have a $n-1$-dimensional boundary and cannot simultaneously lie on both sides of any given part of its boundary. Clearly the domain $(0,T) \times \mathcal{O}$ has this property and moreover it is Lipschitzian. Now we can state a useful result from [1], Theorem 3.22.

Theorem 4.7.1 *Given an arbitrary domain $\Omega \subset \mathbb{R}^n$, if it has the segment property, then the set of restrictions to Ω of functions $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\Omega)$ for $1 \le p < \infty$.*

We can now proceed with our connection with the optimal stopping problem. For $k \in \mathbb{N}$ given, we take $v^{(k)}$ to be as in (4.39). Then in particular one has

$$v^{(k)} \in W^{1\,2,p}((0,T) \times \mathcal{O}),$$

for arbitrary $1 \le p < \infty$. Hence it also holds for $p > n$. The Sobolev embedding theorem holds. Clearly $W^{1\,2,p}((0,T) \times \mathcal{O}) \subset W^{1\,1,p}((0,T) \times \mathcal{O})$. The bounded set $(0,T) \times \mathcal{O}$, understood as a domain in $\mathbb{R}^{n+1}$, has the *cone property*[1]. If we now denote $\mathcal{O}' = (0,T) \times \mathcal{O}$, we can write

$$v^{(k)} \in W^{1,p}(\mathcal{O}'), \qquad p > n,$$

and then for the Sobolev embedding theorem (cf. [1], Chapter V) we have that $W^{1,p}(\mathcal{O}') \hookrightarrow C(\mathcal{O}')$. From now on we consider $v^{(k)}$ to be the continuous version in the equivalence class $W^{1,p}(\mathcal{O}')$.

From the result of Theorem 4.7.1 we can choose a sequence $\{v_j^{(k)}\}_{j=1}^\infty$, such that $v_j^{(k)} \in C_c^\infty(\mathbb{R}^{n+1})$ for all $j \in \mathbb{N}$ and

$$\|v_j^{(k)} - v^{(k)}\|_{W^{1\,2,p}((0,T)\times\mathcal{O})} \to 0, \qquad \text{as } j \to \infty. \tag{4.45}$$

This is due to the fact that the approximation is obtained by partitioning the domain and then on each element of the partition one adopts the standard mollification. The usual properties of the mollifiers and the fact that ∂_t, ∇ and ∇^2 are closed operators in L^p guarantees that the limit holds. Moreover, for the continuity of $v^{(k)}$ up to a suitable extension to $\mathbb{R}^{n+1}$, one also has uniform convergence on any compact $\mathcal{O}''$ such that $[0,T] \times \bar{\mathcal{O}} \subset \mathcal{O}''$, i.e.

$$\|v_j^{(k)} - v^{(k)}\|_{L^\infty} \to 0, \qquad \text{on } \mathcal{O}'', \text{ as } j \to \infty.$$

[1]There exists a finite cone C, such that each point $(t,x) \in (0,T) \times \mathcal{O}$ is the vertex of a finite cone $C_{t,x}$ contained in $(0,T) \times \mathcal{O}$ and congruent to C.

This fact obviously implies that $\sup_{j\in\mathbb{N}} \|v_j^{(k)}\|_{L^\infty} < \infty$. We now show that the solution $v^{(k)}$ of the regularized problem coincides with an appropriate optimal stopping functional. In order to do so we recall [4], Lemma 8.1, Chapter 2, Section 8

Lemma 4.7.1 *Given the SDE (4.1), if $g\,(t,z) \in L^2((0,T)\times\mathcal{O})$ then for any stopping time $\tau \le \tau_{\mathcal{O}}$, $\mathbb{P}$-a.s. there exists $C_{T,\mathcal{O}} > 0$ only depending on T and on the size of the domain, such that*

$$\left|\mathbb{E}\left[\int_t^\tau g\,(s, Z_s^{t,z})ds\right]\right| \le C_{T,\mathcal{O}}\|g\,\|_{L^2((0,T)\times\mathcal{O})}. \tag{4.46}$$

PROOF: The proof follows standard arguments in [4],

$$\left|\mathbb{E}\left[\int_t^\tau g\,(s, Z_s^{t,z})ds\right]\right| = \left|\mathbb{E}\left[\int_t^{\tau\wedge T} I(s \le \tau_{\mathcal{O}})g\,(s, Z_s^{t,z})ds\right]\right|$$

$$= \left|\mathbb{E}\left[\int_t^{\tau\wedge T} I(Z_s^{t,z} \in \mathcal{O})g\,(s, Z_s^{t,z})ds\right]\right| \le \mathbb{E}\left[\int_t^T I(Z_s^{t,z} \in \mathcal{O})|g\,(s, Z_s^{t,z})|ds\right]$$

$$= \int_{\mathbb{R}^n}\int_t^T I(y \in \mathcal{O})|g\,(s,y)|\mathbb{P}(Z_s^{t,z} \in dy)ds$$

$$\le \left(\int_{\mathcal{O}}\int_t^T |g\,(s,y)|^2 dy\,ds\right)^{\frac{1}{2}} \left(\int_{\mathcal{O}}\int_t^T \mathbb{P}(Z_s^{t,z} \in dy)ds\right)^{\frac{1}{2}}$$

$$\le C_{T,\mathcal{O}}\|g\,\|_{L^2((0,T)\times\mathcal{O})}.$$

■

We can now state the verification theorem

Theorem 4.7.2 *Let $v^{(k)}(t,z)$ be the unique strong solution of (4.39). Then*

$$v^{(k)}(t,z) = \sup_{t\le\tau\le T} \mathbb{E}\left[\hat{\Psi}_k(\tau\wedge\tau_{\mathcal{O}}, Z^{t,z}_{\tau\wedge\tau_{\mathcal{O}}})\right]. \tag{4.47}$$

Moreover, the optimal stopping time is characterized as:

$$\tau_k^* := \inf\{s \ge t \,:\, v^{(k)}(s, Z_s^{t,z}) = \hat{\Psi}_k(s, Z_s^{t,z})\} \wedge T.$$

PROOF: The proof develops along the lines of [4], Chapter 3, Section 4. If we now consider $v_j^{(k)}(s, Z_s^{t,z})$, for $s \geq t$ and $Z^{t,z}$ the solution of (4.1), we can apply the Itô's formula in the random time interval $[0, \tau \wedge \tau_{\mathcal{O}}]$ for τ a stopping time in $[0,T]$:

$$\begin{aligned} v_j^{(k)}(\tau \wedge \tau_{\mathcal{O}}, Z^{t,z}_{\tau\wedge\tau_{\mathcal{O}}}) &= v_j^{(k)}(t,z) + \int_t^{\tau\wedge\tau_{\mathcal{O}}} \left[\frac{\partial v_j^{(k)}}{\partial s}(s, Z_s^{t,z}) + \mathcal{L} v_j^{(k)}(s, Z_s^{t,z}) \right] ds \\ &+ \sum_{i,l=1}^{n} \int_t^{\tau\wedge\tau_{\mathcal{O}}} \frac{\partial v_j^{(k)}}{\partial z_i}(s, Z_s^{t,z}) g_{\,i,l}(Z_s^{t,z}) dW_s^l. \end{aligned}$$

If we take into account that $\frac{\partial v_j^{(k)}}{\partial z_i}$ and $g(z)$ are bounded in $[0,T] \times \mathcal{O}$, we can take the average of both left and right side and the stochastic integral part will vanish. Then

$$\mathbb{E}\left[v_j^{(k)}(\tau \wedge \tau_{\mathcal{O}}, Z^{t,z}_{\tau\wedge\tau_{\mathcal{O}}}) \right] = v_j^{(k)}(t,z) + \mathbb{E}\left[\int_t^{\tau\wedge\tau_{\mathcal{O}}} \left(\frac{\partial v_j^{(k)}}{\partial s} + \mathcal{L} v_j^{(k)} \right)(s, Z_s^{t,z}) ds \right]. \tag{4.48}$$

We want now to take the limit as $j \to \infty$. For the term on the left hand side and for the first term on the right hand side we can exploit the uniform convergence stated above together with dominated convergence. For the term involving the integral we rely on the previous lemma. This fact implies that

$$\left| \mathbb{E}\left[\int_t^{\tau\wedge\tau_{\mathcal{O}}} \left(\frac{\partial v_j^{(k)}}{\partial s} + \mathcal{L} v_j^{(k)} \right)(s, Z_s^{t,z}) ds \right] - \mathbb{E}\left[\int_t^{\tau\wedge\tau_{\mathcal{O}}} \left(\frac{\partial v^{(k)}}{\partial s} + \mathcal{L} v^{(k)} \right)(s, Z_s^{t,z}) ds \right] \right|$$

$$\leq C_{T,\mathcal{O}} \left\| \frac{\partial v_j^{(k)}}{\partial t} - \frac{\partial v^{(k)}}{\partial t} + \mathcal{L} v_j^{(k)} - \mathcal{L} v^{(k)} \right\|_{L^2((0,T)\times\mathcal{O})}.$$

Now conditions (4.29) on the coefficients and the convergence result (4.45) allow us to take the limit as $j \to \infty$ and conclude that

$$\mathbb{E}\left[\int_t^{\tau\wedge\tau_{\mathcal{O}}} \left(\frac{\partial v_j^{(k)}}{\partial s} + \mathcal{L} v_j^{(k)} \right)(s, Z_s^{t,z}) ds \right] \to \mathbb{E}\left[\int_t^{\tau\wedge\tau_{\mathcal{O}}} \left(\frac{\partial v^{(k)}}{\partial s} + \mathcal{L} v^{(k)} \right)(s, Z_s^{t,z}) ds \right]$$

So we can take the limit in (4.48) and obtain that

$$\mathbb{E}\left[v^{(k)}(\tau \wedge \tau_{\mathcal{O}}, Z^{t,z}_{\tau\wedge\tau_{\mathcal{O}}}) \right] = v^{(k)}(t,z) + \mathbb{E}\left[\int_t^{\tau\wedge\tau_{\mathcal{O}}} \left(\frac{\partial v^{(k)}}{\partial s} + \mathcal{L} v^{(k)} \right)(s, Z_s^{t,z}) ds \right]. \tag{4.49}$$

Given that the diffusion matrix $g(z)$ is uniformly elliptic (4.30) we have that the law of $Z^{t,z}$ is absolutely continuous with respect to the Lebesgue measure on $(0,T) \times \mathcal{O}$. Since from (4.39) we have

$$\left(\frac{\partial v^{(k)}}{\partial s} + \mathcal{L} v^{(k)} \right)(t,z) \leq 0, \qquad \text{a.e on } (0,T) \times \mathcal{O},$$

and

$$v^{(k)}(t,z) \geq \hat{\Psi}_k(t,z), \qquad \text{a.e on } (0,T) \times \mathcal{O},$$

then

$$v^{(k)}(t,z) \geq \mathbb{E}\left[v^{(k)}(\tau \wedge \tau_{\mathcal{O}}, Z^{t,z}_{\tau \wedge \tau_{\mathcal{O}}}) \right] \geq \mathbb{E}\left[\hat{\Psi}_k(\tau \wedge \tau_{\mathcal{O}}, Z^{t,z}_{\tau \wedge \tau_{\mathcal{O}}}) \right].$$

We can now prove the optimality of the stopping time τ_k^*. Indeed choosing $\tau = \tau_k^*$ in expression (4.49) and taking into account the fact that

$$\max \left\{ \frac{\partial v^{(k)}}{\partial t} + \mathcal{L} v^{(k)} \, , \, \hat{\Psi}_k - v^{(k)} \right\} = 0,$$

holds almost everywhere, we conclude that the integral term in (4.49) is equal to zero for $s \leq \tau_k^*$. Hence

$$v^{(k)}(t,z) = \mathbb{E}\left[v^{(k)}(\tau_k^* \wedge \tau_{\mathcal{O}}, Z^{t,z}_{\tau_k^* \wedge \tau_{\mathcal{O}}}) \right] = \mathbb{E}\left[\hat{\Psi}_k(\tau_k^* \wedge \tau_{\mathcal{O}}, Z^{t,z}_{\tau_k^* \wedge \tau_{\mathcal{O}}}) \right]. \tag{4.50}$$

The last term derives from the definition of τ_k^* if $\tau_k^* \leq \tau_{\mathcal{O}}$ and from the boundary condition in the variational inequality if $\tau_k^* > \tau_{\mathcal{O}}$. It then concludes the proof since the maximum is attainted at τ_k^* and then this stopping time is the optimal stopping time in the problem (4.47). ■

As a straightforward consequence of setting the variational problem in the weak sense we obtain the following Corollary.

Corollary 4.7.1 *Let $v^{(k)}(t,z)$ denote the unique weak solution of (4.39), interpreted in terms of (4.44). Then*

$$v^{(k)}(t,z) = \sup_{t \leq \tau \leq T} \mathbb{E}\left[\hat{\Psi}_k(\tau \wedge \tau_{\mathcal{O}}, Z^{t,z}_{\tau \wedge \tau_{\mathcal{O}}}) \right]. \tag{4.51}$$

Moreover, the following regularity holds

$$v^{(k)} \in L^2(0,T; H^1(\mathcal{O})) \cap C((0,T) \times \mathcal{O}), \tag{4.52}$$

and

$$v^{(k)}(T) = \hat{\Psi}_k(T), \text{ on } \mathcal{O}$$

$$v^{(k)} = \hat{\Psi}_k \text{ on } (0,T) \times \partial\mathcal{O}.$$

The optimal stopping time is

$$\tau_k^* := \inf\{s \geq t \, : \, v^{(k)}(s, Z_s^{t,z}) = \hat{\Psi}_k(s, Z_s^{t,z})\} \wedge T. \tag{4.53}$$

Our next step will be to extend the variatonal inequality to the whole space $\mathbb{R}^n$. In what follows the order of regularization of the gain function is fixed. Hence since $k \in \mathbb{N}$ is always the same we can simplify the notation and replace $\hat{\Psi}_k = \hat{\Theta}$ and $v^{(k)} = v$. For the closed convex set we adopt the same simplification and hence $\mathcal{K}_\Psi^{(k)} = \mathcal{K}_\Theta$. Moreover we also denote $\mathcal{T}_{\psi,k} = \mathcal{T}_\theta$. All the indexes will be recovered in the final steps of our infinite dimensional analysis.

Chapter 5

Variational inequality on an unbounded domain

In this Chapter we extend the previous results to the finite dimensional unbounded domain $\mathbb{R}^n$. In order to do so we introduce a suitable Gaussian measure that turns out to be a good one when we consider the problem at its original infinite dimensional level. We provide a number of estimates about the solution of the variational inequality on bounded domain. The main issue throughout this chapter is to prove a *universal* estimate for the bilinear form appearing in the variational inequality on unbounded domain. This estimate does not depend on the order of the Yosida approximation nor on the finite dimensional approximation. Thanks to this estimate and to the ones about the value function we can pass to the limit and prove existence (and uniqueness) of a weak solution of the variational problem on unbounded domain. We also show the connection with the optimal stopping problem and the characterization of the optimal stopping time.

5.1 Finite dimensional unbounded domain

In order to extend the EVI we have solved in the previous chapter we are supposed to set a measure $\mu_n(dz)$ on $\mathbb{R}^n$. This is quite a delicate issue, because our aim is to pass to infinite dimensions in the last step. Looking forward to that, the first problem that we have to tackle is the lack of any analogue of the Lebesgue measure on $\mathcal{H}$. To our knowledge the natural choice to extend the concept of weak derivative to the infinite dimensional framework is the one of Gaussian measures, cf. [11, 18, 22].

We start defining a gaussian weight $\mu_n(dz)$ which will play a fundamental role in setting a proper Gauss measure on $\mathcal{H}$. We denote by λ_i, $i = 1, 2, \ldots, n$ a sequence of positive real numbers. They will represent the first n eigenvalues of the operator Q of Definition 3.2.1. We define the Gaussian weight as

$$\mu_n(dz) := \frac{1}{\sqrt{(2\pi)^n \lambda_1 \lambda_2 \cdots \lambda_n}} \exp\left(-\sum_{i=1}^{n} \frac{z_i^2}{\lambda_i}\right) dz. \tag{5.1}$$

As usual we introduce the weighted L^2-norm

$$\|u\|_{L^2_\mu(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |u(z)|^2 \mu_n(dx),$$

and the Hilbert space $L^2_\mu(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R} \,|\, \|u\|_{L^2_\mu(\mathbb{R}^n)} < \infty\}$. We can now introduce a weighted Sobolev space which has an infinite dimensional analogue under suitable conditions for the sequence of parameters λ_i. Some authors refer to this space as to Gauss-Sobolev space [11, 16] and we denote it by $W^{1,2}(\mathbb{R}^n, \mu_n)$. In particular

$$W^{1,2}(\mathbb{R}^n, \mu_n) := \{u \in L^2_\mu(\mathbb{R}^n) \,:\, \|\nabla u\|_{L^2_\mu(\mathbb{R}^n)} < \infty\}, \tag{5.2}$$

where

$$\|\nabla u\|^2_{L^2_\mu(\mathbb{R}^n)} := \sum_{i=1}^{n} \int_{\mathbb{R}^n} \left|\frac{\partial u(z)}{\partial z_i}\right|^2 \mu_n(dz).$$

It is now clear that the sets inclusion $H^1(\mathbb{R}^n) \subset W^{1,2}(\mathbb{R}^n, \mu_n)$ holds. We can consider the zero extension outside $\mathcal{O}$ of our solution $u(t)$ to equation (4.40), for all $t \in [0, T]$. We denote the zero extension again by $u(t)$, and then it is an element of $W^{1,2}(\mathbb{R}^n, \mu_n)$. We would like to set the problem in Theorem 4.6.1 into the framework of this weighted Sobolev space. We will partially rely on the approach of [4], Chapter 3, Section 1.11. In order to do so we simply redefine the test function $w \in \mathcal{K}$. Let us denote by $\tilde{w}$ the original test function of our EVI and let us define a new class of functions through

$$\tilde{w}(z) - u(z) := \frac{1}{\sqrt{(2\pi)^n \lambda_1 \lambda_2 \cdots \lambda_n}} \exp\left(-\sum_{i=1}^{n} \frac{z_i^2}{\lambda_i}\right) (w - u)(z).$$

It is easy to check that if $\tilde{w} \in \mathcal{K}$ then also $w \in \mathcal{K}$.

We have now to adapt all the terms of our EVI to this new setting. If we simply substitute $\tilde{w}$ into the definition of weak solution as given in Section 4.4.2, equation (4.28), we see that we can replace

$$\int_0^T \left[-(\frac{\partial \tilde{w}}{\partial t}, \tilde{w} - u) \right] dt + \frac{1}{2}|\tilde{w}(T)|^2_{L^2} = \int_0^T \left[-(\frac{\partial w}{\partial t}, w - u)_{L^2_\mu(\mathcal{O})} \right] dt + \frac{1}{2}|w(T)|^2_{L^2_\mu(\mathcal{O})}.$$

Notice that although u and w are now extended to $\mathbb{R}^n$ we still keep memory of the fact that the scalar product is over $\mathcal{O}$. We will get rid of it shortly. Pluging w into the bilinear form $a(t; u, \tilde{w} - u)$ and performing all the derivatives we obtain a new bilinear form,

$$\begin{aligned} a_\mu(t; u, w-u)|_{\mathcal{O}} \;:=\; & \frac{1}{2} \sum_{i,j=1}^n \int_{\mathcal{O}} (g\, g^*)_{i,j}(z) \frac{\partial u}{\partial z_i} \frac{\partial}{\partial z_j}(w-u)(t,z)\mu_n(dz) \\ & + \sum_{i=1}^n \int_{\mathcal{O}} \left(\frac{1}{2} \sum_{j=1}^n \frac{\partial (g\, g^*)_{i,j}}{\partial z_j}(z) - b_i(z) \right) \frac{\partial u}{\partial z_i}(w-u)(t,z)\mu_n(dz) \\ & - \sum_{i,j=1}^n \frac{1}{2} \int_{\mathcal{O}} (g\, g^*)_{i,j}(z) \frac{z_j}{\lambda_j} \frac{\partial u}{\partial z_i}(t,z)(w-u)(t,z)\mu_n(dz) \end{aligned}$$

By analogy we also have a new dual pairing

$$\langle \mathcal{T}_\theta(t), w-u \rangle^{\mathcal{O}}_{W^{1,2}_\mu, W^{1,2\,*}_\mu} := (\frac{\partial \hat{\Theta}}{\partial t}(t), w-u)_{L^2_\mu(\mathcal{O})} - a_\mu(t; \hat{\Theta}(t), w-u)|_{\mathcal{O}}, \tag{5.3}$$

and we denote it for simplicity as $\langle \mathcal{T}_\theta(t), w-u \rangle_\mu|_{\mathcal{O}}$. We also consider a new convex set relative to the weighted space,

$$\mathcal{K}_\mu := \{ w : \; w \in L^2(0,T; W^{1,2}(\mathbb{R}^n, \mu_n)), \frac{\partial w}{\partial t} \in L^2(0,T; L^2_\mu(\mathbb{R}^n)), w(t,z) \geq 0 \text{ a.e. in } (0,T)\times\mathbb{R}^n \}. \tag{5.4}$$

Now we can restate our variational inequality saying that $u \in L^2(0,T; W^{1,2}(\mathbb{R}^n, \mu_n))$ is the unique solution of

$$\begin{aligned} \int_0^T \left[-(\frac{\partial w}{\partial t}, w-u)_{L^2_\mu(\mathcal{O})} + a_\mu(t; u, w-u)|_{\mathcal{O}} - \langle \mathcal{T}_\theta(t), w-u \rangle_\mu|_{\mathcal{O}} \right] dt & \\ + \frac{1}{2}|w(T)|^2_{L^2_\mu(\mathcal{O})} \geq 0, & \end{aligned}$$

for all $w \in \mathcal{K}_\mu$.

It is worth noticing that the restriction to $\mathcal{O}$ in some sense keeps the memory of the zero boundary condition. If we now define $\mathcal{O}_l = \{z \in \mathbb{R}^n : |z| < l\}$ we can denote as $u^{(l)}$ the zero extension over $\mathbb{R}^n$ of the unique weak solution of the EVI on $\mathcal{O}_l$. If moreover we set $w^{(l)} \in \mathcal{K}_\mu \cap \{w : w|_{\mathcal{O}_l} = 0\}$ then we have

$$\int_0^T \left[-(\frac{\partial w^{(l)}}{\partial t}, w^{(l)} - u^{(l)})_{L^2_\mu(\mathbb{R}^n)} + a_\mu(t; u^{(l)}, w^{(l)} - u^{(l)}) - \langle \mathcal{T}_\theta(t), w^{(l)} - u^{(l)} \rangle_\mu \right] dt$$
$$+ \frac{1}{2} |w^{(l)}(T)|^2_{L^2_\mu(\mathbb{R}^n)} \geq 0.$$

Here we have extended all the integrals over $\mathbb{R}^n$. Our aim is now to provide some a priori estimates on $u^{(l)}$ and on each term of the EVI in order to consider the limit as $l \to \infty$.

5.2 Uniform estimates for the solution of the variational inequality

We derive the estimates on $\|u^{(l)}\|_{L^2(0,T;W^{1,2}(\mathbb{R}^n,\mu_n))}$ from its probabilistic representation. This is mainly due to the fact that our main aim is to take the limit to infinite dimensions. For such a reason we cannot rely on the coerciveness of the bilinear form. It prevents us from providing standard estimates on the gradient, uniformly with respect to the number of dimensions. Moreover the probabilistic representation gives some insights that will enable us to set the EVI in a suitable smaller closed convex set.

We recall Proposition 4.1.1 and refine a bit the results therein. It is clear that all the properties which hold for Ψ, hold for $\hat{\Psi}_k = \hat{\Theta}$ as well. Our first observation is that since $\mu(\mathbb{R}^n) = 1$, $n \in \mathbb{N}$, the Lipschitz constant L_1 provides a uniform bound on $\nabla \hat{\Theta}$ in $L^2_\mu(\mathbb{R}^n)$. Even though the result that we are presenting holds in wider generality, we specialize our study to the case in which the following holds

Assumption 5.2.1 *For* $0 \leq s \leq t \leq S$ *we have*

$$|\Psi(t,x) - \Psi(s,x)| \leq L_2 \|x\|_{\mathcal{H}} \, |t-s|, \qquad \forall x \in \mathcal{H},$$

or analogously

$$|\hat{\Psi}(t,z) - \hat{\Psi}(s,z)| \leq L_2 |z|_{\mathbb{R}^n} \, |t-s|, \qquad \forall z \in \mathbb{R}^n.$$

The same holds for $\hat{\Theta}$, as well. Now from analogous rationale as before and from Proposition 4.1.1 we also obtain that

$$\int_0^T \| \frac{\partial \hat{\Theta}}{\partial t}(t) \|^2_{L^2_\mu(\mathbb{R}^n)} dt \leq L_2^2 \int_{\mathbb{R}^n} |z|^2_{\mathbb{R}^n} \mu_n(dz) \cdot T.$$

We now recognize the necessity of the hypothesis about Q. Indeed

$$\int_{\mathbb{R}^n} |z|^2_{\mathbb{R}^n} \mu_n(dz) = \sum_{i=1}^n \lambda_i \leq \sum_{i=1}^\infty \lambda_i = Tr(Q) < \infty.$$

We recall the estimate (4.8) and set $\hat{C}_\Theta = C_\Psi + L_2^2 Tr(Q)$. Hence we obtain

$$\int_0^T \left[\| \frac{\partial \hat{\Theta}}{\partial t}(t) \|^2_{L^2_\mu(\mathbb{R}^n)} + \| \hat{\Theta}(t) \|^2_{W^{1,2}(\mathbb{R}^n, \mu_n)} \right] dt \leq \hat{C}_\Theta \, T. \tag{5.5}$$

We can now exploit these regularity to infer about the regularity of our solution $u^{(l)}$. We know in fact that $u^{(l)}$ is the zero extension outside $\mathcal{O}_l$ of $v^{(l)} - \hat{\Theta}$, where now

$$v^{(l)}(t,z) = U_\Theta^{\mathcal{O}_l}(t,z) = \sup_{t \leq \tau \leq T} \mathbb{E}\left[\hat{\Theta}(\tau \wedge \tau_{\mathcal{O}_l}, Z^{t,z}_{\tau \wedge \tau_{\mathcal{O}_l}}) \right]. \tag{5.6}$$

We simplify the notation once more and set $U_\Theta^{\mathcal{O}_l} = U^{\mathcal{O}_l}$. We can discuss the regularity of $U^{\mathcal{O}_l}$ and then deduce the regularity of $u^{(l)}$ from it. We summarize the results in the following proposition.

Proposition 5.2.1 *For $U^{\mathcal{O}_l}(t,z)$ there exists a uniform bound $C_U > 0$ such that:*

$$\int_0^T \| U^{\mathcal{O}_l}(t) \|^2_{W^{1,2}(\mathbb{R}^n, \mu_n)} dt < C_U \, T. \tag{5.7}$$

The bound does not depend on the size of the domain neither on the dimensions of the space or on the order of the Yosida approximation.

PROOF: We know from Proposition 2.2.2 that the value function of the original problem is uniformly bounded by $\overline{\Psi}$ and Lipschitz with respect to the space variable with Lipschitz constant L_1^V. From Remark 3.3.1 we know that the same properties hold for the approximating finite dimensional value function. Then they also hold for $U^{\mathcal{O}_l}(t,z)$ when $(t,z) \in [0,T] \times \mathcal{O}_l$. Outside $\mathcal{O}_l$ we have $U^{\mathcal{O}_l} = \hat{\Theta}$ and hence uniform boundedness and Lipschitz property hold. From the same arguments as in Proposition 4.1.1, since $U^{\mathcal{O}_l}(t) \in W^{1,2}(\mathbb{R}^n, \mu_n)$ for all $t \in [0,T]$, then we can approximate it with a sequence of smooth functions $\{U_j^{\mathcal{O}_l}(t)\}_{j=1}^{\infty}$. Clearly we obtain $\|\nabla U_j^{\mathcal{O}_l}(t)\|_{L^2_\mu(\mathbb{R}^n)} \leq L_1 \vee L_1^V$ and hence passing to the limit $\|\nabla U^{\mathcal{O}_l}(t)\|_{L^2_\mu(\mathbb{R}^n)} \leq L_1 \vee L_1^V$. Setting $C_U = \overline{\Psi}^2 + (L_1 \vee L_1^V)^2$ we easily obtain the thesis. ∎

These results give us a uniform bound on $u^{(l)}$, indeed we have

$$\int_0^T \|u^{(l)}(t)\|^2_{W^{1,2}(\mathbb{R}^n,\mu_n)} dt \leq (C_\Psi + C_U)\, T. \tag{5.8}$$

Since $L^2(0,T;W^{1,2}(\mathbb{R}^n,\mu_n))$ is a Hilbert space, there exists a subsequence which we will denote again by $u^{(l)}$ such that

$$u^{(l)} \rightharpoonup \bar{u}, \quad \textit{in } L^(0,T;W^{1,2}(\mathbb{R}^n,\mu_n)). \tag{5.9}$$

We notice that since τ is a stopping time bounded by T, then $\tau \wedge \tau_{\mathcal{O}_l} \to \tau$, $\mathbb{P}$-a.s. as $l \to \infty$. Moreover from continuity of the process and of the gain function we have

$$\hat{\Theta}(\tau \wedge \tau_{\mathcal{O}_l}, Z^{t,z}_{\tau\wedge\tau_{\mathcal{O}_l}}) \to \hat{\Theta}(\tau, Z^{t,z}_\tau), \quad \text{as } l \to \infty, \mathbb{P}\text{-a.s.}$$

We prove now a technical lemma about the solution of the SDE (4.1) which will be useful in what follows.

Lemma 5.2.1 *Let $Z^{t,z}$ be the unique strong solution for equation (4.1). Then for any stopping time τ such that $\tau \in [t,T]$, it holds*

$$\mathbb{E}\left[\sup_{t\leq s\leq T} \|Z^{t,z}_s - Z^{t,z}_\tau\|^2 I(s>\tau)\right] \leq C_T(1+\|z\|^2)\mathbb{E}\left[(T-\tau)^+\right]^{\frac{1}{2}}.$$

PROOF: On the set $\{s > \tau\}$ we have

$$\begin{aligned} \|Z^{t,z}_s - Z^{t,z}_\tau\|^2 &= \left\| \int_\tau^s b(Z^{t,z}_u)du + \int_\tau^s g(Z^{t,z}_u)dW_u \right\|^2 \\ &\leq 2\left\|\int_\tau^s b(Z^{t,z}_u)du\right\|^2 + 2\left\|\int_\tau^s g(Z^{t,z}_u)dW_u\right\|^2. \end{aligned}$$

Since τ is a stopping time we can pass to the indicator variables and use Hölder inequality in the first term. Then

$$\|Z_s^{t,z} - Z_\tau^{t,z}\|^2 \leq 2(T-t)\int_t^s I(\tau \leq u)\left\|b(Z_u^{t,z})\right\|^2 du + 2\left\|\int_t^s I(\tau \leq u) g(Z_u^{t,z}) dW_u\right\|^2.$$

Hence we have

$$\begin{aligned}
&\|Z_s^{t,z} - Z_\tau^{t,z}\|^2 I(s>\tau) \\
&\leq 2\left(T\int_t^s I(\tau \leq u)\left\|b(Z_u^{t,z})\right\|^2 du + 2\left\|\int_t^s I(\tau \leq u) g(Z_u^{t,z}) dW_u\right\|^2\right) I(s>\tau) \\
&\leq 2T\int_t^s I(\tau \leq u)\left\|b(Z_u^{t,z})\right\|^2 du + 2\left\|\int_t^s I(\tau \leq u) g(Z_u^{t,z}) dW_u\right\|^2.
\end{aligned}$$

We can take the supremum over all $s \in [t,T]$ and take into account that the integrand in the stochastic integral is bounded. So for a suitable constant $\gamma_T > 0$ we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\leq s\leq T}\|Z_s^{t,z} - Z_\tau^{t,z}\|^2 I(s>\tau)\right] \\
&\leq 2T\mathbb{E}\left[\int_t^T I(\tau \leq u)\left\|b(Z_u^{t,z})\right\|^2 du\right] + 8\mathbb{E}\left[\int_t^T I(\tau \leq u)\left\|g(Z_u^{t,z})\right\|^2 du\right] \\
&\leq \gamma_T\mathbb{E}\left[\int_t^T I(\tau \leq u)(1+\|Z_u^{t,z}\|^2) du\right] \\
&\leq \sqrt{2}\gamma_T\mathbb{E}\left[\left(\int_t^T I(\tau \leq u) du\right)^{\frac{1}{2}}\left(\int_t^T (1+\|Z_u^{t,z}\|^4) du\right)^{\frac{1}{2}}\right] \\
&\leq \gamma_T\sqrt{2T}\,\mathbb{E}\left[\sqrt{(T-\tau)^+}\left(1+\sup_{t\leq u\leq T}\|Z_u^{t,z}\|^4\right)^{\frac{1}{2}}\right] \\
&\leq \gamma_T\sqrt{2T}\,\mathbb{E}\left[(T-\tau)^+\right]^{\frac{1}{2}}\mathbb{E}\left[1+\sup_{t\leq u\leq T}\|Z_u^{t,z}\|^4\right]^{\frac{1}{2}}.
\end{aligned}$$

Once again, choosing a suitable constant $C_T > 0$ we get

$$\mathbb{E}\left[\sup_{t\leq s\leq T}\|Z_s^{t,z} - Z_\tau^{t,z}\|^2 I(s>\tau)\right] \leq C_T(1+\|z\|^2)\mathbb{E}\left[(T-\tau)^+\right]^{\frac{1}{2}}.$$

■

Notice that this lemma is a bit different from the usual a priori estimates for solutions of SDEs because it is adapted for stopping times.

We can now state the convergence of the sequence of value functions on bounded domains.

