On gradients of odd Theta functions in genus 2 and related topics

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Introduction

The theory of automorphic functions rose around the second half of the nineteenth century as the study of functions being invariant under the action of a group on their domain and consequently well defined on the quotient space.

The so-called elliptic functions, for which the first systematical exposition was given by Karl Weierstrass, are one of the simplest examples of automorphic functions. These are meromorphic functions of one variable with two independent periods and can be thus regarded as functions on the complex torus $\mathbb{C}/L$, where L is the lattice generated by the two periods [1].

A first classical generalization of this notion is suggested by working with functions defined on isomorphism classes of complex tori. Since a bijection exists between the isomorphism classes of complex tori and the points of the quotient space $\mathbb{H}/SL(2,\mathbb{Z})$, $\mathbb{H}$ being the complex upper half-plane [2], a remarkable example pertaining to the theory of automorphic functions is provided by holomorphic functions on $\mathbb{H}$, that are invariant under the action of $SL(2,\mathbb{Z})$; this is indeeed what is meant by modular function [3].

More generally, the classical theory of the so-called elliptic modular forms points to the study of holomorphic functions on $\mathbb{H}$, transforming with a multiplying factor satisfying the 1-cocycle condition under the action of a discrete subgroup of $SL(2,\mathbb{R})$. In the first half of the twentieth century Carl Ludwig Siegel was the first to generalize the elliptic modular theory to the case of more variable, by discovering some prominent examples of automorphic functions in several complex variables; these are named after him Siegel modular forms. In this theory ([F], [Kl], [VdG]), $\mathbb{H}$ is generalized by the upper half-plane $\mathfrak{S}_g := \{\tau \in Sym(g,\mathbb{C}) \mid Im\tau > 0\}$ and a transitive action of the symplectic group $Sp(2g,\mathbb{R})$ is defined on $\mathfrak{S}_g$, thus generalizing the action of $SL(2,\mathbb{R})$ on $\mathbb{H}$:

$$Sp(2g,\mathbb{R}) \times \mathfrak{S}_g \longrightarrow \mathfrak{S}_g$$

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tau\right) \to (A\tau + B)\cdot(C\tau + D)^{-1}$$

[1] A classical result completely describes these functions by means of the Weierstrass $\wp$-function and its derivative; the elementary theory is detailed, for instance, in [FB]

[2] See, for instance, [FB].

[3] A classical result states the field of modular functions is generated by the so-called absolute modular invariant, which was discovered by Felix Klein in 1879 (see Chapter 2, Example 2.3)

The arithmetic subgroup $\Gamma_g := Sp(2g, \mathbb{Z})$ plays particularly a notable geometrical role with respect to this action, as the points of the quotient space $A_g := \mathfrak{S}_g/\Gamma_g$ are set in correspondence with the isomorphism classes of principally polarized abelian varieties [4].

Modular forms can be used as coordinates in order to construct suitable projective immersions of level quotient spaces with respect to the action of remarkable subgroups of Γ_g. Riemann Theta functions with characteristics:

$$\theta_m(\tau, z) := \sum_{n\in\mathbb{Z}^g} e^{\pi i\left[{}^t\left(n+\frac{m'}{2}\right)\tau\left(n+\frac{m'}{2}\right)+2\,{}^t\left(n+\frac{m'}{2}\right)\left(z+\frac{m''}{2}\right)\right]}$$

are, in particular, are a good instrument at disposal of the theory, to construct suitable modular forms, since the so-called Theta constants:

$$\theta_m(\tau) := \theta_m(\tau, 0)$$

and the Jacobian determinants

$$D(n_1, \ldots, n_g)(\tau) := \frac{1}{\pi^g}\begin{vmatrix} \frac{\partial}{\partial z_1}\theta_{n_1}|_{z=0}(\tau) & \ldots & \frac{\partial}{\partial z_g}\theta_{n_1}|_{z=0}(\tau) \\ \vdots & & \vdots \\ \frac{\partial}{\partial z_1}\theta_{n_g}|_{z=0}(\tau) & \ldots & \frac{\partial}{\partial z_g}\theta_{n_g}|_{z=0}(\tau) \end{vmatrix}$$

turn out to be modular forms with respect to a technical level subgroup $\Gamma_g(4, 8) \subset \Gamma_g$. A remarkable map can be defined in particular on the level moduli space $A_g^{4,8} := \mathfrak{S}_g/\Gamma_g(4, 8)$

$$\mathbb{P}grTh : A_g^{4,8} \longrightarrow Gr_{\mathbb{C}}(g, 2^{g-1}(2^g - 1))$$

whose image lies in the Grassmannian of g- dimensional complex subspaces in $\mathbb{C}^{2^{g-1}(2^g-1)}$, the Jacobian determinants being its Plücker coordinates. This map is somehow a generalization of that studied by Lucia Caporaso and Edoardo Sernesi in [CS1], which sends a plane quartic to the set of its 28 bitangents.
In their paper [GSM] Samuel Grushevski and Riccardo Salvati Manni proved the map $\mathbb{P}grTh$ is generically injective whenever $g \geq 3$ and injective on tangent spaces when $g \geq 2$. The map is conjectured to be injective whenever $g \geq 3$, albeit it has not been proved yet. When $g = 2$, instead, the map:

$$\mathbb{P}grTh : A^{4,8} \longrightarrow \mathbb{P}^{14}$$

is known not to be injective.

In this thesis, a description is provided for the congruence subgroup Γ containing $\Gamma_2(4, 8)$ and being such that the map $\mathbb{P}grTh$ in genus 2 is still well defined

[4] See, for instance, [GH], [De] or [SU]

and also injective on the correspondent level moduli space $A_\Gamma := \mathfrak{S}_g/\Gamma$. Γ is also proved to be a normal subgroup of Γ_2 with no fixed points, the level moduli space A_Γ being thus smooth.
A structure theorem is also proved for the graded ring of modular forms $A(\Gamma)$ and for the ideal of cusp forms $S(\Gamma)$ (namely the modular forms that vanish on the boundary of Satake's compactification) with respect to Γ. Alas, a geometrical description still misses, due to the the not simple algebraic structure of $A(\Gamma)$ and $S(\Gamma)$.

Chapters 1 and 2 are designed to provide an outline of the basic theory; the reader is obviously referred to the bibliography for a deeper exposition of the topics.

Chapter 3 is mainly focused on Theta constants and related topics. Section 3.6 is particularly devoted to provide a new geometric description for a classical modular form of weight 30 in genus 2, being characterized by transforming with a non-trivial character under the action of Γ_2.

Chapter 4 is, finally, centered around the above mentioned results concerning the group Γ.

The author wishes to thank Professor Salvati Manni for his help and also seizes the opportunity to thank Professor Freitag for having performed some necessary computations by means of a Singular program devised by himself.

Notation

For each set A the symbol $|A|$ will stand for its cardinality. The notation $A \subset B$ will mean that A is a not necessarily proper subset of B; to state the set A is a proper subset of B, the symbol $A \subsetneq B$ will be, though, used.

For any associative ring with unity R, the symbol $M(g, R)$ will denote the $g \times g$ matrices, whose entries are elements of R, while the symbol $Sym_g(R)$ will stand for the symmetric $g \times g$ matrices.

For any field F, $GL(g, F)$ and $SL(g, F)$ will stand respectively for the general linear group of degree n and for the special linear group of degree n on F.

The symbol S_n will mean the symmetric group of order $n!$.

The exponential function $e^{\pi i w}$ will be also denoted by the symbol $\exp(w)$, where such a notation will be needed.

Chapter 1

The Siegel Upper Half-space and the Symplectic Group

A more exhaustive discussion around these topics can be found in Siegel's classical work [Si], as well as in Freitag's [F] and Klingen's [Kl] books, where a focus on Minkowski's reduction theory is also available. Most of the topics are also outlined in Namikawa's Lecture notes [Na], where Satake's and Mumford's compactifications are also dealt with.

1.1 The Symplectic Group $Sp(g, \mathbb{R})$

Definition 1.1. *The* **symplectic group of degree** *g is the following subgroup of $GL(2g, \mathbb{R})$:*

$$Sp(g, \mathbb{R}) := \left\{ \gamma \in M(2g, \mathbb{R}) \mid {}^t\gamma \cdot J_g \cdot \gamma = J_g \right\}$$

where:

$$J_g = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$$

is the so called **symplectic standard form** *of degree g.*

The symplectic group naturally arises as the group of automorphisms of the lattice $\mathbb{Z}^{2g}$, provided with the form J_g [1]; as it clearly turns out from the definition, it is an algebraic group.
A generic element of the symplectic group can be depicted in a standard block notation as:

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \qquad a_\gamma, b_\gamma, c_\gamma, d_\gamma \in M(g, \mathbb{R})$$

[1] More in general, a symplectic group $Sp(g, \Lambda)$ can be defined, whose elements are the linear transformations preserving a given non-degenerate skew-symmetric bilinear form Λ of degree 2; such transformations are in fact named symplectic transformations.

By expliciting the conditions describing the elements of the group, one gains:

$$\begin{aligned} Sp(g,\mathbb{R}) &= \left\{ \gamma \in GL(2g,\mathbb{R} \mid \begin{matrix} {}^t a_\gamma c_\gamma = {}^t c_\gamma a_\gamma \\ {}^t b_\gamma d_\gamma = {}^t d_\gamma b_\gamma \end{matrix} \;\; ; \;\; {}^t a_\gamma d_\gamma - {}^t c_\gamma b_\gamma = 1_g \right\} = \\ &= \left\{ \gamma \in Gl(2g,\mathbb{R}) \mid \gamma^{-1} = J_g^{-1}\, {}^t\gamma J_g = \begin{pmatrix} {}^t d_\gamma & -{}^t b_\gamma \\ -{}^t c_\gamma & {}^t a_\gamma \end{pmatrix} \right\} \end{aligned} \tag{1.1}$$

The symplectic group is self-evidently stable under the transposition $\gamma \to {}^t\gamma$; hence, an equivalent characterization follows by expliciting the conditions for the transposed element:

$$Sp(g,\mathbb{R}) = \left\{ \gamma \in GL(2g,\mathbb{R} \mid \begin{matrix} a_\gamma {}^t b_\gamma = b_\gamma {}^t a_\gamma \\ c_\gamma {}^t d_\gamma = d_\gamma {}^t c_\gamma \end{matrix} \;\; ; \;\; a_\gamma {}^t d_\gamma - b_\gamma {}^t c_\gamma = 1_g \right\} \tag{1.2}$$

A remarkable subgroup of $Sp(g,\mathbb{R})$ the theory mainly focuses on is the so-called Siegel modular group:

Definition 1.2. *The subgroup:*

$$\Gamma_g := Sp(g,\mathbb{Z}) = \left\{ \gamma \in Sp(g,\mathbb{R}) \mid a_\gamma, b_\gamma, c_\gamma, d_\gamma \in M(g,\mathbb{Z}) \right\} = Sp(g,\mathbb{R}) \cap M(2g,\mathbb{Z})$$

is called the **Siegel modular group** *of degree g* [2]*. When $g = 1$, $\Gamma_1 = SL(2,\mathbb{Z})$ is also known as the* **elliptic modular group** [3]

Proposition 1.1. *The set:*

$$S := \left\{ \; J_g \quad , \quad \left\{ \begin{pmatrix} 1_g & S \\ 0 & 1_g \end{pmatrix} \right\}_{S \in Sym_g(\mathbb{Z})} \; \right\}$$

is a set of generators for the modular group Γ_g.

Proof. For each $\eta \in \Gamma_1$ and $h = 1, \ldots g$, let $A^{(g)}_{\eta,h}$ be the $2g \times 2g$ matrix, whose entries are:

$$(A^{(g)}_{\eta,h})_{ij} = \begin{cases} a_\eta - 1 & \text{if} \quad i = j = h; \\ b_\eta & \text{if} \quad i = h, j = h + g; \\ c_\eta & \text{if} \quad i = h + g, j = h; \\ d_\eta - 1 & \text{if} \quad i = j = h + g; \\ 0 & \text{otherwise} \end{cases}$$

Then:

$$\gamma^{(g)}_{\eta,h} := 1_{2g} + A^{(g)}_{\eta,h} \in \Gamma_g \qquad \forall \eta \in \Gamma_1, \quad \forall h = 1, \ldots g$$

[2] Γ_g is then an example of arithmetic subgroup of $Sp(g,\mathbb{R})$

[3] A classical result states that $\Gamma_1/\{\pm 1\}$ is the group of biholomorphic automorphisms of the Riemann sphere $\hat{\mathbb{C}}$.

By multiplying $\gamma \in \Gamma_g$ by suitable elements of the type $\gamma^{(g)}_{\eta,h}$ from the left, an element $u \in GL_g(\mathbb{Z})$ is found to exist, being such that the $(g+1)$-th column of the matrix:

$$N_\gamma = \begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix} \prod_{\eta,h} \gamma^{(g)}_{\eta,h} \gamma$$

is made by the unit vector e_{g+1}. Since $\gamma \in \Gamma_g$, the first row of N_γ must be made by e_1. Then, by induction, the group Γ_g is seen to be generated by:

$$\left\{ \quad \{\gamma^{(g)}_{\eta,h}\}_{\eta,h} \quad , \left\{ \begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix} \right\}_{u \in GL_g(\mathbb{Z})} \quad , \quad \left\{ \begin{pmatrix} 1_g & S \\ 0 & 1_g \end{pmatrix} \right\}_{S \in Sym_g(\mathbb{Z})} \quad \right\}$$

However, the elements $\gamma^{(g)}_{\eta,h}$ are plainly checked to be generated by the other elements and J_g. The modular group itself is, therefore, generated by the set:

$$\left\{ \quad J_g \quad , \quad \left\{ \begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix} \right\}_{u \in GL_g(\mathbb{Z})} \quad , \quad \left\{ \begin{pmatrix} 1_g & S \\ 0 & 1_g \end{pmatrix} \right\}_{S \in Sym_g(\mathbb{Z})} \quad \right\}$$

Furthermore, for each $u \in GL_g(\mathbb{R})$ one has:

$$\begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix} = \begin{pmatrix} 1_g & u \\ 0 & 1_g \end{pmatrix} \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} \begin{pmatrix} 1 & u^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} \begin{pmatrix} 1_g & u \\ 0 & 1_g \end{pmatrix} \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$$

hence, the thesis follows. □

Corollary 1.1. *The modular group Γ_g is also generated by matrices of the form:*

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ 0 & d_\gamma \end{pmatrix} \quad , \quad {}^t\gamma^{-1} = \begin{pmatrix} d_\gamma & 0 \\ -b_\gamma & a_\gamma \end{pmatrix} \qquad {}^t b_\gamma d_\gamma = {}^t d_\gamma b_\gamma \ ; \ {}^t a_\gamma d_\gamma = 1_g$$

Proof. Thanks to Proposition 1.1, one only needs to check the set S is generated by such matrices. However, since the following identity holds:

$$J_g = \begin{pmatrix} 1_g & 0 \\ -1_g & 1_g \end{pmatrix} \begin{pmatrix} 1_g & 1_g \\ 0 & 1_g \end{pmatrix} \begin{pmatrix} 1_g & 0 \\ -1_g & 1_g \end{pmatrix}$$

the thesis plainly follows.

□

1.2 Congruence Subgroups of the Siegel Modular Group

This section aims to introduce remarkable subgroups of the Siegel modular group Γ_g, which turn out to be the natural setting to generalize the notion of modular function.

Definition 1.3. *For each $n \in \mathbb{N}$ let $\Gamma_g(n)$ be the kernel of the natural homomorphism $\Gamma_g \longrightarrow Sp(g, \mathbb{Z}/n\mathbb{Z})$, induced by the canonical projection. The group $\Gamma_g(n)$ is known as the* **principal congruence subgroup** *of level n.*

As a kernel of a group homomorphism, $\Gamma_g(n)$ is a normal subgroup of Γ_g. Moreover, since:

$$\Gamma_g(n) = \{\gamma \in \Gamma_g \mid \gamma \equiv 1_{2g} \bmod n\}$$

an immediate characterization is derived for this group:

$$\Gamma_g(n) = \left\{\gamma \in M(2g, \mathbb{Z}) \mid \gamma \equiv 1_{2g} + nM_\gamma\right\}$$

$$\text{with} \quad M_\gamma = \begin{pmatrix} a_M & b_M \\ c_M & d_M \end{pmatrix} \quad \text{s.t.} \quad \begin{cases} {}^t b_M = b_M + n({}^t d_M b_M - {}^t b_M d_M) \\ \\ {}^t c_M = c_M + n({}^t a_M c_M - {}^t c_M a_M) \\ \\ d_M + {}^t a_M + n({}^t a_M d_M - {}^t c_M b_M) = 0 \end{cases} \tag{1.3}$$

This subgroups are of finite index in the Siegel modular group; in particular, the following Lemma holds:

Lemma 1.1. *For each $n \in \mathbb{N}$:*

$$[\Gamma_g : \Gamma_g(n)] = n^{g(2g+1)} \prod_{p|n} \prod_{1 \le k \le g} \left(1 - \frac{1}{p^{2k}}\right)$$

Proof. A proof can be found in [Ko]. □

As concerns principal congruence subgroups with even level, some elementary lemmas can be stated:

Lemma 1.2. *If $\gamma \in \Gamma_g(2n)$, then $\gamma^2 \in \Gamma_g(4n)$.*

Proof. Let $\gamma_1 = 1_{2g} + 2nM_1, \gamma_2 = 1_{2g} + 2nM_2 \in \Gamma_g(2n)$ with reference to the notation introduced in (1.3). Then, one has:

$$\gamma_1\gamma_2 = 1_{2g} + 2n(M_1 + M_2 + 2nM_1M_2) \tag{1.4}$$

hence, the thesis follows. □

Lemma 1.3. *Whenever $\gamma \in \Gamma_g(2n)$, one has:*

$$diag({}^t a_\gamma c_\gamma) \equiv diag(c_\gamma) \bmod 4n \qquad diag({}^t b_\gamma d_\gamma) \equiv diag(b_\gamma) \bmod 4n$$

Proof. Using again the notation introduced in (1.3), the following simple chain of congruences is derived:

$$\begin{aligned} diag({}^t a_\gamma c_\gamma) &= diag[{}^t(1_g + 2na_M) \cdot 2nc_M] = diag[2nc_M + 4n^2({}^t a_M c_M)] = \\ &= 2n\, diag(c_M) + 4n^2 diag({}^t a_M c_M) \equiv 2n\, diag(c_M) \bmod 4n \end{aligned}$$

thus proving the first relation, as $c_\gamma = 2nc_M$. Likewise, the second relation is proved. □

Definition 1.4. *A subgroup $\Gamma \subset \Gamma_g$, such that $\Gamma_g(n) \subset \Gamma$ for some $n \in \mathbb{N}$ is called a* **congruence subgroup** *of level n.*

Congruence subgroups are all the subgroups of finite index in Γ_g. Remarkable examples of proper congruence subgroups, namely congruence subgroups being not principal, are given by the following family:

$$\Gamma_{g,0}(n) := \left\{ \gamma \in \Gamma_g \mid c_\gamma \equiv 0 \bmod n \right\} \tag{1.5}$$

These are plainly congruence subgroups of level n being not normal in Γ_g.

