CHARACTERISTIC POLYNOMIALS, ASSOCIATED TO THE ENERGY GRAPH OF THE NON LINEAR SCHRÖDINGER EQUATION

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Abstract. We study the irreducibility and the separation of characteristic polynomials, associated to the energy graph of the non-linear Schrödinger equation. This fact will be useful in the study of stability of a class of normal forms of the completely resonant non–linear Schrödinger equation on a torus described in [11]. The problem can be also considered as an independent interesting algebraic combinatorial problem.

Contents

1. Introduction

The main object in this work is the study of an algebraic and combinatorial problem (cf. Theorem 1.1) which arises from the study of non linear Schrödinger equation (NLS for short) on an n–dimensional torus:

$$-\mathrm{i}u_t + \triangle u = \kappa |u|^{2q} u, q \geq 1 \in \mathbb{N} \tag{1}$$

where $\kappa \in \mathbb{R}$, $u = u(t, \varphi), \varphi \in \mathbb{T}^n$, The case $q = 1$ is associated to the *cubic* NLS.

The NLS is an example of a universal nonlinear model that describes many physical nonlinear systems. The equation can be applied to hydrodynamics, nonlinear optics, nonlinear acoustics, quantum condensates, heat pulses in solids and various other nonlinear instability phenomena.

Remark 1.1. *One can rescale u to get $\kappa = \pm(q+1)$.*

Proof. See Appendix 9 ☐

So one can restrict to the NLS of this form:

$$-\mathrm{i}u_t + \triangle u = \pm(q+1)|u|^{2q} u, q \geq 1 \in \mathbb{N} \tag{2}$$

We fix the sign to be + since in our treatment it does not play any particular role.

1.1. **Some related literature.** The cubic NLS in dimension 1 has a long history. It is one of the simplest partial differential equations (PDEs) with completely integrability and several its explicit solutions are known (see [13], [10], [2]). Moreover by [9] it has a convergent normal form. In higher dimensions we loose the complete integrability and all techniques associated to it, but we can still use the following well-known fact

Proposition 1. *The NLS (2) can be written as an infinite dimensional Hamiltonian dynamical system $\dot{u} = \{H, u\}$, where the symplectic variables are Fourier coefficients of the functions*

$$u(t,\varphi) = \sum_{k\in\mathbb{Z}^n} u_k(t)e^{\mathrm{i}(k,\varphi)}. \tag{3}$$

the symplectic form is $\mathrm{i}\sum_{k\in\mathbb{Z}^n} du_k \wedge d\bar{u}_k$ *and the Hamiltonian is*

$$H := \sum_{k\in\mathbb{Z}^n} |k|^2 u_k\bar{u}_k + \sum_{k\in\mathbb{Z}^n:\sum_{i=1}^{2q+2}(-1)^i k_i=0} u_{k_1}\bar{u}_{k_2}u_{k_3}\bar{u}_{k_4}...u_{2q+1}\bar{u}_{2q+2} \tag{4}$$

Proof. See Appendix 10. □

By formula (4) we can write equation (1) as an infinite dimensional Hamiltonian system, where the quadratic term consists of infinitely many independent oscillators with rational frequencies and hence completely resonant (all the bounded solutions are periodic). The presence of the nonlinear part couples oscillators and modulates the frequencies so that one expects the existence of small-periodic (and almost-periodic) solutions for appropriate choices of the initial data. In order to prove the existence of such quasi-periodic solutions for Hamiltonian PDEs there are two main methods used in the literature: one by KAM theory and the other by using Lyapunov-Schmidt decomposition and then Nash-Moser implicit function theorems. In particular in [5] Bourgain studied the cubic NLS in dimension two and proved the existence of quasi-periodic solutions with two frequencies by using the second method (the so-called Craig-Wayne-Bourgain approach, see [6], [5] and a more recent paper [4]). Meanwhile the KAM algorithm was used by Geng-Yi in [7] for the NLS in dimension one with the nonlinearity $|u|^4u$ and by Geng-You and Xu in [8] to prove the existence(but not stability) of quasi-periodic solutions for the cubic NLS in dimension two. It is important to notice however that for both approaches it is necessary start from a suitably non degenerate *normal form* (see Definition 3.4) and the existence of a such normal form is not obvious for equation (1).

Recently in the paper [11] C.Procesi and M. Procesi have studied *A Normal Form for the Schrödinger equation with analytic non–linearities.*

In this paper the normal form is described by an infinite dimensional Hamiltonian which determines a linear operator $ad(N)$, depending on a finite number of parameters ξ_i (the actions of certain excited frequencies), on a certain infinite dimensional vector space $F^{(0,1)}$ (see Definition 4.1).

Stability for this infinite dimensional operator will be interpreted in the same way as it appears for finite dimensional linear systems, that is the property that the linear operator is semisimple with distinct eigenvalues.

This was shown in [12] to be true for cubic NLS outside a zero measure set of parameters and on a smaller set of positive measure it was shown that the dynamic is elliptic. This condition in a more precise quantitative form (which will be discussed elsewhere) in the Theory of dynamical systems is referred to as the *second Melnikov condition*(see 3.6). This fact will be useful in [14] in order to prove, by a KAM algorithm, the existence and stability of quasi–periodic solutions for the NLS (not just the normal form). The fact that this property makes at all sense depends upon the results in [11], where it is shown that this linear operator decomposes into an infinite direct sum of finite dimensional blocks. Furthermore, these finite dimensional blocks are described by translating, with suitable scalars, a finite number of combinatorially defined matrices, constructed from certain combinatorial objects called *marked colored graphs* with vertices certain integral vectors (cf. Definition 4.3 and Remark 4.2).

The characteristic polynomials $\det(tI - ad(N)_\Gamma)$ of the operator $ad(N)$ restricted to the infinitely many blocks Γ are all polynomials in the variables ξ_i and t with integer coefficients. The issue is thus to prove that a rather complicated infinite list of polynomials in a variable t, of degree increasing with the space dimension, and with coefficients polynomials in the parameters ξ_i have distinct roots for *generic* (see Appendix 12) values of the parameters.

In general, following the classical Theory of Sylvester in order to prove that a single polynomial has distinct roots, one has to prove the non–vanishing of its discriminant (see Definition 11.2), for two polynomials to have different roots the condition is the non–vanishing of the resultant (see Definition 11.1).

Although both the discriminant and the resultant can be computed by explicit formulas above (see (219), (220)) a proof of their non–vanishing for the infinite list of complicated polynomials appearing seems to be a hopeless task.

We thus followed a different approach. Remark that, if we have a list of different polynomials in one variable t, with coefficients in a field F, a sufficient condition that all their roots (in the algebraic closure $\overline{F}$ of F) be distinct is that they are all *irreducible* (over F).This follows immediately from the fact that an irreducible polynomial $f(t)$ is uniquely determined as the minimal polynomial of each of its roots (cf. [1]) and, in characteristic 0, its derivative $f'(t)$ is non–zero. By the irreducibility of $f(t)$ the greatest common divisor between $f(t), f'(t)$ is 1 so all the roots of $f(t)$ are distinct.

In our case we can consider all the characteristic polynomials as having coefficients in the field of rational functions in the parameters ξ_i, its algebraic closure is a *field of algebraic functions*. Thus the resultant of two distinct irreducible polynomials in $\mathbb{Q}(\xi_1, \dots, \xi_m)[t]$ is non–zero as a polynomial in the ξ and thus outside a real hypersurface the two polynomials have distinct roots.

The way in which we shall attack this problem is by showing that

Theorem 1.1. *(Separation and Irreducibility Theorem)Polynomials* $\det(tI - ad(N)_\Gamma)$ *are all distinct and irreducible as polynomials with integer coefficients.*

The proof of this proposition is the content of Part 2 and Part 3, and requires a rather tedious and lengthy case analysis.

1.2. **The plan of the thesis.** The thesis is devoted to prove Theorem 1.1. It is composed of three parts. The first part explains why we need to study the problem. The second part considers the case of cubic NLS in all dimensions, meanwhile the third part considers higher degree NLS in low dimensions.

Part 1. Some background

Abstract. *This part is a short summary of some of the results of* [11] *for all q which explain the nature of the matrices which will be analyzed in the second and the third part.*

We work on the scale of complex Hilbert spaces

$$\bar{\ell}^{(a,p)} := \{u = \{u_k\}_{k\in\mathbb{Z}^n} | \sum_{k\in\mathbb{Z}^n} |u_k|^2 e^{2a|k|} |k|^{2p} := \|u\|^2_{a,p} < \infty\}, a > 0, p > n/2 \tag{5}$$

equipped with the symplectic structure $\mathrm{i}\sum_{k\in\mathbb{Z}^n} du_k\wedge d\bar{u}_k$. These choices are rather standard in the literature:

Remark 1.2. *The condition imposed on u by* (5) *means that:*

- *We restrict our study to functions which extend to analytic functions in the domain of the complex torus $\mathbb{C}^n/2\pi\mathbb{Z}^n$ where $(z_1,\dots,z_n)\in\mathbb{C}^n,\ Im(z_i)\le a$.*
- *The functions on the boundary are in the Sobolev space H^p.*
- *The condition $p>n/2$ implies that the function space under consideration embeds in L^∞. In particular the following uniform bound holds for each $u\in\bar{\ell}^{(\mathbf{a},\mathbf{p})}$:*

$$|u_k|\le C(s,a)\frac{\|u\|_{a,p}e^{-a|k|}}{\langle k\rangle^{p-n/2}},\quad \langle k\rangle:=\max(1,|k|). \tag{6}$$

In fact this implies that $\bar{\ell}^{(\mathbf{a},\mathbf{p})}$ has a Hilbert algebra structure.

Remark 1.3. *For any function $f(u,\bar{u})$ we have:*

$$\dot{f}=\sum_k\Big(\frac{\partial f}{\partial u_k}\dot{u}_k+\frac{\partial f}{\partial\bar{u}_k}\dot{\bar{u}}_k\Big)=\sum_k\Big(\frac{\partial f}{\partial u_k}\mathrm{i}\frac{\partial H}{\partial\bar{u}_k}+\frac{\partial f}{\partial\bar{u}_k}(-\mathrm{i}\frac{\partial H}{\partial u_k})\Big)=\{H,f\} \tag{7}$$

2. Conservation laws

We may write, for any d

$$[u]^{2d}:=\sum_{k_i\in\mathbb{Z}^n}u_{k_1}\bar{u}_{k_2}u_{k_3}\bar{u}_{k_4}...u_{k_{2d-1}}\bar{u}_{k_{2d}}=\sum_{\alpha,\beta\in(\mathbb{Z}^n)^{\mathbb{N}}:|\alpha|_1=|\beta|_1=d}\binom{d}{\alpha}\binom{d}{\beta}u^\alpha\bar{u}^\beta \tag{8}$$

where $\alpha:k\mapsto\alpha_k\in\mathbb{N}$, same for β. It is easy to see that for any d $[u]^{2d}$ is an analytic function of $u,\bar{u}$.

Remark 2.1. *All the terms in the right hand side of* (8) *Poisson commute with L. The terms which Poisson with the momentum M are the ones which satisfy $\mathrm{i}\sum_k k(\alpha_k-\beta_k)=0$, meanwhile the terms which Poisson with the quadratic energy $K:=\sum_k|k|^2u_k\bar{u}_k$ are the ones which satisfy $\sum_k|k|^2(\alpha_k-\beta_k)=0$.*

Proposition 2. *(Conservation laws) Our Hamiltonian H(see Formula* (4)*) has $(n+1)$ conserved quantities: the n-vector momentum $M=\sum_k k|u_k|^2$, the scalar mass $L=\sum_k|u_k|^2$.*

Proof. (Proof of Proposition 2 and Remark 2.1)

Since by Remark 1.3 $\dot{M}=\{H,M\},\dot{L}=\{H,L\}$, it is enough to prove that M,L Poisson commute with H.

We get easily

$$\{u_k\bar{u}_k,u_h\}=\begin{cases}0 & \text{if } k\ne h\\ \mathrm{i}u_h & \text{if } k=h.\end{cases}$$

and

$$\{u_k\bar{u}_k,\bar{u}_h\}=\begin{cases}0 & \text{if } k\ne h\\ -\mathrm{i}\bar{u}_h & \text{if } k=h.\end{cases}$$

Hence

$$\{M,u_h\}=\mathrm{i}hu_h,\{M,\bar{u}_h\}=-\mathrm{i}h\bar{u}_h,\{L,u_h\}=\mathrm{i}u_h,\{L,\bar{u}_h\}=-\mathrm{i}\bar{u}_h,\\ \{K,u_h\}=\mathrm{i}|h|^2u_h,\{K,\bar{u}_h\}=-\mathrm{i}|h|^2\bar{u}_h \tag{9}$$

We have:

$$\{L,u^\alpha\}=\{L,\prod_k u_k^{\alpha_k}\}=\sum_k\prod_{j\neq k}u_j^{\alpha_j}\{L,u_k^{\alpha_k}\}=\sum_k\prod_{j\neq k}u_j^{\alpha_j}\alpha_k u_k^{\alpha_k-1}\{L,u_k\}= \tag{10}$$

$$=\sum_k\prod_{j\neq k}u_j^{\alpha_j}\alpha_k u_k^{\alpha_k-1}\mathrm{i}u_k=\mathrm{i}\sum_k\alpha_k u^\alpha.$$

Similarly,

$$\{L,\bar{u}^\beta\}=-\mathrm{i}\sum_{\beta_k}\beta_k\bar{u}^\beta \tag{11}$$

$$\{M,u^\alpha\}=\mathrm{i}\sum_k k\alpha_k u^\alpha \tag{12}$$

$$\{M,\bar{u}^\beta\}=-\mathrm{i}\sum_k k\beta_k\bar{u}^\beta \tag{13}$$

From (10) and (11) we have:

$$\{L,u^\alpha\bar{u}^\beta\}=\{L,u^\alpha\}\bar{u}^\beta+u^\alpha\{L,\bar{u}^\beta\}=\mathrm{i}\sum_k(\alpha_k-\beta_k)u^\alpha\bar{u}^\beta \tag{14}$$

and from (12) and (13)

$$\{M,u^\alpha\bar{u}^\beta\}=\mathrm{i}\sum_k k(\alpha_k-\beta_k)u^\alpha\bar{u}^\beta \tag{15}$$

Similarly,

$$\{K,u^\alpha\bar{u}^\beta\}=\imath\sum_k|k|^2(\alpha_k-\beta_k)u^\alpha\bar{u}^\beta \tag{16}$$

From (11),(12),(13) we have Remark 2.1 and $\{L,u_h\bar{u}_h\}=\{M,u_h\bar{u}_h\}=\{K,u_h\bar{u}_h\}=0\ \forall h\in\mathbb{Z}^n$. The term

$$\sum_{\substack{k_1,k_2,\dots,k_{2q+1},k_{2q+2}\in\mathbb{Z}^n\\ \sum_{i=1}^{2q+2}(-1)^ik_i=0}}u_{k_1}\bar{u}_{k_2}...u_{k_{2q+1}}\bar{u}_{2q+2}$$

in Formula (4) can be written in this form:

$$\sum_{\substack{k_1,k_2,\dots,k_{2q+1},k_{2q+2}\in\mathbb{Z}^n\\ \sum_{i=1}^{2q+2}(-1)^ik_i=0}}u_{k_1}\bar{u}_{k_2}...u_{k_{2q+1}}\bar{u}_{2q+2}=\sum_{\substack{\alpha,\beta\in(\mathbb{Z}^n)^{\mathbb{N}}:|\alpha|_1=|\beta|_1=q+1,\\ \sum_k k\alpha_k-\sum_k k\beta_k=0}}\begin{pmatrix}q+1\\ \alpha\end{pmatrix}\begin{pmatrix}q+1\\ \beta\end{pmatrix}u^\alpha\bar{u}^\beta \tag{17}$$

Since

$$\sum_{\substack{k_1,k_2,\dots,k_{2q+1},k_{2q+2}\in\mathbb{Z}^n\\ \sum_{i=1}^{2q+2}(-1)^ik_i=0}}u_{k_1}\bar{u}_{k_2}...u_{k_{2q+1}}\bar{u}_{2q+2}$$

contain the terms $u^\alpha\bar{u}^\beta$ with $|\alpha|_1=\sum_k\alpha_k=|\beta|_1=\sum_k\beta_k$, from (14) we get

$$\{L,\sum_{\substack{k_1,k_2,\dots,k_{2q+1},k_{2q+2}\in\mathbb{Z}^n\\ \sum_{i=1}^{2q+2}(-1)^ik_i=0}}u_{k_1}\bar{u}_{k_2}...u_{k_{2q+1}}\bar{u}_{2q+2}\}=0.$$

From (15) and (17) we get

$$\{M, \sum_{\substack{k_1,k_2,\ldots,k_{2q+1},k_{2q+2}\mathbb{Z}^n \\ \sum_{i=1}^{2q+2}(-1)^i k_i=0}} u_{k_1}\bar{u}_{k_2}\ldots u_{k_{2q+1}}\bar{u}_{2q+2}\} = 0 \tag{18}$$

We have proved that every term in formula (4) of H Poisson commutes with L, M, hence $\{L, H\} = \{M, H\} = 0$. □

3. The nonlinear Schrödinger equation as an infinite dimensional Hamiltonian equation

In [11] C. Procesi and M. Procesi used a standard instrument called the "resonant Birkhoff normal form"(see [3]).

In Formula (4) denote by $K = \sum_{k\in\mathbb{Z}^n} |k|^2 u_k\bar{u}_k$. The first step of "resonant Birkhoff normal form" is the sympletic change of variables which reduces Hamiltonian H to

$$H = H_{Res} + H^{(2q+4)}; H_{Res} = K + H^{(2q+2)}_{res}(u,\bar{u}),$$

where $H^{(2q+4)}$ is an analytic function of degree at least $2q+4$, while $H^{(2q+2)}_{res}$ is of degree $2q+2$ and consists exactly of the degree $2q+2$ terms of (4) which Poisson commute with K. Then one wants to treat the *truncated system* $H_{Res} = K + H^{(2q+2)}_{res}(u,\bar{u})$, as the new unperturbed system and $H^{(2q+4)}$ as a small perturbation. Although the truncated system is very complicated(see Formula (19)) they showed that it admits infinitely many invariant subspaces (see 3.2), defined by requiring $u_k = 0$ for all $k \notin S$ where $S = \{v_1, \ldots v_m\}$, *tangential sites*, it is some (arbitrarily large) subset of $\mathbb{Z}^n$ satisfying the *completeness condition* (see Proposition 3). By momentum conservation for any set $S \subset \mathbb{Z}^n$, the subspace $u_k = 0$ for all $k \notin Span(S)$ is invariant(not only for H_{Res} but also for full Hamiltonian H). They restricted to this subspace and denoted by $S^c = Span(S) \setminus S$ the *normal sites*. Collecting the terms by the degree (denoted by $\sharp S^c$) in the variables $u_k, \bar{u}_k, k \in S^c$, one has:

$$H_{Res} = H_S + H_{\sharp S^c=1} + H_{\sharp S^c=2} + H_{\sharp S^c>2}$$

by definition the *completeness* is equivalent to the fact that $H_{\sharp S^c=1} = 0$. Then they showed that the term $H_{\sharp S^c>2}$ is negligible and gave an explicit formula for $H_{\sharp S^c=2}$ described by an infinite dimensional matrix (cf. Formula (41)).

3.1. **One step of Birkhoff normal form.** By (16) the monomial $u^\alpha\bar{u}^\beta$ Poisson commutes with K if and only if $\sum_k |k|^2(\alpha_k - \beta_k) = 0$. We apply one step of Birkhoff normal form, by which we cancel all the monomials of degree $2(q+1)$ which do not Poisson commute with K. This is done by constructing an analytic change of variables with generating function

$$A := \sum_{\substack{\alpha,\beta\in(\mathbb{Z}^n)^{\mathbb{N}}:|\alpha|=|\beta|=q+1; \\ \sum_k(\alpha_k-\beta_k)k=0, \sum_k(\alpha_k-\beta_k)|k|^2\neq 0}} \begin{pmatrix} q+1 \\ \alpha \end{pmatrix}\begin{pmatrix} q+1 \\ \beta \end{pmatrix} \frac{u^\alpha\bar{u}^\beta}{\sum_k(\alpha_k-\beta_k)|k|^2}.$$

We denote the change of variables by $\Psi^{(1)} = e^{adA}$ and notice that it is well defined and analytic: $B_{\epsilon_0} \times B_{\epsilon_0} \to B_{\epsilon_0}$, with $\epsilon_0 = (2c_{a,p})^{-1}$ (here B_r denotes the open ball of radius r, $c_{a,p}$ is the algebra constant of the space $\bar{\ell}^{(a,p)}$).

By the construction Ψ^1 brings (4) to the form $H = H_{Res} + H^{2(q+2)}(u)$, where

$$H_{Res} := \sum_{k\in\mathbb{Z}^n} |k|^2 u_k \bar{u}_k + \sum_{\substack{\alpha,\beta\in(\mathbb{Z}^n)^{\mathbb{N}}:|\alpha|=|\beta|=q+1;\\ \sum_k(\alpha_k-\beta_k)k=0,\sum_k(\alpha_k-\beta_k)|k|^2=0}} \begin{pmatrix} q+1 \\ \alpha \end{pmatrix} \begin{pmatrix} q+1 \\ \beta \end{pmatrix} u^\alpha \bar{u}^\beta \tag{19}$$

and $H^{2(q+2)}(u)$ is analytic of degree at least $2(q+2)$ in u, it is analytic and satisfies the bound

$$sup_{(u,\bar{u})\in B_\epsilon\times B_\epsilon} \parallel X_{H^{2(q+2)}} \parallel_{a,p} \leq cost\epsilon^{2q+3}, \forall\epsilon < \epsilon_0 \tag{20}$$

where cost denotes a universal constant (depending only on $q, c_{a,p}$ and the function G).

Remark 3.1. *The three constraints in the second summand of the formula* (19) *express the conservation of L, M and the quadratic energy K.*

Definition 3.1. *We say that a list $k_1, ..., k_{2d}$ of vectors in $\mathbb{Z}^n$ is* resonant *if, up to reordering we have:*

$$k_1+k_3+...+k_{2d-1} = k_2+k_4+...+k_{2d}, |k_1|^2+|k_3|^2+...+|k_{2d-1}|^2 = |k_2|^2+|k_4|^2+...+|k_{2d}|^2.$$

We say that the list is integrable *if furthermore, up to reordering, we have $k_{2i-1} = k_{2i}, i = 1, ..., d$. A subset of $\mathbb{Z}^n$ is called* integrable *if all the list of $2q+2$ vectors which are resonant are also integrable.*

The resonant list with $d = q+1$ describe *resonant monomials*, that is those monomials which Poisson commute with K, which appear in H_{Res}. The integrable list describe the monomials in $|u_h|^2$.

Example 3.1. *When $q = 1$:$k_1 + k_3 = k_2 + k_4, |k_1|^2 + |k_3|^2 = |k_2|^2 + |k_4|^2$ is equivalent to*

$$k_1 + k_3 = k_2 + k_4, (k_1 - k_2, k_3 - k_2) = 0$$

This means that the points k_1, k_2, k_3, k_4 are vertices of a rectangle.

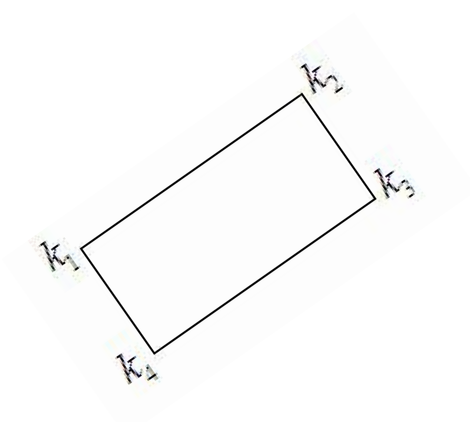


FIGURE 1. A resonant quadruple k_1, k_2, k_3, k_4

3.2. **Invariant subspaces.** Given any set $S \subset \mathbb{Z}^n$, set

$$\bar{\ell}_S^{(a,p)} := \{u \in \bar{\ell}^{(a,p)} : u_k = 0, \forall k \notin Span(S)\}.$$

Then by the conservation of momentum $\bar{\ell}_S^{(a,p)} \times \bar{\ell}_S^{(a,p)}$ is an invariant set for the dynamics. We want to study H_{Res} on the invariant subspaces $\bar{\ell}_S^{(a,p)}$ for suitable choices of S.

Definition 3.2. *A subset $S \subset \mathbb{Z}^n$ is called* complete *if the Hamiltonian vector field $X_{H_{Res}}$ is tangent to the subspace V_S of equations*

$$u_k = 0 = \bar{u}_k, \forall k \in S^c = Span(S) \setminus S$$

(this of course implies that this subspace is stable under the dynamics).

From the definitions one immediately deduces

Proposition 3. *S is complete if and only if, for any choice of $2q+1$ vectors $v_i \in S$ the following holds: if there exists a further vector $w \in \mathbb{Z}^n$ such that the list $v_1, ..., v_{2q+1}, w$ is resonant then $w \in S$.*

Proof. By the definition the tangent space of V_S at the point $v \in V_S$ is

$$T_v(V_S) = Span_{k\in S}(\frac{\partial}{\partial u_k}|_v, \frac{\partial}{\partial \bar{u}_k}|_v) \tag{21}$$

By the definition of the Hamiltonian vector field we have

$$X_{H_{Res}} = -\mathrm{i}\sum_k(\frac{\partial H_{Res}}{\partial \bar{u}_k}\frac{\partial}{\partial u_k} - \frac{\partial H_{Res}}{\partial u_k}\frac{\partial}{\partial \bar{u}_k}) \tag{22}$$

From (19),(22) and since we work on $\bar{\ell}_S^{(a,p)}$ we get

$$\begin{aligned} (23)\quad X_{H_{Res}} = -\mathrm{i}\sum_{k\in Span(S)}((|k|^2u_k+ \\ +(q+1)\sum_{\alpha,\hat{\beta}\in(\mathbb{Z}^n)^{\mathbb{N}}:|\alpha|_1=q+1,|\hat{\beta}|_1=q,\sum_l l(\alpha_l-\hat{\beta}_l)=k,\sum_l |l|^2(\alpha_l-\hat{\beta}_l)=|k|^2}\begin{pmatrix} q+1 \\ \alpha\end{pmatrix}\begin{pmatrix} q \\ \hat{\beta}\end{pmatrix}u^\alpha\bar{u}^{\hat{\beta}})\frac{\partial}{\partial u_k}- \\ (|k|^2\bar{u}_k+\sum_{\hat{\alpha},\beta\in(\mathbb{Z}^n)^{\mathbb{N}}:|\hat{\alpha}|_1=q,|\beta|_1=q+1,\sum_l l(\hat{\alpha}_l-\beta_l)=-k,\sum_l |l|^2(\hat{\alpha}_l-\beta_l)=-|k|^2}\begin{pmatrix} q \\ \hat{\alpha}\end{pmatrix}\begin{pmatrix} q+1 \\ \beta\end{pmatrix}u^{\hat{\alpha}}\bar{u}^{\beta})\frac{\partial}{\partial \bar{u}_k}), \end{aligned}$$

where $\hat{\alpha}_i = \alpha_i, \hat{\beta}_i = \beta_i$ for all $i \neq k$, $\hat{\alpha}_k = \alpha_k - 1, \hat{\beta}_k = \beta_k - 1$.

Notice that

$$\begin{aligned} (24)\quad \sum_{\alpha,\hat{\beta}\in(\mathbb{Z}^n)^{\mathbb{N}}:|\alpha|_1=q+1,|\hat{\beta}|_1=q,\sum_l l(\alpha_l-\hat{\beta}_l)=k,\sum_l |l|^2(\alpha_l-\hat{\beta}_l)=|k|^2}\begin{pmatrix} q+1 \\ \alpha\end{pmatrix}\begin{pmatrix} q \\ \hat{\beta}\end{pmatrix}u^\alpha\bar{u}^{\hat{\beta}}) = \\ = \sum_{k_1,...,k_{2q+1}\in\mathbb{Z}^n:\sum_{i=1}^{2q+1}(-1)^{i+1}k_i=k,\sum_{i=1}^{2q+1}(-1)^{i+1}|k_i|^2=|k|^2} u_{k_1}\bar{u}_{k_2}...u_{k_{2q-1}}\bar{u}_{k_{2q}}u_{k_{2q+1}} \end{aligned}$$

and

$$\begin{aligned} (25)\quad \sum_{\hat{\alpha},\beta\in(\mathbb{Z}^n)^{\mathbb{N}}:|\hat{\alpha}|_1=q,|\beta|_1=q+1,\sum_l l(\alpha_l-\hat{\beta}_l)=-k,\sum_l |l|^2(\alpha_l-\hat{\beta}_l)=-|k|^2}\begin{pmatrix} q \\ \hat{\alpha}\end{pmatrix}\begin{pmatrix} q \\ \beta\end{pmatrix}u^{\hat{\alpha}}\bar{u}^{\beta}) = \\ = \sum_{k_1,...,k_{2q},k_{2q+2}\in\mathbb{Z}^n:\sum_i(-1)^{i+1}k_i=-k,\sum_i(-1)^{i+1}|k_i|^2=-|k|^2} u_{k_1}\bar{u}_{k_2}...u_{k_{2q-1}}\bar{u}_{k_{2q}}\bar{u}_{k_{2q+2}} \end{aligned}$$

-If there exists a resonant list $k_1, ..., k_{2q+1}, k$ such that $k_1, ..., k_{2q+1} \in S$ but $k \notin S$, then from (23), (24) and (25) we see that $X_{H_{Res}}$ contains the term $u_{k_1}\bar{u}_{k_2}...u_{k_{2q-1}}\bar{u}_{k_{2q}}u_{k_{2q+1}}\frac{\partial}{\partial u_k}, k \notin S$. Then by (21) $X_{H_{Res}}$ is not tangent to the subspace V_S.
-Inversely, if S satisfy the condition of Proposition 3, then for every $v \in V_S$ since $u_{k,v} = \bar{u}_{k,v} = 0$ for all $k \in S^c$ we see from (23) that $X_{H_{Res}}$ is a linear combination of $\frac{\partial}{\partial u_k}|_v, \frac{\partial}{\partial \bar{u}_k}|_v, k \in S$. Hence $X_{H_{Res}} \in T_v(V_S)$. □

Remark 3.2. *A sufficient condition for S to be integrable is the following:* Set $S = \{v_1, ..., v_m\}$, introduce variables $e_1, ..., e_m$. For any choice of $2q+2$ elements $e_{i_1}, ..., e_{i_{2q+2}}$ if the expression

$$e_{i_1} + ... + e_{i_{2q+1}} - (e_{i_2} + ... + e_{i_{2q+2}})$$

is not zero then

$$v_{i_1} + ... + v_{i_{2q+1}} - (v_{i_2} + ... + v_{i_{2q+2}}) \neq 0.$$

Proof. In fact if a list of $2q+2$ vectors $v_{i_1}, ..., v_{i_{2q+2}} \in S$ is resonant, then we have $v_{i_1} + ... + v_{i_{2q+1}} - (v_{i_2} + ... + v_{i_{2q+2}}) = 0$, so

$$e_{i_1} + ... + e_{i_{2q+1}} - (e_{i_2} + ... + e_{i_{2q+2}}) = 0.$$

Since $e_1, ..., e_m$ are variables, one deduces that up to reordering $i_1 = i_2, ..., i_{2q+1} = i_{2q+2}$, and hence up to reordering $v_{i_1} = v_{i_2}, ..., v_{i_{2q+1}} = v_{i_{2q+2}}$. □

Example 3.2. $q = 1, n = 2, m = 4$*: Four vectors v_1, v_2, v_3, v_4 in the plane are not complete if they form a picture of type*

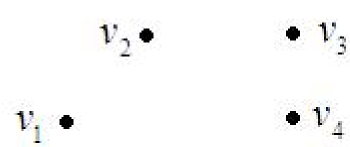


FIGURE 2

that we have the a right triangle which is not completed to a rectangle. The list

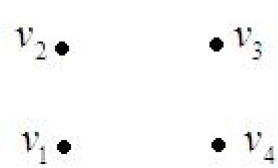


FIGURE 3

is complete but not integrable. Finally, the list

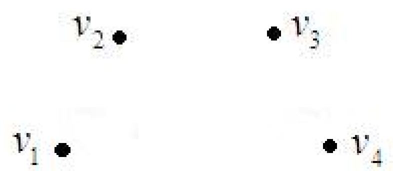


FIGURE 4

is complete and integrable.