Proposition 5.2.2 *Let*

$$U_\Theta(t,z) = \sup_{t\le\tau\le T} \mathbb{E}\left[\hat{\Theta}(\tau, Z_\tau^{t,z})\right].$$

The pointwise convergence $U^{\mathcal{O}_l}(t,z) \to U_\Theta(t,z)$ *holds for* $(t,z) \in [0,T]\times\mathbb{R}^n$. *Moreover the convergence is uniform on every compact subset* $[0,T]\times\mathcal{K} \subset [0,T]\times\mathbb{R}^n$ *and* $U_\Theta \in C([0,T]\times\mathbb{R}^n)$.

PROOF: Notice that for any $(t,z) \in [0,T]\times\mathbb{R}^n$ there exists $L > 0$ such that $(t,z) \in [0,T]\times\mathcal{O}_l$ for all $l \ge L$. Then it makes sense to compute $U^{\mathcal{O}_l}(t,z) - U_\Theta(t,z)$ for $l \ge L$. We obtain

$$\sup_{t\le\tau\le T} \mathbb{E}\left[\hat{\Theta}(\tau\wedge\tau_{\mathcal{O}_l}, Z_{\tau\wedge\tau_{\mathcal{O}_l}}^{t,z})\right] - \sup_{t\le\sigma\le T} \mathbb{E}\left[\hat{\Theta}(\sigma, Z_\sigma^{t,z})\right] \le 0,$$

because stopping at $\tau\wedge\tau_{\mathcal{O}_l}$ is sub-optimal in the second term. The reverse estimate produces

$$\begin{aligned}
&\sup_{t\le\sigma\le T} \mathbb{E}\left[\hat{\Theta}(\sigma, Z_\sigma^{t,z})\right] - \sup_{t\le\tau\le T}\left[\hat{\Theta}(\tau\wedge\tau_{\mathcal{O}_l}, Z_{\tau\wedge\tau_{\mathcal{O}_l}}^{t,z})\right]\\
&= \sup_{t\le\sigma\le T}\inf_{t\le\tau\le T} \mathbb{E}\left[\hat{\Theta}(\sigma, Z_\sigma^{t,z}) - \hat{\Theta}(\tau\wedge\tau_{\mathcal{O}_l}, Z_{\tau\wedge\tau_{\mathcal{O}_l}}^{t,z})\right]\\
&\le \sup_{t\le\sigma\le T} \mathbb{E}\left[\hat{\Theta}(\sigma, Z_\sigma^{t,z}) - \hat{\Theta}(\sigma\wedge\tau_{\mathcal{O}_l}, Z_{\sigma\wedge\tau_{\mathcal{O}_l}}^{t,z})\right]\\
&= \sup_{t\le\sigma\le T} \mathbb{E}\left[\left(\hat{\Theta}(\sigma, Z_\sigma^{t,z}) - \hat{\Theta}(\sigma\wedge\tau_{\mathcal{O}_l}, Z_{\sigma\wedge\tau_{\mathcal{O}_l}}^{t,z})\right) I(\sigma > \tau_{\mathcal{O}_l})\right]\\
&= \sup_{t\le\sigma\le T} \mathbb{E}\left[\left(\hat{\Theta}(\sigma, Z_\sigma^{t,z}) - \hat{\Theta}(\tau_{\mathcal{O}_l}, Z_{\tau_{\mathcal{O}_l}}^{t,z})\right) I(\sigma > \tau_{\mathcal{O}_l})\right]\\
&\le \sup_{t\le\sigma\le T} \mathbb{E}\left[\left|\hat{\Theta}(\sigma, Z_\sigma^{t,z}) - \hat{\Theta}(\tau_{\mathcal{O}_l}, Z_\sigma^{t,z})\right| I(\sigma > \tau_{\mathcal{O}_l})\right]\\
&\quad + \sup_{t\le\sigma\le T} \mathbb{E}\left[\left|\hat{\Theta}(\tau_{\mathcal{O}_l}, Z_\sigma^{t,z}) - \hat{\Theta}(\tau_{\mathcal{O}_l}, Z_{\tau_{\mathcal{O}_l}}^{t,z})\right| I(\sigma > \tau_{\mathcal{O}_l})\right]\\
&\le L_2 \sup_{t\le\sigma\le T} \mathbb{E}\left[\|Z_\sigma^{t,z}\|\,|\sigma - \tau_{\mathcal{O}_l}| I(\sigma > \tau_{\mathcal{O}_l})\right]\\
&\quad + L_1 \sup_{t\le\sigma\le T} \mathbb{E}\left[\|Z_\sigma^{t,z} - Z_{\tau_{\mathcal{O}_l}}^{t,z}\| I(\sigma > \tau_{\mathcal{O}_l})\right]\\
&\le L_2 \mathbb{E}\left[\sup_{t\le s\le T} \|Z_s^{t,z}\|\,(T - \tau_{\mathcal{O}_l})^+\right] + L_1 \mathbb{E}\left[\sup_{t\le s\le T} \|Z_s^{t,z} - Z_{\tau_{\mathcal{O}_l}}^{t,z}\| I(s > \tau_{\mathcal{O}_l})\right].
\end{aligned}$$

Now from Lemma 5.2.1 and from Hölder inequality we obtain

$$
\begin{aligned}
&\sup_{t\le\sigma\le T}\mathbb{E}\left[\hat{\Theta}(\sigma,Z^{t,z}_{\sigma})\right]-\sup_{t\le\tau\le T}\left[\hat{\Theta}(\tau\wedge\tau_{\mathcal{O}_l},Z^{t,z}_{\tau\wedge\tau_{\mathcal{O}_l}})\right]\\
&\le L_2\mathbb{E}\left[\sup_{t\le s\le T}\|Z^{t,z}_s\|^2\right]^{\frac{1}{2}}\mathbb{E}\left[(T-\tau_{\mathcal{O}_l})^{+\;2}\right]^{\frac{1}{2}}+C_T(1+\|z\|^2)\mathbb{E}\left[(T-\tau_{\mathcal{O}_l})^{+}\right]^{\frac{1}{2}}.
\end{aligned}
$$

From this and from the previous estimate we can conclude for a suitable constant $K_T>0$

$$
|U^{\mathcal{O}_l}(t,z)-U_\Theta(t,z)| \;\le\; K_T(1+\|z\|)(1+\|z\|^2)\left(\mathbb{E}\left[(T-\tau_{\mathcal{O}_l})^{+\;2}\right]^{\frac{1}{2}}+\mathbb{E}\left[(T-\tau_{\mathcal{O}_l})^{+}\right]^{\frac{1}{2}}\right).
$$

From dominated convergence and from the fact that $(T-\tau_{\mathcal{O}_l})^+\to 0$ as $l\to\infty$ we obtain pointwise convergence. We can prove uniform convergence for example by similar equicontinuity arguments as in Theorem 3.1.2. Yet, notice that from the time-homogeneity of the process we have $\tau^{t,x}_{\mathcal{O}_l}=t+\tau^{0,x}_{\mathcal{O}_l}$. Moreover, since Z^z is a non degenerate diffusion in $\mathbb{R}^n$ and $\{\mathcal{O}_l\}_{l\ge1}$ is a sequence of spherical domains, the first exit time from $\mathcal{O}_l$ is a continuous function of the initial data, i.e. $z\mapsto\tau^{0,z}_{\mathcal{O}_l}$ is $\mathbb{P}$-a.s. continuous (cf. [4], Lemma 3.2, p.332). Hence if we define

$$
M_l(t,z):=K_T(1+\|z\|)(1+\|z\|^2)\left(\mathbb{E}\left[(T-\tau_{\mathcal{O}_l})^{+2}\right]^{\frac{1}{2}}+\mathbb{E}\left[(T-\tau_{\mathcal{O}_l})^{+}\right]^{\frac{1}{2}}\right),
$$

the sequence $\{M_l\}_{l\ge1}$ is a decreasing sequence of continuous functions converging to zero. From Theorem C.0.3 we get uniform convergence on compact sets and in particular, for any $\mathcal{K}\subset\mathbb{R}^n$ we have

$$
\lim_{l\to\infty}\sup_{(t,z)\in[0,T]\times\mathcal{K}}|U^{\mathcal{O}_l}(t,z)-U_\Theta(t,z)|=0.
$$

Since all the $U^{\mathcal{O}_l}$ are continuous then U_Θ has to be continuous on every compact subset $[0,T]\times\mathcal{K}$. Given that we are now in finite dimensional space this is enough for global continuity. ∎

Remark 5.2.1 *It is worth noticing that this result holds at any step of the finite dimensional approximating optimal stopping problem, i.e. $U^{(n)}_\Theta\in C([0,T]\times\mathbb{R}^n)$ for all $n\ge1$. This fact together with the uniform convergence of Assumption 4.6.1 and other considerations allowed us to prove the continuity also at the infinite dimensional level, cf. Theorem 3.2.2.*

From dominated convergence we have for $1 \le p < \infty$

$$U^{\mathcal{O}_l} \to U_\Theta \quad \text{in } L^2(0,T;L^p_\mu(\mathbb{R}^n)) \text{ as } l \to \infty. \tag{5.10}$$

Since the limit has to be unique we have from (5.9) that $\bar{u} = U_\Theta - \hat{\Theta}$. Given that the sequence $u^{(l)}$ is uniformly bounded then clearly $u^{(l)} \in L^2(0,T;L^p_\mu(\mathbb{R}^n))$ for all $l \ge 1$. We can introduce a subset $\hat{\mathcal{K}}^p_\mu \subset \mathcal{K}_\mu$ of our convex set, namely for $2 < p < \infty$

$$\hat{\mathcal{K}}^p_\mu := \{w : \; w \in \mathcal{K}_\mu \text{ and } w \in L^2(0,T;L^p_\mu(\mathbb{R}^n))\}. \tag{5.11}$$

This is a closex convex set and $\hat{\mathcal{K}}^p_\mu \neq \varnothing$. Our EVI holds in the form:
$u^{(l)} \in L^2(0,T;W^{1,2}(\mathbb{R}^n,\mu_n) \cap L^p_\mu(\mathbb{R}^n)) \cap C([0,T]\times\mathbb{R}^n)$ is the unique solution to

$$\int_0^T \left[-(\frac{\partial w^{(l)}}{\partial t}, w^{(l)} - u^{(l)})_{L^2_\mu(\mathbb{R}^n)} + a_\mu(t;u^{(l)},w^{(l)}-u^{(l)}) - \langle \mathcal{T}_\theta(t), w^{(l)}-u^{(l)}\rangle_\mu \right] dt \\ +\frac{1}{2}|w^{(l)}(T)|^2_{L^2_\mu(\mathbb{R}^n)} \ge 0,$$

for all $w^{(l)} \in \hat{\mathcal{K}}^p_\mu \cap \{w \,:\, w|_{\partial\mathcal{O}_l} = 0\}$. We can now pass to the estimates on the bilinear form and it will be clear why the setting in the L^p-space is the natural one for our problem.

5.3 Estimates for the bilinear form

Let $2 < p < \infty$ and let us denote $\mathcal{V}^p_n := W^{1,2}(\mathbb{R}^n,\mu_n) \cap L^p_\mu(\mathbb{R}^n)$. Consider $u(t), w(t) \in \mathcal{V}^p_n$, a.e. $t \in [0,T]$. We study in some detail the expression

$$\begin{aligned} a_\mu(t;u,w) &= \underbrace{\frac{1}{2}\sum_{i,j=1}^n \int_{\mathbb{R}^n} (g\,g^*)_{i,j}(z)\frac{\partial u}{\partial z_i}\frac{\partial w}{\partial z_j}(t,z)\mu_n(dz)}_{I} \\ &+ \underbrace{\sum_{i=1}^n \int_{\mathbb{R}^n} \left(\frac{1}{2}\sum_{j=1}^n \frac{\partial (g\,g^*)_{i,j}}{\partial z_j}(z) - b_i(z) \right) \frac{\partial u}{\partial z_i} w(t,z)\mu_n(dz)}_{II} \\ &- \underbrace{\sum_{i,j=1}^n \frac{1}{2}\int_{\mathbb{R}^n} (g\,g^*)_{i,j}(z)\frac{z_j}{\lambda_j}\frac{\partial u}{\partial z_i}(t,z)w(t,z)\mu_n(dz)}_{III}. \end{aligned}$$

It is usefull to rewrite all the terms in a more compact form, recalling the notation of the infinite dimensional setting. We recall from Section 4.2 that the first term (I) can be written as

$$\begin{aligned}(I) &= \frac{1}{2}\int_{\mathbb{R}^n}\sum_{i,j=1}^{n}\left(\langle\sigma^{(n)}(z),\varphi_i\rangle_{\mathcal{H}}\langle\sigma^{(n)}(z),\varphi_j\rangle_{\mathcal{H}}+\delta_{i,j}\epsilon_n^2\right)\frac{\partial u}{\partial z_i}\frac{\partial w}{\partial z_j}(t,z)\mu_n(dz)\\ &= \frac{1}{2}\int_{\mathbb{R}^n}\langle\sigma^{(n)}(z),D_z u(t,z)\rangle_{\mathcal{H}}\langle\sigma^{(n)}(z),D_z w(t,z)\rangle_{\mathcal{H}}\mu_n(dz)\\ &\quad+\frac{1}{2}\int_{\mathbb{R}^n}\epsilon_n^2\langle D_z u(t,z),D_z w(t,z)\rangle_{\mathcal{H}}\mu_n(dz),\end{aligned}$$

or equivalently, for more general volatilities, as

$$(I) = \frac{1}{2}\int_{\mathbb{R}^n}\langle\sigma^{(n)}\sigma^{(n)*}(z)D_z u(t,z),D_z w(t,z)\rangle_{\mathcal{H}}\mu_n(dz)+\frac{1}{2}\int_{\mathbb{R}^n}\epsilon_n^2\langle D_z u(t,z),D_z w(t,z)\rangle_{\mathcal{H}}\mu_n(dz).$$

We have set $D_z u = (\frac{\partial u}{\partial z_1},\dots,\frac{\partial u}{\partial z_n})$. For the second term a bit more care is needed, in particular we split it in two terms

$$(II) = \underbrace{\int_{\mathbb{R}^n}\sum_{i,j=1}^{n}\frac{1}{2}\frac{\partial(g\,g^*)_{i,j}}{\partial z_j}(z)\frac{\partial u}{\partial z_i}w(t,z)\mu_n(dz)}_{IIa}-\underbrace{\int_{\mathbb{R}^n}\sum_{i=1}^{n}b_i(z)\frac{\partial u}{\partial z_i}w(t,z)\mu_n(dz)}_{IIb}.$$

Then again from results in Section 4.2,

$$\begin{aligned}(IIa) &= \int_{\mathbb{R}^n}\sum_{i,j=1}^{n}\frac{1}{2}\langle P_n D\sigma^{(n)}(z)\varphi_j,\varphi_i\rangle_{\mathcal{H}}\langle\sigma^{(n)}(z),\varphi_j\rangle_{\mathcal{H}}\frac{\partial u}{\partial z_i}w(t,z)\mu_n(dz)\\ &\quad+\int_{\mathbb{R}^n}\sum_{i,j=1}^{n}\frac{1}{2}\langle P_n D\sigma^{(n)}(z)\varphi_j,\varphi_j\rangle_{\mathcal{H}}\langle\sigma^{(n)}(z),\varphi_i\rangle_{\mathcal{H}}(z)\frac{\partial u}{\partial z_i}w(t,z)\mu_n(dz)\\ &= \frac{1}{2}\int_{\mathbb{R}^n}\langle P_n D\sigma^{(n)}(z)\sum_{j=1}^{n}\langle\sigma^{(n)}(z),\varphi_j\rangle_{\mathcal{H}}\varphi_j,D_z u(t,z)\rangle_{\mathcal{H}}w(t,z)\mu_n(dz)\\ &\quad+\frac{1}{2}\int_{\mathbb{R}^n}Tr[P_n D\sigma^{(n)}(z)]_{\mathcal{H}}\langle\sigma^{(n)}(z),D_z u(t,z)\rangle_{\mathcal{H}}w(t,z)\mu_n(dz)\end{aligned}$$

We now recognize that

$$P_n D\sigma^{(n)}(z)\sum_{j=1}^{n}\langle\sigma^{(n)}(z),\varphi_j\rangle_{\mathcal{H}}\varphi_j = P_n D\sigma^{(n)}(z)\cdot\sigma^{(n)}(z),$$

denotes the action of the linear operator $P_n D\sigma^{(n)}(z) \in \mathcal{L}(\mathcal{H},\mathcal{H})$ on the element $\sigma^{(n)}(z)$ and hence we conclude

$$(IIa) = \frac{1}{2}\int_{\mathbb{R}^n} \langle P_n D\sigma^{(n)}(z)\cdot\sigma^{(n)}(z), D_z u(t,z)\rangle_{\mathcal{H}} w(t,z)\mu_n(dz)$$

$$+\frac{1}{2}\int_{\mathbb{R}^n} Tr[P_n D\sigma^{(n)}(z)]_{\mathcal{H}}\langle \sigma^{(n)}(z), D_z u(t,z)\rangle_{\mathcal{H}} w(t,z)\mu_n(dz).$$

For the next term we have to take care of the action of the unbounded operator A. Even though at this stage of our algorithm we are dealing with the Yosida approximation A_α we want to remove any dependency on α from our estimates. This will enable us, in Chapter 6, to take the limit as $\alpha \to \infty$. It is convenient to write the term in this form

$$(IIb) = \int_{\mathbb{R}^n} \sum_{i,j=1}^n z_j \langle A_\alpha \varphi_j, \varphi_i\rangle_{\mathcal{H}} \frac{\partial u}{\partial z_i} w(t,z)\mu_n(dz)$$

$$= \sum_{j=1}^n \int_{\mathbb{R}^n} \langle A_\alpha\varphi_j, D_z u(t,z)\rangle_{\mathcal{H}} z_j w(t,z)\mu_n(dz).$$

Concluding we rewrite the last term following the same rationale as in (I). In particular we notice that for $z \in \mathbb{R}^n$ we have $Q_n^{-1}z = (\frac{z_1}{\lambda_1}, \frac{z_2}{\lambda_2}, \ldots, \frac{z_n}{\lambda_n})$. So we obtain

$$(III) = \sum_{i,j=1}^n \frac{1}{2}\int_{\mathbb{R}^n} \left(\langle\sigma^{(n)}(z),\varphi_i\rangle_{\mathcal{H}}\langle\sigma^{(n)}(z),\varphi_j\rangle_{\mathcal{H}} + \delta_{i,j}\epsilon_n^2\right)\frac{z_j}{\lambda_j}\frac{\partial u}{\partial z_i}(t,z)w(t,z)\mu_n(dz)$$

$$= \frac{1}{2}\int_{\mathbb{R}^n} \langle\sigma^{(n)}(z), D_z u(t,z)\rangle_{\mathcal{H}}\langle\sigma^{(n)}(z), Q_n^{-1}z\rangle_{\mathcal{H}} w(t,z)\mu_n(dz)$$

$$+\epsilon_n^2\frac{1}{2}\int_{\mathbb{R}^n}\langle Q_n^{-1}z, D_z u(t,z)\rangle_{\mathcal{H}} w(t,z)\mu_n(dz),$$

or equivalently for more general volatility structures

$$(III) = \frac{1}{2}\int_{\mathbb{R}^n}\langle\sigma^{(n)}\sigma^{(n)*}(z)Q_n^{-1}z, D_z u(t,z)\rangle_{\mathcal{H}} w(t,z)\mu_n(dz)$$

$$+\epsilon_n^2\frac{1}{2}\int_{\mathbb{R}^n}\langle Q_n^{-1}z, D_z u(t,z)\rangle_{\mathcal{H}} w(t,z)\mu_n(dz).$$

The estimates we are going to provide now are important not only in order to extend the finite dimensional EVI to unbounded domains but they will be useful also when taking the limit as the dimensions of the space n and the Yosida parameter α go to infinity. Having this perspective in mind it turns out that the most delicate term is (IIb) because it involves the unbounded linear operator A.

We will adopt the following notation: let $f : \mathcal{H}^{(n)} \to \mathbb{R}$ be a generic function on $\mathcal{H}^{(n)} \sim \mathbb{R}^n$. We always think in terms of the isometry between the Euclidean norm and the $\mathcal{H}$-norm, i.e. we will equivalently write $\| \cdot \|_{\mathbb{R}^n}$ or $\| \cdot \|_{\mathcal{H}}$. Then

$$\begin{aligned}\|f\|_{L^p_\mu(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} |f(z)|^p \mu_n(dz)\right)^{\frac{1}{p}},\\ \|Df\|_{L^p_\mu(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} \|Df(z)\|^p_{\mathcal{H}}\, \mu_n(dz)\right)^{\frac{1}{p}}.\end{aligned}$$

5.3.1 Estimates - part 1

The necessity to control the action of the operator A on the basis functions will determine the choice of the operator Q. We start with the following estimate

$$\begin{aligned}|(IIb)| &= \left|\sum_{j=1}^n \int_{\mathbb{R}^n} \langle A_\alpha \varphi_j, D_z u(t,z)\rangle_{\mathcal{H}} z_j w(t,z) \mu_n(dz)\right|\\ &\le \sum_{j=1}^n \int_{\mathbb{R}^n} |\langle A_\alpha \varphi_j, D_z u(t,z)\rangle_{\mathcal{H}}|\, |z_j|\, |w(t,z)| \mu_n(dz)\\ &\le \sum_{j=1}^n \|A_\alpha \varphi_j\|_{\mathcal{H}} \int_{\mathbb{R}^n} \|D_z u(t,z)\|_{\mathcal{H}}\, |z_j|\, |w(t,z)| \mu_n(dz).\end{aligned}$$

From Appendix B we know that since the semigroup $\{S(t),\, t \ge 0\}$ is uniformly bounded by M, we get

$$\|A_\alpha \varphi_j\|_{\mathcal{H}} = \alpha \|R(\alpha; A) A\varphi_j\|_{\mathcal{H}} \le \alpha \|R(\alpha; A)\|_{\mathcal{L}}\, \|A\varphi_j\|_{\mathcal{H}} \le M \cdot \|A\varphi_j\|_{\mathcal{H}}.$$

Then we obtain an estimate which does not depend on the order of approximation with respect to to the Yosida parameter, i.e.

$$|(IIb)| \le M \sum_{j=1}^{n} \|A\varphi_j\|_{\mathcal{H}} \int_{\mathbb{R}^n} \|D_z u(t,z)\|_{\mathcal{H}}\, |z_j|\, |w(t,z)| \mu_n(dz).$$

We use Hölder inequality twice. In particular we consider $q, r > 1$ such that $\frac{1}{q} + \frac{1}{r} = 1$ and get

$$\begin{aligned} |(IIb)| &\le M \|D_z u(t)\|_{L^2_\mu(\mathbb{R}^n)} \|w(t)\|_{L^{2q}_\mu(\mathbb{R}^n)} \sum_{j=1}^{n} \|A\varphi_j\|_{\mathcal{H}} \left(\int_{\mathbb{R}^n} |z_j|^{2r} \mu_n(dz) \right)^{\frac{1}{2r}} \\ &= C_r\, M \|D_z u(t)\|_{L^2_\mu(\mathbb{R}^n)} \|w(t)\|_{L^{2q}_\mu(\mathbb{R}^n)} \sum_{j=1}^{n} \|A\varphi_j\|_{\mathcal{H}} \sqrt{\lambda_j}. \end{aligned}$$

Here $C_r > 0$ is the constant that we obtain calculating the $2r$-th moment of a centered gaussian distribution. Now we recall that $u, w \in \mathcal{V}^p_n$ for $2 < p < \infty$. Hence it makes sense to take $q = \frac{p}{2}$ and $r = \frac{p}{p-2}$. So relabelling $C_r =: C_p$ we obtain

$$|(IIb)| \le C_p\, M \|D_z u(t)\|_{L^2_\mu(\mathbb{R}^n)} \|w(t)\|_{L^p_\mu(\mathbb{R}^n)} \sum_{j=1}^{n} \|A\varphi_j\|_{\mathcal{H}} \sqrt{\lambda_j}.$$

The choice of the Banach space $\mathcal{V}^p_n$ is now justified. It is also clear that looking forward to the infinite dimensional limit, we choose $\{\lambda_j\}_{j=1}^{\infty}$ such that

$$\sum_{j=1}^{\infty} \|A\varphi_j\|_{\mathcal{H}} \sqrt{\lambda_j} < \infty.$$

For instance it means $\sqrt{\lambda_j} \sim 1/(j^2 \vee \|A\varphi_j\|_{\mathcal{H}})$. This constraint deserves some qualitative discussions. It is indeed not surprising at all that we explicitly obtain this rate of decrease for the eigenvalues of the Q matrix. We can compare this result with the one in [18], Chapter 10, Section 4, where the appropriate matrix Q is related to the unbounded operator through $A := -\frac{1}{2}Q^{-1}$. We see that in our case $Q^{\frac{1}{2}}(\mathcal{H}) \subset D(A)$ [1], but our bilinear form (Dirichlet Form) is quite more complicated than the one in [18]. Moreover in that book the author starts with a simple Dirichlet Form

[1] Let $h \in \mathcal{H}$ be given, $v_n := Q^{\frac{1}{2}} P_n h = \sum_{i=1}^{n} \sqrt{\lambda_i} h_i \varphi_i$. Then $v_n \to v := Q^{\frac{1}{2}} h$ as $n \to \infty$. Moreover $v_n \in D(A)$ and, for $n > m$, we get $\|Av_n - Av_m\|_{\mathcal{H}} \le \left(\sum_{i=m+1}^{n} \lambda_i \|A\varphi_i\|^2 \right)^{\frac{1}{2}} \|h\|_{\mathcal{H}}$. Hence the sequence is Cauchy and $Av_n \to f$ for some $f \in \mathcal{H}$. From closedness of A we get $f = Av = AQ^{\frac{1}{2}} h$ and hence $Q^{\frac{1}{2}}(\mathcal{H}) \subset D(A)$.

on a given Gauss-Sobolev space and then associates to it a diffusion process. It means that he has no constraints on the form of A and hence he defines it starting from the matrix Q. In the present case, things are somehow reversed. We can also discuss this result in terms of the Assumption 3.2.1 about the regularity of the volatility structure. We remark that the assumption $\sigma(x) = Q\gamma(x)$ means that $\sigma : \mathcal{H} \to Q(\mathcal{H}) \subset D(A^2)$. It is interesting to stress that since $\ker\{Q\} = 0$, then $Q(\mathcal{H})$ is dense in $\mathcal{H}$. There are in literature assumptions about the volatility structure which seems comparable with ours. In the framework of finite dimensional realizations of the forward rate curves, cf. for instance [10, 28], they often assume $\sigma(x) \in D(A^\infty)$. Nevertheless it is worth stressing that the aim of those papers is completely different from ours.

Summarizing from now on we assume

Assumption 5.3.1 *Let Q be such that*

$$\sum_{j=1}^{\infty} \|A\varphi_j\|_{\mathcal{H}} \sqrt{\lambda_j} < \infty,$$

holds.

It is worth to remark that an important consequence of this assumption is that the operator $Q^{\frac{1}{2}}$ is trace class itself.

5.3.2 Estimates - part 2

We want to estimate now the term (IIa). First of all it is worth remarking that by adopting the same rationale as in Section 4.2 we deduce that $D\sigma(x) = QD\gamma(x)$ for $x \in \mathcal{H}$. At the finite dimensional level we have $D\sigma^{(n)}(x) = P_nQP_nD\gamma^{(n)}(x)$, where clearly $\gamma^{(n)} = P_n(\gamma \circ P_n)$. In order to carry out our estimates we will make use of [22], Proposition 1.1.1, which guarantees that for $x \in \mathcal{H}$ if Q is trace class and $D\gamma(x)$ is bounded linear operator on $\mathcal{H}$, then $QD\gamma(x)$ is also a trace class operator. Moreover if we denote by $\|\cdot\|_{tc}$ the operatorial norm of trace class operators and by $\|\cdot\|_{\mathcal{L}}$ the operatorial norm on $\mathcal{L}(\mathcal{H};\mathcal{H})$ it holds

$$\|QD\gamma(x)\|_{tc} \leq \|Q\|_{tc}\|D\gamma(x)\|_{\mathcal{L}}.$$

In the general case one has $|Tr\, QD\gamma(x)| \le \|QD\gamma(x)\|_{tc}$. We also notice that since Q diagonal and positive we also have $\|Q\|_{tc} = Tr\, Q$. We have now all the tools we need to perform our estimate.

$$\begin{aligned}
|(IIa)| &\le \left| \frac{1}{2} \int_{\mathbb{R}^n} \langle P_n D\sigma^{(n)}(z) \cdot \sigma^{(n)}(z), D_z u(t,z) \rangle_{\mathcal{H}} w(t,z) \mu_n(dz) \right| \\
&\quad + \left| \frac{1}{2} \int_{\mathbb{R}^n} Tr[P_n D\sigma^{(n)}(z)]_{\mathcal{H}} \langle \sigma^{(n)}(z), D_z u(t,z) \rangle_{\mathcal{H}} w(t,z) \mu_n(dz) \right| \\
&\le \frac{1}{2} \int_{\mathbb{R}^n} |\langle P_n D\sigma^{(n)}(z) \cdot \sigma^{(n)}(z), D_z u(t,z) \rangle_{\mathcal{H}}|\, |w(t,z)| \mu_n(dz) \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^n} |Tr[P_n D\sigma^{(n)}(z)]_{\mathcal{H}}|\, |\langle \sigma^{(n)}(z), D_z u(t,z) \rangle_{\mathcal{H}}|\, |w(t,z)| \mu_n(dz) \\
&\le \frac{1}{2} \int_{\mathbb{R}^n} \|P_n D\sigma^{(n)}(z)\|_{\mathcal{L}}\, \|\sigma^{(n)}(z)\|_{\mathcal{H}}\, \|D_z u(t,z)\|_{\mathcal{H}}\, |w(t,z)| \mu_n(dz) \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^n} \|P_n D\sigma^{(n)}(z)\|_{tc}\, \|\sigma^{(n)}(z)\|_{\mathcal{H}}\, \|D_z u(t,z)\|_{\mathcal{H}}\, |w(t,z)| \mu_n(dz).
\end{aligned}$$

We now rely on our assumptions about $\sigma \in C_b^2(\mathcal{H}, \mathcal{H})$. We know indeed the following

i) $\|\sigma^{(n)}(z)\|_{\mathcal{H}} \le \sup_{x \in \mathcal{H}} \|\sigma(x)\|_{\mathcal{H}} \le b_\sigma$,

ii) $\|P_n D\sigma^{(n)}(z)\|_{\mathcal{L}} \le \sup_{x \in \mathcal{H}} \|D\sigma(x)\|_{\mathcal{L}} \le B_\sigma$,

iii) $\|P_n D\sigma^{(n)}(z)\|_{tc} \le \sup_{x \in \mathcal{H}} \|D\sigma(x)\|_{tc} \le \|Q\|_{tc} \sup_{x \in \mathcal{H}} \|D\gamma(x)\|_{\mathcal{L}} \le B_\gamma \|Q\|_{tc}$.