A notable family of congruence subgroups this work will mainly focus on, is the following one:

$$\begin{aligned}\Gamma_g(n,2n) &:= \left\{ \gamma \in \Gamma_g(n) \mid diag({}^t a_\gamma c_\gamma) \equiv diag({}^t b_\gamma d_\gamma) \equiv 0 \bmod 2n \right\} = \\ &= \left\{ \gamma \in \Gamma_g(n) \mid diag(a_\gamma {}^t b_\gamma) \equiv diag(c_\gamma {}^t d_\gamma) \equiv 0 \bmod 2n \right\}\end{aligned}$$

Since $\Gamma_g(2n) \subset \Gamma_g(n,2n)$, such subgroups are congruence subgroups of level $2n$. Furthermore, Lemma 1.3 implies the following characterization:

$$\Gamma_g(2n,4n) = \left\{ \gamma \in \Gamma_g(2n) \mid diag(b_\gamma) \equiv diag(c_\gamma) \equiv 0 \bmod 4n \right\} \tag{1.6}$$

Congruence subgroups as in (1.6) satisfy some remarkable properties:

Lemma 1.4. *If $\gamma \in \Gamma_g(2n,4n)$, then $\gamma^2 \in \Gamma_g(4n,8n)$.*

Proof. Let $\gamma = 1_{2g} + 2nM$ be as in (1.3); then, the thesis follows from (1.4), since $diag(b_M) \equiv diag(c_M) \equiv 0 \bmod 2$ by hypothesis. □

Proposition 1.2. *$\Gamma_g(2n,4n)$ is normal in Γ_g for each n. Furthermore:*

$$[\Gamma_g(2n) : \Gamma_g(2n,4n)] = 2^{2g}$$

Proof. Let $\gamma = 1_{2g} + 2nM$ be, as in (1.3), the generic element of $\Gamma_g(2n)$; then, for each n one can define the map:

$$\begin{array}{cccl} D_n & : & \Gamma_g(2n) & \longrightarrow \mathbb{Z}_2^g \times \mathbb{Z}_2^g \\ & & \gamma & \longrightarrow (diag(b_M) \bmod 2,\, diag(c_M) \bmod 2) \end{array}$$

Due to (1.4), D_n is a group homomorphism. Moreover, $\Gamma_g(2n,4n)$ is the kernel of D_n, the condition $D_n(\gamma) = 0$ being equivalent to $diag(b_\gamma) \equiv diag(c_\gamma) \equiv 0 \bmod 4n$; in particular, $\Gamma_g(2n,4n)$ is normal in $\Gamma_g(2n)$. To prove that $\Gamma_g(2n,4n)$ is normal in Γ_g, one has to show that $\eta\gamma\eta^{-1} \in KerD_n$ whenever $\gamma \in \Gamma_g(2n,4n)$ and $\eta \in \Gamma_g$. One has:

$$\eta\gamma\eta^{-1} = \begin{pmatrix} a_\eta & b_\eta \\ c_\eta & d_\eta \end{pmatrix} \left[\begin{pmatrix} 1_g & 0 \\ 0 & 1_g \end{pmatrix} + 2n \begin{pmatrix} a_M & b_M \\ c_M & d_M \end{pmatrix} \right] \begin{pmatrix} {}^t d_\eta & -{}^t b_\eta \\ -{}^t c_\eta & {}^t a_\eta \end{pmatrix} = 1_{2g} + 2n \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

where, for instance, $b' = a_\eta b_M {}^t a_\eta - b_\eta c_M {}^t b_\eta + {}^t(a_\eta {}^t d_M {}^t b_\eta) - (a_\eta a_M {}^t b_\eta)$; by (1.3) $a_M + {}^t d_m \equiv 0 \bmod 2n$, hence the following identity arises:

$$diag(b') \equiv a_\eta \cdot diag(b_M) + b_\eta \cdot diag(c_M) \bmod 2$$

Since $\gamma \in KerD_n$, then $diag(b') \equiv 0 \bmod 2$; the identity $diag(c') \equiv 0 \bmod 2$ is likewise gained
To prove the second part of the statement, one observes that D_n is also surjective. In fact, for $(\epsilon_1, \epsilon_2) \in \mathbb{Z}_2^g \times \mathbb{Z}_2^g$, by (1.3) $\gamma_1 \in \Gamma_g(2n)$ can be chosen in such a way that $a_{M_1} = c_{M_1} = d_{M_1} = 0$ and $diag(b_{M_1}) \equiv \epsilon_1 \bmod 2$, while $\gamma_2 \in \Gamma_g(2n)$ can be such that $a_{M_2} = b_{M_2} = d_{M_2} = 0$ and $diag(c_{M_2}) \equiv \epsilon_2 \bmod 2$. Then, $D(\gamma_1) = (\epsilon_1, 0)$ and $D(\gamma_2) = (0, \epsilon_2)$, hence the surjectivity of D_n follows, for $D(\gamma_1\gamma_2) = (\epsilon_1, \epsilon_2)$. Therefore, the following isomorphism is gained:

$$\Gamma_g(2n)/\Gamma_g(2n, 4n) \cong (\mathbb{Z}_2)^{2g}$$

hence $[\Gamma_g(2n) : \Gamma_g(2n, 4n)] = 2^{2g}$. □

Proposition 1.3. *$\Gamma_g(2n, 4n)/\Gamma_g(4n, 8n)$ is a $g(2g+1)$-dimensional vector space on $\mathbb{Z}_2$.*

Proof. Lemma 1.4 implies that each element in $\Gamma_g(2n, 4n)/\Gamma_g(4n, 8n)$, which differs from identity, has order 2. $\Gamma_g(2n, 4n)/\Gamma_g(4n, 8n)$ is, in particular, an abelian group. By Lemma 1.1 and Proposition 1.2, one immediately gains $[\Gamma_g(2n, 4n) : \Gamma_g(4n, 8n)] = 2^{g(2g+1)}$. Then:

$$\Gamma_g(2n, 4n)/\Gamma_g(4n, 8n) \cong \mathbb{Z}_2^{g(2g+1)} \tag{1.7}$$

which concludes the proof. □

Proposition 1.4. *For each couple of indices $1 \leq i, j \leq g$ one can denote by $\tilde{O}_{ij}$ the matrix $g \times g$, whose coordinates are $\tilde{O}_{ij}^{(hk)} = \delta_{ih}\delta_{jk}$. Then, the following matrices:*

$$A_{ij} := \begin{pmatrix} a_{ij} & 0 \\ 0 & {}^t a_{ij}^{-1} \end{pmatrix} \qquad (1 \leq i, j \leq g) \qquad \textit{with} \quad a_{ij} := \begin{cases} 1_g + 2\tilde{O}_{ij} & \textit{if } i \neq j \\ 1_g - 2\tilde{O}_{ij} & \textit{if } i = j \end{cases}$$

$$B_{ij} := \begin{pmatrix} 1_g & b_{ij} \\ 0 & 1_g \end{pmatrix} \qquad (1 \leq i \leq j \leq g) \qquad \textit{with} \quad b_{ij} := \begin{cases} 2\tilde{O}_{ij} + 2\tilde{O}_{ji} & \textit{if } i \neq j \\ 2\tilde{O}_{ij} & \textit{if } i = j \end{cases}$$

$$C_{ij} := {}^t B_{ij} \qquad (1 \leq i \leq j \leq g)$$

are a set of generators for $\Gamma_g(2)$.

Proof. A proof can be found in [I3]. □

As a consequence of Proposition 1.4 one has the following:

Corollary 1.2. *The $g(2g+1)$ elements A_{ij} (for $i, j \neq g$), B_{ij}, C_{ij} (for $i < j$), $B_{ii}^2, C_{ii}^2, -1_{2g}$ are a basis for the $\mathbb{Z}^2$ - vector space $\Gamma_g(2,4)/\Gamma_g(4,8)$.*

Proof. Thanks to the characterization (1.6), the elements described in the statement are plainly found to belong to $\Gamma_g(2,4)$. More precisely, such elements are checked to belong to distinct independent cosets of $\Gamma_g(4,8)$ in $\Gamma_g(2,4)$. Then, the thesis follows from Proposition 1.3. □

Another important family of remarkable congruence subgroups this work will focus on is defined by:

$$\Gamma_g(n,2n,4n) := \left\{\gamma \in \Gamma_g(2n,4n) \mid Tr(a_\gamma) \equiv g \bmod n\right\} \tag{1.8}$$

In the next section the main features of the action of these subgroups on a notable tube domain of the complex euclidean space will be reviewed.

1.3 The Action of $Sp(g,\mathbb{R})$ on the Siegel Upper Half-space

Definition 1.5. *The* **Siegel upper half-space** *of degree g is the following subset:*

$$\mathfrak{S}_g := \left\{\tau \in Sym_g(\mathbb{C}) \mid Im\tau > 0\right\} \tag{1.9}$$

It is self-evidently a generalization of the complex upper half-plane $\mathbb{H} := \{\tau \in \mathbb{C} \,|\, Im\tau > 0\} = \mathfrak{S}_1$.

An action of $Sp(g,\mathbb{R})$ can be defined on $\mathfrak{S}_g$ by means of biholomorphic automorphisms as a generalization of the classical action of $SL(2,\mathbb{R})$ on $\mathbb{H}$:

$$\begin{aligned} &Sp(g,\mathbb{R}) \times \mathfrak{S}_g \longrightarrow \mathfrak{S}_g \\ &(\gamma,\tau) \rightarrow (a_\gamma\tau + b_\gamma)\cdot(c_\gamma\tau + d_\gamma)^{-1} \end{aligned} \tag{1.10}$$

Proposition 1.5. *The action in (1.10) is well defined and transitive.*

Proof. For each $\gamma \in Sp(g,\mathbb{R})$ and $\tau \in Sym_g(\mathbb{C})$ the following identities are easily found to hold:

$${}^t(a_\gamma\tau + b_\gamma)(c_\gamma\tau + d_\gamma) - {}^t(c_\gamma\tau + d_\gamma)(a_\gamma\tau + b_\gamma) = \tau - {}^t\tau = 0 \tag{1.11}$$

$${}^t(a_\gamma\tau + b_\gamma)\overline{(c_\gamma\tau + d_\gamma)} - {}^t(c_\gamma\tau + d_\gamma)\overline{(a_\gamma\tau + b_\gamma)} = \tau - \overline{\tau} = 2i(Im\tau) \tag{1.12}$$

In order to be sure the expression in (1.10) is well defined, $c_\gamma\tau + d_\gamma$ must be proved to be invertible for each $\tau \in \mathfrak{S}_g$ and for each $\gamma \in Sp(g,\mathbb{R})$. Then, if $c_\gamma\tau + d_\gamma$ were such an element not being invertible, a nonzero vector $z \in \mathbb{C}^g$ would exist, such that $(c_\gamma\tau + d_\gamma)z = 0$; hence:

$$0 = {}^tz^t(a_\gamma\tau + b_\gamma)\overline{(c_\gamma\tau + d_\gamma)}\overline{z} = {}^tz^t(c_\gamma\tau + d_\gamma)\overline{(a_\gamma\tau + b_\gamma)}\overline{z}$$

and (1.12) would imply $2i^tz(Im\tau)\overline{z} = 0$, thus contradicting the hypothesis $\tau \in \mathfrak{S}_g$. Now, one needs to prove that $\gamma\tau := (a_\gamma\tau + b_\gamma)\cdot(c_\gamma\tau + d_\gamma)^{-1} \in \mathfrak{S}_g$ for each $\tau \in \mathfrak{S}_g$

and for each $\gamma \in Sp(g,\mathbb{R})$. Since $c_\gamma\tau + d_\gamma$ is invertible under this hypothesis, (1.11) is equivalent to $\gamma\tau \in Sym_g(\mathbb{C})$. This assertion and (1.12) imply:

$$\begin{aligned} Im(\gamma\tau) &= \frac{1}{2i}[{}^t(\gamma\tau) - (\overline{\gamma\tau})] = \\ &= \frac{1}{2i}{}^t(c_\gamma\tau + d_\gamma)^{-1}[{}^t(a_\gamma\tau + b_\gamma)(\overline{c_\gamma\tau + d_\gamma}) - {}^t(c_\gamma\tau + d_\gamma)(\overline{a_\gamma\tau + b_\gamma})](\overline{c_\gamma\tau + d_\gamma})^{-1} = \\ &= {}^t(c_\gamma\tau + d_\gamma)^{-1} \cdot Im\tau \cdot (\overline{c_\gamma\tau + d_\gamma})^{-1} \end{aligned}$$

from which $Im(\gamma\tau) > 0$ follows, since $\tau \in \mathfrak{S}_g$.
The law (1.10) is plainly checked to satisfy the properties defining action. Finally, thanks to the Cholesky decomposition (Corollary A.2), for each $\tau \in \mathfrak{S}_g$ there exists a matrix $u \in GL(g,\mathbb{R})$ such that $Im\tau = u^t u$. Hence:

$$\tau = \begin{pmatrix} 1_g & Re\tau \\ 0 & 1_g \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix} (-i1_g)$$

and also the transitivity of the action is proved. □

The action of the symplectic group on $\mathfrak{S}_g$ provides a complete description for the group $Aut(\mathfrak{S}_g)$ of the biholomorphic automorphisms of $\mathfrak{S}_g$:

Proposition 1.6. *$Sp(g,\mathbb{R})/\{\pm 1_g\} \cong Aut(\mathfrak{S}_g)$.*

Proof. The action (1.10) allows to define for each $\gamma \in Sp(g,\mathbb{R})$ the holomorphic maps $T_\gamma : \tau \to \gamma\tau$ on $\mathfrak{S}_g$ to itself. Each map T_γ is clearly invertible with inverse $T_{\gamma^{-1}}$; a group homomorphism is therefore defined:

$$\begin{array}{cccc} T : & Sp(g,\mathbb{R}) & \to & Aut(\mathfrak{S}_g) \\ & \gamma & \to & T_\gamma \end{array} \tag{1.13}$$

whose kernel is precisely $\{\pm 1_g\}$. The proof of the surjectivity of T is provided in [Si], by applying a generalized version of Schwartz lemma for several complex variables. □

Some remarkable properties of the Siegel upper half-space related to the action of the symplectic group can be stated here.

Proposition 1.7. *$\mathfrak{S}_g$ is a symmetric space.*

Proof. One needs to prove that each point of $\mathfrak{S}_g$ admits a symmetry. For such a purpose, one can consider the generator:

$$\gamma = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} \in Sp(g,\mathbb{R})$$

and the related holomorphic map T_γ as in (1.13).
T_γ is an involution of $\mathfrak{S}_g$, for $T_\gamma^2 = Id_{\mathfrak{S}_g}$. Moreover, $T_\gamma(i1_g) = i1_g$, hence T_γ is a symmetry for the point $i1_g \in \mathfrak{S}_g$. Since by Proposition 1.5 $Sp(g,\mathbb{R})$ acts transitively on $\mathfrak{S}_g$, for each $\tau \in \mathfrak{S}_g$ there exists $\eta \in Sp(g,\mathbb{R})$ such that $\tau = \eta i1_g$. $T_{\eta\gamma\eta^{-1}}$ is thus a symmetry for τ. □

To prove other important properties of this action, a classical result regarding the group action on topological spaces have to be recalled:

Theorem 1.1. *Let G be a second-countable locally compact Hausdorff topological group acting continuously and transitively on a locally compact Hausdorff topological space X. Then, for each $x \in X$ the homeomorphism $G/St_x \cong X$ holds between the two topological spaces.*

Proof. The following map:

$$\begin{aligned} T: \quad & G/St_x \to X \\ & gSt_x \to gx \end{aligned}$$

is clearly well defined and injective; it is also surjective, for the action of G is transitive. To prove T is indeed a homeomorphism, one needs to show the map $g \mapsto gx$, which is continuous by hypothesis, is actually an open map.
Let thus $U \subset G$ be an open set; $gU = \{gx \,|\, g \in U\}$ will be proved to be an open set. Then, let $gx \in gU$ and let V be a compact neighbourhood of the identity $e \in G$ such that $V^{-1} = V$ and $gV^2 \subset U$. Since G is second-countable, there exists a collection of elements $\{g_n\}_{n \in \mathbb{N}} \subset G$ such that $G = \cup_{n=1}^{\infty} g_n V$, hence $X = \cup_{n=1}^{\infty} g_n Vx$. For each n, the set $g_n Vx$ is a closed set, for it is compact in X; furthermore, the interiors of $g_n Vx$ can not be all empty, since X is a Baire space, for it is a locally compact Hausdorff space. Therefore, there exists $n_0 \in \mathbb{N}$ such that $g_{n_0} Vx$ has interior points, hence the interior of Vx is not empty, Vx being homeomorphic to $g_{n_0} Vx$. Then, let $x_0 \in X$ such that $x_0 x \in Int(g_{n_0} Vx)$; one has:

$$gx \in gx_0^{-1} Int(g_{n_0} Vx) \subset gV^2 x \subset Ux$$

hence gx is an interior point of gU; since each point of gU is interior, gU is an open set, which concludes the proof. □

Proposition 1.8. *$\mathfrak{S}_g$ is a homogeneous space.*

Proof. The subgroup:

$$U(g) = \left\{ \gamma \in Sp(g, \mathbb{R}) \,|\, d_\gamma = a_\gamma,\ c_\gamma = -b_\gamma,\ a_\gamma{}^t a_\gamma + b_\gamma{}^t b_\gamma = 1_g \right\}$$

is checked to be the stabilizer St_{i1_g} of the point $i1_g$. As a consequence of Theorem 1.1 one gains, therefore, the homeomorphism:

$$\mathfrak{S}_g \cong Sp(g, \mathbb{R})/U(g) \tag{1.14}$$

□

In general, discrete subgroups on homogeneous spaces act as follows:

Proposition 1.9. *Let $X \cong G/K$ a homogeneous space. Then, each discrete subgroup of G acts properly discontinuously on X.*

Proof. It follows from the fact $\pi : G \to G/K$ is a proper map. □

The following Corollary for the action of the Siegel modular group is thus immediately derived:

Corollary 1.3. *The Siegel modular group Γ_g acts properly discontinuously on $\mathfrak{S}_g$ by (1.10).*

Proof. Since Γ_g is a discrete subgroup of $Sp(g, \mathbb{R})$, the statement follows by plainly applying Proposition 1.9. □

Siegel provided in [Si] an explicit description for a fundamental domain for the action of Γ_g on $\mathfrak{S}_g$:

$$\mathfrak{F}_g = \left\{ \tau \in \mathfrak{S}_g \;\middle|\; \begin{array}{l} n\, Im\tau\, {}^t n \geq Im\tau_{kk} \quad \forall\, n = (n_1, \cdots, n_g) \in \mathbb{Z}^g \ k\text{-admissible} \\ Im\tau_{k,k+1} \geq 0 \quad \forall\, k \\ |\det(c_\gamma \tau + d_\gamma)| \geq 1 \quad \forall\, \gamma \in \Gamma \\ |Re\tau| \leq 1/2 \end{array} \right\}$$

where $n = (n_1, \cdots, n_g) \in \mathbb{Z}^g$ is called k-admissible for $1 \leq k \leq g$ whenever $n_k, \cdots, n_g$ are coprime. This domain is known as the **Siegel fundamental domain** of degree g, and will be henceforward denoted by the symbol $\mathfrak{F}_g$ [4].