We introduce

$$A_r(\xi_1, ..., \xi_m) = \sum_{\sum_i k_i = r} \begin{pmatrix} r \\ k_1, ..., k_m \end{pmatrix}^2 \prod_i \xi_i^{k_i} \tag{26}$$

Denote by H_S the restricted Hamiltonian to the subspace V_S. We have

Proposition 4. *If $S = \{v_1, ..., v_m\}$ is complete and integrable the restricted Hamiltonian is :*

$$H_S = \sum_{i=1}^m |v_i|^2 |u_{v_i}|^2 + A_{q+1}(|u_{v_1}|^2, ..., |u_{v_m}|^2) = \tag{27}$$

$$= \sum_{i=1}^m |v_i|^2 |u_{v_i}|^2 + \sum_{\sum_i s_i = q+1} \begin{pmatrix} q+1 \\ s_1, ..., s_m \end{pmatrix}^2 \prod_i |u_{v_i}|^{2s_i}$$

Proof. From Formula (19) , the definition of V_S, the completeness of S we have:

$$H_S = \sum_{i=1}^m |v_i|^2 |u_{v_i}|^2 + \sum_{k_i \in S : \sum_i (-1)^i k_i = 0, \sum_i (-1)^i |k_i|^2 = 0} u_{k_1} \bar{u}_{k_2} ... u_{k_{2q+1}} \bar{u}_{k_{2q+2}} \tag{28}$$

Since S is integrable, we have $k_1 = k_2, ..., k_{2q+1} = k_{2q+2}$ (up to reordering). So:

(29)

$$H_S = \sum_{i=1}^m |v_i|^2 |u_{v_i}|^2 + \sum_{k_i \in S} (|u_{k_1}| ... |u_{k_{2q+1}}|)^2 = \sum_{i=1}^m |v_i|^2 |u_{v_i}|^2 + \sum_{\sum_i s_i = q+1} \begin{pmatrix} q+1 \\ s_1, ..., s_m \end{pmatrix}^2 \prod_i |u_{v_i}|^{2s_i}.$$

□

3.3. Tangential sites in action variables.

We set

$$u_k := z_k, k \in S^c, u_{v_i} := \sqrt{\xi_i + y_i} e^{\mathrm{i} x_i} = \sqrt{\xi_i}(1 + \frac{y_i}{2\xi_i} + \ldots) e^{\mathrm{i} x_i} \text{ for } i = 1, \ldots, m, \tag{30}$$

considering ξ_i as parameters,$|y_i| < \xi_i$, while $y, x, w := (z, \bar{z})$ are dynamical variables.

Definition 3.3. *We denote by $\ell^{(a,p)}$ the subspace of $\bar{\ell}^{(a,p)} \times \bar{\ell}^{(a,p)}$ generated by the indices in S^c with coordinates $w = (z, \bar{z})$.*

For all $\varepsilon > 0$ and for all

$$\xi \in A_{\varepsilon^2} := \{\xi : \frac{1}{2}\varepsilon^2 \le \xi_i \le \varepsilon^2\}, \tag{31}$$

Formula (30) is a well known analytic and symplectic change of variables $\Psi_\xi^{(2)}$ in the domain
(32)

$$D_{(a,p)}(s,r) = D(s,r) := \{x, y, w : x \in \mathbb{T}^m_s, |y| < r^2, \| w \|_{(a,p)} < r\} \subset \mathbb{T}^m_s \times \mathbb{C}^m \times \ell^{(a,p)}.$$

Here $\varepsilon > 0, s > 0$ and $0 < r < \varepsilon/2$ are auxiliary parameters. $\mathbb{T}^m_s$ denote the open subset of the complex torus $\mathbb{T}^m_{\mathbb{C}} := \mathbb{C}^m/2\pi\mathbb{Z}^m$ where $x \in \mathbb{C}^m, |Im(s)| < s$. Moreover if

$$\sqrt{2m}(max(|v_i|))^p e^{s+amax(|v_i|)}\varepsilon < \epsilon_0 \tag{33}$$

the change of variables sends $D(s,r) \to B_{\epsilon_0}$ so we can apply it to our Hamiltonian. We thus assume that parameters ε, r, s satisfy (33). Formula (30) puts in action angle variables $(y; x) = (y_1, ..., y_m; x_1, ...x_m)$ the tangential sites, close to the action $\xi = \xi_1, ...\xi_m$ which are parameters for the system. From $u_k = 0 \forall k \notin Span(S)$ and Formula (30) the symplectic form now becomes

$$\begin{aligned} i\sum_{k\in\mathbb{Z}^n} du_k \wedge d\bar{u}_k &= i\sum_{i=1}^m du_{v_i} \wedge d\bar{u}_{v_i} + i\sum_{k\in S^c} du_k \wedge d\bar{u}_k = \\ &= \sum_{i=1}^m dy_i \wedge dx_i + i\sum_{k\in S^c} dz_k \wedge d\bar{z}_k = dy \wedge dx + i\sum_{k\in S^c} dz_k \wedge d\bar{z}_k. \end{aligned} \tag{34}$$

In the new variables

$$\begin{aligned} M = \sum_i \xi_i v_i + \sum_i y_i v_i + \sum_{k\in S^c} k|z_k|^2, \quad L = \sum_i \xi_i + \sum_i y_i + \sum_{k\in S^c} |z_k|^2 \\ \sum_{k\in\mathbb{Z}^n} |k|^2 u_k\bar{u}_k = K = (\omega_0, \xi + y) + \sum_{k\in S^c} |k|^2|z_k|^2, \omega_0 = (|v_1|^2, ..., |v_m|^2). \end{aligned} \tag{35}$$

Remark 3.3. *The terms $\sum_i \xi_i, \sum_i \xi_i v_i$ and $\sum_i \xi_i |v_i|^2$ are constants and can be dropped, renormalizing M, L, K.*

We formalize the momentum and mass by two linear maps

$$\pi : \mathbb{Z}^m \to Span(S), \pi(e_i) = v_i : \text{momentum}; \eta : \mathbb{Z}^m \to \mathbb{Z}, \eta(e_i) = 1 : \text{mass} \tag{36}$$

where $\{e_1, ..., e_m\}$ be a basis of $\mathbb{Z}^m$.

3.4. **A normal form.**

Definition 3.4. *(Normal form) We separate $H_{Res} + P^{2(q+2)}(u) = H = N + P$ where N is called the* normal *form and collects all the terms of H_{Res} (as series in y, w) of degree ≤ 2 in the variables y, w.*

The series P collects all terms of $P^{2(q+2)}(u)$ and all the terms of H_{Res} of degree > 2 in the variables y, w. It is called the *pertubation*.

Definition 3.5. *(edges) Consider the elements:*

$$X_q := \{\ell = \sum_{j=1}^{2q} \pm e_{i_j} = \sum_{i=1}^m \ell_i e_i, \ell \neq 0, -2e_i, \eta(\ell) \in \{0, -2\}\} \tag{37}$$

The support *of an edge $\ell = \sum_i n_i e_i$ is the set of indices i with $n_i \neq 0$.*

We have $\sum_i |\ell_i| \le 2q$ and have imposed the mass constraint $\sum_i \ell_i = \eta(\ell) \in \{0, -2\}$. We call all the elements respectively the *black*, $\eta(\ell) = 0$ and *red* $\eta(\ell) = -2$ *edges* and denote them by X_q^0, X_q^{-2} respectively.

Notice that by our constraints the support of an edge contains at least 2 elements.

Constraint 3.1. *(1) We assume that $\sum_{j=1}^m n_j v_j \neq 0$ for all $n_i \in \mathbb{Z}$, $\sum_i |n_i| \le 2q+2$.*

(2) $|\sum_i n_i v_i|^2 - \sum_i n_i |v_i|^2 \neq 0$ when $n_i \in \mathbb{Z}, \sum_i n_i = 1, 1 < \sum_i |n_i| \le 2q+1$.

(3) We assume that $\sum_{j=1}^m \ell_j v_j \neq 0$, when $u := \sum_j \ell_j v_j$ is either an edge or a sum or a difference of two distinct edges.

(4) $2\sum_{j=1}^m \ell_j |v_j|^2 + |\sum_{j=1}^m \ell_j v_j|^2 \neq 0$ for all edges $\ell = \sum_j \ell_j e_j$ in X_q^{-2}.

We now recall Lemma 2 and Proposition 4 in [11]

Lemma 3.1. *Constraint 1 is an* integrability *constraint. Constraint 2 is a* completeness *constraint. Constraint 3 means that an edge $\ell = \sum_{j=1}^m \ell_j v_j$ is determined by the associated vector $\pi(\ell) = \sum_{j=1}^m \ell_j v_j$.*

Proof. -The first statement follows from Remark 3.2.

-Using Proposition 4 under Constraint **??** it is enough to show that we can not find $2q+1$ vectors $u_j = v_{i_j}$ for which there is a further vector $w\mathbb{Z}^m$ with $u_1, .., u_{2q+1}, w$ resonant. Otherwise $w = \sum_i n_i v_i$ is a linear combination with ± 1 coefficients of the v_i, hence it is a vector satisfying the hypotheses of item 2, but the quadratic condition in the same item implies that the list is not resonant.

-Constraint 3 implies that $\pi(u-v) \neq 0 \implies \pi(u) \neq \pi(v)$ if u, v are two distinct edges. Hence the last statement is true. □

Proposition 5. *Under the previous constraints we have*

$$N = (\omega(\xi), y) + \sum_{k \in S^c} |k|^2 |z_k|^2 + \mathcal{Q}(x, w) \tag{38}$$

where

$$\omega = \omega_0 + \nabla_\xi A_{q+1}(\xi) - (q+1)^2 A_q(\xi)\underline{1}, \omega_0 = (|v_1|^2, ..., |v_m|^2). \tag{39}$$

does not depend on the dynamical variables. Here $\underline{1} \in \mathbb{N}^m$ denotes the vector with all coordinates equal to 1,$\mathcal{Q}$ is given by formula (41).

Definition 3.6.
- *When $\ell \in X_q^0$, we define $\mathcal{P}_\ell$ as the set of pairs k, h satisfying* (43).
- *When $\ell \in X_q^{-2}$, we define $\mathcal{P}_\ell$ as the set of unordered pairs $\{h, k\}$ satisfying* (44).

For every edge ℓ, set $\ell = \ell^+ - \ell^-$ and define

(40)
$$c(\ell) = c_q(\ell) := \begin{cases} (q+1)^2 \xi^{\frac{\ell^+ + \ell^-}{2}} \sum_{\alpha \in \mathbb{N}^m; |\alpha + \ell^+|_1 = q} \begin{pmatrix} q \\ \ell^+ + \alpha \end{pmatrix} \begin{pmatrix} q \\ \ell^- + \alpha \end{pmatrix} \xi^\alpha, & \ell \in X_q^0; \\ (q+1) q \xi^{\frac{\ell^+ + \ell^-}{2}} \sum_{\alpha \in \mathbb{N}^m; |\alpha + \ell^+|_1 = q-1} \begin{pmatrix} q+1 \\ \ell^- + \alpha \end{pmatrix} \begin{pmatrix} q-1 \\ \ell^+ + \alpha \end{pmatrix} \xi^\alpha, & \ell \in X_q^{-2} \\ c_q(\ell) = c_q(-\ell), & \ell \in X_q^2. \end{cases}$$

$$\mathcal{Q}(x, w) = \sum_{\ell \in X_q^0} c(\ell) e^{\mathrm{i}(\ell, x)} \sum_{(h,k) \in \mathcal{P}_\ell} z_h \bar{z}_k + \sum_{\ell \in X_q^{-2}} c(\ell) \sum_{h,k \in \mathcal{P}_\ell} (e^{\mathrm{i}(\ell, x)} z_h z_k + e^{-\mathrm{i}(\ell, x)} \bar{z}_h \bar{z}_k) \tag{41}$$

Proof. (Proof of Proposition 5) By the definition the normal form collects all the terms of H_{Res} (as series in y, w) of degree ≤ 2 in the variables y, w. In turn H is the sum of the quadratic term $K = \sum_k u_k \bar{u}_k$ and of the terms of degree $2q+2$ in the original variables $u, \bar{u}$.

From Remark 3.3 the quadratic term K contributes to N the terms

$$(\omega_0, y) + \sum_{k \in S^c} |k|^2 |z_k|^2$$

The remaining terms $u_{k_1} \bar{u}_{k_2} ... u_{k_{2q+1}} \bar{u}_{k_{2q+2}}$ satisfy the constraint:

$$\sum_i (-1)^i k_i = 0, \sum_i (-1)^i |k_i|^2 = 0. \tag{42}$$

These terms may contribute to terms of N only if they are of total degree ≤ 2 in y, w. We analyze three possible cases of degree 0,1,2 in w:

- *degree 0* If all the k_i are in S the momentum $\sum_i (-1)^i k_i$ is a linear combination $\sum_j m_j v_j$. From momentum conservation and constraint 1 we must have $m_j = 0, \forall j$. This implies that we can pair the even and odd k's and, as shown in proposition 4, this gives a contribution $A_{q+1}(\xi + y)$. In this expression the terms of degree ≤ 2 give a constant(which we ignore) and the term $(\nabla_\xi A_{q+1}(\xi), y)$.
- *degree 1* One and only one of the $k_i = k \in S^c$. Formula (42) becomes

  $$k - \sum_i n_i v_i = 0, |k|^2 - \sum_i n_i |v_i|^2 = 0$$

  where $\sum_i n_i v_i$ satisfies the hypotheses of constraint 2. Thus these terms do not occur and S is complete .
- *degree 2* Given $h, k \in S^c$ we compute the coefficients of $z_h \bar{z}_k$ or $z_h z_k$ or $\bar{z}_h \bar{z}_k$. These terms are obtained when all but two of the k_i are in S. Each k_i in S contributes $\sqrt{\xi_i + y_i} e^{\pm x_i}$, giving a coefficient $\sqrt{\prod_{j=1}^m \xi_j^{\ell_j}} e^{\mathrm{i}(\ell, x)}$, whenever:

$$(z_h \bar{z}_k) : \sum_{j=1}^m \ell_j v_j + h - k = 0; \sum_{j=1}^m \ell_j |v_j|^2 + |h|^2 - |k|^2 = 0, \ell \in X_q^0 \tag{43}$$

$$(z_h z_k) : \sum_{j=1}^m \ell_j v_j + k + h = 0; \sum_{j=1}^m \ell_j |v_j|^2 + |k|^2 + |h|^2 = 0, \ell \in X_q^{-2} \tag{44}$$

$$(\bar{z}_h \bar{z}_k) \sum_{j=1}^m \ell_j v_j - h - k = 0, \sum_{j=1}^m \ell_j |v_j|^2 - |h|^2 - |k|^2 = 0, \ell \in X_q^2 \tag{45}$$

Constraint 3, where u is the sum or difference of two edges, implies that h, k fix ℓ uniquely. In Formulas (44) and (45) we see that we cannot have $\ell = \mp 2e_i$, since the equations in these Formulas have the only solution $h = k = v_i \in S$. This explains why in Definition we exclude $\pm 2e_i$ as edges. Constraint 4 implies that $h \neq k$ in Formulas (44), (45). By Constraint 3 where u is an edge, in (43) $h = k$ implies $\ell = 0$. This contributes a term $(q+1)^2 A_q(\xi) \sum_{k \in S^c} |z_k|^2$. It is convenient to write

$$\sum_k (q+1)^2 A_q(\xi) |z_k|^2 = (q+1)^2 A_q(\xi) (\sum_k |z_k|^2 + \sum_i y_i) - (q+1)^2 A_q(\xi) (\sum_i y_i)$$

and notice that $(q+1)^2 A_q(\xi)(\sum_k |z_k|^2+\sum_i y_i)$ is a mass term (hence a constant of motion for the whole Hamiltonian) and can be dropped from the Hamiltonian, so we change N into:

$$N = K + (\nabla_\xi A_{q+1}(\xi) - (q+1)^2 A_q(\xi)\underline{1}, y) + \mathcal{Q}(x,w), K = (\omega_0, y) + \sum_k |k|^2 |z_k|^2. \tag{46}$$

where $\underline{1}$ denotes the vectors with all coordinates equal to 1.

Let us now compute $\mathcal{Q}(x,w)$, given an edge ℓ set $\ell = \ell^+ - \ell^-$ Formula (40) comes from the expansion

(47)
$$c_q(\ell) := \begin{cases} (q+1)^2 \sum_{e_{h_1}-e_{k_1}+e_{h_2}\ldots+e_{h_q}-e_{k_q}=\ell} \prod_{i=1}^q (\xi_{h_i}\xi_{k_i})^{1/2}, & \ell \in X_q^0; \\ (q+1)q \sum_{e_{h_1}-e_{k_1}+e_{h_2}\ldots+e_{h_{q-1}}-e_{k_{q-1}}-e_{h_q}-e_{k_q}=\ell} \prod_{i=1}^q (\xi_{h_i}\xi_{k_i})^{1/2}, & \ell \in X_q^{-2}; \\ c_q(-\ell) = c_q(\ell) & \end{cases}$$

□

$\mathcal{Q}$ is a very complicated infinite dimensional quadratic Hamiltonian, one needs to decompose this infinite dimensional system into infinitely many decoupled finite dimensional systems.

3.5. **The new Hamiltonian.**

Following Theorem 1 in [11] for all ε, r, s satisfy (33) and for all $\xi \in A_{\varepsilon^2}$ there exist an analytic symplectic change of variables

$$\Phi_\xi : (y,x) \times (z,\bar{z}) \implies (u,\bar{u})$$

from $D(s,r/2) \implies B_{2\epsilon_0}$ such that the Hamiltonian (4) in the new variables is analytic and has the form

$$H \circ \Phi_\xi = (\omega(\xi), y) + \sum_{k\in S^c} \Omega_k |z_k|^2 + \tilde{\mathcal{Q}}(\xi, w) + \tilde{P}(\xi, y, x, w) \tag{48}$$

where $\tilde{\Omega}_k = |k|^2 + \sum_{i=1}^m |v_i|^2 L^{(i)}(k)$, $L^{(i)}(k) \in \mathbb{Z}$ satisfy $|L^{(i)}(k)| \le 4nq$, $\tilde{P}$ is small.

Moreover, following Corrolary 1 in the same paper there exists an algebraic hypersurface $\mathcal{A}$ such that on the open region $A_{\epsilon^2} \setminus \mathcal{A}$ there is a further analytic change of coordinates taking $\tilde{\mathcal{Q}}$ into a diagonal form with constant coefficients plus a form $\bar{\mathcal{Q}}$ with constant coefficients depending only on finitely many variables $z_k, \bar{z}_k, k \in A$. The Hamiltonian is then

$$H_{fin} = (\omega(\xi), y) + \sum_{k\in S^c} \bar{\Omega}_k |z_k|^2 + \bar{\mathcal{Q}} + P(\xi, y, x, w) \tag{49}$$

where

$$\bar{\Omega}_k = \begin{cases} \tilde{\Omega}_k + \lambda_k(\xi), & \forall k \in S^c \setminus A; \\ \tilde{\Omega}_k, & k \in A. \end{cases} \tag{50}$$

The correction $\lambda_k(\xi)$ is chosen in a finite list, say

$$\lambda_k(\xi) \in \{\lambda^{(1)}(\xi), \ldots, \lambda^{(K)}(\xi)\}, K = K(n,m), \tag{51}$$

of different (real) analytic functions of ξ.

3.6. **KAM scheme.** An interesting application of the results for this normal form is to prove the existence and stability of *quasi-periodic* solution by a KAM scheme (see [14] and also [15] for an existence result). This kind of scheme is based on verification of the following hypotheses:

(1) A *regularity/ smallness condition* on the perturbation P, namely that $\| X_P \| \lll \varepsilon^2$.
(2) A *regularity condition* namely $\omega(\xi)$ must be a diffeomorphism and $\bar{\Omega}_k(\xi) - |k|^2$ must be a bounded Lipschitz function.
(3) A *non-degeneracy condition*, that is three Melnikov resonances

$$(\omega(\xi), \nu) = 0, (\omega(\xi), \nu) + \bar{\Omega}_k(\xi) = 0, (\omega(\xi), \nu) + \bar{\Omega}_k(\xi) + \sigma\bar{\Omega}_h(\xi) = 0 \tag{52}$$

hold in a set of measure 0.
(4) A Quasi-Töplitz condition to control the measure estimates in *the second Melnikov condition.*

4. The operator $ad(N)$

4.1. The map π.

Definition 4.1. *Denote by* $\mathbb{Z}^m := \{\sum_{i=1}^m a_i e_i,\ a_i \in \mathbb{Z}\}$ *the lattice with basis the elements* e_i. *Set* $\pi : \mathbb{Z}^m \to \mathbb{Z}^n,\ \pi : e_i \mapsto v_i$.

At this point it is useful to formalize the idea of *energy transfer* in a combinatorial way. Let $S^2[\mathbb{Z}^m] := \{\sum_{i,j=1}^m a_{i,j} e_i e_j\},\ a_{i,j} \in \mathbb{Z}$ be the polynomials of degree 2 in the e_i with integer coefficients. We extend the map π and introduce a linear map $a \mapsto a^{(2)}$ as:

$$\pi(e_i) = v_i, \quad \pi(e_i e_j) := (v_i, v_j), \quad *^{(2)} : \mathbb{Z}^m \to S^2(\mathbb{Z}^m),\ e_i \mapsto e_i^2.$$

We have $\pi(AB) = (\pi(A), \pi(B)), \forall A, B \in \mathbb{Z}^m$.

Remark 4.1. *Notice that we have* $a^{(2)} = a^2$ *if and only if* a *equals 0 or one of the variables* e_i.

4.2. The spaces $V^{i,j}$ and $F^{0,1}$.

Definition 4.2. *We denote by* $V^{i,j}$ *the space of functions spanned by elements of total degree* i *in* y *and* j *in* w *and* $V^h = \sum_{i+j=h} V^{i,j}, V^\infty = \sum_{i,j} V^{i,j}$.

Denote by $F^{0,1}$ *the subspace of* $V^{0,1}$ *commuting with momentum.*

The space $V^{0,1}$ has a basis over $\mathbb{C}$ given by the elements $\{e^{\mathrm{i}\sum_j \nu_j x_j} z_k, \quad e^{-\mathrm{i}\sum_j \nu_j x_j} \bar{z}_k\}$, where $\nu \in \mathbb{Z}^m, k \in S^c$. The space $F^{0,1}$ has as basis, which we call *frequency basis*, the set F_B of elements

$$F_B = \{e^{\mathrm{i}\sum_j \nu_j x_j} z_k, \quad e^{-\mathrm{i}\sum_j \nu_j x_j} \bar{z}_k\}; \quad \sum_j \nu_j v_j + k = \pi(\nu) + k = 0\,, \quad k \in S^c. \tag{53}$$

An element of F_B is completely determined by the value of ν and the fact that the z variable may or may not be conjugated, thus sometimes we refer to $e^{\mathrm{i}\sum_j \nu_j x_j} z_{-\pi(\nu)}$ as $(\nu, +)$ and to $e^{-\mathrm{i}\sum_j \nu_j x_j} \bar{z}_{-\pi(\nu)}$ as $(\nu, -)$. By construction $\nu \in \mathbb{Z}^m_c$ where

$$\mathbb{Z}^m_c := \{\mu \in \mathbb{Z}^m \mid -\pi(\mu) \in S^c\}\,, \tag{54}$$

We can further decompose the space $F^{0,1} = \oplus F^{0,1}_\ell$ by the eigenspaces of the mass operator $ad(L)$. Notice that the *mass* of $e^{\mathrm{i}\sum_j \nu_j x_j} z_k$ is $\ell = \sum_i \nu_i + 1$, thus on the subspace commuting with L we have $-1 = \sum_i \nu_i$ for $(\nu, \pm)$. Now the blocks for $ad(N)$ appear in a natural matrix representation on the space $F^{0,1}$ as infinitely many matrices with coefficients quadratic polynomials in the variables $\sqrt{\xi_i}$. One easily sees that in the characteristic polynomial of each one of these matrices the square roots disappear.

4.3. **The Cayley graphs.** We recall how we have found useful to cast some of the description of the operator $ad(N)$ into the language of group theory and in particular of the *Cayley graph.* In fact to a matrix $C = (c_{i,j})$ we can always associate a graph with vertices the indices of the matrix and an edge between i, j if and only if $c_{i,j} \neq 0$. For the matrix of $ad(N)$ in the frequency basis the relevant graph comes from a special Cayley graph.

Let G be a group and $X = X^{-1} \subset G$ a subset.

Definition 4.3. *An X–marked graph is an oriented graph Γ such that each oriented edge is marked with an element $x \in X$.*

$$a \xrightarrow{x} b \qquad\qquad a \xleftarrow{x^{-1}} b$$

We mark the same edge, with opposite orientation, with x^{-1}. Notice that if $x^2 = 1$ we may drop the orientation of the edge.

A typical way to construct an X–marked graph is the following. Consider an action $G \times A \to A$ of G on a set A, we then define.

Definition 4.4 (Cayley graph)**.** *The graph A_X has as vertices the elements of A and, given $a, b \in A$ we join them by an oriented edge $a \xrightarrow{x} b$, marked x, if $b = xa$, $x \in X$.*

In our setting the relevant group is the group $G := \mathbb{Z}^m \rtimes \mathbb{Z}/(2)$ the *semidirect product*, denote by $\tau := (0, -1)$ so $G = \mathbb{Z}^m \cup \mathbb{Z}^m\tau$. We think of an element $a = e^{\mathrm{i}\sum_j \nu_j x_j} z_k$ as being associated to the group element which, by abuse of notation, we still denote by $a = \sum_j \nu_j e_j \in \mathbb{Z}^m$. Then $\bar{a} = e^{-\mathrm{i}\sum_j \nu_j x_j} \bar{z}_k$ is associated to the group element $a\tau = (\sum_j \nu_j e_j)\tau \in \mathbb{Z}^m\tau$. Thus the frequency basis is indexed by elements of $G^1 \setminus \bigcup_{i=1}^m \{-e_i, -e_i\tau\}$, where

$$G^1 := \{a, a\tau,\ a \in \mathbb{Z}^m \mid \eta(a) = -1\}.$$

We now consider the Cayley graph G_X of G with respect to the elements $X_q^0 \cup X_q^{-2}$(see Definition 3.4). If $p \in \mathbb{Z}$ it is easily seen that the set $G_p := \{a,\ \eta(a) = 0,\ a\tau \mid \eta(a) = p\}$ form a subgroup. In particular

Remark 4.2. *G_{-2} is generated by the elements $X := X_q^0 \cup X_q^{-2}$ and it is a connected component of the Cayley graph.*

We distinguish the edges by color*, as X^0 to be* black *and X^{-2}* red*, hence the Cayley graph is accordingly colored.*

G^1 is also a coset of G_{-2} and it is also a connected component of the Cayley graph.

If G acts on two sets A_1 and A_2 and $\pi : A_1 \to A_2$ is a map compatible with the G action then π is also a morphism of marked graphs.

A special case is obtained when G acts on itself by left (resp. right) multiplication and we have the Cayley graph G_X^l (resp. G_X^r). We concentrate on G_X^l which we just denote by G_X. One then immediately sees that

Lemma 4.1. *If G acts on a set A and $a \in A$ the* orbit map *$g \mapsto ga$ is compatible with the graph structure.*

The graph G_X is preserved by right multiplication by elements of G, that is if a, b are joined by an edge marked g then also ah, bh are so joined, for all $h \in G$.

The graphs G_X^l, G_X^r are isomorphic with opposite orientations under the map $g \mapsto g^{-1}$.

The graph G_X is connected if and only if X generates G, otherwise its connected components are the right cosets in G of the subgroup H generated by X.

4.3.1. *The linear rules.* Denote by $\mathbb{Z}^m := \{\sum_{i=1}^m a_i e_i,\ a_i \in \mathbb{Z}\}$ the lattice with basis the elements e_i.

We consider the group $G := \mathbb{Z}^m \rtimes \mathbb{Z}/(2)$ semi–direct product. Its elements are pairs (a, σ) with $a \in \mathbb{Z}^m$, $\sigma = \pm 1$. It will be notationally convenient to identify by a the element $(a, +1)$ and by τ the element $(0, -1)$. Note the commutation rules $a\tau = \tau(-a)$. Sometimes we refer to the elements $a = (a, +1)$ as *black* and $a\tau = (a, -1)$ as *red.*

Consider the *mass*[1] $\eta : \mathbb{Z}^m \to \mathbb{Z},\ \eta(e_i) := 1$.

Definition 4.5. *We set Λ to be the Cayley graph associated to the elements $X_q := X_q^0 \cup X_q^{-2}$.*

We give a definition useful to describe the graphs that appear in our construction.

Definition 4.6. *A* complete marked graph*, on a set $A \subset \mathbb{Z}^m \rtimes \mathbb{Z}/(2)$ is the full sub–graph generated by the vertices in A.*

Definition 4.7.
- *A graph A with $k+1$ vertices is said to be of* dimension k.
- *We call the dimension of the affine space spanned by A in $\mathbb{R}^m$ the* rank, $\operatorname{rk} A$*, of the graph A.*
- *If the rank of A is strictly less than the dimension of A we say that A* is degenerate.

4.4. **The matrix description of** $ad(N)$**.** Define iM is the matrix of $ad(N)$ in the frequency basis $e^{i\mu x} z_k, e^{-i(\mu,x)}\bar{z}_k, \pi(\mu) + k = 0, \eta(\mu) = -1$. We now compute i$M$. Recall that we have the rules of Poisson bracket:

$$\{y_i, y_j\} = \{x_i, x_j\} = 0, \{y_i, x_j\} = \delta^i_j, \{y_i, z_k\} = \{x_j, z_k\} = 0 \\ \{z_h, z_k\} = \{\bar{z}_h, \bar{z}_k\} = 0, \{\bar{z}_h, z_k\} = \mathrm{i}\delta^h_k. \tag{55}$$

We have:

$$\{y_i, e^{\mathrm{i}\sum_j \mu_j x_j} z_l\} = e^{\mathrm{i}\sum_j \mu_j x_j}\{y_i, z_l\} + e^{\mathrm{i}\sum_j \mu_j x_j}\mathrm{i}\sum_j \mu_j z_l \{y_i, x_j\} = \\ = \mathrm{i}e^{\mathrm{i}\sum_j \mu_j x_j} z_l \sum_j \mu_j \delta^i_j = \mathrm{i}\mu_i e^{\mathrm{i}\sum_j \mu_j x_j} z_l \tag{56}$$

Hence

$$\{(\omega_0, y), e^{\mathrm{i}\sum_j \mu_j x_j} z_l\} = \{\sum_{i=1}^m |v_i|^2 y_i, e^{\mathrm{i}\sum_j \mu_j x_j} z_l\} = \mathrm{i}\sum_{i=1}^m \mu_i |v_i|^2 e^{\mathrm{i}\sum_j \mu_j x_j} z_l. \tag{57}$$

and

$$\{(\nabla_\xi A_{q+1}(\xi) - (q+1)^2 A_q(\xi)\underline{1}, y), e^{\mathrm{i}\sum_j \mu_j x_j} z_l\} = \\ = \mathrm{i}(\sum_{i=1}^m \mu_i \frac{\partial A_q + 1(\xi)}{\partial \xi_i} - (q+1)^2 A_q(\xi) \sum_{i=1}^m \mu_i) e^{\mathrm{i}\sum_j \mu_j x_j} z_l \tag{58}$$

$$\{|z_k|^2, e^{\mathrm{i}\sum_j \mu_j x_j} z_l\} = \{z_k \bar{z}_k, e^{\mathrm{i}\sum_j \mu_j x_j} z_l\} = z_k e^{\mathrm{i}\sum_j \mu_j x_j}\{\bar{z}_k, z_l\} = z_k e^{\mathrm{i}\sum_j \mu_j x_j}\mathrm{i}\delta^k_l. \tag{59}$$

[1] the name comes from dynamical considerations

(60)
$$\implies \{\sum_{k\in S^c}|k|^2|z_k|^2, e^{\mathrm{i}\sum_j \mu_j x_j} z_l\} = \mathrm{i}|l|^2 e^{\mathrm{i}\sum_j \mu_j x_j} z_l = \mathrm{i}|\pi(\mu)|^2 e^{\mathrm{i}\sum_j \mu_j x_j} z_l = \mathrm{i}|\sum_j \mu_j v_j|^2 e^{\mathrm{i}\sum_j \mu_j x_j} z_l.$$

Now consider the operator $ad(\mathcal{Q})$. It is easy to see that

$$\{e^{\mathrm{i}(\ell,x)} z_h \bar z_k, e^{\mathrm{i}\sum_j \mu_j x_j} z_l\} = \mathrm{i}e^{\mathrm{i}\sum_j(\ell_j+\mu_j)x_j} z_h \delta^k_l, \tag{61}$$

$$\{e^{\mathrm{i}(\ell,x)} z_h z_k, e^{\mathrm{i}\sum_j \mu_j x_j} z_l\} = 0, \tag{62}$$

$$\{e^{-\mathrm{i}(\ell,x)} \bar z_h \bar z_k, e^{\mathrm{i}\sum_j \mu_j x_j} z_l\} = \mathrm{i}\bar z_k e^{-\mathrm{i}\sum_j(\ell_j-\mu_j)x_j}\delta^h_l + \mathrm{i}\bar z_h e^{-\mathrm{i}\sum_j(\ell_j-\mu_j)x_j}\delta^k_l. \tag{63}$$

And we get easily similar formulas for the action of terms of N in Formula (38) on $e^{-\mathrm{i}\sum_j \mu_j x_j}\bar z_l$. Finally, from (57)-(63) we get the following: Given $a = \sum_i \mu_i e_i,\ \sigma = \pm 1$ set

$$C((a,\sigma)) := \frac{\sigma}{2}(a^2 + a^{(2)}) = \frac{\sigma}{2}((\sum_i \mu_i e_i)^2 + \sum_i \mu_i e_i^2), \tag{64}$$
$$K((a,\sigma)) := \pi(C(u)) = \frac{\sigma}{2}(|\sum_i \mu_i v_i|^2 + \sum_i \mu_i |v_i|^2).$$

Sometimes we call $K(u)$ the *quadratic energy* of u, notice that $C(u)$ has integer coefficients. In particular if $a\in\mathbb{Z}^m$ we have $K(a\tau) = -K(a)$ and we have for $a,b\in\mathbb{Z}^m$

$$M_{a,a} = K(a) + \sum_i \mu_i \frac{\partial A_{q+1}(\xi)}{\partial \xi_i} - \sum_i \mu_i (q+1)^2 A_q(\xi), \tag{65}$$
$$M_{a\tau,a\tau} = K(a\tau) - \sum_i \mu_i \frac{\partial A_{q+1}(\xi)}{\partial \xi_i} + \sum_i \mu_i (q+1)^2 A_q(\xi)$$

$$M_{a\tau,b\tau} = -c(\ell),\ M_{a,b} = c(\ell), \quad \text{if } a,\ b \text{ are connected by a black edge } \ell \tag{66}$$

$$M_{a,b\tau} = -c(\ell),\ M_{a\tau,b} = c(\ell), \quad \text{if } a,\ b\tau \text{ are connected by a red edge} \ell \tag{67}$$

We have shown in [11] that the blocks M on $F^{0,1}$ come into pairs of conjugate Lagrangian blocks $\Gamma,\Gamma\tau$. With respect to the frequency basis the blocks are described as the connected components of a graph Λ_S which we now describe.