We then obtain

$$\begin{aligned}
|(IIa)| &\le \frac{1}{2} B_\sigma \cdot b_\sigma \|D_z u(t)\|_{L^2_\mu(\mathbb{R}^n)}\, \|w(t)\|_{L^2_\mu(\mathbb{R}^n)} \\
&\quad + \frac{1}{2} B_\gamma \cdot b_\sigma \cdot \|Q\|_{tc}\, \|D_z u(t)\|_{L^2_\mu(\mathbb{R}^n)}\, \|w(t)\|_{L^2_\mu(\mathbb{R}^n)}.
\end{aligned}$$

We can now pass to another term

5.3.3 Estimates - part 3

The estimate of the term (I) is quite simple and proceeds as follows

$$
\begin{aligned}
|(I)| \;&\leq\; \left|\frac{1}{2}\int_{\mathbb{R}^n} \langle \sigma^{(n)}\sigma^{(n)*}(z) D_z u(t,z), D_z w(t,z)\rangle_{\mathcal{H}} \mu_n(dz)\right| \\
&\quad + \left|\frac{1}{2}\int_{\mathbb{R}^n} \epsilon_n^2 \langle D_z u(t,z), D_z w(t,z)\rangle_{\mathcal{H}} \mu_n(dz)\right| \\
&\leq\; \frac{1}{2}\int_{\mathbb{R}^n} |\langle \sigma^{(n)}\sigma^{(n)*}(z) D_z u(t,z), D_z w(t,z)\rangle_{\mathcal{H}}|\, \mu_n(dz) \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n} \epsilon_n^2 |\langle D_z u(t,z), D_z w(t,z)\rangle_{\mathcal{H}}|\, \mu_n(dz) \\
&\leq\; \frac{1}{2}\int_{\mathbb{R}^n} \|\sigma^{(n)}\sigma^{(n)*}(z)\|_{\mathcal{L}}\, \|D_z u(t,z)\|_{\mathcal{H}}\, \|D_z w(t,z)\|_{\mathcal{H}}\, \mu_n(dz) \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n} \epsilon_n^2 \|D_z u(t,z)\|_{\mathcal{H}}\, \|D_z w(t,z)\|_{\mathcal{H}}\, \mu_n(dz).
\end{aligned}
$$

We now notice that[2] $\|\sigma^{(n)}\sigma^{(n)*}(z)\|_{\mathcal{L}} \leq \sup_{x\in\mathcal{H}} \|\sigma\sigma^*(x)\|_{\mathcal{L}} \leq b_\sigma^2$ and hence

$$
|(I)| \leq \frac{1}{2}\left(b_\sigma^2 + \epsilon_n^2\right)\, \|D_z u(t)\|_{L^2_\mu(\mathbb{R}^n)}\, \|D_z w(t)\|_{L^2_\mu(\mathbb{R}^n)}.
$$

The dependence on n is of course irrelevant, since $\epsilon_n \to 0$, then without losing in generality we can simply write

$$
|(I)| \leq \frac{1}{2}\left(1 + b_\sigma^2\right)\, \|D_z u(t)\|_{L^2_\mu(\mathbb{R}^n)}\, \|D_z w(t)\|_{L^2_\mu(\mathbb{R}^n)}.
$$

5.3.4 Estimates - part 4

In this last section we estimate (III) and doing so we will clarify the importance about the assumptions on the volatility structure. As an auxiliary result we will also obtain the rate of convergence

[2] For $x \in \mathcal{H}$ given and fixed, the adjoint $\sigma^*(x)$ is a bounded linear functional on $\mathcal{H}$, i.e. $\sigma^*(x) : \mathcal{H} \to \mathbb{R}$. Moreover it is easy to verify that for any $h \in \mathcal{H}$ the dual pairing satisfies $\langle \sigma^*(x), h\rangle_{\mathcal{H},\mathcal{H}^*} = \langle \sigma(x), h\rangle_{\mathcal{H}}$.

needed by $\{\epsilon_n\}_{n=1}^{\infty}$ for the whole approximating process to be well posed. We exploit the fact that $\sigma^{(n)}\sigma^{(n)*}(z)Q_n^{-1}$ is selfadjoint and we obtain

$$\begin{aligned}
|(III)| &\le \frac{1}{2}\left|\int_{\mathbb{R}^n}\langle\sigma^{(n)}\sigma^{(n)*}(z)Q_n^{-1}z, D_z u(t,z)\rangle_{\mathcal{H}} w(t,z)\mu_n(dz)\right| \\
&\quad +\epsilon_n^2\frac{1}{2}\left|\int_{\mathbb{R}^n}\langle Q_n^{-1}z, D_z u(t,z)\rangle_{\mathcal{H}} w(t,z)\mu_n(dz)\right| \\
&= \frac{1}{2}\left|\int_{\mathbb{R}^n}\langle z, Q_n^{-1}\sigma^{(n)}\sigma^{(n)*}(z) D_z u(t,z)\rangle_{\mathcal{H}} w(t,z)\mu_n(dz)\right| \\
&\quad +\epsilon_n^2\frac{1}{2}\left|\int_{\mathbb{R}^n}\langle Q_n^{-1}z, D_z u(t,z)\rangle_{\mathcal{H}} w(t,z)\mu_n(dz)\right|.
\end{aligned}$$

We consider separately the two terms. First we focus on the term involving ϵ_n, this will give us a suitable rate of convergence for the removal of degeneracy. We obtain

$$\begin{aligned}
&\epsilon_n^2\frac{1}{2}\left|\int_{\mathbb{R}^n}\langle Q_n^{-1}z, D_z u(t,z)\rangle_{\mathcal{H}} w(t,z)\mu_n(dz)\right| \\
&= \epsilon_n^2\frac{1}{2}\left|\sum_{i=1}^n\int_{\mathbb{R}^n}\frac{z_i}{\lambda_i}\langle\varphi_i, D_z u(t,z)\rangle_{\mathcal{H}} w(t,z)\mu_n(dz)\right| \\
&\le \epsilon_n^2\frac{1}{2}\sum_{i=1}^n\frac{1}{\lambda_i}\int_{\mathbb{R}^n}|z_i|\,|\langle\varphi_i, D_z u(t,z)\rangle_{\mathcal{H}}|\,|w(t,z)|\mu_n(dz)
\end{aligned}$$

Again we remember that $2 < p < \infty$ and proceed as above using Hölder inequality twice. In particular the second time we consider $q = \frac{p}{2}$ and $q = \frac{p}{p-2}$. Hence

$$\begin{aligned}
&\epsilon_n^2\frac{1}{2}\left|\int_{\mathbb{R}^n}\langle Q_n^{-1}z, D_z u(t,z)\rangle_{\mathcal{H}} w(t,z)\mu_n(dz)\right| \\
&\le \epsilon_n^2\frac{1}{2}\|D_z u(t)\|_{L^2_\mu(\mathbb{R}^n)}\|w(t)\|_{L^p_\mu(\mathbb{R}^n)}\sum_{i=1}^n\frac{1}{\lambda_i}\left(\int_{\mathbb{R}^n}|z_i|^{2\frac{p-2}{p}}\mu_n(dz)\right)^{\frac{p}{2(p-2)}} \\
&= C_p\,\epsilon_n^2\frac{1}{2}\|D_z u(t)\|_{L^2_\mu(\mathbb{R}^n)}\|w(t)\|_{L^p_\mu(\mathbb{R}^n)}\sum_{i=1}^n\frac{1}{\sqrt{\lambda_i}}.
\end{aligned}$$

It is clear that ϵ_n controls the term involving $Tr[Q_n^{-\frac{1}{2}}]_{\mathcal{H}}$ when $n \to \infty$. Since $\lambda_1 \geq \lambda_2 \geq \dots$ we have the estimate

$$\epsilon_n^2 \sum_{i=1}^{n} \frac{1}{\sqrt{\lambda_i}} \leq \epsilon_n^2 \frac{1}{\sqrt{\lambda_n}} \cdot n.$$

One possible choice would be $\epsilon_n = \lambda_n^{\frac{1}{4}}/n$. We then obtain

$$\epsilon_n^2 \frac{1}{2} \left| \int_{\mathbb{R}^n} \langle Q_n^{-1} z, D_z u(t,z) \rangle_{\mathcal{H}} w(t,z) \mu_n(dz) \right| \leq \frac{1}{2n} C_p \, \|D_z u(t)\|_{L^2_\mu(\mathbb{R}^n)} \|w(t)\|_{L^p_\mu(\mathbb{R}^n)}.$$

It is worth noticing that for the finite dimensional approximation to be convergent we also require $n\,\epsilon_n \to 0$ as $n \to \infty$ and it is verified with our choice of ϵ_n.

We now analyze the last term remaining and conclude our estimates. It is worth recalling that since Q is continuous then

$$\sigma^{(n)}(x) = P_n Q \gamma(P_n x) = P_n \Big(\sum_{i=1}^{\infty} \lambda_i \langle \gamma(P_n x), \varphi_i \rangle_{\mathcal{H}} \varphi_i\Big) = P_n Q P_n \gamma(P_n x) = Q_n \gamma^{(n)}(x).$$

We can provide the following estimate

$$\begin{aligned}
&\frac{1}{2} \left| \int_{\mathbb{R}^n} \langle z, Q_n^{-1} \sigma^{(n)} \sigma^{(n)*}(z) D_z u(t,z) \rangle_{\mathcal{H}} w(t,z) \mu_n(dz) \right| \\
&\quad \leq \sum_{i=1}^{n} \frac{1}{2} \left| \int_{\mathbb{R}^n} \langle z, \varphi_i \rangle_{\mathcal{H}} \langle \varphi_i, Q_n^{-1} \sigma^{(n)} \sigma^{(n)*}(z) D_z u(t,z) \rangle_{\mathcal{H}} w(t,z) \mu_n(dz) \right| \\
&\quad \leq \sum_{i=1}^{n} \frac{1}{2} \int_{\mathbb{R}^n} |\langle z, \varphi_i \rangle_{\mathcal{H}}| \, |\langle \varphi_i, Q_n^{-1} \sigma^{(n)} \sigma^{(n)*}(z) D_z u(t,z) \rangle_{\mathcal{H}}| \, |w(t,z)| \mu_n(dz) \\
&\quad = \sum_{i=1}^{n} \frac{1}{2} \int_{\mathbb{R}^n} |\langle z, \varphi_i \rangle_{\mathcal{H}}| \, |\langle \varphi_i, Q_n^{-1} \sigma^{(n)}(z) \rangle_{\mathcal{H}}| \, |\langle \sigma^{(n)}(z), D_z u(t,z) \rangle_{\mathcal{H}}| \, |w(t,z)| \mu_n(dz) \\
&\quad = \sum_{i=1}^{n} \frac{1}{2} \int_{\mathbb{R}^n} |\langle z, \varphi_i \rangle_{\mathcal{H}}| \, |\langle \varphi_i, \gamma^{(n)}(z) \rangle_{\mathcal{H}}| \, |\langle \sigma^{(n)}(z), D_z u(t,z) \rangle_{\mathcal{H}}| \, |w(t,z)| \mu_n(dz).
\end{aligned}$$

Now the estimate is straightforward, indeed denoting $b_\gamma := \sup_{x\in\mathcal{H}} \|\gamma(x)\|_{\mathcal{H}}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\left|\int_{\mathbb{R}^n} \langle z, Q_n^{-1}\sigma^{(n)}\sigma^{(n)*}(z)D_z u(t,z)\rangle_{\mathcal{H}} w(t,z)\mu_n(dz)\right| \\
&\quad \le b_\gamma b_\sigma \sum_{i=1}^{n} \frac{1}{2}\int_{\mathbb{R}^n} |z_i|\, \|D_z u(t,z)\|_{\mathcal{H}}\, |w(t,z)|\mu_n(dz) \\
&\quad \le \frac{1}{2} C_p\, b_\gamma\, b_\sigma \left(\sum_{i=1}^{\infty}\sqrt{\lambda_i}\right) \|D_z u(t)\|_{L^2_\mu(\mathbb{R}^n)}\, \|w(t)\|_{L^p_\mu(\mathbb{R}^n)}.
\end{aligned}$$

This concludes all the estimates on our bilinear form. We can now draw some further observations.

5.4 The variational inequality on unbounded domain

We endowe $\mathcal{V}_n^p = W^{1,2}(\mathbb{R}^n,\mu_n)\cap L^p_\mu(\mathbb{R}^n)$ with the norm $|||\cdot|||_{n,p}$ defined as

$$|||f|||_{n,p} := \|f\|_{L^p_\mu(\mathbb{R}^n)} + \|Df\|_{L^2_\mu(\mathbb{R}^n)}. \tag{5.12}$$

The space $\mathcal{V}_n^p$ is a Banach space with respect to this norm. The estimates that we have carried out above can be summarized as follows

Proposition 5.4.1 *Let $u(t), w(t) \in \mathcal{V}_n^p$, a.e. $t\in[0,T]$. There exists a constant $C_\mu > 0$ only depending on the choice of the Gaussian measure over $\mathcal{H}$ and on the bounds on the coefficients of the bilinear form, such that*

$$\int_0^T |a_\mu(t;u(t),w(t))|dt \le C_\mu \left(\int_0^T |||u(t)|||^2_{n,p}dt\right)^{\frac{1}{2}} \left(\int_0^T |||w(t)|||^2_{n,p}dt\right)^{\frac{1}{2}} \tag{5.13}$$

As a consequence of this proposition and of Proposition 4.1.1 we obtain the following

Corollary 5.4.1 *The map $\mathcal{T}_\theta(t) : \mathcal{V}_n^p \to \mathbb{R}$ is a continuous linear functional, i.e., if we denote again by $\langle\,\cdot\,,\,\cdot\,\rangle_\mu$ the dual pairing between $\mathcal{V}_n^p$ and $\mathcal{V}_n^{p*}$ we have*

$$|\langle \mathcal{T}_\theta(t), w\rangle_\mu| \le \left(\|\frac{\partial\hat{\Theta}}{\partial t}(t)\|^2_{L^2_\mu(\mathbb{R}^n)} + C_\mu \left|\left|\left|\hat{\Theta}(t)\right|\right|\right|_{n,p}\right) |||w|||_{n,p}\,.$$

It then implies that

$$\|\mathcal{T}_\theta(t)\|_{\mathcal{V}_n^{p*}} = \left(\|\frac{\partial\hat{\Theta}}{\partial t}(t)\|^2_{L^2_\mu(\mathbb{R}^n)} + C_\mu \left|\left|\left|\hat{\Theta}(t)\right|\right|\right|_{n,p}\right). \tag{5.14}$$

We have a sequence of EVI indexed by l each of which refers to an increasing sequence of domains over $\mathbb{R}^n$. We recall that $u^{(l)} \in L^2(0,T;\mathcal{V}_n^p) \cap C([0,T]\times\mathbb{R}^n)$ is the unique solution of

$$\int_0^T \left[-(\frac{\partial w^{(l)}}{\partial t}, w^{(l)} - u^{(l)})_{L^2_\mu(\mathbb{R}^n)} + a_\mu(t; u^{(l)}, w^{(l)} - u^{(l)}) - \langle \mathcal{T}_\theta(t), w^{(l)} - u^{(l)} \rangle_\mu \right] dt \\ + \frac{1}{2} |w^{(l)}(T)|^2_{L^2_\mu(\mathbb{R}^n)} \geq 0,$$

for all $w^{(l)} \in \hat{\mathcal{K}}^p_\mu \cap \{w \,:\, w|_{\partial\mathcal{O}_l} = 0\}$. For each $w \in \hat{\mathcal{K}}^p_\mu$ we can choose a sequence $w^{(l)} \in \hat{\mathcal{K}}^p_\mu \cap \{w \,:\, w|_{\partial\mathcal{O}_l} = 0\}$ such that

$$\int_0^T \|\frac{\partial w^{(l)}}{\partial t}(t) - \frac{\partial w}{\partial t}(t)\|^2_{L^2_\mu(\mathbb{R}^n)} dt + \int_0^T |||w^{(l)}(t) - w(t)|||^2_{n,p}\, dt \to 0, \qquad \text{as } n \to \infty.$$

For instance at each l we associate an open regular set $\mathcal{O}'_l \subset\subset \mathcal{O}_l$ and assume the sequence $\{\mathcal{O}'_l\}_l$ to be increasing. Following for instance [1], Chapter 3, Theorem 3.22 we can define a sequence of non-negative functions $\{f_l\}_{l\geq 1} \subset C^1([0,T]; C^\infty_c(\mathbb{R}^n))$, $f_l \geq 0$, such that for $M > 0$ independent of l we obtain

$$\begin{aligned} & f_l(t,z) = 1, && \textit{on } [0,T] \times \{z \in \overline{\mathcal{O}'}_l\}, \\ & f_l(t,z) = 0, && \textit{on } [0,T] \times \{z \in \partial\mathcal{O}_l\}, \\ & \left|\tfrac{\partial f_l}{\partial t}\right| + \textstyle\sum_{i=1}^n \left|\tfrac{\partial f_l}{\partial z_i}\right| \leq M && \textit{on } [0,T] \times \mathbb{R}^n. \end{aligned}$$

For each l we take $w^{(l)} := f_l\, w$ and hence $w^{(l)} = w$ on $[0,T] \times \mathcal{O}'_l$, $w^{(l)}(t)|_{\partial\mathcal{O}} = 0$, a.e. $t \in [0,T]$ and $w^{(l)} \geq 0$.

We recall from Proposition 5.2.1 and the following observations, that $u^{(l)} \rightharpoonup \bar{u}$ in $L^2(0,T; W^{1,2}(\mathbb{R}^n,\mu_n))$ and $u^{(l)} \to \bar{u}$ in $L^2(0,T;L^p_\mu(\mathbb{R}^n))$. This is the same as saying that $u^{(l)} \rightharpoonup \bar{u}$ in $L^2(0,T;\mathcal{V}^p_n)$ and strongly in $L^2(0,T;L^p_\mu(\mathbb{R}^n))$. When $l \to \infty$ the following are then straightforward from [14], Chapter 3, Proposition 3.5 (iv):

$$\int_0^T (\frac{\partial w^{(l)}}{\partial t}, w^{(l)} - u^{(l)})_{L^2_\mu(\mathbb{R}^n)} dt \;\; \to \;\; \int_0^T (\frac{\partial w}{\partial t}, w - \bar{u})_{L^2_\mu(\mathbb{R}^n)} dt,$$

$$\int_0^T \langle \mathcal{T}_\theta(t), w^{(l)} - u^{(l)} \rangle_\mu dt \;\; \to \;\; \int_0^T \langle \mathcal{T}_\theta(t), w - \bar{u} \rangle_\mu dt,$$

$$|w^{(l)}(T)|^2_{L^2_\mu(\mathbb{R}^n)} \;\; \to \;\; |w(T)|^2_{L^2_\mu(\mathbb{R}^n)}.$$

A bit more explanations are needed for the term involving the bilinear form. In what follows we denote by $(I)(u^{(l)})$ the term (I) of the bilinear form when $w = u^{(l)}$. The same notation is considered for all the other terms. Let us first notice that collecting all the previous estimates we obtain

$$a_\mu(t; u^{(l)}, u^{(l)}) \;\; = \;\; (I)(u^{(l)}) + (IIa)(u^{(l)}) + (IIb)(u^{(l)}) + (III)(u^{(l)})$$

We can consider $\Lambda > 0$ large enough and such that

$$(I)(u^{(l)}) \geq 0,$$

$$(IIa)(u^{(l)}) + (IIb)(u^{(l)}) + (III)(u^{(l)}) \geq -(|(IIa)(u^{(l)})| + |(IIb)(u^{(l)})| + |(III)(u^{(l)})|) \geq$$

$$\geq -\Lambda \|D_z u^{(l)}\|_{L^2_\mu(\mathbb{R}^n)} \|u^{(l)}\|_{L^p_\mu(\mathbb{R}^n)}.$$

We exploit now the uniform bound on the gradient and we obtain

$$a_\mu(t; u^{(l)}, u^{(l)}) \;\; \geq \;\; -\hat{\Lambda} \, \|u^{(l)}(t)\|_{L^p_\mu(\mathbb{R}^n)}, \tag{5.15}$$

where now $\hat{\Lambda} := \Lambda(C_\Psi + C_U)$. We have then

$$a_\mu(t; u^{(l)}, w^{(l)} - u^{(l)}) = a_\mu(t; u^{(l)}, w^{(l)}) - a_\mu(t; u^{(l)}, u^{(l)}).$$

The first term converges for the same arguments as above

$$\int_0^T a_\mu(t;u^{(l)}(t),w^{(l)}(t))dt \to \int_0^T a_\mu(t;\bar{u}(t),w(t))dt.$$

For the second term we carry out some calculations. We easily obtain the following expression.

$$\int_0^T a_\mu(t;u^{(l)}(t),u^{(l)}(t))dt = \int_0^T a_\mu(t;u^{(l)}(t)-\bar{u}(t),u^{(l)}(t)-\bar{u}(t))dt +$$

$$+\int_0^T a_\mu(t;\bar{u}(t),u^{(l)}(t))dt + \int_0^T a_\mu(t;u^{(l)}(t)-\bar{u}(t),\bar{u}(t))dt.$$

From the same arguments as above, the second and third term converge respectively to

$$\int_0^T a_\mu(t;\bar{u}(t),u^{(l)}(t))dt \quad \to \quad \int_0^T a_\mu(t;\bar{u}(t),\bar{u}(t))dt,$$

and

$$\int_0^T a_\mu(t;u^{(l)}(t)-\bar{u}(t),\bar{u}(t))dt \quad \to \quad 0.$$

For the first term, from equation (5.15), it holds

$$\int_0^T a_\mu(t;u^{(l)}(t)-\bar{u}(t),u^{(l)}(t)-\bar{u}(t))dt \geq -\hat{\Lambda}\int_0^T \|u^{(l)}(t)-\bar{u}(t)\|_{L^p_\mu(\mathbb{R}^n)}dt \to 0.$$

It is worth noticing that in the last expression we are implicitly using the fact that $\|\bar{u}\| \leq \sqrt{C_\Psi + C_U}$ which we know, both from the probabilistic representation and from the lower semicontinuity of the weak convergence, that is

$$\|\bar{u}\|_{W^{1,2}(\mathbb{R}^n,\mu_n)} \leq \liminf_{l\to\infty} \|u^{(l)}\|_{W^{1,2}(\mathbb{R}^n,\mu_n)} \leq \sqrt{C_\Psi + C_U}.$$

Now summarizing the last few rows we have

$$\lim_{l\to\infty}\int_0^T a_\mu(t;u^{(l)}(t),u^{(l)}(t))dt \geq \int_0^T a_\mu(t;\bar{u}(t),\bar{u}(t))dt.$$

When taking the limit in the sequence of EVI's and from the limit in equation (5.10), given the uniqueness of the limit we obtain the following theorem

Theorem 5.4.1 *There exists at least a solution $\bar{u} \in L^2(0,T;\mathcal{V}_n^p)$ to the EVI*

$$\int_0^T \left[-(\frac{\partial w}{\partial t}, w-\bar{u})_{L^2_\mu(\mathbb{R}^n)} + a_\mu(t;\bar{u}, w-\bar{u}) - \langle \mathcal{T}_\theta(t), w-\bar{u}\rangle_\mu\right] dt \\ +\frac{1}{2}|w(T)|^2_{L^2_\mu(\mathbb{R}^n)} \geq 0,$$

for all $w \in \hat{\mathcal{K}}^p_\mu$. Moreover, such a solution can be represented in terms of the value function of the optimal stopping problem as:

$$\bar{u}(t,z) = U_\Theta(t,z) - \hat{\Theta}(t,z), \tag{5.16}$$

where

$$U_\Theta(t,z) = \sup_{t\leq\tau\leq T} \mathbb{E}\left[\hat{\Theta}(\tau, Z^{t,z}_\tau)\right]. \tag{5.17}$$

It implies $\bar{u} \in L^2(0,T;\mathcal{V}_n^p) \cap C([0,T]\times\mathbb{R}^n)$.

It follows a straightforward corollary which characterizes the value function of the finite dimensional Optimal Stopping problem on the unbounded domain. It is simply derived removing the homogeneization with respect to the obstacle. In order to properly set the non homogeneous problem we introduce the convex set

$$\hat{\mathcal{K}}^{\Theta,p}_\mu := \{w:\ w(t) \in L^2(0,T;\mathcal{V}_n^p),\ \frac{\partial w}{\partial t}(t) \in L^2(0,T;L^2_\mu(\mathbb{R}^n)),\ w \geq \hat{\Theta} \text{ a.e. } [0,T]\times\mathbb{R}^n\}.$$

Corollary 5.4.2 *There exists at least a solution $\bar{v} \in L^2(0,T;\mathcal{V}_n^p)$ to the EVI*

$$\int_0^T \left[-(\frac{\partial w}{\partial t}, w-\bar{v})_{L^2_\mu(\mathbb{R}^n)} + a_\mu(t;\bar{v}, w-\bar{v})\right] dt \\ +\frac{1}{2}|w(T) - \hat{\Psi}(T)|^2_{L^2_\mu(\mathbb{R}^n)} \geq 0,$$

for all $w \in \hat{\mathcal{K}}^{\Theta,p}_\mu$. Moreover, such a solution can be represented in terms of the value function of the optimal stopping problem as:

$$\bar{v}(t,z) = U_\Theta(t,z) = \sup_{t\leq\tau\leq T} \mathbb{E}\left[\hat{\Theta}(\tau, Z^{t,z}_\tau)\right]. \tag{5.18}$$

It implies $\bar{v} \in L^2(0,T;\mathcal{V}_n^p) \cap C([0,T]\times\mathbb{R}^n)$.

We separately state the existence of the optimal stopping time.

Theorem 5.4.2 *The optimal stopping time for the problem* (5.18) *is defined as*

$$\tau^*_{t,z} := \inf\{s \geq t \,:\, U_\Theta(s, Z^{t,z}_s) = \Theta(s, Z^{t,z}_s)\} \wedge T \tag{5.19}$$

and for all stopping times $\tau \leq \tau^*_{t,z}$ *the following holds*

$$U_\Theta(t,z) = \mathbb{E}\left[U_\Theta(\tau, Z^{t,z}_\tau)\right]. \tag{5.20}$$

PROOF: If we denote by $\tau^*_{t,z,l}$ the optimal stopping time (4.53) associated to the region $\mathcal{O}_l$, then the sequence $\{\tau^*_{t,z,l}\}_{l\geq 0}$ is optimal with respect to the sequence of optimal stopping problems on bounded domains. Thanks to this optimality and to Proposition 5.2.2 we can prove that $\tau^*_{t,z,l} \wedge \tau^*_{t,z} \to \tau^*_{t,z}$ $\mathbb{P}$-a.s., as $l \to \infty$. The proof is a simpler version of the one we will produce for Lemma 6.4.1. Moreover, equation (5.20) is a simple application of the dynamic programming principle and can be explicitly obtained. In fact if we replace τ^*_k in (4.50) by $\tau^*_{t,z,l} \wedge \tau^*_{t,z}$ the first equality still holds, i.e. we have

$$U^{\mathcal{O}_l}(t,z) = \mathbb{E}\left[U^{\mathcal{O}_l}(\tau^*_{t,z,l} \wedge \tau^*_{t,z}, Z^{t,z}_{\tau^*_{t,z,l}\wedge\tau^*_{t,z}})\right].$$

We take the limit as $l \to \infty$ and exploit the convergence of Proposition 5.2.2. Following the rationale in the proof of Theorem 6.4.1 we obtain (5.19). The whole algorithm would clearly hold for $\tau \leq \tau^*_{t,z,l}$ and hence (5.20) is verified. ∎

This concludes the extension of our EVI to the whole domain $\mathbb{R}^n$. The next step will be to take the limit when the number of dimensions goes to infinity.

Remark 5.4.1 *It is worth to stress that here the uniqueness might be recovered through the same rationale as on the bounded domain. Nevertheless we would lose it when taking the limit to infinite dimensions. Then it seems more interesting at this stage to find the infinite dimensional representation of our EVI and later to discuss the uniqueness.*

Chapter 6

Variational inequality on a Hilbert space

In this Chapter we achieve the main result of this Thesis. We extend the variational inequality in finite dimensions to the infinite dimensional setting. In particular, we prove the existence of a weak solution to the infinite dimensional variational inequality that is obtained as the limit for $n \to \infty$ and $\alpha \to \infty$. This solution turns out to be the value function of the infinite dimensional optimal stopping problem that was introduced in Chapter 2. The optimal stopping time is characterized and some regularity results are provided. First we consider the limit as the number of dimensions n goes to infinity and then the one as the Yosida parameter α goes to infinity.

6.1 Extending the Gaussian measure

We are going to show that the results obtained in the finite dimensional setting can be extended to the infinite dimensional one. In order to do so, some preliminary considerations are needed. In this and the next sections the notation will get a bit cumbersome. Hence in order to reduce the difficulties we suppress the Θ index in the value function, i.e. $U_\Theta = U$. We will recover this notation later when we analyze the limiting behaviour of the smoothing procedure for the gain function.

First of all we shall reintroduce the index n denoting the order of the finite dimensional approximation. Yet we keep $\alpha > 0$ fixed and then we recall the notation $U^{(n)}_\alpha(t,z) := U(t,z)$, where

$$U_\alpha^{(n)}(t,z) := \sup_{t\le\tau\le T} \mathbb{E}\left[\hat{\Theta}^{(n)}(\tau, Z_\tau^{(\alpha,n)\,t,z})\right].$$

We recall that for $x \in \mathcal{H}$ and $x^{(n)} = P_n x \in \mathcal{H}^{(n)}$ we have defined the isometry $x^{(n)} \sim z$, where $z \in \mathbb{R}^n$. Then $\hat{\Theta}^{(n)}(t,z) = \hat{\Psi}_k^{(n)}(t,z) = \Psi_k^{(n)}(t,x^{(n)}) =: \Theta^{(n)}(t,x^{(n)})$ and as a matter of fact $\Theta^{(n)}(t,x^{(n)}) = \Theta(t,P_n x)$. For the value function we adopt the notation $V_\Theta = V$. Then we have

$$U_\alpha^{(n)}(t,z) = V_\alpha^{(n)}(t,x^{(n)}) := \sup_{t\le\tau\le T} \mathbb{E}\left[\Theta(\tau, X_\tau^{(\alpha,n)\,t,x})\right].$$

Although this different notations can look a bit clumsy, it is worth to understand that they simply represent the same mathematical object from different perspectives. In particular it is equivalent to think of $U_\alpha^{(n)}$ as a function $U_\alpha^{(n)} : [0,T]\times\mathbb{R}^n \to \mathbb{R}$, as we used to do, or as $U_\alpha^{(n)} : [0,T]\times\mathcal{H} \to \mathbb{R}$. When considering the second notation we implicitly mean $U_\alpha^{(n)}(t)(\cdot) = U_\alpha^{(n)}(t)\circ P_n(\cdot)$, i.e. the input of $U_\alpha^{(n)}(t)(\cdot)$ is an element of $\mathcal{H}$ and the function considers only its finite dimensional projection on the subset spanned by $\{\varphi_1,\dots,\varphi_n\}$.

The value function obtained in Corollary 5.4.2 can hence be renamed as $\bar{v} = \bar{v}_\alpha^{(n)}$ and according to our discussion can be understood as a function $\bar{v}_\alpha^{(n)} : [0,T]\times\mathcal{H} \to \mathbb{R}$ in the sense that $\bar{v}_\alpha^{(n)}(t)(\cdot) = \bar{v}_\alpha^{(n)}(t)\circ P_n(\cdot)$. This notation is helpful to embed the finite dimensional EVI's into an infinite dimensional framework.

Let us consider now a generic function $f : \mathcal{H}\to\mathbb{R}$, and let us assume $\exists \hat{f} : \mathbb{R}^n \to \mathbb{R}$ such that $f = \hat{f}\circ P_n$. These functions are often referred to as *cylindrical functions* in $\mathbb{R}^n$. We again denote by $z\in\mathbb{R}^n$ the isometric vector to $x^{(n)} \in \mathcal{H}^{(n)} \subset \mathcal{H}$. Let us assume f regular enough, in particular $f \in L^2(\mathbb{R}^n, \mu_n)$. We have (cf. [22], Remark 9.2.6) that

$$\int_{\mathbb{R}^n} |\hat{f}(z)|^2 \mu_n(dz) = \int_{\mathcal{H}} |f(x)|^2 \mu(dx),$$

where μ is the Gaussian measure over the Hilbert space $\mathcal{H}$. A detailed exposition about Gaussian measures over Hilbert spaces can be found in [11, 18, 22], but the rough idea is the following. A possible way of defining the Gaussian measure over $\mathcal{H}$ is to obtain it as the limit for $n\to\infty$ of the measure

$$\mu_n(dx) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\lambda_i}} e^{-\frac{x_i^2}{2\lambda_i}} dx_i,$$

where as usual $x_i = \langle x, \varphi_i \rangle_{\mathcal{H}}$. The infinite product measure is restricted to the set of vectors $x \in \mathbb{R}^\infty$ such that $\sum_{i=1}^{\infty} x_i^2 < \infty$. Now if we think of this measure as

$$\mu(dx) = \prod_{i=1}^{\infty} \frac{1}{\sqrt{2\pi\lambda_i}} e^{-\frac{x_i^2}{2\lambda_i}} dx_i,$$

it is clear that

$$\int_{\mathcal{H}} |f(x)|^2 \mu(dx) = \int_{\mathbb{R}^n} |\hat{f}(x_1, \ldots, x_n)|^2 \mu_n(dx),$$

since all the integrals not involving the function f sum to one.

The importance of adopting the Gaussian measure μ over $\mathcal{H}$ stems from the fact that it represents the natural substitute of the Lebesgue measure on finite dimensional spaces. Moreover one can prove (cf. Appendix E) that the operator of directional derivative (Friedrichs derivative) is closable under this measure. Hence a suitable analogue of the weak derivative can be found in Hilbert spaces and furthermore a concept of Sobolev space is meaningful. In particular for a function $g : \mathcal{H} \to \mathbb{R}$ we denote by $Dg(x)$, $x \in \mathcal{H}$, the closure of the directional derivative evaluated at the point x. Obviously $Dg(x) \in \mathcal{H}^*$ and hence after indentifying $\mathcal{H} \approx \mathcal{H}^*$, we have $Dg(x) \in \mathcal{H}$. We now define the Gauss Sobolev space $W^{1,2}(\mathcal{H}, \mu)$ as

$$W^{1,2}(\mathcal{H}, \mu) := \{g : \mathcal{H} \to \mathbb{R} \, : \, \|g\|_{L^2(\mathcal{H},\mu)} < \infty \text{ and } \|Dg\|_{L^2(\mathcal{H},\mu;\mathcal{H})} < \infty\},$$

where

$$\|g\|^2_{L^2(\mathcal{H},\mu)} = \int_{\mathcal{H}} |g(x)|^2 \mu(dx),$$

and

$$\|Dg\|^2_{L^2(\mathcal{H},\mu;\mathcal{H})} = \int_{\mathcal{H}} \|Dg(x)\|^2_{\mathcal{H}} \mu(dx).$$

In the present analysis it is rather natural to introduce the space $\mathcal{V}^p := W^{1,2}(\mathcal{H}, \mu) \cap L^p(\mathcal{H}, \mu)$ endowed with the norm

$$|||g|||_p := \|g\|_{L^p(\mathcal{H},\mu)} + \|Dg\|_{L^2(\mathcal{H},\mu;\mathcal{H})}.$$

Here the main idea is that for a cylindrical function in $\mathbb{R}^n$, namely f, there is a complete equivalence between the spaces $\mathcal{V}^p$ and $\mathcal{V}^p_n$. In particular the following hold

$$\|f\|_{L^p_\mu(\mathbb{R}^n)} = \|f\|_{L^p(\mathcal{H},\mu)}, \qquad 1 \le p < \infty,$$

$$\|f\|_{W^{1,2}(\mathbb{R}^n,\mu)} = \|f\|_{W^{1,2}(\mathcal{H},\mu)},$$

and eventually

$$|||f|||_{n,p} = |||f|||_p \, .$$

It is then clear that $\mathcal{V}^p_n \subset \mathcal{V}^p$ and in particular $\mathcal{V}^p_n$ is the subset of $\mathcal{V}^p$ of cylindrical functions in $\mathbb{R}^n$. This fact will allow us to establish a clear relation between the finite dimensional EVI and its infinite dimensional counterpart.

Before discussing this relations we need to pay some attention to the convergence of the diffusion coefficients and of its derivatives.

6.1.1 Some basic convergence results

The aim of this section is to prove some convergence results for the diffusion coefficients. We summarize them in the next proposition.