Example 1.1. *The Siegel fundamental domain in the case $g = 1$ is:*

$$\mathfrak{F}_1 = \{\tau \in \mathbb{H} \mid |Re\tau| \leq 1/2, |\tau| \geq 1\}$$

The following property is a remarkable consequence of Corollary 1.3:

Corollary 1.4. *The coset space $A_g := \mathfrak{S}_g/\Gamma_g$ admits a normal analytic space structure.*

Proof. It is a straightforward application of Cartan's Theorem about the existence of an analytic space structure for quotients by the action of a finite group (cf. [Ca]). □

The coset space $A_g := \mathfrak{S}_g/\Gamma_g$ plays an outstanding role in the theory of abelian varieties, for its points can be set in a one-to-one correspondence with the classes of isomorphic principally polarized abelian varieties (see [De] and [GH]).

Some Lemmas will be outlined here to prove a useful property of the Siegel fundamental domain $\mathfrak{F}_g$, that is to say it is contained in generalized vertical strip.

Lemma 1.5. *Whenever $\tau \in \mathfrak{F}_g$, one has:*

1. $Im\tau_{kk} \leq Im\tau_{k+1,k+1} \qquad \forall k \in \{1, \ldots, g-1\}$
2. $|2Im\tau_{kl}| \leq Im\tau_{kk} \qquad \forall k < l$
3. $\exists\, c > 0$ *such that:*

$$\det Im\tau \leq \prod_{i=1}^{g} Im\tau_{ii} \leq c \det Im\tau$$

[4] The conditions $Im\tau_{k,k+1} \geq 0$ and $n Im\tau^t n \geq Im\tau_{kk}$ for each n which is k-admissible are traditionally expressed by stating the matrix $Im\tau$ is reduced in the sense of Minkowski (or, equivalently, it belongs to a Minkowski reduced domain)

Proof. Let $1 \le k \le g-1$ be fixed. Since $n\, Im\tau^t n \ge Im\tau_{kk}$ for each k-admissible n, condition 1. is obtained by choosing $n = e_{k+1}$. By setting $n = e_k \pm e_l$ for each l such that $k < l \le g$, one has:

$$Im\tau_{kk} + Im\tau_{l,l} \pm (Im\tau_{kl} + Im\tau_{l,k}) = Im\tau_{kk} + Im\tau_{l,l} \pm 2Im\tau_{kl} \ge Im\tau_{kk}$$

hence condition 2. is also fulfilled. Finally, condition 3. is a consequence of Hermite inequality, which holds for real positive definite matrices $M \in Sym_n(\mathbb{R})$:

$$\min_{\substack{k\in\mathbb{Z}^n\\ k\neq 0}} {}^t kMk \le c \det M^{\frac{1}{n}}$$

where $c > 0$ is a constant depending only on n (see [Kl] for details). □

By using Proposition A.4, the following technical statement can be derived by Lemma 1.5 (cf. [Kl]):

Lemma 1.6. *For each $\tau \in \mathfrak{F}_g$ let $Im\tau^D$ be the diagonal matrix*

$$Im\tau^D := \begin{pmatrix} Im\tau_{11} & 0 & \cdots & 0 \\ 0 & Im\tau_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & Im\tau_{gg} \end{pmatrix} \tag{1.15}$$

Then, there exists $c > 0$ such that:

$$c\, Im\tau - Im\tau^D > 0$$

Thanks to this Lemma, the Siegel fundamental domain $\mathfrak{F}_g$ is proved to be contained in a generalized vertical strip:

Lemma 1.7. *There exists $\lambda > 0$ such that:*

$$\mathfrak{F}_g \subset \{\tau \in \mathfrak{S}_g \mid Im\tau - \lambda 1_g \ge 0\}$$

Proof. Let $\tau \in \mathfrak{F}_g$, $Im\tau^D$ as in (1.15) and η the element:

$$\eta := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1_{g-1} & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{g-1} \end{pmatrix} \in \Gamma_g$$

Since $\tau \in \mathfrak{F}_g$, then, in particular, $|\det(c_\eta\tau + d_\eta)| = |\tau_{11}| > 1$; moreover, $|Re\tau_{11}| \le 1/2$, hence $Im\tau_{11} \ge \sqrt{3}/2$. Since for construction $Im\tau^D_{ii}$ is diagonal and $Im\tau^D_{ii} = Im\tau_{ii}$ for each i, (by condition 1. in Lemma 1.5), one has $(Im\tau^D - \lambda_1 1_g) \ge 0$ with $\lambda_1 := \sqrt{3}/2$. Then, let $c > 0$ be as in Lemma 1.6; by setting $\lambda = \lambda_1 c^{-1}$, one has $Im\tau - \lambda 1_g = (Im\tau - c^{-1}Im\tau^D) + (c^{-1}Im\tau^D - \lambda 1_g) \ge 0$, and the statement is proved. □

Regarding level moduli spaces, it is important to recall a notable family of such spaces admits a complex structure:

Proposition 1.10. *Let $n \geq 3$. The action $\Gamma_g(n)$ on the Siegel upper half-space $\mathfrak{S}_g$ is free; $\mathfrak{S}_g/\Gamma_g(n)$ is, therefore, a $g(g+1)/2$-dimensional complex manifold.*

Proof. A proof can be found in [Se]. □

The following properties can be particularly derived:

Proposition 1.11. *Let $\gamma \in \Gamma_g(4,8)$ an element having fixed points on $\mathfrak{S}_g$; then $\gamma = 1_g$. In particular, the so-called level moduli space $A_g^{4,8} := \mathfrak{S}_g/\Gamma_g(4,8)$ is smooth.*

Proposition 1.12. *An element $\gamma \in \Gamma_g$ having fixed point on $\mathfrak{S}_g$, is of finite order.*

A useful consequence is expressed by the following:

Corollary 1.5. *An element $\gamma \in \Gamma_g(2,4)$ having fixed points on $\mathfrak{S}_g$, has order 2.*

Proof. Let $\gamma \in \Gamma_g(2,4)$ such an element. By Proposition 1.12, γ is of finite order; moreover, $\gamma^2 \in \Gamma_g(4,8)$ by Lemma 1.4; then, Proposition 1.11 implies $\gamma^2 = 1$. □

Chapter 2

Siegel Modular Forms

2.1 Definition and Examples

This section aims to introduce a brief overview on the basic aspects of the theory of Siegel modular forms.
For a detailed exposition of this topic, Freitag's book [F] is one of the main references. Klingen's introductory book [Kl] and Van der Geer's lectures [VdG] are also important references to be quoted, together with Mumford's lectures [Mf] and the Andrianov and Zhurav's book [An]. Other works concerning with this topic can be found in the ending bibliography.

Definition 2.1. *Let $k \in \mathbb{Z}$ and let Γ be a congruence subgroup. A classical* **Siegel modular form** *of weight k with respect to Γ is a function $f : \mathfrak{S}_g \to \mathbb{C}$, satisfying the following conditions:*

1. *f is holomorphic on $\mathfrak{S}_g$*
2. $f(\gamma\tau) = \det(c_\gamma \tau + d_\gamma)^k f(\tau) \qquad \forall \gamma \in \Gamma, \quad \forall \tau \in \mathfrak{S}_g$
3. *When $g = 1$, the function $\tau \mapsto \det(c_\gamma\tau + d_\gamma)^{-k} f(\gamma\tau)$ is bounded on $\mathfrak{F}_1$ for each $\gamma \in \Gamma_1$* [1]

For any fixed $k \in \mathbb{Z}$ the function:

$$(\gamma|_k f)(\tau) := \det(c_{\gamma^{-1}}\tau + d_{\gamma^{-1}})^{-k} f(\gamma^{-1}\tau) \tag{2.1}$$

can be defined for each $\gamma \in Sp(g, \mathbb{R})$. Since:

$$det(c_{\gamma\gamma'}\tau + d_{\gamma\gamma'})^k = det(c_\gamma \gamma'\tau + d_\gamma)^k det(c_{\gamma'}\tau + d_{\gamma'})^k \qquad \forall \gamma, \gamma' \in Sp(g, \mathbb{R})$$

one has:

$$\gamma|_k(\gamma'|_k f) = \gamma\gamma'|_k f \qquad 1_{2g}|_k f = f$$

[1] This condition, which is equivalent to the request for the function f to be holomorphic on ∞ when $\Gamma = \Gamma_1$, is indeed redundant when $g > 1$ (see Corollary 2.1)

Hence (2.1) defines an action of $Sp(g,\mathbb{R})$ on the space of the holomorphic function on $\mathfrak{S}_g$ [2]
By means of the notation introduced in (2.1), the conditions appearing in Definition 2.1 can be translated into the following ones:

1. f is holomorphic on $\mathfrak{S}_g$;
2. $\gamma^{-1}|_k f = f \qquad \forall \gamma \in \Gamma$
3. When $g = 1$, $\gamma^{-1}|_k f$ is bounded on $\mathfrak{F}_1$ for each $\gamma \in \Gamma_1$

Example 2.1. (Eisenstein series of weight $k \geq 3$) *For each $k \geq 3$ the series:*

$$E_k(\tau) := \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ MCD(c,d)=1}} \frac{1}{(c\tau + d)^k}$$

converges absolutely. Moreover it uniformly converges on the compact sets of the complex upper half-plane $\mathbb{H}$; therefore, when $k \geq 3$ the series E_k defines a holomorphic function, which is easily seen to be a modular form of weight k with respect to the Siegel modular group (see, for instance, [DS]). This modular form is called the Eisenstein series of weight k.

Example 2.2. (Generalized Eisenstein series) *The series*

$$E_k^{(g)}(\tau) := \sum_{\substack{c,d\in Sym_g(\mathbb{Z}) \\ c,d\ coprime}} \det(c\tau + d)^{-k}$$

is seen to be uniformly convergent on each compact and also a modular form with respect to the Siegel modular group Γ_g of weight k whenever $k > g + 1$ (see, for instance, [An]); hence, it defines a modular form which is the generalization of that introduced in Example 2.1.

Besides Eisenstein series other remarkable Siegel modular forms can be constructed by means of the so-called Theta constants, as shown in Chapter 3.

The additional condition in case $g = 1$ is not redundant, as the following counterexample shows:

Example 2.3. (The absolute modular invariant J). *By setting:*

$$e_4 := 60E_4 \qquad\qquad e_6 := 140E_6$$

[2] More in general, if M is a complex manifold on which a group G acts biholomorphically, a non vanishing function $R : G \times M \to \mathbb{C}$, which is holomorphic on M, is called a factor of automorphy if:

$$R(gg', p) = R(g, g'p)R(g', p) \qquad \forall g, g' \in G, \quad \forall p \in M$$

Then, for each function f which is holomorphic on M, one can set:

$$(g \cdot f)(p) := R(g^{-1}, p)^{-1} f(g^{-1}p)$$

and an action of G turns out to be thus defined on the space of holomorphic functions on M.

where E_4 and E_6 are the Eisenstein series respectively of weight 4 and 6, as described in Example 2.1, one can define the modular invariant by:

$$J(\tau) := 1728\frac{e_4^3(\tau)}{\Delta(\tau)}$$

where $\Delta(\tau) := (e_4^3(\tau) - 27e_6^2(\tau))$. The function J clearly verifies condition 1. and condition 2. in Definition 2.1 for $k = 0$ and $\Gamma = \Gamma_g$. However, J is not bounded on the Siegel fundamental domain $\mathfrak{F}_1$ (cf. Example 1.1), since:

$$\lim_{t\to\infty} |J(it)| = \infty$$

Condition 3. does not hold therefore for J.

Henceforward, this work will refer to classical Siegel modular forms simply as modular forms.

The set of modular forms of weight k with respect to a congruence subgroup Γ is naturally provided with a complex vector space structure; throughout this work this complex vector space will be denoted by $A_k(\Gamma)$.

Definition 2.2. *Let Γ be a congruence subgroup. The graded ring $A(\Gamma) = \bigoplus_{k\in\mathbb{Z}} A_k(\Gamma)$ is called the* **ring of the modular forms with respect to** Γ.

As it will be proved in the following Section, this ring is a positively graded ring. Due to the definition, $A(\Gamma') \subset A(\Gamma)$ whenever $\Gamma' \subset \Gamma$.

In Sections 3.5 and 3.7 some remarkable examples of rings of modular forms will be reviewed.

This Section concludes by noting that for each weight k and for each character χ of Γ'/Γ, where Γ is a normal subgroup of Γ', a complex vector space is defined:

$$A_k(\Gamma, \chi) := \{f \in O(\mathfrak{S}_g) \mid f(\gamma\tau) = \chi(\gamma)det(c_\gamma\tau + d_\gamma)^k f(\tau) \;\; \forall\gamma \in \Gamma\}$$

the symbol $O(\mathfrak{S}_g)$ standing for the space of holomorphic functions on $\mathfrak{S}_g$, also satisfying condition 3. in Definition 2.1. If Γ is a fixed subgroup of the Siegel modular group Γ_g, and $\Gamma' \subset \Gamma_g$ is such that Γ is a normal subgroup of Γ', the transformation properties of the Siegel modular forms under the action of Γ' induce a decomposition of the homogeneous part $A_k(\Gamma)$ of the ring $A(\Gamma)$:

$$A_k(\Gamma) = \bigoplus_{\chi\in\hat{G}} A_k(\Gamma', \chi) \tag{2.2}$$

where $\hat{G}$ is the group of characters of Γ'/Γ.

2.2 Fourier Series of a Modular Form

First and foremost, one needs to recall a Laurent series exapansion can be obtained for holomorphic functions on Reinhardt's domains as a consequence of Cauchy's formula in several complex variables [3]; in particular, one has:

Theorem 2.1. *Let f be a holomorphic function on a Reinhardt domain R. Then, for each z in the product of annuli $A(r_1, a_1) \times \cdots \times A(r_n, a_n) \subset R$, one has:*

$$f(z) = \sum_{k_1,\dots k_n=-\infty}^{\infty} c_{k_1,\dots k_n}(z_1 - a_1)_1^k \cdots (z_n - a_n)_n^k$$

with:

$$c_{k_1,\dots k_n} = \left(\frac{1}{2\pi i}\right)^n \int_{\partial \overline{D}_{r_n}} \cdots \int_{\partial \overline{D}_{r_1}} \frac{f(\xi_1, \dots \xi_n)}{(\xi_1 - a_1)^{k_1+1} \cdots (\xi_n - a_n)^{k_n+1}} d\xi_1 \cdots d\xi_n$$

and the series converges absolutely and uniformely on the compact sets contained in the cartesian product of annuli $A(r_1, a_1) \times \cdots \times A(r_n, a_n)$ [4]

Definition 2.3. *A matrix $N \in Sym_g(\mathbb{Q})$ is called a half-integer matrix whenever $2N \in Sym_g(\mathbb{Z})$ and $diag(2N) \equiv 0 \, mod 2$.*

Henceforth the symbol $Sym_g^s(\mathbb{Q}) \subset Sym_g(\mathbb{Q})$ will conventionally denote the set of half-integer matrices.

Proposition 2.1. *Let be $n \in \mathbb{N}$ and let $f : \mathfrak{S}_g \to \mathbb{C}$ be a holomorphic function such that $f(\tau + nN) = f(\tau)$ for each $N \in Sym_g(\mathbb{Z})$. Then, the function f admits a Fourier expansion:*

$$f(\tau) = \sum_{N \in Sym_g^s(\mathbb{Q})} a(S) \, e^{\frac{2\pi i}{n} Tr(N\tau)} \tag{2.3}$$

with coefficients:

$$a(N) = \int_{[0,n]^K} f(x + iy) e^{\frac{2\pi i}{n} Tr[N(x+iy)]} dx \qquad \forall y > 0 \tag{2.4}$$

where $K = \frac{g(g+1)}{2}$. In particular, the series (2.3) converges absolutely on $\mathfrak{S}_g$ and uniformly on each compact in $\mathfrak{S}_g$.

Proof. One can consider the holomorphic map:

$$\begin{aligned} e_n : \quad & \mathfrak{S}_g \to \mathbb{C}^N \\ & \tau \mapsto \{e^{\frac{2\pi}{n} \tau_{ij}}\}_{i \le j} \end{aligned} \tag{2.5}$$

[3] See, for instance, [Ra]

[4] More in general, the convergence domain of the Laurent series expansion is a logaritmically convex and relatively complete Reinhardt's domain. See, for instance, [GR], [O], or [Sh]

Due to the definition of e_n, the range $A := \mathrm{Ran}\, e_n$ is a Reinhardt's domain. Moreover, for a fixed $q \in A$, $\tau_1, \tau_2 \in e^{-1}(q)$ if and only if $\tau_1^{(i,j)} - \tau_2^{(i,j)} \in \mathbb{Z}$ for each (i, j); since the periodicity condition on f implies, in particular, that f is periodic with period n in each variable $\tau^{(i,j)}$, a function $g : A \to \mathbb{C}$ is well defined by:

$$g(q) := \hat{f}(\hat{e}_n{}^{-1}(q)) \qquad \forall q \in A$$

where $\hat{f}$ e $\hat{e}_n$ are the functions respectively induced on the quotient by f and e_n:

$$\hat{f} : \mathfrak{S}_g/Sym_g(\mathbb{Z}) \to \mathbb{C} \qquad \hat{e}_n : \mathfrak{S}_g/Sym_g(\mathbb{Z}) \to A$$

The function g satisfies $g \cdot e_n = f$ and is, therefore, a holomorphic function. In particular, g admits a Laurent expansion on each product of annuli contained in A (Theorem 2.1):

$$g(q) = \sum_{n_1 \dots n_K = -\infty}^{\infty} c_{n_1 \dots n_K} q_1^{n_1} \cdots q_K^{n_K} \tag{2.6}$$

with:

$$\begin{aligned} c_{n_1,\dots n_K} &= \left(\frac{1}{2\pi i}\right)^K \int_{\partial \overline{D}_{r_K}} \cdots \int_{\partial \overline{D}_{r_1}} \frac{g(\xi_1, \dots \xi_K)}{\xi_1^{n_1+1} \cdots \xi_K^{n_K+1}} d\xi_1 \cdots d\xi_K = \\ &= \int_{[0,n]^K} g(r_1 e^{\frac{2\pi i}{n} t_1}, \dots, r_K e^{\frac{2\pi i}{n} t_K}) \left(\prod_{j=1}^{K} r_j^{-n_j} e^{\frac{2\pi i}{n} n_j t_j} \right) dt_1 \cdots dt_K \end{aligned}$$

Hence, by setting:

$$r_j = e^{\frac{2\pi}{n} y_j}; \qquad x_j = t_j; \qquad N = \begin{pmatrix} n_1 & \frac{n_2}{2} & \cdots & \frac{n_g}{2} \\ \frac{n_2}{2} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{n_g}{2} & \cdots & \cdots & n_K \end{pmatrix};$$

one has:

$$\begin{aligned} c_{n_1,\dots n_K} &= \int_{[0,n]^K} g(e^{\frac{2\pi i}{n}(x_1 + i y_1)}, \dots, e^{\frac{2\pi i}{n}(x_K + i y_K)})\, e^{\frac{2\pi i}{n} \sum_{j=1}^{K} n_j (x_j + i y_j)} dx_1 \cdots dx_K = \\ &= \int_{[0,n]^K} f(x + iy) e^{\frac{2\pi i}{n} Tr(S\tau)} dx_1 \cdots dx_K \end{aligned}$$

Since $g \cdot e_n = f$, (2.6) implies (2.3) with $a(N) = c_{n_1 \dots n_K}$ and one is supplied with the desired properties of convergence by Theorem 2.1. □

Theorem 2.2. (Götzky-Koecher Principle)[5] *Let $\Gamma \subset \Gamma_g$ be a congruence subgroup and let $n > 1$ be such that $\Gamma_g(n) \subset \Gamma$. If $f \in A_k(\Gamma)$, then f admits the following Fourier expansion:*

$$f(\tau) = \sum_{\substack{N \in Sym_g^s(\mathbb{Q}) \\ N \geq 0}} a(N)\, e^{\frac{2\pi i}{n} Tr(N\tau)} \tag{2.7}$$

with coefficients $a(N)$ as in (2.4).