Definition 4.8. *Given an edge* $u \xrightarrow{x} v$, $u=(a,\sigma), v=(b,\rho)=xu,\ x\in X_q$, *we say that the edge is* compatible *with* S *or* π *if* $K(u)=K(v)$.

Remark now that $K(-e_i) = K(-e_i)\tau = 0$. We call the elements $\{-e_i, -e_i\tau\}$ the *special component*. Let $\Theta = Ker(\pi)$.

Definition 4.9. *The graph* Λ_S *is the subgraph of* $G^1\setminus\bigcup_i\{-e_i+\Theta, (-e_i+Theta)\tau\}$ *in which we only keep the compatible edges.*

We then have

Theorem 4.1. *The indecomposable blocks of the matrix* M *in the frequency basis correspond to the connected components of the graph* Λ_S.

The entries of M *are given by* (65), (66), (67).

The fact that in the graph Λ_S we keep only compatible edges implies in particular that the *scalar part* $\pm\frac{1}{2}[\sum_j \mu_j(|v_j|^2 + |\sum_j \mu_j v_j|^2]$ (which is an integer) is constant on each block. On the other hand, in general, there are infinitely many blocks with the same scalar part.

Remark 4.3. *One of the main ingredients of our work is to understand the possible connected components of the graph Λ_S, we do this by analyzing such a component as a translation $\Gamma = Au$ where A is come complete subgraph of the Cayley graph but contained in G_{-2} and containing the element 0. In particular we have shown (cf.* [11], *§9) that A can be chosen among a finite number of graphs which we call* combinatorial.

4.5. **Geometric graph Γ_S.** In order to understand the possible components of the graph Λ_S we need to study a purely geometric graph. We define a graph on $\mathbb{R}^n$ using formulas (43) and (44).

Definition 4.10. *An edge $\ell \in X_q^{-2}$ defines a sphere S_ℓ through the relation*

$$|x|^2 + (x, \sum_i \ell_i v_i) = \frac{-1}{2}(|\sum_i \ell_i v_i|^2 + \sum_i \ell_i |v_i|^2). \tag{68}$$

An edge $\ell \in X_q^0$ defines a plane H_ℓ through the relation

$$(x, \sum_i \ell_i v_i) = \frac{1}{2}(|\sum_i \ell_i v_i|^2 + \sum_i \ell_i |v_i|^2). \tag{69}$$

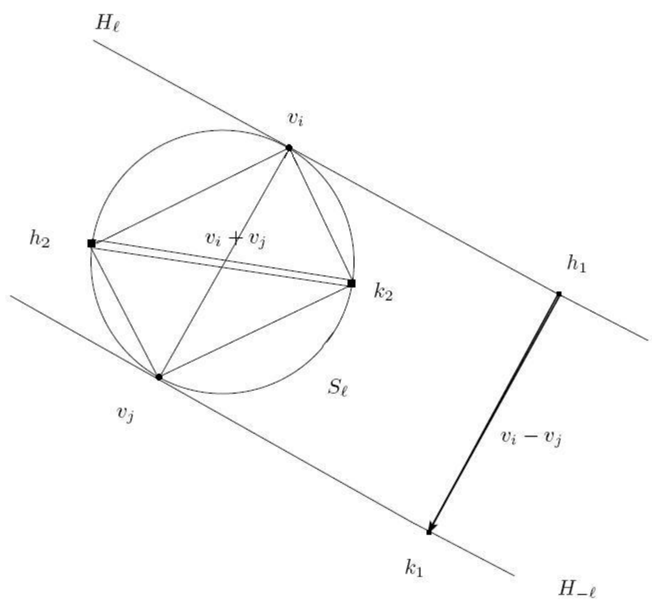


FIGURE 5. The plane H_ℓ with $\ell = e_j - v_i$ and the sphere S_ℓ with $\ell = -e_i - e_j$. The points h_1, k_1, v_j, v_i form the vertices of a rectangle. Same for the points h_2, v_i, k_2, v_j.

Definition 4.11. *Each $\ell \in S_\ell$ is joined by a* red *unoriented edge to $-x - \sum_i \ell_i v_i \in S_\ell$. Each $x \in H_\ell$ is joined by a* black *oriented edge to $x - \sum_i \ell_i v_i \in H_{-\ell}$. We construct the geometric graph $|Gamma_S$ with vertices all the points of $\mathbb{R}^n$ and edges the black and edges described.*

It is convenient to mark each edge of the graph with the element $-\pi(\ell)$ from which it comes from. Remark that Constraint 1 implies that the edge ℓ is uniquely determined by the vector $-\pi(\ell)$.

Remark 4.4. *It is immediate by the definitions that the points in S are all pairwise connected by black and red edges and it is not hard to see that, the completeness constraint 1 implies that the set S is itself a connected component which we call the* special component.

4.6. **From the combinatorial to the geometric graph.** In our geometric setting, we have chosen a list S of vectors v_i and we then define $\pi : \mathbb{Z}^m \to \mathbb{R}^n$ by $\pi : e_i \mapsto v_i$.

We then think of G also as linear operators on $\mathbb{R}^n$ by setting

$$ak := -\pi(a) + k,\ k \in \mathbb{R}^n,\ a \in \mathbb{Z}^m\,, \quad \tau k = -k \tag{70}$$

We extend $\pi : \mathbb{Z}^m \to \mathbb{R}^n$ to $\mathbb{Z}^m \rtimes \mathbb{Z}/(2)$ by setting $\pi(a\tau) := \pi(a)$ so that $-\pi$ is just the orbit map of 0 associated to the action (70) (the sign convention is suggested by the conservation of momentum in the NLS).

We then have

Remark 4.5. *X defines also a Cayley graph on $\mathbb{R}^n$. and in fact the graph Γ_S is a subgraph of this graph.*

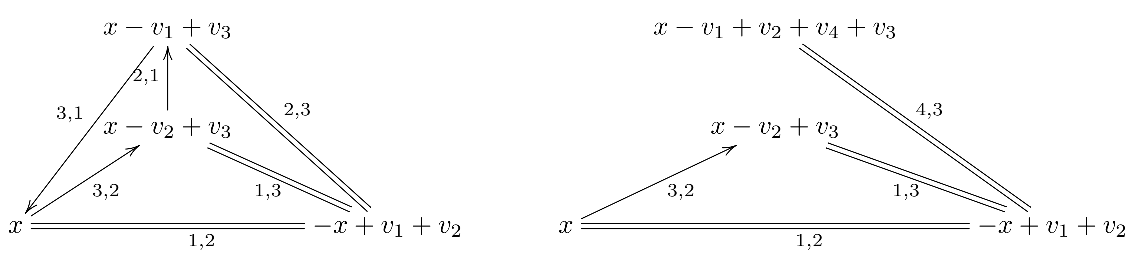
There are symmetries in the graph. The symmetric group S_m of the $m!$ permutations of the elements e_i preserves the graph. By Lemma 4.1 we have the right actions of G, on the graph:

$$(b,\sigma) \mapsto (b,\sigma)\tau = b\sigma\tau, \quad (b,\sigma) \mapsto (b,\sigma)a = (b + \sigma a, \sigma),\ \forall a, b \in \mathbb{Z}^m. \tag{71}$$

Up to the G action any subgraph can be translated to one containing 0.

Each connected component of the graph Γ_S has a combinatorial description based on (68) and (69) which encodes the information on the various types of edges which connect the vertices of the component.

Example 4.1.



the equations that x has to satisfy are:

$$\begin{aligned}(x, v_2 - v_3) &= |v_2|^2 - (v_2, v_3)\\ |x|^2 - (x, v_1 + v_2) &= -(v_1, v_2)\\ (x, v_1 - v_3) &= |v_1|^2 - (v_2, v_3)\end{aligned}$$

$$\begin{gathered}(x, v_2 - v_3) = |v_2|^2 - (v_2, v_3)\\ |x|^2 + (x, v_1 + v_2) = -(v_1, v_2)\\ (x, v_1 - v_2 - v_3 - v_4) - |v_1|^2 + (v_1, v_2) + (v_1, v_3)\\ -(v_2, v_3) + (v_1, v_4) - (v_2, v_4) - (v_3, v_4)\end{gathered}$$

4.7. **The correspondence of Γ_S with Λ_S.** This correspondence comes from the fact that

Remark 4.6. *Equations which define edges in the graph Γ_S are exactly the ones which define compatible edges in Λ_S, in other words, set $a, b \in \mathbb{Z}^m$ such that $-\pi(a) = x, -\pi(b) = y$, we have*

(1) *$x, y \in S_\ell$ are connected by a red edge marked by $-pi(\ell)$ if and only if $a, b\tau$ are connected by a red edge marked by ℓ and $K(a) = K(b)$.*

(2) *$x \in H_\ell, y \in H_{-\ell}$ are connected by a black edge marked by $-pi(\ell)$ if and only if a, b are connected by a black edge marked by ℓ and $K(a) = K(b)$*

Proof. We will prove 1. The proof for 2 is similar. i) Let $x = \sum_{j=1}^m \mu_j v_j \in S_\ell$. We have $a \in \mathbb{Z}^m : a = -\sum_{j=1}^m \mu_j e_j$ such that $-\pi(a) = x$. By Definition 4.11 x is joined by a red edge marked by $-\pi(\ell)(\ell = \sum_{j=1}^m \ell_j e_j \in X_q^{-2})$ with y if and only if $y = -x - \sum_{j=1}^m \ell_j v_j$ and we have $b \in \mathbb{Z}^m : b = \sum_{j=1}^m (\mu_j + \ell_j) e_j$ such that $-\pi(b) = y$. Since $a+b = \sum_{j=1}^m \ell_j e_j \in X_q^{-2}$, $a, b\tau$ will be connected a red edge marked by ℓ. We have

$$K(a) = \frac{1}{2}(|-\sum_{j=1}^m \mu_j v_j|^2 - \sum_{j=1}^m \mu_j |v_j|^2), \tag{72}$$

$$K(b\tau) = -\frac{1}{2}(|\sum_{j=1}^m (\mu_j \ell_j) v_j|^2 + \sum_{j=1}^m (\mu_j + \ell_j)|v_j|^2). \tag{73}$$

(74)
$$K(b\tau) = -\frac{1}{2}(|\sum_j \mu_j v_j|^2 + |\sum_j \ell_j v_j|^2 + 2(\sum_j \mu_j v_j, \sum_j \ell_j v_j) + \sum_j \mu_j |v_j|^2 + \sum_j \ell_j |v_j|^2)$$

From (72) and (74) we get

$$\begin{aligned} K(a) = K(b\tau) &\Leftrightarrow 2|\sum_j \mu_j v_j|^2 + 2(\sum_j \mu_j v_j, \sum_j \ell_j v_j) = -(|\sum_j \ell_j v_j|^2 + \sum_j \ell_j |v_j|^2) \\ &\Leftrightarrow |x|^2 + (x, \sum_j \ell_j v_j) = -\frac{1}{2}(|\sum_j \ell_j v_j|^2 + \sum_j \ell_j |v_j|^2) \end{aligned} \tag{75}$$

The last equation in (75) is exactly the equation (68) which defines S_ℓ. □

Therefore we have:

Remark 4.7. *The map $-\pi$ gives an isomorphism between connected components of Λ_S to its image in Γ_S.*

In application of the KAM algorithm to our Hamiltonian a main point is to prove the validity of the second Melnikov condition. The problem arises in the study of the second Melnikov equation where we have to understand when it is that two eigenvalues are equal or opposite. The condition for a polynomial to have distinct roots is the non–vanishing of the discriminant while the condition for two polynomials to have a root in common is the vanishing of the resultant. In our case these resultants and discriminants are polynomials in the parameters ξ_i so, in order to make sure that the singularities are only in measure 0 sets (in our case even an algebraic hypersurface), it is necessary to show that these polynomials are formally non–zero. This is a purely algebraic problem involving, in each dimension n, only finitely many explicit polynomials and so it can be checked by a finite algorithm. The problem is that, even in dimension 3, the total number of these polynomials is quite high (in the order of the hundreds or thousands) so that the algorithm becomes quickly non practical. In order to avoid this we have experimented with a conjecture which is stronger than the mere non-vanishing of the desired polynomials. We expect our polynomials to be irreducible and separated, in the sense that the connected component of the graph giving rise to the block and its polynomial can be recovered from the associated characteristic polynomial.

4.8. **Characteristic polynomials of complete color marked graphs.** For every complete colored marked graph $\mathcal{G}$ we will consider the matrix $C_{\mathcal{G}}$ indexing by vertices of $\mathcal{G}$ as computed in (65), (66), (67). The irreducibility property of characteristic polynomials is invariant under translations (see Theorem 4.2) so in the proof of the irreducibility can assume that the graph contains 0. Hence every vertex has mass equal to 0 or -2 and we have constant $K(a) = K(0) = 0 \forall a$ (since we keep only compatible edges). So the matrix $C_{\mathcal{G}}$ will be as follows: Given $(a, \sigma), a = \sum_{i=1}^{m} n_i e_i$ set

$$(q+1)a(\xi) := \sum_{i=1}^{m} n_i \frac{\partial}{\partial \xi_i} A_{q+1}(\xi) \tag{76}$$

then

- In the diagonal at the position $(a, \sigma), a = \sum_{i=1}^{m} n_i e_i$ we put

$$\begin{cases} (q+1)a(\xi) & \text{if } \sigma = 1 (\implies \eta(a) = \sum_i n_i = 0) \\ -(q+1)a(\xi) - 2(q+1)^2 A_q(\xi) & \text{if } \sigma = -1 (\implies \eta(a) = \sum_i n_i = -2) \end{cases} \tag{77}$$

- At the position $((a, \sigma_a), (b, \sigma_b))$ we put 0 if they are not connected, otherwise we put $\sigma_b c(\ell)$ (c. f. 40, where ℓ is the edge connecting a, b.

Define $\chi_{\mathcal{G}} = \chi_{C_{\mathcal{G}}}(t) = det(tI - C_{\mathcal{G}})$- the characteristic polynomial of $C_{\mathcal{G}}$.

Theorem 4.2.

$$C_{\tau_c(G)} = c(\xi)I + C_G, \quad C_{\bar{G}} = -C_G. \tag{78}$$

where τ_c is the translation map by vector c, $\bar{G}$ is the image of G under the sign change (see (71)*).*

Consequence 4.1.

$$\chi_{\tau_c(G)}(t) = \chi_G(t - c(\xi)) \tag{79}$$

Proof. We have by theorem 4.2

$$\chi_{\tau_c(G)}(t) = det(tI - C_{\tau_c(G)}) = det((t - c(\xi))I - C_G) = \chi_G(t - c(\xi)).$$

□

As we said in 1 in order to check the second Melnikov condition we expect that for connected colored marked graphs $\mathcal{G}$ $\chi_{\mathcal{G}}$ are irreducible over $\mathbb{Z}$ and separated.

Remark 4.8. *In the proof of separation we do not assume that the quadratic energy $K(a)$ is zero. And in fact in our proof of the separation we use only the induction, the constant $K(a)$ does not play any role.*

Lemma 4.2. *For any $a \in \mathbb{Z}^m$: $a(\xi)$ has integer coefficients.*

Proof. Let $a = \sum_i n_i e_i$. We have

$$\frac{\partial}{\partial \xi_i} A_{q+1}(\xi) = \sum_{\beta \in \mathbb{N}^m; |\beta|_1 = q+1; \beta_i \geq 1} \binom{q+1}{\beta}^2 \beta_i \xi_1^{\beta_1} ... \xi_i^{\beta_i - 1} ... \xi_m^{\beta_m}$$

$$\binom{q+1}{\beta}^2 \beta_i = \binom{q+1}{\beta} \binom{q}{\beta_1, ..., \beta_i - 1, ..., \beta_m} (q+1)$$

is divisible by $q+1$. □

Hence all diagonal elements of $C_{\mathcal{G}}$ are divisible by $q+1$. Besides by the formula 40 all off-diagonal elements of $C_{\mathcal{G}}$ are also divisible by $q+1$. Thus we can write:

$C_{\mathcal{G}} = (q+1)\tilde{C}_{\mathcal{G}} \Rightarrow \chi_{C_{\mathcal{G}}}(t) = det(tI - C_{\mathcal{G}}) = det((q+1)\tilde{t}I - (q+1)\tilde{C}_{\mathcal{G}}) = (q+1)^{n+1}\chi_{\tilde{C}_{\mathcal{G}}}(\tilde{t})$

So in order to prove the irreducibility of the polynomials $\chi_{C_{\mathcal{G}}}$ it is enough to prove the irreducibility and the separation of the polynomials $\chi_{\tilde{C}_{\mathcal{G}}}$. For simplicity we will denote $\chi_{\tilde{C}_{\mathcal{G}}}$ also by $\chi_{\mathcal{G}}$, and we will redefine $c(\ell)$ by division the right hand sides of (40) by $q+1$:

(80)

$$c(\ell) = c_q(\ell) := \begin{cases} (q+1)\xi^{\frac{\ell^+ + \ell^-}{2}} \sum_{\alpha\in\mathbb{N}^m; |\alpha+\ell^+|_1 = q} \begin{pmatrix} q \\ \ell^+ + \alpha \end{pmatrix} \begin{pmatrix} q \\ \ell^- + \alpha \end{pmatrix} \xi^\alpha, & \ell \in X_q^0; \\ q\xi^{\frac{\ell^+ + \ell^-}{2}} \sum_{\alpha\in\mathbb{N}^m; |\alpha+\ell^+|_1 = q-1} \begin{pmatrix} q+1 \\ \ell^- + \alpha \end{pmatrix} \begin{pmatrix} q-1 \\ \ell^+ + \alpha \end{pmatrix} \xi^\alpha, & \ell \in X_q^{-2}. \end{cases}$$

Take a complete colored marked graph $\mathcal{A}$ and compute its characteristic polynomial $\chi_{\mathcal{A}}(t)$. We have:

Theorem 4.3. *When we set a variable $\xi_i = 0$ in $\chi_{\mathcal{A}}(t)$ we obtain the product of the polynomials $\chi_{\mathcal{A}_i}(t)$ where the A_i are the connected components of the graph obtained from $\mathcal{A}$ by deleting all the edges in which i appears as index, with the induced markings (with $\xi_i = 0$).*

Proof. This is immediate from the form of the matrices. □

Part 2. The separation and irreducibility of characteristic polynomials, associated to the cubic NLS

Abstract. *This part is the proof of Theorem 1.1 for the cubic NLS. It requires a lengthy and complicated analysis. One needs to classify graphs by the appearance of indices and apply induction on the size of matrices and on the number of variables ξ_i.*

The cubic NLS is the equation of the form (1) when $q = 1$. In this case:

(81)

$$A_{q+1}(\xi) = A_2(\xi) = \sum_{j=1}^m \xi_j^2 + 4\sum_{j\neq k}\xi_j\xi_k \implies \frac{\partial}{\partial\xi_i}A_2(\xi) = 2\xi_i + 4\sum_{j\neq i}\xi_j = -2\xi_i + 4\sum_{j=1}^m \xi_j.$$

$$A_q(\xi) = A_1(\xi) = \sum_{k=1}^m \xi_k$$

$X^0 := X_1^0 = \{e_i - e_j,\ i \neq j \in [1, \dots, m]\}, \quad X^{-2} := X_1^{-2}\{(-e_i - e_j)\tau,\ i \neq j \in [1, \dots, m]\}$.

Let $(a,\sigma), (b,\rho) \in \mathbb{Z}^m \rtimes \mathbb{Z}/(2)$.

- We join $(a,\sigma), (b,\rho)$ with an oriented black edge, marked (i,j) if
$$\sigma = \rho,\ b = a + e_i - e_j,\ \iff\ a = b + e_j - e_i.$$
- We join $(a,\sigma), (b,\rho)$ with an unoriented red edge, marked (i,j) if
$$\sigma\rho = \tau,\ b + a + e_i + e_j = 0.$$

$$b = a + e_i - e_j \qquad a \xrightarrow{(i,j)} b \qquad \iff \qquad a \xrightarrow{e_i - e_j} b$$

$$c + d + e_j + e_i = 0 \qquad c \overset{(i,j)}{=\!=\!=} d \qquad \iff \qquad c \overset{(-e_i - e_j)\tau}{\rule{2em}{0.4pt}} d$$

From Formula (80) for $q = 1$ we get $c(\ell) = 2\sqrt{\xi_i\xi_j}$ if $\ell = e_i - e_j$ or $\ell = -e_i - e_j$.

For every connected component G of Γ_S we will consider the matrix C_G indexing by vertices of G.Given $(a, \sigma), a = \sum_{i=1}^m n_i e_i$, by Formula (169) in the case $q = 1$ we have $a(\xi) := \frac{1}{2}\sum_i n_i(-2\xi_i + 4\sum_k \xi_k) = \begin{cases} -\sum_i n_i\xi_i, & \text{if } \sigma_a = 1, \eta(a) = 0; \\ -\sum_i n_i\xi_i - 4\sum_k \xi_k, & \text{if } \sigma_a = -1, \eta(a) = -2 \end{cases}$

Hence we get easily

Lemma 4.3. *The entries of the matrix C_G, over the indexing set of the vertices of G, are:*

- *In the diagonal at the vertex (a, σ) equals to $-\sigma\sum_{i=1}^n n_i\xi_i$.*
- *At the position $(a, \sigma), (b, \tau)$ we put 0 unless they are connected by an oriented edge $e = ((a, \sigma), (b, \tau))$ marked with (i, j). In this case we place*

$$C(e) := 2\tau\sqrt{\xi_i\xi_j}. \tag{82}$$

It is easily verified that when we expand the characteristic polynomial of such a matrix the square roots disappear and we get a polynomial, denoted $\chi_A(t)$ monic in t and with coefficients polynomials in the variables ξ_i with integral coefficients. Our goal is to prove that each of these polynomials is irreducible (as polynomial in $\mathbb{Z}[t, \xi]$) , this we call *irreducibility theorem* and furthermore that the graph A is determined by $\chi_A(t)$, this we call the *separation lemma*.

In fact in this form the statement is not true, we need to restrict to the subspace of $F^{(0,1)}$ where mass is conserved. This is enough for the dynamical consequences. In algebraic terms the conservation of mass consists in restricting to the coset of G_2 (one of the connected components of the Cayley graph) of elements $a, a\tau \in G,\ a \in \mathbb{Z}^m,\ \eta(a) = -1$. Moreover, in [12] we have proved

Theorem 4.4. *For generic choices of S (see the redefinition of genericity in Appendix 12) the connected components of graph Γ_S, different from the special component, are formed by affinely independent points.*

In particular each component has at most $n + 1$ points.

5. The irreducibility and separation

5.1. **Preliminaries.** Observe first that, given $a \in \mathbb{Z}^m, A \subset \mathbb{Z}^m$ we have that $\chi_A(t)$ is irreducible if and only if $\chi_{A+a}(t)$ is irreducible.

Consider a projection $\pi_i : \mathbb{Z}^m \to \mathbb{Z}^{m-1}$ where $\pi_i(a_1, \dots, a_m) \mapsto (a_1, \dots, \check{a}_i, \dots a_m)$ (we remove the i^{th} coordinate). Take now a set $A \subset \mathbb{Z}^m$ of vertices and consider the graph obtained from Γ_A by removing all the edges which contain i in its marking, call this new graph Γ_A^i. Even if A is connected this new graph Γ_A^i may well not be connected. We now claim

Proposition 6. *If A is connected the map π_i, restricted to Γ_A^i, is injective and a graph isomorphism with $\Gamma_{\pi_i(A)}$, a graph in $\mathbb{Z}^{m-1}$.*

If A is non degenerate each connected component of $\Gamma_{\pi_i(A)}$ is non degenerate.

Proof. We know that the augmentation $\ell = \eta(a)$ depends only on the color of a so that we have $a_i = \eta(a) - \eta(\pi_i(a))$ and thus if a, b are black vertices (or red vertices), $\pi_i(a) = \pi_i(b)$:$\eta(a) = \eta(b)$ hence $a_i = b_i \implies a = b$. Otherwise, if a is black, b is red then it is clearly $\pi_i(a) \neq \pi_i(b)$ because $\pi_i(a)$ is black, $\pi_i(b)$ is red. If we decompose $X = X_m$ into the elements containing the index i and the complement X_m^i we see that π_i establishes a 1–1 correspondence between X_m^i and X_{m-1} from which the second claim. The third claim follows easily from the definitions.

□

A simple corollary of this proposition is that.

Corollary 5.1. *If we set $\xi_i = 0$ in the matrix C_A we have the matrix $C_{\pi_i(A)}$, hence*

$$\chi_A(t)|_{\xi_i=0} = \chi_{\pi_i(A)}(t)$$

Let $B_1, \dots, B_k$ be the connected components of $\pi_i(A)$. We have

$$\prod_{j=1}^{k} \chi_{B_j}(t) = \chi_{\pi_i(A)}(t) = \chi_A(t)|_{\xi_i=0}.$$

As a consequence, we have the following inductive step.

Corollary 5.2. *Assume that A is non degenerate and that we have already proved the irreducibility theorem for $m-1$ or for $n < |A|$. We deduce that the factors $\chi_{B_j}(t)$ of $\chi_{\pi_i(A)}(t)$ are the irreducible monic factors of $\chi_A(t)|_{\xi_i=0}$.*

We want to prove Theorem 1.1 by induction as follows. We assume irreducibility and separation in dimension $n-1$ and prove first the separation in dimension n and finally irreducibility in dimension n.

Take a connected A and let ℓ be the augmentation of a black vertex of A, then the augmentation of a red vertex is $-2-\ell$.

Lemma 5.1 (Parity test). *(1) If we compute t at a number $g \not\cong \ell \mod (2)$, we have $\chi_A(g) \neq 0$.*

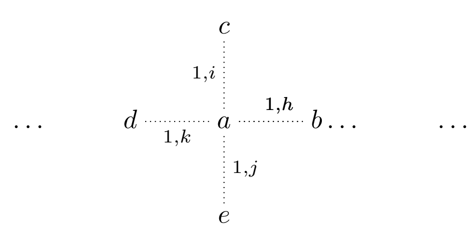
(2) If a linear form $t + \sum_i a_i\xi_i$, $a_i \in \mathbb{Z}$ divides $\chi_A(t)$ we must have $\sum_i a_i \cong \ell \mod (2)$.

Proof. i) We compute modulo 2 and set all $\xi_i = 1$, we get $\chi_A(t) \cong (t+\ell)^m \mod (2)$, hence $\chi_A(g) \cong (g+\ell)^m \cong g+\ell \mod (2)$.

ii) A linear form $t+\sum_i a_i\xi_i$, $a_i \in \mathbb{Z}$ divides $\chi_A(t)$ if and only if we have $\chi_A(-\sum_i a_i\xi_i) = 0$, then set $\xi_i = 1$ and use the first part. □

We shall use the parity test as follows.

Lemma 5.2. *Suppose we have a connected set A in $\mathbb{Z}^m$, in which we find a vertex a and an index, say 1, so that the graph Γ_A has the following properties:*



we have:

- *1 appears in all and only the edges having a as vertex.*
- *When we remove a (and the edges meeting a) we have a connected graph $\mathcal{A}$ with at least 2 vertices.*
- *When we remove the edges associated to any index, the factors described in Corollary 5.1 are irreducible.*

Then the polynomial $\chi_A(t)$ is irreducible.

Proof. We take a as root, and translate the set A so that $a = 0$. Setting $\xi_1 = 0$ we have by Corollary 5.1 and the hypotheses, that $\chi_A(t) = t\,P(t)$ with $P = \chi_{\mathcal{A}}(t)$ irreducible of degree > 1. Thus, if the polynomial $\chi_A(t)$ factors, then it must factor into a linear $t - L(\xi)$ times an irreducible polynomial of degree > 1.

Moreover modulo $\xi_1 = 0$ we have that 0 and ℓ coincide, thus $L(\xi)$ is a multiple of ξ_1.

Take another index $i \neq 1, h$ if a is an end and the only edge from a is marked $(1, h)$ otherwise just different from 1 and set $\xi_i = 0$. Now the polynomial $\chi_A(t)$ specializes to the product $\prod_j \chi_{A_j}(t)$ where the A_j are the connected component of the graph obtained from A by removing all edges in which i appears as marking. By hypothesis $\{a\}$ is not one of the A_j.

If no factor is linear we are done. Otherwise there is an isolated vertex $d \neq a$ so that $\{d\}$ is one of the connected components A_j. The linear factor associated is $t + d(\xi)|_{\xi_i=0}$. Clearly we have that the coefficient of ξ_1 in $d(\xi)$ is ± 1 (since the marking 1 appears only once). This implies that $L(\xi) = \pm\xi_1$ and this is not possible by the parity test. □

By Theorem 4.4 we need to consider only the graphs formed by affinely independent vertices.

6. The separation lemma

Let be given a colored marked graphs G. We define the graph $\tau G = \{(-a, -\sigma) | (a, \sigma) \in G\}$.

Remark 6.1. *τG is a connected graph, if and only if G contains only black edges.*

Proof. If there exists a red edge marked i, j that connects two vertices a, b then $a + b = -e_i - e_j \Rightarrow -a - b = e_i + e_j$, then $-a, -b$ are not connected in τG. If $b - a = e_i - e_j \Rightarrow -b - (-a) = a - b = e_j - e_i$, $-a, -b$ are connected by a black edge marked j, i in τG. □

Lemma 6.1. *(Separation lemma) Given two connected colored marked graphs G_1, G_2 if $\chi_{G_1} = \chi_{G_2}$, then $G_1 = G_2$ or $G_1 = \tau G_2$.*

Since if G is of mass -1 we have that τG is of mass 1, we deduce that a connected color marked graph G of mass -1 can be recovered from its characteristic polynomial.

Proof. We will prove this lemma by induction. When $n = 0 : \chi_G(t) = t + a$, it is easy to see that $G = \{(a, +)\}$ or $G = \{(-a, -)\}$.
Induction process: $n > 1$. Suppose that we have the separation and the irreducibility for graphs of dimensions $k \leq n - 1$. Take a connected colored marked graph $G = \{(v_1, \sigma_1), \dots, (v_{n+1}, \sigma_m)\}$, $(v_i, \sigma_i) \in \mathbb{Z}^m \rtimes \mathbb{Z}/(2)$, the associated matrix C_G and its characteristic polynomial χ_G. We want to show that G can be uniquely (up to the sign) reconstructed by χ_G.

First associate to G the list of vectors $w_i := \sigma_i v_i$, we see that these vectors are affinely independent. If the w_i have all the same mass then the graph G has only black edges and then it is either the graph with vertices w_i or with vertices τw_i as seen before, if they have different masses then the masses are of type k for black vertices and $k + 2$ for red and the graph G is thus reconstructed.