Proposition 6.1.1 *Let Assumptions 3.2.1 hold, then for $1 \le p < \infty$ the following convergence results are verified*

i) $\int_{\mathcal{H}} \|\sigma^{(n)}(x) - \sigma(x)\|^p_{\mathcal{H}} \mu(dx) \to 0$ *as* $n \to \infty$,

ii) $\int_{\mathcal{H}} \|\sigma^{(n)}\sigma^{(n)*}(x) - \sigma\sigma^*(x)\|^p_{\mathcal{L}} \mu(dx) \to 0$ *as* $n \to \infty$,

iii) $\int_{\mathcal{H}} \|D\sigma^{(n)}(x) - D\sigma(x)\|^p_{\mathcal{L}} \mu(dx) \to 0$ *as* $n \to \infty$,

iv) $\int_{\mathcal{H}} \|D\sigma^{(n)}(x) \cdot \sigma^{(n)}(x) - D\sigma(x) \cdot \sigma(x)\|^p_{\mathcal{H}} \mu(dx) \to 0$ *as* $n \to \infty$,

v) $\int_{\mathcal{H}} \left|Tr[P_n D\sigma^{(n)}(x)]_{\mathcal{H}} - Tr[D\sigma(x)]_{\mathcal{H}}\right|^p \mu(dx) \to 0$ *as* $n \to \infty$.

Moreover, all the above results hold pointwisely as well.

PROOF: We can easily prove the pointwise convergence of the approximating coefficient $\sigma^{(n)}(x)$ for $x \in \mathcal{H}$. If indeed $\sigma \in C_b^2(\mathcal{H};\mathcal{H})$ we then have

$$\|\sigma^{(n)}(x) - \sigma(x)\|_{\mathcal{H}} \leq \|P_n\sigma(P_nx) - P_n\sigma(x)\|_{\mathcal{H}} + \|(1-P_n)\sigma(x)\|_{\mathcal{H}} \leq$$
$$\leq \|\sigma(P_nx) - \sigma(x)\|_{\mathcal{H}} + \|(1-P_n)\sigma(x)\|_{\mathcal{H}} \to 0, \quad \text{as } n \to \infty.$$

The first term converges for the continuity of σ and the second by definition of ortoghonal projection. This result can be extended by means of dominated convergence (or also monotone convergence) for $1 \leq p < \infty$ to

$$\int_{\mathcal{H}} \|\sigma^{(n)}(x) - \sigma(x)\|_{\mathcal{H}}^p \mu(dx) \to 0 \quad \text{as } n \to \infty.$$

As a consequence of this result we obtain pointwise convergence

$$\|\sigma^{(n)}\sigma^{(n)*}(x) - \sigma\sigma^*(x)\|_{\mathcal{L}} \to 0, \qquad x \in \mathcal{H}. \tag{6.1}$$

Indeed for any $h \in \mathcal{H}$ and x given and fixed, it holds

$$\|\sigma^{(n)}\sigma^{(n)*}(x)h - \sigma\sigma^*(x)h\|_{\mathcal{H}} = \|\sigma^{(n)}(x)\langle\sigma^{(n)}(x), h\rangle_{\mathcal{H}} - \sigma(x)\langle\sigma(x), h\rangle_{\mathcal{H}}\|_{\mathcal{H}}$$

$$\leq |\langle\sigma^{(n)}(x) - \sigma(x), h\rangle_{\mathcal{H}}|\, \|\sigma^{(n)}(x)\|_{\mathcal{H}} - |\langle\sigma(x), h\rangle_{\mathcal{H}}|\, \|\sigma^{(n)}(x) - \sigma(x)\|_{\mathcal{H}}$$

$$\leq \|\sigma^{(n)}(x) - \sigma(x)\|_{\mathcal{H}} \left(\|\sigma^{(n)}(x)\|_{\mathcal{H}} - \|\sigma(x)\|_{\mathcal{H}}\right) \|h\|_{\mathcal{H}}.$$

Then

$$\|\sigma^{(n)}\sigma^{(n)*}(x) - \sigma\sigma^*(x)\|_{\mathcal{L}} = \|\sigma^{(n)}(x) - \sigma(x)\|_{\mathcal{H}} \left(\|\sigma^{(n)}(x)\|_{\mathcal{H}} - \|\sigma(x)\|_{\mathcal{H}}\right),$$

and hence the limit (6.1) holds. Again from dominated convergence we get, for $1 \leq p < \infty$

$$\int_{\mathcal{H}} \|\sigma^{(n)}\sigma^{(n)*}(x) - \sigma\sigma^*(x)\|_{\mathcal{L}}^p \mu(dx) \to 0 \quad \text{as } n \to \infty.$$

Similar results hold for the derivative, i.e. recalling that $D\sigma^{(n)}(x) = P_n D\sigma(P_n x)$

$$\|D\sigma^{(n)}(x) - D\sigma(x)\|_{\mathcal{L}} \le \|D\sigma(P_n x) - D\sigma(x)\|_{\mathcal{L}} + \|(1-P_n)D\sigma(x)\|_{\mathcal{L}}.$$

Then pointwise convergence $\|D\sigma^{(n)}(x) - D\sigma(x)\|_{\mathcal{H}} \to 0$ holds because the derivative is continuous in $\mathcal{L}(\mathcal{H};\mathcal{H})$ and again from dominated convergence we obtain

$$\int_{\mathcal{H}} \|D\sigma^{(n)}(x) - D\sigma(x)\|_{\mathcal{L}}^p \mu(dx) \to 0 \quad \text{as } n \to \infty,\ 1 \le p < \infty.$$

The previous results allow us to prove the next one:

$$\begin{aligned}
&\|D\sigma^{(n)}(x)\cdot\sigma^{(n)}(x) - D\sigma(x)\cdot\sigma(x)\|_{\mathcal{H}} \\
&\quad \le \|D\sigma^{(n)}(x)\cdot(\sigma^{(n)}(x) - \sigma(x))\|_{\mathcal{H}} + \|(D\sigma^{(n)}(x) - D\sigma(x))\cdot\sigma(x)\|_{\mathcal{H}} \\
&\quad \le \|D\sigma^{(n)}(x)\|_{\mathcal{L}}\cdot\|\sigma^{(n)}(x) - \sigma(x)\|_{\mathcal{H}} + \|D\sigma^{(n)}(x) - D\sigma(x)\|_{\mathcal{L}}\cdot\|\sigma(x)\|_{\mathcal{H}}.
\end{aligned}$$

Now the pointwise convergence holds for the previous results and also by dominated convergence we have

$$\int_{\mathcal{H}} \|D\sigma^{(n)}(x)\cdot\sigma^{(n)}(x) - D\sigma(x)\cdot\sigma(x)\|_{\mathcal{H}}^p \mu(dx) \to 0 \quad \text{as } n \to \infty,\ 1 \le p < \infty.$$

The result (v) relies on the trace class property of the derivative of the diffusion coefficient. Indeed

$$\begin{aligned}
\left|Tr[P_n D\sigma^{(n)}(x)]_{\mathcal{H}} - Tr[D\sigma(x)]_{\mathcal{H}}\right| \le & \left|Tr[P_n(D\sigma(P_n x) - D\sigma(x))]_{\mathcal{H}}\right| \\
& + \left|Tr[(1-P_n)D\sigma(x)]_{\mathcal{H}}\right|,
\end{aligned}$$

and the second term goes to zero as usual. For the first term we have the esimate

$$\begin{aligned}
\|P_n(D\sigma(P_n x) - D\sigma(x))\|_{tc} &\le \|D\sigma(P_n x) - D\sigma(x)\|_{tc} \\
&\le \|Q\|_{tc}\cdot\|D\gamma(P_n x) - D\gamma(x)\|_{\mathcal{L}},
\end{aligned}$$

and then for $x \in \mathcal{H}$ given and fixed,

$$\left|Tr[P_n D\sigma^{(n)}(x)]_{\mathcal{H}} - Tr[D\sigma(x)]_{\mathcal{H}}\right| \to 0, \quad \text{as } n \to \infty.$$

We can now conclude that since $|Tr[P_n D\sigma^{(n)}(x)]_{\mathcal{H}}| \le B_\gamma\|Q\|_{tc}$, also

$$\int_{\mathcal{H}} \left|Tr[P_n D\sigma^{(n)}(x)]_{\mathcal{H}} - Tr[D\sigma(x)]_{\mathcal{H}}\right|^p \mu(dx) \to 0 \quad \text{as } n \to \infty,\ 1 \le p < \infty.$$

■

6.2 Extending the bilinear form

The very first step we have to make in order to properly set our problem in an infinite dimensional setting is to redefine the convex set of test functions. We now rename the convex set in $\mathbb{R}^n$ by

$$\hat{\mathcal{K}}^{\Theta,p}_{\mu,n} := \{w:\ w \in L^2(0,T;\mathcal{V}^p_n),\ \frac{\partial w}{\partial t} \in L^2(0,T;L^2_\mu(\mathbb{R}^n)),\ w \geq \Theta^{(n)} \text{ a.e. } [0,T]\times\mathbb{R}^n\}.$$

The natural extension to the infinite dimensional setting is

$$\hat{\mathcal{K}}^{\Theta,p}_{\mu,\infty} := \{w:\ w \in L^2(0,T;\ \mathcal{V}^p\ ),\ \frac{\partial w}{\partial t} \in L^2(0,T;L^2(\mathcal{H},\mu)),\ w \geq \Theta \text{ a.e. } [0,T]\times\mathcal{H}\}.$$

We adopt the same rationale to embed the solution of our EVI into the infinite dimensional framework. In particular we have $\bar{v} = \bar{v}^{(n)} \in L^2(0,T;\mathcal{V}^p_n) \subset L^2(0,T;\ \mathcal{V}^p\ )$. We also rename the bilinear form. Let $u^{(n)}(t), w^{(n)}(t)$ be generic functions in $\mathcal{V}^p_n \subset \mathcal{V}^p$, then it is easy to verify from the arguments above that our bilinear form reads

$$\begin{aligned} a^{(n)}_\mu(t;u^{(n)}(t),w^{(n)}(t)) \quad = \quad & \underbrace{\frac{1}{2}\int_{\mathcal{H}} \langle \sigma^{(n)}\sigma^{(n)*}(x) D_x u^{(n)}(t,x), D_x w^{(n)}(t,x)\rangle_{\mathcal{H}}\mu(dx)}_{A} \\ & + \frac{1}{2}\epsilon_n^2 \int_{\mathcal{H}} \langle D_x u^{(n)}(t,x), D_x w^{(n)}(t,x)\rangle_{\mathcal{H}}\mu(dx) \\ & \underbrace{+\frac{1}{2}\int_{\mathcal{H}} \langle P_n D_x \sigma^{(n)}(x)\cdot\sigma^{(n)}(x), D_x u^{(n)}(t,x)\rangle_{\mathcal{H}} w^{(n)}(t,x)\mu(dx)}_{B} \\ & \underbrace{+\frac{1}{2}\int_{\mathcal{H}} Tr[P_n D_x \sigma^{(n)}(x)]_{\mathcal{H}} \langle \sigma^{(n)}(x), D_x u^{(n)}(t,x)\rangle_{\mathcal{H}} w^{(n)}(t,x)\mu(dx)}_{C} \\ & \underbrace{-\int_{\mathcal{H}} \langle A_{\alpha,n}x, D_x u^{(n)}(t,x)\rangle_{\mathcal{H}} w^{(n)}(t,x)\mu(dx)}_{D} \end{aligned}$$

$$+\underbrace{\frac{1}{2}\int_{\mathcal{H}}\langle\sigma^{(n)}\sigma^{(n)*}(x)Q_n^{-1}x, D_x u^{(n)}(t,x)\rangle_{\mathcal{H}} w^{(n)}(t,x)\mu(dx)}_{E}$$

$$+\epsilon_n^2\frac{1}{2}\int_{\mathcal{H}}\langle Q_n^{-1}x, D_x u^{(n)}(t,x)\rangle_{\mathcal{H}} w^{(n)}(t,x)\mu(dx).$$

Our aim of course would be to give a meaning to "$\lim_{n\to\infty} a_\mu^{(n)}(t;u^{(n)},w^{(n)})$".

6.2.1 Convergence of the bilinear form

The first crucial observation is that all the estimates we carried out in the previous section regarding the bilinear form were independent of n and α. In this sense we can say that they are *universal* esimates. In particular it means that even for functions $u(t), w(t) \in \mathcal{V}^p$ the bilinear form $a_\mu^{(n)}(t;u(t),w(t))$ is completely well defined. Indeed we have

$$|a_\mu^{(n)}(t;u(t),w(t))| \le C_\mu \, |||u(t)|||_p \cdot |||w(t)|||_p \, .$$

We disregard of the time dependence for a while and consider $t \in [0,T]$ given and fixed. Let now $u \in \mathcal{V}^p$ be given. Let then $\{w^{(n)}\}_{n=1}^{\infty}$ be a sequence of functions such that $w^{(n)} \in \mathcal{V}_n^p$ and $w^{(n)} \to w$ in $\mathcal{V}^p$. Clearly the maps $u \mapsto a_\mu^{(n)}(t;u,w^{(n)})$, $n \ge 1$ represent a sequence of bounded linear functionals on $\mathcal{V}^p$.

We will analyze term by term the bilinear form. We can disregard of the terms depending on ϵ_n because they will vanish as $n \to \infty$. We start with the term (A) and in particular we want to estimate

$$\left|\int_{\mathcal{H}}\langle\sigma^{(n)}\sigma^{(n)*}(x)D_x u(x), D_x w^{(n)}(x)\rangle_{\mathcal{H}}\mu(dx) - \int_{\mathcal{H}}\langle\sigma\sigma^*(x)D_x u(x), D_x w(x)\rangle_{\mathcal{H}}\mu(dx)\right|$$
$$\le \int_{\mathcal{H}}\left|\langle\sigma^{(n)}\sigma^{(n)*}(x)D_x u(x), D_x w^{(n)}(x) - D_x w(x)\rangle_{\mathcal{H}}\right|\mu(dx)$$

$$+\int_{\mathcal{H}} \left|\langle(\sigma^{(n)}\sigma^{(n)*}(x)-\sigma\sigma^*(x))D_xu(x),D_xw(x)\rangle_{\mathcal{H}}\right| \mu(dx)$$

$$\leq \int_{\mathcal{H}} \|\sigma^{(n)}\sigma^{(n)*}(x)\|_{\mathcal{L}} \|D_xu(x)\|_{\mathcal{H}} \|D_xw^{(n)}(x)-D_xw(x)\|_{\mathcal{H}}\mu(dx)$$

$$+\int_{\mathcal{H}} \|\sigma^{(n)}\sigma^{(n)*}(x)-\sigma\sigma^*(x)\|_{\mathcal{L}} \|D_xu(x)\|_{\mathcal{H}} \|D_xw(x)\|_{\mathcal{H}}\mu(dx)$$

$$\leq b_\sigma^2 \left(\int_{\mathcal{H}} \|D_xu(x)\|^2_{\mathcal{H}}\mu(dx)\right)^{\frac{1}{2}} \left(\int_{\mathcal{H}} \|D_xw^{(n)}(x)-D_xw(x)\|^2_{\mathcal{H}}\mu(dx)\right)^{\frac{1}{2}}$$

$$+\left(\int_{\mathcal{H}} \|\sigma^{(n)}\sigma^{(n)*}(x)-\sigma\sigma^*(x)\|^2_{\mathcal{L}} \|D_xw(x)\|^2_{\mathcal{H}}\mu(dx)\right)^{\frac{1}{2}} \left(\int_{\mathcal{H}} \|D_xu(x)\|^2_{\mathcal{H}}\mu(dx)\right)^{\frac{1}{2}}.$$

Notice that all the integrals are well defined since $\|\sigma^{(n)}\sigma^{(n)*}(x)-\sigma\sigma^*(x)\|^2_{\mathcal{L}} \leq 2b_\sigma^2$. Then from dominated convergence and the convergence results discussed in Proposition 6.1.1 we obtain

$$\int_{\mathcal{H}} \langle\sigma^{(n)}\sigma^{(n)*}(x)D_xu(x),D_xw^{(n)}(x)\rangle_{\mathcal{H}}\mu(dx) \to$$
$$\to \int_{\mathcal{H}} \langle\sigma\sigma^*(x)D_xu(x),D_xw(x)\rangle_{\mathcal{H}}\mu(dx),$$

as $n\to\infty$.

We deal now with the (B) term in our bilinear form. Notice that from equation (4.12) we know that

$$P_nD_x\sigma^{(n)}(x)\cdot\sigma^{(n)}(x) = D_x\sigma^{(n)}(x)\cdot\sigma^{(n)}(x).$$

So we also have

$$\left|\int_{\mathcal{H}} \langle D_x\sigma^{(n)}(x)\cdot\sigma^{(n)}(x),D_xu(x)\rangle_{\mathcal{H}}w^{(n)}(x)\mu(dx)\right.$$
$$\left.-\int_{\mathcal{H}} \langle D_x\sigma(x)\cdot\sigma(x),D_xu(x)\rangle_{\mathcal{H}}w(x)\mu(dx)\right| \leq$$

$$\leq \int_{\mathcal{H}} \left|\langle D_x\sigma^{(n)}(x)\cdot\sigma^{(n)}(x),D_xu(x)\rangle_{\mathcal{H}}(w^{(n)}-w)(x)\right|\mu(dx)$$
$$+\int_{\mathcal{H}} \left|\langle(D_x\sigma^{(n)}(x)\cdot\sigma^{(n)}(x)-D_x\sigma(x)\cdot\sigma(x)),D_xu(x)\rangle_{\mathcal{H}}w(x)\right|\mu(dx)$$

$$\leq \int_{\mathcal{H}} \|D_x\sigma^{(n)}(x)\cdot\sigma^{(n)}(x)\|_{\mathcal{H}}\,\|D_x u(x)\|_{\mathcal{H}}\,|w^{(n)}(x)-w(x)|\mu(dx)$$
$$+\int_{\mathcal{H}} \|D_x\sigma^{(n)}(x)\cdot\sigma^{(n)}(x)-D_x\sigma(x)\cdot\sigma(x)\|_{\mathcal{H}}\,\|D_x u(x)\|_{\mathcal{H}}\,|w(x)|\mu(dx).$$

In light of this estimate and from the same arguments as for the (A) term we conclude that

$$\int_{\mathcal{H}} \langle D_x\sigma^{(n)}(x)\cdot\sigma^{(n)}(x), D_x u(x)\rangle_{\mathcal{H}} w^{(n)}(x)\mu(dx) \to$$
$$\to \int_{\mathcal{H}} \langle D_x\sigma(x)\cdot\sigma(x), D_x u(x)\rangle_{\mathcal{H}} w(x)\mu(dx).$$

Similarly for the (C) term we obtain the estimate

$$\Bigg|\int_{\mathcal{H}} Tr[P_n D_x\sigma^{(n)}(x)]_{\mathcal{H}}\langle\sigma^{(n)}(x), D_x u(x)\rangle_{\mathcal{H}} w^{(n)}(x)\mu(dx)$$
$$-\int_{\mathcal{H}} Tr[D_x\sigma(x)]_{\mathcal{H}}\langle\sigma(x), D_x u(x)\rangle_{\mathcal{H}} w(x)\mu(dx)\Bigg|$$
$$\leq \Bigg|\int_{\mathcal{H}} Tr[P_n D_x\sigma^{(n)}(x)]_{\mathcal{H}}\langle\sigma^{(n)}(x), D_x u(x)\rangle_{\mathcal{H}} (w^{(n)}-w)(x)\mu(dx)\Bigg|$$
$$+\Bigg|\int_{\mathcal{H}} Tr[P_n D_x\sigma^{(n)}(x)]_{\mathcal{H}}\langle\sigma^{(n)}(x)-\sigma(x), D_x u(x)\rangle_{\mathcal{H}} w(x)\mu(dx)\Bigg|$$
$$+\Bigg|\int_{\mathcal{H}} Tr[P_n D_x\sigma^{(n)}(x)-D_x\sigma(x)]_{\mathcal{H}}\langle\sigma(x), D_x u(x)\rangle_{\mathcal{H}} w(x)\mu(dx)\Bigg|$$
$$\leq b_\sigma\, B_\gamma\,\|Q\|_{tc}\int_{\mathcal{H}} \|D_x u(x)\|_{\mathcal{H}}|w^{(n)}-w|(x)\mu(dx)$$
$$+B_\gamma\,\|Q\|_{tc}\int_{\mathcal{H}} \|\sigma^{(n)}(x)-\sigma(x)\|_{\mathcal{H}}\,\|D_x u(x)\|_{\mathcal{H}}|w(x)|\mu(dx)$$
$$+b_\sigma\int_{\mathcal{H}} \big|Tr[P_n D_x\sigma^{(n)}(x)-D_x\sigma(x)]_{\mathcal{H}}\big|\,\|D_x u(x)\|_{\mathcal{H}}|w(x)|\mu(dx),$$

and hence from the same arguments as above we obtain the convergence

$$\int_{\mathcal{H}} Tr[P_n D_x\sigma^{(n)}(x)]_{\mathcal{H}}\langle\sigma^{(n)}(x), D_x u(x)\rangle_{\mathcal{H}} w^{(n)}(x)\mu(dx) \to$$
$$\to \int_{\mathcal{H}} Tr[D_x\sigma(x)]_{\mathcal{H}}\langle\sigma(x), D_x u(x)\rangle_{\mathcal{H}} w(x)\mu(dx)$$

The analysis of the (D) term is quite simple at this stage, since we are dealing with the bounded linear operator A_α. More sophisticated issues will arise when taking the limit as $\alpha\to\infty$. We have the following estimate

$$
\begin{aligned}
&\left|\int_{\mathcal{H}} \langle A_{n,\alpha}x, D_x u(x)\rangle_{\mathcal{H}} w^{(n)}(x)\mu(dx) - \int_{\mathcal{H}} \langle A_{\alpha}x, D_x u(x)\rangle_{\mathcal{H}} w(x)\mu(dx)\right| \\
&\quad \le \left|\int_{\mathcal{H}} \langle A_{n,\alpha}x, D_x u(x)\rangle_{\mathcal{H}} (w^{(n)} - w)(x)\mu(dx)\right| \\
&\quad + \left|\int_{\mathcal{H}} \langle (A_{n,\alpha} - A_{\alpha})x, D_x u(x)\rangle_{\mathcal{H}} w(x)\mu(dx)\right| \\
&\quad \le \|A_{\alpha}\|_{\mathcal{L}} \left(\int_{\mathcal{H}} \|x\|_{\mathcal{H}}^4 \mu(dx)\right)^{\frac{1}{4}} \|w^{(n)} - w\|_{L^p(\mathcal{H},\mu)} \|D_x u\|_{L^2(\mathcal{H},\mu)} \\
&\quad + \|(A_{n,\alpha} - A_{\alpha})\|_{\mathcal{L}} \left(\int_{\mathcal{H}} \|x\|_{\mathcal{H}}^4 \mu(dx)\right)^{\frac{1}{4}} \|w\|_{L^p(\mathcal{H},\mu)} \|D_x u\|_{L^2(\mathcal{H},\mu)}
\end{aligned}
$$

From identical arguments as above and from the convergence $\|A_{n,\alpha} - A_{\alpha}\|_{\mathcal{L}} \to 0$ as $n \to \infty$ we simply get the result

$$
\int_{\mathcal{H}} \langle A_{n,\alpha}x, D_x u(x)\rangle_{\mathcal{H}} w^{(n)}(x)\mu(dx) \to \int_{\mathcal{H}} \langle A_{\alpha}x, D_x u(x)\rangle_{\mathcal{H}} w(x)\mu(dx),
$$

as $n \to \infty$. It is worth noticing that from monotone convergence we have

$$
\int_{\mathcal{H}} \|x\|_{\mathcal{H}}^4 \mu(dx) = \int_{\mathcal{H}} \sum_{i,j=1}^{\infty} |x_i|^2 \, |x_j|^2 \mu(dx) = \sum_{i,j=1}^{\infty} \int_{\mathcal{H}} |x_i|^2 \, |x_j|^2 \mu(dx) = [TrQ]^2.
$$

The estimates on the (E) term rely on the same arguments as above. We have

$$
\begin{aligned}
&\Bigg| \int_{\mathcal{H}} \langle \sigma^{(n)}\sigma^{(n)*}(x) Q_n^{-1} x, D_x u(x)\rangle_{\mathcal{H}} w^{(n)}(x)\mu(dx) \\
&\qquad - \int_{\mathcal{H}} \langle \sigma\sigma^*(x) Q^{-1} x, D_x u(x)\rangle_{\mathcal{H}} w(x)\mu(dx)\Bigg| \\
&= \Bigg| \int_{\mathcal{H}} \langle \sigma^{(n)}(x), D_x u(x)\rangle_{\mathcal{H}} \langle \gamma^{(n)}(x), x\rangle_{\mathcal{H}} w^{(n)}(x)\mu(dx) \\
&\quad - \int_{\mathcal{H}} \langle \sigma(x), D_x u(x)\rangle_{\mathcal{H}} \langle \gamma(x), x\rangle_{\mathcal{H}} w(x)\mu(dx)\Bigg| \\
&\le \int_{\mathcal{H}} \left|\langle \sigma^{(n)}(x), D_x u(x)\rangle_{\mathcal{H}} \langle \gamma^{(n)}(x), x\rangle_{\mathcal{H}} (w^{(n)} - w)(x)\right| \mu(dx) \\
&\quad + \int_{\mathcal{H}} \left|\langle \sigma^{(n)}(x) - \sigma(x), D_x u(x)\rangle_{\mathcal{H}} \langle \gamma^{(n)}(x), x\rangle_{\mathcal{H}} w(x)\right| \mu(dx) \\
&\quad + \int_{\mathcal{H}} \left|\langle \sigma(x), D_x u(x)\rangle_{\mathcal{H}} \langle \gamma^{(n)}(x) - \gamma(x), x\rangle_{\mathcal{H}} w(x)\right| \mu(dx).
\end{aligned}
$$

All the terms converge to zero from the same arguments used above. Then we have

$$\int_{\mathcal{H}} \langle \sigma^{(n)}\sigma^{(n)*}(x)Q_n^{-1}x, D_x u(x)\rangle_{\mathcal{H}} w^{(n)}(x)\mu(dx) \to$$
$$\to \int_{\mathcal{H}} \langle \sigma\sigma^*(x)Q^{-1}x, D_x u(x)\rangle_{\mathcal{H}} w(x)\mu(dx).$$

For $t \in [0,T]$ given and for any $w \in \mathcal{V}^p$, we define $\mathcal{A}^w_{t,n} \in \mathcal{V}^{p*}$ as $\mathcal{A}^w_{t,n}(\cdot) := a^{(n)}_\mu(t;\,\cdot\,, w^{(n)})$. Moreover we define $\mathcal{A}^w_t \in \mathcal{V}^{p*}$ as $\mathcal{A}^w_t(\cdot) := a_\mu(t;\,\cdot\,, w)$, where

$$\begin{aligned} a_\mu(t;u,w) &= \frac{1}{2}\int_{\mathcal{H}} \langle \sigma\sigma^*(x) D_x u(t,x), D_x w(t,x)\rangle_{\mathcal{H}} \mu(dx) \\ &+\frac{1}{2}\int_{\mathcal{H}} \langle D_x\sigma(x)\cdot\sigma(x), D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx) \\ &+\frac{1}{2}\int_{\mathcal{H}} Tr[D_x\sigma(x)]_{\mathcal{H}} \langle \sigma(x), D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx) \\ &-\int_{\mathcal{H}} \langle A_\alpha x, D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx) \\ &+\frac{1}{2}\int_{\mathcal{H}} \langle \sigma\sigma^*(x)Q^{-1}x, D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx). \end{aligned}$$

We can state the following proposition,

Proposition 6.2.1 *For any $w \in \mathcal{V}^p$ given, and for $\mathcal{A}^w_t \in \mathcal{V}^{p*}$ defined as $\mathcal{A}^w_t(\cdot) := a_\mu(t;\,\cdot\,, w)$, the sequence $\{\mathcal{A}^w_{t,n}\}_{n=1}^\infty$ converges in the following sense*

$$\lim_{n\to\infty} \|\mathcal{A}^w_{t,n} - \mathcal{A}^w_t\|_{\mathcal{V}^{p*}} = 0 \qquad \forall t \in [0,T].$$

PROOF: For $u \in \mathcal{V}^p$, if we take into account also the terms involving ϵ_n, we can summarize the previous estimates as follows

$$|\mathcal{A}^w_{t,n}(u) - \mathcal{A}^w_t(u)| \le (\eta_n + o(\epsilon_n))\, |||u|||_p\,.$$

where $\eta_n \to 0$ as $n \to \infty$. In fact we have

$$\begin{aligned}
\eta_n \;=\; & (b_\sigma + 2b_\sigma \cdot B_\gamma \cdot \|Q\|_{tc} + \|A_\alpha\|_{\mathcal{L}}\|Q\|_{tc}^{\frac{1}{2}} + b_\sigma \cdot b_\gamma \cdot \|Q\|_{tc}^{\frac{1}{2}}) \left|\left|\left|w^{(n)} - w\right|\right|\right|_p \\
& + \left(\int_{\mathcal{H}} \|\sigma^{(n)}\sigma^{(n)*}(x) - \sigma\sigma^*(x)\|_{\mathcal{L}}^4 \mu(dx)\right)^{\frac{1}{4}} |||w|||_p \\
& + \left(\int_{\mathcal{H}} \|D_x\sigma^{(n)}(x) \cdot \sigma^{(n)}(x) - D_x\sigma(x) \cdot \sigma(x)\|_{\mathcal{H}}^4 \mu(dx)\right)^{\frac{1}{4}} |||w|||_p \\
& + B_\gamma \cdot \|Q\|_{tc} \left(\int_{\mathcal{H}} \|\sigma^{(n)}(x) - \sigma(x)\|_{\mathcal{H}}^4 \mu(dx)\right)^{\frac{1}{4}} |||w|||_p \\
& + b_\sigma \left(\int_{\mathcal{H}} \left|Tr[P_n D_x\sigma^{(n)}(x) - D_x\sigma(x)]_{\mathcal{H}}\right|^4 \mu(dx)\right)^{\frac{1}{4}} |||w|||_p \\
& + \|(A_{n,\alpha} - A_\alpha)\|_{\mathcal{L}}\|Q\|_{tc}^{\frac{1}{2}} |||w|||_p \\
& + b_\gamma \left(\int_{\mathcal{H}} \|\sigma^{(n)}(x) - \sigma(x)\|_{\mathcal{H}}^2 \|x\|_{\mathcal{H}}^2 |w(x)|^2 \mu(dx)\right)^{\frac{1}{2}} \\
& + b_\sigma \left(\int_{\mathcal{H}} \|\gamma^{(n)}(x) - \gamma(x)\|_{\mathcal{H}}^2 \|x\|_{\mathcal{H}}^2 |w(x)|^2 \mu(dx)\right)^{\frac{1}{2}}.
\end{aligned}$$

This concludes the proof. ∎

If we reintroduce the time dependence again, we obtain the same results as above provided that for any $w \in L^2(0,T;\mathcal{V}^p)$ and $\frac{\partial w}{\partial t} \in L^2(0,T;L^2(\mathcal{H},\mu))$ there exists a sequence $\{w_n\}_{n=1}^\infty$, $w_n \in L^2(0,T;\mathcal{V}_n^p)$ and $\frac{\partial w_n}{\partial t} \in L^2(0,T;L^2_\mu(\mathbb{R}^n))$ such that

$$w_n \to w, \qquad \textit{in } L^2(0,T;\mathcal{V}_n^p)$$

and

$$\frac{\partial w_n}{\partial t} \to \frac{\partial w}{\partial t} \qquad \textit{in } L^2(0,T;L^2_\mu(\mathbb{R}^n)),$$

cf. Section 6.3.1. Then from natural extension of the previous results we obtain the following propositions.

Proposition 6.2.2 *For any* $w \in L^2(0,T;\mathcal{V}^p)$ *and* $\frac{\partial w}{\partial t} \in L^2(0,T;L^2(\mathcal{H},\mu))$ *given, and for* $\mathcal{A}^w \in L^2(0,T;\mathcal{V}^{p*})$ *defined as*

$$\mathcal{A}^w(\cdot) := \int_0^T a_\mu(t;\,\cdot\,,w(t))dt,$$

the sequence $\{\mathcal{A}_n^w\}_{n=1}^\infty$ *defined as*

$$\mathcal{A}_n^w(\cdot) := \int_0^T a_\mu^{(n)}(t;\,\cdot\,, w^{(n)}(t))dt,$$

converges in the strong sense

$$\lim_{n\to\infty} \|\mathcal{A}_n^w - \mathcal{A}^w\|_{L^2(0,T;\mathcal{V}^{p\,*})} = 0.$$

An analogous result holds for the dual pairing term.