[5] The following basic property was discovered by Götzky in 1928 for particular Hilbert modular forms. In 1954 Koecher provided a general demonstration ([Ko]).

Proof. Since:

$$\gamma_S^{(n)} := \begin{pmatrix} 1_g & nS \\ 0_g & 1_g \end{pmatrix} \in \Gamma \qquad \forall S \in Sym_g(\mathbb{Z})$$

in particular, one has $f(\tau + nS) = f(\gamma_S^{(n)}\tau) = f(\tau)$ for each $\tau \in \mathfrak{S}_g$ (namely f is periodic with period n in each variable $\tau^{(i,j)}$); therefore, the function f admits a Fourier expansion as in (2.3), for it fulfills the hypothesis in Proposition 2.1. Furthermore, the series in (2.3) absolutely converges for each $\tau \in \mathfrak{S}_g$; by evaluating it on $\tau = i1_g$, the following convergent numerical series is gained:

$$\sum_{N \in Sym_g^s(\mathbb{Q})} |a(N)| e^{-\frac{2\pi}{n} Tr(N)}$$

Therefore, there exists a constant $C \geq 0$ such that:

$$|a(N)| \leq C e^{\frac{2\pi}{n} Tr(N)} \qquad \forall N \in Sym_g^s(\mathbb{Q}) \tag{2.8}$$

A peculiar transformation formula also holds for the Fourier coefficients $a(N)$; thanks to the modularity condition stated on f, a matrix $u \in Gl(g, \mathbb{Z})$ such that $u \equiv 1_g \bmod n$ is found to satisfy for each $N \in Sym_g^s(\mathbb{Q})$ the identity:

$$a({}^t uNu) = det(u)^k a(N) \tag{2.9}$$

In fact, by defining for such a matrix u the element:

$$\gamma_u := \begin{pmatrix} {}^t u & 0 \\ 0 & u^{-1} \end{pmatrix} \in \Gamma_g(n)$$

one has:

$$\begin{aligned} a({}^t uNu) &= \int_{[0,n]^K} f(x+iy) e^{\frac{2\pi i}{n} Tr[{}^t uNu(x+iy)]} dx = \\ &= det(u)^k \int_{[0,n]^K} f(\gamma_u(x+iy)) e^{\frac{2\pi i}{n} Tr[{}^t uNu(x+iy)]} dx = \\ &= det(u)^k \int_{[0,n]^K} f({}^t u(x+iy)u) e^{\frac{2\pi i}{n} Tr[Nu(x+iy){}^t u]} dx = det(u)^k a(N) \end{aligned}$$

Now, the estimate (2.8) and the transformation law (2.9) can be used to show that $a(N) = 0$ for each $N \in Sym_g^s(\mathbb{Q})$ which is not positive semi-definite. Let, then, $N \in Sym_g^s(\mathbb{Q})$ be non positive semi-definite; then, there exists a primitive vector $v \in \mathbb{Z}^g$ such that ${}^t vNv < 0$. Thanks to Corollary A.1, the vector v can be completed to a matrix $u' \in GL(g, \mathbb{Z})$; in particular, the $(1,1)$-entry of the matrix ${}^t u'Nu'$ is negative, due to the choice of the vector v. By elementary operations on the columns of u', one can obtain a new matrix u with $\det u' = detu = 1$, in such a way that $u \equiv 1_g \bmod n$ and the matrix $N' := {}^t uNu$ still satisfies $N'_{11} < 0$. Then, for each $h \in \mathbb{Z}$ one can define the matrix:

$$M_h := \begin{pmatrix} 1 & nh & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & & \\ \vdots & \vdots & & 1_{g-2} & & \\ 0 & 0 & & & & \end{pmatrix} \in GL(g, \mathbb{Z})$$

which also satisfies $M_h \equiv 1_g \bmod n$ for each $h \in \mathbb{Z}$. Then:

$$\lim_{h\to\infty} Tr({}^tM_h N' M_h) = \lim_{h\to\infty}[Tr(M_h) + N'_{11}n^2h^2 + 2N'_{12}nh] = -\infty \tag{2.10}$$

Therefore, by reiterating (2.9) and applying the estimate (2.8), one obtains:

$$\begin{aligned} |\,a(N)\,| = |\,a(N')\,| &= |\,\det({}^tM_h)|^{-1}\,|\,a({}^tM_hN'M_h)\,| = \\ &= |\,a({}^tM_hN'M_h)\,| \leq Ce^{\frac{2\pi}{n}Tr({}^tM_hN'M_h)} \qquad \forall h \in \mathbb{Z} \end{aligned}$$

from which $a(N) = 0$ follows, because of (2.10).

□

The following Proposition is an immediate consequence of the Götzky-Koecher Principle:

Proposition 2.2. *Let Γ be a congruence subgroup of the Siegel modular group Γ_g for $g > 1$ and let $f \in A_k(\Gamma)$. Then f is bounded on each set of the kind:*

$$\mathfrak{S}_g^\lambda := \{\tau \in \mathfrak{S}_g \,|\, Im\tau - \lambda 1_g > 0\}$$

with $\lambda \geq 0$.

Proof. Let $n_0 > 1$ be such that $\Gamma_g(n_0) \subset \Gamma$; by applying the Götzky-Koecher Principle, one has:

$$f(\tau) = \sum_{N\geq 0} a(N)e^{\frac{2\pi i}{n_0}Tr(N\tau)}$$

Moreover, for each $\tau \in \mathfrak{S}_g^\lambda$ and for each $N \in Sym_g^s(\mathbb{Q})$ such that $N \geq 0$, one has:

$$|\,a(N)\,e^{\frac{2\pi i}{n_0}Tr(N\tau)}| = |\,a(N)\,|\,e^{-\frac{2\pi}{n_0}Im[Tr(N\tau)]} \leq |\,a(N)\,|\,e^{-\frac{2\pi}{n_0}Tr(N\lambda)}$$

Hence:

$$|f(\tau)| \leq \sum_{N\geq 0} |a(N)|\,e^{-\frac{2\pi}{n_0}Tr(N\lambda)} \qquad \forall \tau \in \mathfrak{S}_g^\lambda$$

and the numerical series on the right is convergent, for it is the absolute Fourier expansion of f in $i1_g \in \mathfrak{S}_g$ □

Corollary 2.1. *Let Γ be a congruence subgroup of the Siegel modular group Γ_g for $g > 1$ and let $f \in A_k(\Gamma)$. Then $\gamma^{-1}|_k f$ is bounded on Siegel's fundamental domain $\mathfrak{F}_g$ for each $\gamma \in \Gamma_g$.*

Proof. It follows from Proposition 2.2 and Lemma 1.7. □

Corollary 2.1 shows that condition 3. in Definition 2.1 is a consequence of both conditions 1. and 2. when $g > 1$.

Another important consequence of Götzky-Koecher Principle is the following:

Corollary 2.2. *Modular forms of negative weight vanish.*

Proof. Let Γ be a congruence subgroup of the Siegel modular group Γ_g and let $n_0 > 0$ be such that $\Gamma(n_0) \subset \Gamma$.
If $f \in A_k(\Gamma)$, the function f is bounded on Siegel's fundamental domain $\mathfrak{F}_g$ (by definition when $g = 1$ and by Corollary 2.1 when $g > 1$).
Moreover, if $k < 0$, the function $\tau \mapsto |\det Im\tau|^{\frac{k}{2}}$ is bounded on $\mathfrak{F}_g$ by Lemma 1.7. Therefore, a constant $C > 0$ exists, such that:

$$|\det Im\tau|^{\frac{k}{2}} f(\tau) \leq C \qquad \forall \tau \in \mathfrak{F}_g$$

The coefficients $a(N)$ of the Fourier expansion of f satisfy thus for each $y > 0$:

$$\begin{aligned} |a(N)|e^{-\frac{2\pi}{n}Tr(Ny)} &\leq \int_{[0,n]^K} |f(x+iy)||e^{\frac{2\pi i}{n}Tr(Nx)}|dx \leq \\ &\leq \sup_{x \in [0,n]^K} f(x+iy) \leq C|\det Im\tau|^{-\frac{k}{2}} \end{aligned}$$

Then, by letting y tend to zero, one obtains $a(N) = 0$ whenever $N \geq 0$. □

Corollary 2.3. *The ring of modular forms with respect to a congruence subgroup Γ is a positive graded ring $A(\Gamma) = \bigoplus_{k \geq 0} A_k(\Gamma)$.*

An important theorem states the algebraic dependence of suitably many modular forms of given weight (see [VdG] for details):

Theorem 2.3. *Let $\Gamma \subset \Gamma_g$ be a congruence subgroup. Then, for each $k \in \mathbb{Z}^+$ the complex vector space $A_k(\Gamma)$ is finite-dimensional.*

The existence of non-vanishing modular forms, though, is not a trivial question. Regarding the modularity with respect to the Siegel modular group Γ_g, Eisenstein series are examples of non trivial modular forms of even weight. Other outstanding examples of modular forms with respect to the remarkable congruence subgroups introduced in the Section 1.2 will be studied in the following chapter.
The next section will be, instead, devoted to the introduction of a remarkable kind of modular forms.

2.3 Cusp Forms

Proposition 2.3. *Let be $1 \leq k \leq g$ and $\{\tau_n\}_{n \in \mathbb{N}} \subset \mathfrak{S}_g$ a sequence such that:*

$$\tau_n = \begin{pmatrix} \tau' & u_n \\ {}^t u_n & w_n \end{pmatrix} \qquad \forall n \in \mathbb{N} \tag{2.11}$$

where $\tau' \in \mathfrak{S}_k$ is a fixed point, $\{u_n\}_{n \in \mathbb{N}} \subset Sym_{k,g-k}(\mathbb{C})$ is bounded, and $\{w_n\}_{n \in \mathbb{N}}$ is such that all the eigenvalues of $Im(w_n)$ tend to infinity. Then, the limit:

$$\lim_{n \to \infty} f(\tau_n)$$

exists and is finite for each $f \in A(\Gamma_g)$

Proof. Let $f \in A(\Gamma_g)$ and let:

$$f(\tau_n) = \sum_{\substack{N \in Sym_g^s(\mathbb{Z}) \\ N \geq 0}} a(N) e^{2\pi i Tr(N\tau_n)} \qquad \forall n \in \mathbb{N}$$

be its Fourier expansion. By hypothesis, the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ is contained in a set on which the Fourier series of f uniformly converges. Since:

$$Tr(N\tau) := \sum_{i=1}^{g} N_{ii}\tau_{ii} + 2 \sum_{1\leq i\leq j\leq g} N_{ij}\tau_{ij}$$

one has:

$$\lim_{n\to\infty} f(\tau_n) = \sum_{\substack{N \in Sym_k^s(\mathbb{Z}) \\ N \geq 0}} \lim_{n\to\infty} a(N) e^{2\pi i Tr(N\tau)} = \sum_{\substack{N' \in Sym_r^s(\mathbb{Z}) \\ N' \geq 0}} a\begin{pmatrix} N' & 0 \\ 0 & 0 \end{pmatrix} e^{2\pi i Tr(N'\tau')} \tag{2.12}$$

Then the thesis follows, since $\tau' \in \mathfrak{S}_k$ implies the convergence of the series on the right. □

One has to observe that for a given couple g, k with $1 \leq k \leq g$, and $\tau \in \mathfrak{S}_k$ fixed, there always exists a sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathfrak{S}_g$ of the kind in (2.11) with $\tau' = \tau$. Any sequence of points in $\mathfrak{S}_g$ satisfying such a property will be therefore denoted by $\{\tau_n^\tau\}$.

For each couple g, k with $1 \leq k \leq g$, Proposition 2.3 allows to define an operator acting on modular forms by setting:

$$(\Phi_{g,k} f)(\tau) := \lim_{n\to\infty} f(\tau_n^\tau) \qquad \forall f \in A(\Gamma_g) \tag{2.13}$$

Proposition 2.4. *The law (2.13) defines an operator $\Phi_{g,k} : A(\Gamma_g) \to A(\Gamma_k)$ which preserves the weight.*

Proof. Let $f \in A_h(\Gamma_g)$. For each $\tau \in \mathfrak{S}_k$, $(\Phi_{g,k} f)(\tau)$ does not depend on the choice of the sequence $\{\tau_n^\tau\} \subset \mathfrak{S}_g$, because of (2.12); moreover, as the series in (2.12) is uniformly convergent on each compact, (2.13) defines a holomorphic function on $\mathfrak{S}_k$, which in the $g = 1$ case is also bounded on the Siegel fundamental domain $\mathfrak{F}_1$. By setting:

$$\gamma_\eta := \begin{pmatrix} a_\eta & 0 & b_\eta & 0 \\ 0 & 1_{g-k} & 0 & 0 \\ c_\eta & 0 & d_\eta & 0 \\ 0 & 0 & 0 & 1_{g-k} \end{pmatrix} \in \Gamma_g \qquad \forall \eta \in \Gamma_k$$

the following transformation law is gained for each $\eta \in \Gamma_k$, $\tau' \in \mathfrak{S}_k$ and $\lambda > 0$, since $f \in A_h(\Gamma_g)$:

$$f\begin{pmatrix} \eta\tau' & 0 \\ 0 & i\lambda 1_{g-k} \end{pmatrix} = \det(c_\eta\tau' + d_\eta)^h f\begin{pmatrix} \tau' & 0 \\ 0 & i\lambda 1_{g-k} \end{pmatrix}$$

Hence, when $\lambda \to \infty$, one has:

$$\Phi_{g,k} f(\eta\tau') = \det(c_\eta \tau' + d_\eta)^h \Phi_{g,k} f(\tau') \qquad \forall\, \eta \in \Gamma_k\,, \quad \forall \tau' \in \mathfrak{S}_k$$

and consequently $\Phi_{g,k} f \in A_h(\Gamma_k)$, which concludes the proof. ◻

Definition 2.4. *The operator defined in (2.13) is called the* **Siegel operator***.*

The simplest way to describe the action of the Siegel operator is the following:

$$\Phi_{g,k}(f)(\tau) = \lim_{\lambda\to\infty} f\begin{pmatrix} \tau & 0 \\ 0 & i\lambda \end{pmatrix} \qquad \forall f \in A(\Gamma_g), \;\; \forall \tau \in \mathfrak{S}_k \tag{2.14}$$

The Siegel operator can be used to define the so-called cusp forms:

Definition 2.5. *The elements of the complex vector space:*

$$S_k(\Gamma) := \{ f \in A_k(\Gamma) \,|\, \Phi_{g,g-1}(\gamma^{-1}|_k f) = 0 \quad \forall \gamma \in \Gamma_g \}$$

are called **cusp forms** *of weight k.*

More generally, one can define the ideal of cusp forms with respect to a congruence subgroup Γ in $A(\Gamma)$:

Definition 2.6. *Let be Γ a congruence subgroup. The ideal $S(\Gamma) := \bigoplus_{k\geq 0} S_k(\Gamma) \subset A(\Gamma)$ is called the* **ideal of the cusp forms** *with respect to Γ.*

This definition characterizes in fact the cusp forms with respect to Γ as the modular forms that vanish on the boundary of the Satake's compactification of A_Γ [6].

As well as for the rings of modular forms, one clearly has $S(\Gamma') \subset S(\Gamma)$ whenever $\Gamma' \subset \Gamma$.

Example 2.4. *The function Δ introduced in Example 2.3 is a modular form of weight* 12*. It is easily checked that:*

$$\lim_{t\to+\infty} \Delta(it) = 0$$

The modular form Δ is, therefore, a cusp form.

[6] See Appendix B, (B.10) for the case of full modularity.

Chapter 3

Theta Constants

3.1 Characteristics

This section is devoted to introduce the notion of characteristic, which is used to parametrize the so-called Theta constants.

Definition 3.1. *A g-characteristic (or simply a characteristic, when no misunderstanding is allowed) is a vector column* $\begin{bmatrix} m' \\ m'' \end{bmatrix}$ *with* $m', m'' \in \mathbb{Z}_2^g$.

Definition 3.2. *Let* $m = \begin{bmatrix} m' \\ m'' \end{bmatrix}$ *be a characteristic. The function:*

$$e(m) = (-1)^{{}^t m' m''} \tag{3.1}$$

is called the **parity** *of* m. *A characteristic* m *is called* **even** *if* $e(m) = 1$ *and* **odd** *if* $e(m) = -1$.

Henceforward, the symbol $C^{(g)}$ will stand for the set of g-characteristics. Needless to say, $C^{(g)}$ is isomorphic to $\mathbb{Z}_2^g \times \mathbb{Z}_2^g$ as a ring, hence each g-characteristic m satisfies, in particular, $m + m = 0$. The symbols $C_e^{(g)}$ and $C_o^{(g)}$ will stand respectively for the subset of even g-characteristics and the subset of odd g-characteristics. Their cardinalities are easily computed by introducing the notation:

$$\tilde{m}_\delta = \begin{bmatrix} m' & \delta' \\ m'' & \delta'' \end{bmatrix} \in C^{(g)} \qquad \forall\, m = \begin{bmatrix} m' \\ m'' \end{bmatrix} \in C^{(g-1)}, \quad \forall\, \delta = \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} \in C^{(1)}$$

and by noting that, whenever m is even, $\tilde{m}_\delta$ is even or odd, depending on whether the 1-characteristic δ is respectively even or odd; on the other hand, whenever m is odd, $\tilde{m}_\delta$ is even or odd, depending on whether δ is respectively odd or even. Since there are three even 1-characteristics

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and only an odd one

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

one has:

$$|C_e^{(g)}| = 3|C_e^{(g-1)}| + |C_o^{(g-1)}|$$

$$|C_o^{(g)}| = |C_e^{(g-1)}| + 3|C_o^{(g-1)}|$$

Hence, $|C_e^{(g)}| - |C_o^{(g)}| = 2(|C_e^{(g-1)}| - |C_o^{(g-1)}|)$. Then, one gains by induction $|C_e^{(g)}| - |C_o^{(g)}| = 2^g$, as $|C_e^{(1)}| - |C_o^{(1)}| = 2$. Since $|C_e^{(g)}| + |C_o^{(g)}| = 2^{2g}$, one has, therefore:

$$|C_e^{(g)}| = \frac{1}{2}(2^{2g} + 2^g) = 2^{g-1}(2^g + 1)$$

$$|C_o^{(g)}| = \frac{1}{2}(2^{2g} - 2^g) = 2^{g-1}(2^g - 1)$$

Thus, to sum up, there are $2^{g-1}(2^g + 1)$ even g-characteristics and $2^{g-1}(2^g - 1)$ odd g-characteristics.