Therefore we need to show that from the characteristic polynomial we can recover the list $L := \{w_1, \dots, w_n\}$. Before starting the proof let us make a useful remark, the characteristic polynomial gives as information the trace of the matrix C_G and thus in particular the sum $\sum_{i=1}^n w_i(\xi)$ and the mass $s := \sum_{i=1}^n \eta(w_i)$. If we have a elements in the list of mass k and $(n-a)$ of mass $k+2$ we have that $s = nk+2b = n(k+2)-2(n-b)$. Thus if we know that a certain number h is the mass of a vertex we can deduce

Lemma 6.2. *If $s = nh$ then all vertices in G have the same color. If $nh < s$ then h is the mass of the black vertices and there are b red vertices where $s = nh+2b$. Similarly if $nh > s$ then h is the mass of the red vertices and there are b red vertices where $s = nh-2(n-b)$.*

We set one of the variables $\xi_i = 0$ for instance $\xi_1 = 0$. We know that the matrix C_G specializes to the direct sum of the matrices C_{G_i} where the G_i correspond to the various connected components of the graph G which are obtained by removing all edges in which 1 appears as marking and dropping in each component the first coordinate of the various vertices. We have that specializing $\xi_1 = 0$ we specialize the polynomial χ_G to $\prod_i \chi_{G_i}$. Since we are assuming irreducibility in dimensions less than $n-1$ the factors χ_{G_i} are all irreducible and thus can be determined by the unique factorization of polynomials. Therefore all the vectors of $\pi_1(L)$, that is the w_i with the first coordinate removed can be recovered uniquely (up to the sign) by induction and we obtain a list of n vectors $L^1 : \{u_i = (*, b_i, c_{3,i}, ..., c_{m,i})\}$.
Now we set another variable, say $\xi_2 = 0$. By similar arguments as above all the w_i with the second coordinate removed can be recovered by induction giving a list $L^2 : \{t_i = (a_i, *, c_{3,i}, ..., c_{m,i})\}$.

Now our problem is this: if we know the vectors obtained from L after removing the first or the second coordinate can we recover the given vectors? We shall need to perform a case analysis.

1) Recovering the list L:

We thus consider the vectors $L^{1,2}$ obtained from L by dropping the first two coordinates $(*, *, c_3, ..., c_m)$ and collect the ones where $c_3, \ldots, c_m; \sigma$ are fixed. The first remark is that, if in this list a given vector $(*, *, c_3, ..., c_m)$ appears only once then we know exactly from which vector it comes from the two lists L^1, L^2 and so we can reconstruct the vector v in L from which it arises. Then by Lemma 6.2 we can determine if in the graph all vertices have the same color or if this is not the case which is the mass of the black end red vertices and how many there are.

Next since the vectors in the graph, by assumption, are affinely independent, we have at most 3 vectors in L, giving the same vector $(*, *, c_3, ..., c_m)$ in $L^{1,2}$ since 4 of such vectors lie in a 2–dimensional plane so they are not affinely independent.

a) Assume we have 3 vectors $v_1, v_2, v_3 \in L$ giving the same vector $\underline{c} = (*, *, c_3, ..., c_m)$ in $L^{1,2}$ and let $c = \eta(\underline{c})$. We claim that v_1, v_2, v_3 cannot have the same color, in fact this would imply that they have the same mass and then they lie in a line and cannot be affinely independent. Let then a_1, a_2, a_3 resp. b_1, b_2, b_3 be the first, resp. second coordinates of these vectors (deduced from the two lists L^1, L^2) we need to be able to reconstruct the 3 vectors $v_1, v_2, v_3 \in L$ by matching the a_i with the b_j. First observe that we know the total mass m of v_1, v_2, v_3. This is $m = 3k+2$ or $m = 3k+4$ depending if we have two or 1 black vertices among v_1, v_2, v_3. Since $3k+2$ is congruent to 2 modulo 3 while $3k+4$ is congruent to 1 modulo 3, we can deduce both k and the number of black vertices from m.

Call $l := k - c$, now consider one of the vectors in L^1, start from $(a_1, *, \underline{c})$, if there is no b_i with $a_1 + b_i = l$ then there must necessarily be one, say b_1 with $a_1 + b_1 = l + 2$ and then $(a_1, *, \underline{c})$ comes from the red vector $(a_1, b_1, \underline{c})$. Similarly if there is no b_i with $a_1 + b_i = l+2$ then there must necessarily be one, say b_1 with $a_1 + b_1 = l$ and then $(a_1, *, \underline{c})$ comes from the black vector $(a_1, b_1, \underline{c})$. In this case we can easily see how to match the other two vectors, in case the other two vectors have the same color we must match them so that $a_2+b_i=l', a_3+b_j=l'$ where $l'=l$ if the color is black and $l+2$ if red. We claim that only one match is possible, in fact if we had $a_2+b_3=a_3+b_2=a_2+b_2=a_3+b_3$ we would have that the two vectors v_2, v_3 coincide.

Suppose now we know that the two colors are distinct, then as before, if there is no $b_j, j=2,3$ such that $a_2+b_j=l$ we know that there is one, say b_2 for which $a_2+b_2=l+2$ and we have reconstructed the two vectors $(a_2,b_2,\underline{c}),(a_3,b_3,\underline{c})$. Finally it is possible that $b_3=b_2+2$ and $a_2+b_2=l$ then we have $a_3+b_3=l+2$ which implies $a_3=a_2=a$ and again we reconstruct the two vectors (actually by Proposition **??** this is not allowed).

It remains to analyze the case in which none of the a_i satisfies the condition that it cannot be paired uniquely.

So let us assume that, up to reordering b_1 is maximum, since there is one a_i which must be paired with b_1 and we are assuming that it can also be paired with another b_i giving a different color we must necessarily have that this a_i which we may assume reordering to be $a_1=l+2-b_1$ and we have recovered a red vector $(a_1,b_1,\underline{c})$. The rest of the analysis follows as before.

b) There are in $L^{1,2}$ only 2 vectors of the form $(*,*,c_3,...,c_m)$ with $c_3,...,c_m$ fixed. For simplicity we denote $\underline{c}:=(c_3,..,c_m)$ and their sum by c. We know then two vectors in $L^{1,2}$ of the form $(a_1,*,\underline{c}),(a_2,*,\underline{c})$ and two vectors in L^2 of the form $(*,b_1,\underline{c}),(*,b_2,\underline{c})$ which specialize in $L^{1,2}$ to the given vectors.

A priori in L we can either have $(a_1,b_1,\underline{c}),(a_2,b_2,\underline{c})$ or $(a_1,b_2,\underline{c}),(a_2,b_1,\underline{c})$. The first pair gives two vertices of the same color if and only if $a_1+b_1=a_2+b_2$, similarly for the second. If we have $a_1+b_1=a_2+b_2, a_1+b_2=a_2+b_1$ we deduce that $a_1=a_2, b_1=b_2$ and this is impossible since implies that in L we have two equal vectors, therefore in at least one of the two pairs we have different colors. We may thus assume (changing the indices if necessary) that $a_1+b_2=a_2+b_1+2 \implies a_1-a_2=b_1-b_2+2$ then write $a_1+b_1=a_2+b_2+x,\ x\in(-2,0,2)$ and thus $2(b_1-b_2)=x-2$. If $x=-2$ we have $b_1-b_2=-2, a_1=a_2$ and we argue as before, this case is impossible.

If $x=2$ we have $b_1=b_2=b, a_1=a_2+2=a+2$ we have in the possible list of vectors $(a+2,b,\underline{c}),(a,b,\underline{c})$. We know that this list is not allowed by Proposition **??**. Assume that $x=0$ thus $b=b_1, b_2=b+1, a=a_2, a_1=a+1$ we have the two possibilities 1) $(a+1,b,\underline{c}),(a,b+1,\underline{c})$ or 2) $(a+1,b+1,\underline{c}),(a,b,\underline{c})$. In this case both cases are a priori possible, in fact if the graph were just a single edge marked e_1-e_2 or $-e_1-e_2$ the two cases cannot be recovered by the two specializations but only from the full characteristic polynomial.

$$G_1 = (e_1,+) \xrightarrow{e_2-e_1} (e_2,+) \qquad G_2 = (0,+) \overset{-e_2-e_1}{=\!=\!=} (-e_1-e_2,-)\ ,$$

$$C_{G_1} = \begin{vmatrix} -\xi_1 & 2\sqrt{\xi_1\xi_2} \\ 2\sqrt{\xi_1\xi_2} & -\xi_2 \end{vmatrix}, \quad C_{G_2} = \begin{vmatrix} 0 & -2\sqrt{\xi_1\xi_2} \\ 2\sqrt{\xi_1\xi_2} & -\xi_1-\xi_2 \end{vmatrix} \tag{83}$$

The characteristic polynomials are distinct:

$$t^2+(\xi_1+\xi_2)t-3\xi_1\xi_2, \quad t^2+(\xi_1+\xi_2)t+4\xi_1\xi_2$$

but the two specializations coincide.

So we need a deeper analysis. First let us assume that we know if all the vectors have the same mass or we know the mass of black and red vertices.

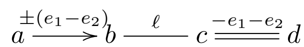
If we know that all vertices have the same mass then case 2) is excluded. Suppose then that we know the mass k of a black vertex.

If case 1) holds we must have that $a+b+c$ is either $k-1$ or $k+1$, if case 2) holds we must have that $a+b+c=k$. Thus we can determine in which case we are.

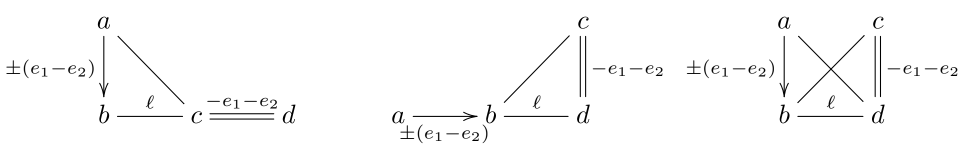
The other possibility is that we do not have the previous information but by the previous analysis this means that in the list $L^{1,2}$ each vector appears twice. If the list consists of just two vectors we can conclude by the explicit formulas of the characteristic polynomial.

Assume we have at least two pairs one u_1, u_2 giving $(*,*,\underline{c})$ the other v_1, v_2 giving $(*,*,\underline{d})$. In each case we know that the two vertices are connected either by the edge e_1-e_2 or by $-e_1-e_2$. We deduce that the only possibility at this point is that there are only two such lists so L has 4 elements and we must have both edges e_1-e_2 and $-e_1-e_2$.

The two edges involve two disjoint pairs of vertices so that the graph must be of the form



if ℓ does not contain any of the indices $1,2$ or possibly of the form



if ℓ contains one of the indices $1,2$. The edge l can have either color (which determines the color of the further edge).

In particular the graph has either 3 black and one red vertex or 3 red and one black vertex so either $s=4k+6=4(k+1)+2$ or $s=4k+2$.

This gives two possible values for the mass of black vertices, k or $k+1$. Finally specializing to $\xi_i=0$ where $i\neq 1,2$ appears in ℓ and to $\xi_1=0$ (or $\xi_2=0$) if 1 resp. 2 does not appear in ℓ we see that of the 4 vectors in $L^{1,2}$ at least one appears only once and we are back in the previous case which we have treated.

□

7. Irreducibility theorem

We prove this by induction. Suppose the separation and irreducibility in all dimensions less than n, we will prove the irreducibility in dimension n. Since this property is invariant under translation we often choose a vertex as the root and assume that it corresponds to 0.

Let G be connected marked graph and take a maximal tree T of G. So T consists of n linearly independent edges. We must have at least n distinct indices appearing in the edges, otherwise these edges span a subspace of dimension less than n. In total on the n edges of T appear $2n$ indices counted with multiplicity. If no index appears with multiplicity 1 we must have that all the indices appear with multiplicity 2.

If only one index appears with multiplicity 1, the remaining $k\geq n-1$ cannot all have multiplicity ≥ 3 since $3(n-1)>2n-1$ unless $n\leq 2$, in which case this can also be excluded since no edge is of the form $-2e_i$. Thus we may have one index of multiplicity 1 and another of multiplicity 2. If only two indices appear with multiplicity 1 and in the same edge the remaining indices must still be $k\geq n-1$ since they give $n-1$ linearly independent edges. Thus they cannot all have multiplicity ≥ 3 by the previous argument.

We thus have to treat 3 cases.

Remark 7.1.
- *Dash lines mean that they may be black or red.*
- *Black edges are denoted by single lines, red edges-by double lines.*
- *$\bar{A}$ denotes the completed graph obtained from the graph A.*

Sometimes given a combinatorial graph G by a *block* A of G we mean a connected complete subgraph A of G.

Lemma 7.1. *If in T there are two blocks A, B and two indices i, j such that:*

(1) i, j do not appear in the edges of the blocks A, B.
(2)

$$\chi_{\bar{A}}|_{\xi_i=\xi_j=0} = \chi_{\bar{B}}|_{\xi_i=\xi_j=0}, \tag{84}$$

then $|B| = |A| = 1,\ A = \{(a, \sigma_1)\}, B = \{(b, \sigma_2)\},\ a, b \in \mathbb{Z}^m$ and $\sigma_2 b + n_i e_i + n_j e_j = \sigma_1 a$. The numbers n_i, n_j are determined by the path in T from a to b.

Proof. Since the degree of the characteristic polynomial is the number of vertices by assumption $|B| = |A|$. Choose the root in A. This gives to each vertex v a sign σ_v. Let $A = \{(a_1, \sigma_1), ..., (a_r, \sigma_r)\}; B = \{(b_1, \delta_1), ..., (b_r, \delta_r)\}$, then to these graphs we associate as in §**??** the list L of vectors $v_h = \sigma_h a_h$ and $w_h = \delta_h b_h$. Since i, j do not appear in A (resp. B), the vectors v_h have the same i-th and j-th coordinates and we can write $v_h = \bar{v}_h + a$, similarly for B the vectors $w_h = \bar{w}_h + b$ where a, b are linear combinations of e_i, e_j and $\bar{v}_h, \bar{w}_h$ are linear combinations of the $e_s, s \neq i, j$.

The list of vectors $\bar{v}_h$ is the one associated to the graph $\bar{A}$ once we set equal to 0 the elements e_i, e_j hence it is the list of vectors associated to the polynomial $\chi_{\bar{A}}|_{\xi_i=\xi_j=0}$ similarly $\bar{w}_h$ is the one associated to $\chi_{\bar{B}}|_{\xi_i=\xi_j=0}$. Hence by the separation lemma up to reordering we may assume that $\bar{v}_h = \bar{w}_h$ hence $v_h = w_h + c,\ c = a - b = n_i e_i + n_j e_j$.

Clearly if $r > 1$ we have that $w_r = w_1 - v_1 + v_r$ so that the vectors (v_h, w_k) are not affinely independent contrary to the hypotheses.

We have thus proved that $|B| = |A| = 1$ hence $A = \{(a, \sigma_1)\}, B = \{(b, \sigma_2)\}$ and finally $\sigma_2 b + n_i e_i + n_j e_j = \sigma_1 a$. Of course $n_i e_i + n_j e_j$ is the value up to sign of the path joining a, b. ☐

Suppose T is a maximal tree in a graph Γ and ℓ be an edge in T containing the indices i, j. We have two connected components A, B of T obtained by removing ℓ.

Lemma 7.2. *Assume that the two connected components A, B do not have the index i in any edge. Then any other edge in Γ connecting A, B must contain the index i.*

Proof. In a path which is a circuit you cannot have that an index appears only once (or even an odd number of times). ☐

We now consider two edges ℓ_1, ℓ_2 containing the indices i, h and i, k respectively. When we remove these edges in T we have 3 connected components in T

$$A \overset{i,h}{\therefore} B \overset{i,k}{\therefore} C$$

in the complete graph $\bar{T}$ once we remove all the edges containing i the graph $\bar{B}$ is a connected component. Then we may either have other 2 components $\bar{A}, \bar{C}$ or a connected component $\overline{A \cup C}$.

Lemma 7.3. *If there exists a pair of indices, say $(1, i)$, such that 1 appears only once in the maximal tree T and T has the form:*

$$A \overset{1,h}{-\!-\!-} B$$

FIGURE 6

where $i \neq h$, and i appears only in the block B. Then χ_G is irreducible.

Proof. Let the root be in A. Since 1 appears only once in T, every edge in G that connects A and B must have 1 in the indexing. We have:

$$\chi_G|_{\xi_1=0} = \chi_{\bar{A}}\chi_{\bar{B}}|_{\xi_1=0}. \tag{85}$$

By induction assumption and since 1 does not appear in B, the polynomials $\chi_{\bar{A}}, \chi_{\bar{B}}|_{\xi_1=0}$ are irreducible. Hence, if χ_G is not irreducible, it must factor into two irreducible polynomials: $\chi_G = UV$ such that:

$$U|_{\xi_1=0} = \chi_{\bar{A}}. \tag{86}$$

Let $B_1, ..., B_s$ be the connected components obtained from B by deleting all the edges which have i in the indexing, B_1 be the component that is connected with A. We have:

$$\chi_G|_{\xi_i=0} = \chi_{\overline{A \cup B_1}}\chi_{\bar{B}_2}|_{\xi_i=0} \cdots \chi_{\bar{B}_s}|_{\xi_i=0}. \tag{87}$$

Remark that $deg(U) = |A| < deg(\chi_{\overline{A \cup B_1}}) = |A| + |B_1|$. $U|_{\xi_1=\xi_i=0} = \chi_{\bar{A}}$ is irreducible, then $U|_{\xi_i=0}$ must be irreducible. Hence

$$U|_{\xi_i=0} = \chi_{\bar{B}_j}|_{\xi_i=0} \text{ for some } j \in \{2, ..., s\} \tag{88}$$

From (86) and (88) we get $\chi_{\bar{A}} = \chi_{\bar{B}_j}|_{\xi_1=\xi_i=0}$. So, by lemma 7.1, $|A| = |B_j| = 1$. Let $A = \{a\}$. Then by lemma 5.2, for the vertex a and the index 1, χ_G is irreducible. □

Corollary 7.1. *If there are two indices which appear only once and not in the same edge in the maximal tree then χ_G is irreducible.*

7.1. **Two indices which appear only once and in the same edge.** Let these two indices be $1, 2$. If there exists another index, say 3, which appears only once, then we can replace 2 by 3 and we are back in the case of Corollary 7.1. Otherwise all other indices, different from 1, 2 appear at least twice. Due to the dimension we must have at least $n-1$ distinct indices, different from $1, 2$. Since we have all together $2n$ indices (with repetition), we have exactly $n-1$ distinct indices different from 1, 2 and they appear twice. Take one of these indices, say 3. If we cannot apply lemma 7.3 we must be in the case, in which the maximal tree T has the form

$$A \overset{3,k}{-\!-\!-} B \overset{1,2}{-\!-\!-} C \overset{3,h}{-\!-\!-} D$$

FIGURE 7

where the indices 1 and 3 do not appear elsewhere in the tree. Consider the case of figure (7). By inspection all edges in G which connect A and C contain $1, 3$ in the indexing, all edges in G which connect B and D contain $1, 3$ in the indexing. Then we have:

$$\chi_G|_{\xi_1=0} = \chi_{\overline{A \cup B}} \cdot \chi_{\overline{C \cup D}}|_{\xi_1=0}. \tag{89}$$

$$\chi_G|_{\xi_3=0} = \chi_{\bar{A}} \cdot \chi_{\overline{B \cup C}}|_{\xi_3=0} \cdot \chi_{\bar{D}}|_{\xi_3=0} \quad \text{or} \quad \chi_G|_{\xi_3=0} = \chi_{\overline{A \cup D}} \cdot \chi_{\overline{B \cup C}}|_{\xi_3=0}. \tag{90}$$

The second case holds when A, D are joined by some edge which does not contain 3. From (89) we see that if χ_G is not irreducible, then it must factor into two irreducible polynomials: $\chi_G = UV$, $U|_{\xi_1=0} = \chi_{\overline{A \cup B}}$. Comparing (89) and (90) by degree and using the irreducibility of $\chi_{\bar{A}}, \chi_{\bar{D}}|_{\xi_3=0}$ we get the following possibilities in the first case of (90)

(1)

$$U|_{\xi_3=0} = \chi_{\bar{A}} \chi_{\bar{D}}|_{\xi_3=0} \implies \chi_{\overline{A \cup B}}|_{\xi_3=0} = U|_{\xi_1=\xi_3=0} = \chi_{\bar{A}} \chi_{\bar{D}}|_{\xi_3=\xi_1=0}$$

On the other hand:

$$\chi_{\overline{A \cup B}}|_{\xi_3=0} = \chi_{\bar{A}} \cdot \chi_{\bar{B}}|_{\xi_3=0}$$
$$\implies \chi_{\bar{B}}|_{\xi_3=0} = \chi_{\bar{D}}|_{\xi_3=\xi_1=0} \tag{91}$$

Hence by lemma 7.1 we must have: $|B| = |D| = 1$ and $d \pm b = n_1 e_1 + n_3 e_3$. But in fact by figure (7) we see $d \pm b = \pm e_2 + \sum_{i \neq 2} n_i e_i$, a contradiction.

(2)

$$U|_{\xi_3=0} = \chi_{\overline{B \cup C}}|_{\xi_3=0} \implies \chi_{\overline{A \cup B}}|_{\xi_3=0} = U|_{\xi_3=\xi_1=0} \chi_{\overline{B \cup C}}|_{\xi_3=\xi_1=0} \tag{92}$$

$$\implies \chi_{\bar{C}}|_{\xi_3=\xi_1=0} = \chi_{\bar{A}} \tag{93}$$

Hence by lemma 7.1 we get $|A| = |C| = 1, A = \{0\}, C = \{c\}$ $c = \pm e_1 \pm e_3$, but in fact by figure (7) we see $c = \pm e_2 + \sum_{i \neq 2} n_i e_i$, contradiction.

In the second case of (90) we arrive at the same conclusions.

7.2. **There is only one index, say 1, which appears once in the tree.** Other indices appear at least twice in the tree. We have exactly $n-1$ indices, different from 1, since if there are more than $n-1$, then they exhaust $2n$ indices (with repetition). From this we see that there is only one index, say 3, which appears three times. All other indices, different from 1, 3, appear twice.

7.2.1. *When 1, 3 appear together in one edge.* If T has the form as in figure (8) then, by lemma 7.3, χ_G is irreducible.

$$A \overset{1,3}{-\,-\,-} B \overset{2,k_1}{-\,-\,-} C \overset{2,k_2}{-\,-\,-} D$$

FIGURE 8

Therefore, assume that T has the form as in figure (9)

$$A \overset{2,k_1}{-\,-\,-} B \overset{1,3}{-\,-\,-} C \overset{2,k_2}{-\,-\,-} D$$

FIGURE 9

1) If A, D are not joined by an edge then:

$$\chi_G|_{\xi_1=0} = \chi_{\overline{A \cup B}} \chi_{\overline{C \cup D}}|_{\xi_1=0}, \tag{94}$$

$$\chi_G|_{\xi_2=0} = \chi_{\bar{A}} \chi_{\overline{B \cup C}}|_{\xi_2=0} \chi_{\bar{D}}|_{\xi_2=0}. \tag{95}$$

2) If A, D are joined by an edge, this edge contains 1 and we have $\chi_G|_{\xi_2=0} = \chi_{\overline{B\cup C}}|_{\xi_2=0}\chi_{\overline{A\cup D}}|_{\xi_2=0}$.

From (94) we see that if χ_G is not irreducible, it must factor into 2 irreducible polynomials: $\chi_G = UV$. Choose the root in A to be 0 so that:

$$U|_{\xi_1=0} = \chi_{\overline{A\cup B}}. \tag{96}$$

Hence $deg(U) = |A| + |B|$. In case 1), from (95) we get the following possibilities:

a)

$$U|_{\xi_2=0} = \chi_{\overline{B\cup C}}|_{\xi_2=0} \implies \chi_{\overline{A\cup B}}|_{\xi_2=0} = \chi_{\overline{B\cup C}}|_{\xi_1=\xi_2=0} \\ \implies \chi_{\bar{A}}\chi_{\bar{B}}|_{\xi_2=0} = \chi_{\bar{B}}|_{\xi_2=0}\chi_{\bar{C}}|_{\xi_1=\xi_2=0} \implies \chi_{\bar{A}} = \chi_{\bar{C}}|_{\xi_1=\xi_2=0}. \tag{97}$$

b)

(98)

$$\chi_{\overline{A\cup B}}|_{\xi_2=0} = \chi_{\bar{A}}\chi_{\bar{D}}|_{\xi_1=\xi_2=0} \implies \chi_{\bar{A}}\chi_{\bar{B}}|_{\xi_2=0} = \chi_{\bar{A}}\chi_{\bar{D}}|_{\xi_1=\xi_2=0} \implies \chi_{\bar{B}}|_{\xi_2=0} = \chi_{\bar{D}}|_{\xi_1=\xi_2=0}$$

In case 2) we arrive at the same conclusions. By symmetry we need to consider only case (97). By lemma 7.1 we get get $|A| = |C| = 1, A = \{0\}, C = \{c\}, c = \tau_{n_1e_1+n_2e_2}(0)$. By inspection of Figure (9) $n_1, n_2 \in \{\pm 1\}$.

$$\eta(c) \in \{0, -2\} \implies c = \pm(e_1 - e_2), -e_1 - e_2 \tag{99}$$

i. e. there exists an edge marked $(1, 2)$ that connects 0 and c. Moreover, all indices, different from 1, 2 must appear an even number of times in every path from 0 to c. Consider the index k_1.

i) If $k_1 \neq 3$, then k_1 must appear once more in the block B like:

$$0 \overset{2,k_1}{---} B_1 \overset{k_1,s}{---} B_2 \overset{1,3}{---} c \overset{2,k_2}{---} D$$

Now we can apply 7.3 to the pair $(1, k_1)$ and get the irreducibility of χ_G.

ii) If $k_1 = 3$, consider the index k_2.

A) If $k_2 \neq 3$, then either k_2 appears in the block D as in figure (10), or it appears in the block B as in figure (11).

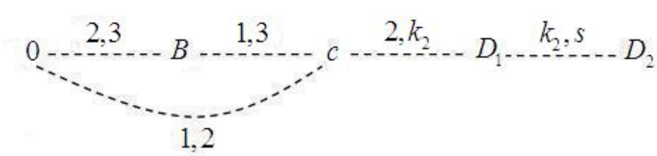


FIGURE 10

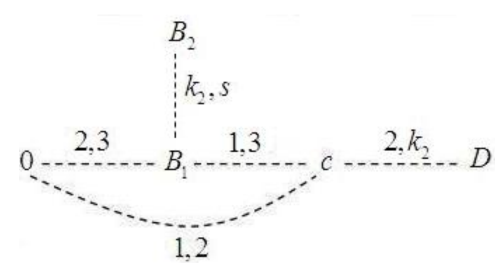


FIGURE 11

In the case of figure (10), by lemma 7.3 for the pair $(1, k_2)$, χ_G is irreducible.
Now consider the case of figure (11).

$$\chi_G|_{\xi_1=0} = \chi_{\overline{0\cup B_1\cup B_2}}\chi_{\overline{c\cup D}}|_{\xi_1=0}, \tag{100}$$

$$\chi_G|_{\xi_{k_2}=0} = \chi_{\overline{0\cup B_1\cup C}}\chi_{\bar{B_2}}|_{\xi_{k_2}=0}\chi_{\bar{D}}|_{\xi_{k_2}=0} \tag{101}$$

We have assumed that $\chi_G = UV$ with U, V irreducible and $U|_{\xi_1=0} = \chi_{\overline{0\cup B_1\cup B_2}}$. From (101) if we have $U|_{\xi_{k_2}=0} = \chi_{\overline{0\cup B_1\cup c}} \implies \chi_{\overline{0\cup B_1\cup B_2}}|_{\xi_{k_2}=0} = \chi_{\overline{0\cup B_1\cup c}}|_{\xi_1=0} \implies \chi_{\bar{B_2}}|_{\xi_{k_2}=0} = \chi_c|_{\xi_1=0}$. Then by lemma 7.1 we have $B_2 = \{b_2\}, c = \tau_{\pm e_1 \pm e_{k_2}}(\pm b_2)$. We have in the case $\sigma_{b_2} = \sigma_c \implies c = b_2 \pm (e_1 - e_{k_2})$, i. e. there exists a black edge with the marking $(1, k_2)$ that connects c and b_2; and in the case $\sigma_{b_2} = -\sigma_c \implies \eta(b_2 + c) = -2 \implies c = -b_2 - e_1 - e_{k_2}$, i. e. there exists a red edge with the marking $(1, k_2)$ that connects c and b_2.

+) If $s = 3$ and $B_1 = \{b_1\}$, then, by lemma 5.2 for the vertex b_1 and the index 3, χ_G is irreducible.

+) If $s = 3$ and $|B_1| > 1$, let i be an index that appears in the block B_1. If i appears twice in the block B_1 then by Lemma 7.3 for the pair $(1, k_2)$, χ_G is irreducible. Hence, since i appears only twice, we need to consider the case, when i appears once in the block B_1 and once in the block D as in figure (12).

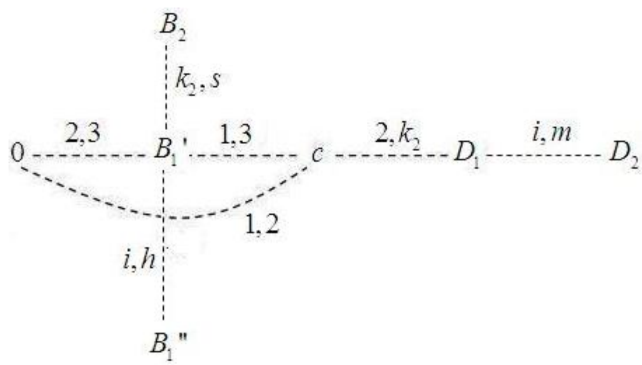


Figure 12

Compare the factorizations of $\chi_G|_{\xi_1=0}$ and

$$\chi_G|_{\xi_i=0} = \chi_{\overline{0\cup b_2\cup c\cup B_1'\cup D_1}}\chi_{\bar{B_1''}}|_{\xi_i=0}\chi_{\bar{D_2}}|_{\xi_i=0}.$$

We have that $U_{\xi_1=\xi_i=0} = \chi_{\overline{0\cup b_2\cup B_1'}}\chi_{\overline{B_1''}}$. If $U_{\xi_i=0} = \chi_{\overline{0\cup b_2\cup c\cup B_1'\cup D_1}}$ we get $\chi_{\overline{c\cup D_1}}|_{\xi_1=0} = \chi_{\bar{B_1''}}|_{\xi_i=0}$ (by Lemma 7.1 this implies $|c \cup D_1| = 1$, which is impossible). The other cases can also be similarly excluded, for instance $\chi_{\bar{D_2}}|_{\xi_1=\xi_i=0} = \chi_{\overline{0\cup b_2\cup B_1'}}$ (by Lemma 7.1 this implies $|0 \cup b_2 \cup B_1'| = 1$, which is impossible).

B) If $k_2 = 3$ and $|B| > 1$. Let i be an index that appears in B. If i appears twice in B, then, by lemma 7.3 we get the irreducibility of χ_G. Otherwise, i appears in this form:

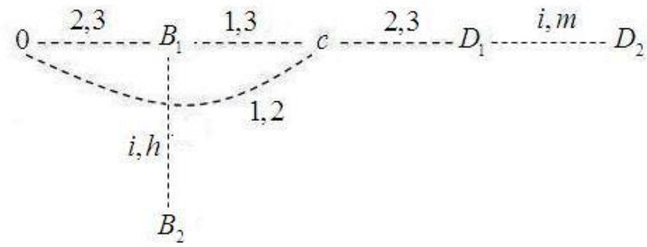


FIGURE 13

Consider the factorizations of $\chi_G|_{\xi_1=0}$ and $\chi_G|_{\xi_i=0}$ we get easily either $\chi_{\overline{c\cup D_1}}|_{\xi_1=0} = \chi_{\bar{B_2}}|_{\xi_i=0}$ (by Lemma 7.1 this implies $|c \cup D_1| = 1$, that is impossible), or $\chi_{\bar{D_2}}|_{\xi_1=\xi_i=0} = \chi_{\overline{0\cup B_1}}$ (by Lemma 7.1 this implies $|0 \cup B_1| = 0$, that is impossible). The situation when $|D| > 1$ is treated similarly. So now we have to consider only the case, when $|B| = |D| = 1$.

C) $k_2 = 3, |B| = |D| = 1$. Up to symmetry, we have 4 subcases, displayed in figures (14)-(17).

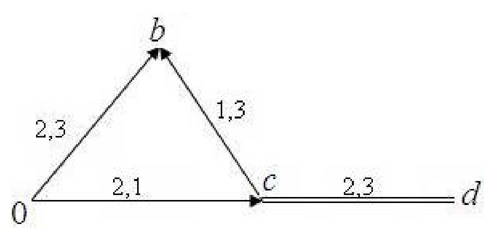


FIGURE 14

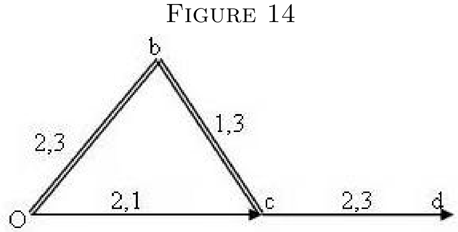


FIGURE 15

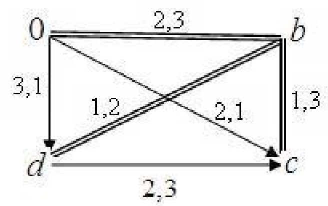


FIGURE 16

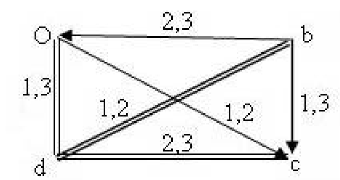


FIGURE 17

By using the program Mathematica we have verified that the characteristic polynomials of these graphs are irreducible.