Proposition 6.2.3 *Let* $\mathcal{A}^\Theta \in L^2(0,T;\mathcal{V}^{p\,*})$ *be defined as*

$$\mathcal{A}^\Theta(\cdot) := \int_0^T \left[(\frac{\partial \Theta}{\partial t}(t),\,\cdot\,)_{L^2(\mathcal{H},\mu)} - a_\mu(t;\Theta(t),\,\cdot\,)\right] dt.$$

The sequence $\{\mathcal{A}_n^\Theta\}_{n=1}^\infty$ *defined as*

$$\mathcal{A}_n^\Theta(\cdot) := \int_0^T \left[(\frac{\partial \Theta^{(n)}}{\partial t}(t),\,\cdot\,)_{L^2(\mathcal{H},\mu)} - a_\mu^{(n)}(t;\Theta^{(n)}(t),\,\cdot\,)\right] dt,$$

converges in the strong sense

$$\lim_{n\to\infty} \|\mathcal{A}_n^\Theta - \mathcal{A}^\Theta\|_{L^2(0,T;\mathcal{V}^{p\,*})} = 0.$$

PROOF: In light of the results of Proposition 6.2.2 it is enough to prove that $\Theta^{(n)} \to \Theta$ in $L^2(0,T;\mathcal{V}^p)$ and $\frac{\partial \Theta^{(n)}}{\partial t} \to \frac{\partial \Theta}{\partial t}$ in $L^2(0,T;L^2(\mathcal{H},\mu))$. At this point we realize the importance of the smoothing procedure for the gain function. Indeed we exploit the fact that for some $k \in \mathbb{N}$ fixed earlier we have $\Theta = \Psi_k \in C^1([0,T]\times\mathcal{H})$. Then at the final dimensional level we have $\Theta^{(n)}(t,x) = \Psi_k^{(n)}(t,x) = \Psi_k(t,P_n x) \in C^1([0,T]\times\mathbb{R}^n) \subset C^1([0,T]\times\mathcal{H})$. We denote the linear operator of first derivative as $D_{t,x} := (\frac{\partial}{\partial t}, D)$. The continuity of Ψ_k and of its first derivatives guarantee the pointwise convergences $\Psi_k(t,P_n x) \to \Psi_k(t,x) = \Theta(t,x)$ and $D_{t,x}\Psi_k(t,P_n x) \to D_{t,x}\Psi_k(t,x) = D_{t,x}\Theta(t,x)$, as $n\to\infty$. The uniform estimates obtained in the proof of Proposition 4.1.1 enable us to use dominated convergence to obtain $\Theta^{(n)} \to \Theta$ in $L^2(0,T;\mathcal{V}^p)$ and $\frac{\partial \Theta^{(n)}}{\partial t} \to \frac{\partial \Theta}{\partial t}$ in $L^2(0,T;L^2(\mathcal{H},\mu))$. ■

6.3 Infinite dimensional variational inequality - part 1

Before stating the main result of this section we recall the properties of the value function $\bar{v}_\alpha^{(n)}$ which we characterized as a weak solution of the n-th finite dimensional EVI. From both analytical and probabilistic results we know that $\bar{v}_\alpha^{(n)} \in L^2(0,T;\mathcal{V}^p)$ and in particular that $\{\bar{v}_\alpha^{(n)}\}_{n=1}^\infty$ is a uniformly bounded sequence in the set $L^2(0,T;W^{1,2}(\mathcal{H},\mu))$. In fact from analogous rationale as in Proposition 2.2.2 we get uniform Lipschitz property and then the same arguments as in Proposition 5.2.1 guarantee

$$\int_0^T \int_{\mathcal{H}} |\bar{v}_\alpha^{(n)}(t,x)|^2 \mu + \int_0^T \int_{\mathcal{H}} \|D\bar{v}_\alpha^{(n)}(t,x)\|^2_{\mathcal{H}} \mu(dx)dt \leq (\overline{\Psi}^2 + L_V^2)T.$$

Then there exists $\bar{v}_\alpha \in L^2(0,T;W^{1,2}(\mathcal{H},\mu))$ such that there exists a subsequence $\bar{v}_\alpha^{(n_j)} \rightharpoonup \bar{v}_\alpha$ in $L^2(0,T;W^{1,2}(\mathcal{H},\mu))$ as $j \to \infty$. From Corollary 3.2.2 we also know that $\bar{v}_\alpha^{(n)} \to V_\alpha$ in $L^2(0,T;L^p(\mathcal{H},\mu))$, $1 \leq p < \infty$ and hence from uniqueness of the limit

$$\bar{v}_\alpha(t,x) = V_\alpha(t,x) = \sup_{t\leq\tau\leq T} \mathbb{E}\left[\Theta(\tau, X_\tau^{(\alpha)\,t,x})\right], \quad \forall(t,x) \in [0,T]\times\mathcal{H}.$$

Another important issue that we obtain from the probabilistic analysis (cf. Theorem 3.2.1) is that any sequence $\{\bar{v}_\alpha^{(n_j)}\}_{j=1}^\infty$ is uniformly converging on any compact subset $[0,T]\times\mathcal{K}$ of the whole space. From now on we will denote such a subsequence simply by $\{\bar{v}_\alpha^{(n)}\}_{n=1}^\infty$. It is straightforward to see that in the homogenized problem we have $\bar{u}_\alpha^{(n)} = \bar{v}_\alpha^{(n)} - \Theta^{(n)}$ and the same convergence results stated above hold.

We rename the convex sets for the homogeneous obstacle problems as

$$\hat{\mathcal{K}}^p_{\mu,n} := \{w : w \in L^2(0,T;\mathcal{V}^p_n),\ \frac{\partial w}{\partial t} \in L^2(0,T;L^2_\mu(\mathbb{R}^n)),\ w \geq 0\ \textit{a.e.}\ [0,T]\times\mathbb{R}^n\}$$

and

$$\hat{\mathcal{K}}^p_{\mu,\infty} := \{w : w \in L^2(0,T;\mathcal{V}^p),\ \frac{\partial w}{\partial t} \in L^2(0,T;L^2(\mathcal{H},\mu)),\ w \geq 0\ \textit{a.e.}\ [0,T]\times\mathcal{H}\}.$$

From the results of Proposition 6.2.2 and 6.2.3, in order to pass to the limit in the homogenized variational inequality, it is fundamental to provide a suitable sequence of test functions $w^{(n)}$. Each of these functions has to be in the n-th convex set $\hat{\mathcal{K}}^p_{\mu,n}$.

6.3.1 Sequence of test functions

Let w be an arbitrary element of the convex set $\hat{\mathcal{K}}^p_{\mu,\infty}$. From the discussion in Appendix E and adopting the notation therein we know that $\hat{\mathcal{K}}^p_{\mu,\infty} \subset W^{1,2}([0,T]\times\mathcal{H}, \lambda(dt)\times\mu(dx))$. We introduce the set

$$\mathcal{E}([0,T]\times\mathcal{H}) := \text{span}\{\mathscr{R}e(\phi_{\eta,h}),\ \mathscr{I}m(\phi_{\eta,h}),\ \phi_{\eta,h}(t,x) = e^{i\eta t + i\langle h,x\rangle_{\mathcal{H}}},\ (\eta,h)\in\mathbb{R}\times\mathcal{H}\}.$$

Such a set is dense in $W^{1,2}([0,T]\times\mathcal{H}, \lambda(dt)\times\mu(dx))$ and in $L^2(0,T;L^p(\mathcal{H},\mu))$, cf. Appendix E. Hence we know that there exists a sequence $\{\phi^{(k)}\}_{k=1}^{\infty}$, $\phi^{(k)}\in\mathcal{E}([0,T]\times\mathcal{H})$, such that

$$\begin{gathered}\int_0^T \|\phi^{(k)}(t) - w(t)\|^2_{\mathcal{V}^p} dt \to 0,\\ \int_0^T \|\frac{\partial\phi^{(k)}}{\partial t}(t) - \frac{\partial w}{\partial t}(t)\|^2_{L^2(\mathcal{H},\mu)} dt \to 0,\end{gathered} \tag{6.2}$$

as $k\to\infty$. Moreover, up to a subsequence, the convergence holds in the a.e. sense.

Let us now fix $k\in\mathbb{N}$, then there exist $N_k\in\mathbb{N}$, $a,b,\eta\in\mathbb{R}^{N_k}$ and a sequence $\{h_i\}_{i=1}^{N_k}\subset\mathcal{H}$, such that

$$\phi^{(k)}(t,x) = \sum_{i=1}^{N_k}[a_i\cos(\eta_i t + \langle h_i,x\rangle_{\mathcal{H}}) + b_i\sin(\eta_i t + \langle h_i,x\rangle_{\mathcal{H}})].$$

We can take a finite dimensional projection. Let $n\in\mathbb{N}$, then let us define $\phi^{(k)}_n(t,x) := \phi^{(k)}(t,P_n x)$ as

$$\phi^{(k)}(t,P_n x) = \sum_{i=1}^{N_k}[a_i\cos(\eta_i t + \langle h_i,P_n x\rangle_{\mathcal{H}}) + b_i\sin(\eta_i t + \langle h_i,P_n x\rangle_{\mathcal{H}})].$$

If $|\cdot|_k$ represents the Euclidean norm in $\mathbb{R}^{N_k}$, the uniform boundedness holds

$$\sup_{(t,x)\in[0,T]\times\mathcal{H}} |\phi^{(k)}(t,P_n x)| \le C(k,|a|_k,|b|_k),$$

independent of the order n. It is worth noticing also that, if we define $D_{t,x} := (\frac{\partial}{\partial t}, D_x)$ and $\|\cdot\|_{\mathbb{R}\times\mathcal{H}} := |\cdot| + \|\cdot\|_{\mathcal{H}}$, then it is easy to prove

$$\sup_{(t,x)\in[0,T]\times\mathcal{H}} \|D_{t,x}\cos(\eta_i t + \langle h_i,P_n x\rangle_{\mathcal{H}})\|_{\mathbb{R}\times\mathcal{H}} \le C(k,|\eta_i|,\|h_i\|_{\mathcal{H}}).$$

The bound does not depend on n. It clearly extends to finite linear combinations and hence for $\|h\|_k := \sum_{i=1}^{N_k} \|h_i\|_{\mathcal{H}}$, we have

$$\sup_{(t,x)\in[0,T]\times\mathcal{H}} \|D_{t,x}\phi^{(k)}(t, P_n x)\|_{\mathbb{R}\times\mathcal{H}} \leq C(k, |a|_k, |b|_k, |\eta|_k, \|h\|_k).$$

Now from dominated convergence and continuity of $\phi^{(k)}$ and $D_{t,x}\phi^{(k)}$ we obtain

$$\int_0^T \|\phi_n^{(k)}(t) - \phi^{(k)}(t)\|^2_{\mathcal{V}^p} dt + \int_0^T \|\frac{\partial \phi_n^{(k)}}{\partial t}(t) - \frac{\partial \phi^{(k)}}{\partial t}(t)\|^2_{L^2(\mathcal{H},\mu)} dt \to 0, \quad \text{as } n \to \infty. \tag{6.3}$$

We define $\phi_{n,0}^{(k)} := 0 \vee \phi_n^{(k)} = (\phi_n^{(k)})^+$. The importance of this sequence of functions is clear if we notice that for k fixed the function $\phi_{n,0}^{(k)}$ belongs to the convex sets $\hat{\mathcal{K}}^p_{\mu,n}$, for all $n \in \mathbb{N}$ and $1 \leq p < \infty$. Actually we have got an even nicer property; in fact for $k, n \in \mathbb{N}$ both fixed, $\phi_{n,0}^{(k)} \in \hat{\mathcal{K}}^p_{\mu,d}$ for all $d \geq n$. We want to study now the convergence properties with respect to both the indexes.

We easily get

$$|\phi_{n,0}^{(k)}(t,x)| \leq |\phi_n^{(k)}(t,x)| \leq C(k, |a|_k, |b|_k), \qquad (t,x) \in [0,T] \times \mathcal{H}, \tag{6.4}$$

and

$$\sup_{(t,x)\in[0,T]\times\mathcal{H}} |\phi_{n,0}^{(k)}(t,x)| \leq \sup_{(t,x)\in[0,T]\times\mathcal{H}} |\phi^{(k)}(t,x)| \leq C(k, |a|_k, |b|_k). \tag{6.5}$$

From general results about weighted Sobolev spaces in $\mathbb{R}^n$ (cf. [54], Corollary 2.1.8, Chapter 2) we also know that

$$D_{t,x}\phi_{n,0}^{(k)} = \begin{cases} D_{t,x}\phi_n^{(k)} & \text{on } \{\phi_n^{(k)} \geq 0\}, \\ 0 & \text{elsewhere}. \end{cases}$$

Equation (6.4) guarantees that

$$\int_0^T \left\|\phi_{n,0}^{(k)}(t)\right\|^2_{L^p(\mathcal{H},\mu)} dt \leq 2\|\phi_n^{(k)}\|_{L^2(0,T;L^p(\mathcal{H},\mu))}.$$

From the convergence (6.3), for $k \in \mathbb{N}$ fixed, we can provide the bound

$$\int_0^T \|\phi_n^{(k)}(t)\|_{\mathcal{V}^p}^2 dt + \int_0^T \|\frac{\partial \phi_n^{(k)}}{\partial t}(t)\|_{L^2(\mathcal{H},\mu)}^2 dt \leq C(k).$$

As usual we use the isometry $\|\cdot\|_{\mathbb{R}^n} = \|\cdot\|_{\mathcal{H}}$.

$$\begin{aligned}
&\int_{[0,T]\times\mathbb{R}^n} \left[\left|\frac{\partial \phi_{n,0}^{(k)}}{\partial t}(t,x)\right|^2 + \|D_x \phi_{n,0}^{(k)}(t,x)\|_{\mathbb{R}^n}^2 \right] \mu(dx)dt \\
&= \int_{[0,T]\times\mathbb{R}^n} \left[\left|\frac{\partial \phi_{n,0}^{(k)}}{\partial t}(t,x)\right|^2 + \|D_x \phi_{n,0}^{(k)}(t,x)\|_{\mathcal{H}}^2 \right] \mu(dx)dt \\
&= \int_{\{[0,T]\times\mathbb{R}^n\}\cap\{\phi_n^{(k)}\geq 0\}} \left[\left|\frac{\partial \phi_n^{(k)}}{\partial t}(t,x)\right|^2 + \|D_x \phi_n^{(k)}(t,x)\|_{\mathcal{H}}^2 \right] \mu(dx)dt \\
&\leq \int_0^T \|\phi_n^{(k)}(t)\|_{\mathcal{V}^p}^2 dt + \int_0^T \|\frac{\partial \phi_n^{(k)}}{\partial t}(t)\|_{L^2(\mathcal{H},\mu)}^2 dt \leq C(k).
\end{aligned}$$

From equation (6.5) and pointwise convergence we also get

$$\lim_{n\to\infty} \int_0^T \|\phi_{n,0}^{(k)}(t) - 0 \vee \phi^{(k)}(t)\|_{L^p(\mathcal{H},\mu)}^2 dt = 0.$$

Then summarizing we have

$$\|\phi_{n,0}^{(k)}\|_{W^{1,2}([0,T]\times\mathcal{H},\lambda\times\mu)}^2 \leq C(k),$$

hence there exists $f \in W^{1,2}([0,T]\times\mathcal{H},\lambda\times\mu)$ and a subsequence $\{\phi_{n_j,0}^{(k)}\}_{j=1}^{\infty}$, such that $\phi_{n_j,0}^{(k)} \rightharpoonup f$ in $W^{1,2}([0,T]\times\mathcal{H},\lambda\times\mu)$, as $j\to\infty$. Moreover $\phi_{n,0}^{(k)} \to 0\vee\phi^{(k)}$ in $L^p([0,T]\times\mathcal{H},\lambda\times\mu)$ as $n\to\infty$ and hence $f = 0\vee\phi^{(k)}$. It is worth stressing that the subsequence $\{n_j\}_{j\in\mathbb{N}}$ depends on the index k, i.e. to be more precise we should write $\{n_j^k\}_{j\in\mathbb{N}}$. Yet, since the dependence on k is explicitly taken into account in the upper index of the approximating function, we denoted $n_j := n_j^k$ and consequently $\phi_{n_j^k,0}^{(k)} := \phi_{n_j,0}^{(k)}$.

From the same arguments as in Appendix A and [42] we know that the following representation holds

$$D_{t,x}(\phi^{(k)})^+(t,x) = I_{\{\phi^{(k)}\geq 0\}}(t,x) D_{t,x}\phi^{(k)}(t,x), \quad \lambda\times\mu\text{-a.e. } (t,x)\in[0,T]\times\mathcal{H}.$$

Hence from the same arguments as above and convergence (6.2) we obtain

$$\begin{aligned}
&\int_{[0,T]\times\mathcal{H}} \left[\left| \frac{\partial(\phi^{(k)})^+}{\partial t}(t,x) \right|^2 + \|D_x(\phi^{(k)})^+(t,x)\|_{\mathcal{H}}^2 \right] \mu(dx)dt \\
&= \int_{[0,T]\times\mathcal{H}} I_{\{\phi^{(k)}\geq 0\}}(t,x) \left[\left| \frac{\partial\phi^{(k)}}{\partial t}(t,x) \right|^2 + \|D_x\phi^{(k)}(t,x)\|_{\mathcal{H}}^2 \right] \mu(dx)dt \\
&\leq \int_0^T \|\phi^{(k)}(t)\|_{\mathcal{V}^p}^2 dt + \int_0^T \left\| \frac{\partial\phi^{(k)}}{\partial t}(t) \right\|_{L^2(\mathcal{H},\mu)}^2 dt \leq C.
\end{aligned}$$

It then implies that there exists $g \in W^{1,2}([0,T]\times\mathcal{H}, \lambda\times\mu)$ and a subsequence $\{\phi^{(k_j)}\}_{j=1}^{\infty}$ such that $0 \vee \phi^{(k_j)} \rightharpoonup g$ as $j \to \infty$ in $W^{1,2}([0,T]\times\mathcal{H}, \lambda\times\mu)$. We also have that $0 \vee \phi^{(k_j)} \to 0 \vee w$, as $j \to \infty$ in $L^2(0,T; L^p(\mathcal{H},\mu))$. So we conclude that $g = 0 \vee w = w$.

Summarizing, we have that, up to suitable subsequences, taking the limit with respect to n first and with respect to k later, we get

$$\phi_{n,0}^{(k)} \rightharpoonup w, \qquad \text{in } W^{1,2}([0,T]\times\mathcal{H}, \lambda\times\mu),$$

$$\phi_{n,0}^{(k)} \to w, \qquad \text{in } L^2(0,T; L^p(\mathcal{H},\mu)),$$

$$\phi_{n,0}^{(k)}(t) \to w(t), \qquad \text{in } L^p(\mathcal{H},\mu), \text{ a.e. } t \in [0,T].$$

6.3.2 The limit to infinite dimensions

We are now ready to present the main theorem of this section. We recall the explicit expression for the bilinear form

$$\begin{aligned}
a_\mu^{(\alpha)}(t;\, u, w) \;=\; & \frac{1}{2}\int_{\mathcal{H}} \langle \sigma\sigma^*(x) D_x u(t,x), D_x w(t,x)\rangle_{\mathcal{H}} \mu(dx) \\
& + \frac{1}{2}\int_{\mathcal{H}} \langle D_x\sigma(x)\cdot\sigma(x), D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx) \\
& + \frac{1}{2}\int_{\mathcal{H}} Tr[D_x\sigma(x)]_{\mathcal{H}} \langle \sigma(x), D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx)
\end{aligned}$$

$$-\int_{\mathcal{H}} \langle A_\alpha x, D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx)$$

$$+\frac{1}{2}\int_{\mathcal{H}} \langle \sigma\sigma^*(x)Q^{-1}x, D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx).$$

The theorem is as follows:

Theorem 6.3.1 *The value function of the optimal stopping problem*

$$V_\alpha(t,x) = \sup_{t\le\tau\le T} \mathbb{E}\left[\Theta(\tau, X_\tau^{(\alpha)\,t,x})\right],$$

is a weak solution of the variational problem: find $v \in L^2(0,T;\mathcal{V}^p)$, $v \geq \Theta$ *such that*

$$-\int_0^T \big(\frac{\partial w}{\partial t}, w-v\big)_{L^2(\mathcal{H},\mu)}dt + \int_0^T a_\mu^{(\alpha)}(t;v,w-v)dt + \frac{1}{2}\|w(T)-\Theta(T)\|^2_{L^2(\mathcal{H},\mu)} \geq 0,$$

for all $w \in \hat{\mathcal{K}}^{\Theta,p}_{\mu,\infty}$, $1 \le p < \infty$.

PROOF: We prove the convergence of the variational inequality for the homogenized case and then pass to the non homogeneous one. For arbitrary $w \in \hat{\mathcal{K}}^p_{\mu,\infty}$, we take the approximating sequence $\{\phi^{(k)}_{n,0}\}_{k,n}$ introduced above. For k,n fixed the function $\phi^{(k)}_{n,0}$ belongs to $\hat{\mathcal{K}}^p_{\mu,d}$ for all $d \geq n$. As usual we denote $\Theta^{(d)}(t) := \Theta(t)\circ P_d$, for the gain function of the d-dimensional problem. Hence for any $d \geq n$ we have

$$-\int_0^T \big(\frac{\partial \phi^{(k)}_{n,0}}{\partial t}, \phi^{(k)}_{n,0} - \bar{u}^{(d)}_\alpha\big)_{L^2(\mathcal{H},\mu)}dt + \int_0^T a_\mu^{(\alpha,d)}(t;\bar{u}^{(d)}_\alpha, \phi^{(k)}_{n,0} - \bar{u}^{(d)}_\alpha)dt$$
$$-\mathcal{A}^\Theta_d(\phi^{(k)}_{n,0} - \bar{u}^{(d)}_\alpha) + \frac{1}{2}\|\phi^{(k)}_{n,0}(T)\|^2_{L^2(\mathcal{H},\mu)} \geq 0.$$

Here $\mathcal{A}^\Theta_d$ is the one introduced in Proposition 6.2.3. We keep k and n fixed and take the limit as $d \to \infty$. We exploit the results of Propositions 6.2.2 and 6.2.3 and similar arguments to those adopted to prove Theorem 5.4.1. We are allowed to do so because all of our estimates on the bilinear form and on the bounds for the solutions are independent of the number of dimensions in space. It is worth noticing that in particular

$$a_\mu^{(\alpha,d)}(t;\bar{u}^{(d)}_\alpha,\bar{u}^{(d)}_\alpha) = a_\mu^{(\alpha,d)}(t;\bar{u}^{(d)}_\alpha - \bar{u}_\alpha,\bar{u}^{(d)}_\alpha - \bar{u}_\alpha) + a_\mu^{(\alpha,d)}(t;\bar{u}_\alpha,\bar{u}^{(d)}_\alpha) + a_\mu^{(\alpha,d)}(t;\bar{u}^{(d)}_\alpha - \bar{u}_\alpha,\bar{u}_\alpha).$$

Since from lower semicontinuity of the weak limit $D\bar{u}_\alpha$ is bounded in $L^2(\mathcal{H},\mu;\mathcal{H})$, recalling equation (5.15), we obtain

$$a_\mu^{(\alpha,d)}(t;\bar{u}_\alpha^{(d)},\bar{u}_\alpha^{(d)}) \geq -\Lambda\|\bar{u}_\alpha^{(d)}-\bar{u}_\alpha\|_{L^p(\mathcal{H},\mu)} + a_\mu^{(\alpha,d)}(t;\bar{u}_\alpha,\bar{u}_\alpha^{(d)}) + a_\mu^{(\alpha,d)}(t;\bar{u}_\alpha^{(d)}-\bar{u}_\alpha,\bar{u}_\alpha).$$

Hence from weak convergence of solutions $\bar{u}_\alpha^{(d)}$ we obtain, up to a subsequence, that

$$\begin{aligned}
\int_0^T \big(\frac{\partial\phi_{n,0}^{(k)}}{\partial t},\phi_{n,0}^{(k)}-\bar{u}_\alpha^{(d)}\big)_{L^2(\mathcal{H},\mu)}dt &\to \int_0^T \big(\frac{\partial\phi_{n,0}^{(k)}}{\partial t},\phi_{n,0}^{(k)}-\bar{u}_\alpha\big)_{L^2(\mathcal{H},\mu)}dt,\\
\int_0^T a_\mu^{(\alpha,d)}(t;\bar{u}_\alpha^{(d)},\phi_{n,0}^{(k)})dt &\to \int_0^T a_\mu^{(\alpha)}(t;\bar{u}_\alpha,\phi_{n,0}^{(k)})dt,\\
\mathcal{A}_d^\Theta(\phi_{n,0}^{(k)}-\bar{u}_\alpha^{(d)}) &\to \mathcal{A}^\Theta(\phi_{n,0}^{(k)}-\bar{u}_\alpha),\\
\lim_{d\to\infty}\int_0^T a_\mu^{(\alpha,d)}(t;\bar{u}_\alpha^{(d)},\bar{u}_\alpha^{(d)})dt &\geq \int_0^T a_\mu^{(\alpha)}(t;\bar{u}_\alpha,\bar{u}_\alpha)dt.
\end{aligned}$$

Then in summary we have

$$\begin{aligned}
-\int_0^T \big(\frac{\partial\phi_{n,0}^{(k)}}{\partial t},\phi_{n,0}^{(k)}-\bar{u}_\alpha\big)_{L^2(\mathcal{H},\mu)}dt + \int_0^T a_\mu^{(\alpha)}(t;\bar{u}_\alpha,\phi_{n,0}^{(k)}-\bar{u}_\alpha)dt&\\
-\mathcal{A}^\Theta(\phi_{n,0}^{(k)}-\bar{u}_\alpha) + \frac{1}{2}\|\phi_{n,0}^{(k)}(T)\|^2_{L^2(\mathcal{H},\mu)} &\geq 0.
\end{aligned}$$

Now we take the limits in the order

$$\lim_{k\to\infty}\lim_{n\to\infty}$$

along the subsequences discussed in Section 6.3.1. Exploiting the results therein, we obtain

$$\begin{aligned}
-\int_0^T \big(\frac{\partial w}{\partial t},w-\bar{u}_\alpha\big)_{L^2(\mathcal{H},\mu)}dt + \int_0^T a_\mu^{(\alpha)}(t;\bar{u}_\alpha,w-\bar{u}_\alpha)dt&\\
-\mathcal{A}^\Theta(w-\bar{u}_\alpha) + \frac{1}{2}\|w(T)\|^2_{L^2(\mathcal{H},\mu)} &\geq 0.
\end{aligned}$$

We now substitute $\bar{u}_\alpha = \bar{v}_\alpha - \Theta$ and obtain the non homogeneous variational inequality

$$\begin{aligned}
-\int_0^T \big(\frac{\partial w}{\partial t},w-\bar{v}_\alpha\big)_{L^2(\mathcal{H},\mu)}dt + \int_0^T a_\mu^{(\alpha)}(t;\bar{v}_\alpha,w-\bar{v}_\alpha)dt&\\
+\frac{1}{2}\|w(T)-\Theta(T)\|^2_{L^2(\mathcal{H},\mu)} &\geq 0,
\end{aligned}$$

for all $w \in \hat{\mathcal{K}}^{\Theta,p}_{\mu,\infty}$. From the probabilistic representation we already know that $\bar{v}_\alpha = V_\alpha$ and the the proof is complete. ■

It is worth noticing that the pointwise evaluation $w(T)$ still makes sense. In fact for a general Banach space B the inclusion $W^{1,2}(0,T;B) \subset C([0,T];B)$ holds (cf. [25], Section 5.9.2).

6.4 Connection with the optimal stopping in infinite dimensions

The main aim of this section is to characterize the optimal stopping time for the infinite dimensional problem. The main idea of the proof is quite similar to the one in Section 4.7, but we also exploit some complementary results. Let us state the theorem.

Theorem 6.4.1 *The optimal stopping time for the problem*

$$V_\alpha(t,x) = \sup_{t \le \tau \le T} \mathbb{E}\left[\Theta(\tau, X^{(\alpha)\,t,x}_\tau)\right], \tag{6.6}$$

is $\tau^*_{\alpha,t,x}$ defined as

$$\tau^*_{\alpha,t,x} := \inf\{s \ge t \,:\, V_\alpha(s, X^{(\alpha)\,t,x}_s) = \Theta(s, X^{(\alpha)\,t,x}_s)\} \wedge T. \tag{6.7}$$

In order to prove this theorem we need to prove the following lemma.

Lemma 6.4.1 *Let $(t,x) \in [0,T] \times \mathcal{H}$ be given. Let $\tau^*_{\alpha,n,t,x}$ be the stopping time*

$$\tau^*_{\alpha,n,t,x} := \inf\{s \ge t \,:\, V^{(n)}_\alpha(s, X^{(\alpha,n)\,t,x}_s) = \Theta^{(n)}(s, X^{(\alpha,n)\,t,x}_s)\} \wedge T. \tag{6.8}$$

Then there exists a subsequence $\{\tau^*_{\alpha,n_j,t,x}\}_{j=1}^\infty$ such that the following convergence holds

$$\lim_{j\to\infty} (\tau^*_{\alpha,t,x} \wedge \tau^*_{\alpha,n_j,t,x})(\omega) = \tau^*_{\alpha,t,x}(\omega), \quad \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{6.9}$$

PROOF: The proof of this Lemma is adapted from [4], Chapter 3, Section 3, Theorem 3.7 (cf. in particular p. 322). First we notice that from Proposition 3.2.1 there exists $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega \setminus \Omega_0) = 0$ and a subsequence $\{n_j\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty} \sup_{0\le t\le T} \left\| X^{(\alpha,n_j)\,x}_t(\omega) - X^{(\alpha)\,x}_t(\omega) \right\|_{\mathcal{H}} \to 0, \quad \forall \omega \in \Omega_0. \tag{6.10}$$

It is enough to prove the Lemma in the case of a diffusion starting at time zero. Let us consider a fixed initial data and let us simplify the notation without losing in generality, i.e. let us define

$$\tau^*_\alpha := \inf\{t \geq 0 \,:\, V_\alpha(t, X^{(\alpha)\,x}_t) = \Theta(t, X^{(\alpha)\,x}_t)\}.$$

We know from Theorem 5.4.2 that the stopping time

$$\tau^*_{\alpha,n} := \inf\{t \geq 0 \,:\, V^{(n)}_\alpha(t, X^{(\alpha,n)\,x}_t) = \Theta^{(n)}(t, X^{(\alpha,n)\,x}_t)\},$$

is optimal for the n-th approximating problem. For those $\omega \in \Omega_0$ s.t. $\tau^*_\alpha(\omega) = 0$ there is nothing to prove. Let us now take $\omega \in \Omega_0$ such that $\tau^*_\alpha(\omega) > \delta$ for some $\delta > 0$. Then as usual

$$V_\alpha(t, X^{(\alpha)\,x}_t(\omega)) > \Theta(t, X^{(\alpha)\,x}_t(\omega)), \qquad t \in [0, \tau^*_\alpha(\omega) - \delta].$$

The map $t \mapsto X^{(\alpha)\,x}_t(\omega)$ is continuous and $[0, \tau^*_\alpha(\omega) - \delta]$ is a compact set. Hence the range of the process, denoted by $\chi^\delta_\omega := \{y \in \mathcal{H} \,:\, y = X^{(\alpha)\,x}_t(\omega)\,,\, t \in [0, \tau^*_\alpha(\omega) - \delta]\}$, is a compact subset of $\mathcal{H}$. From Theorem 3.2.2 we know that the map $(t, x) \mapsto V_\alpha(t, x) - \Theta(t, x)$ is continuous and then it attains a minimum on $[0, \tau^*_\alpha(\omega) - \delta] \times \chi^\delta_\omega$, i.e. there exists $\eta(\delta, \omega) > 0$ such that

$$\eta(\delta, \omega) := \min\{V_\alpha(t, X^{(\alpha)\,x}_t(\omega)) - \Theta(t, X^{(\alpha)\,x}_t(\omega)),\, t \in [0, \tau^*_\alpha(\omega) - \delta]\},$$

and

$$V_\alpha(t, X^{(\alpha)\,x}_t(\omega)) \geq \Theta(t, X^{(\alpha)\,x}_t(\omega)) + \eta(\delta, \omega), \qquad t \in [0, \tau^*_\alpha(\omega) - \delta].$$

Our analysis is only concerned with the convergence of the approximation scheme on the compact set $[0, \tau^*_\alpha(\omega) - \delta] \times \chi^\delta_\omega$. We have uniform convergence of both $\{V^{(n)}_\alpha\}^\infty_{n=1}$ and $\{\Theta^{(n)}\}^\infty_{n=1}$ on compact subsets. Hence there exists $N(\delta, \omega; x) > 0$, such that for all $n \geq N(\delta, \omega; x)$ it simultaneously holds

$$V^{(n)}_\alpha(t, x^{(n)}) \geq V_\alpha(t, x) - \frac{\eta}{4}(\delta, \omega),$$
$$\Theta^{(n)}(t, x^{(n)}) \leq \Theta(t, x) + \frac{\eta}{4}(\delta, \omega),$$

for all $(t, x) \in [0, \tau^*_\alpha(\omega) - \delta] \times \chi^\delta_\omega$. This implies

$$V^{(n)}_\alpha(t, x^{(n)}) \geq \Theta^{(n)}(t, x^{(n)}) + \frac{\eta}{2}(\delta, \omega), \qquad (t, x) \in [0, \tau^*_\alpha(\omega) - \delta] \times \chi^\delta_\omega.$$

In terms of our diffusion it reads

$$V_\alpha^{(n)}(t, P_n X_t^{(\alpha)\,x}(\omega)) \geq \Theta^{(n)}(t, P_n X_t^{(\alpha)\,x}(\omega)) + \frac{\eta}{2}(\delta, \omega), \qquad t \in [0, \tau_\alpha^*(\omega) - \delta].$$

This is not exactly what we are looking for. In fact we would like to have $X^{(\alpha,n)\,x}$ as the argument of the functions above rather than $X^{(\alpha)\,x}$. We restrict our attention to the subsequence in equation (6.10). We use the Lipschitz property of the value function (cf. Proposition 2.2.2) and the fact that $P_n X^{(\alpha,n)\,x} = X^{(\alpha,n)\,x}$ to obtain the estimates

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left|V_\alpha^{(n_j)}(t, P_{n_j} X_t^{(\alpha)\,x}(\omega)) - V_\alpha^{(n_j)}(t, X_t^{(\alpha,n_j)\,x}(\omega))\right| \\
&\qquad\qquad\qquad\qquad \leq L_V^1 \sup_{0\leq t\leq T}\left\|X_t^{(\alpha,n_j)\,x}(\omega) - X_t^{(\alpha)\,x}(\omega)\right\|_{\mathcal{H}}, \\
&\sup_{0\leq t\leq T}\left|\Theta^{(n_j)}(t, P_{n_j} X_t^{(\alpha)\,x}(\omega)) - \Theta^{(n_j)}(t, X_t^{(\alpha,n_j)\,x}(\omega))\right| \\
&\qquad\qquad\qquad\qquad \leq L_1 \sup_{0\leq t\leq T}\left\|X_t^{(\alpha,n_j)\,x}(\omega) - X_t^{(\alpha)\,x}(\omega)\right\|_{\mathcal{H}}.
\end{aligned}$$

Then from equation (6.10), for $n_j \geq N(\delta, \omega; x)$ large enough, we obtain

$$V_\alpha^{(n_j)}(t, X_t^{(\alpha,n_j)\,x}(\omega)) > \Theta^{(n_j)}(t, X_t^{(\alpha,n_j)\,x}(\omega)), \qquad t \in [0, \tau_\alpha^*(\omega) - \delta].$$

In other words for any δ, ω, x given there exists a number $N(\delta, \omega; x) > 0$ such that $\tau_{\alpha,n_j}^*(\omega) > \tau_\alpha^*(\omega) - \delta$ for all $n_j \geq N(\delta, \omega; x)$ belonging to the subsequence such that (6.10) holds. Since $\delta > 0$ is arbitrarily small we have $(\tau_{\alpha,n_j}^* \wedge \tau_\alpha^*)(\omega) \to \tau_\alpha^*(\omega)$. Notice that the subsequence is independent of δ, ω, hence the convergence holds for all $\omega \in \Omega_0$, as $j \to \infty$. ■

Before giving the proof of the main theorem we also recall a simple consequence of the probabilistic representation of $V_\alpha^{(n)}$ and V_α.