An action of the modular group Γ_g on the set $C^{(g)}$ can be defined by setting:

$$\gamma \begin{bmatrix} m' \\ m'' \end{bmatrix} := \left[\begin{pmatrix} d_\gamma & -c_\gamma \\ -b_\gamma & a_\gamma \end{pmatrix} \begin{pmatrix} m' \\ m'' \end{pmatrix} + \begin{pmatrix} diag(c_\gamma {}^t d_\gamma) \\ diag(a_\gamma {}^t b_\gamma) \end{pmatrix} \right] mod 2 \tag{3.2}$$

A straightforward computations yields:

$$\gamma(\gamma' m) = (\gamma \cdot \gamma')m, \qquad 1_g m = m$$

Moreover $e(\gamma m) = e(m)$ for each γ in Γ_g. Hence, one can state:

Lemma 3.1. *The law in (3.2) defines an action on $C^{(g)}$, preserving the parity of the characteristics; in particular, this action is not transitive.*

More precisely, one has (cf.[I3] or [I6]):

Lemma 3.2. *$C^{(g)}$ decomposes into two orbits by the action in (3.2). These two orbits consist of the set of even characteristics and the set of odd characteristics.*

The action defined in (3.2) is an affine transformation of $C^{(g)}$; the congruence subgroup $\Gamma_g(1, 2)$, in particular, acts linearly on $C^{(g)}$ by definition. Moreover:

Lemma 3.3. *The action of the principal congruence subgroup $\Gamma_g(2)$ on $C^{(g)}$ is trivial.*

Proof. Let $\gamma \in \Gamma_g(2)$. Then, with reference to the notation introduced in (1.3), one has:

$$\gamma \begin{bmatrix} m' \\ m'' \end{bmatrix} = \left[\begin{pmatrix} 1_{2g} + 2d_M & -2c_M \\ -2b_M & 1_{2g} + 2a_M \end{pmatrix} \begin{pmatrix} m' \\ m'' \end{pmatrix} \right] \mathrm{mod} 2 = \begin{bmatrix} m' \\ m'' \end{bmatrix}$$

□

Corollary 3.1. *An action of $\Gamma_g/\Gamma_g(2) \cong Sp(g,\mathbb{Z}_2)$ is defined on $C^{(g)}$.*

The parity function can be used to classify remarkable k-plets of characteristics with respect to the action introduced in (3.2). To pursue this purpose, a parity can be also introduced for triplets:

$$e(m_1,m_2,m_3) := e(m_1)e(m_2)e(m_3)e(m_1+m_2+m_3) \tag{3.3}$$

Definition 3.3. *A triplet (m_1,m_2,m_3) is called* **azygetic** *if $e(m_i,m_j,m_h) = -1$ and* **syzygetic** *if $e(m_i,m_j,m_h) = 1$*

Regarding the parity of the sum of an odd sequence of g-characteristics, the following formula holds (cf. [I8]):

$$e\left(\sum_{i=1}^{2k+1}\right) = \left(\prod_{i=1}^{2k+1} e(m_i)\right)\left(\prod_{1<i<j\leq 2k+1} e(m_1,m_i,m_j)\right) \tag{3.4}$$

The parity is involved in characterizing the orbits of K-plets of g-characteristics under the action described in (3.2), as the following Proposition states (cf. [I6]):

Proposition 3.1. *Let $(m_1,\ldots m_K)$ and $(n_1,\ldots n_K)$ be two ordered K-plets of g-characteristics. Then, there exists $\gamma \in \Gamma_g$ such that $\gamma m_i = n_i$ for each $i = 1,\ldots n$ if and only if:*

1. *$e(m_i) = e(n_i)$ for each $i = 1,\ldots,K$;*
2. *$e(m_i,m_j,m_k) = e(n_i,n_j,n_k)$ for each $1 \leq i < j < k \leq K$;*
3. *Whenever $\{m_1,\ldots,m_{2k}\} \subset \{m_1,\ldots,m_K\}$ is such that $m_1 + \cdots m_{2k} \neq 0$, then $n_1 + \cdots n_{2k} \neq 0$;*

Corollary 3.2. *Let $(m_1,\ldots m_g)$ and $(n_1,\ldots n_g)$ be two different orderings of the set $C_e^{(g)}$ such that for each l,r,s one has:*

$$e(m_l,m_r,m_s) = e(n_l,n_r,n_s)$$

Then, there exists an element $[\gamma] \in \Gamma_g/\Gamma_g(2)$ such that $[\gamma]m_i = n_i$ for each $i = 1,\ldots g$.

Proof. Let $(m_1,\ldots m_g)$ and $(n_1,\ldots n_g)$ be as in the hypothesis, and let $\{m_1,\ldots,m_{2h}\} \subset \{m_1,\ldots,m_g\}$ be such that $m_1 + \cdots m_{2h} = 0$; then, by (3.4) one has:

$$\begin{aligned} 1 = e(m_{2h}) = e\left(\sum_{i=1}^{2h-1} m_i\right) = \\ = \prod_{1<i<j\leq 2h-1} e(m_1,m_i,m_j) = \prod_{1<i<j\leq 2h-1} e(n_1,n_i,n_j) = e\left(\sum_{i=1}^{2h-1} n_i\right) \end{aligned}$$

hence, for each $1 \le l < r \le g$:

$$e\left(n_l, n_r, \sum_{i=1}^{2h-1} n_i\right) = e\left(n_l + n_r + \sum_{i=1}^{2h-1} n_i\right) = e(n_1, n_l, n_r) \prod_{1<i<j\le 2h-1} e(n_1, n_i, n_j) =$$

$$= e(m_1, m_l, m_r) \prod_{1<i<j\le 2h-1} e(m_1, m_i, m_j) = e\left(m_l + m_r + \sum_{i=1}^{2h-1} m_i\right) =$$

$$= e\left(m_l, m_r, \sum_{i=1}^{2h-1} m_i\right) = e(m_l, m_r, m_{2h}) = e(n_l, n_r, n_{2h})$$

Then $\sum_{i=1}^{2h-1} n_i = n_{2h}$. Hence, whenever $\{m_1, \ldots, m_{2h}\} \subset \{m_1, \ldots, m_g\}$ is such that $m_1 + \cdots m_{2h} = 0$, $\{n_1, \ldots, n_{2h}\} \subset \{n_1, \ldots, n_g\}$ also verifies $n_1 + \cdots n_{2h} = 0$. Then, the thesis follows from Proposition 3.1. □

The notion of azygeticity can be also given for K-plets of characteristics:

Definition 3.4. *Let be $\{m_1, \ldots, m_K\}$ a K-plet of characteristics. $\{m_1, \ldots, m_K\}$ is called* **azygetic** *if each sub3-plet $\{m_i, m_j, m_h\}$ is azygetic.*

Since the $g = 2$ case will be chiefly dealt with, a focus on the main properties of k-plets of even 2-characteristics under the action (3.2) will be needed. The next section is thus centered around this technical feature.

3.2 K-plets of Even 2-Characteristics

A specific notation will be introduced here for subsets of even 2-characteristics and henceforward referred to. The symbols $C_1 := C_e^{(2)}$ and $\tilde{C}_1 := C_o^{(2)}$ will denote respectively the set of even 2-characteristics and the set of odd 2-characteristics, so that $C^{(2)} = C_1 \cup \tilde{C}_1$ with $|C_1| = 10$ and $|\tilde{C}_1| = 6$. By the symbols C e $\tilde{C}$ the parts of C_1 and the parts of $\tilde{C}_1$ will be respectively meant. More in general, $C_k \in C$ will stand for the set of non ordered k-plets of even 2-characteristics and $\tilde{C}_k \in \tilde{C}$ for the set of non ordered k-plets of odd 2-characteristics.

For each subset $M \subset C_1$ its complementary set $C_1 - M$ in C_1 will be denoted by the symbol M^c. The so-called symmetric difference will be represented by the classic notation:

$$M_i \vartriangle M_j := (M_i \cup M_j) - (M_i \cap M_j) \in C \qquad \forall\, M_i, M_j \in C$$

Therefore, for each couple M_i, $M_j \subset C_1$, the set C_1 can be written as a disjoint union of sets:

$$C_1 = \left(M_i^c \cap M_j^c\right) \cup \left(M_i \cap M_j\right) \cap \left(M_i \vartriangle M_j\right) \tag{3.5}$$

This section is devised to point out some combinatorial features concerning with remarkable subsets of C_k, arising as orbits under the action of $\Gamma_2/\Gamma_2(2)$. In fact, when $g = 2$, the group $\Gamma_2/\Gamma_2(2)$ both acts on C_1 and $\tilde{C}_1$ due to Corollary

3.1 and Lemma 3.2. By focusing on the action on $\tilde{C}_1$, one notes that the group homomorphism naturally defined by:

$$\psi_P : \Gamma_2/\Gamma_2(2) \mapsto S_6 \tag{3.6}$$

is injective, for the identity in $\Gamma_2/\Gamma_2(2)$ is the only element fixing all the six odd 2-characteristics. Moreover, $\Gamma_2/\Gamma_2(2) \cong Sp(2, \mathbb{Z}_2)$ and S_6 have the same order, hence the homomorphism (3.6) is also surjective. Then:

$$\Gamma_2/\Gamma_2(2) \cong S_6 \tag{3.7}$$

An action of this group is naturally defined on each C_k by means of the one defined on C_1. While the action on C_2 is transitive, each C_k for $k > 2$ turns out to be decomposed into orbits, which can be described by using Corollary 3.2. Importing from [VGVS] a useful notation, one can easily denote such orbits.The set C_3 decomposes into two orbits C_3^- and C_3^+, respectively consisting of azygetic and syzygetic triplets:

$$C_3^- = \{\{m_1, m_2, m_3\} \in C_3 \mid m_1 + m_2 + m_3 \in \tilde{C}_1\}$$

$$C_3^+ = \{\{m_1, m_2, m_3\} \in C_3 \mid m_1 + m_2 + m_3 \in C_1\}$$

with $|C_3^-| = |C_3^+| = 60$.

The set C_4 decomposes into three orbits C_4^+, C_4^* and C_4^-:

$$C_4^- = \{\{m_1, \dots, m_4\} \in C_4 \mid \{m_i, m_j, m_k\} \in C_3^- \ \ \forall \{m_i, m_j, m_k\} \subset \{m_1, \dots, m_4\}\}$$

$$C_4^+ = \{\{m_1, \dots, m_4\} \in C_4 \mid \{m_i, m_j, m_k\} \in C_3^+ \ \ \forall \{m_i, m_j, m_k\} \subset \{m_1, \dots, m_4\}\}$$

with $|C_4^-| = |C_4^+| = 15$.

The set C_5 also decomposes into three orbits C_5^+, C_5^* and C_5^-, where:

$$C_5^- = \{\{m_1, \dots m_5\} \in C_5 \mid \{m_1, \dots m_5\} \text{ contains a unique element of } C_4^-\}$$

$$C_5^+ = \{\{m_1, \dots m_5\} \in C_5 \mid \{m_1, \dots m_5\} \text{ contains a unique element of } C_4^+\}\}$$

with $|C_5^-| = |C_5^+| = 90$.

The orbit decomposition for C_k when $k > 5$ can be described by taking the complementary sets. In particular, the following orbits of C_6 will be also focused on henceforward:

$$C_6^- = \{\{m_1, \dots, m_6\} \in C_6 \mid \{m_1, \dots, m_6\}^c \in {C_4}^+\}$$

$$C_6^+ = \{\{m_1, \dots, m_6\} \in C_6 \mid \{m_1, \dots, m_6\}^c \in {C_4}^-\}$$

The behaviour of this action is completely displayed in a diagram in [VGVS]. Here, some specific properties will be briefly reviewed.

Pertaining to the orbits of C_3, one has the following:

Lemma 3.4. *Let $m_1, m_2 \in C_1$ be distinct. If $M = \{m_1, m_2\}^c \subset C_1$, then $M = \{n_1, n_2, n_3, n_4, h_1, h_2, h_3, h_4\}$, with $\{m_1, m_2, n_i\} \in C_3^-$ and $\{m_1, m_2, h_i\} \in C_3^+$ for each i.*

Concerning with the orbits of C_4, one has, in particular:

Lemma 3.5. *Let* $\{m_1, m_2, m_3\} \in C_3^-$. *There is exactly one characteristic* $n \in C_1$ *such that* $\{m_1, m_2, m_3, n\} \in C_4^-$.

Lemma 3.6. *Let* $\{m_1, m_2, m_3\} \in C_3^+$. *There exists exactly one characteristic* $n \in C_1$ *such that* $\{m_1, m_2, m_3, n\} \in C_4^+$, *namely* $n = m_1 + m_2 + m_3$.

As a straight consequence of Lemma 3.6 one has the following:

Corollary 3.3. $C_4^+ \subset C_4$ *is the set of the* 4*-plets* $\{m_1, m_2, m_3, m_4\}$ *satisfying the condition* $m_1 + m_2 + m_3 + m_4 = 0$.

Regarding the existence of common couples of characteristics for elements in C_4^-, Lemma 3.4 implies the following:

Corollary 3.4. *Let* $h_1, h_2 \in C_1$ *be distinct characteristics. There are exactly two elements in* C_4^-, *containing both them.*

More precisely, one has:

Lemma 3.7. *If* $\{m_1, m_2, h, k\}, \{m_3, m_4, h, k\} \in C_4^-$ *are distinct,* $\{m_1, m_2, m_3, m_4\} \in C_4^-$.

Lemma 3.8. *If* $\{m_1, m_2, m_3, n\}, \{m_4, m_5, m_6, n\} \in C_4^-$ *are such that* m_i, n *are all distinct,* $\{m_7, m_8, m_9, n\} \in C_4^-$, *where* $\{m_7, m_8, m_9\} = \{m_1, m_2, m_3, m_4, m_5, m_6, n\}^c$

The following property holds for C_5^-:

Proposition 3.2. *If* $M \notin C_5^-$, *M does not contain any element of* C_4^-.

For the orbits of C_6, one has:

Proposition 3.3. $M \in C_6^-$ *if and only if M contains exactly six elements of* C_5^-.

Moreover, Corollary 3.3 implies the following:

Corollary 3.5. $M \in C_6^-$ *if and only if* $\sum_{m \in M} m = 0$.

Lemma 3.9. $M \in C_6^-$ *if and only if M does not contain any element of* C_4^+.

Proof. A 4-plets $\{m_1, m_2, m_3, m_4\} \in C_4^-$ can not belong to an element of C_6^-, for $\sum_{i=1}^4 m_i = 0$. □

By taking the complementary sets, one obtains:

Lemma 3.10. $M \in C_6^+$ *if and only if M does not contain any element of* C_4^-.

For elements M_1, M_2 belonging both to C_4^+ or C_6^- Corollaries 3.3 and 3.5 immediately provide a peculiar property for their symmetric difference:

$$0 = \sum_{m \in M_1} m + \sum_{m \in M_2} m = \sum_{m \in M_1 \triangle M_2} m \tag{3.8}$$

The behaviour of the symmetric difference of elements of C_4^- and C_6^+ can be described more precisely by means of the results gathered as yet.

Proposition 3.4. *Let $M_1, M_2 \in C_4^-$. Then $M_1 \cap M_2 \neq \emptyset$ and:*

$$1) \quad M_1 \vartriangle M_2 \in C_6^+ \quad if \quad |M_1 \cap M_2| = 1$$

$$2) \quad M_1 \vartriangle M_2 \in C_4^- \quad if \quad |M_1 \cap M_2| = 2$$

$$3) \quad M_1 = M_2 \quad if \quad |M_1 \cap M_2| > 2$$

Proof. Lemma 3.10 implies $M_2 \not\subset M_1^c$, hence $M_1 \cap M_2 \neq \emptyset$. The possible cases are thus to be checked:

1) If $M_1 \cap M_2 = \{n\}$, one can set $M_1 = \{m_1, m_2, m_3, n\}$ and $M_2 = \{m_4, m_5, m_6, n\}$. Then, by Lemma 3.8, $M_1 \vartriangle M_2 = \{m_1, m_2, m_3, m_4, m_5, m_6\} = \{m_7, m_8, m_9, n\}^c \in C_6^+$.

2) If $M_1 \cap M_2 = \{h, k\}$, one can set $M_1 = \{m_1, m_2, h, k\}$ and $M_2 = \{m_3, m_4, h, k\}$. Then, by Lemma 3.7, $M_1 \vartriangle M_2 = \{m_1, m_2, m_3, m_4\} \in C_4^-$.

3) It is a straight consequence of Lemma 3.5. □

Proposition 3.5. *Let $M_1, M_2 \in C_6^+$. Then, the only possible cases are:*

$$1) \quad M_1 \vartriangle M_2 \in C_6^+ \quad if \quad |M_1 \cap M_2| = 3$$

$$2) \quad M_1 \vartriangle M_2 \in C_4^- \quad if \quad |M_1 \cap M_2| = 4$$

$$3) \quad M_1 = M_2 \quad if \quad |M_1 \cap M_2| > 4$$

Proof. Since $|C_1| = 10$, obviously $|M_1 \cap M_2| \geq 2$. Moreover (3.5) and the relation $12 = |M_1| + |M_2| = |M_1 \cup M_2| + |M_1 \cap M_2|$ imply:

$$|M_1 \cap M_2| - |M_1^c \cap M_2^c| = 2 \tag{3.9}$$

Therefore $|M_1 \cap M_2| > 2$, since $M_1^c \cap M_2^c \neq \emptyset$ by Proposition 3.4. The cases described in the statement are thus the only ones to be checked.

1) If $M_1 \cap M_2 = \{h, k, l\}$, one can set $M_1 = \{m_1, m_2, m_3, h, k, l\}$ and $M_2 = \{m_4, m_5, m_6, h, k, l\}$. Then, $M_1^c = \{m_4, m_5, m_6, n\} \in C_4^-$ and $M_2^c = \{m_1, m_2, m_3, n\} \in C_4^-$, where $n \neq m_i, h, k, l$. Therefore, by Lemma 3.8, $M_1 \vartriangle M_2 = \{h, k, l, n\}^c \in C_6^+$.

2) If $M_1 \cap M_2 = \{n, h, k, l\}$, one can set $M_1 = \{m_1, m_2, n, h, k, l\}$ and $M_2 = \{m_3, m_4, n, h, k, l\}$. Then, $M_1^c = \{m_3, m_4, i, j\} \in C_4^-$ and $M_2^c = \{m_1, m_2, i, j\} \in C_4^-$, where $i, j \neq m_i, , n, h, k, l$. Hence, by Lemma 3.7, $M_1 \vartriangle M_2 = \{m_1, m_2, m_3, m_4\} \in C_4^-$.