7.2.2. *When 1, 3 do not appear together in any edge:* We have three possible cases (figures (18), (19), (24)).

1) When T up to symmetry has the form as in figure (18):

$$A \overset{1,2}{---} B$$

FIGURE 18

where 3 appears only in the block B then, by lemma 7.3, for the pair $(1,3)$, χ_G is irreducible.

2) When T up to symmetry has the form as in the figure (19):

$$A \overset{3,k_1}{---} B \overset{1,2}{---} C \overset{3,k_2}{---} D \overset{3,k_3}{---} E$$

FIGURE 19

We have

$$\chi_G|_{\xi_1=0} = \chi_{\overline{A\cup B}}\chi_{\overline{C\cup D\cup E}}|_{\xi_1=0} \tag{102}$$

$$\chi_G|_{\xi_3=0} = \begin{cases} \chi_{\bar{A}}\chi_{\overline{B\cup C}}|_{\xi_3=0}\chi_{\bar{D}}|_{\xi_3=0}\chi_{\bar{E}}|_{\xi_3=0} \\ \chi_{\overline{A\cup D}}\chi_{\overline{B\cup C}}|_{\xi_3=0}\chi_{\bar{E}}|_{\xi_3=0} \\ \chi_{\overline{A\cup D}}\chi_{\overline{B\cup C\cup E}}|_{\xi_3=0} \\ \chi_{\overline{A}}\chi_{\overline{D}}\chi_{\overline{B\cup C\cup E}}|_{\xi_3=0} \end{cases} \tag{103}$$

Arguing as in previous cases, if χ_G factors then we can factor it as UV with $U_{\xi_1=0} = \chi_{\overline{A\cup B}}$. Analyzing the possible values of $U_{\xi_3=0}$ we have, comparing (102) and (103) and setting $\xi_1 = \xi_3 = 0$, the following possibilities (shown in equations (104)-(108)):

$$U_{\xi_3=0} = \chi_{\overline{B\cup C}}|_{\xi_3=0} \implies \chi_{\bar{A}} = \chi_{\bar{C}}|_{\xi_1=\xi_3=0}, \tag{104}$$

$$U_{\xi_3=0} = \begin{cases} \chi_{\overline{A\cup D}}|_{\xi_3=0} \\ \chi_{\bar{A}}\chi_{\bar{D}}|_{\xi_3=0} \end{cases} \implies \chi_{\bar{B}}|_{\xi_3=0} = \chi_{\bar{D}}|_{\xi_1=\xi_3=0}, \tag{105}$$

$$U_{\xi_3=0} = \chi_{\bar{D}}|_{\xi_3=0}\chi_{\bar{E}}|_{\xi_3=0} \implies \text{either } \chi_{\bar{A}} = \chi_{\bar{D}}|_{\xi_1=\xi_3=0}, \chi_{\bar{B}}|_{\xi_3=0} = \chi_{\bar{E}}|_{\xi_1=\xi_3=0}, \text{or} \tag{106}$$

$$\chi_{\bar{A}} = \chi_{\bar{E}}|_{\xi_1=\xi_3=0}, \chi_{\bar{B}}|_{\xi_3=0} = \chi_{\bar{D}}|_{\xi_1=\xi_3=0}, \tag{107}$$

$$U_{\xi_3=0} = \chi_{\bar{A}}\chi_{\bar{E}}|_{\xi_3=0} \implies \chi_{\bar{B}}|_{\xi_3=0} = \chi_{\bar{E}}|_{\xi_1=\xi_3=0}.. \tag{108}$$

We see that (106) implies (108), (107) implies (105). So we need to show that (104),(105) and (108) cannot hold.

-If (104) holds, by lemma 7.1 and by inspection we deduce that $A = \{0\}, C = \{c\}$ and $c = \pm(e_1 - e_3), -e_1 - e_3$. Hence there is an edge that connects 0 and c and all indices, different from $1, 3$, must appear an even number of times in any path that connects 0 and c. In particular, k_1 must appear in B or $k_1 = 2$.

a) If $k_1 \in B$ we can apply lemma 7.3 replacing i by k_1.

b) If $k_1 = 2$, consider the positions of the index k_2.

i) If $k_2 \in D \cup E$ or $k_2 = k_3$, then, by lemma 7.3 for the pair $(1, k_2)$, χ_G is irreducible.

ii) If $k_2 \in B$ then it must appear in the form:

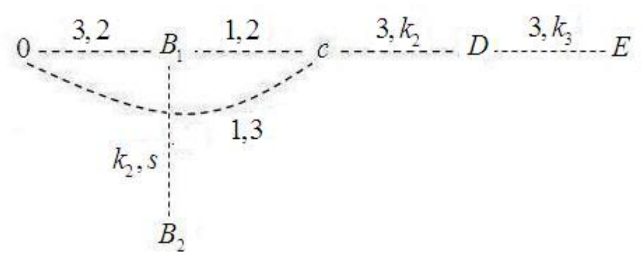


Figure 20

Then:

$$\chi_G|_{\xi_1=0} = \chi_{\overline{0\cup B_1\cup B_2}}\chi_{\overline{c\cup D\cup E}}|_{\xi_1=0}, \tag{109}$$

$$\chi_G|_{\xi_{k_2}=0} = \chi_{\overline{0\cup B_1\cup c}}\chi_{\bar{B}_2}|_{\xi_{k_2}=0}\chi_{\overline{D\cup E}}|_{\xi_{k_2}=0}. \tag{110}$$

Comparing (109) and (110) and setting $\xi_1 = \xi_{k_2} = 0$ we have $\chi_{\bar{B}_2}|_{\xi_{k_2}=0} = \chi_c|_{\xi_1=0}$. By lemma 7.1 we have $B_2 = \{b_2\}, b_2 \pm c = n_1 e_1 + n_{k_2} e_{k_2}$, but this is not possible, since by the inspection of figure (20), $b_2 \pm c = \pm e_2 + \sum_{m\neq 2} n_m e_m$.

- If (105) holds, then, by lemma 7.1 $B = \{b\}, D = \{d\}$ and $d \pm b = n_1 e_1 + n_3 e_3$. This case is treated similarly as the case of (104).

- If (108) holds, then, by lemma 7.1 $B = \{0\}, E = \{d\}$ and $e \pm b = n_1 e_1 + n_3 e_3$. By the form of T in the figure (19) we have $n_1 = \pm 1, n_3 \in \{0, \pm 2\}$. It is easy to check that there does not exist a such pair (n_1, n_3) in order to get $\eta(e\pm) \in \{0, \pm 2\}$, a contradiction.

iii) When T has the form:

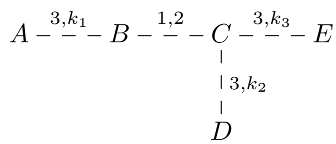


FIGURE 21

$$\chi_G|_{\xi_1=0} = \chi_{\overline{A\cup B}}\chi_{\overline{C\cup D\cup E}}|_{\xi_1=0} \tag{111}$$

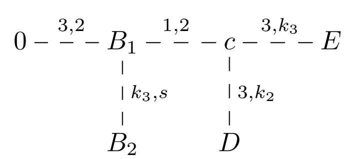
$$\chi_G|_{\xi_3=0} = \begin{cases} \chi_{\bar{A}}\chi_{\overline{B\cup C}}|_{\xi_3=0}\chi_{\bar{D}}|_{\xi_3=0}\chi_{\bar{E}}|_{\xi_3=0} \\ \chi_{\overline{A\cup D}}\chi_{\overline{B\cup C}}|_{\xi_3=0}\chi_{\bar{E}}|_{\xi_3=0} \\ \chi_{\overline{A\cup E}}\chi_{\overline{B\cup C}}|_{\xi_3=0}\chi_{\bar{D}}|_{\xi_3=0} \\ \chi_{\bar{A}}\chi_{\overline{B\cup C}}|_{\xi_3=0}\chi_{\overline{D\cup E}}|_{\xi_3=0} \\ \chi_{\overline{A\cup D\cup E}}\chi_{\overline{B\cup C}}|_{\xi_3=0} \end{cases} \tag{112}$$

From (111) we see that if χ_G is not irreducible, then $\chi_G = UV$, where U, V are irreducible, $U|_{\xi_1=0} = \chi_{A\bar{\cup}B}$. See (112), there are the three following subcases:

1)$\chi_{\bar{C}}|_{\xi_1=\xi_3=0} = \chi_{\bar{A}}$, by Lemma 7.1, $A = \{0\}, C = \{c\}, c = n_1e_1 + n_3e_3$. Hence, all indices, different from $(1,3)$ must appear an even number of times in any path from 0 to c.

-If $k_1 \neq 2$, then k_1 must appear in B, then by lemma 7.3 for the pair $(1, k_1)$, χ_G is irreducible.

-If $k_1 = 2$, consider k_3. If $k_3 \in D \cup E$ or $k_2 = k_3$, then we use Lemma 7.3 for the pair $(1, k_3)$ to get the irreducibility. Otherwise k_3 appears as follows:



Considering specializations $\chi_G|_{\xi_1=0}$ and $\chi_G|_{\xi_{k_3}=0}$, we get easily either $|c\cup D| = |B_2| = 1$, or $|0 \cup B_1| = |E| = 1$. Both of them are not possible.

2) $\chi_{\bar{B}}|_{\xi_3=0} = \chi_{\bar{D}}|_{\xi_1=\xi_3=0}$ by lemma 7.1 $\implies |B| = |D| = 1, B = \{b\}, D = \{d\}, d \pm b = n_1e_1 + n_3e_3$. Hence, all indices, different from 1, 3, must appear an even number of times in any path from b to d.

-In particular, if $k_2 \neq 2$, k_2 must appear in the block C. Then, by lemma 7.3 for the pair $(1, k_2)$ χ_G is irreducible.

-If $k_2 = 2$, consider positions of k_3:

+) If $k_3 \in C \cup E$, then, by lemma 7.3 for the pair $(1, k_3)$, χ_G is irreducible.

+) If $k_3 \in A$:

$$\begin{array}{c} A_1 \overset{k_3,s}{- - -} A_2 \overset{3,k_1}{- - -} b \overset{1,2}{- - -} C \overset{3,k_3}{- - -} E \\ \hphantom{A_1 - - - A_2 - - - b - - - } \vdots \scriptstyle{3,2} \hphantom{- - - E} \\ \hphantom{A_1 - - - A_2 - - - b - - - } d \hphantom{- - - E} \end{array}$$

by lemma for the pair $(1, k_3)$ we get either $|A_2 \cup b| = 1$, either $|C \cup d| = 1$. Both of them are not possible.

+) If $k_3 = k_1$:

$$\begin{array}{c} A \overset{3,k_1}{- - -} b \overset{1,2}{- - -} C \overset{3,k_1}{- - -} E \\ \hphantom{A - - - b - - - } \vdots \scriptstyle{3,2} \hphantom{- - - E} \\ \hphantom{A - - - b - - - } d \hphantom{- - - E} \end{array}$$

By lemma 7.3 for the pair $(1, k_1)$ we get either $|C \cup d| = |A| = 1$ (which is not possible), or $E = \{e\}, e \pm b = \pm(e_1 - e_{k_1})$ (this is not possible since by inspection $e \pm b = \pm e_3 + \sum_{m \neq 3} n_m e_m$.)

3) $\chi_{\bar{B}}|_{\xi_3=0} = \chi_{\bar{E}}|_{\xi_1=\xi_3=0}$. By lemma 7.1 we get $|B| = |E| = 1, B = \{b\}, E = \{e\}, e \pm b = n_1 e_1 + n_3 e_3$. This case is treated by similar way as in 2), changing the role of k_2 and k_3.

7.3. **Every index appears twice in the tree.** We start with some special cases:

7.3.1. $n = 2$.

$$-e_1 - e_2 =\!= 0 \longrightarrow e_1 - e_2$$

$$C_G = \begin{pmatrix} -\xi_1 - \xi_2 & 2\sqrt{\xi_1\xi_2} & 0 \\ -2\sqrt{\xi_1\xi_2} & 0 & 2\sqrt{\xi_1\xi_2} \\ 0 & 2\sqrt{\xi_1\xi_2} & \xi_2 - \xi_1 \end{pmatrix}$$

determinant

$$(-\xi_1 - \xi_2)(-4\xi_1\xi_2) + 4\xi_1\xi_2(\xi_2 - \xi_1) = 8\xi_1\xi_2^2$$

$$\chi_G(t) = det(tI - C_G) = det \begin{pmatrix} t + \xi_1 + \xi_2 & -2\sqrt{\xi_1\xi_2} & 0 \\ 2\sqrt{\xi_1\xi_2} & t & -2\sqrt{\xi_1\xi_2} \\ 0 & -2\sqrt{\xi_1\xi_2} & t - \xi_2 + \xi_1 \end{pmatrix}$$

if it is not irreducible it is divisible by a linear form, set $\xi_1 = 0$ get $t(t + \xi_2)(t - \xi_2)$ set $\xi_2 = 0$ get $t(t + \xi_1)^2$ so the possible linear factor can be

$$t, t + \xi_1, t \pm \xi_2$$

On the other hand:

$$\chi_G(t) = t^3 + 2\xi_1 t^2 + (\xi_1^2 - \xi_2^2)t - 8\xi_1\xi_2^2. \tag{113}$$

Then we have:

$$\chi_G(0) = -8\xi_1\xi_2^2, \tag{114}$$

$$\chi_G(-\xi_1) = -\xi_1^3 + 2\xi_1^3 + (\xi_1^2 - \xi_2^2)(-\xi_1) = -7\xi_1\xi_2^2, \tag{115}$$

$$\chi_G(\xi_2) = \xi_2^3 + 2\xi^1\xi_2^2 + (\xi_1^2 - \xi_2^2)\xi_2 = \xi_1^2\xi_2 - 6\xi_1\xi_2^2 \tag{116}$$

$$\chi_G(-\xi_2) = -\xi_1^2\xi_2 - 6\xi_1\xi_2^2. \tag{117}$$

So χ_G does not have any linear factor, hence it is irreducible.

7.3.2. $n = 3$. T has the form as in figure (22) or as in figure (23):

$$0 - \overset{1,2}{-} - b - \overset{2,3}{-} - c - \overset{1,3}{-} - d$$

FIGURE 22

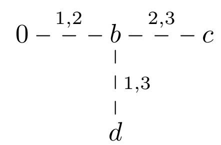


FIGURE 23

Remark 7.2. *If all edges in T are black, or there are exactly two red edges then the edges are linearly dependent.*

1) When the maximal tree T has the form as in figure (22)

Remark 7.3.
- *If in $\bar{T}$ there is an edge marked $(1,3)$ that connects 0 and c, then, by lemma 5.2 for the vertex b and the index 2, χ_G is irreducible.*
- *If in $\bar{T}$ there is an edge marked $(1,2)$ that connects b and d, then, by lemma 5.2 for the vertex c and the index 3, χ_G is irreducible.*

a) If all edges are red, then $G = \bar{T}$ has the form:

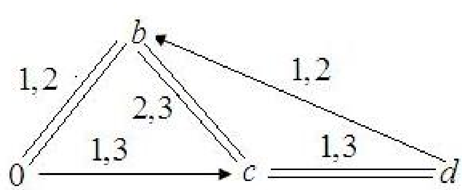


FIGURE 24

By lemma 5.2 for the vertex b and the index 2, χ_G is irreducible.
We need to consider the cases, when in T there is one red and two black edges.
b) When the red edge connects 0 and b:
b1) When T has the form:

$$0 \overset{1,2}{=\!=\!=} b \xrightarrow{2,3} c \overset{1,3}{\text{———}} d$$

We have

$$b = -e_1 - e_2, c - b = e_2 - e_3 \implies c = -e_1 - e_2.$$

Hence in $G = \bar{T}$ there is a red edge marked $(1,2)$ that connects 0 and c. Hence by remark 8.1 χ_G is irreducible.
b2) If T has the form:

$$0 \overset{1,2}{=\!=\!=} b \xleftarrow{2,3} c \xrightarrow{1,3} d$$

We have $b - c = e_1 - e_3, d - c = e_1 - e_3 \implies d - b = e_1 - e_2$, i. e. in G there is a black edge marked $(1,2)$ that connects b and d, hence by remark 8.1 χ_G is irreducible.
b3) If T has the form:

$$0 \overset{1,2}{=\!=\!=} b \xleftarrow{2,3} c \xleftarrow{1,3} d$$

$$\chi_G = det \begin{pmatrix} t & 2\sqrt{\xi_1\xi_2} & 0 & 0 \\ -2\sqrt{\xi_1\xi_2} & t+\xi_1+\xi_2 & 2\sqrt{\xi_2\xi_3} & 0 \\ 0 & 2\sqrt{\xi_2\xi_3} & t+\xi_1+2\xi_2-\xi_3 & 2\sqrt{\xi_1\xi_3} \\ 0 & 0 & 2\sqrt{\xi_1\xi_3} & t+2\xi_1+2\xi_2-2\xi_3 \end{pmatrix} \tag{118}$$

By using the program Mathematica we computed χ_G and verified that it is irreducible.
c) When the red edge connects b and c:
c1) If T has the form:

$$0 \overset{1,2}{-\!-\!-} b \overset{2,3}{=\!=\!=} c \xleftarrow{1,3} d$$

we have $b + c = -e_2 - e_3, c - d = e_1 - e_3 \implies b + d = -e_1 - e_2$, i. e. there is a red edge marked $(1,2)$ that connects b and d, hence by remark 8.1 χ_G is irreducible.
c2) If T has the form

$$0 \xrightarrow{1,2} b \overset{2,3}{=\!=\!=} c \overset{1,3}{-\!-\!-} d$$

we have $b = e_1 - e_2, b + c = -e_2 - e_3 \implies c = e_1 - e_3$, i. e. there is a black edge marked $(1,3)$ that connects 0 and c, hence by remark 8.1 χ_G is irreducible.
c3) If T has the form:

$$0 \xleftarrow{1,2} b \overset{2,3}{=\!=\!=} c \xrightarrow{1,3} d$$

we have

$$\chi_G = det \begin{pmatrix} t & -2\sqrt{\xi_1\xi_2} & 0 & 0 \\ -2\sqrt{\xi_1\xi_2} & t-\xi_1+\xi_2 & 2\sqrt{\xi_2\xi_3} & 0 \\ 0 & -2\sqrt{\xi_2\xi_3} & t-\xi_1+2\xi_2+\xi_3 & 2\sqrt{\xi_1\xi_3} \\ 0 & 0 & 2\sqrt{\xi_1\xi_3} & t-2\xi_1+2\xi_2+2\xi_3 \end{pmatrix} \tag{119}$$

We use the program Mathematica to compute χ_G and to verify that it is irreducible.
2) When T has the form as in figure (23):
a) When in T there are 3 red edges, then $G = \bar{T}$ has the form:

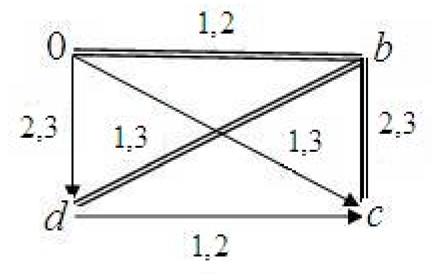


FIGURE 25

This figure can be obtained from figure (16) by exchanging the role of indices (i. e. the role of variables ξ_1, ξ_2, ξ_3). Hence χ_G is irreducible.
b) When in T there is only one red edge, by the symmetry property of T we may suppose that this red edge connects 0 and b.
b1) If T has the form:

$$\begin{array}{ccc} 0 \overset{1,2}{=\!=\!=} & b & \xrightarrow{2,3} c \\ & \Big| {\scriptstyle 1,3} & \\ & d & \end{array}$$

we have $b = -e_1 - e_2, c - b = e_2 - e_3 \implies c = e_2 - e_3 + b = -e_1 - e_3$. Hence in G there is a red edge marked $(1,3)$ that connects 0 and c. There is another maximal tree of G:

$$c \overset{1,3}{=\!=\!=} 0 \overset{1,2}{=\!=\!=} b \overset{1,3}{\text{———}} d$$

in which the index 2 appears once, the index 1 appears three times. So χ_G is irreducible by the subsection 7.2.
b2) If T has the form:

$$\begin{array}{ccc} 0 \overset{1,2}{=\!=\!=} & b & \overset{2,3}{\text{———}} c \\ & \Big\downarrow {\scriptstyle 1,3} & \\ & d & \end{array}$$

we have $b = -e_1 - e_2, d - b = e_1 - e_3 \implies d = -e_2 - e_3$, hence in G there is a red edge marked $(2,3)$ that connects 0 and d. There is another maximal tree of G:

$$d \overset{2,3}{=\!=\!=} 0 \overset{1,2}{=\!=\!=} b \overset{2,3}{\text{———}} c$$

in which 1 appears once, 2 appears three times. So χ_G is irreducible by the subsection 7.2.
b3) If T has the form:

$$\begin{array}{ccc} 0 \overset{1,2}{=\!=\!=} & b & \xleftarrow{2,3} c \\ & \Big\uparrow {\scriptstyle 1,3} & \\ & d & \end{array}$$

we have $b - c = e_2 - e_3, b - d = e_1 - e_3 \implies d - c = e_2 - e_1$, hence there is a black edge marked $(2, 1)$ that connects c and d. There is another maximal tree of G:

$$0 \xlongequal{1,2} b \xleftarrow{2,3} c \xrightarrow{2,1} d$$

in which 3 appears once, 2 appears three times. So χ_G is irreducible by the subsection 7.2.

7.4. $n \geq 4$. At this point we are assuming that we have $n \geq 4$ edges in a maximal tree T and n indices, each appearing twice. Thus given an index, say 1, it appears in two edges paired with at most two other indices, thus we can find another index, say 2 which is not in these two edges. Up to symmetry we may have six cases displayed in figures (26)-(31):

$$A \overset{1,h}{- - -} B \overset{2,k}{- - -} C \overset{2,i}{- - -} D \overset{1,j}{- - -} E$$

FIGURE 26

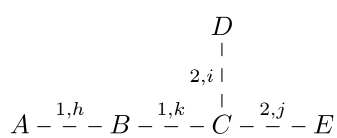


FIGURE 27

$$A \overset{1,h}{- - -} B \overset{2,k}{- - -} C \overset{1,i}{- - -} D \overset{2,j}{- - -} E$$

FIGURE 28

$$A \overset{1,h}{- - -} B \overset{1,k}{- - -} C \overset{2,i}{- - -} D \overset{2,j}{- - -} E$$

FIGURE 29

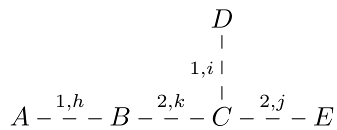


FIGURE 30

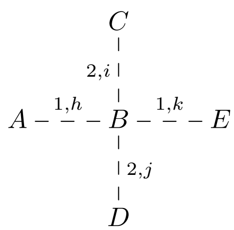


FIGURE 31

What is common to all these cases is that when we put $\xi_1 = 0$ or $\xi_2 = 0$ we may have at most 3 connected components in the graph, so by induction we deduce that, if the characteristic polynomial is not irreducible it can factor in at most 3 factors. If we have exactly 3 factors we see that in each case we have two pairs of disjoint blocks which give under specialization the same characteristic polynomials. At first we start with several lemmas which will be useful for further consideration of all figures.

Lemma 7.4. *If there exist two indices 1, 2, such that T is of the form as in figure (29), $0 \in A$ then either χ_G is irreducible, or $B = \{b\}, D = \{d\}, d \pm b = \pm e_1 \pm e_2$ or $A = \{0\}, E = \{e\}, e = \pm 2(e_1 - e_2)$.*

Proof.
(120)

$$\chi_G|_{\xi_1=0} = \begin{cases} \chi_{\bar{A}}\chi_{\bar{B}}|_{\xi_1=0}\chi_{\overline{C\cup D\cup E}}|_{\xi_1=0}, \text{if in } \bar{T} \text{ there is no edge that connects } A \text{ with } C\cup D\cup E \\ \chi_{\overline{A\cup C\cup D\cup E}}\chi_{\bar{B}}|_{\xi_1=0}, \text{if in } \bar{T} \text{ there is an edge that connects } A \text{ with } C\cup D\cup E \end{cases}$$

(121)

$$\chi_G|_{\xi_2=0} = \begin{cases} \chi_{\overline{A\cup B\cup C}}\chi_{\bar{D}}|_{\xi_2=0}\chi_{\bar{E}}|_{\xi_2=0}, \text{if in } \bar{T} \text{ there is no edge that connects } A\cup B\cup C \text{ with } E \\ \chi_{\overline{A\cup B\cup C\cup E}}\chi_{\bar{D}}|_{\xi_2=0}, \text{if in } \bar{T} \text{ there is an edge that connects } A\cup B\cup C \text{ with } E \end{cases}$$

Suppose that χ_G is not irreducible, then its factors under the specializations $\xi_1 = 0$ and $\xi_2 = 0$ give (120) and (121).
1) If there is a factor U which under $\xi_1 = 0$ gives $\chi_{\bar{A}}$ or $\chi_{\bar{B}}|_{\xi_1=0}$, then U under $\xi_2 = 0$ gives either $\chi_{\bar{D}}|_{\xi_2=0}$ or $\chi_{\bar{E}}|_{\xi_2=0}$. We get the following sub-cases:

(122)
either $\chi_{\bar{B}}|_{\xi_1=0} = \chi_{\bar{D}}|_{\xi_1=\xi_2=0}$, by lemma 7.1, $|B| = |D| = 1, B = \{b\}, D = \{d\}, d = \tau_{n_1e_1+n_2e_2}(\pm b)$

By inspection $n_1 \in \{\pm 1\}, n_2 \in \{\pm 2\} \implies d \pm b = \pm e_1 \pm e_2$.

$$\text{or } \chi_{\bar{B}}|_{\xi_1=0} = \chi_{\bar{E}}|_{\xi_1=\xi_2=0}. \tag{123}$$

We get $B = \{b\}, E = \{e\}, e \pm b = n_1e_1 + n_2e_2$, where $n_1 \in \{\pm 1\}, n_2 \in \{0, \pm 2\} \implies \eta(e \pm b) \in \{\pm 1, \pm 3\}$, a contradiction.

$$\text{or } \chi_{\bar{A}} = \chi_{\bar{D}}|_{\xi_1=\xi_2=0} \implies D = \tau_{n_1e_1+n_2e_2}(A) \tag{124}$$

$\implies |A| = |D| = 1, A = \{0\}, D = \{d\}, d = \tau_{n_1e_1+n_2e_2}(0) = n_1e_1+n_2e_2, \sigma_d = \sigma_0 = 1 \implies \eta(d) = \eta(0) = 0.$

But in fact by inspection $n_1 \in \{0, \pm 2\}, n_2 \in \{\pm 1\} \implies \eta(d) \in \{\pm 1, \pm 3\} \implies \eta(d) \neq 0$, a contradiction.

$$\text{or } \chi_{\bar{A}} = \chi_{\bar{E}}|_{\xi_1=\xi_2=0} \implies E = \tau_{n_1e_1+n_2e_2}(A) \tag{125}$$

$\implies |A| = |E| = 1, A = \{0\}, E = \{e\}, e = \tau_{n_1e_1+n_2e_2}(0) = n_1e_1+n_2e_2, \sigma_e = \sigma_0 = 1 \implies \eta(e) = \eta(0) = 0.$

By inspection $n_1 \in \{0, \pm 2\}, n_2 \in \{0, \pm 2\}$. Then in order to have $\eta(e) \in \{0, -2\}, e \neq 0$ we must have $e = \pm(2e_1 - 2e_2)$.

2) If we have $\chi_G = UV$, $U_{\xi_1=0} = \chi_{\overline{C \cup D \cup E}}|_{\xi_1=0}$, $V_{\xi_1=0} = \chi_{\bar{A}}\chi_{\bar{B}}|_{\xi_1=0}$. We must then have that $V_{\xi_2=0} = \chi_{\bar{D}}|_{\xi_2=0}\chi_{\bar{E}}|_{\xi_2=0}$ and we are back in one of the previous cases. ☐

Lemma 7.5. *If there is a pair of indices, say $(1, 2)$, such that T has the form as in figure (30), $0 \in A$, then at least one of the following statements is true:*

- *χ_G is irreducible*
- *$A = \{0\}, C = \{c\}, c = \pm(e_1 - e_2)$*
- *$B = \{b\}, D = \{d\}, d \pm b = \pm e_1 \pm e_2$*
- *$D = \{d\}, E = \{e\}, e \pm d = \pm e_1 \pm e_2$.*

Proof. We have

$$\chi_G|_{\xi_1=0} = \begin{cases} \chi_{\bar{A}}\chi_{\overline{B \cup C \cup E}}|_{\xi_1=0}\chi_{\bar{D}}|_{\xi_1=0}, \text{if in } \bar{T} \text{ there is no edge that connects } A, D \\ \chi_{\overline{A \cup D}}\chi_{\overline{B \cup C \cup E}}|_{\xi_1=0}, \text{if in } \bar{T} \text{ there is an edge that connects } A, D \end{cases} \tag{126}$$

$$\chi_G|_{\xi_2=0} = \begin{cases} \chi_{\overline{A \cup B}}\chi_{\overline{C \cup D}}|_{\xi_2=0}\chi_{\bar{E}}|_{\xi_2=0}, \text{if in } \bar{T} \text{ there is no edge that connects } B, E \\ \chi_{\overline{A \cup B \cup E}}\chi_{\overline{C \cup D}}|_{\xi_2=0}, \text{if in } \bar{T} \text{ there is an edge that connects } B, E. \end{cases} \tag{127}$$

Suppose that χ_G is not irreducible, then $\chi_G = UV$ (U, V may be irreducible or not). According to (126) and (127), since the roles of U, V are the same, then there are 2 following possibilities:

1)$U|_{\xi_1=0} = \chi_{\bar{A}} \implies U|_{\xi_1=\xi_2=0} = \chi_{\bar{A}}$ is irreducible $\implies U|_{\xi_2=0} = \chi_{\bar{E}}|_{\xi_2=0} \implies \chi_{\bar{A}} = U|_{\xi_1=\xi_2=0} = \chi_{\bar{E}}|_{\xi_1=\xi_2=0}$. Hence by lemma 7.1 we have $A = 0, E = \{e\}, e = n_1e_1 + n_2e_2$. According to figure (30) $n_1 \in \{\pm 1\}, n_2 \in \{0, \pm 2\}$. So $\eta(e) = \pm 1, \pm 3 \notin \{0, -2\}$, a contradiction.

$U|_{\xi_1=0} = \chi_{\bar{D}}|_{\xi_1=0} \implies U|_{\xi_1=\xi_2=0} = \chi_{\bar{D}}|_{\xi_1=\xi_2=0}$ is irreducible, so $U|_{\xi_2=0} = \chi_{\bar{E}}|_{\xi_2=0} \implies \chi_{\bar{D}}|_{\xi_1=\xi_2=0} = U|_{\xi_1=\xi_2=0} = \chi_{\bar{D}}|_{\xi_1=\xi_2=0} = \chi_{\bar{E}}|_{\xi_1=\xi_2=0}$. Hence by lemma 7.1 $D = \{d\}, E = \{e\}, e = \tau_{n_1e_1+n_2e_2}(\pm d)$. Moreover, according to figure (30) $n_1 = \pm 1, n_2 = \pm 1 \implies e \pm d = \pm e_1 \pm e_2$.

2) $U|_{\xi_1=0} = \chi_{\overline{B \cup C \cup E}}|_{\xi_1=0}$. There are 2 subcases:

a) $U|_{\xi_2=0} = \chi_{\overline{A \cup B}}\chi_{\bar{E}}|_{\xi_2=0}$or $\chi_{\overline{A \cup B \cup E}} \implies \chi_{\bar{A}} = \chi_{\bar{C}}|_{\xi_1=\xi_2=0}$, by lemma 7.1 we get $|A| = |C| = 1, A = \{0\}, C = \{c\}, c = \tau_{n_1e_1+n_2e_2}(0)$. According to the figure (30) $n_1 = \pm 1, n_2 = \pm 1$, in order to get $\eta(c) \in \{0, -2\}$ we must have $c = \pm(e_1 - e_2), -e_1 - e_2$.

b) $U|_{\xi_2=0} = \chi_{\overline{C\cup D}}|_{\xi_2=0}\chi_{\bar{E}}|_{\xi_2=0} \implies \chi_{\overline{C\cup D}}|_{\xi_1=\xi_2=0}\chi_{\bar{E}}|_{\xi_1=\xi_2=0} = U|_{x_1=\xi_2=0} = \chi_{\overline{B\cup C\cup E}}|_{\xi_1=\xi_2=0} \implies \chi_{\bar{C}}|_{\xi_1=\xi_2=0}\chi_{\bar{D}}|_{\xi_1=\xi_2=0}\chi_{\bar{E}}|_{\xi_1=\xi_2=0} = \chi_{\bar{B}}|_{\xi_1=0}\chi_{\bar{C}}|_{\xi_1=\xi_2=0}\chi_{\bar{E}}|_{\xi_1=\xi_2=0} \implies \chi_{\bar{D}}|_{\xi_1=\xi_2=0} = \chi_{\bar{B}}|_{\xi_1=0}$, hence by lemma 7.1 $D = \{d\}, B = \{b\}, d = \tau_{n_1e_1+n_2e_2}(\pm b), \sigma(d) = \sigma_b, \eta(d) = \eta(b)$, according to figure (30) $n_1 = \pm 1, n_2 = \pm 1 \implies d \pm b = \pm e_1 \pm e_2$.