Lemma 6.4.2 *Let X and Y be $\mathcal{F}_s$-measurable random variables on $\mathcal{H}$. Then*

$$\sup_{t\leq s\leq T}|V_\alpha^{(n)}(s, X) - V_\alpha(s, Y)| \leq L_1 \sup_{t\leq s\leq T}\mathbb{E}\left[\sup_{s\leq u\leq T}\|X_u^{(\alpha,n)\,s,X} - X_u^{(\alpha)\,s,Y}\|_{\mathcal{H}}\middle|\mathcal{F}_s\right], \quad \mathbb{P}\text{-a.s.}$$

PROOF: We analyze the term in the left hand side of the inequality, i.e. we consider

$$
\begin{aligned}
&V_\alpha^{(n)}(s,X) - V_\alpha(s,Y)\\
&\quad = \operatorname*{ess\,sup}_{s\le\tau\le T} \mathbb{E}\left[\Theta^{(n)}(\tau, X_\tau^{(\alpha,n)\,s,X})\middle|\mathcal{F}_s\right] - \operatorname*{ess\,sup}_{s\le\sigma\le T} \mathbb{E}\left[\Theta(\sigma, X_\sigma^{(\alpha)\,s,Y})\middle|\mathcal{F}_s\right]\\
&\quad \le \operatorname*{ess\,sup}_{s\le\tau\le T} \mathbb{E}\left[|\Theta^{(n)}(\tau, X_\tau^{(\alpha,n)\,s,X}) - \Theta(\tau, X_\tau^{(\alpha)\,s,Y})|\,\middle|\mathcal{F}_s\right].
\end{aligned}
$$

We obtain the same estimate if we reverse the first expression. Then we have

$$
\begin{aligned}
&\sup_{t\le s\le T} |V_\alpha^{(n)}(s,X) - V_\alpha(s,Y)|\\
&\quad \le \sup_{t\le s\le T}\left\{\operatorname*{ess\,sup}_{s\le\tau\le T} \mathbb{E}\left[|\Theta^{(n)}(\tau, X_\tau^{(\alpha,n)\,s,X}) - \Theta(\tau, X_\tau^{(\alpha)\,s,Y})|\,\middle|\mathcal{F}_s\right]\right\}\\
&\quad \le L_1 \sup_{t\le s\le T}\left\{\operatorname*{ess\,sup}_{s\le\tau\le T} \mathbb{E}\left[\|X_\tau^{(\alpha,n)\,s,X} - X_\tau^{(\alpha)\,s,Y}\|_{\mathcal{H}}\,\middle|\mathcal{F}_s\right]\right\}\\
&\quad \le L_1 \sup_{t\le s\le T} \mathbb{E}\left[\sup_{s\le u\le T}\|X_u^{(\alpha,n)\,s,X} - X_u^{(\alpha)\,s,Y}\|_{\mathcal{H}}\,\middle|\mathcal{F}_s\right].
\end{aligned}
$$

■

We now have all the ingredients that we need to carry out the proof of the theorem.

PROOF: For the optimality of τ_α^* we start from Theorem 5.4.2 and in particular from equation (5.20). In fact we simply set $\tau = \tau_\alpha^* \wedge \tau_{\alpha,n}^*$ in that equation and obtain

$$
V_\alpha^{(n)}(t,x^{(n)}) = \mathbb{E}\left[V_\alpha^{(n)}(\tau_\alpha^*\wedge\tau_{\alpha,n}^*, X_{\tau_\alpha^*\wedge\tau_{\alpha,n}^*}^{(\alpha,n)\,t,x})\right]. \tag{6.11}
$$

We can consider the subsequence $\{V_\alpha^{(n_j)}\}_{j=1}^\infty$ such that $\tau_\alpha^*\wedge\tau_{\alpha,n_j}^* \to \tau_\alpha^*$. The aim now is to take the limit when $j\to\infty$. We have pointwise convergence for the left hand side, i.e. $V_\alpha^{(n_j)}(t,x^{(n_j)}) \to V_\alpha(t,x)$. We can make the following estimates on the right hand side

$$
\begin{aligned}
&\left|\mathbb{E}\left[V_\alpha^{(n_j)}(\tau_\alpha^*\wedge\tau_{\alpha,n_j}^*, X_{\tau_\alpha^*\wedge\tau_{\alpha,n_j}^*}^{(\alpha,n_j)\,t,x}) - V_\alpha(\tau_\alpha^*, X_{\tau_\alpha^*}^{(\alpha)\,t,x})\right]\right|\\
&\quad \le \mathbb{E}\left[|V_\alpha^{(n_j)}(\tau_\alpha^*\wedge\tau_{\alpha,n_j}^*, X_{\tau_\alpha^*\wedge\tau_{\alpha,n_j}^*}^{(\alpha,n_j)\,t,x}) - V_\alpha^{(n_j)}(\tau_\alpha^*\wedge\tau_{\alpha,n_j}^*, P_{n_j}X_{\tau_\alpha^*\wedge\tau_{\alpha,n_j}^*}^{(\alpha)\,t,x})|\right]\\
&\quad\ +\mathbb{E}\left[|V_\alpha^{(n_j)}(\tau_\alpha^*\wedge\tau_{\alpha,n_j}^*, P_{n_j}X_{\tau_\alpha^*\wedge\tau_{\alpha,n_j}^*}^{(\alpha)\,t,x}) - V_\alpha(\tau_\alpha^*\wedge\tau_{\alpha,n_j}^*, X_{\tau_\alpha^*\wedge\tau_{\alpha,n_j}^*}^{(\alpha)\,t,x})|\right]\\
&\quad\ +\mathbb{E}\left[|V_\alpha(\tau_\alpha^*\wedge\tau_{\alpha,n_j}^*, X_{\tau_\alpha^*\wedge\tau_{\alpha,n_j}^*}^{(\alpha)\,t,x}) - V_\alpha(\tau_\alpha^*, X_{\tau_\alpha^*}^{(\alpha)\,t,x})|\right]
\end{aligned}
$$

$$\leq \mathbb{E}\left[\sup_{t\leq s\leq T}|V_\alpha^{(n_j)}(s,X_s^{(\alpha,n_j)\,t,x})-V_\alpha^{(n_j)}(s,P_{n_j}X_s^{(\alpha)\,t,x})|\right]$$
$$+\mathbb{E}\left[\sup_{t\leq s\leq T}|V_\alpha^{(n_j)}(s,P_{n_j}X_s^{(\alpha)\,t,x})-V_\alpha(s,X_s^{(\alpha)\,t,x})|\right]$$
$$+\mathbb{E}\left[|V_\alpha(\tau_\alpha^*\wedge\tau_{\alpha,n_j}^*,X^{(\alpha)\,t,x}_{\tau_\alpha^*\wedge\tau_{\alpha,n_j}^*})-V_\alpha(\tau_\alpha^*,X^{(\alpha)\,t,x}_{\tau_\alpha^*})|\right].$$

For the first term in the last expression we can adopt the estimate

$$\mathbb{E}\left[\sup_{t\leq s\leq T}|V_\alpha^{(n_j)}(s,X_s^{(\alpha,n_j)\,t,x})-V_\alpha^{(n_j)}(s,P_{n_j}X_s^{(\alpha)\,t,x})|\right]$$
$$\leq L_1\mathbb{E}\left[\sup_{t\leq s\leq T}\|X_s^{(\alpha,n_j)\,t,x}-P_{n_j}X_s^{(\alpha)\,t,x}\|_{\mathcal{H}}\right]$$
$$\leq L_1\mathbb{E}\left[\sup_{t\leq s\leq T}\|X_s^{(\alpha,n_j)\,t,x}-X_s^{(\alpha)\,t,x}\|_{\mathcal{H}}\right].$$

This term goes to zero as $j\to\infty$. For the second term we recall the result from Lemma 6.4.2 and obtain

$$\mathbb{E}\left[\sup_{t\leq s\leq T}|V_\alpha^{(n_j)}(s,P_{n_j}X_s^{(\alpha)\,t,x})-V_\alpha(s,X_s^{(\alpha)\,t,x})|\right]$$
$$\leq L_1\mathbb{E}\left[\sup_{t\leq s\leq T}\mathbb{E}\left[\sup_{s\leq u\leq T}\|X_u^{(\alpha,n_j)\,s,P_{n_j}X_s^{(\alpha)\,t,x}}-X_u^{(\alpha)\,s,X_s^{(\alpha)\,t,x}}\|_{\mathcal{H}}\,\Big|\mathcal{F}_s\right]\right].$$

Let us concentrate on the inner expectation. We have

$$\mathbb{E}\left[\sup_{s\leq u\leq T}\|X_u^{(\alpha,n_j)\,s,P_{n_j}X_s^{(\alpha)\,t,x}}-X_u^{(\alpha)\,s,X_s^{(\alpha)\,t,x}}\|_{\mathcal{H}}\,\Big|\mathcal{F}_s\right]$$
$$\leq\left(\mathbb{E}\left[\sup_{s\leq u\leq T}\|X_u^{(\alpha,n_j)\,s,P_{n_j}X_s^{(\alpha)\,t,x}}-X_u^{(\alpha)\,s,X_s^{(\alpha)\,t,x}}\|_{\mathcal{H}}^2\,\Big|\mathcal{F}_s\right]\right)^{\frac{1}{2}}.$$

Notice that $X^{(\alpha,n)\,s,P_{n_j}X_s^{(\alpha)\,t,x}}=X^{(\alpha,n)\,s,X_s^{(\alpha)\,t,x}}$ from the definition of the finite dimensional SDE. We know from natural generalization of the arguments in the proof of Proposition 3.2.1, that there exists a suitable constant $C>0$ such that

$$\mathbb{E}\left[\sup_{s\leq u\leq T}\|X_u^{(\alpha,n_j)\,s,P_{n_j}X_s^{(\alpha)\,t,x}}-X_u^{(\alpha)\,s,X_s^{(\alpha)\,t,x}}\|_{\mathcal{H}}^2\,\Big|\mathcal{F}_s\right]\leq C^2\Big\{\|P_{n_j}X_s^{(\alpha)\,t,x}-X_s^{(\alpha)\,t,x}\|_{\mathcal{H}}^2$$
$$+\int_s^T\mathbb{E}\left[\|(I-P_{n_j})A_\alpha X_u^{(\alpha)\,s,X_s^{(\alpha)\,t,x}}\|_{\mathcal{H}}^2\big|\mathcal{F}_s\right]du$$
$$+\int_s^T\mathbb{E}\left[\|(I-P_{n_j})\sigma(X_u^{(\alpha)\,s,X_s^{(\alpha)\,t,x}})\|_{\mathcal{H}}^2\big|\mathcal{F}_s\right]du+n_j^2\epsilon_{n_j}^2T\Big\}$$

$$
\begin{aligned}
&= C^2\Big\{\|P_{n_j}X_s^{(\alpha)\,t,x} - X_s^{(\alpha)\,t,x}\|_{\mathcal{H}}^2 + \mathbb{E}\left[\int_s^T \|(I-P_{n_j})A_\alpha X_u^{(\alpha)\,t,x}\|_{\mathcal{H}}^2 du \big| \mathcal{F}_s\right] \\
&+\mathbb{E}\left[\int_s^T \|(I-P_{n_j})\sigma(X_u^{(\alpha)\,t,x})\|_{\mathcal{H}}^2 du \big| \mathcal{F}_s\right] + n_j^2\epsilon_{n_j}^2 T\Big\} \\
&\leq C^2\Big\{\|P_{n_j}X_s^{(\alpha)\,t,x} - X_s^{(\alpha)\,t,x}\|_{\mathcal{H}}^2 + \mathbb{E}\left[\int_t^T \|(I-P_{n_j})A_\alpha X_u^{(\alpha)\,t,x}\|_{\mathcal{H}}^2 du \big| \mathcal{F}_s\right] \\
&+\mathbb{E}\left[\int_t^T \|(I-P_{n_j})\sigma(X_u^{(\alpha)\,t,x})\|_{\mathcal{H}}^2 du \big| \mathcal{F}_s\right] + n_j^2\epsilon_{n_j}^2 T\Big\}.
\end{aligned}
$$

We take now the supremum and the expectation and obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\leq s\leq T} |V_\alpha^{(n_j)}(s, P_{n_j}X_s^{(\alpha)\,t,x}) - V_\alpha(s, X_s^{(\alpha)\,t,x})|\right] \\
&\leq L_1 \mathbb{E}\left[\sup_{t\leq s\leq T}\left(\mathbb{E}\left[\sup_{s\leq u\leq T}\|X_u^{(\alpha,n_j)\,s,P_{n_j}X_s^{(\alpha)\,t,x}} - X_u^{(\alpha)\,s,X_s^{(\alpha)\,t,x}}\|_{\mathcal{H}}^2 \Big| \mathcal{F}_s\right]\right)^{\frac{1}{2}}\right] \\
&\leq L_1\left(\mathbb{E}\left[\sup_{t\leq s\leq T}\mathbb{E}\Big[\sup_{s\leq u\leq T}\|X_u^{(\alpha,n_j)\,s,P_{n_j}X_s^{(\alpha)\,t,x}} - X_u^{(\alpha)\,s,X_s^{(\alpha)\,t,x}}\|_{\mathcal{H}}^2 \Big| \mathcal{F}_s\Big]\right]\right)^{\frac{1}{2}} \\
&\leq L_1\Bigg(\mathbb{E}\Bigg[\sup_{t\leq s\leq T} C\Big\{\|P_{n_j}X_s^{(\alpha)\,t,x} - X_s^{(\alpha)\,t,x}\|_{\mathcal{H}}^2 + \mathbb{E}\left[\int_t^T \|(I-P_{n_j})A_\alpha X_u^{(\alpha)\,t,x}\|_{\mathcal{H}}^2 du \big| \mathcal{F}_s\right] \\
&+\mathbb{E}\left[\int_t^T \|(I-P_{n_j})\sigma(X_u^{(\alpha)\,t,x})\|_{\mathcal{H}}^2 du \big| \mathcal{F}_s\right] + n_j^2\epsilon_{n_j}^2 T\Big\}\Bigg]\Bigg)^{\frac{1}{2}} \\
&\leq L_1\cdot C\Bigg(\mathbb{E}\Big[\sup_{t\leq s\leq T}\|P_{n_j}X_s^{(\alpha)\,t,x} - X_s^{(\alpha)\,t,x}\|_{\mathcal{H}}^2\Big] \\
&+\mathbb{E}\Big[\sup_{t\leq s\leq T}\mathbb{E}\left[\int_t^T \|(I-P_{n_j})A_\alpha X_u^{(\alpha)\,t,x}\|_{\mathcal{H}}^2 du \big| \mathcal{F}_s\right]\Big] \\
&+\mathbb{E}\Big[\sup_{t\leq s\leq T}\mathbb{E}\left[\int_t^T \|(I-P_{n_j})\sigma(X_u^{(\alpha)\,t,x})\|_{\mathcal{H}}^2 du \big| \mathcal{F}_s\right]\Big] + n_j^2\epsilon_{n_j}^2 T\Bigg)^{\frac{1}{2}}
\end{aligned}
$$

From dominated convergence and Dini's Theorem the first term clearly converges to zero. The last term is not even stochastic and goes to zero. We then recognize that the terms

$$M_s := \mathbb{E}\left[\int_t^T \|(I-P_{n_j})A_\alpha X_u^{(\alpha)\,t,x}\|_{\mathcal{H}}^2 du \big| \mathcal{F}_s\right]$$

and

$$N_s := \mathbb{E}\left[\int_t^T \|(I-P_{n_j})\sigma(X_u^{(\alpha)\,t,x})\|_{\mathcal{H}}^2 du \big| \mathcal{F}_s\right]$$

are square integrable non-negative martingales. Hence from Doob inequality

$$\mathbb{E}\left[\sup_{t\le s\le T}|M_s|\right]\le \mathbb{E}\left[\sup_{t\le s\le T}|M_s|^2\right]^{\frac{1}{2}}\le 2\left(\mathbb{E}\left[|M_T|^2\right]\right)^{\frac{1}{2}}$$
$$=2\left(\mathbb{E}\left[\left|\int_t^T\|(I-P_{n_j})A_\alpha X_u^{(\alpha)\,t,x}\|^2_{\mathcal{H}}du\right|^2\right]\right)^{\frac{1}{2}}.$$

From dominated convergence this term and the other involving N_s go to zero as $j\to\infty$. We use dominated convergence also in the last term. It goes to zero because of the results about the limiting behaviour of the stopping time, Lemma 6.4.1, and the continuity of the value function V_α, i.e.

$$\lim_{j\to\infty}\mathbb{E}\left[|V_\alpha(\tau^*_\alpha\wedge\tau^*_{\alpha,n_j},X^{(\alpha)\,t,x}_{\tau^*_\alpha\wedge\tau^*_{\alpha,n_j}})-V_\alpha(\tau^*_\alpha,X^{(\alpha)\,t,x}_{\tau^*_\alpha})|\right]=0.$$

We take the limit as $n_j\to\infty$ in equation (6.11) and get

$$V_\alpha(t,x)=\mathbb{E}\left[V_\alpha(\tau^*_\alpha,X^{(\alpha)\,t,x}_{\tau^*_\alpha})\right]=\mathbb{E}\left[\Theta(\tau^*_\alpha,X^{(\alpha)\,t,x}_{\tau^*_\alpha})\right]. \tag{6.12}$$

This shows the optimality of τ^*_α and concludes the proof. ■

It is worth noticing that passing through the subsequence n_j allows us to use the particular convergence of the stopping times but the result we get at the does not keep any memories of this algorithm.

6.5 Infinite dimensional variational inequality - part 2

In order to complete the characterization of our value function in terms of an infinite dimensional EVI, we are now interested in taking the limit as $\alpha\to\infty$. From both probabilistic and analytic results we know that $\{\bar{v}_\alpha\}_{\alpha\ge0}$ forms a uniformly bounded sequence in $L^2(0,T;W^{1,2}(\mathcal{H},\mu))$. In fact we can exploit once more the same arguments as in Proposition 2.2.2 and Proposition 5.2.1 to prove that

$$\int_0^T\int_{\mathcal{H}}|\bar{v}_\alpha(t,x)|^2\mu+\int_0^T\int_{\mathcal{H}}\|D\bar{v}_\alpha(t,x)\|^2_{\mathcal{H}}\mu(dx)dt\le(\overline{\Psi}^2+L_V^2)T.$$

Then there exists a function $\bar{v} \in L^2(0,T;\mathcal{V}^p)$ and a subsequence $\{\bar{v}_{\alpha_j}\}_{j=1}^{\infty}$ such that $\bar{v}_{\alpha_j} \rightharpoonup \bar{v}$ in $L^2(0,T;W^{1,2}(\mathcal{H},\mu))$. We also know from Corollary 3.1.1 that $\bar{v}_\alpha \to V$ in $L^2(0,T;L^p(\mathcal{H},\mu))$ and hence from the uniqueness of the limit

$$\bar{v}(t,x) = V(t,x) = \sup_{t\le\tau\le T} \mathbb{E}\left[\Theta(\tau, X^{t,x}_\tau)\right].$$

Another important result is that V is continuous on the whole space (cf. Theorem 3.1.2) and that $\bar{v}_\alpha \to V$ uniformly on compact subsets of $[0,T]\times\mathcal{H}$.

In the EVI of Theorem 6.3.1 the only term depending on α is the bilinear form $a^{(\alpha)}_\mu(t;u,w)$. The first question we want to address is how to characterize the limiting behaviour of this term. In particular we notice that there is actually only one term of the bilinear form which needs to be discussed, namely the one involving the unbounded operator A.

6.5.1 The limit in the Yosida approximation

Let $w \in L^2(0,T;\mathcal{V}^p)$ be given. For any $u \in L^2(0,T;\mathcal{V}^p)$ we define the linear functional $T^{(n)}_{A,w} \in L^2(0,T;\mathcal{V}^{p*})$ as

$$T^{(n)}_{A,w}(u) := \int_0^T \int_{\mathcal{H}} \langle A\, P_n x, D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx)dt.$$

If we denote $C_A := \sum_{j=1}^{\infty} \|A\varphi_j\|_{\mathcal{H}}\sqrt{\lambda_j}$ we know from the estimates of Section 5.3.1 above that

$$|T^{(n)}_{A,w}(u)| \le T\, C_A \left(\int_0^T |||w(t)|||_p^2\, dt\right)^{\frac{1}{2}} \left(\int_0^T |||u(t)|||_p^2\, dt\right)^{\frac{1}{2}}. \tag{6.13}$$

We prove now that the sequence $\{T^{(n)}_{A,w}\}_{n\in\mathbb{N}}$ is Cauchy in $L^2(0,T;\mathcal{V}^{p*})$. Let $n > m$, then

$$\begin{aligned}
|T^{(n)}_{A,w}(u) - T^{(m)}_{A,w}(u)| &= \left|\int_0^T \int_{\mathcal{H}} \langle A(P_n - P_m)x, D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx)dt\right| \\
&= \left|\sum_{j=m+1}^{n} \int_0^T \int_{\mathcal{H}} \langle A\varphi_j, D_x u(t,x)\rangle_{\mathcal{H}} x_j w(t,x)\mu(dx)dt\right| \\
&\le T \left(\int_0^T |||u(t)|||_p^2\, dt\right)^{\frac{1}{2}} \left(\int_0^T |||w(t)|||_p^2\, dt\right)^{\frac{1}{2}} \sum_{j=m+1}^{n} \|A\varphi_j\|_{\mathcal{H}}\sqrt{\lambda_j}.
\end{aligned}$$

Since $\sum_{j=1}^n \|A\varphi_j\|_{\mathcal{H}}\sqrt{\lambda_j}$ converges as $n \to \infty$, this term is clearly Cauchy. Moreover this estimate proves that

$$\|T^{(n)}_{A,w} - T^{(m)}_{A,w}\|_{L^2(0,T;\mathcal{V}^{p*})} \leq T \left(\int_0^T |||w|||_p^2 \, dt\right)^{\frac{1}{2}} \sum_{j=m+1}^{n} \|A\varphi_j\|_{\mathcal{H}}\sqrt{\lambda_j},$$

and hence the sequence is Cauchy in $L^2(0,T;\mathcal{V}^{p*})$. From completeness we know that there exists $\hat{T}_{A,w} \in L^2(0,T;\mathcal{V}^{p*})$ such that $T^{(n)}_{A,w} \to \hat{T}_{A,w}$ as $n \to \infty$.

The second question we address is the explicit form of $\hat{T}_{A,w}$. Let us restrict our analysis for a moment to the set $\mathcal{E}_A([0,T]\times\mathcal{H})$, which is a dense subset of $L^2(0,T;\mathcal{V}^p)$, cf. Appendix E. It is enough to perform our analysis on elements from this set. For any $u \in \mathcal{E}_A([0,T]\times\mathcal{H})$ it is easily verified that $A^*Du \in L^2(0,T;L^2(\mathcal{H},\mu))$ and so we can write

$$\int_0^T \int_{\mathcal{H}} \langle A\,P_n x, D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx)dt = \int_0^T \int_{\mathcal{H}} \langle P_n x, A^* D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx)dt.$$

We can use dominated convergence to obtain

$$T^{(n)}_{A,w}(u) \to \int_0^T \int_{\mathcal{H}} \langle x, A^* D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx)dt, \qquad \text{as } n \to \infty,$$

for $u \in \mathcal{E}_A([0,T]\times\mathcal{H})$. Moreover, we know from (6.13) that

$$\left|\int_0^T \int_{\mathcal{H}} \langle x, A^* D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx)dt\right| \leq T\,C_A \left(\int_0^T |||w(t)|||_p^2 \, dt\right)^{\frac{1}{2}} \left(\int_0^T |||u(t)|||_p^2 \, dt\right)^{\frac{1}{2}}, \quad u \in \mathcal{E}_A([0,T]\times\mathcal{H}). \tag{6.14}$$

Hence we have a linear fuctional $(T_{A,w}, D(T_{A,w}))$ defined as

$$T_{A,w}(u) := \int_0^T \int_{\mathcal{H}} \langle x, A^* D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx)dt.$$

The domain $D(T_{A,w}) \subset L^2(0,T;\mathcal{V}^p)$ is dense in $L^2(0,T;\mathcal{V}^p)$ and $T^{(n)}_{A,w}(u) \to T_{A,w}(u)$, $u \in D(T_{A,w})$. Moreover, given equation (6.14) and the fact that $D(T_{A,w})$ is dense in $L^2(0,T;\mathcal{V}^p)$, we can extend $T_{A,w}$ to the whole space $L^2(0,T;\mathcal{V}^p)$. In particular we denote this extension by $\bar{T}_{A,w}$.

The last question we want to address is whether $\hat{T}_{A,w} = \bar{T}_{A,w}$. Let $u \in L^2(0,T;\mathcal{V}^p)$ and let $\{u_j\}_{j=1}^\infty$ be an approximating sequence in $\mathcal{E}_A([0,T]\times\mathcal{H})$. We then have

$$\begin{aligned}\left(\int_0^T |||u_j(t)-u(t)|||_p^2\,dt\right)^{\frac{1}{2}}\left(\int_0^T |||w(t)|||_p^2\,dt\right)^{\frac{1}{2}} C_A\,T &\geq |T_{A,w}^{(n)}(u_j-u)| \\ &= |T_{A,w}^{(n)}(u_j)-T_{A,w}^{(n)}(u)|.\end{aligned}$$

Taking the limit as $n\to\infty$ we obtain

$$\begin{aligned}\left(\int_0^T |||u_j(t)-u(t)|||_p^2\,dt\right)^{\frac{1}{2}}\left(\int_0^T |||w(t)|||_p^2\,dt\right)^{\frac{1}{2}} C_A\,T &\geq |T_{A,w}(u_j)-\hat{T}_{A,w}(u)| \\ &= |\bar{T}_{A,w}(u_j)-\hat{T}_{A,w}(u)|.\end{aligned}$$

If we now take the limit as $j\to\infty$ we obtain $|\bar{T}_{A,w}(u)-\hat{T}_{A,w}(u)|\leq 0$, and then from the arbitrariness of u we have $\bar{T}_{A,w}=\hat{T}_{A,w}$. We can then conclude that

$$\lim_{n\to\infty} T_A^{(n)} = \bar{T}_A, \qquad \text{in } L^2(0,T;\mathcal{V}^{p\,*}). \tag{6.15}$$

In a similar way we define $T_{A,w}^{(\alpha)} \in L^2(0,T;\mathcal{V}^{p\,*})$ as

$$T_{A,w}^{(\alpha)}(u) := \int_0^T \int_{\mathcal{H}} \langle A_\alpha\, x, D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx)dt,$$

for $u\in L^2(0,T;\mathcal{V}^p)$. We also define the sequence $\{T_{A,w}^{(\alpha,n)}\}_{n\in\mathbb{N}}$ as

$$T_{A,w}^{(\alpha,n)}(u) := \int_0^T \int_{\mathcal{H}} \langle A_\alpha\, P_n x, D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx)dt.$$

From the same arguments as above it is easily verified that

$$\|T_{A,w}^{(\alpha,n)} - T_{A,w}^{(\alpha)}\|_{L^2(0,T;\mathcal{V}^{p\,*})} \leq T\sum_{j=n+1}^{\infty}\|A\varphi_j\|_{\mathcal{H}}\sqrt{\lambda_j}\left(\int_0^T |||w(t)|||_p^2\,dt\right)^{\frac{1}{2}}.$$

This clearly implies the uniform convergence

$$\lim_{n\to\infty}\sup_{\alpha\geq 0}\|T_{A,w}^{(\alpha,n)} - T_{A,w}^{(\alpha)}\|_{L^2(0,T;\mathcal{V}^{p\,*})} = 0. \tag{6.16}$$

We can now prove an important convergence result.

Proposition 6.5.1 *For $T^{(\alpha)}_{A,w}$ and $\bar{T}_{A,w}$ defined as above the following holds*

$$\lim_{\alpha\to\infty} \|T^{(\alpha)}_{A,w} - \bar{T}_{A,w}\|_{L^2(0,T;\mathcal{V}^{p*})} = 0. \tag{6.17}$$

PROOF: Let $\varepsilon > 0$ be an arbitrary constant. From (6.15) and (6.16) that there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\begin{aligned}\|T^{(\alpha)}_{A,w} - \bar{T}_{A,w}\|_{L^2(0,T;\mathcal{V}^{p*})} \leq & \|T^{(\alpha)}_{A,w} - T^{(\alpha,n_\varepsilon)}_{A,w}\|_{L^2(0,T;\mathcal{V}^{p*})} + \|T^{(\alpha,n_\varepsilon)}_{A,w} - T^{(n_\varepsilon)}_{A,w}\|_{L^2(0,T;\mathcal{V}^{p*})} \\ & + \|T^{(n_\varepsilon)}_{A,w} - \bar{T}_{A,w}\|_{L^2(0,T;\mathcal{V}^{p*})} \\ < & \varepsilon + \|T^{(\alpha,n_\varepsilon)}_{A,w} - T^{(n_\varepsilon)}_{A,w}\|_{L^2(0,T;\mathcal{V}^{p*})}.\end{aligned}$$

From the same calculations as above it is easy to verify that

$$\|T^{(\alpha,n_\varepsilon)}_{A,w} - T^{(n_\varepsilon)}_{A,w}\|_{L^2(0,T;\mathcal{V}^{p*})} \leq T \sum_{j=1}^{\infty} \|(A_\alpha - A)\varphi_j\|_{\mathcal{H}} \sqrt{\lambda_j} \left(\int_0^T |||w(t)|||_p^2 \, dt \right)^{\frac{1}{2}}.$$

Clearly the sum is well definite and moreover

$$\sup_{\alpha\geq 0} \left| \sum_{j=1}^{k} \|(A_\alpha - A)\varphi_j\| \sqrt{\lambda_j} - \sum_{j=1}^{\infty} \|(A_\alpha - A)\varphi_j\|_{\mathcal{H}} \sqrt{\lambda_j} \right| \leq 2 \sum_{j=k+1}^{\infty} \|A\varphi_j\|_{\mathcal{H}} \sqrt{\lambda_j}.$$

Then the series is convergent uniformly with respect to α. We also know from [49] that

$$\lim_{\alpha\to\infty} A_\alpha \varphi_j = A\varphi_j, \quad j \in \mathbb{N}.$$

A well known result from analysis tells us that

$$\lim_{\alpha\to\infty} \sum_{j=1}^{\infty} \|(A_\alpha - A)\varphi_j\|_{\mathcal{H}} \sqrt{\lambda_j} = 0.$$

This implies that

$$\lim_{\alpha\to\infty} \|T^{(\alpha)}_{A,w} - \bar{T}_{A,w}\|_{L^2(0,T;\mathcal{V}^{p*})} < \varepsilon. \tag{6.18}$$

Given that ε is arbitrary this concludes the proof. ■

This fact allows us to state the final result about our variational inequality in infinite dimensions. We are able indeed to characterize the final bilinear form as

$$\int_0^T a_\mu(t;\, u(t)\,, w(t))dt \;=\; \frac{1}{2}\int_0^T \int_{\mathcal{H}} \langle \sigma\sigma^*(x) D_x u(t,x), D_x w(t,x)\rangle_{\mathcal{H}} \mu(dx)dt$$

$$+\frac{1}{2}\int_0^T \int_{\mathcal{H}} \langle D_x\sigma(x)\cdot\sigma(x), D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx)dt$$

$$+\frac{1}{2}\int_0^T \int_{\mathcal{H}} Tr[D_x\sigma(x)]_{\mathcal{H}}\langle \sigma(x), D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx)dt$$

$$-\bar{T}_{A,w}(u)$$

$$+\frac{1}{2}\int_0^T \int_{\mathcal{H}} \langle \sigma\sigma^*(x) Q^{-1}x, D_x u(t,x)\rangle_{\mathcal{H}} w(t,x)\mu(dx)dt.$$

Theorem 6.5.1 *The value function of the optimal stopping problem*

$$V(t,x) = \sup_{t\le\tau\le T} \mathbb{E}\left[\Theta(\tau, X_\tau^{t,x})\right],$$

is a weak solution of the variational problem: find $v \in L^2(0,T;\mathcal{V}^p)$*,* $v \ge \Theta$ *such that*

$$-\int_0^T \Big(\frac{\partial w}{\partial t}, w - v\Big)_{L^2(\mathcal{H},\mu)} dt + \int_0^T a_\mu(t; v, w-v)dt + \frac{1}{2}\|w(T)-\Theta(T)\|^2_{L^2(\mathcal{H},\mu)} \ge 0,$$

for all $w \in \hat{\mathcal{K}}^{\Theta,p}_{\mu,\infty}$.