3) If $|M_1 \cap M_2| > 4$, then (3.9) implies $|M_1^c \cap M_2^c| > 2$; therefore, $M_1^c = M_2^c$ by Proposition 3.4, hence $M_1 = M_2$. □

Proposition 3.6. *Let $M_1 \in C_6^+$ and $M_2 \in C_4^-$. If $M_1^c \neq M_2$ the only possible cases are:*

$$1) \quad M_1 \vartriangle M_2 \in C_4^- \quad if \quad |M_1 \cap M_2| = 3$$

$$2) \quad M_1 \vartriangle M_2 \in C_6^+ \quad if \quad |M_1 \cap M_2| = 2$$

Proof. Obviously $|M_1 \cap M_2| \leq 4$; moreover, Lemma 3.10 implies $M_2 \not\subset M_1$, hence $|M_1 \cap M_2| \leq 3$. If $M_1^c \neq M_2$, Lemma 3.5 implies $|M_1^c \cap M_2| < 3$ and consequently $|M_1 \cap M_2| > 1$. Therefore, the only cases to be checked, when $M_1^c \neq M_2$, are the cases 1) and 2):

1) If $M_1 \cap M_2 = \{h, k, l\}$, one can set $M_1 = \{m_1, m_2, m_3, h, k, l\}$ and $M_2 = \{m_4, h, k, l\}$. Then $M_2^c = \{m_1, m_2, m_3, r, s, t\} \in C_6^+$ with $r, s, t \neq m_i, h, k, l$, and by Proposition 3.5 (case 1) one has:

$$M_1 \vartriangle M_2 = \{m_1, m_2, m_3, m_4\} = \{h, k, l, r, s, t\}^c = (M_1 \vartriangle M_2^c)^c \in C_4^-$$

2) If $M_1 \cap M_2 = \{h, k\}$, one can set $M_1 = \{m_1, m_2, m_3, m_4, h, k\}$ and $M_2 = \{m_5, m_6, h, k\}$. Then $M_1^c = \{m_5, m_6, i, j\} \in C_4^-$ with $i, j \neq m_i, h, k$, and by Proposition 3.4 (case 2) one has:

$$M_1 \vartriangle M_2 = \{m_1, m_2, m_3, m_4, m_5, m_6\} = \{i, j, h, k\}^c = (M_1^c \vartriangle M_2)^c \in C_6^+$$

□

3.3 Theta Constants

Theta constants are strictly involved in the construction of several modular forms. The aim of this section is to introduce these remarkable functions as well as to outline their main basic properties. The prominent references for this topic are Igusa's classical works [I3], [I4] and [I6], Freitag's book [F], Farkas and Kra's book [FK] and Mumford's lectures [Mf].

Definition 3.5. *For each $m = (m', m'') \in \mathbb{Z}^g \times \mathbb{Z}^g$, the function $\theta_m : \mathfrak{S}_g \times \mathbb{C}^g \to \mathbb{C}$, defined by:*

$$\theta_m(\tau, z) := \sum_{n \in \mathbb{Z}^g} exp\left\{ {}^t\left(n + \frac{m'}{2}\right)\tau\left(n + \frac{m'}{2}\right) + 2^t\left(n + \frac{m'}{2}\right)\left(z + \frac{m''}{2}\right)\right\} \tag{3.10}$$

is called **Riemann Theta function with characteristic** $m = (m', m'')$.

The series in (3.10) is easily seen to be absolutely convergent and uniformly convergent on each compact of $\mathfrak{S}_g \times \mathbb{C}^g$; therefore, for each m it defines a holomorphic function.

By setting:

$$\tilde{\tau} := (\tau \, 1_g) \in Sym_{g,2g(\mathbb{C})} \qquad \forall \tau \in \mathfrak{S}_g \tag{3.11}$$

for each $m = (m', m'')$, $n = (n', n'') \in \mathbb{Z}^g \times \mathbb{Z}^g$ the Theta function with characteristic m is plainly found to satisfy the equation [1]

$$\theta_m(\tau, z + \tilde{\tau}n) = (-1)^{{}^tm'n'' - {}^tm''n'} \exp\left\{2^t\left(-{}^tn'z - \frac{1}{2}{}^tn\tau n'\right)\right\}\theta_m(\tau, z) \tag{3.12}$$

[1] One proves that any other holomorphic function $\theta : \mathbb{C}^n \to \mathbb{C}$ which is solution of (3.12) differs from the Theta function with characteristic by a multiplicative factor; for any fixed $\tau \in \mathfrak{S}_g$, the function $z \mapsto \theta_m(\tau, z)$ is, therefore, characterized as an analytic function on $\mathbb{C}^n$ by being solution of the equation (3.12).

As proved through term by term differentiation, the Theta function with characteristic also satisfies the heat equation: [2].

$$\sum_{j,k=1}^{g} \sigma_{jk} \frac{\partial \theta_m}{\partial z_j z_k} = 2\pi i \sum_{j,k=1}^{g} \sigma_{jk} \frac{\partial \theta_m}{\partial \tau_{jk}} \tag{3.13}$$

With reference to the actions of Γ_g described in (1.10) and (3.2) , if $n = \gamma m$ with $\gamma \in \Gamma_g$, it easily turns out that:

$$\theta_n(\gamma\tau, {}^t(c_\gamma\tau + d_\gamma)^{-1}z) = K(m, \gamma, \tau) \exp\left\{2\left(-\frac{1}{2}{}^t z(c_\gamma\tau + d_\gamma)^{-1} c_\gamma z\right)\right\} \theta_m(\tau, z) \tag{3.14}$$

where $K : \mathbb{Z}^{2g} \times \Gamma_g \times \mathfrak{S}_g \to \mathbb{C}$ is a non-vanishing function, which does not depend on the variable z.

Theta functions can be parametrized by g-characteristics; the definition (3.10) implies, indeed:

$$\theta_{m+2n}(\tau, z) = (-1)^{{}^t m' n''} \theta_m(\tau, z)$$

for each $m = (m', m'')$, $n = (n', n'') \in \mathbb{Z}^g \times \mathbb{Z}^g$. Hence, in order to parametrize distinct Theta functions, one only needs to focus on those related to g-characteristics.

Definition 3.6. *For each characteristic m the holomorphic function $\theta_m : \mathfrak{S}_g \to \mathbb{C}$, defined by:*

$$\theta_m(\tau) := \theta_m(\tau, 0) \tag{3.15}$$

is called **Theta constant** *with characteristic m.*

From (3.10) it follows that:

$$\theta_m(\tau, -z) = e(m)\, \theta_m(\tau, z) \tag{3.16}$$

where $e(m)$ is the parity of the characteristic m introduced in (3.1) [3] By (3.16) Theta constants related to odd characteristics vanish. On the converse, one has $\lim_{\lambda\to\infty} \theta_0(i\lambda 1_g) = 1$; hence, the Theta constant θ_0, related to the null characteristic, does not vanish. Lemma 3.2 and the non-vanishing property of the function K imply that each Theta constant θ_m associated to an even characteristic m, does not vanish.

In order to determine an explicit expression for the function K, the following differential equation, derived from (3.13) for even characteristics, can be used:

$$\sum_{1\le j,k\le g} \sigma_{jk} \frac{\partial \log K}{\tau_{jk}} = \frac{1}{2} \sum_{1\le j,k\le g} \sigma_{jk}\mu_{jk}$$

[2] One also proves that as a holomorphic function $\theta_m : \mathfrak{S}_g \times \mathbb{C}^g \to \mathbb{C}$, the Theta function is characterized by being solution of (3.12) and (3.13).

[3] The definition of this function as a parity is indeed due to (3.16).

This is shown to imply:

$$K(m,\gamma,\tau) = C(m,\gamma)\det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}$$

where C is a function not depending on τ. In particular, one has to observe that, albeit the root $\det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}$ is not unique, it is well defined as an analytic function on $\mathfrak{S}_g$. Now, for each $n = (n', n'') \in C^{(g)}$ e $\tilde{\tau} = (\tau\, 1_g)$ as in (3.11), one has:

$$\theta_m\left(\tau, z + \frac{1}{2}\tilde{\tau}n\right) = \exp\left\{2\left[-\frac{1}{8}{}^tn'\tau n' - \frac{1}{2}{}^tn'\left(z + \frac{1}{2}(m'' + n'')\right)\right]\right\}\theta_{m+n}(\tau, z)$$

Then, by selecting the branch of $\det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}$ whose sign turns to be positive when $Re\tau = 0$, and applying (3.14), one has for each $\gamma \in \Gamma_g$:

$$\theta_{\gamma 0}(\gamma\tau, {}^t(c_\gamma\tau + d_\gamma)^{-1}z) = C(m,\gamma)\det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}\exp\left\{{}^tz(c_\gamma\tau + d_\gamma)^{-1}c_\gamma\right\}\theta_0(\tau, z)$$

Operating for each even characteristic m the substitution $z \mapsto z + \frac{1}{2}\tilde{\tau}m$, the multiplicative factor of C, depending only on m, can be explicited; hence one is supplied with the following transformation law for the Theta function:

$$\begin{gathered}\theta_{\gamma m}(\gamma\tau\,,{}^t(c_\gamma\tau + d_\gamma)^{-1}z) = \kappa(\gamma)e^{\pi i\left\{\frac{1}{2}{}^tz\,[(c_\gamma\tau + d_\gamma)^{-1}c_\gamma]\,z + 2\phi_m(\gamma)\right\}}det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}\theta_m(\tau, z)\\ \forall\gamma \in \Gamma_g, \quad \forall m \in C^{(g)}, \quad \forall\tau \in \mathfrak{S}_g, \quad \forall z \in \mathbb{C}^g\end{gathered} \tag{3.17}$$

where:

$$\begin{aligned}\phi_m(\gamma) = &-\frac{1}{8}({}^tm'{}^tb_\gamma d_\gamma m' + {}^tm''{}^ta_\gamma c_\gamma m'' - 2{}^tm'{}^tb_\gamma c_\gamma m'') +\\ &-\frac{1}{4}{}^tdiag(a_\gamma{}^tb_\gamma)(d_\gamma m' - c_\gamma m'')\end{aligned} \tag{3.18}$$

and $\det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}$ is the root chosen in the described way [4].

As concerns the function:

$$\Phi(m,\gamma,\tau,z) := \kappa(\gamma)\exp\left\{\frac{1}{2}{}^tz\,[(c_\gamma\tau + d_\gamma)^{-1}c_\gamma]\,z + 2\phi_m(\gamma)\right\}$$

$\Phi|_{z=0}$ does not depend on τ; for this reason the functions θ_m defined in (3.15) are called Theta constants. The transformation law (3.17) becomes then for Theta constants:

$$\begin{gathered}\theta_{\gamma m}(\gamma\tau) = \kappa(\gamma)\chi_m(\gamma)det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}\theta_m(\tau)\\ \forall\gamma \in \Gamma_g, \quad \forall m \in C^{(g)}, \quad \forall\tau \in \mathfrak{S}_g,\end{gathered} \tag{3.19}$$

[4]The transformation law (3.17) is due to Jacobi and Hermite for the case $g = 1$ and to Gordan, Clebsch, Thomae and Weber for $g > 1$

where:

$$\chi_m(\gamma) := \Phi(m, \gamma, \tau, 0) = e^{2\pi i \phi_m(\gamma)} \tag{3.20}$$

An important property needs to be stated in order to describe the behaviour of the function κ:

Proposition 3.7. *κ^2 is a character of $\Gamma_g(1,2)$.*

Proof. Since $\chi_0 = 1$ for the null characteristic $m = 0$ (by definition in (3.20) and (3.18)), one obtains:

$$\theta^2_{\gamma 0}(\gamma\tau) = \kappa^2(\gamma) det(c_\gamma \tau + d_\gamma)\theta^2_0(\tau)$$

by applying (3.19) to the Theta constant θ_0 related to the null characteristic. However, $\Gamma_g(1,2)$ acts linearly, hence $\gamma 0 = 0$ whenever $\gamma \in \Gamma_g(1,2)$; then:

$$det(c_\gamma \tau + d_\gamma)^{-1}\theta^2_0(\gamma\tau) = \kappa^2(\gamma)\theta^2_0(\tau) \qquad \forall \gamma \in \Gamma_g(1,2)$$

Therefore, with reference to the action introduced in (2.1), one has:

$$\gamma^{-1}|_1 \theta^2_0 = \kappa^2(\gamma)\theta^2_0 \qquad \forall \gamma \in \Gamma_g(1,2)$$

and then:

$$\kappa^2(\gamma\gamma')\theta^2_0(\tau) = \kappa^2(\gamma)\kappa^2(\gamma')\theta^2_0(\tau) \qquad \forall \gamma, \gamma' \in \Gamma_g(1,2)$$

Therefore, $\kappa^2(\gamma\gamma') = \kappa^2(\gamma)\kappa^2(\gamma')$ and $\kappa^2(1_{1g}) = 1$, for θ_0 does not vanish; hence the thesis follows. □

The following Lemma provides an explicit expression for the function κ on special elements of Γ_g:

Lemma 3.11. *Let:*

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ 0 & d_\gamma \end{pmatrix} \in \Gamma_g$$

Then:

$$\kappa^2(\gamma) = \kappa^2({}^t\gamma^{-1}) = \det(d_\gamma)$$

Proof. Since $\chi_0 = 1$, the transformation law (3.19) implies particularly:

$$\theta^2_{\gamma 0}(\gamma\tau) = \kappa^2(\gamma)\det(d_\gamma)\theta^2_0(\tau)$$

$$\forall\, \gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ 0 & d_\gamma \end{pmatrix} \in \Gamma_g$$

As $\theta^2_{\gamma 0}(\gamma\tau) = \theta^2_0(\tau)$, one has $\kappa^2(\gamma) = \det(d_\gamma)$. Hence, the identity $\kappa^2(\gamma) = \kappa^2({}^t\gamma^{-1})$ only needs to be proved. A suitable decomposition for such an element γ will be then useful. Since $a_\gamma {}^t d_\gamma = 1_g$, one has $\det d_\gamma = \pm 1$ and $a_\gamma = {}^t d^{-1}_\gamma$; moreover, the identities $b_\gamma = s' d_\gamma$ and $b_\gamma = s_\gamma d_\gamma$, where $s_\gamma = {}^t d^{-1 t}_\gamma b_\gamma \in Sym_g(\mathbb{Z})$, imply $s' = s_\gamma$

because of the inveritibility of d_γ. Therefore, for elements $\gamma \in \Gamma_g$ satisfying the property stated in the hypothesis, there exists a unique decomposition:

$$\gamma = \begin{pmatrix} {}^t d_\gamma^{-1} & s_\gamma d_\gamma \\ 0 & d_\gamma \end{pmatrix} = \begin{pmatrix} 1_g & s_\gamma \\ 0 & 1_g \end{pmatrix} \begin{pmatrix} {}^t d_\gamma^{-1} & 0_g \\ 0_g & d_\gamma \end{pmatrix}$$

with $s_\gamma \in Sym_g(\mathbb{Z})$. Then, by operating with J_g as in (2.1), one has:

$$J_g|_1 \theta_0^2 = \kappa^2(J_g^{-1})\, \theta_0^2$$

since $\phi_0(J_g) = 0$. Moreover, by setting $m_0 := -\begin{bmatrix} 0 \\ {}^t d_\gamma \cdot diag(s_\gamma) \end{bmatrix}$ the solution of $\gamma m_0 = 0$, one has $\phi_{m_0}(\gamma) = 0$, hence:

$$\gamma^{-1}|_1 \theta_0^2 = \kappa^2(\gamma)\theta_{m_0}^2$$

Finally, by setting $n_0 := \begin{bmatrix} {}^t d_\gamma \cdot diag(s_\gamma) \\ 0 \end{bmatrix}$ the solution of $\gamma n_0 = m_0$, one has then $\phi_{n_0}(\gamma) = 0$ and consequently:

$$J_g^{-1}|_1 \theta_{m_0}^2 = \kappa^2(J_g)\theta_{n_0}^2$$

Since ${}^t\gamma = J_g^{-1}\gamma^{-1}J_g$, one gains, therefore:

$${}^t\gamma|_1 \theta_0^2 = \kappa^2(J_g)\kappa^2(\gamma)\kappa^2(J_g^{-1})\theta_{n_0}^2$$

However, $\kappa^2(J_g)\kappa^2(J_g^{-1}) = 1$ by Proposition 3.7, since $J_g \in \Gamma_g(1,2)$; then, the following identity holds:

$$({}^t\gamma^{-1})^{-1}|_1 \theta_0^2 = \kappa^2(\gamma)\theta_{n_0}^2$$

Moreover, since ${}^t\gamma^{-1}n_0 = 0$ and $\phi_{n_0}({}^t\gamma^{-1}) = 0$, another identity rises:

$$({}^t\gamma^{-1})^{-1}|_1 \theta_0^2 = \kappa^2({}^t\gamma^{-1})\theta_{n_0}^2$$

and the remaining part of the thesis is thus proved by comparing the two identities.

□

Thanks to this Lemma, a characterization for κ is supplied:

Proposition 3.8. *For each $\gamma \in \Gamma$, $\kappa(\gamma)$ is an eight root of the unity.*

Proof. By definition in (3.20) and (3.18), one clearly has $\chi_m^8 = 1$. Then (3.19) implies:

$$det(c_\gamma \tau + d_\gamma)^{-4}\theta^8_{\gamma m}(\gamma\tau) = \kappa^8(\gamma)\theta^8_m(\tau)$$

$$\forall\gamma \in \Gamma_g, \quad \forall m \in C^{(g)}$$

Again with reference to the action introduced in (2.1), one has:

$$\gamma^{-1}|_4\theta^8_{\gamma m} = \kappa^8(\gamma)\theta^8_m \qquad\qquad \forall\gamma \in \Gamma_g, \quad \forall m \in C^{(g)}$$

Hence:

$$\kappa^8(\gamma\gamma')\theta^8_m(\tau) = \kappa^8(\gamma)\kappa^8(\gamma')\theta^8_m(\tau) \qquad \forall\gamma,\gamma' \in \Gamma_g \quad \forall m \in C^{(g)}$$

Then, κ^8 is a character of Γ_g. Moreover, by Lemma 3.11, one has:

$$\kappa^4(\gamma) = 1 \qquad\qquad \kappa^4({}^t\gamma^{-1}) = 1$$

whenever $\gamma \in \Gamma_g$ is such that $c_\gamma = 0$; since Γ_g is generated by these elements (Corollary 1.1), one has:

$$\kappa^8(\gamma) = 1 \qquad \forall\gamma \in \Gamma_g \tag{3.21}$$

□

An explicit expression for κ^4, which can be found in [I3], is recalled in the following statement:

Proposition 3.9. *For each $\gamma \in \Gamma_g$ the following expression holds:*

$$\kappa(\gamma)^4 = e^{\pi Tr({}^t b_\gamma c_\gamma)i} \tag{3.22}$$

A simple expression for κ^2 on suitable congruence subgroups is also obtained, by using Proposition 3.7:

Proposition 3.10. *For each $\gamma \in \Gamma_g(2)$ the following expression holds:*

$$\kappa(\gamma)^2 = e^{\frac{\pi}{2}Tr(a_\gamma - 1_g)i} \tag{3.23}$$

Proof. Let $\gamma, \gamma' \in \Gamma_g(2)$. With reference to the notation in (1.3) one has:

$$a_{\gamma\gamma'} = 1_g + 2(a_M + a_{M'}) + 4a_M a_{M'} + 4b_M c_{M'}$$

Therefore, by setting:

$$g(\gamma) := e^{\frac{\pi}{2}Tr(a_\gamma - 1_g)i}$$

one has:

$$g(\gamma\gamma') = e^{\frac{\pi}{2}2Tr(a_M + a_{M'})i} = e^{\frac{\pi}{2}2Tra_M i}e^{\frac{\pi}{2}2Tra_{M'})i} =$$
$$= e^{\frac{\pi}{2}Tr(a_\gamma - 1_g)i}e^{\frac{\pi}{2}Tr(a_{\gamma'} - 1_g)i} = g(\gamma)g(\gamma')$$

The function g is thus a character of $\Gamma_g(2)$. However, κ^2 is also a character of $\Gamma_g(2)$, by Proposition 3.7. Then, by using Lemma 3.11, it is plainly found that $g(\gamma) = \kappa^2(\gamma)$ and $g({}^t\gamma^{-1}) = \kappa^2({}^t\gamma^{-1})$, whenever γ is such that $c_\gamma = 0$; since by Proposition 1.4 such elements generates $\Gamma_g(2)$, the characters g and κ^2 coincide on $\Gamma_g(2)$. □

Thanks to (3.23) one has, in particular:

$$k^2(\gamma) = 1 \qquad \forall\, \gamma \in \Gamma(4,8) \tag{3.24}$$

and

$$\kappa^2(-1_{2g}) = \begin{cases} 1 & \text{if } g \text{ is even} \\ -1 & \text{if } g \text{ is odd} \end{cases} \tag{3.25}$$

(3.25) means, in particular, that the function κ^2 is well defined on the quotient group $\Gamma_g(2)/\{\pm 1_{2g}\}$ whenever g is even. Then, Proposition 3.7 implies:

Corollary 3.6. *When g is even, κ^2 is a character of $\Gamma_g(2)/\{\pm 1_{2g}\}$.*

Moreover, one has:

Corollary 3.7. *When g is even, κ^2 is a character of the group $\Gamma_g(2,4)/\{\pm\Gamma_g(4,8)\}$.*

Proof. When g is even, Corollary 3.6 implies, in particular, that κ^2 is a character of $\Gamma_g(2,4)/\{\pm 1_{2g}\}$; hence the thesis follows, since κ^2 is well defined on the quotient $\Gamma_g(2,4)/\{\pm\Gamma_g(4,8)\}$ due to (3.24). □

As concerns the function χ_m introduced in (3.20), a straightforward computation yields:

Lemma 3.12.

$$\chi_m(-1_{2g}) = \begin{cases} 1 & \textit{if } m \textit{ is even} \\ -1 & \textit{if } m \textit{ is odd} \end{cases}$$

Moreover:

Lemma 3.13. *Let A_{ij} $(i, j \neq g)$, B_{ij}, C_{ij} $(i < j)$, B^2_{ii}, C^2_{ii}, $-1_{2g} \in \Gamma_g(2,4)$ be as in Proposition 1.4. Then, for each characteristic $m = \begin{bmatrix} m' \\ m'' \end{bmatrix}$ one has:*

$$\chi_m(A_{ij}) = (-1)^{m'_i m''_j}$$

$$\chi_m(B_{ij}) = (-1)^{m'_i m'_j}$$

$$\chi_m(C_{ij}) = (-1)^{m'''_i m''_j}$$

$$\chi_m(B^2_{ii}) = (-1)^{m'^2_i}$$

$$\chi_m(C^2_{ii}) = (-1)^{m''^2_i}$$

where m'_i and m''_i denote respectively the i-th coordinate of m' and the i-th coordinate of m''.