□

Now we will prove the irreducibility of χ_G in each case, displayed in figures (26)-(31).

7.4.1. *Figure* (26).

Lemma 7.6. *If there are two indices, say 1, 2, such that T is of the form as in figure (26), then χ_G is irreducible.*

Proof.
(128)

$$\chi_G|_{\xi_1=0} = \begin{cases} \chi_{\bar{A}}\chi_{\overline{B\cup C\cup D}}|_{\xi_1=0}\chi_{\bar{E}}|_{\xi_1=0}, \text{if in } \bar{T} \text{ there is no edge that connects } A, E \\ \chi_{\overline{A\cup E}}\chi_{\overline{B\cup C\cup D}}|_{\xi_1=0}, \text{if in } \bar{T} \text{ there is an edge that connects } A, E \end{cases}$$

We have the following two cases:
1) In $\bar{T}$ there is no edge that connects $A\cup B$ with $D\cup E$.
We have:

$$\chi_G|_{\xi_2=0} = \chi_{\overline{A\cup B}}\chi_{\bar{C}}|_{\xi_2=0}\chi_{\overline{D\cup E}}|_{\xi_2=0}. \tag{129}$$

Suppose that χ_G is not irreducible, $\chi_G = UV$ (U, V may be irreducible or not). Comparing (128) and (129), since the roles of U, V are the same, we get the following possibilities ((130)-(133))

$$U|_{\xi_1=0} = \chi_{\bar{A}}, U|_{\xi_2=0} = \chi_{\bar{C}}|_{\xi_2=0} \implies \chi_{\bar{A}} = \chi_{\bar{C}}|_{\xi_1=\xi_2=0} \tag{130}$$

$$U|_{\xi_1=0} = \chi_{\bar{E}}|_{\xi_1=0}, U|_{\xi_2=0} = \chi_{\bar{C}}|_{\xi_2=0} \implies \chi_{\bar{C}}|_{\xi_1=\xi_2=0} = \chi_{\bar{E}}|_{\xi_1=\xi_2=0} \tag{131}$$

$$\begin{aligned} U|_{\xi_1=0} = \chi_{\overline{B\cup C\cup D}}|_{\xi_1=0}, U|_{\xi_2=0} = \chi_{\overline{A\cup B}}|_{\xi_2=0}\chi_{\bar{C}}|_{\xi_2=0} \\ \implies U|_{\xi_1=\xi_2=0} = \chi_{\overline{B\cup C\cup D}}|_{\xi_1=\xi_2=0} = \chi_{\overline{A\cup B}}|_{\xi_1=0}\chi_{\bar{C}}|_{\xi_1=\xi_2=0} \\ \implies \chi_{\bar{B}}|_{\xi_1=0}\chi_{\bar{C}}|_{\xi_1=\xi_2=0}\chi_{\bar{D}}|_{\xi_1=\xi_2=0} = \chi_{\bar{A}}\chi_{\bar{B}}|_{\xi_1=0}\chi_{\bar{C}}|_{\xi_1=\xi_2=0} \\ \implies \chi_{\bar{D}}|_{\xi_1=\xi_2=0} = \chi_{\bar{A}} \end{aligned} \tag{132}$$

$$\begin{aligned} U|_{\xi_1=0} = \chi_{\overline{B\cup C\cup D}}|_{\xi_1=0}, U|_{\xi_2=0} = \chi_{\overline{D\cup E}}|_{\xi_2=0}\chi_{\bar{C}}|_{\xi_2=0} \\ \implies U|_{\xi_1=\xi_2=0} = \chi_{\overline{B\cup C\cup D}}|_{\xi_1=\xi_2=0} = \chi_{\overline{D\cup E}}|_{\xi_1=0}\chi_{\bar{C}}|_{\xi_1=\xi_2=0} \\ \implies \chi_{\bar{B}}|_{\xi_1=0}\chi_{\bar{C}}|_{\xi_1=\xi_2=0}\chi_{\bar{D}}|_{\xi_1=\xi_2=0} = \chi_{\bar{D}}|_{\xi_1=\xi_2=0}\chi_{\bar{E}}|_{\xi_1=0}\chi_{\bar{C}}|_{\xi_1=\xi_2=0} \\ \implies \chi_{\bar{B}}|_{\xi_1=0} = \chi_{\bar{E}}|_{\xi_1=\xi_2=0} \end{aligned} \tag{133}$$

By symmetry we need to consider only the cases (130) and (132).
a) Consider case (130), by lemma 7.1 we have $|C| = |A| = 1, C = \{c\}, A = \{0\}, c = \tau_{\pm e_1 \pm e_2}(0)$, since $\eta(c) =\in \{0, -2\} \implies c = \pm(e_1 - e_2), -e_1 - e_2$. Hence there is an edge marked $(1, 2)$ connecting 0 and c. All indices, different from 1, 2 must appear an even number of times in the path connecting 0 and c. In particular, k appears in this path in B or $k = h$.
i) If k appears in B:

$$0 \overset{1,h}{-\,-} B_1 \overset{s,k}{-\,-} B_2 \overset{2,k}{-\,-} c \overset{2,i}{-\,-} D \overset{1,j}{-\,-} E$$

Since U is a linear polynomial looking at $\chi_G|_{\xi_k=0}$, we see that we can only have $U|_{\xi_k=0} = \chi_{B_2}|_{\xi_k=0}$ hence $|B_2| = 1, B_2 = \{b_2\}$. Then the vertex b_2 and the index k satisfy the conditions of lemma 5.2, hence χ_G is irreducible.

ii) If $k = h$, consider the index i, i may appear as:

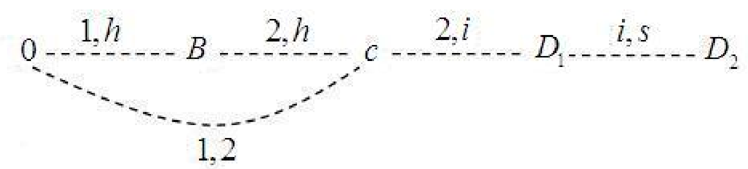


Figure 32

or as:

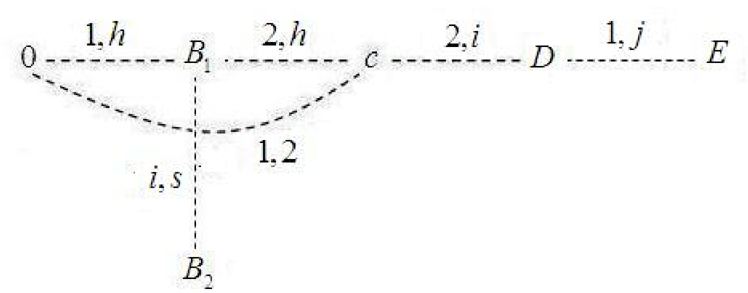


Figure 33

A) Consider figure 32. By lemma 7.4 for the pair of indices (h, i) we get the following two possibilities:
+) $B = \{b\}, D_1 = \{d_1\}, d_1 \pm b = \pm e_h \pm e_i$. By lemma 5.2, for the vertex b and the index h, χ_G is irreducible.
+) $D_2 = \{d_2\}, d_2 = \pm 2(e_h - e_i)$. But in fact, if we look at the path $(0, c, D_1, d_2)$ we see that the h-th coordinate of d_2 is zero.
B) Consider figure (33). We have

$$\chi_G|_{\xi_h=0} = \chi_{\overline{0 \cup c \cup D \cup E}} \chi_{\overline{B_1 \cup B_2}}|_{\xi_h=0}, \tag{134}$$

$$\chi_G|_{\xi_i=0} = \chi_{\overline{0 \cup B_1 \cup c}} \chi_{\bar{B}_2}|_{\xi_i=0} \chi_{\overline{D \cup E}}|_{\xi_i=0}. \tag{135}$$

From (134) we see that if χ_G is not irreducible, then it must factor into 2 irreducible polynomials: $\chi_G = UV$, $U|_{\xi_h=0} = \chi_{\overline{B_1 \cup B_2}}|_{\xi_h=0}$ hence $U|_{\xi_i=\xi_h=0} = \chi_{\overline{B_1}}|_{\xi_i=\xi_h=0} \chi_{\overline{B_2}}|_{\xi_i=\xi_h=0}$. Then for (135) we have the only possibility that $U|_{\xi_i=0} = \chi_{\overline{D \cup E}}|_{\xi_i=0} s$. This implies $\chi_{\overline{B_1 \cup B_2}}|_{\xi_h=\xi_i=0} = \chi_{\overline{D \cup E}}|_{\xi_h=\xi_i=0} \implies \chi_{\bar{B}_1}|_{\xi_i=\xi_h=0} \chi_{\bar{B}_2}|_{\xi_i=\xi_h=0} = \chi_{\overline{D \cup E}}|_{\xi_i=\xi_h=0}$. But this is not possible, since i does not appear in $D \cup E$, $\chi_{\overline{D \cup E}}$ remains irreducible by induction assumption.
b) Consider the case (132). By lemma 7.1 we get $|D| = |A| = 1, A = \{0\}, D = \{d\}$

and $d = n_1e_1 + n_2e_2$, where by figure (26) $n_1 \in \{\pm 1\}, n_2 \in \{0, \pm 2\}$. So we have $\eta(d) \in \{\pm 1, \pm 3\}$, a contradiction.
2) In $\bar{T}$ there is an edge that connects $A \cup B$ with $D \cup E$.
We have:

$$\chi_G|_{\xi_2=0} = \chi_{\overline{A\cup B\cup D\cup E}}\chi_{\bar{C}}|_{\xi_2=0} \tag{136}$$

From (136) we deduce that if χ_G is not irreducible, it will factor in exactly 2 irreducible factors: $\chi_G = UV$, one of them, say U, under the specialization $\xi_2 = 0$ gives $\chi_{\bar{C}}|_{\xi_2=0}$. Then $deg(U) = |C| < |B| + |C| + |D| = deg(\chi_{\overline{B\cup C\cup D}})$, so according to (128) $U|_{\xi_1=0}$ must be equal to $\chi_{\bar{A}}$ or $\chi_{\bar{E}}|_{\xi_1=0}$. Then we have following cases:

$$\chi_{\bar{C}}|_{\xi_1=\xi_2=0} = \chi_{\bar{A}}, \tag{137}$$

$$\text{or } \chi_{\bar{C}}|_{\xi_1=\xi_2=0} = \chi_{\bar{E}}|_{\xi_1=0} \tag{138}$$

In any case by lemma 7.1 we get $|C| = 1, C = \{c\}$. Then we can apply lemma 5.2 for the vertex c and the index 2 and get the result.

□

7.4.2. *Figure* (27).

Lemma 7.7. *If there are two indices, say* $1, 2$ *such that* T *is of the form as in figure* (27)*, then* χ_G *is irreducible.*

Proof. We have:

(139)
$$\chi_G|_{\xi_1=0} = \begin{cases} \chi_{\bar{A}}\chi_{\bar{B}}|_{\xi_1=0}\chi_{\overline{C\cup D\cup E}}|_{\xi_1=0}, \text{if in } \bar{T} \text{ there is no edge that connects } A \text{ with } C\cup D\cup E \\ \chi_{\overline{A\cup C\cup D\cup E}}\chi_{\bar{B}}|_{\xi_1=0}, \text{if in } \bar{T} \text{ there is an edge that connects } A \text{ with } C\cup D\cup E \end{cases}$$

(140)
$$\chi_G|_{\xi_2=0} = \begin{cases} \chi_{\overline{A\cup B\cup C}}\chi_{\bar{D}}|_{\xi_2=0}\chi_{\bar{E}}|_{\xi_2=0}, \text{if in } \bar{T} \text{ there is no edge that connects } D, E \\ \chi_{\overline{A\cup B\cup C}}\chi_{\overline{D\cup E}}|_{\xi_2=0}, \text{if in } \bar{T} \text{ there is an edge that connects } D, E \end{cases}$$

Suppose that χ_G is not irreducible. Comparing (139) and (140) and by a simple analysis we have only the following possibilities:

$$\chi_{\bar{A}} = \chi_{\bar{D}}|_{\xi_1=\xi_2=0}, \tag{141}$$

$$\text{or } \chi_{\bar{A}} = \chi_{\bar{E}}|_{\xi_1=\xi_2=0}, \tag{142}$$

$$\text{or } \chi_{\bar{B}}|_{\xi_1=0} = \chi_{\bar{D}}|_{\xi_1=\xi_2=0}, \tag{143}$$

$$\text{or } \chi_{\bar{B}}|_{\xi_1=0} = \chi_{\bar{E}}|_{\xi_1=\xi_2=0}. \tag{144}$$

By the symmetry property we need to consider only (141) and (143).
1) Consider (141). We get by lemma 7.1 $|A| = |D| = 1, A = \{0\}, D = \{d\}, d = n_1e_1+n_2e_2$, where $n_1 \in \{0, \pm 2\}, n_2 = \{\pm 1\}$. Then $\eta(d) \notin \{0, -2\}$, a contradiction.
2) Consider (143), we get by lemma 7.1 $B = \{b\}, D = \{d\}, d \pm b = \pm e_1 \pm e_2$. From this, we have in the case $\sigma_b = \sigma_d \implies \eta(b) = \eta(d) \implies d - b = \pm(e_1 - e_2)$, i. e. there is a black edge marked $(1, 2)$ that connects b and d. In the case $\sigma_b = -\sigma_d \implies \eta(b) + \eta(d) = -2 \implies d + b = -e_1 - e_2$, i. e. there exists a red edge marked $(1, 2)$ that connects b and d. Then in any case i must appear twice in the path from b to d. There are the following subcases:
a) If $i \neq k$ then i must appear as

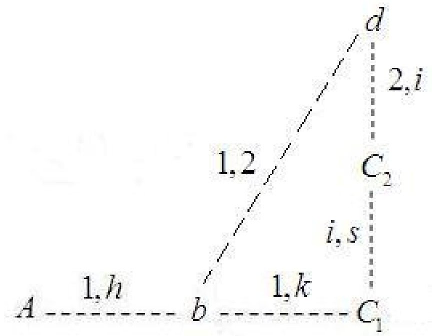


Figure 34

Applying lemma 7.4 for the pair of indices $(1, i)$ we get either

$$C_2 = \{c_2\}, c_2 \pm b = \pm e_1 \pm e_i, \tag{145}$$

or

$$A = \{0\}, d = \pm 2(e_1 - e_i). \tag{146}$$

-If (145) holds then the vertex c_2 and the index i satisfy all conditions of lemma 5.2, hence χ_G is irreducible.
-(146) cannot hold, since if we look at the path from 0 to d in figure (34), we will see that the second coordinate of d is equal to 1 or -1.
b) If $i = k$, consider the index j. There are the following possibilities (figures (35)-(38)):
i) If j appears as:

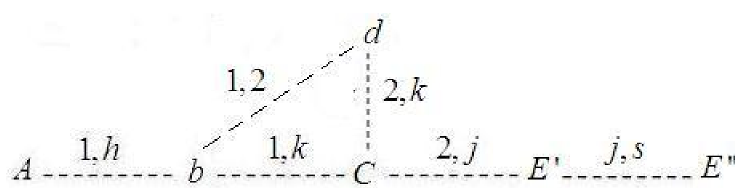


Figure 35

Applying lemma 7.4 for the pair of indices $(1, j)$ we get either $E' = \{e'\}, e' \pm b = \pm e_1 \pm e_j$ (this is impossible, since by inspection $e' \pm b = \pm e_2 + \sum_{m \neq 2} n_m e_m$), or $E'' = \{e''\}, A = \{0\}, e'' = \pm 2(e_1 - e_j)$ (this is also impossible, since by inspection $e'' = \pm e_2 + \sum_{m \neq 2} n_m e_m$).
ii) If j appears as

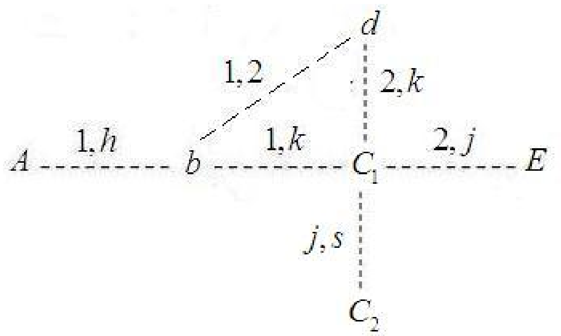


FIGURE 36

Applying to the pair $(1, j)$ the process that we have done for the pair $(1, 2)$ at the beginning of this proof, we get: either $C_2 = \{c_2\}, c_2 \pm b = \pm e_1 \pm e_j$ (this is impossible, since according to figure (36): $c_2 \pm b = \pm e_k + \sum_{m \neq k} n_m e_m$), or $E = \{e\}, e \pm b = \pm e_1 \pm e_j$ (this is impossible, since by inspection $e \pm b = \pm e_2 + \sum_{m \neq 2} n_m e_m$).
iii) If j appears as:

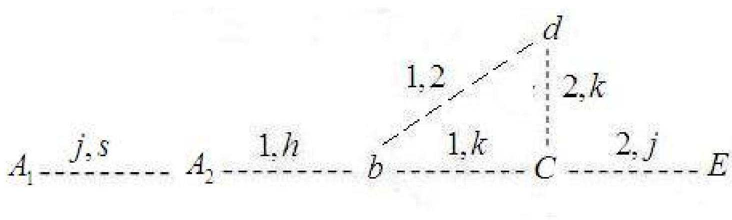


FIGURE 37

then, by lemma 7.6 for the pair $(j, 1)$, χ_G is irreducible.
iv) If $j = h$:

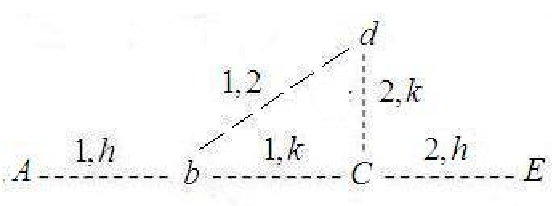


FIGURE 38

$$\chi_G|_{\xi_k=0} = \chi_{\overline{A \cup b \cup d}} \chi_{\overline{C \cup E}}|_{\xi_k=0}, \tag{147}$$

$$\chi_G|_{\xi_h=0} = \chi_{\bar{A}} \chi_{\overline{b \cup C \cup d}}|_{\xi_h=0} \chi_{\bar{E}}|_{\xi_h=0}. \tag{148}$$

From (147) we see that if χ_G is not irreducible, then $\chi_G = PQ$, where P, Q are irreducible: $P|_{\xi_k=0} = \chi_{\overline{A\cup b\cup d}}$. From (148) we get $P|_{\xi_h=0} = \chi_{\overline{b\cup C\cup d}}|_{\xi_h=0} \implies \chi_{\overline{A\cup b\cup d}}|_{\xi_h=0} = \chi_{\overline{b\cup C\cup d}}|_{\xi_k=\xi_h=0} \implies \chi_{\bar{A}}\chi_{b\cup d}|_{\xi_h=0} = \chi_{\bar{C}}|_{\xi_k=\xi_k=0}\chi_{b\cup d}|_{\xi_h=0} \implies \chi_{\bar{A}} = \chi_{\bar{C}}|_{\xi_h=\xi_k=0}$. Hence by lemma 7.1 $A = \{0\}, C = \{c\}, c = \tau_{n_h e_h + n_k e_k}(0)$ where according to figure (38) $n_h, n_k \in \{\pm 1\}$, $\eta(c) \in \{0, -2\} \implies c = \pm(e_h - e_k), -e_h - e_k$. So in G there is an edge marked (h, k) that connects 0 and c. Then the vertex b and the index 1 satisfy all conditions of lemma 5.2, χ_G is irreducible.

□

7.4.3. *Figure* (28).

Lemma 7.8. *If there exist two indices, say 1, 2, such that T is of the form as in the figure* (28)*, then χ_G is irreducible.*

Proof. We have:

(149)

$$\chi_G|_{\xi_1=0} = \begin{cases} \chi_{\bar{A}}\chi_{\overline{B\cup C}}|_{\xi_1=0}\chi_{\overline{D\cup E}}|_{\xi_1=0}, \text{if in } \bar{T} \text{ there is no edge that connects } A \text{ with } D\cup E \\ \chi_{\overline{A\cup D\cup E}}\chi_{\overline{B\cup C}}|_{\xi_1=0}, \text{if in } \bar{T} \text{ there is an edge that connects } A \text{ with } D\cup E \end{cases}$$

(150)

$$\chi_G|_{\xi_2=0} = \begin{cases} \chi_{\overline{A\cup B}}\chi_{\overline{C\cup D}}|_{\xi_2=0}\chi_{\bar{E}}|_{\xi_2=0}, \text{if in } \bar{T} \text{ there is no edge that connects } A\cup B \text{ with } E \\ \chi_{\overline{A\cup B\cup E}}\chi_{\overline{C\cup D}}|_{\xi_2=0}, \text{if in } \bar{T} \text{ there is an edge that connects } A\cup B \text{ with } E \end{cases}.$$

Comparing (149) and (150) and by a simple analysis we get the following possibilities:

Assume χ_G is not irreducible:$\chi_G = UV$. Since U, V play the same role, by (149) and (150) we may suppose $U|_{\xi_1=0} = \chi_{\overline{A}}$ or $U|_{\xi_1=0}$ equals $\chi_{\overline{B\cup C}}|_{\xi_1=0}$ or $\chi_{\overline{D\cup E}}|_{\xi_1=0}$.If $U|_{\xi_1=0} = \chi_{\overline{A}}$ we must have $U|_{\xi_2=0} = \chi_{\overline{E}}$ and

(151) $$\chi_{\bar{A}} = \chi_{\bar{E}}|_{\xi_1=\xi_2=0}.$$

Otherwise if $U|_{\xi_1=0}$ equals $\chi_{\overline{B\cup C}}|_{\xi_1=0}$ or $\chi_{\overline{D\cup E}}|_{\xi_1=0}$.

We may have $U|_{\xi_2=0}$ equals $\chi_{\overline{A\cup B}}|_{\xi_2=0}$ or $\chi_{\overline{C\cup D}}|_{\xi_2=0}$. We deduce, respectively:

(152) $$\chi_{\bar{A}} = \chi_{\bar{C}}|_{\xi_1=\xi_2=0}$$

(153) $$\text{or } \chi_{\bar{C}}|_{\xi_1=\xi_2=0} = \chi_{\bar{E}}|_{\xi_1=\xi_2=0}$$

(154) $$\text{or } \chi_{\bar{B}}|_{\xi_1=0} = \chi_{\bar{D}}|_{\xi_1=\xi_2=0}$$

(155) $$\text{or } \chi_{\overline{D}}|_{\xi_1=\xi_2=0} = \chi_{\overline{A}}|_{\xi_1=\xi_2=0},\ \chi_{\overline{E}}|_{\xi_1=\xi_2=0} = \chi_{\overline{B}}|_{\xi_1=\xi_2=0}.$$

By symmetry (153) is similar to (152).

1) Consider case (152). This happens if $\chi_G = UV$ with $U|_{\xi_1=0} = \chi_{\overline{B\cup C}}$, $U|_{\xi_2=0} = \chi_{\overline{A\cup B}}$. By lemma 7.1 we get $|A| = |C| = 1, A = \{0\}, C = \{c\}, c = \tau_{n_1e_1+n_2e_2}(0)$. According to figure (28) $n_1, n_2 \in \{\pm 1\} \implies c = \pm(e_1 - e_2), -e_1 - e_2$. Hence there is an edge marked $(1, 2)$ that connects 0 and c and all indices, different from 1, 2, either do not appear or appear twice in any path from 0 to c. We now divide this case into 4 sub–cases

$$a) : h \neq k,\ b) : h = k, i \neq j,\ i \in B,\ c) : h = k, i \neq j,\ i \in D,\ d) : h = k,\ i = j \text{ or } i \in E.$$

a) If $k \neq h$, then k must appear once in an edge of the block B which belongs to the path that connects 0 and c, T has the form :

$$0 \overset{1,h}{--} B_1 \overset{s,k}{--} B_2 \overset{2,k}{--} c \overset{1,i}{--} D \overset{2,j}{--} E$$

We apply lemma 7.6 for the pair of indices $(1,k)$ and get the irreducibility of χ_G.

b) If $k=h$, $i\neq j$, $i\in B$, then T has the form (39) or the form (40).

$$\begin{array}{l} 0 \overset{1,h}{---} B_1 \overset{2,h}{---} c \overset{1,i}{---} D \overset{2,j}{---} E \\ \quad\;\; \mid \\ \quad\;\; \mid\, i,s \\ \quad\;\; \mid \\ \quad\;\; B_2 \end{array}$$

FIGURE 39

$$0 \overset{1,h}{---} B_1 \overset{i,s}{---} B_2 \overset{2,h}{---} c \overset{1,i}{---} D \overset{2,j}{---} E$$

FIGURE 40

-Consider the case of figure (39). From the fact that $U|_{\xi_2=0}=\chi_{\overline{A\cup B}}=\chi_{\overline{0\cup B_1\cup B_2}}$ and $\chi_G|_{\xi_i=0}=\chi_{\overline{0\cup B_1\cup c}}|_{\xi_i=0}\chi_{\overline{B_2}}|_{\xi_i=0}\chi_{\overline{D\cup E}}|_{\xi_i=0}$ or $\chi_G|_{\xi_i=0}=\chi_{\overline{0\cup B_1\cup c}}|_{\xi_i=0}\chi_{\overline{B_2\cup D\cup E}}|_{\xi_i=0}$ we may have $U|_{\xi_i=0}=\chi_{\overline{D\cup E}}|_{\xi_i=0}$, or $U|_{\xi_i=0}=\chi_{\overline{B_2\cup D\cup E}}|_{\xi_i=0}$ or $U|_{\xi_i=0}=\chi_{\overline{0\cup B_1\cup c}}|_{\xi_i=0}$. We see that $U|_{\xi_i=0}=\chi_{\overline{D\cup E}}|_{\xi_i=0}$ is incompatible with $U|_{\xi_2=0}=\chi_{\overline{A\cup B}}=\chi_{\overline{0\cup B_1\cup B_2}}$ implying $|0\cup B_1|=1$.

$U|_{\xi_i=0}=\chi_{\overline{B_2\cup D\cup E}}|_{\xi_i=0}$ is also incompatible with $U|_{\xi_2=0}=\chi_{\overline{A\cup B}}=\chi_{\overline{0\cup B_1\cup B_2}}$ implying $\chi_{\bar{D}}|_{\xi_2=\xi_i=0}\chi_{\bar{E}}|_{\xi_2=\xi_i=0}=\chi_{\overline{0\cup B_1}}$ (that is an equality between product of 2 polynomials and an irreducible polynomial).
Hence the only case to consider is: $U|_{\xi_i=0}=\chi_{\overline{0\cup B_1\cup c}}|_{\xi_i=0}$. Since $U|_{\xi_2=0}=\chi_{\overline{A\cup B}}=\chi_{\overline{0\cup B_1\cup B_2}}$ we deduce $\chi_{\bar{B_2}}|_{\xi_i=0}=\chi_c|_{\xi_i=\xi_2=0}\implies |B_2|=1, B_2=\{b_2\}, c\pm b_2=\pm e_i\pm e_2$. But this is not possible since by inspection of figure (39) $c\pm b_2=\pm e_h+\sum_{m\neq h}n_m e_m$.

-The case of figure (40) is treated similarly as the case (39) by considering factorizations of $\chi_G|_{\xi_i=0},\chi_G|_{\xi_2=0}$.

c) If $k=h, i\neq j, i\in D$, then T has the form (41) or the form (42).

$$0 \overset{1,h}{---} B \overset{2,h}{---} c \overset{1,i}{---} D_1 \overset{i,s}{---} D_2 \overset{2,j}{---} E$$

FIGURE 41

$$\begin{array}{l} 0 \overset{1,h}{-\!-\!-} B \overset{2,h}{-\!-\!-} c \overset{1,i}{-\!-\!-} D_1 \overset{2,j}{-\!-\!-} E \\ \qquad\qquad\qquad\qquad\quad\ \ \mid i,s \\ \qquad\qquad\qquad\qquad\quad\ D_2 \end{array}$$

FIGURE 42

Consider case (41). If χ_G is not irreducible, then, by lemma 7.4 for the pair of indices (h,i) we get either $|D_2 \cup E| = 1$ (that is impossible); or $B = \{b\}, D_1 = \{d_1\}, d_1 \pm b = \pm e_h \pm e_i$ (that is impossible, since according to figure (41) $d_1 \pm b = \pm e_1 + \sum_{m\neq 1} n_m e_m$).

Consider case (42). By lemma 7.4 we get either $|D_2| = 1, D_2 = \{d_2\}, d_2 = \pm 2(e_h - e_i)$, (that is not possible, since by inspection $d_2 = \pm e_2 + \sum_{m\neq 2} n_m e_m$); or $|D_1 \cup E| = |B| = 1$ (that is not possible, since $|D_2 \cup E| \geq 2$).
d) If $k = h, i \in E$ (or $i = j$), then T has the form:

$$0 \overset{1,h}{-\!-\!-} B \overset{2,h}{-\!-\!-} c \overset{1,i}{-\!-\!-} E_1 \overset{s,i}{-\!-\!-} E_2$$

FIGURE 43

Applying lemma 7.4 for the pair of indices (i,h) we get either $E_2 = \{e_2\}, e_2 = \pm 2(e_i - e_h)$ (that is not possible, since $c = \pm(e_1 - e_2)$ and h does not appear elsewhere in a path from c to e_2, or $B = \{b\}, E_1 = \{e_1\}, e_1 \pm b = \pm e_i \pm e_h$ (that is not possible, since according to figure (43) $e_1 \pm b = \pm e_1 + \sum_{m\neq 1} n_m e_m$).

2) Consider case (154) This happens if $\chi_G = UV$ with $U|_{\xi_1=0} = \chi_{\overline{B\cup C}}$, $U|_{\xi_2=0} = \chi_{\overline{C\cup D}}$. This implies $B = \{b\}, D = \{d\}, d \pm b = \pm e_1 \pm e_2$. Hence all indices, different from 1, 2 appear an even number of times in any path connecting b and d. We have 4 cases a),b),c),d) depending on the values and positions of h, i, k.
a) If $k \neq i$, k must appear once more in the path from b to d as:

$$A \overset{1,h}{-\!-\!-} b \overset{2,k}{-\!-\!-} C_1 \overset{k,s}{-\!-\!-} C_2 \overset{1,i}{-\!-\!-} d \overset{2,j}{-\!-\!-} E$$

Then we apply lemma 7.6 for the pair $(1,k)$ and get the irreducibility of χ_G.
b) If $k = i$, $h \in C$, then T has the form:

$$\begin{array}{l} A \overset{1,h}{-\!-\!-} b \overset{2,i}{-\!-\!-} C_1 \overset{1,i}{-\!-\!-} d \overset{2,j}{-\!-\!-} E \\ \qquad\qquad\qquad\ \ \mid h,s \\ \qquad\qquad\qquad\ C_2 \end{array}$$

Consider factorizations of $U|_{\xi_h=0}$ and $U|_{\xi_i=0}$. Since $U|_{\xi_1=\xi_h=0} = \chi_{\overline{b\cup C_1}}|_{\xi_1=\xi_h=0}\chi_{\overline{C_2}}|_{\xi_1=\xi_h=0}$ and $\chi_G|_{\xi_h=0} = \begin{cases} \chi_{\overline{A}}\chi_{\overline{C_2}}|_{\xi_h=0}\chi_{\overline{b\cup C_1\cup d\cup E}}|_{\xi_h=0} \\ \chi_{\overline{A\cup C_2}}\chi_{\overline{b\cup C_1\cup d\cup E}}|_{\xi_h=0} \end{cases}$.