PROOF: We simply take the limit in the EVI of Theorem 6.3.1 as $\alpha \to \infty$ (up to a suitable subsequence) and exploit the results above. ■

6.6 The optimal stopping time for the infinite dimensional problem

The proof of the optimality of the stopping time follows exactly the same arguments as in the previous section. For this reason we will only sketch the proofs. We first state the analogue of Lemma 6.4.1.

Lemma 6.6.1 *For $(t,x) \in [0,T] \times \mathcal{H}$ given and fixed, we denote by $\tau^*_{t,x}$ the stopping time*

$$\tau^*_{t,x} := \inf\{s \geq t \,:\, V(s, X^{t,x}_s) = \Theta(s, X^{t,x}_s)\} \wedge T. \tag{6.19}$$

Then there exists a subsequence $\{\tau^*_{\alpha_j,t,x}\}_{j=1}^{\infty}$ such that the following convergence holds

$$\lim_{j\to\infty} (\tau^*_{t,x} \wedge \tau^*_{\alpha_j,t,x})(\omega) = \tau^*_{t,x}(\omega), \quad \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{6.20}$$

PROOF: The rationale of the proof is exactly the same as in Lemma 6.4.1, provided that the fundamental properties of the value functions involved still hold. From Theorem 3.1.2 we know that $V \in C_b([0,T] \times \mathcal{H})$ and moreover $V_\alpha \to V$ uniformly on every compact subset $[0,T] \times \mathcal{K}$. From Proposition 3.1.1 we know that there exists a subsequence $\{\alpha_j\}_{j=1}^{\infty}$ such that

$$\lim_{\alpha_j\to\infty} \sup_{0\leq t\leq T} \left\| X^{(\alpha)\,x}_t - X^x_t \right\|_{\mathcal{H}} = 0, \quad \mathbb{P}\text{-a.s.}$$

Finally the Lipschitz continuity of the value function and of the obstacle hold in the form

$$\sup_{0\leq t\leq T} \left| V_\alpha(t, X^x_t) - V_\alpha(t, X^{(\alpha)\,x}_t) \right| \leq L^1_V \sup_{0\leq t\leq T} \left\| X^x_t - X^{(\alpha)\,x}_t \right\|_{\mathcal{H}}, \quad \mathbb{P}\text{-a.s.}$$
$$\sup_{0\leq t\leq T} \left| \Theta(t, X^x_t) - \Theta(t, X^{(\alpha)\,x}_t) \right| \leq L_1 \sup_{0\leq t\leq T} \left\| X^x_t - X^{(\alpha)\,x}_t \right\|_{\mathcal{H}}, \quad \mathbb{P}\text{-a.s..}$$

Hence the proof can be carried out by means of the same arguments as in Lemma 6.4.1. ∎

Theorem 6.6.1 *The optimal stopping time for the problem*

$$V(t,x) = \sup_{t\leq\tau\leq T} \mathbb{E}\left[\Theta(\tau, X^{t,x}_\tau)\right], \tag{6.21}$$

is $\tau^*_{t,x}$ defined as

$$\tau^*_{t,x} := \inf\{s \geq t \,:\, V(s, X^{t,x}_s) = \Theta(s, X^{t,x}_s)\} \wedge T. \tag{6.22}$$

PROOF: The whole analysis might start from equation (6.12). Again we simplify the notation $\tau^* = \tau^*_{t,x}$. Indeed in the analysis of the previous section we might substitute τ^*_α by $\tau^* \wedge \tau^*_\alpha$ and carry out the same arguments in order to obtain

$$V_\alpha(t,x) = \mathbb{E}\left[V_\alpha(\tau^* \wedge \tau^*_\alpha, X^{(\alpha)\,t,x}_{\tau^*\wedge\tau^*_\alpha})\right]. \tag{6.23}$$

We can restrict our analysis to the subsequence $\{V_{\alpha_j}\}_{j=1}^{\infty}$ which guarantees the convergence of the stopping times in Lemma 6.6.1. Then taking the limit in the left hand side of the previous equation, we get $V_{\alpha_j}(t,x) \to V(t,x)$ pointwisely. For the right hand side we can perform the following estimates

$$\begin{aligned}
&\left|\mathbb{E}\left[V_{\alpha_j}(\tau^* \wedge \tau^*_{\alpha_j}, X^{(\alpha_j)\,t,x}_{\tau^*\wedge\tau^*_{\alpha_j}}) - V(\tau^*, X^{t,x}_{\tau^*})\right]\right| \\
&\quad \leq \mathbb{E}\left[\left|V_{\alpha_j}(\tau^* \wedge \tau^*_{\alpha_j}, X^{(\alpha_j)\,t,x}_{\tau^*\wedge\tau^*_{\alpha_j}}) - V_{\alpha_j}(\tau^* \wedge \tau^*_{\alpha_j}, X^{t,x}_{\tau^*\wedge\tau^*_{\alpha_j}})\right|\right] \\
&\quad\; + \leq \mathbb{E}\left[\left|V_{\alpha_j}(\tau^* \wedge \tau^*_{\alpha_j}, X^{t,x}_{\tau^*\wedge\tau^*_{\alpha_j}}) - V(\tau^* \wedge \tau^*_{\alpha_j}, X^{t,x}_{\tau^*\wedge\tau^*_{\alpha_j}})\right|\right] \\
&\quad\; + \leq \mathbb{E}\left[\left|V(\tau^* \wedge \tau^*_{\alpha_j}, X^{t,x}_{\tau^*\wedge\tau^*_{\alpha_j}}) - V(\tau^*, X^{t,x}_{\tau^*})\right|\right].
\end{aligned}$$

For the first term on the right hand side we have

$$\begin{aligned}
&\mathbb{E}\left[\left|V_{\alpha_j}(\tau^* \wedge \tau^*_{\alpha_j}, X^{(\alpha_j)\,t,x}_{\tau^*\wedge\tau^*_{\alpha_j}}) - V_{\alpha_j}(\tau^* \wedge \tau^*_{\alpha_j}, X^{t,x}_{\tau^*\wedge\tau^*_{\alpha_j}})\right|\right] \\
&\quad \leq L_V \mathbb{E}\left[\sup_{t\leq s\leq T}\left\|X^{(\alpha_j)\,t,x}_s - X^{t,x}_s\right\|_{\mathcal{H}}\right],
\end{aligned}$$

hence this converges to zero. For the second term explicit calculations can get very cumbersome so we adopt a different rationale in order to prove convergence. We have

$$\begin{aligned}
&\mathbb{E}\left[\left|V_{\alpha_j}(\tau^* \wedge \tau^*_{\alpha_j}, X^{t,x}_{\tau^*\wedge\tau^*_{\alpha_j}}) - V(\tau^* \wedge \tau^*_{\alpha_j}, X^{t,x}_{\tau^*\wedge\tau^*_{\alpha_j}})\right|\right] \\
&\quad \leq \mathbb{E}\left[\sup_{t\leq s\leq T}\left|V_{\alpha_j}(s, X^{t,x}_s) - V(s, X^{t,x}_s)\right|\right].
\end{aligned}$$

In the time interval $s \in [t,T]$ the process $s \mapsto X^{t,x}_s(\omega)$ ranges in a compact subset of $\mathcal{H}$. We define such a subset as $\mathcal{K}^x_{t,T}(\omega) := \{y : X^{t,x}_s(\omega) = y,\ s \in [t,T]\}$ and then we can write

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\leq s\leq T}\left|V_{\alpha_j}(s, X^{t,x}_s) - V(s, X^{t,x}_s)\right|\right] \\
&\quad \leq \mathbb{E}\Bigg[\sup\left\{\left|V_{\alpha_j}(s,y) - V(s,y)\right|,\ (s,y) \in [t,T] \times \mathcal{K}^x_{t,T}(\omega)\right\}\Bigg].
\end{aligned}$$

For dominated convergence we can carry the limit under the expectation and then use the uniform convergence of the value functions on compact subsets, cf. Theorem 3.1.2. Hence this term also goes to zero as $j \to \infty$. The last term simply converges to zero for dominated convergence and the continuity of the value function.

Summarizing, we obtain

$$V(t,x) = \mathbb{E}\left[V(\tau^*, X^{t,x}_{\tau^*})\right] = \mathbb{E}\left[\Theta(\tau^*, X^{t,x}_{\tau^*})\right], \tag{6.24}$$

and hence the optimality of τ^* is proven. ■

The very last thing that we discuss in order to conclude our analysis is the removal of the smoothing on the gain function.

6.7 Removal of the regularization on the gain function

We recall that $\Theta = \Psi_k$ for some $k \in \mathbb{N}$. In particular $\Psi_k \in C^\infty([0,T] \times \mathcal{H})$ and from Assumption 4.6.1 we know that $\Psi_k \to \Psi$ in $W^{1,2}([0,T] \times \mathcal{H}, \lambda \times \mu)$ and uniformly. Hence we have to reintroduce the index k and in particular for the value function we have $V = V_\Theta = V_k$, where clearly

$$V_k(t,x) = \sup_{t \le \tau \le T} \mathbb{E}\left[\Psi_k(\tau, X^{t,x}_\tau)\right].$$

Here we can fully understand the importance of having set our whole analysis in the framework of weak solutions for the variational inequality. We know, from Proposition 5.2.1 and from the analysis we carried out to this point, that $\{V_k\}_{k\in\mathbb{N}}$ forms an equibounded sequence in $L^2(0,T;W^{1,2}(\mathcal{H},\mu))$ and $L^2(0,T;L^p(\mathcal{H},\mu))$, $1 \le p < \infty$. This clearly implies that there exists $\bar{V} \in L^2(0,T;W^{1,2}(\mathcal{H},\mu))$ and a subsequence $\{V_{k_j}\}_{j\in\mathbb{N}}$ such that we have the weak convergence $V_{k_j} \rightharpoonup \bar{V}$ in $L^2(0,T;W^{1,2}(\mathcal{H},\mu))$. Moreover from the uniform convergence of Ψ_k and from Proposition A.0.3 we also have the uniform convergence $V_k \to V$, where

$$V(t,x) = \sup_{t \le \tau \le T} \mathbb{E}\left[\Psi(\tau, X^{t,x}_\tau)\right].$$

This fact, the continuity and boundedness of all V_k and the Theorem C.0.4 guarantee the following result

Proposition 6.7.1 *The value function V is bounded and continuous, that is $V \in C_b([0,T] \times \mathcal{H})$.*

From dominated convergence we conclude that $V_k \to V$ in $L^2(0,T; L^p(\mathcal{H},\mu))$, $1 \le p < \infty$ and hence from uniqueness of the limit that $\bar{V} = V$. We can now prove the theorem connecting the optimal stopping functional with the variational inequality in infinite dimensions.

Theorem 6.7.1 *The value function of the optimal stopping problem*

$$V(t,x) = \sup_{t \le \tau \le T} \mathbb{E}\left[\Psi(\tau, X_\tau^{t,x})\right],$$

is a weak solution of the variational problem: find $v \in L^2(0,T;\mathcal{V}^p)$, $v \ge \Psi$ such that

$$-\int_0^T \big(\frac{\partial w}{\partial t}, w - v\big)_{L^2(\mathcal{H},\mu)} dt + \int_0^T a_\mu(t; v, w-v) dt + \frac{1}{2}\|w(T) - \Psi(T)\|^2_{L^2(\mathcal{H},\mu)} \ge 0,$$

for all $w \in \hat{\mathcal{K}}^{\Psi,p}_{\mu,\infty}$.

PROOF: We first prove the result for the homogenized variational inequality. The advantage in doing so is that the closed convex set $\hat{\mathcal{K}}^p_{\mu,\infty}$ does not vary when taking the limit as $k \to \infty$. For any $k \in \mathbb{N}$ we set $\bar{v}_k := V_k$ and

$$\mathcal{A}^k(\,\cdot\,) := \int_0^T \left[(\frac{\partial \Psi_k}{\partial t}(t), \,\cdot\,)_{L^2(\mathcal{H},\mu)} - a_\mu(t; \Psi_k(t), \,\cdot\,) \right] dt.$$

From the strong convergence $\Psi_k \to \Psi$ in $W^{1,2}([0,T] \times \mathcal{H}, \lambda \times \mu)$ and in $L^2(0,T;L^p(\mathcal{H},\mu))$, and from the same arguments as in Proposition 6.2.3 we can easily prove

$$\lim_{k \to \infty} \|\mathcal{A}^k - \mathcal{A}\|_{L^2(0,T;\mathcal{V}^{p\,*})} = 0,$$

where

$$\mathcal{A}(\,\cdot\,) := \int_0^T \left[(\frac{\partial \Psi}{\partial t}(t), \,\cdot\,)_{L^2(\mathcal{H},\mu)} - a_\mu(t; \Psi(t), \,\cdot\,) \right] dt.$$

We have that $\bar{u}_k = \bar{v}_k - \Psi_k$ solves

$$-\int_0^T \big(\frac{\partial w}{\partial t}, w - \bar{u}_k\big)_{L^2(\mathcal{H},\mu)} dt + \int_0^T a_\mu(t; \bar{u}_k, w - \bar{u}_k) dt \\ -\mathcal{A}^k(w - \bar{u}_k) + \frac{1}{2}\|w(T)\|^2_{L^2(\mathcal{H},\mu)} \ge 0,$$

for all $w \in \hat{\mathcal{K}}^p_{\mu,\infty}$. We fix $w \in \hat{\mathcal{K}}^p_{\mu,\infty}$ and take the limit as $k \to \infty$. The same arguments as in Theorem 6.3.1 and Theorem 5.4.1 allow us to conclude that

$$\begin{aligned} \int_0^T \big(\frac{\partial w}{\partial t}, \bar{u}_k\big)_{L^2(\mathcal{H},\mu)} dt &\to \int_0^T \big(\frac{\partial w}{\partial t}, \bar{u}\big)_{L^2(\mathcal{H},\mu)} dt, \\ \int_0^T a_\mu(t; \bar{u}_k, w) dt &\to \int_0^T a_\mu(t; \bar{u}, w) dt, \\ \mathcal{A}^k(\bar{u}_k) &\to \mathcal{A}(\bar{u}), \\ \lim_{k\to\infty} \int_0^T a_\mu(t; \bar{u}_k, \bar{u}_k) dt &\geq \int_0^T a_\mu(t; \bar{u}, \bar{u}) dt \end{aligned}$$

Hence we have

$$\begin{aligned} -\int_0^T \big(\frac{\partial w}{\partial t}, w - \bar{u}\big)_{L^2(\mathcal{H},\mu)} dt + \int_0^T a_\mu(t; \bar{u}, w - \bar{u}) dt \\ -\mathcal{A}(w - \bar{u}) + \frac{1}{2}\|w(T)\|^2_{L^2(\mathcal{H},\mu)} \geq 0, \end{aligned}$$

where clearly $\bar{u} = \bar{v} - \Psi$. Moreover the inequality holds for all $w \in \hat{\mathcal{K}}^p_{\mu,\infty}$. We substitute $\bar{u} = \bar{v} - \Psi$ in the variational inequality and obtain

$$-\int_0^T \big(\frac{\partial w}{\partial t}, w - v\big)_{L^2(\mathcal{H},\mu)} dt + \int_0^T a_\mu(t; v, w - v) dt + \frac{1}{2}\|w(T) - \Psi(T)\|^2_{L^2(\mathcal{H},\mu)} \geq 0,$$

for all $w \in \hat{\mathcal{K}}^{\Psi,p}_{\mu,\infty}$. From the probabilistic results we know that $\bar{v} = V$ and

$$V(t,x) = \sup_{t \leq \tau \leq T} \mathbb{E}\left[\Psi(\tau, X^{t,x}_\tau)\right],$$

This concludes the proof. ∎

We can now proceed in the characterization of the optimal stopping time. We have the following lemma

Lemma 6.7.1 *Let $\tau^*_{t,x,k}$ be the stopping time defined as*

$$\tau^*_{t,x,k} := \inf\{s \geq t \,:\, V_k(s, X^{t,x}_s) = \Psi_k(s, X^{t,x}_s)\} \wedge T.$$

Similarly let $\tau^*_{t,x}$ be the stopping time

$$\tau^*_{t,x} := \inf\{s \geq t \,:\, V(s, X^{t,x}_s) = \Psi(s, X^{t,x}_s)\} \wedge T.$$

Then

$$\lim_{k\to\infty}(\tau^*_{t,x,k}\wedge\tau^*_{t,x})(\omega)=\tau^*_{t,x}(\omega),\quad \mathbb{P}\text{-}a.e.\,\omega\in\Omega.$$

PROOF: The proof in this case is a simpler version of those carried out previously. It is worth analyzing why in this case we do not rely on a subsequence. As usual let us prove this result in the case of diffusions starting at time zero. We consider the initial data given and fixed and simplify the notation without losing in generality, i.e.

$$\tau^*:=\inf\{t\geq 0\,:\,V(t,X^x)=\Psi(t,X^x)\}.$$

We know from the previous section that τ^*_k is optimal for the k-th regularized problem. . For those $\omega\in\Omega$ s.t. $\tau^*(\omega)=0$ there is nothing to prove. Let us now take $\omega\in\Omega$ such that $\tau^*(\omega)>\delta$ for some $\delta>0$. Then as usual

$$V(t,X^x_t(\omega))>\Psi(t,X^x_t(\omega)),\qquad t\in[0,\tau^*(\omega)-\delta].$$

The map $t\mapsto X^x_t(\omega)$ is continuous, $[0,\tau^*(\omega)-\delta]$ is a compact set and $(t,x)\mapsto V(t,x)-\Psi(t,x)$ is continuous as well. There exists $\eta(\delta,\omega)>0$ such that

$$\eta(\delta,\omega):=\min\{V(t,X^x_t(\omega))-\Psi(t,X^x_t(\omega)),\,t\in[0,\tau^*(\omega)-\delta]\},$$

and

$$V(t,X^x_t(\omega))\geq\Psi(t,X^x_t(\omega))+\eta(\delta,\omega),\qquad t\in[0,\tau^*(\omega)-\delta].$$

From the uniform convergences $\Psi_k\to\Psi$ and $V_k\to V$ we can conclude that for $K(\delta,\omega;x)\in\mathbb{N}$ large enough and for all $k\geq K(\delta,\omega;x)$ we have

$$V_k(t,X^x_t(\omega))>\Psi_k(t,X^x_t(\omega)),\qquad t\in[0,\tau^*(\omega)-\delta].$$

This clearly implies that for any δ,ω,x given there exists a number $K(\delta,\omega;x)>0$ such that $\tau^*_k(\omega)>\tau^*(\omega)-\delta$ for all $k\geq K(\delta,\omega;x)$. We then have $(\tau^*_k\wedge\tau^*)(\omega)\to\tau^*(\omega)$. The convergence holds $\mathbb{P}$-a.e. $\omega\in\Omega$, as $j\to\infty$. ■

We can conclude our analysis proving the optimality of the stopping time.

Theorem 6.7.2 *The stopping time*

$$\tau^*_{t,x} := \inf\{s \geq t \,:\, V(s, X^{t,x}_s) = \Psi(s, X^{t,x}_s)\} \wedge T,$$

is optimal for the optimal stopping problem

$$V(t,x) = \sup_{t \leq \tau \leq T} \mathbb{E}\left[\Psi(\tau, X^{t,x}_\tau)\right].$$

PROOF: Once more we start from equation (6.24) which now reads

$$V_k(t,x) = \mathbb{E}\left[V_k(\tau^*_k, X^{t,x}_{\tau^*_k})\right].$$

The same clearly holds when we substitute $\tau^*_k \wedge \tau^*$ in it, i.e.

$$V_k(t,x) = \mathbb{E}\left[V_k(\tau^*_k \wedge \tau^*, X^{t,x}_{\tau^*_k \wedge \tau^*})\right].$$

We are interested in taking the limit as $k \to \infty$. The left hand side converges to $V(t,x)$ by uniform convergence. The right hand side can be treated as follows

$$\begin{aligned}
&\left|\mathbb{E}\left[V_k(\tau^*_k \wedge \tau^*, X^{t,x}_{\tau^*_k \wedge \tau^*}) - V(\tau^*, X^{t,x}_{\tau^*})\right]\right| \\
&\quad \leq \mathbb{E}\left[\left|V_k(\tau^*_k \wedge \tau^*, X^{t,x}_{\tau^*_k \wedge \tau^*}) - V(\tau^*_k \wedge \tau^*, X^{t,x}_{\tau^*_k \wedge \tau^*})\right|\right] \\
&\quad + \mathbb{E}\left[\left|V(\tau^*_k \wedge \tau^*, X^{t,x}_{\tau^*_k \wedge \tau^*}) - V(\tau^*, X^{t,x}_{\tau^*})\right|\right].
\end{aligned}$$

The first term converges to zero from uniform convegrence. The second one converges to zero because of the continuity of V and the result in Lemma 6.7.1. Hence we get

$$V(t,x) = \mathbb{E}\left[V(\tau^*, X^{t,x}_{\tau^*})\right] = \mathbb{E}\left[\Psi(\tau^*, X^{t,x}_{\tau^*})\right].$$

This proves the optimality of τ^* and concludes the proof. ■

In the next section we will shortly discuss the uniqueness of the solution.

6.8 Some remarks about uniqueness

It is worth noticing that all the results in this chapter and in the previous ones hold in the case of the diffusion (1.9) and with stochastic discount factor as in (1.15). This is due to the fact that, as long as the non-linear drift term has the form (1.7), all the important regularity properties are preserved. Moreover, as long as the discount factor is bounded from below, the algorithm implemented in solving the variational inequality remains the same. Notice also that for $u, v \in \mathcal{V}$ it holds

$$\int_{\mathcal{H}} (x)^+ u(x) v(x) \mu(dx) < C \; |||u||| \; |||v||| \, .$$

Hence the variational inequality is well posed, because the bilinear form is still continuous.

We were able to prove a new existence result for the solution of the infinite dimensional variational inequality. Yet, it was not possible to prove uniqueness of the solution or at least maximality/minimality. Similarly, in [56], Theorem 9, the Author does not claim the uniqueness when he introduces the application of his results to the American Bond option problem, although he considers a simpler model with deterministic diffusion coefficient. In [2] the explicit results for the American Bond option are not even discussed. Hence, our results are compatible with the ones in [2] and [56] and indeed provide new insights for the theory of infinite dimensional variational inequalities.

In our case the main problem in proving the uniqueness or the maximality/minimality is due to the degeneracy of the bilinear form. In some finite dimensional cases (cf. [43, 44]) it is possible to recover the coerciveness of the bilinear form even though the diffusion is degenerate. In our case this seems to be impossible due to several problems arising from the unbounded terms of the bilinear form. The key ingredient for the uniqueness in some infinite dimensional cases (see [2] and [56]) is the fact that μ is assumed to be an excessive measure for the semigroup generated by the process. This fact is the infinite dimensional analogue of the recovery of coerciveness in [43, 44].

Even though an excessive measure always exists for any diffusion process, unfortunately the Gaussian measure we need seems not to fulfill this requirement. Notice that the excessive measure can be written explicitly in rather few cases while we are able to give a precise notion of our measure μ. In some sense, the non-uniqueness can be considered as the drawback when writing the measure explicitly. If we simplify the SDE and choose deterministic constant volatility $\sigma(X_t) =$

$\sigma \in \mathcal{H}$ then a Gaussian invariant measure for the HJM dynamics exists (cf. [21], Chapter 9, Theorem 9.3.1). Hence, it seems natural to choose such Gaussian measure to perform our analysis. Since an invariant measure is also excessive we might be able to recover uniqueness of our solution by means of a suitable modification of the proof in [4]. This issue remains rather delicate and deserves further investigations.

In the general case of non-constant diffusion coefficient the problem gets far more complicated and the issue of uniqueness represents an extremely hard task. Maximality/minimality of the solution cannot be recovered approaching the problem via the penalization techniques described in [4]. In fact even in finite dimensions the coerciveness is a necessary condition for the proof (cf. [45, 51]). Only in very special cases the uniqueness might be recovered. For instance, considering the Goldys-Musiela-Sondermann model [34], it should be possible to remove the unbounded term in the SDE of the forward rate (cf. [31]). Then, the variational inequality might be solved in a different convex set in which uniqueness would be recovered.

In general we expect that further characterizations of the solution of the variational inequality (if available) arise choosing a proper convex set which has to be determined on a case by case basis.

Chapter 7

Asymptotic properties of the continuation region

In this chapter we show the connection between the continuation/stopping regions of the sequence of finite dimensional optimal stopping problems and the one in infinite dimensions. In particular, we prove that in a simple case, when the Yosida approximation is not required, it is possible to fully characterize the shape of the infinite dimensional continuation/stopping region as the limit, in a proper sense, of the continuation/stopping regions at the finite dimensional level.

7.1 A simplified setting

The aim of the analysis we carry out here is to find a connection between the optimality regions of the approximating stopping problems and the original one. The simplest case is the one in which the approximating procedure can be reduced to a single index approximation. In practice we can obtain rather explicit results when Assumption 3.3.1 is fulfilled. It is clear from the analysis of the previous sections that, under this assumption, the approximating algorithm can be obtained by means of the finite dimensional reduction only. Hence

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |V^{(n)}(t,x^{(n)}) - V(t,x)| = 0, \quad x \in \mathcal{H}.$$

Moreover the convergence is uniform on compact subsets of $\mathcal{H}$ and it also holds in $L^p(\mathcal{H},\mu)$-norms. It is worth noticing that the regularization of the gain function is not being taken into account. We only attempt to analyze how the behaviour of the continuation and stopping regions at the finite dimensional level affects the one at the infinite dimensional level. Clearly we are restricting our attention to the simplest case.

We now denote by $\mathcal{C}$ and $\mathcal{S}$ respectively the continuation and the stopping regions for the original problem. Then we can set

$$\mathcal{C} := \{(t,x) \in [0,T] \times \mathcal{H} \,:\, V(t,x) > \Psi(t,x)\},$$

$$\mathcal{S} := \{(t,x) \in [0,T] \times \mathcal{H} \,:\, V(t,x) = \Psi(t,x)\}.$$

In particular we focus on the t-section of the continuation region $\mathcal{C}_t$ defined as

$$\mathcal{C}_t := \{x \in \mathcal{H} \,:\, V(t,x) > \Psi(t,x)\},$$

for $t \in [0,T]$ given. We now set $\varepsilon > 0$ arbitrary and fixed and define the ε-optimal stopping strategy by characterizing the sets $\mathcal{C}_t^{(\varepsilon)}$ and $\mathcal{S}_t^{(\varepsilon)}$ as

$$\mathcal{C}_t^{(\varepsilon)} := \{x \in \mathcal{H} \,:\, V(t,x) > \Psi(t,x) + \varepsilon\},$$

$$\mathcal{S}_t^{(\varepsilon)} := \{h \in \mathcal{H} \,:\, V(t,x) \leq \Psi(t,x) + \varepsilon\}.$$

For $\varepsilon > 0$ given, it is easy to verify the sets inclusions $\mathcal{C}_t^{(\varepsilon)} \subset \mathcal{C}_t$ and $\mathcal{S}_t \subset \mathcal{S}_t^{(\varepsilon)}$, moreover $\mathcal{C}_t^{(\varepsilon)} \uparrow \mathcal{C}_t$ and $\mathcal{S}_t^{(\varepsilon)} \downarrow \mathcal{S}_t$ as $\varepsilon \to 0$. Similarly, we can define a continuation and a stopping region for the approximating optimal stopping problem $\mathcal{C}^{(n)}$ and $\mathcal{S}^{(n)}$ as

$$\mathcal{C}^{(n)} := \{(t,y) \in [0,T] \times \mathcal{H}^{(n)} \,:\, V^{(n)}(t,y) > \Psi^{(n)}(t,y)\},$$

$$\mathcal{S}^{(n)} := \{(t,y) \in [0,T] \times \mathcal{H}^{(n)} \,:\, V^{(n)}(t,y) = \Psi^{(n)}(t,y)\}.$$

For $t \in [0,T]$ given the t-section of the continuation region then holds to be

$$\mathcal{C}_t^{(n)} := \{y \in \mathcal{H}^{(n)} \,:\, V^{(n)}(t,y) > \Psi^{(n)}(t,y)\},$$

and adopting the same rationale as before we can prove the following sets inclusions $\mathcal{C}_t^{(n,\varepsilon)} \subset \mathcal{C}_t^{(n)}$ and $\mathcal{S}_t^{(n)} \subset \mathcal{S}_t^{(n,\varepsilon)}$. Here clearly

$$\mathcal{C}_t^{(n,\varepsilon)} := \{y \in \mathcal{H}^{(n)} \,:\, V^{(n)}(t,y) > \Psi^{(n)}(t,y) + \varepsilon\},$$

$$\mathcal{S}_t^{(n,\varepsilon)} := \{y \in \mathcal{H}^{(n)} \,:\, V^{(n)}(t,y) \leq \Psi^{(n)}(t,y) + \varepsilon\}.$$

Remark 7.1.1 *By the continuity property of the gain function and of the value function, both in the original optimal stopping problem and in the approximating problems, we know that the sets $\mathcal{C}_t$, $\mathcal{C}_t^{(\varepsilon)}$, $\mathcal{C}_t^{(n)}$ and $\mathcal{C}_t^{(n\,,\,\varepsilon)}$ are open sets whilst $\mathcal{S}_t$, $\mathcal{S}_t^{(\varepsilon)}$, $\mathcal{S}_t^{(n)}$ and $\mathcal{S}_t^{(n\,,\,\varepsilon)}$ are closed sets*

We are now ready to prove a result which is intuitively reasonable but not completely obvious and constitutes a technical lemma useful to prove the convergence of the approximating continuation regions.

Lemma 7.1.1 *Let us assume that for $\varepsilon > 0$ given and fixed $\mathcal{C}_t^{(\varepsilon)} \neq \varnothing$, then for any $x \in \mathcal{C}_t^{(\varepsilon)}$there exists $n \geq 0$ large enough and such that $x^{(n)} \in \mathcal{C}_t^{(\varepsilon)}$.*

PROOF: The crucial fact is that for any $t \in [0,T]$ the map $x \mapsto (V(t,x) - \Psi(t,x))$ is continuous. The subset (ε, ∞) is an open subset of the real axis and then $\mathcal{C}_t^{(\varepsilon)}$ has to be an open set in $\mathcal{H}$ because it is the inverse image of (ε, ∞) through $(V(t,x) - \Psi(t,x))$. Then for any $x \in \mathcal{C}_t^{(\varepsilon)}$ there exists a $\delta_\varepsilon > 0$ and an open sphere $B_{\delta_\varepsilon}(x)$ of radius δ_ε such that $y \in \mathcal{C}_t^{(\varepsilon)}$ for all $y \in B_{\delta_\varepsilon}(x)$. Since $x^{(n)}$ represents a polynomial expansion of x, there must exists $N_{\delta_\varepsilon} \geq 0$ such that $\|x - x^{(n)}\|_{\mathcal{H}} < \delta_\varepsilon$ for all $n \geq N_{\delta_\varepsilon}$. It then concludes the proof. ∎

This fact in particular implies that if $\mathcal{C}_t^{(\varepsilon)} \neq \varnothing$ then we can choose $n \geq 0$ to be large enough in order for $\mathcal{C}_t^{(\varepsilon)} \cap \mathcal{H}^{(n)} \neq \varnothing$ to be granted. We can now prove that the t-sections of the approximating continuation regions converge in some appropriate sense to the t-section of the continuation region for the original problem. In order to do so we introduce the sets

$$\mathcal{D}_t := \bigcup_{n \geq 1} \left(\mathcal{C}_t \cap \mathcal{H}^{(n)} \right), \tag{7.1}$$

$$\mathcal{D}_t^{(\varepsilon)} := \bigcup_{n \geq 1} \left(\mathcal{C}_t^{(\varepsilon)} \cap \mathcal{H}^{(n)} \right). \tag{7.2}$$

Clearly $\mathcal{D}_t \subset \mathcal{C}_t$, $\mathcal{D}_t^{(\varepsilon)} \subset \mathcal{C}_t^{(\varepsilon)}$ and $\mathcal{D}_t^{(\varepsilon)} \uparrow \mathcal{D}_t$ as $\varepsilon \downarrow 0$ given that $\mathcal{C}_t^{(\varepsilon)} \uparrow \mathcal{C}_t$. Our interest for these sets is motivated by the following lemma

Lemma 7.1.2 *The set inclusion $\mathcal{C}_t \subset \overline{\mathcal{D}}_t$ holds.*

PROOF: Let $\varepsilon > 0$ be given and let $x \in \mathcal{C}_t^{(\varepsilon)}$. Then there exists a sequence $\{x^{(n)}\}_{n=1}^{\infty}$ with $x^{(n)} := P_n x$ and hence $x^{(n)} \in \mathcal{H}^{(n)}$, such that $\|x^{(n)} - x\|_{\mathcal{H}} \to 0$ as $n \to \infty$. Then for the continuity of V and Ψ there exists $N(\varepsilon, x) \in \mathbb{N}$ such that

$$|V(t,x) - V(t,x^{(n)})| < \frac{\varepsilon}{2},$$

$$|\Psi(t,x) - \Psi(t,x^{(n)})| < \frac{\varepsilon}{2},$$

for all $n \geq N(\varepsilon, x)$. We have implicitly used the fact that $\Psi^{(n)}(t,x^{(n)}) = \Psi(t,x^{(n)})$. These two inequalities imply that

$$V(t,x^{(n)}) > V(t,x) - \frac{\varepsilon}{2} > \Psi(t,x) + \varepsilon - \frac{\varepsilon}{2} > \Psi(t,x^{(n)}) - \frac{\varepsilon}{2} + \varepsilon - \frac{\varepsilon}{2} = \Psi(t,x^{(n)}),$$

for all $n \geq N(\varepsilon, x)$. It means that $V(t,x^{(n)}) > \Psi(t,x^{(n)})$ for all but finitely many $x^{(n)}$. This also implies that $x^{(n)} \in \mathcal{C}_t \cap \mathcal{H}^{(n)}$ for all $n \geq N(\varepsilon, x)$ and hence $x^{(n)} \in \bigcup_{k \geq 1} \left(\mathcal{C}_t \cap \mathcal{H}^{(n)} \right)$ for all $n \geq N(\varepsilon, x)$. In summary for $\varepsilon > 0$ given and fixed and for any $x \in \mathcal{C}_t^{(\varepsilon)}$, the elements of the convergent sequence $\{x^{(n)}\}_{n=1}^{\infty}$, stay definitely in $\mathcal{D}_t$. One can then conclude that $x \in \overline{\mathcal{D}_t}$. It then implies that

$$\mathcal{C}_t^{(\varepsilon)} \subset \overline{\mathcal{D}_t}.$$

The set inclusion does not depend on ε. Hence taking the limit as $\varepsilon \to 0$ it turns out that $\mathcal{C}_t \subset \overline{\mathcal{D}_t}$.