Proof. By applying (3.18) to the generic element in $\Gamma_g(2,4)$, one finds:

$$\chi_m(\gamma) = \exp\left\{\tfrac{1}{2}{}^t m_1(-b_\gamma m' + a_\gamma m'' - m'')\right\}\exp\left\{-\tfrac{1}{4}({}^t m'{}^t b_\gamma d_\gamma m' + {}^t m''{}^t a_\gamma c_\gamma m'')\right\}$$
$$\forall\, \gamma \in \Gamma_g(2,4) \tag{3.26}$$

thanks to which the values in the statement can be computed. □

By means of Corollary 1.2, the following statements can be straightly proved using Lemmas 3.12 and 3.13:

Lemma 3.14. $\chi^2_m(\gamma) = 1$ *for each* $\gamma \in \Gamma_g(2,4)$

Lemma 3.15. *For each characetristic $m \in C^{(g)}$, χ_m is a character of $\Gamma_g(2,4)$.*

Lemma 3.16. $\chi_m(\gamma) = 1$ *for each* $\gamma \in \Gamma_g(4,8)$.

By Lemmas 3.15 and 3.16 one has, in particular:

Corollary 3.8. *For each characteristic $m \in C^{(g)}$, the function χ_m is well defined on the quotient group:*

$$\chi_m : \Gamma_g(2,4)/\Gamma_g(4,8) \longrightarrow \mathbb{C}^*$$

Moreover χ_m is a character of $\Gamma_g(2,4)/\Gamma_g(4,8)$ [5].

Corollary 3.8 and Lemma 3.12 imply, in particular:

[5]Likewise, one can observe that Lemma 3.14, allows to define χ^2_m on the quotient group $\Gamma_g(2)/\Gamma_g(2,4)$, hence χ^2_m is a character of this group. (cf. [SM2])

Corollary 3.9. *Let $m, n \in C^{(g)}$. If m and n are both even or odd, the function $\chi_m\chi_n$ is well defined on the quotient group:*

$$\chi_m\chi_n : \Gamma_g(2,4)/\{\pm\Gamma_g(4,8)\} \longrightarrow \mathbb{C}^*$$

and is a character of $\Gamma_g(2,4)/\{\pm\Gamma_g(4,8)\}$ [6].

In general, one can prove the following:

Proposition 3.11. *The set $\{\chi_m\}$, parametrized by even characteristics m, is a set of generators for the group of characters of $\Gamma_g(2,4)/\{\pm\Gamma_g(4,8)\}$.*

Proof. A proof can be found in [SM2]. □

Due to (3.19), the action of the modular group Γ_g can be defined on couples of characters $\chi_m\chi_n$. In fact, if one defines for each $\gamma \in \Gamma_g$ the function:

$$\gamma \cdot \chi_m(\eta) := \chi_m(\gamma\eta\gamma^{-1}) \tag{3.27}$$

the following statement holds:

Proposition 3.12. *The law (3.27) defines an action of the modular group Γ_g on the products $\chi_m\chi_n$ satisfying:*

$$\gamma(\chi_m\chi_n) = (\gamma \cdot \chi_m)(\gamma \cdot \chi_n) = \chi_{\gamma^{-1}m}\chi_{\gamma^{-1}n} \tag{3.28}$$

Proof. With reference to the action introduced in (2.1), (3.18) and (3.19) imply for each $\gamma \in \Gamma$:

$$\gamma^{-1}|_1\theta_m\theta_n = \kappa^2(\gamma)e^{2\pi i(\phi_{\gamma^{-1}m}(\gamma)+\phi_{\gamma^{-1}n}(\gamma))}\theta_{\gamma^{-1}m}\theta_{\gamma^{-1}n} \tag{3.29}$$

Now, let be m, n both even. Then (3.29) yields:

$$\kappa^2(\gamma)\kappa^2(\gamma^{-1})e^{2\pi i(\phi_{\gamma^{-1}m}(\gamma)+\phi_{\gamma^{-1}n}(\gamma))}e^{2\pi i(\phi_{\gamma m}(\gamma^{-1})+\phi_{\gamma n}(\gamma^{-1}))} = 1 \tag{3.30}$$

Hence, for each $\gamma \in \Gamma_g$ and $\eta \in \Gamma_g(2,4)$:

$$((\gamma\eta\gamma^{-1})^{-1}|_1\theta_m\theta_n)(\tau) = \kappa^2(\gamma\eta\gamma^{-1})^2\chi_m(\gamma\eta\gamma^{-1})\chi_n(\gamma\eta\gamma^{-1})\theta_m(\tau)\theta_n(\tau) \tag{3.31}$$

But since one also has:

$$\gamma^{-1}|_1\theta_m\theta_n = \kappa^2(\gamma)\chi_m\chi_n\theta_{\gamma^{-1}m}\theta_{\gamma^{-1}n} \qquad \forall\gamma \in \Gamma_g(2,4) \tag{3.32}$$

then (3.30) implies

$$((\gamma\eta\gamma^{-1})^{-1}|_1\theta_m\theta_n)(\tau) = \kappa^2(\eta)\chi_{\gamma^{-1}m}(\eta)\chi_{\gamma^{-1}n}(\eta)\theta_m(\tau)\theta_n(\tau) \tag{3.33}$$

Then the thesis follows from (3.31) and (3.33), as (3.23) implies:

$$\kappa^2(\gamma\eta\gamma^{-1}) = \kappa^2(\eta) \qquad \forall\gamma \in \Gamma_g, \qquad \forall\eta \in \Gamma_g(2,4)$$

Regarding the remaining cases, for each odd characteristic n the holomorphic maps

$$\mathrm{grad}_z^0\theta_n := \mathrm{grad}_z\theta_n|_{z=0} = \left(\frac{\partial}{\partial z_1}\theta_n|_{z=0}, \ldots, \frac{\partial}{\partial z_g}\theta_n|_{z=0}\right)$$

[6]More in general $\chi_m\chi_n$ is proved to be a character of $\Gamma_g(2)/\{\pm 1_g\}$.

satisfy the following transformation law:

$$\begin{aligned}&\mathrm{grad}_z^0\theta_n(\gamma\tau) = \det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}(c_\gamma\tau + d_\gamma)\cdot \mathrm{grad}_z^0\theta_n(\tau)\\ &\forall\gamma\in\Gamma_g(4,8), \quad \forall\tau\in\mathfrak{S}_g\end{aligned} \tag{3.34}$$

Hence, the same argument on $\theta_m \mathrm{grad}_z^0\theta_n$ and $\mathrm{grad}_z^0\theta_m \mathrm{grad}_z^0\theta_n$ proves likewise the statement respectively when m is even and n is odd and when both m and n are odd. □

Thanks to Proposition 3.12 the group Γ_g acts on even sequences of characters χ_m.

The transformation law in (3.34) means, in particular, that gradients of odd Theta functions can be regarded as modular forms with respect to $\Gamma_g(4,8)$ under the representation [7]:

$$T_0(A) := \det(A)^{1/2}\dot{A} \tag{3.35}$$

Such a behaviour allows to define the map, on which this work will focus in the next chapter.

Thanks to the results gathered here, the modular property of the Theta constants can be proved:

Proposition 3.13. *The product of two Theta constants $\theta_m\theta_n$ is a modular form of weight 1 with respect to $\Gamma_g(4,8)$.*

Proof. First of all, by Lemma 3.3 the law (3.19) turns out to be:

$$\begin{aligned}&\theta_m(\gamma\tau) = \kappa(\gamma)\chi_m(\gamma)\det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}\theta_m(\tau)\\ &\forall\gamma\in\Gamma_g(2), \quad \forall m\in C^{(g)}, \quad \forall\tau\in\mathfrak{S}_g,\end{aligned} \tag{3.36}$$

Then, by applying (3.24) and Lemma 3.16, the following transformation law arises:

$$\begin{aligned}&\theta_m\theta_n(\gamma\tau) = \det(c_\gamma\tau + d_\gamma)\theta_m\theta_n(\tau)\\ &\forall\gamma\in\Gamma_g(4,8), \quad \forall m\in C_g, \quad \forall\tau\in\mathfrak{S}_g,\end{aligned} \tag{3.37}$$

which concludes the proof. □

A useful criterion of modularity with respect to $\Gamma_g(2,4)$ can be more generally stated for products of an even sequence of Theta constants:

Proposition 3.14. *Let $M = (m_1,\dots,m_{2k})$ be a sequence of even characteristics. The product $\theta_{m_1}\cdots\theta_{m_{2k}}$ is a modular form with respect to $\Gamma_g(2,4)$ if and only if the following condition holds:*

$$M\,{}^tM \equiv k\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \; mod\, 2 \tag{3.38}$$

[7]The classical modular forms of weight k, which are the only ones discussed in this work, are indeed modular forms under the representation $T(A) := \det(A)^k$

Proof. Let $M = (m_1, \dots, m_{2k})$ be a sequence of even characteristics; one can set:

$$m_h = \begin{bmatrix} m'_h \\ m''_h \end{bmatrix}$$

for each characteristic m_h appearing in the sequence M, and $m'_{h,(i)}$ and $m''_{h,(i)}$ respectively for the i-th coordinate of m'_h and m''_h.
Now, by (3.36) and (3.23), if k is even, $\theta_{m_1} \cdots \theta_{m_{2k}}$ is a modular form with respect to $\Gamma_g(2,4)$ if and only if the character $\chi_M := \chi_{m_1} \cdots \chi_{m_{2k}}$ is trivial on $\Gamma_g(2,4)$. Due to Corollary 1.2, this is true if and only if the elements A_{ij} (for $i, j \neq g$), B_{ij}, C_{ij} (for $i < j$), B^2_{ii}, and C^2_{ii} belong to the kernel $Ker\chi_M$. Then, Lemma 3.13 allows to translate this criterion into conditions for the sequence M:

$$\begin{array}{llll}
a) & \sum_{h=1}^k (m'_{h,(i)})^2 = 0 & \forall\, i & (\ \chi_M(B^2_{ii}) = 1\ ) \\
b) & \sum_{h=1}^k (m''_{h,(i)})^2 = 0 & \forall\, i & (\ \chi_M(C^2_{ii}) = 1\ ) \\
c) & \sum_{h=1}^k m'_{h,(i)} m'_{h,(j)} = 0 & \forall\, i < j & (\ \chi_M(B_{ij}) = 1\ ) \\
d) & \sum_{h=1}^k m''_{h,(i)} m''_{h,(j)} = 0 & \forall\, i < j & (\ \chi_M(C_{ij}) = 1\ ) \\
e) & \sum_{h=1}^k m'_{h,(i)} m''_{h,(j)} = 0 & \forall\, i, j \neq g & (\ \chi_M(A_{ij}) = 1\ )
\end{array}$$

Therefore, for k even, one gains the criterion:

$$M\,{}^tM \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \bmod 2$$

Now, if k is odd, (3.36) and (3.23) imply that $\theta_{m_1} \cdots \theta_{m_{2k}}$ is a modular form with respect to $\Gamma_g(2,4)$ if and only if $\kappa^2\chi_M$ is trivial on $\Gamma_g(2,4)$. Using the definition of these elements (see Proposition 1.4), one can apply (3.23) and verify the A_{ii} are the only elements on which the function κ^2 assumes the value -1. Then conditions a) b) c) and d) keep unchanged, while condition e) is replaced by a twofold condition:

$$\begin{array}{llll}
e') & \sum_{h=1}^k m'_{h,(i)} m''_{h,(j)} = 0 & \forall\, i \neq j & (\ \kappa^2\chi_M(A_{ij}) = 1\ ) \\
e'') & \sum_{h=1}^k m'_{h,(i)} m''_{h,(i)} = 1 & \forall\, i & (\ \kappa^2\chi_M(A_{ij}) = 1\ )
\end{array}$$

Hence, for k odd, one obtains the criterion:

$$M\,{}^tM \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bmod 2$$

and the proof ends. □

As concerns the product of all the non-vanishing Theta constants:

$$\chi^{(g)} := \prod_{m \,\text{even}} \theta_m \tag{3.39}$$

$\chi^{(g)}$ is known to be a modular form of weight $2^{g-2}(2^g+1)$ with respect to $\Gamma_g(4,8)$ for $g \leq 2$ by virtue of Proposition 3.13. However, for each $\gamma \in \Gamma_g$ one has by (3.17):

$$\prod_{m \text{ even}} \theta_m(\gamma\tau) = \prod_{m \text{ even}} \theta_{\gamma m}(\gamma\tau) =$$
$$= \left(\kappa^{2^{g-1}(2^g+1)}(\gamma) \prod_{m \text{ even}} \chi_m(\gamma)\right) \det(c_\gamma\tau + d_\gamma)^{2^{g-2}(2^g+1)} \prod_{m \text{ even}} \theta_m(\tau)$$

and the character appearing is found to be trivial whenever $g \geq 3$. Therefore:

Proposition 3.15. *When $g \geq 3$, $\chi^{(g)}$ is a modular form of weight $2^{g-2}(2^g+1)$ with respect to the full modular group Γ_g.*

The case $g = 2$ is a special one; in fact, the product

$$\chi_5 := \chi^{(2)} := \prod_{m \in C_1} \theta_m \tag{3.40}$$

satisfying the transformation law:

$$\chi_5(\gamma\tau) = \kappa^2(\gamma) \textstyle\prod_{m \in C_1} \chi_m(\gamma) \det(c_\gamma\tau + d_\gamma)^5 \chi_5(\tau) \qquad \forall \gamma \in \Gamma_g \tag{3.41}$$

is not a modular form with respect to Γ_2, since the character appearing in this transformation is not trivial. However, the function:

$$\chi_{10} := \chi_5^2 = \left(\prod_{m \in C_1} \theta_m\right)^2 \tag{3.42}$$

is a modular form of weight 10 with respect to Γ_2.

Other remarkable modular forms with respect to suitable congruence subgroups can be built by the following notable functions:

Definition 3.7. *For each $m' \in \mathbb{Z}_2^g$ the holomorphic function $\Theta_{m'} : \mathfrak{S}_g \times \mathbb{C}^g \to \mathbb{C}$, defined by:*

$$\Theta_{m'}(\tau, z) := \theta_{\left[\begin{smallmatrix} m' \\ 0 \end{smallmatrix}\right]}(2\tau, 2z) \tag{3.43}$$

is called a **second order Theta function**.

In particular, one has the following:

Definition 3.8. *For each $m' \in \mathbb{Z}_2^g$ the holomorphic function:*

$$\Theta_{m'}(\tau) := \theta_{\left[\begin{smallmatrix} m' \\ 0 \end{smallmatrix}\right]}(2\tau, 0) = \theta_{\left[\begin{smallmatrix} m' \\ 0 \end{smallmatrix}\right]}(2\tau) \tag{3.44}$$

is named **second order Theta constant**.

A transformation law for second order Theta constants can be easily derived by focusing on the congruence subgroup $\Gamma_{g,0}(2)$ introduced in (1.5). For each $\gamma \in \Gamma_g$ one can define:

$$\tilde{\gamma} := \begin{pmatrix} a_\gamma & 2b_\gamma \\ \frac{1}{2}c_\gamma & d_\gamma \end{pmatrix} \in Sp(g, \mathbb{Q})$$

Then, $2\gamma\tau = \tilde{\gamma}2\tau$ for each $\tau \in \mathfrak{S}_g$, because of the definition in (1.10). Since $\tilde{\gamma} \in \Gamma_g$ whenever $\gamma \in \Gamma_{g,0}(2)$, one can apply (3.19) to obtain [8]:

$$\begin{gathered} \Theta_{m'}(\gamma\tau) = \theta_{\left[\substack{m' \\ 0}\right]}(\tilde{\gamma}2\tau) = \kappa(\tilde{\gamma})\chi_{\left[\substack{m' \\ 0}\right]}(\tilde{\gamma}) \det(c_\gamma\tau + d_\gamma)^{\frac{1}{2}}\Theta_{m'}(\tau) \\ \forall\gamma \in \Gamma_{g,0}(2), \qquad \forall m' \in \mathbb{Z}_2^g, \qquad \forall\tau \in \mathfrak{S}_g \end{gathered} \tag{3.45}$$

With reference to the notation introduced in (1.3) one has:

$$\tilde{\gamma} = \begin{pmatrix} 1_g + 2a_M & 4b_M \\ c_M & 1_g + 2d_M \end{pmatrix} \qquad \forall\gamma \in \Gamma_g(2)$$

Then, by (3.18) and (3.20) one has:

$$\chi_{\left[\substack{m' \\ 0}\right]}(\tilde{\gamma}) = e^{-\pi i[{}^t m'{}^t b_M(1+2d_M)m']} = e^{-\pi i[{}^t m'{}^t b_M m']} \qquad \forall\gamma \in \Gamma_g(2) \tag{3.46}$$

Hence, $\chi_{\left[\substack{m' \\ 0}\right]}(\tilde{\gamma}) = 1$ whenever $\gamma \in \Gamma_g(2,4)$; therefore, (3.45) implies [9]:

$$\begin{gathered} \Theta_{m'}\Theta_{n'}(\gamma\tau) = \kappa^2(\tilde{\gamma}) \det(c_\tau + d_\gamma)\Theta_{m'}\Theta_{n'}(\tau) \\ \forall\gamma \in \Gamma_g(2,4), \qquad \forall m' \in \mathbb{Z}_2^g, \qquad \forall\tau \in \mathfrak{S}_g \end{gathered} \tag{3.47}$$

Proposition 3.16. *Let be $g \geq 2$. Then, the product of all the second order Theta constants:*

$$F^{(g)} := \prod_{m' \in \mathbb{Z}^g} \Theta_{m'} \tag{3.48}$$

is a modular form of weight 2^{g-1} with respect to $\Gamma_g(2)$.