We may have $U|_{\xi_h=0} = \begin{cases} \chi_{\overline{A}}\chi_{\overline{C_2}}|_{\xi_h=0} \\ \chi_{\overline{A\cup C_2}} \end{cases}$ or $U|_{\xi_h=0} = \chi_{\overline{b\cup C_1\cup d\cup E}}$. Both are incompatible, the first gives $|b\cup C_1| = 1$ and the second $|d\cup E| = 1$.

c) If $k = i, h \in E$ or $h = j$, then T has the form:

$$A \overset{1,h}{-\,-\,-} b \overset{2,i}{-\,-\,-} C \overset{1,i}{-\,-\,-} d \overset{2,j}{-\,-\,-} E_1 \overset{h,s}{-\,-\,-} E_2$$

or

$$A \overset{1,h}{-\,-\,-} b \overset{2,i}{-\,-\,-} C \overset{1,i}{-\,-\,-} d \overset{2,h}{-\,-\,-} E$$

then by lemma 7.6 for the pair (h, i), χ_G is irreducible.

d) If $k = i, h \in A$:

$$A_1 \overset{h,s}{-\,-\,-} A_2 \overset{1,h}{-\,-\,-} b \overset{2,i}{-\,-\,-} C \overset{1,i}{-\,-\,-} d \overset{2,j}{-\,-\,-} E$$

suppose χ_G is not irreducible then by lemma 7.4 for the pair (h, i) we get either $A_2 = \{a_2\}, C = \{c\}, c \pm a_2 = \pm e_h \pm e_i$(which is not possible since by inspection $c \pm a_2 = \pm e_1 + \sum_{k\neq 1} n_k e_k$); or $|d \cup E| = |A_1| = 1$, a contradiction.

3) Consider case (151). We have a factor U so that $U|_{\xi_1=0} = \chi_{\overline{A}}$, $U|_{\xi_2=0} = \chi_{\overline{E}}$. This implies $A = \{0\}, E = \{e\}, e = \pm 2e_1 \pm 2e_2$. Hence all indices, different from 1, 2 appear twice in the path from 0 to e. Consider the possible positions of the index i.

a) If i appears in one edge of C or D in the path from 0 to e, i. e. T has the form

$$0 \overset{1,h}{-\,-\,-} B \overset{2,k}{-\,-\,-} C_1 \overset{i,s}{-\,-\,-} C_2 \overset{1,i}{-\,-\,-} D \overset{2,j}{-\,-\,-} e$$

or

$$0 \overset{1,h}{-\,-\,-} B \overset{2,k}{-\,-\,-} C \overset{1,i}{-\,-\,-} D_1 \overset{i,s}{-\,-\,-} D_2 \overset{2,j}{-\,-\,-} e$$

then, by lemma 7.6 for the pair $(2, i)$, χ_G is irreducible.

b) If i appears in one edge of B in the path from 0 to e:

$$0 \overset{1,h}{-\,-\,-} B_1 \overset{i,s}{-\,-\,-} B_2 \overset{2,k}{-\,-\,-} C \overset{1,i}{-\,-\,-} D \overset{2,j}{-\,-\,-} e$$

Since U is a linear polynomial this is impossible as setting $\xi_i = 0$ the factorization of $\chi_G|_{\xi_i=0}$ has no linear polynomial.

c) If $i = j$, consider the positions of the index h.

i) If h appears in one edge of B in the path from 0 to e:

$$0 \overset{1,h}{-\,-\,-} B_1 \overset{h,s}{-\,-\,-} B_2 \overset{2,k}{-\,-\,-} C \overset{1,i}{-\,-\,-} D \overset{2,i}{-\,-\,-} e$$

then, by lemma 7.4 for the pair $(h, 2)$ we get either $|C \cup D| = |B_1| = 1$ (that is certainly not possible, since $|C \cup D| \geq 2$); or $e = \pm 2e_h \pm 2e_2$) that contradicts the fact that $e = \pm 2e_1 \pm 2e_2$.

ii) If $h \in C$:

$$0 \overset{1,h}{-\,-\,-} B \overset{2,k}{-\,-\,-} C_1 \overset{h,s}{-\,-\,-} C_2 \overset{1,i}{-\,-\,-} D \overset{2,i}{-\,-\,-} e$$

then applying lemma 7.4 for the pair (h, i) we get either $|B \cup C_1| = |D| = 1$ (that is not possible, since $|B \cup C_1| \geq 2$), or $e = \pm 2e_h \pm 2e_i$ (that contradicts the fact that $e = \pm 2e_1 \pm 2e_2$).

iii) If $h \in D$:

$$0 \overset{1,h}{-\,-\,-} B \overset{2,k}{-\,-\,-} C \overset{1,i}{-\,-\,-} D_1 \overset{h,s}{-\,-\,-} D_2 \overset{2,i}{-\,-\,-} e$$

Considering the factorizations of $\chi_G|_{\xi_h=0}$ and $\chi_G|_{\xi_2=0}$ we get the following possibilities:
-$\chi_0 = \chi_{\overline{C\cup D_1}}|_{\xi_h=\xi_2=0}$ or $\chi_{\overline{C\cup D_1}}|_{\xi_h=\xi_2=0} = \chi_e|_{\xi_h=\xi_2=0}$. Both of them imply that $|C\cup D_1| = 1$, a contradiction.
-$\chi_B|_{\xi_h=0} = \chi_{\bar{D_2}}|_{\xi_h=\xi_2=0}$. By lemma 7.1 we get $B = \{b\}, D_2 = \{d_2\}, d_2 \pm b = n_h e_h + n_2 e_2$, but by inspection $d_2 \pm b = \pm e_1 + \sum_{m\neq 1} n_m e_m$.
iv) If $h = k$:

$$0 \overset{1,k}{---} B \overset{2,k}{---} C \overset{2,i}{---} D \overset{1,j}{---} e$$

FIGURE 44

Applying lemma 7.4 for the pair (k, i) we get following possibilities:
-$B = \{b\}, D = \{d\}, d \pm b = \pm e_k \pm e_i$. But according to figure (44) $d \pm b = \pm e_1 + \sum_{m\neq 1} n_m e_m$, a contradiction.
$e = \pm 2(e_k - e_i)$, this contradicts $e = \pm 2(e_1 - e_2)$. □

7.4.4. *Figure* (29). By lemma 7.4 we have 2 subcases:
1)

$$B = \{b\}, D = \{d\}, d \pm b = \pm e_1 \pm e_2, \quad \text{or} \quad A = \{0\}, E = \{e\}, e = \pm 2(e_1 - e_2). \tag{156}$$

In the second case all indices appear twice in the path from 0 to e. Consider the possible positions of the index h.
a) If h appears in an edge of B in the path from 0 to e, or $h = k$, T will have the form:

$$0 \overset{1,h}{---} B_1 \overset{h,s}{---} B_2 \overset{1,k}{---} C \overset{2,i}{---} D \overset{2,j}{---} e$$

By lemma 7.4 for the pair $(h, 2)$ there are 2 possibilities:
i)$B_1 = \{b_1\}, D = \{d\}, d \pm b_1 = \pm e_h \pm e_2$, but by inspection: $d \pm b_1 = \pm e_1 + \sum_{m\neq 1} n_m e_m$.
ii) $e = \pm 2(e_h - e_2)$, this contradicts (156).
b) If h appears in an edge of C in the path from 0 to e:

$$0 \overset{1,h}{---} B \overset{1,k}{---} C_1 \overset{h,s}{---} C_2 \overset{2,i}{---} D \overset{2,j}{---} e$$

by lemma 7.4 for the pair of indices $(h, 2)$ there are 2 possibilities:
i) $|B \cup C_1| = |D| = 1$, a contradiction, since $|B \cup C_1| \geq 2$
ii) $e = \pm 2(e_h - e_2)$, this contradicts (156).
c) If h appears in an edge of D in the path from 0 to e:

$$0 \overset{1,h}{---} B \overset{1,k}{---} C \overset{2,i}{---} D_1 \overset{h,s}{---} D_2 \overset{2,j}{---} e$$

then, by lemma 7.8 for the pair $(h, 2)$ we get the irreducibility of χ_G.
d) If $h = i$, consider the index k.
i) If $k \in B$ or $k \in C$, then by lemma 7.6 for the pair (h, k) we get the irreducibility of χ_G.
ii) If $k \in D$ or $k = j$, then by lemma 7.8 for the pair (h, k) we get the irreducibility of χ_G.
e) If $h = j$, then for any case: $k \in B$ or $k \in C$ or $k \in D$ or $k = i$, by lemma 7.6 for the pair (h, k) χ_G is irreducible.
ii) $e = \pm 2(e_k - e_2)$, this contradicts (156).
2)

$$B = \{b\}, D = \{d\}, d \pm b = \pm e_1 \pm e_2 \tag{157}$$

We get $d-b=\pm(e_1-e_2)$ in the case $\sigma_d=\sigma_b, \eta(d)=\eta(b)$, and $d+b=-e_1-e_2$ in the case $\sigma_d=-\sigma_b, \eta(d)+\eta(b)=-2$. Then in $\bar{T}$ there is an edge marked $(1,2)$ that connects b and d and all indices, different from 1, 2 must appear an even number of times in the path from b to d. Consider the possible positions of the index k:$k\in C$ or $k=i$. If $k\in C$

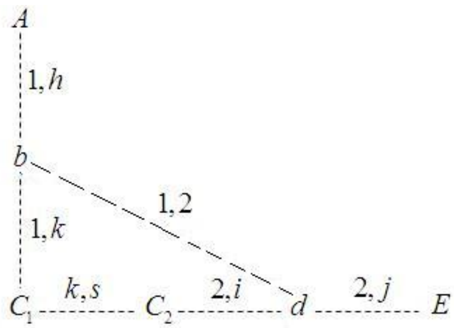


FIGURE 45

By lemma 7.4 for the pair $(k,2)$ we get 2 possibilities:
i) $C_1=\{c_1\}$. Then the vertex c_1 and the index k satisfy all conditions of lemma 5.2, so χ_G is irreducible.
ii) $|A\cup b|=|E|=1$, a contradiction since $|A\cup b|\geq 2$.
b) If $k=i$, consider the possible positions of the index j: $j\in A, j=h, j\in C$ or $j\in E$.
i) If $j\in C$ then j must appear as:

$$\begin{array}{c} A \overset{1,h}{- - -} b \overset{1,i}{- - -} C_1 \overset{2,i}{- - -} d \overset{2,j}{- - -} e \\ \hphantom{A \overset{1,h}{- - -} b \overset{1,i}{- - -}} \Big| \, {\scriptstyle j,s} \hphantom{\overset{2,i}{- - -} d \overset{2,j}{- - -} e} \\ \hphantom{A \overset{1,h}{- - -} b \overset{1,i}{- - -}} C_2 \hphantom{\overset{2,i}{- - -} d \overset{2,j}{- - -} e} \end{array}$$

then, by lemma 7.7 for the pair $(1,j)$, χ_G is irreducible.
ii) If $j\in A$ or $j=h$, then, by lemma 7.6 for the pair (j,i), χ_G is irreducible.
iii) If $j\in E$:

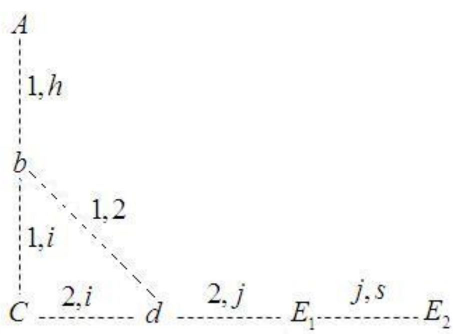
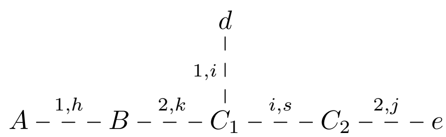


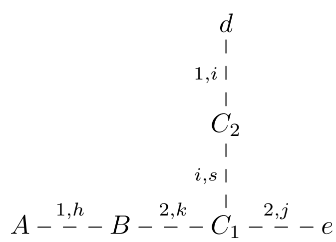
FIGURE 46

by lemma 7.4 for the pair (i, j) there are 2 possibilities:
- $C = \{c\}, E_1 = \{e_1\}$. Then the vertex c and the index i satisfy all conditions of 5.2, so χ_G is irreducible.
-$|A \cup b| = |E_2| = 1$, a contradiction, since $|A \cup b| \geq 2$.

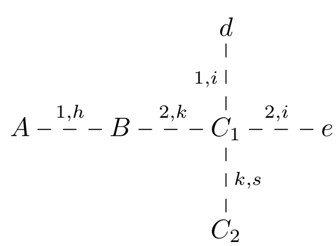
7.4.5. *Figure* (30). By lemma 7.5 we have to consider 4 subcases:
1) When $D = \{d\}, E = \{e\}, e \pm d = \pm e_1 \pm e_2$, all indices, different from 1, 2 must appear an even number of times in the path of T from d to e. In particular, $i \in C$ or $i = j$.
a) If i appears in C as:



then, by lemma 7.5 for the pair $(2, i)$ and since $|A \cup B| > 1$ we get the only possibility $C_1 = \{c_1\}, e \pm c_1 = \pm e_1 \pm e_2$. So $j = s$ or j appears in one edge of C_2 in the path from e to c_1. Hence, by lemma 7.7 for the pair $(j, 1)$, χ_G is irreducible.
b) If i appears in C as:



then, by lemma 7.7 for the pair $(i, 2)$, χ_G is irreducible.
c) If $i = j$, consider the positions of the index k:
i) If $k \in A$ or $k \in B$ or $k = h$, then, by lemma 7.7 for the pair (k, i), χ_G is irreducible.
ii) If $k \in C$, then k must appear as



by lemma 7.5 for the pair $(1, k)$ and since $|C_1 \cup e| > 1$ we get 2 possibilities:
+) $B = \{b\}, d \pm b = \pm e_1 \pm e_k$, but this is not possible since by inspection $d \pm b = \pm e_2 + \sum_{m \neq 2} n_m e_m$.
+) $C_2 = \{c_2\}, c_2 \pm d = \pm e_1 \pm e_k$, but this is not possible since by inspection $c_2 \pm d = \pm e_1 + \sum_{m \neq 1} n_m e_m$.

2) When $A = \{0\}, C = \{c\}, c = \pm(e_1 - e_2)$, all indices, different from $1, 2$, must appear an even number of times in the path from 0 to c. In particular, $h \in B$ or $h = k$.

a) If $h \in B$, then h must appear as:

$$\begin{array}{ccccccccc} & & & & & & D & & \\ & & & & & & {\scriptstyle 1,i}\,| & & \\ 0 & \overset{1,h}{--} & B_1 & \overset{h,s}{--} & B_2 & \overset{2,k}{--} & c & \overset{2,j}{--} & E \end{array}$$

It is easy to see that T has the form of figure (29), replacing $(1, 2)$ by $(h, 2)$. Hence we have already shown that χ_G is irreducible.
b) If $h = k$, consider the index i.
i) If $i \in D$, then T has the form of figure (29), replacing $(1, 2)$ by (k, i). Hence χ_G is irreducible.
ii) If $i \in E$ or $i = j$, then by lemma 7.7 for the pair (k, i), χ_G is irreducible.
iii)If $i \in B$, then i must appear as:

$$\begin{array}{ccccccc} & & & & D & & \\ & & & & {\scriptstyle 1,i}\,| & & \\ 0 & \overset{1,k}{--} & B_1 & \overset{2,k}{--} & c & \overset{2,j}{--} & E \\ & & |\,{\scriptstyle i,s} & & & & \\ & & B_2 & & & & \end{array}$$

By lemma 7.5 for the pair (k, i) and since $|c \cup E| > 1$ there are only the two following possibilities:
⋄ $B_2 = \{b_2\}, b_2 = \pm(e_k - e_i)$, but this is not possible, since by inspection $b_2 = \pm e_1 + \sum_{m \neq 1} n_m e_m$.
⋄ $B_1 = \{b_1\}, D = \{d\}, d \pm b_1 = \pm e_k \pm e_i$, but this is not possible, since by inspection $d \pm b_1 = \pm e_1 + \sum_{m \neq 1} n_m e_m$.

3) When $B = \{b\}, D = \{d\}, d \pm b = \pm e_1 \pm e_2$, then it is easy to deduce that in any case there is an edge marked $(1, 2)$ that connects b and d. All indices different from 1, 2, must appear an even number of times in the path from b to d.

$$\begin{array}{ccccccc} & & & & d & & \\ & & & & {\scriptstyle 1,i}\,| & & \\ A & \overset{1,h}{--} & b & \overset{2,k}{--} & C & \overset{2,j}{--} & E \end{array}$$

a) If k appears in one edge of C in the path of T from b to d, then by lemma 7.6 for the pair $(1, k)$, χ_G is irreducible.
b) If $k = i$, consider the index h.There are 4 possibilities:
i) If $h \in A$, then T has the form (29), replacing $(1, 2)$ by $(h, 2)$.
ii) If $h \in C$, then h must appear as:

$$\begin{array}{ccccccc} & & & & d & & \\ & & & & {\scriptstyle 1,i}\,| & & \\ A & \overset{1,h}{--} & b & \overset{2,k}{--} & C_1 & \overset{2,j}{--} & E \\ & & & & |\,{\scriptstyle h,s} & & \\ & & & & C_2 & & \end{array}$$

By lemma 7.5 for the pair (h,i) and since $|C_1 \cup E| > 1$ there are only two possibilities:
+) $C_2 = \{c_2\}, c_2 \pm b = \pm e_h \pm e_i$, but this is not possible since by inspection $c_2 \pm b = \pm e_2 + \sum_{m \neq 2} n_m e_m$.
+) $C_2 = \{c_2\}, c_2 \pm d = \pm e_h \pm e_i$, but this is not possible, since by inspection $c_2 \pm d = \pm e_1 + \sum_{m \neq 1} n_m e_m$.
iii) If $h \in E$:

$$\begin{array}{c} d \\ | \\ 1,i\ | \\ | \\ A \overset{1,h}{---} b \overset{2,k}{---} C \overset{2,j}{---} E_1 \overset{h,s}{---} E_2 \end{array}$$

then, by lemma 7.6 for the pair $(h,2)$, χ_G is irreducible.
iv) If $h = j$:

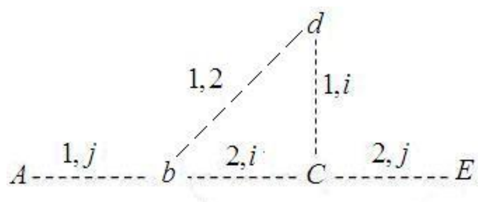


FIGURE 47

then by lemma 7.5 for the pair (h,i) the only possibility is $A = \{0\}, C = \{c\}, c = \pm(e_h - e_i)$. But this is not possible, since according to figure (47) $c = \pm e_1 + \sum_{m \neq 1} n_m e_m$.

7.4.6. *Figure* (31). Let $0 \in A$. We distinguish four cases:
1) When in the complete graph there is an edge that connects C, D and an edge that connects A, E:

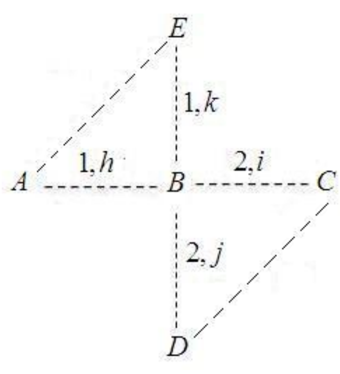


FIGURE 48

we have:

$$\chi_G|_{\xi_1=0} = \chi_{\overline{A \cup E}} \chi_{\overline{C \cup B \cup D}}|_{\xi_1=0}, \tag{158}$$

$$\chi_G|_{\xi_2=0} = \chi_{\overline{A \cup B \cup E}} \chi_{\overline{C \cup D}}|_{\xi_2=0}. \tag{159}$$

From (158) and (159) we deduce that $\chi_G = UV$, where U, V are irreducible and:$U|_{\xi_1=0} = \chi_{\overline{A\cup E}}, U|_{\xi_2=0} = \chi_{\overline{C\cup D}}|_{\xi_2=0}$. Hence:$\chi_{\overline{A\cup E}} = U|_{\xi_1=\xi_2=0} = \chi_{\overline{C\cup D}}|_{\xi_1=\xi_2=0}$. By lemma 7.1 we get $|C\cup D| = |A\cup E| = 1$, a contradiction.
2) When in the complete graph there is an edge that connects C, D and there is no edge that connects A, E:

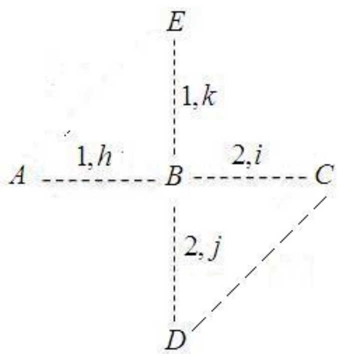


FIGURE 49

we have

$$\chi_G|_{\xi_1=0} = \chi_{\bar{A}}\chi_{\bar{E}}|_{\xi_1=0}\chi_{\overline{C\cup B\cup D}}|_{\xi_1=0}, \tag{160}$$

$$\chi_G|_{\xi_2=0} = \chi_{\overline{A\cup B\cup E}}\chi_{\overline{C\cup D}}|_{\xi_2=0}. \tag{161}$$

Comparing (160) and (161) we get easily $\chi_{\overline{C\cup D}}|_{\xi_1=\xi_2=0} = \chi_{\bar{A}}\chi_{\bar{E}}|_{\xi_1=0}$. But since $1, 2$ do not appear elsewhere in $C\cup D$, $\chi_{\overline{C\cup D}}|_{\xi_1=\xi_2=0}$ is irreducible, then we get a contradiction.

3) The case when in the complete graph there is no edge that connects C, D and there is an edge that connects A, E, is absolutely similar to the previous case.

4) When in the complete graph there is no edge that connects C, D and there is no edge that connects A, E, we have:

$$\chi_G|_{\xi_1=0} = \chi_{\bar{A}}\chi_{\overline{C\cup B\cup D}}|_{\xi_1=0}\chi_{\bar{E}}|_{\xi_1=0}, \tag{162}$$

$$\chi_G|_{\xi_2=0} = \chi_{\overline{A\cup B\cup E}}\chi_{\bar{C}}|_{\xi_2=0}\chi_{\bar{D}}|_{\xi_2=0}. \tag{163}$$

Suppose that χ_G is not irreducible, then its factors under specializations $\xi_1 = 0$ and $\xi_2 = 0$ give (162) and (163) respectively. Comparing (162) and (163) and by a simple analysis we get only the following subcases:

$$\chi_{\bar{A}} = \chi_{\bar{C}}|_{\xi_1=\xi_2=0}, \tag{164}$$

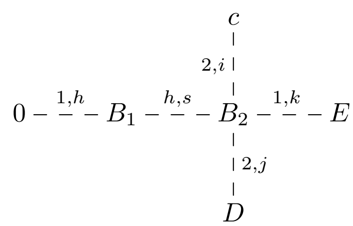
$$\text{or } \chi_{\bar{A}} = \chi_{\bar{D}}|_{\xi_1=\xi_2=0}, \tag{165}$$

$$\text{or } \chi_{\bar{E}}|_{\xi_1=\xi_2=0} = \chi_{\bar{D}}|_{\xi_1=\xi_2=0}, \tag{166}$$

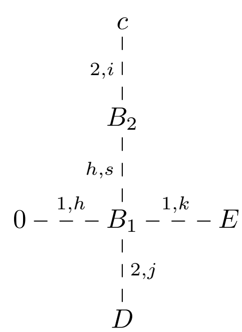
$$\text{or } \chi_{\bar{E}}|_{\xi_1=0} = \chi_{\bar{C}}|_{\xi_1=\xi_2=0}. \tag{167}$$

By the symmetry of the tree in figure (31), we need consider only case (164). We get easily by lemma 7.1 $|A| = |C| = 1, A = \{0\}, C = \{c\}, c = \pm(e_1 - e_2), -e_1 - e_2$. Hence all indices, different from 1,2, must appear an even number of times in any path from 0 to c.
a) If $h \neq i$, h must appears once more in the block B.

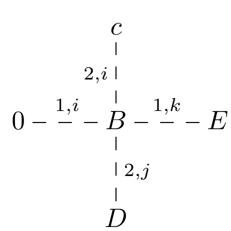
-If h appears in B as:



then we can apply lemma 7.7 for the pair of indices $(h, 2)$ and get the result.
-If h appears in B as :



then T has the form of figure (30), replacing $(1, 2)$ by $(h, 2)$. Hence we have already shown that χ_G is irreducible.
b) If $h = i$:



consider the index j.
i) If $j \in D$ then, by lemma 7.7 for the pair (j, i) χ_G is irreducible.
ii) If $j \in B \cup E$ or $j = k$, then in $\bar{T}$ there is the following subgraph:

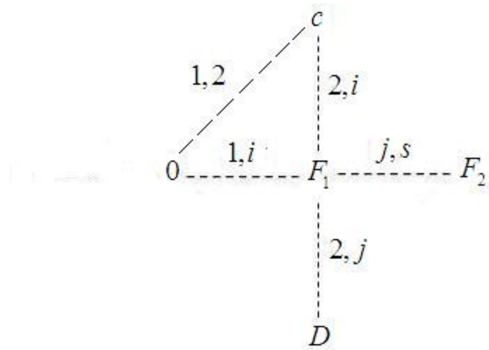


FIGURE 50

In this case the pair (i, j) plays the role of the pair $(1, 2)$ in parts 1), 2), 3) of this subsubsection, hence χ_G is irreducible.

Part 3. The separation and irreducibility of characteristic polynomial, associated to higher degree NLS

ABSTRACT. *In the previous part we proved completely theorem 1.1 for the cubic NLS(i.e. the equation* (1) *in the case* $q = 1$*). For bigger* q *we do not have the affine independence between vertices of every connected component* $\mathcal{G}$ *of* Γ_S*. So we shall prove the separation and irreducibility theorem directly by arithmetical arguments.*

As we said in the last part of subsection 4.8 for every complete colored marked graph $\mathcal{G}$ we will consider the matrix $C_{\mathcal{G}}$ indexing by vertices of $\mathcal{G}$.Given $(a, \sigma), a = \sum_{i=1}^m n_i e_i$ set

$$(q+1)a(\xi) := \sum_{i=1}^{m} n_i \frac{\partial}{\partial \xi_i} A_{q+1}(\xi) \tag{168}$$

then

- In the diagonal at the position $(a, \sigma), a = \sum_{i=1}^m n_i e_i$ we put

$$\begin{cases} a(\xi) & \text{if } \sigma = 1 \\ -a(\xi) - 2(q+1)A_q(\xi) & \text{if } \sigma = -1 \end{cases} \tag{169}$$

- At the position $((a, \sigma_a), (b, \sigma_b))$ we put 0 if they are not connected, otherwise we put $\sigma_b c(\ell)$ (c. f. (80)), where ℓ is the edge connecting a, b.

Define $\chi_{\mathcal{G}} = \chi_{C_{\mathcal{G}}}(t) = det(tI - C_{\mathcal{G}})$- the characteristic polynomial of $C_{\mathcal{G}}$.

Remark 7.4.

$$\frac{\partial}{\partial \xi_i} A_{q+1}(\xi)|_{\xi_i = \xi_j} = \frac{\partial}{\partial \xi_j} A_{q+1}(\xi)|_{\xi_i = \xi_j} \forall i, j \tag{170}$$

Remark 7.5. *Let* $b = \sum_{i=1}^k n_i e_i, n_i \neq 0; \sum_{i=1}^k n_i = 0$*. Then:*

$$b(\xi)|_{\xi_1 = \xi_2 = \ldots = \xi_k} = 0 \tag{171}$$

Proof. By the remark 7.4 we have:

$$b(\xi)|_{\xi_1=\xi_2=\ldots=\xi_k} = \sum_{i=1}^{k} n_i \frac{\partial}{\partial \xi_i} A_{q+1}(\xi)|_{\xi_1=\xi_2=\ldots=\xi_k} = \frac{\partial}{\partial \xi_1} A_{q+1}(\xi)|_{\xi_1=\xi_2=\ldots=\xi_k} \sum_{i=1}^{k} n_i = 0$$

. □

Remark 7.6. *Let $\ell = \ell^+ - \ell^-$ be an edge. We have:*
i) If ℓ is a black edge, then $|\ell^+|_1 = |\ell^-|_1 \le q$.
ii) If ℓ is a red edge, then $|\ell^+|_1 \le q-1, |\ell^-|_1 \le q+1$.

Proof. By the definition of edges we have :

$$|\ell^+|_1 + |\ell^-|_1 \le 2q. \tag{172}$$

On the other hand:
i) If ℓ is a black edge, then

$$|\ell^+|_1 - |\ell^-|_1 = 0. \tag{173}$$

From (172) and (173) we get $|\ell^+|_1 = |\ell^-|_1 \le q$.
ii) If ℓ is red edge, then

$$|\ell^+|_1 - |\ell^-|_1 = -2. \tag{174}$$

From (172) and (174) we get $|\ell^+| \le q-1, |\ell^-|_1 \le q+1$. □

Remark 7.7. *: Let $\ell = \sum_{i=1}^{k} n_i e_i = \ell^+ - \ell^-, n_i \ne 0$, be an edge.*
i) If ℓ is a black edge and $k = m$, then $|\ell^+|_1 = |\ell^-|_1 = q$ and $c(\ell) = (q+1)\xi^{(\ell^+ + \ell^-)/2} \begin{pmatrix} q \\ \ell^+ \end{pmatrix} \begin{pmatrix} q \\ \ell^- \end{pmatrix}$.
ii) If ℓ is a red edge and $k = m$, then $|\ell^+|_1 = q-1, |\ell^-|_1 = q+1$ and $c(\ell) = q\xi^{(\ell^+ + \ell^-)/2} \begin{pmatrix} q+1 \\ \ell^- \end{pmatrix} \begin{pmatrix} q-1 \\ \ell^+ \end{pmatrix}$.

Proof. Since $S = \{v_1, ..., v_m\}$ is some arbitrarily large set, we may suppose $m \ge 2q$. If $k = m$ then $|\ell^+|_1 + |\ell^-|_1 = \sum_{i=1}^{m} n_i \ge m \ge 2q$. Moreover, by definition of edges $\sum_{i=1}^{m} n_i \le 2q$. Hence:

$$|\ell^+|_1 + |\ell^-|_1 = \sum_{i=1}^{m} n_i = 2q. \tag{175}$$

i) When ℓ is a black edge, we have

$$|\ell^+|_1 - |\ell^-|_1 = 0 \tag{176}$$

From (175) and (176) we get $|\ell^+|_1 = |\ell^-|_1 = q$. By formula (80) we obtain $c(\ell) = (q+1)\xi^{(\ell^+ + \ell^-)/2} \begin{pmatrix} q \\ \ell^+ \end{pmatrix} \begin{pmatrix} q \\ \ell^- \end{pmatrix}$.
ii) When ℓ is a red edge, we have

$$|\ell^+|_1 - |\ell^-|_1 = -2 \tag{177}$$

From (175) and (177) we get $|\ell^+|_1 = q-1, |\ell^-|_1 = q+1$. By formula (80) we obtain $c(\ell) = q\xi^{(\ell^+ + \ell^-)/2} \begin{pmatrix} q+1 \\ \ell^- \end{pmatrix} \begin{pmatrix} q-1 \\ \ell^+ \end{pmatrix}$. □

We finally recall Proposition 14 of [11]

Proposition 7. *(i) For $n = 1$ and for generic choices of S, all the connected components of Γ_S are either vertices or single edges.*

(ii) For $n = 2$, and for every m there exist infinitely many choices of generic tangential sites $S = \{v_1, \dots, v_m\}$ such that, if A is a connect component of the geometric graph Γ_S, then A is either a vertex or a single edge.

Obtained results: For graphs reduced to one vertex the statement is trivial. At the moment we are able to prove the irreducibility and separation in dimension 1, and dimension 2, under the assumptions of Proposition 7 for all q since all graphs which appear have at most one edge.

8. One edge

8.1. **Separation.** In this case we have immediately the separation of the characteristic polynomial by the same analysis as in 1) a) of 6 since in this case in the graph there are only two vertices.

8.2. **Irreducibility.**

Theorem 8.1. *For any q and any connected colored marked graph with one edge the characteristic polynomial is irreducible.*

Proof. We choose the root so that the graph has one of the forms:

$$0 \xrightarrow[black]{\ell} \ell \qquad \text{or} \qquad 0 \overset{\ell}{\underset{red}{=\!=\!=}} \ell$$

Let $\ell = \sum_{i=1}^{k} n_i e_i, n_i \neq 0$. We have

(178)
$$\ell(\xi) = \frac{1}{q+1}\sum_{i=1}^{k} n_i \frac{\partial}{\partial \xi_i} A_{q+1}(\xi) = \sum_{i=1}^{k} n_i \sum_{\beta\in\mathbb{N}^m;|\beta|_1=q+1;\beta_i\geq 1} \binom{q+1}{\beta}\binom{q}{\beta_1,\dots,\beta_i-1,\dots,\beta_m} \xi_1^{\beta_1}\dots\xi_i^{\beta_i-1}\dots\xi_m^{\beta_m}$$

Set $\bar\ell(\xi) := \ell(\xi)$ if $\eta(\ell) = 0$ and $\bar\ell(\xi) := -\ell(\xi) - 2(q+1)A_q(\xi)$ if $\eta(\ell) = -2$.