■

By definition we also have $\mathcal{D}_t \subset \mathcal{C}_t$ and thanks to the lemma we conclude thats

$$\mathcal{D}_t \subset \mathcal{C}_t \subset \overline{\mathcal{D}_t}.$$

We want now characterize $\mathcal{D}_t$ in terms of the sets $\mathcal{C}^{(n)}$. We will make it through two popositions.

Proposition 7.1.1 *The following set inclusion holds*

$$\mathcal{D}_t \subset \bigcup_{n\geq 1}\bigcap_{p\geq n} \mathcal{C}_t^{(p)}. \tag{7.3}$$

PROOF: Let $\varepsilon > 0$ be given and let $x \in \mathcal{D}_t^{(\varepsilon)}$. There exists $N \in \mathbb{N}$ such that $x \in \mathcal{C}_t^{(\varepsilon)} \cap \mathcal{H}^{(N)}$. This implies that $x \in \bigcap_{n\geq N}\left(\mathcal{C}_t^{(\varepsilon)} \cap \mathcal{H}^{(n)}\right)$. Moreover by the convergence results there exists $M(\varepsilon, x) \in \mathbb{N}$ such that

$$|V^{(n)}(t, x^{(n)}) - V(t, x)| < \varepsilon, \qquad \forall n \geq M(\varepsilon, x).$$

Denoting by $N(\varepsilon, x) = N \vee M(\varepsilon, x)$, we obtain

$$V^{(n)}(t, x) > V(t, x) - \varepsilon > \Psi(t, x) + \varepsilon - \varepsilon, \qquad \forall n \geq N(\varepsilon, x),$$

i.e. $V^{(n)}(t, x) > \Psi(t, x)$ and $x \in \mathcal{H}^{(n)}$, $\forall n \geq N(\varepsilon, x)$. This fact implies that

$$x \in \bigcap_{k\geq N(\varepsilon,x)} \mathcal{C}_t^{(k)} \subset \bigcup_{n\geq 1}\bigcap_{p\geq n} \mathcal{C}_t^{(p)}.$$

Since it holds for any $x \in \mathcal{D}_t^{(\varepsilon)}$ we have

$$\mathcal{D}_t^{(\varepsilon)} \subset \bigcup_{n\geq 1}\bigcap_{p\geq n} \mathcal{C}_t^{(p)},$$

and since the inclusion is uniform with respect to ε the proof is completed taking the limit as $\varepsilon \to 0$. ■

A similar estimate is provided in the following proposition.

Proposition 7.1.2 *The following set inclusion holds*

$$\bigcap_{n\geq 1}\bigcup_{p\geq n}\mathcal{C}_t^{(p)}\subset\mathcal{D}_t. \tag{7.4}$$

PROOF: Let $\varepsilon>0$ be given and let $x\in\bigcap_{n\geq 1}\bigcup_{p\geq n}\mathcal{C}_t^{(p,\varepsilon)}$. Then x belongs to infinitely many of $\mathcal{C}_t^{(p,\varepsilon)}$. It follows that there exists $N(\varepsilon,x)\in\mathbb{N}$ such that

$$|V^{(n)}(t,x^{(n)})-V(t,x)|<\varepsilon,\qquad\forall n\geq N(\varepsilon,x),$$

and $x\in\mathcal{C}_t^{(p,\varepsilon)}$ for infinitely many indexes $p>N(\varepsilon,x)$. The latter consideration implies that $x\in\mathcal{H}^{(n)}$ for $n\geq N(\varepsilon,x)$. We can then consider $\bar{p}>N(\varepsilon,x)$ and $x\in\mathcal{C}_t^{(\bar{p},\varepsilon)}$. Recalling that $x^{(\bar{p})}=P_{\bar{p}}x=x$ we obtain

$$V(t,x)>V^{(\bar{p})}(t,x)-\varepsilon>\Psi(t,x)+\varepsilon-\varepsilon=\Psi(t,x).$$

We then have that $x\in\mathcal{C}_t$ and $x\in\mathcal{H}^{(\bar{p})}$, i.e.

$$x\in\mathcal{C}_t\cap\mathcal{H}^{(\bar{p})}\subset\bigcup_{n\geq 1}\left(\mathcal{C}_t\cap\mathcal{H}^{(n)}\right).$$

For the arbitrariness of x, this implies that

$$\bigcap_{n\geq 1}\bigcup_{p\geq n}\mathcal{C}_t^{(p,\varepsilon)}\subset\mathcal{D}_t,$$

and then taking the limit as $\varepsilon\to 0$ we conclude the proof. ■

We have shown that

$$\limsup_{n\to\infty}\mathcal{C}_t^{(n)}=\bigcap_{n\geq 1}\bigcup_{p\geq n}\mathcal{C}_t^{(p)}\subset\mathcal{D}_t\subset\bigcup_{n\geq 1}\bigcap_{p\geq n}\mathcal{C}_t^{(p)}=\liminf_{n\to\infty}\mathcal{C}_t^{(n)}.$$

It is now clear that

$$\mathcal{D}_t=\bigcap_{n\geq 1}\bigcup_{p\geq n}\mathcal{C}_t^{(p)}=\bigcup_{n\geq 1}\bigcap_{p\geq n}\mathcal{C}_t^{(p)}.$$

We have proved the following theorem

Theorem 7.1.1 *The limiting behavior of the continuation regions can be characterized as follows*

$$\mathcal{D}_t \subset \mathcal{C}_t \subset \overline{\mathcal{D}_t}, \tag{7.5}$$

where

$$\mathcal{D}_t = \bigcap_{n\geq 1} \bigcup_{p\geq n} \mathcal{C}_t^{(p)} = \bigcup_{n\geq 1} \bigcap_{p\geq n} \mathcal{C}_t^{(p)}. \tag{7.6}$$

When we remove the Assumption 3.3.1, then the same arguments hold if instead of $\mathcal{C}$ we consider the continuation region of the Yosida approximating stopping problem, i.e.

$$\mathcal{C}_\alpha := \{(t,x) \in [0,T] \times \mathcal{H} \,:\, V_\alpha(t,x) > \Psi(t,x)\}.$$

The connection between the finite dimensional problems and the original one should pass through an intermediate relation between the latter and the Yosida approximation. At this stage it is not completely clear whether such a connection can be explicitly established and if it would provide a meaningful description of the final optimality regions.

Appendix A

Regularization of the gain function

Here we introduce a smoothing procedure that represents a generalization of the results proven in [42], Chapter 4, Lemma 4.1. We first focus on the particular problem of regularizing the gain function of the Put option on a Bond and then we shortly discuss some further extensions.

We can define a family $\{\Phi_t\}_{t\in[0,\hat{T}]} \subset \mathcal{H}^*$, as

$$\Phi_t(h) := -\int_0^{\hat{T}-t} h(x)dx \tag{A-1}$$

It is easy to see that

$$|\Phi_t(h)| \leq C_{\hat{T}} \|h\|_{\mathcal{H}}, \qquad h \in \mathcal{H}^*,$$

and hence the family $\{\Phi_t\}_{t\in[0,\hat{T}]} \subset \mathcal{H}^*$ is uniformly bounded by the constant $C_{\hat{T}}$. Notice that

$$\Psi(t,h) = (K - e^{\Phi_t(h)})^+,$$

can be understood as the composition of $f : [0,T] \times \mathcal{H}_w \to \mathbb{R}$,

$$f(t,h) := K - e^{\Phi_t(h)},$$

and $g : \mathbb{R} \to \mathbb{R}_+$, where $g(z) := (z)^+$. In practice $\Psi(t,h) = g \circ f\,(t,h)$. We also remark that $Im(f) = (-\infty, K)$ and hence $g : (-\infty, K) \to [0, K)$.We denote $I := (-\infty, K)$ and it is easy to verify that $g \in W^{1,p}(I)$ for all $1 \leq p \leq \infty$. We mollify g by means of standard mollifiers (cf. [14], Chapter 4) and define the sequence $\{g_k\}_{k\geq 1} \subset C_c^\infty(I)$, as $g_k := \rho_k \star g$. Then the convergence

$g_k \to g$ in $W^{1,p}(I)$, $1 \le p < \infty$ holds. It is easy to prove that $g'_k = \rho_k \star g'$, where g' represents the weak derivative of g. Notice that $g_k \to g$ and $g'_k \to g'$ pointwise. If we consider g on the whole $\mathbb{R}$ the convergence is also locally uniform, i.e. $\|g_k - g\|_{L^\infty(\hat{I})} \to 0$, as $k \to \infty$ for any compact $\hat{I} \subset \mathbb{R}$. Since g and its weak derivative g' are both uniformly bounded on I, we can easily verify that $\|g_k\|_{L^\infty(I)} \le \|g\|_{L^\infty(I)}$ and $\|g'_k\|_{L^\infty(I)} \le \|g'\|_{L^\infty(I)}$.

We see that $f \in C^\infty([0,\hat{T}] \times \mathcal{H}_w)$. Let μ be a finite measure on $\mathcal{H}_w$ and let D be the closure in $L^2(\mathcal{H}_w, \mu)$ of the directional derivative on $\mathcal{H}_w$ (cf. Appendix E). Hence we have

$$D(g_k \circ f)(t,h) = g'_k(f(t,h))\, Df(t,h).$$

From pointwise convergence and dominated convergence theorem we can conclude that

$$g_k \circ f \to g \circ f, \quad \text{in } L^2(0,\hat{T}; L^2(\mathcal{H}_w, \mu)),$$

$$D(g_k \circ f) \to g'(f)\, Df, \quad \text{in } L^2(0,\hat{T}; L^2(\mathcal{H}_w, \mu; \mathcal{H}_w)),$$

and from the closedness of D we have $D(g \circ f) = g'(f)\, Df$. We have implicitly exploited the fact that when we integrate over $[0,\hat{T}] \times \mathcal{H}_w$ the function f ranges over $I \subset \mathbb{R}$ and hence g_k and g'_k remain uniformly bounded.

We now have a smooth approximation of our gain function, in fact $\Psi_k := g_k \circ f$ is in $C_b^\infty([0,\hat{T}] \times \mathcal{B}_w)$ for any $\mathcal{B}_w$ bounded subset of $\mathcal{H}_w$[1]. Moreover Ψ_k inherits the Lipschitz properties of Ψ discussed above. In particular we can exploit the uniform Lipschitz continuity with respect to the space variable to obtain a stronger convergence. In fact

$$\begin{aligned}|g_k(f(t,h)) - g(f(t,h))| &= \left| \int_{\mathbb{R}} \rho_k(f(t,h) - z) g(z) dz - g(f(t,h)) \right| \\ = \left| \int_{\mathbb{R}} \rho_k(y) \left[g(f(t,h) - y) - g(f(t,h)) \right] dy \right| &\le \int_{\mathbb{R}} \rho_k(y) \left| g(f(t,h) - y) - g(f(t,h)) \right| dy.\end{aligned}$$

[1] Identifying $\mathcal{H}_w$ and its dual we have that $\Phi_t = (\Phi_t^1, \Phi_t^2, \ldots)$ and $\|\Phi_t\|^2_{\mathcal{H}_w} = \sum_{i=1}^\infty (\Phi_t^i)^2$. Hence $\|Df(t,h)\|^2_{\mathcal{H}_w} = \sum_{i=1}^\infty (D_i f(t,h))^2 = e^{2\Phi_t(h)} \sum_{i=1}^\infty (\Phi_t^i)^2 = e^{2\Phi_t(h)} \|\Phi_t\|_{\mathcal{H}_w}$. Similarly $\langle D^2 f(t,h)\, u, v \rangle \le e^{\Phi_t(h)} \|\Phi_t\|^2_{\mathcal{H}_w} \|u\|_{\mathcal{H}_w} \|v\|_{\mathcal{H}_w}$. With the same rationale one proves infinite differentiability.

We simply use the fact that $|(t-y)^+ - (t)^+| \leq |y|$, which is indeed a weaker property than the one pointed out in Proposition 1.4.1. We also exploit the fact that $\text{supp}\{\rho_k\} = [-\frac{1}{k}, \frac{1}{k}]$. Hence

$$|g_k(f(t,h)) - g(f(t,h))| \leq \int_{\mathbb{R}} \rho_k(y)\,|y|\,dy = \int_{[-\frac{1}{k},\frac{1}{k}]} \rho_k(y)\,|y|\,dy \leq \frac{1}{k}.$$

This result implies that

$$\sup_{(t,h)\in[0,\hat{T}]\times\mathcal{H}_w} |\Psi_k(t,h) - \Psi(t,h)| \leq \frac{1}{k},$$

and hence the convergence is also uniform over the whole space $[0,\hat{T}]\times\mathcal{H}_w$. The fundamental consequence of this fact is summarized in the next proposition.

Proposition A.0.3 *The sequence of value functions $\{V_k\}_{k\geq 1}$, associated to the mollified gain functions $\{\Psi_k\}_{k\geq 1}$, converges uniformly to V, i.e.*

$$\lim_{k\to\infty} \sup_{(t,r)\in[0,T]\times\mathcal{H}_w} \left|V_k(t,r) - V(t,r)\right| = 0.$$

PROOF: We provide an estimate for the difference

$$\begin{aligned}
&V_k(t,r) - V(t,r) \\
&= \sup_{t\leq\tau\leq T} \mathbb{E}\left[e^{-\int_t^\tau (r_s^{t,r}(0))^+ ds}\Psi_k(\tau, r_\tau^{t,r})\right] - \sup_{t\leq\sigma\leq T} \mathbb{E}\left[e^{-\int_t^\sigma (r_s^{t,r}(0))^+ ds}\Psi(\sigma, r_\sigma^{t,r})\right] \\
&= \sup_{t\leq\tau\leq T} \inf_{t\leq\sigma\leq T} \mathbb{E}\left[e^{-\int_t^\tau (r_s^{t,r}(0))^+ ds}\Psi_k(\tau, r_\tau^{t,r}) - e^{-\int_t^\sigma (r_s^{t,r}(0))^+ ds}\Psi(\sigma, r_\sigma^{t,r})\right] \\
&\leq \sup_{t\leq\tau\leq T} \mathbb{E}\left[e^{-\int_t^\tau (r_s^{t,r}(0))^+ ds}\left(\Psi_k(\tau, r_\tau^{t,r}) - \Psi(\tau, r_\tau^{t,r})\right)\right] \\
&\leq \mathbb{E}\left[\sup_{t\leq u\leq\hat{T}} \left|\Psi_k(u, r_u^{t,r}) - \Psi(u, r_u^{t,r})\right|\right] \leq \frac{1}{k}.
\end{aligned}$$

The same holds for $V(t,r) - V_k(t,r)$ and hence we conclude the proof by observing that

$$\sup_{(t,r)\in[0,T]\times\mathcal{H}_w} \left|V_k(t,r) - V(t,r)\right| \leq \frac{1}{k}.$$

■

A crucial feature of this regularization algorithm is that the bounds of Proposition 1.4.1 holds independently of the order of the approximation.

It is worth noticing that the whole procedure holds in a number of different cases. Whenever we deal with a function $\Psi : [0,T] \times \mathcal{H} \to \mathbb{R}$ that may be written as a composition $\Psi := g \circ f$, the feasibility of the regularization depends clearly on the properties of $g : \mathbb{R} \to \mathbb{R}$ and $f : [0,T] \times \mathcal{H} \to \mathbb{R}$. A feasible case, simpler than the one discussed above but still quite general, is the one in which g is Lipschitz and bounded. For the Put option on a Bond, g is not bounded from above but indeed f does and hence, globally, the composition Ψ is bounded and Lipschitz.

Appendix B

Properties of C_0-semigroups

Here we summarize some fundamental results in semigroup theory which can be found in [49], Chapter 1, or in a shorter form in [19], Appendix A. For simplicity we refer to a generic Hilbert space $\mathcal{H}$, but the same results hold in wider generality.

First we recall that by definition the *resolvent* set $\rho(A)$ of the operator A is the set of all complex numbers α for which $(\alpha I - A)$ is invertible, i.e. $(\alpha I - A)^{-1}$ is a bounded operator in $\mathcal{H}$. The family of bounded linear operators $R(\alpha; A) := (\alpha I - A)^{-1}$, $\alpha \in \rho(A)$ is called resolvent of A.

We recall a fundamental theorem (cf. [49], Chapter 1, Sec. 5).

Theorem B.0.2 *Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a linear closed operator*[1]*. Then the following are equivalent*

i) *A is the infinitesimal generator of a C_0-semigroup $\{S(t), t \geq 0\}$ such that*

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{\omega t}, \qquad \forall t \geq 0,$$

ii) *$D(A)$ is dense in $\mathcal{H}$, the resolvent set $\rho(A)$ contains the interval $(\omega, +\infty)$ and the following estimates hold*

$$\|R^k(\alpha; A)\|_{\mathcal{L}(\mathcal{H})} \leq \frac{M}{(\alpha - \omega)^k}, \qquad k = 1, 2, \ldots$$

[1] A is closed if its graph $\mathcal{G}_A := \{(x, y) \in \mathcal{H} \times \mathcal{H} \,:\, x \in D(A),\, y = Ax\}$ is closed in $\mathcal{H} \times \mathcal{H}$. For closed operators one usually endowes the domain $D(A)$ with the graph norm $\|x\|_{D(A)} = \|x\|_{\mathcal{H}} + \|Ax\|_{\mathcal{H}}$ for $x \in D(A)$.

Moreover, if either (i) or (ii) holds then we have an explicit representation of the resolvent, i.e.

$$R(\alpha; A)x = \int_0^\infty e^{-\alpha t} S(t)x dt, \quad x \in \mathcal{H},\, \alpha > \omega.$$

Finally

$$S(t)x = \lim_{\alpha \to \infty} e^{tA_\alpha} x, \qquad \forall x \in \mathcal{H},$$

where $A_\alpha = \alpha A R(\alpha; A)$ and the estimate holds

$$\| e^{tA_\alpha} \|_{\mathcal{L}} \leq M e^{\frac{\alpha \omega t}{\alpha - \omega}}, \quad \forall t \geq 0,\, \alpha > \omega.$$

As a definition the operator $A_\alpha = \alpha A R(\alpha; A)$ is the Yosida approximation of A. It is a bounded linear operator on $\mathcal{H}$ and then generates a uniformly continuous semigroup which is indeed e^{tA_α}. We also mention that $t \mapsto S(t)x$ is a continuous map and hence the integral defining $R(\alpha; A)$ is an improper Riemann integral. Thanks to this fact and to the fact that A is a closed operator it is possible to prove that $AR(\alpha; A)x = R(\alpha; A)Ax$ for $x \in D(A)$. It is then easy to see that the following inequality holds

$$\| A_\alpha x \|_{\mathcal{H}} = \| R(\alpha; A) A x \|_{\mathcal{H}} \leq \frac{M}{\alpha} \| A x \|_{\mathcal{H}}, \quad x \in D(A).$$

Since we are interested in the cases when $\alpha \to \infty$ there is no loss in generality considering $\alpha > M$. Hence we get $\| A_\alpha x \|_{\mathcal{H}} < \| A x \|_{\mathcal{H}}$ for $x \in D(A)$ and $\alpha > M$.
The fundamental properties of the Yosida approximant are summarized in the next proposition.

Proposition B.0.4 *Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be the infinitesimal generator of a C_0-semigroup. Then*

$$\lim_{\alpha \to \infty} \alpha R(\alpha; A)x = x, \qquad \forall x \in \mathcal{H},$$

$$\lim_{\alpha \to \infty} A_\alpha x = Ax, \qquad \forall x \in D(A).$$

A remark which turns out to be crucial at some point of our analysis is the following:

Remark B.0.2 *Notice that if* $\sup_{t\in[0,S]} \|S(t)\|_{\mathcal{L}} \le M$ *then* $\sup_{t\in[0,S]} \|e^{tA_\alpha}\|_{\mathcal{L}} \le M$. *Moreover if we define* $A_{\alpha,n} = P_n A_\alpha P_n$ *then also* $\sup_{t\in[0,S]} \|e^{tA_{\alpha,n}}\|_{\mathcal{L}} \le M$. *In particular the latter can be proved defining the equivalent norm*

$$|x| := \sup_{t\in[0,T]} \|S(t)x\|_{\mathcal{H}}.$$

It is easy to verify that $\|x\|_{\mathcal{H}} \le |x| \le M\|x\|_{\mathcal{H}}$ *and* $|S(t)| := \sup_{|x|\le 1} |S(t)x| \le 1$. *This also implies* $|R(\alpha; A)| := \sup_{|x|\le 1} |R(\alpha; A)x| \le 1/\alpha$. *Then in terms of the new norm we obtain*

$$\left\|e^{tA_{\alpha,n}}x\right\|_{\mathcal{H}} \le \left|e^{tA_{\alpha,n}}x\right| = e^{-t\,\alpha}\left|e^{t\,\alpha^2 P_n R(\alpha;A)P_n}x\right| \le e^{-t\,\alpha}\sum_{k=1}^{\infty}\frac{1}{k!}\alpha^{2k}t^k|(P_n R(\alpha;A)P_n)^k x|$$
$$\le e^{-t\,\alpha}\sum_{k=1}^{\infty}\frac{1}{k!}\alpha^{2k}t^k|P_n R(\alpha;A)P_n|^k|x| \le e^{-t\,\alpha}\sum_{k=1}^{\infty}\frac{1}{k!}\alpha^{k}t^k|x| = |x| \le M\|x\|.$$

This simply implies

$$\sup_{t\in[0,T]} \left\|e^{tA_{\alpha,n}}\right\|_{\mathcal{L}(\mathcal{H})} \le M.$$

Appendix C

Basic convergence theorems

We recall here three useful results about convergence of bounded continuous functions. In this section we always consider real valued functions defined on a generic metric space. Yet some of these results hold in wider generality, cf. [24], Chapter 7.

The first theorem is the so called Dini's theorem (cf. [24], Chapter 7, Sec.2, Th. 7.2.2).

Theorem C.0.3 *Let E be a compact metric space. If an increasing (resp. dercreasing) sequence $\{f_n\}_{n=1}^{\infty}$ of real valued continuous functions converges simply*[1] *to a continuous function g, it converges uniformly to g.*

The second theorem is about uniform limit of continuous functions (cf. [24], Chapter 7, Sec. 2, Th. 7.2.1).

Theorem C.0.4 *Let E be a metric space. A uniform limit of bounded continuous functions on E is continuous.*

The last theorem connects pointwise and uniform convergences for the class of equicontinuous functions (cf. [24], Chapter 7, Sec. 5, Th. 7.5.6)

Theorem C.0.5 *Let E be a compact metric space, $(f_n)_{n\geq 1}$ an equicontinuous sequence in $C(E;\mathbb{R})$. If $(f_n)_{n\geq 1}$ converges simply to g in E, it converges uniformly to g.*

[1] Pointwise convergence.

Appendix D

Vector valued distributions

The solution to the variational inequality $u(t,x)$ can be understood as a function of time taking values in the Sobolev space $H_0^1(\mathcal{O})$, it is then useful to rely on the theory of vector valued distributions. For a complete treatment about this subject one can refer to [23], Chapter VIII. In particular if we denote by Y a Banach space then we denote by $L^1_{loc}(0,T;Y)$ the equivalence class of functions $t \to u(t)$, $u:(0,T) \to Y$ such that, u is $\lambda(dt)$-measurable and

$$\|u\|_{L^p(0,T;Y)} = \left(\int_0^T \|u(t)\|_Y^p dt \right)^{\frac{1}{p}} < \infty.$$

In order to give a meaning to the derivative of such a function u, we introduce the set $\mathcal{D}'(0,T;Y)$ of Y-valued distributions over the set of test functions $\varphi \in C_c^\infty(0,T)$, i.e. of the linear continuous mappings $f : C_c^\infty(0,T) \to Y$. For any $f \in \mathcal{D}'(0,T;Y)$, we define the m-th derivative of f as the distribution

$$\varphi \to (-1)^m f\left(\frac{d^m\varphi}{dt^m}\right), \qquad \varphi \in C_c^\infty(0,T).$$

We therefore have

$$\frac{d^m f}{dt^m} \in \mathcal{D}'(0,T;Y),$$

and

$$\frac{d^m f}{dt^m}(\varphi) = (-1)^m f\left(\frac{d^m\varphi}{dt^m}\right), \qquad \forall\varphi \in C_c^\infty(0,T).$$

In particular if $u \in L^2(0,T;Y)$, then first derivative is the distribution defined as

$$\frac{du}{dt}(\varphi) = -\int_0^T u(t)\varphi'(t)dt,$$

for all $\varphi \in C_c^\infty(0,T)$. In conclusion we say that $\frac{du}{dt} \in L^2(0,T;Y)$ if there exists $v \in L^2(0,T;Y)$ such that for all $\varphi \in C_c^\infty(0,T)$, one has $v(\varphi) = -u(\varphi')$, i.e.

$$\int_0^T v(t)\varphi(t)dt = -\int_0^T u(t)\varphi'(t)dt.$$

We introduce now the set $W(0,T;H_0^1(\mathcal{O}),H^{-1}((O)))$ which turns out to be the right set to look for a solution to our variational problem:

$$W(0,T;H_0^1(\mathcal{O}),H^{-1}(\mathcal{O})) = \{u:\ u \in L^2(0,T;H_0^1(\mathcal{O})),\ \frac{du}{dt} \in L^2(0,T;H^{-1}(\mathcal{O}))\}.$$

It is worth noticing that in the case $u = u(t,x) \in L^1_{loc}(0,T;L^p(\mathcal{O}))$ the distributional derivative presented above is equivalent to the partial distributional derivative $\frac{\partial u}{\partial t}$ of u in $\mathcal{D}'((0,T)\times\mathcal{O})$.

Appendix E

Dense subsets in $W^{1,2}(\mathcal{H}, \mu)$

The construction of the Gauss-Sobolev Space can rely on different approaches. A very general survey of this topic can be found in [11], Chapter 5. Nevertheless in this thesis we mostly refer to the approach of [18], Chapter 10 and [22], Chapter 9, which are substantially the same. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{E}(\mathcal{H})$ be defined as

$$\mathcal{E}(\mathcal{H}) := \text{span}\{\mathscr{R}e(\phi_h),\ \mathscr{I}m(\phi_h),\ \phi_h(x) = e^{i\langle h,x\rangle_{\mathcal{H}}},\ h \in \mathcal{H}\}.$$

By applying the rationale of [22], Proposition 1.2.5, and the simple results in [18], Propositions 1.6 and 1.7, it is easy to prove that $\mathcal{E}(\mathcal{H}) \subset L^2(\mathcal{H}, \mu)$ and in particular $\mathcal{E}(\mathcal{H})$ is dense in this space. Similarly one defines the subset $\mathcal{E}_A(\mathcal{H}) \subset \mathcal{E}(\mathcal{H})$ as

$$\mathcal{E}_A(\mathcal{H}) := \text{span}\{\mathscr{R}e(\phi_h),\ \mathscr{I}m(\phi_h),\ \phi_h(x) = e^{i\langle h,x\rangle_{\mathcal{H}}},\ h \in D(A^*)\}.$$

This set has the same properties as $\mathcal{E}(\mathcal{H})$ (cf. [22] pag.205). Moreover it is easy to prove that $D\phi_h(x) \in D(A^*)$ for all $\phi_h \in \mathcal{E}_A(\mathcal{H})$.

These results might be improved recalling a general fact from measure theory, cf.[38], Lemma 1.35, Chapter 1.

Lemma E.0.3 *Given a metric space $\mathcal{H}$ with Borel σ-field $\mathcal{B}$, a bounded measure μ on $(\mathcal{H}, \mathcal{B})$ and a constant $p > 0$, the set of bounded continuous functions on $\mathcal{H}$ is dense in $L^p(\mathcal{H}, \mu)$.*

Although $\mathcal{E}(\mathcal{H})$ is not dense in $C_b(\mathcal{H})$ the following theorem holds, cf. [18], Lemma 8.1, Chapter 8.

Lemma E.0.4 *For all* $\varphi \in C_b(\mathcal{H})$*, there exists a two-index sequence* $(\phi_{k,n}) \subset \mathcal{E}(\mathcal{H})$ *such that*

$$\lim_{k\to\infty}\lim_{n\to\infty}\phi_{k,n}(x) = \varphi(x), \qquad \forall x \in \mathcal{H},$$
$$\sup_{x\in\mathcal{H}}|\phi_{k,n}(x)| \le \sup_{x\in\mathcal{H}}|\varphi(x)| + \frac{1}{n}, \forall n, k \in \mathbb{N}.$$

The set $W^{1,2}(\mathcal{H},\mu)$ is obtained by proving that the gradient operator $(D, dom(D))$ is closable in $L^2(\mathcal{H},\mu)$. As a matter of fact $W^{1,2}(\mathcal{H},\mu)$ is built as the closure of $\mathcal{E}(\mathcal{H})$ (or equivalently of $\mathcal{E}_A(\mathcal{H})$) in the norm of $W^{1,2}(\mathcal{H},\mu)$, cf. [18], Proposition 10.3, Chapter 10. Sometimes it is useful to denote by $\overline{D}$ the closure of D. From this definition, the domain of $\overline{D}$ is $W^{1,2}(\mathcal{H},\mu)$.

Then as an obvious consequence we know that for any $\psi \in W^{1,2}(\mathcal{H},\mu)$ there exists a sequence $\{\phi_k\}_{k=1}^{\infty}$ made of elements of $\mathcal{E}_A(\mathcal{H})$ such that $\phi_k \to \psi$ in $W^{1,2}(\mathcal{H},\mu)$ as $k \to \infty$.

This construction might be extended to time dependent functions, indeed, if we consider the product space $[0,T]\times\mathcal{H}$ equipped with the product measure $\lambda(dt)\times\mu(dx)$ we can define

$$\mathcal{E}([0,T]\times\mathcal{H}) := \text{span}\{\mathscr{R}e(\phi_{\alpha,h}),\ \mathscr{I}m(\phi_{\alpha,h}),\ \phi_{\alpha,h}(t,x) = e^{i\alpha t + i\langle h,x\rangle_{\mathcal{H}}},\ (\alpha,h)\in\mathbb{R}\times\mathcal{H}\}.$$

This set is dense in $L^2([0,T]\times\mathcal{H},\lambda(dt)\times\mu(dx))$ from the same arguments as above and moreover one can deduce $W^{1,2}([0,T]\times\mathcal{H},\lambda(dt)\times\mu(dx))$ with the same rationale as in the stationary case. It is remarkable that, thanks to Lemmas E.0.4 and E.0.3, we might repeat all the arguments above to prove that the derivative, as a linear mapping,

$$D : \mathcal{E}(\mathcal{H}) \subset L^p(\mathcal{H},\mu) \to L^2(\mathcal{H},\mu;\mathcal{H}),\ \varphi \mapsto D\varphi,$$

is closable. Hence for any $\psi \in W^{1,2}(\mathcal{H},\mu)\cap L^p(\mathcal{H},\mu)$ there exists a sequence $(\phi_k)\subset\mathcal{E}(\mathcal{H})$ such that $\phi_k \to \psi$ in $W^{1,2}(\mathcal{H},\mu)\cap L^p(\mathcal{H},\mu)$.

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Acknowledgments

There are many people I wish to thank for the help and support they gave me during my Ph.D.
I would like to thank my advisor Prof. M.B. Chiarolla for her guide through this branch of maths which was completely unknown to me and that now I like so much.
Thanks to Prof. G. Peskir who was my temporary supervisor in Manchester and with whom I had numerous interesting discussions.
Thanks to G. Ferrari and A. Martire for being my friends and colleagues and making my days at the university easy and interesting all the time.
Thanks to G. Salvi and I. Poggio for all the nice time we spent together and for their patience, especially in the final days of this thesis when I was far too nervous :-)
Thanks to N. Nescatelli for his enthusiasm and friendship.
Thanks to Dr. F. Aurelio, Dr. F. Cesarone, Dr. S. Patrì, Dr. M. Rinaldi, Dr. G. Stabile and Dr. I. Valdivia for helping and listening to me constantly.
I owe my gratitude to Prof. F. Flandoli and Prof. C. Saccon for their kindness and patience. I appreciated a lot their help, comments and advices.
I wish to thank Prof. L. Petrella for numerous helpful advices.
I am thankful to M. Rosestolato for being a smart friend and for listening to me a number of times.
A special thank goes to my family for their love and their constant support in all of my activities. Their open mind and the way they blindly believe in me give me the strength and the enthusiasm to keep doing what I feel is right for myself.
Thanks to my dear Elena for her love, patience and for teaching me a lot about life and maths. We

both know that without her I would have not been able to complete this thesis and I would be much worse a person than I am :-)

Thanks to my closest friends with whom I spend my days and share my feelings. Their advices, their support and their company when drinking a beer late at night are absolutely invaluable. I will not list them all because they know who they are and thanks to God they are a lot :-)