Remark 8.1. *For every i in the support of ℓ the polynomial $\bar\ell(\xi)$ contains the term ξ_i^q with non zero coefficient.*

Proof. In the formula of $\ell(\xi)$ there is the monomial:

$$(n_i + (q+1)\sum_{h\neq i} n_h)\xi_i^q,$$

since $\sum_h n_h = \eta(\ell)$ this equals

$$-qn_i\xi_i^q \quad \text{if } \eta(\ell) = 0$$

and

$$[n_i + (q+1)(-2-n_i)]\xi_i^q \quad \text{if } \eta(\ell) = -2$$

In $A_q(\xi)$ the monomial ξ_i^q appears with coefficient 1,so we get in $\bar\ell$ the coefficient of ξ_i^q is:

$$-n_i + (q+1)(2+n_i) - 2(q+1) = qn_i \tag{179}$$

which is non zero since i is in the support of ℓ, $n_i \neq 0$. ∎

We now compute with the matrix

$$C_{\mathcal{G}} = \begin{pmatrix} 0 & \sigma_\ell c(\ell) \\ c(\ell) & \bar{\ell}(\xi) \end{pmatrix}$$

$$\chi_{\mathcal{G}}(t) = det \begin{pmatrix} t & -\sigma_\ell c(\ell) \\ -c(\ell) & t-\bar{\ell}(\xi) \end{pmatrix} = t^2 - \bar{\ell}(\xi)t - \sigma_\ell c(\ell)^2. \tag{180}$$

Suppose that $\chi_{\mathcal{G}}$ is not irreducible, then:

$$\chi_{\mathcal{G}}(t) = (t + r(\xi))(t - \bar{\ell}(\xi) - r(\xi)). \tag{181}$$

Compare the free coefficients in 180 and 181 we get

$$r(\xi)(-\bar{\ell}(\xi) - r(\xi)) = -\sigma_\ell c(\ell)^2. \tag{182}$$

By the formula 40 $c(\ell)^2$ is divisible by $\xi_i^{|n_i|}, \forall i = 1, ..., k$.
For any i if $r(\xi)$ is divisible by ξ_i, by remark 8.1 $\bar{\ell}(\xi)$ is not divisible by ξ_i, then $-\bar{\ell}(\xi) - r(\xi)$ is not divisible by ξ_i. And inversely, if $-\bar{\ell}(\xi) - r(\xi)$ is divisible by ξ_i, then $r(\xi)$ is not divisible by ξ_i. Hence we have:

$$r(\xi) = \xi_i^{|n_i|} s_i, i \in A \tag{183}$$

$$-\bar{\ell}(\xi) - r(\xi) = \xi_j^{|n_j|} u_j, j \in B. \tag{184}$$

where $A \cup B = \{1, ..., k\}; A \cap B = \emptyset$.

(1) If $A \neq \emptyset$ and $B \neq \emptyset$, then for some couple i, j we have:

$$\bar{\ell}(\xi) = -(\xi_i^{|n_i|} s_i + \xi_j^{|n_j|} u_j) \tag{185}$$

From remark 8.1 we must have $n_h = 0, \forall h \neq i, j$,

(a) **When ℓ is a black edge:**
We have $\sigma_\ell = 1$ and by the definition of edge (cf. 3.4) $\ell = ne_i - ne_j; 2|n| \leq 2q$. We may suppose $i = 1, j = 2, n > 0$. We have $\bar{\ell}(\xi) = \ell(\xi)$ and:

$$\ell(\xi) = n\Big(\sum_{\beta \in \mathbb{N}^m; |\beta|_1 = q+1, \beta_1 \geqslant 1} \binom{q+1}{\beta}\binom{q}{\beta_1 - 1, \beta_2, ..., \beta_m} \xi_1^{\beta_1 - 1}\xi_2^{\beta_2}...\xi_m^{\beta_m} - \tag{186}$$
$$- \sum_{\beta' \in \mathbb{N}^m; |\beta'|_1 = q+1; \beta'_2 \geqslant 1} \binom{q+1}{\beta'}\binom{q}{\beta'_1, \beta'_2 - 1, ..., \beta'_m} \xi_1^{\beta'_1}\xi_2^{\beta'_2 - 1}...\xi_m^{\beta'_m}\Big)$$

Remark that

$$\xi_1^{\beta_1 - 1}\xi_2^{\beta_2}...\xi_m^{\beta_m} = \xi_1^{\beta'_1}\xi_2^{\beta'_2 - 1}...\xi_m^{\beta'_m} \Leftrightarrow \beta_1 - 1 = \beta'_1, \beta_2 = \beta'_2 - 1, \beta_i = \beta'_i \forall i \geqslant 3$$

Then:

$$\ell(\xi) = n \sum_{\beta \in \mathbb{N}^m, |\beta|_1 = q+1, \beta_1 \geqslant 1} \frac{q!}{(\beta_1 - 1)!\beta_2!...\beta_m!} \frac{(q+1)!}{\beta_1!...\beta_m!}(1 - \frac{\beta_1}{\beta_2 + 1})\xi_1^{\beta_1 - 1}\xi_2^{\beta_2}...\xi_m^{\beta_m} \tag{187}$$

By 185 we must have

$$\ell(\xi) = -(\xi_1^n s_1 + \xi_2^n u_2). \tag{188}$$

(i) If $n > 1$, we take $\beta_1 = 1, \beta_2 = n - 1, \beta_3 = q + 1 - n, \beta_4 = ... = \beta_m = 0$, then in the formula (187) of $\ell(\xi)$, there is the monomial

$$n\frac{q!}{(n-1)!(q+1-n)!} \frac{(q+1)!}{(n-1)!(q+1-n)!}(1 - \frac{1}{n})\xi_2^{n-1}\xi_3^{q+1-n} \neq 0$$

and it is not divisible by ξ_1^n or ξ_2^n. This contradicts (188).

(ii) $n=1$. We have $\ell^+=(1,0,...,0);\ell^-=(0,1,...,0)$. Then from 80 we get

(189)
$$c(\ell)^2=(q+1)^2\xi_1\xi_2(\sum_{\alpha\in\mathbb{N}^m:\sum_i a_i+1=q}\binom{q}{\alpha_1+1,\alpha_2,...,\alpha_m}\binom{q}{\alpha_1,\alpha_2+1,...,\alpha_m}\xi^\alpha)^2$$

Let p be a prime divisor of $q+1$:$q+1=p^ku, g.c.d(p,u)=1$. We have:

$$\chi_{\mathcal{G}}=t(t-\ell(\xi))(\text{mod } p)\Rightarrow \chi_{\mathcal{G}}=(t+ps)(t-ps-\ell(\xi)) \tag{190}$$

By (180)and (189) the free coefficient of $\chi_{\mathcal{G}}$ must be divisible by p^{2k}:

$$p^{2k}|ps(-\ell(\xi)-ps) \tag{191}$$

By formula (101) we see that the coefficient of the term ξ_1^q is $-q$, the coefficient of the term ξ_2^q is q. One deduces that $\ell(\xi)$ is not divisible by p since $g.c.d(q,q+1)=1$. Hence $(-\ell(\xi)-ps)$ is not divisible by p. So by (191) we must have $p^{2k-1}|s$. Now take $\xi_1=\xi_2\Rightarrow\ell(\xi)=0$, then the free coefficient of $\chi_{\mathcal{G}}$ when $\xi_1=\xi_2$ is divisible by p^{4k}. But in (180) when $\xi_1=\xi_2$ the free coefficient of $\chi_{\mathcal{G}}$ is $-c(\ell)^2|_{\xi_1=\xi_2}$, it is not divisible by p^{4k}, since in 189 if we take $\alpha_1=\alpha_2=0,\alpha_3=q-1$, we have the monomial:

$$(q+1)^2\xi_1^2(q^2\xi_3^{q-1})^2$$

is not divisible by p^{4k}.

(b) When ℓ is a red edge: we may suppose $\ell=ne_1-(n+2)e_2, n>0$. By Remark 7.6 we must have $n\le q-1$. From (188) we have:

$$\bar{\ell}(\xi)=-\xi_1^n s_1-\xi_2^{n+2}u_2. \tag{192}$$

On the other hand, by computations we get easily:

$$\ell(\xi)=\sum_{\substack{\beta\in\mathbb{N}^m,|\beta|_1=q+1,\\ \beta_1\geqslant 1}}\frac{q!}{(\beta_1-1)!\beta_2!...\beta_m!}\frac{(q+1)!}{\beta_1!...\beta_m!}(n-(n+2)\frac{\beta_1}{\beta_2+1})\xi_1^{\beta_1-1}\xi_2^{\beta_2}...\xi_m^{\beta_m} \tag{193}$$

Hence:

$$\bar{\ell}(\xi)=-\ell(\xi)-2(q+1)A_q(\xi)= \tag{194}$$

$$=-(\sum_{\substack{\beta\in\mathbb{N}^m,|\beta|_1=q+1,\\ \beta_1\geqslant 1}}\frac{q!}{(\beta_1-1)!\beta_2!...\beta_m!}\frac{(q+1)!}{\beta_1!...\beta_m!}(n-(n+2)\frac{\beta_1}{\beta_2+1})+2(q+1)(\frac{q!}{(\beta_1-1)!...\beta_m!})^2)\xi_1^{\beta_1-1}\xi_2^{\beta_2}...\xi_m^{\beta_m}=$$

$$=-\sum_{\substack{\beta\in\mathbb{N}^m,|\beta|_1=q+1,\\ \beta_1\geqslant 1}}(\frac{q!}{(\beta_1-1)!\beta_2!...\beta_m!})^2(q+1)(\frac{1}{\beta_1}(n-\frac{(n+2)\beta_1}{\beta_2+1})+2)\xi_1^{\beta_1-1}\xi_2^{\beta_2}\dots\xi_m^{\beta_m}.$$

If we take $\beta:\beta_1=1,\beta_2=n+1,\beta_3=q-n-1,\beta_4=...=\beta_m=0$, then in Formula (194) for $\bar{\ell}(\xi)$ there is the monomial

$$-(\frac{q!}{(n+1)!(q-n-1)!})^2(q+1)(n+1)\xi_2^{n+1}\xi_3^{q-n-1}$$

which is not divisible by ξ_1^n or ξ_2^{n+2}. This contradicts (192).

(2) If $B=\emptyset$, then $A=\{1,...,k\}$

$$r(\xi)=\xi_1^{|n_1|}...\xi_k^{|n_k|}s \tag{195}$$

.

(a) When ℓ is a black edge: Take $\xi_1=...=\xi_k$, by the remark we have $\ell(\xi)|_{\xi_1=...\xi_k}=0$, hence

$$\chi_{\mathcal{G}}(t)|_{\xi_1=...=\xi_k}=(t+r(\xi)|_{\xi_1=...=\xi_k})(t-r(\xi)|_{\xi_1=...=\xi_k})= \\ =t^2-r(\xi)|^2_{\xi_1=...=\xi_k}. \tag{196}$$

By 196 the free coefficient of $\chi_{\mathcal{G}}|_{\xi_1=...=\xi_k}$ is divisible by $\xi_1^{2\sum_{i=1}^k |n_i|}$. But by 180 the free coefficient of $\chi_G|_{\xi_1=...=\xi_k}$ is $-c(\ell)^2|_{\xi_1=...=\xi_k}$.

-If $k=m$, then by remark 7.7 $-c(\ell)^2|_{\xi_1=...=\xi_k}=-(q+1)^2\xi_1^{\sum_{i=1}^k |n_i|}\begin{pmatrix} q \\ \ell^+ \end{pmatrix}^2\begin{pmatrix} q \\ \ell^- \end{pmatrix}^2$ is not divisible by $\xi_1^{2\sum_{i=1}^k |n_i|}$.

-If $k<m$, then

$$-c(\ell)^2|_{\xi_1=...=\xi_k}=-(q+1)^2\xi_1^{\sum_{i=1}^k |n_i|}(\sum_{\alpha\in\mathbb{N}^m:|\ell^++\alpha|_1=q}(\begin{matrix} q \\ \ell^++\alpha\end{matrix})(\begin{matrix} q \\ \ell^-+\alpha\end{matrix})\xi_1^{\sum_{i=1}^k \alpha_i}\xi_{k+1}^{\alpha_{k+1}}...\xi_m^{\alpha_m})^2 \tag{197}$$

Take $\alpha_1=...=\alpha_k=0,\alpha_{k+1}=q-|\ell^+|_1$, we see that $-c(\ell)^2|_{\xi_1=...\xi_k}$ contains the term $\xi_1^{\sum_{i=1}^k |n_i|}\xi_{k+1}^{2(q-|\ell^+|_1)}$ with the coefficient $-(q+1)^2(\begin{matrix} q \\ \ell^++\alpha\end{matrix})^2(\begin{matrix} q \\ \ell^-+\alpha\end{matrix})^2$.

Hence $-c(\ell)^2|_{\xi_1=...=\xi_k}$ is not divisible by $\xi_1^{2\sum_{i=1}^k |n_i|}$.

(b) When ℓ is a red edge: Take $\xi_1=...=\xi_k$, we have

$$\frac{\partial}{\partial\xi_i}A_{q+1}(\xi)=\frac{\partial}{\partial\xi_j}A_{q+1}(\xi)\forall i,j\Rightarrow \ell(\xi)|_{\xi_1=...=\xi_k}=\sum_{i=1}^k n_i\frac{\partial}{\partial\xi_1}A_{q+1}(\xi)= \\ =-2\frac{\partial}{\partial\xi_1}A_{q+1}(\xi)=-2\sum_{|\alpha|_1=q+1,\alpha_1\geq 1}\frac{1}{q+1}\begin{pmatrix} q+1 \\ \alpha\end{pmatrix}^2\alpha_1\xi_1^{\alpha_1+\alpha_2+...+\alpha_k-1}\xi_{k+1}^{\alpha_{k+1}}...\xi_m^{\alpha_m} \tag{198}$$

$$A_q(\xi)|_{\xi_1=...=\xi_k}=\sum_{\beta:|\beta|_1=q}\begin{pmatrix} q \\ \beta\end{pmatrix}^2\xi_1^{\beta_1+...+\beta_k}\xi_{k+1}^{\beta_{k+1}}...\xi_m^{\beta_m}. \tag{199}$$

From (198) and (96) we have

$$-\bar{\ell}(\xi)|_{\xi_1=...=\xi_k}=(\ell(\xi)+2(q+1)A_q(\xi))|_{\xi_1=...=\xi_k}= \\ =-2\sum_{\alpha:|\alpha|_1=q+1;\alpha_1\geq 1}(\frac{\alpha_1}{q+1}\begin{pmatrix} q+1 \\ \alpha\end{pmatrix}^2-(q+1)\begin{pmatrix} q \\ \alpha_1-1,...,\alpha_m\end{pmatrix}^2)\xi_1^{\alpha_1+...+\alpha_k-1}\xi_{k+1}^{\alpha_{k+1}}...\xi_m^{\alpha_m}= \\ =-2\sum_{\alpha:|\alpha|_1=q+1;\alpha_1\geq 1}(\frac{\alpha_1}{q+1}(\frac{(q+1)!}{\alpha_1!...\alpha_m!})^2-(q+1)(\frac{q!}{(\alpha_1-1)!...\alpha_m!})^2)\xi_1^{\alpha_1+...+\alpha_k-1}\xi_{k+1}^{\alpha_{k+1}}...\xi_m^{\alpha_m}= \\ =2\sum_{\alpha:|\alpha|_1=q+1;\alpha_1>1}\frac{q!}{(\alpha_1-1)!...\alpha_m!}\frac{(q+1)!}{\alpha_1!...\alpha_m!}(\alpha_1-1)\xi_1^{\alpha_1+...+\alpha_k-1}\xi_{k+1}^{\alpha_{k+1}}...\xi_m^{\alpha_m}. \tag{200}$$

Hence $-\bar{\ell(\xi)}|_{\xi_1=...=\xi_k}$ is divisible by ξ_1. By (195) $r(\xi)|_{\xi_1=...=\xi_k} = \xi_1^{|n_1|+...+|n_k|}s$ is divisible by ξ_1. Then $(-\bar{\ell(\xi)} - r(\xi))|_{\xi_1=...=\xi_k}$ is divisible by ξ_1.By (182) and (195)we have:

(201)
$$\xi_1^{|n_1|}...\xi_k^{|n_k|}s(-\bar{\ell}(\xi)-r(\xi)) = c(\ell)^2 = \xi_1^{|n_1|}...\xi_k^{|n_k|}(\sum_{\alpha\in\mathbb{N}^m:|\ell^++\alpha|_1=q-1} \begin{pmatrix} q-1 \\ \ell^++\alpha \end{pmatrix} \begin{pmatrix} q+1 \\ \ell^-+\alpha \end{pmatrix} \xi^\alpha)^2$$
$$\implies s(-\bar{\ell}(\xi) - r(\xi)) = (\sum_{\alpha\in\mathbb{N}^m:|\ell^++\alpha|_1=q-1} \begin{pmatrix} q-1 \\ \ell^++\alpha \end{pmatrix} \begin{pmatrix} q+1 \\ \ell^-+\alpha \end{pmatrix} \xi^\alpha)^2$$

$$\implies s(-\bar{\ell}(\xi) - r(\xi)) = (\sum_{\alpha\in\mathbb{N}^m:|\ell^++\alpha|_1=q-1} \begin{pmatrix} q-1 \\ \ell^++\alpha \end{pmatrix} \begin{pmatrix} q+1 \\ \ell^-+\alpha \end{pmatrix} \xi^\alpha)^2. \tag{202}$$

So the right hand side of (202) when $\xi_1 = \xi_2 = ... = \xi_k$ must be divisible by ξ_1. But in fact:
- If $k = m$, then by remark 7.7

$$(\sum_{\alpha\in\mathbb{N}^m:|\ell^++\alpha|_1=q-1} \begin{pmatrix} q-1 \\ \ell^++\alpha \end{pmatrix} \begin{pmatrix} q+1 \\ \ell^-+\alpha \end{pmatrix} \xi^\alpha)^2 = \begin{pmatrix} q-1 \\ \ell^+ \end{pmatrix}^2 \begin{pmatrix} q+1 \\ \ell^- \end{pmatrix}^2$$

is a constant, not divisible by ξ_1.
- If $k < m$, take $\tilde{\alpha}$ such that $\tilde{\alpha}_1 = ... = \tilde{\alpha}_k = \tilde{\alpha}_{k+2} = ... + \tilde{\alpha}_m = 0, \tilde{\alpha}_{k+1} = q - 1 - \ell^+$ then the right hand side of (202) contains the monomial

$$\begin{pmatrix} q-1 \\ \ell^++\tilde{\alpha} \end{pmatrix}^2 \begin{pmatrix} q+1 \\ \ell^-+\tilde{\alpha} \end{pmatrix}^2 \xi_{k+1}^{2(q-1-|\ell^+|_1)}.$$

Hence the right hand side of (202) is not divisible by ξ_1.

(3) The case $A = \emptyset, B = \{1, ..., k\}$ is similar.

□

Part 4. **Appendix**

Abstract. *This part contains proofs of the facts related to the NLS and the Hamiltonian that we described in Section 1, and some useful definitions.*

9. Appendix: Proof of Remark 1.1

Proof. Let $u = \alpha\tilde{u}$. We have: $u_t = \alpha\tilde{u}_t, \triangle u = \alpha\triangle\tilde{u}, |u| = |\alpha||\tilde{u}|$, then (1) is equivalent to

$$-\mathrm{i}\alpha\tilde{u}_t + \alpha\triangle\tilde{u} = \kappa|\alpha|^{2q}\alpha|\tilde{u}|^{2q}\tilde{u} \tag{203}$$

Dividing 2 sides of (203) by α we get

$$-\mathrm{i}\tilde{u}_t + \triangle\tilde{u} = \kappa|\alpha|^{2q}|\tilde{u}|^{2q}\tilde{u} \tag{204}$$

Hence if we take α such that $|\alpha|^{2q} = (q+1)|\kappa|^{-1}$, then in (204) $\kappa|\alpha|^{2q} = (q+1)\kappa|\kappa|^{-1} = \pm(q+1)$. □

10. Appendix: Proof of Proposition 1

Proof. The Poisson bracket, associated to the symplectic form $i\sum_{k\in\mathbb{Z}^n} du_k \wedge d\bar{u}_k$, is:

$$\{f,g\} = -\mathrm{i}\sum_k \left(\frac{\partial f}{\partial u_k}\frac{\partial g}{\partial \bar{u}_k} - \frac{\partial f}{\partial \bar{u}_k}\frac{\partial g}{\partial u_k}\right) \tag{205}$$

We wish to find H so that

$$\dot{u} = \{H,u\} = \mathrm{i}\sum_k \left(-\frac{\partial H}{\partial u_k}\frac{\partial u}{\partial \bar{u}_k} + \frac{\partial H}{\partial \bar{u}_k}\frac{\partial u}{\partial u_k}\right) = \mathrm{i}\frac{\partial H}{\partial \bar{u}_k}\frac{\partial u}{\partial u_k} = \mathrm{i}\sum_k \frac{\partial H}{\partial \bar{u}_k} e^{\mathrm{i}(k,\varphi)}. \tag{206}$$

On the other hand from (3)

$$\dot{u} = \sum_k \dot{u}_k e^{\mathrm{i}(k,\varphi)} \tag{207}$$

From (206) and (207) we get

$$\dot{u}_k = \mathrm{i}\frac{\partial H}{\partial \bar{u}_k} \Leftrightarrow -\mathrm{i}\dot{u}_k = \frac{\partial H}{\partial \bar{u}_k} \forall k \in \mathbb{Z}^n \tag{208}$$

We have

$$\triangle u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial \varphi_j^2} = -\sum_{k\in\mathbb{Z}^n} u_k(t) e^{\mathrm{i}(k,\varphi)} \sum_{j=1}^{n} k_j^2 = -\sum_{k\in\mathbb{Z}^n} |k|^2 u_k(t) e^{\mathrm{i}(k,\varphi)} \tag{209}$$

$$\begin{aligned} |u|^{2q}u = u^{q+1}\bar{u}^q &= \Big(\sum_k u_k e^{\mathrm{i}(k,\varphi)}\Big)^{q+1}\Big(\sum_k \bar{u}_k e^{-\mathrm{i}(k,\varphi)}\Big)^q = \\ &= \sum_{k_1,\dots,k_{2q+1}} u_{k_1}\dots u_{k_{2q+1}}\bar{u}_{k_2}\dots\bar{u}_{k_{2q}} e^{\mathrm{i}(k_1-k_2+k_3-k_4+\dots+k_{2q-1}-k_{2q}+k_{2q+1},\varphi)}. \end{aligned} \tag{210}$$

From (2) we have

$$-\mathrm{i}u_t = -\mathrm{i}\sum_k \dot{u}_k e^{\mathrm{i}(k,\varphi)} = -\triangle u + (q+1)|u|^{2q}u \tag{211}$$

From (209)-(211) we get

$$-\mathrm{i}\dot{u}_k = |k|^2 u_k + (q+1) \sum_{\substack{k_1,\dots,k_{2q+1}\in\mathbb{Z}^n \\ k_1-k_2+k_3-k_4+\dots+k_{2q-1}-k_{2q}+k_{2q+1}=k}} u_{k_1}\dots u_{k_{2q+1}}\bar{u}_{k_2}\dots\bar{u}_{k_{2q}}. \tag{212}$$

From (208) and (212) we have

$$\frac{\partial H}{\partial \bar{u}_k} = |k|^2 u_k + (q+1) \sum_{\substack{k_1,\dots,k_{2q+1}\in\mathbb{Z}^n \\ k_1-k_2+k_3-k_4+\dots+k_{2q-1}-k_{2q}+k_{2q+1}=k}} u_{k_1}\dots u_{k_{2q+1}}\bar{u}_{k_2}\dots\bar{u}_{k_{2q}}.\forall k \in \mathbb{Z}^n \tag{213}$$

We can write:

$$\sum_{\substack{k_1,\dots,k_{2q+1}\in\mathbb{Z}^n \\ k_1-k_2+k_3-k_4+\dots+k_{2q-1}-k_{2q}+k_{2q+1}=k}} u_{k_1}\dots u_{k_{2q+1}}\bar{u}_{k_2}\dots\bar{u}_{k_{2q}} = \sum_{\substack{\alpha,\beta\in(\mathbb{Z}^n)^{\mathbb{N}}:|\alpha|=q+1,|\beta|_1=q, \\ \sum_l l(\alpha_l-\beta_l)=k}} \begin{pmatrix} q+1 \\ \alpha \end{pmatrix}\begin{pmatrix} q \\ \beta \end{pmatrix} u^\alpha \bar{u}^\beta. \tag{214}$$

Then

$$(q+1)\int \sum_{\substack{k_1,\dots,k_{2q+1}\in\mathbb{Z}^n\\ k_1-k_2+k_3-k_4+\dots+k_{2q-1}-k_{2q}+k_{2q+1}=k}} u_{k_1}...u_{k_{2q+1}}\bar{u}_{k_2}...\bar{u}_{k_{2q}}d\bar{u}_k = \tag{215}$$

$$= (q+1)\sum_{\substack{\alpha,\beta\in(\mathbb{Z}^n)^{\mathbb{N}}:|\alpha|=q+1,|\beta|_1=q,\\ \sum_l l(\alpha_l-\beta_l)=k}} \begin{pmatrix} q+1\\ \alpha\end{pmatrix}\begin{pmatrix} q\\ \beta\end{pmatrix} u^\alpha \frac{\bar{u}_k^{\beta_k+1}}{\beta_k+1}\prod_{i\neq k}\bar{u}_i^{\beta_i} =$$

$$= \sum_{\substack{\alpha,\tilde{\beta}\in(\mathbb{Z}^n)^{\mathbb{N}}:|\alpha|_1=|\tilde{\beta}|_1=q+1,\\ \sum_l l(\alpha_l-\tilde{\beta}_l)=0}} \begin{pmatrix} q+1\\ \alpha\end{pmatrix}\begin{pmatrix} q+1\\ \tilde{\beta}\end{pmatrix} u^\alpha \bar{u}^{\tilde{\beta}}.$$

where $\tilde{\beta}_i = \beta_i$ for $i \neq k$ and $\tilde{\beta}_k = \beta_k + 1$.

Hence

$$H = |k|^2 u_k\bar{u}_k + \sum_{\substack{\alpha,\tilde{\beta}\in(\mathbb{Z}^n)^{\mathbb{N}}:|\alpha|_1=|\tilde{\beta}|_1=q+1,\\ \sum_l l(\alpha_l-\tilde{\beta}_l)=0}} \begin{pmatrix} q+1\\ \alpha\end{pmatrix}\begin{pmatrix} q+1\\ \tilde{\beta}\end{pmatrix} u^\alpha \bar{u}^{\tilde{\beta}} + C \tag{216}$$

where $\frac{\partial C}{\partial \bar{u}_k} = 0$. If we compute $\frac{\partial H}{\partial \bar{u}'_k}$ for $k' \neq k$ by (213) and (216) we get $\frac{\partial C}{\partial \bar{u}'_k} = |k'|^2 u_{k'} \implies C = |k'|^2 u_{k'}\bar{u}_{k'} + C'$. If we continue this process for all k we get easily Formula (4). □

11. Appendix: The resultant and discriminant of polynomials

Definition 11.1. *Let $f(t) = a_nt^n + a_{n-1}t^{n-1} + ... + a_1t + a_0$ and $g(t) = b_mt^m + b_{m-1}t^{m-1} + ... + b_1t + b_0$ be two polynomials of degree n and m, respectively, with coefficients in an arbitrary field F. Suppose that in the algebraic closure of F f has n roots $\alpha_1, ..., \alpha_n$, g has m roots $\beta_1, ..., \beta_m$ (not necessary distinct). The resultant of f and g is*

$$R(f,g) = a_n^m b_m^n \prod_{i=1}^{n}\prod_{j=1}^{m}(\alpha_i - \beta_j). \tag{217}$$

Definition 11.2. *Let $f(t) = a_nt^n + a_{n-1}t^{n-1} + ... + a_1t + a_0$ be a polynomial of degree n with coefficients in an arbitrary field F. Suppose that in the algebraic closure of F f has n roots $\alpha_1, ..., \alpha_n$. The discriminant of f is:*

$$D(f) = a_n^{2n-2}\prod_{1\leq i<j\leq n}(\alpha_i - \alpha_j)^2. \tag{218}$$

There are well-known formulas for the resultant and the discriminant:

$$R(f,g) = det\begin{pmatrix}
a_n & a_{n_1} & a_{n-2} & ... & a_1 & a_0 & 0 & 0 & ... & 0\\
0 & a_n & a_{n-1} & ... & a_1 & a_0 & 0 & 0 & ... & 0\\
... & ... & ... & ... & ... & ... & ... & ... & ... & ...\\
0 & 0 & 0 & ... & 0 & a_n & ... & a_1 & a_0 & 0\\
0 & 0 & 0 & ... & 0 & 0 & a_n & ... & a_1 & a_0\\
b_m & b_{m-1} & b_{m-2} & ... & 0 & 0 & 0 & ... & 0 & 0\\
0 & b_m & b_{m-1} & ... & 0 & 0 & ... & ... & 0 & 0\\
... & ... & ... & ... & ... & ... & ... & ... & ... & ...\\
0 & 0 & ... & 0 & b_m & b_{m-1} & ... & b_1 & b_0 & 0\\
0 & 0 & ... & ... & 0 & b_m & ... & b_2 & b_1 & b_0
\end{pmatrix} \tag{219}$$

where the m first rows contain the coefficients $a_n, a_{n-1}, ..., a_0$ of f shifted $0, 1, ..., m-1$ steps and padded with zeros and the n last rows contain the coefficients $b_m, b_{m-1}, ..., b_0$

shifted $0, 1, ..., n-1$ steps and padded with zeros. In other words, the entry at (i, j) equals a_{n+i-j} if $1 \leq i \leq m$ and b_{i-j} if $m+1 \leq i \leq m+n$, with $a_i = 0$ if $i > n$ or $i < 0$ and $b_i = 0$ if $i > m$ or $i < 0$.

$$D(f) = (-1)^{n(n-1)/2} a_n^{-1} R(f, f') \text{for } n \geq 1 \tag{220}$$

12. Appendix: Genericity condition

Definition 12.1. *Given a list* $\mathcal{R} := \{P_1(y), \ldots, P_N(y)\}$ *of non–zero polynomials in* k *vector variables* y_i*, called* resonance polynomials*, we say that a list of vectors* $S = \{v_1, \ldots, v_m\}, v_i \in \mathbb{C}^n$ *is GENERIC relative to* $\mathcal{R}$ *if, for any list* $A = \{u_1, \ldots, u_k\}$ *such that* $u_i \in S$, $\forall i, u_i \neq u_j$*, the evaluation of the resonance polynomials at* $y_i = u_i$ *is non–zero.*

If m is finite this condition is equivalent to requiring that S (considered as a point in $\mathbb{C}^{nm}$) does not belong to the algebraic variety where at least one of the resonance polynomials is zero.

Example 12.1.

$$P_1(y_1, y_2, y_3) = (y_1 - y_2, y_1 - y_3)$$

means that we require

$$(v_i - v_j, v_i - v_k) \neq 0$$

for all $i \neq j \neq k$

In our specific case the required list of *the resonances,* $P_1(y), \ldots, P_N(y)$, are non–zero polynomials with integer coefficients depending on $d = 4q(n+1)$ vector variables $\zeta = (\zeta_1, \ldots, \zeta_d)$ with $\zeta_i = (\zeta_i^1, \ldots, \zeta_i^n)$. The explicit list of these resonances (see Definition 22 in [11]) depends on some non trivial combinatorics, nevertheless it is easy to give a (highly) redundant list of inequalities out of which the resonances appear. There is a constant $C > 0$ depending only on q, n so that we can take resonances the non-zero polynomials of the form:

- Linear inequalities: For all non-zero vectors $(a_1, ..., a_{4q(n+1)})$ with $a_i \in Z, |a_i| \leq C$ we require that
$$\sum_{i=1}^{4q(n+1)} a_i \zeta_i \neq 0.$$
- Quadratic inequalities : Let $(\zeta_i, \zeta_j) = \sum_{h=1}^{n} \zeta_i^h \zeta_j^h$ be the scalar product. For all non-zero matrices $\{a_{i,j}\}_{i,j=1}^{4q(n+1)}$ with $a_{i,j}\mathbb{Z}, |a_{i,j}| \leq C$ we requires
$$\sum_{i,j=1}^{4q(n+1)} a_{i,j} \zeta_i \zeta_j \neq 0.$$
- Determinantal inequalities: Consider n linear combinations u_h out of the list of elements $\mathcal{L} := \sum_{i=1}^{4q(n+1)} a_{h,i} \zeta_i, a_{h,i}\mathbb{Z}, |a_{h,i}| \leq C$.

  The determinantal resonances are contained in the list of the formally non-zero expressions of type $det(u_1, ..., u_n), u_i \in \mathcal{L}$.

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