Proof. With reference to the notation introduced in (1.3) one has:

$$\tilde{\gamma} = \begin{pmatrix} 1_g + 2a_M & 4b_M \\ c_M & 1_g + 2d_M \end{pmatrix} \qquad \forall\,\gamma \in \Gamma_g(2)$$

Then, the thesis follows from (3.47) by applying (3.23), since there are 2^g second order Theta constants appearing in the product (3.48). □

[8] A transformation law for second order Theta constants under the action of the whole Siegel modular group Γ_g can be also derived; however, such an action turns out not to be monomial, since a single second order Theta constant turns into a linear combination of second order Theta constants.

[9] The transformation law (3.47) means the product of two second order Theta constants is a modular form with respect to $\Gamma_g(2,4)$ with a multiplier.

The section ends recalling that second order Theta constants are related to Theta constants by a relation, which can be derived from the so-called Riemann's addition formula (see, for instance, [FK] or [I6]):

Proposition 3.17. *For each $m \in C^{(g)}$ and $\tau \in \mathfrak{S}_g$:*

$$\theta^2_{\left[\begin{smallmatrix} m' \\ m'' \end{smallmatrix}\right]}(\tau, 0) = \sum_{h \in \mathbb{Z}_2^g} (-1)^{t(m'+h)m''} \theta_{\left[\begin{smallmatrix} m'+h \\ 0 \end{smallmatrix}\right]}(2\tau, 0)\theta_{\left[\begin{smallmatrix} h \\ 0 \end{smallmatrix}\right]}(2\tau, 0) \tag{3.49}$$

3.4 Jacobian Determinants

By introducing half-integer weights in the graded ring $A(\Gamma_g(4, 8))$ [10], the transformation law (3.34) for gradients of odd Theta functions can be regarded in terms of Jacobian determinants:

$$D(n_1, \ldots, n_g)(\tau) := \frac{1}{\pi^g} \begin{vmatrix} \frac{\partial}{\partial z_1}\theta_{n_1}|_{z=0}\,(\tau) & \ldots & \frac{\partial}{\partial z_g}\theta_{n_1}|_{z=0}\,(\tau) \\ \vdots & & \vdots \\ \frac{\partial}{\partial z_1}\theta_{n_g}|_{z=0}\,(\tau) & \ldots & \frac{\partial}{\partial z_g}\theta_{n_g}|_{z=0}\,(\tau) \end{vmatrix}$$

Then, (3.34) implies that the functions $D(n_1, \ldots, n_g)$ are modular forms of weight $\frac{g}{2} + 1$ with respect to the congruence subgroup $\Gamma_g(4, 8)$. More in general, (3.17) implies the following:

Proposition 3.18. *Let $N = \{n_1, \ldots n_g\}$ be a sequence of odd g-characteristics and $D(N)$ the correspondent Jacobian determinant; then, one has:*

$$\begin{aligned} &D(\gamma \cdot N)(\gamma\tau) = \kappa(\gamma)^g e^{2\pi i \sum_{i=1}^g \phi_{n_i}(\gamma)} \det(c_\gamma\tau + d_\gamma)^{\frac{g}{2}+1} D(N)(\tau) \\ &\forall \gamma \in \Gamma_g, \quad \forall \tau \in \mathfrak{S}_g, \end{aligned} \tag{3.50}$$

where the functions ϕ_{n_i} are the ones defined in (3.18), corresponding to each characteristic n_i.

The transformation law (3.50) provides a criterion of modularity with respect to the congruence subgroup $\Gamma_g(2, 4)$ for products of Jacobian determinants, which is similar to the one proved in (3.38) for Theta constants:

Proposition 3.19. *Let $N = (N_1, \ldots, N_h)$ be a sequence of g-plets $N_i = \{n_1^i, \ldots n_g^i\}$ of odd g-characteristics. The product $D(N_1) \cdots D(N_h)$ is, then, a modular form with respect to $\Gamma_g(2, 4)$ if and only if the following condition holds:*

$$N\, {}^tN \equiv h \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathit{mod}\, 2 \tag{3.51}$$

[10] A ring can be generally graded by a monoid; for a detailed discussion on graded rings [E] can be consulted.

To point out more properties around the connection between the Theta constants and the Jacobian determinants, it is useful to introduce a function, which is defined on the parts $P(C_e^{(g)})$ of the set $C_e^{(g)}$ of even g-characteristics by mapping sequences of characteristics to the monomial given by the product of the correspondent Theta constants:

$$\begin{aligned} F : P(C_e^g) &\longrightarrow \mathbb{C}[\theta_m] \\ \{m_1, m_2, \dots, m_h\} &\longrightarrow \theta_{m_1}\theta_{m_2}\cdots\theta_{m_h} \end{aligned} \tag{3.52}$$

with $F(\emptyset) := 1$.

Then, the following important Theorem holds (cf. [F] or [I8]):

Theorem 3.1. (Generalized Jacobi's formula) *For $g \leq 5$, one has:*

$$D(n_1, \dots, n_g) = \sum_M \pm F(M)$$

where the summation is extended to all the sequences $M = \{m_1, \dots m_{g+2}\}$ of even characteristics such that $\{n_1, \dots, n_g, m_1, \dots m_{g+2}\}$ is azygetic.

Then, whenever $g \leq 5$, the Jacobian determinants are contained in the ring $\mathbb{C}[\theta_m]$. A similar statement has been conjectured for $g > 5$, but not proved yet.

Example 3.1. *(Case g=1)*
In this case, one obtains the classical Jacobi's derivative formula:

$$\theta'_{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} = -\theta_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}\theta_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\theta_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}$$

Example 3.2. *(Case g=2)*
The 6 odd characteristics yield 15 distinct Jacobian determinants; by Jacobi's derivative formula, these determinants are monomials of degree 4 in the Theta constants. More precisely, for each $M = (m_1, m_2, m_3, m_4) \in C_4^-$ there exists a Jacobian determinant $D(n_i, n_j)$ such that $D(n_i, n_j) = \theta_{m_1}\theta_{m_2}\theta_{m_3}\theta_{m_4}$, and distinct Jacobian determinants correspond to distinct element of C_4^-. Therefore, the map:

$$\begin{aligned} D : C_4^- &\to \mathbb{C}[\theta_m] \\ M = (m_1, m_2, m_3, m_4) &\longmapsto D(M) := \theta_{m_1}\theta_{m_2}\theta_{m_3}\theta_{m_4} \end{aligned} \tag{3.53}$$

is a bijection providing a parametrization for the Jacobian determinants. By suitably enumerating the 6 odd characteristics:

$$\begin{aligned} n^{(1)} &:= \begin{bmatrix} 01 \\ 01 \end{bmatrix} & n^{(2)} &:= \begin{bmatrix} 10 \\ 10 \end{bmatrix} & n^{(3)} &:= \begin{bmatrix} 01 \\ 11 \end{bmatrix} \\ n^{(4)} &:= \begin{bmatrix} 10 \\ 11 \end{bmatrix} & n^{(5)} &:= \begin{bmatrix} 11 \\ 01 \end{bmatrix} & n^{(6)} &:= \begin{bmatrix} 11 \\ 10 \end{bmatrix} \end{aligned}$$

and the 10 *even characteristics:*

$$m^{(1)} := \begin{bmatrix} 00 \\ 00 \end{bmatrix} \quad m^{(2)} := \begin{bmatrix} 00 \\ 01 \end{bmatrix} \quad m^{(3)} := \begin{bmatrix} 00 \\ 10 \end{bmatrix} \quad m^{(4)} := \begin{bmatrix} 00 \\ 11 \end{bmatrix} \quad m^{(5)} := \begin{bmatrix} 01 \\ 00 \end{bmatrix}$$
$$m^{(6)} := \begin{bmatrix} 10 \\ 00 \end{bmatrix} \quad m^{(7)} := \begin{bmatrix} 11 \\ 00 \end{bmatrix} \quad m^{(8)} := \begin{bmatrix} 01 \\ 10 \end{bmatrix} \quad m^{(9)} := \begin{bmatrix} 10 \\ 01 \end{bmatrix} \quad m^{(10)} := \begin{bmatrix} 11 \\ 11 \end{bmatrix}$$

and by denoting the respective non-trivial Theta constants by $\theta_i := \theta_{m^{(i)}}$ *for each* $i = 1, \ldots 10$*, one has:*

$$D(n^{(1)}, n^{(2)}) = \theta_2\theta_3\theta_5\theta_6; \quad D(n^{(1)}, n^{(3)}) = -\theta_6\theta_7\theta_9\theta_{10}; \quad D(n^{(1)}, n^{(4)}) = \theta_1\theta_4\theta_5\theta_9$$
$$D(n^{(1)}, n^{(5)}) = -\theta_3\theta_4\theta_8\theta_{10}; \quad D(n^{(1)}, n^{(6)}) = \theta_1\theta_2\theta_7\theta_8; \quad D(n^{(2)}, n^{(3)}) = -\theta_1\theta_4\theta_6\theta_8$$
$$D(n^{(2)}, n^{(4)}) = -\theta_5\theta_7\theta_8\theta_{10}; \quad D(n^{(2)}, n^{(5)}) = -\theta_1\theta_3\theta_7\theta_9; \quad D(n^{(2)}, n^{(6)}) = \theta_2\theta_4\theta_9\theta_{10}$$
$$D(n^{(3)}, n^{(4)}) = \theta_2\theta_3\theta_8\theta_9; \quad D(n^{(3)}, n^{(5)}) = -\theta_1\theta_2\theta_5\theta_{10}; \quad D(n^{(3)}, n^{(6)}) = \theta_3\theta_4\theta_5\theta_7$$
$$D(n^{(4)}, n^{(5)}) = -\theta_2\theta_4\theta_6\theta_7; \quad D(n^{(4)}, n^{(6)}) = \theta_1\theta_3\theta_6\theta_{10}; \quad D(n^{(5)}, n^{(6)}) = \theta_5\theta_6\theta_8\theta_9$$

3.5 Classical Structure Theorems

This section aims to outline some classical theorems, describing the structure of the ring of modular forms with respect to the Siegel modular group.

A classical result of the elliptic theory states that the modular forms with respect to Γ_1, namely the modular functions, are generated by the Eisenstein series E_4 and E_6 (see Example 2.1):

Theorem 3.2.

$$A(\Gamma_1) \equiv \mathbb{C}[E_4, E_6]$$

Proof. A proof can be found, for instance, in [FB] or [DS]. □

Pertaining to the ideal $S(\Gamma_1)$ of the cusp forms, one has, moreover:

Theorem 3.3. $S(\Gamma_1)$ *is generated by the cusp form* Δ *(cf. Example 2.3).*

The structure of the ring $A(\Gamma_2)$ reveals itself to be more complicated. In fact, the respective generalized Eisenstein series $E_4^{(2)}$ and $E_6^{(2)}$ (see Example 2.2) are not enough to generate the modular form χ_{10} introduced in (3.42); indeed, such form is generated by $E_4^{(2)}$, $E_6^{(2)}$ and $E_{10}^{(2)}$ (cf. [I2] or [VdG]).
In [I1] Juni-ichi Igusa found the generators of the even part $A(\Gamma_2)^{(e)}$ of the graded ring $A(\Gamma_2)$:

Theorem 3.4.

$$A(\Gamma_2)^{(e)} = \mathbb{C}[E_4^{(2)}, E_6^{(2)}, \chi_{10}, \chi_{12}]$$

where:

$$\chi_{12} = \frac{1}{2^{17}3} \sum_{\{m_{i_1},\dots,m_{i_6}\}\in C_6^-} \pm(\theta_{m_{i_1}} \cdots \theta_{m_{i_6}})^4$$

is a modular form of weight 12*, produced by a suitable simmetrization gained by a proper choice of the signs.*

In [I2], Igusa proved that only a modular form χ_{35} of odd weight has to be added, in order to generate the whole ring $A(\Gamma_2)$. With reference to the notation introduced in Section 3.2, a description of such a modular form can be given, as for χ_{12}, in terms of a symmetrization of products of Theta constants (cf. [I2] and [I7]):

$$\chi_{35} = -\frac{i}{2^{39}5^3}\left(\prod_{m\in C_1}\theta_m\right)\left(\sum_{\{m_i,m_j,m_k\}\in C_3^-} \pm(\theta_{m_i}\theta_{m_j}\theta_{m_k})^{20}\right) \tag{3.54}$$

Then, Igusa's structure theorem can be stated:

Theorem 3.5. (Igusa)

$$A(\Gamma_2) = \mathbb{C}[E_4^{(2)}, E_6^{(2)}, \chi_{10}, \chi_{12}, \chi_{35}]$$

In [I2] Igusa also proved a structure theorem for the ideal $S(\Gamma_2)$ of cusp forms:

Theorem 3.6. *$S(\Gamma_2)$ is generated by χ_{10}, χ_{12} and χ_{35}.*

The modular form χ_{35}, in particular, admits a factor:

$$\chi_{30} = \frac{1}{8} \sum_{\{m_i,m_j,m_k\}\in C_3^-} \pm(\theta_{m_i}\theta_{m_j}\theta_{m_k})^{20} \tag{3.55}$$

which is is a modular form of weight 30 with a non trivial character (cf. [I2]). This modular form is also involved as a generator in another structure theorem. By (3.7), the group $\Gamma_2/\Gamma_2(2)$ has only a non trivial irreducible representation of degree 1, corresponding to the sign of the permutation; then, one can denote the respective character by χ_P, so that:

$$\chi_P([\gamma]) := \begin{cases} 1 & \text{if} \quad \psi_P([\gamma]) \quad \text{is an even permutation} \\ \\ -1 & \text{if} \quad \psi_P([\gamma]) \quad \text{is an odd permutation} \end{cases} \tag{3.56}$$

where ψ_P is the isomorphism described in (3.6). Then, by setting $\Gamma_2^+ := Ker\chi_P$, which is of index 2 in Γ_g, the following structure theorem, proved by Igusa in [I2], holds:

Theorem 3.7.

$$A(\Gamma_2^+) = \mathbb{C}[E_4^{(2)}, E_6^{(2)}, \chi_5, \chi_{12}, \chi_{30}]$$

3.6 A New Result: Another Description for χ_{30}

This section is devoted to the presentation of the first new result proposed in this work. A new description will be provided for the modular form χ_{30}, introduced in (3.55); due to this purpose, a focus on a particular construction, described by Bert Van Geemen and Duco Van Straten in their paper [VGVS] will be needed.

When $g = 2$ there are four second order Theta constants, which can be conventionally enumerated for the sake of simplicity:

$$\Theta_1 := \Theta_{[0\,0]} \quad \Theta_2 := \Theta_{[0\,1]} \quad \Theta_3 := \Theta_{[1\,0]} \quad \Theta_4 := \Theta_{[1\,1]} \tag{3.57}$$

Then, (3.49) provides homogeneous relations between the ten first order Theta constants and these four ones; more precisely, with reference to the notation introduced in Example 3.2 for even Theta constants, one has:

$$\begin{aligned}
\theta_1^2 &= \Theta_1^2 + \Theta_2^2 + \Theta_3^2 + \Theta_4^2; & \theta_2^2 &= \Theta_1^2 - \Theta_2^2 + \Theta_3^2 - \Theta_4^2;\\
\theta_3^2 &= \Theta_1^2 + \Theta_2^2 - \Theta_3^2 - \Theta_4^2; & \theta_4^2 &= \Theta_1^2 - \Theta_2^2 - \Theta_3^2 + \Theta_4^2;\\
\theta_5^2 &= 2\Theta_1\Theta_2 + 2\Theta_3\Theta_4; & \theta_6^2 &= 2\Theta_1\Theta_3 + 2\Theta_2\Theta_4;\\
\theta_7^2 &= 2\Theta_1\Theta_4 + 2\Theta_2\Theta_3; & \theta_8^2 &= 2\Theta_1\Theta_2 - 2\Theta_3\Theta_4;\\
\theta_9^2 &= 2\Theta_1\Theta_3 - 2\Theta_2\Theta_4; & \theta_{10}^2 &= 2\Theta_1\Theta_4 - 2\Theta_2\Theta_3;
\end{aligned}$$

A quadratic form Q_m in the variables X_1, X_2, X_3, X_4 is, therefore, associated to each even 2-characteristic $m \in C_1$:

$$m \mapsto Q_m \qquad \theta_m^2 = Q_m(\Theta_1, \Theta_2, \Theta_3, \Theta_4) \tag{3.58}$$

Hence, a related quadric V_m in P^3 corresponds to each $m \in C_1$:

$$V_m := V(Q_m) = \{[X_1, X_2, X_3, X_4] \in P^3 \mid Q_m(X_1, X_2, X_3, X_4) = 0\} \tag{3.59}$$

Proposition 3.20. *With reference to the notation introduced in Section 3.2, for each 4-plet $M \in C_4^+$, the set:*

$$\bigcap_{m \in M^c} V_m \subset P^3 \tag{3.60}$$

contains exactly of four points.

Proof. The statement needs solely to be proved for a specific element $M \in C_4^+$,C_4^+ being an orbit under the action of Γ_g. By focusing, then, on the 4-plet:

$$M_1 := \{m^{(1)}, m^{(2)}, m^{(3)}, m^{(4)}\} = \left\{\begin{bmatrix}00\\00\end{bmatrix}, \begin{bmatrix}00\\01\end{bmatrix}, \begin{bmatrix}00\\10\end{bmatrix}, \begin{bmatrix}00\\11\end{bmatrix}\right\} \in C_4^+ \tag{3.61}$$

one has:

$$\bigcap_{m\in M^c} V_m = \{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$$

which concludes the proof. □

By virtue of Proposition 3.20, the four points described in (3.60) uniquely determine for each $M \in C_4^+$ a configuration of four hyperplanes in P^3, each of them being characterized by passing through all except one of this points; hence, a collection of four linear forms, describing these four hyperplanes, is determined by each $M \in C_4^+$:

$$\begin{cases} \psi_1^M := \psi_1^M(X_1,X_2,X_3,X_4) \\ \psi_2^M := \psi_2^M(X_1,X_2,X_3,X_4) \\ \psi_3^M := \psi_3^M(X_1,X_2,X_3,X_4) \\ \psi_1^M := \psi_1^M(X_1,X_2,X_3,X_4) \end{cases} \tag{3.62}$$

A tetrahedron T_M in the projective space $\mathbb{P}^3$ is thus uniquely associated to each 4-plet $M = (m_1,m_2,m_3,m_4) \in C_4^+$:

$$M = (m_1,m_2,m_3,m_4) \mapsto T_M \tag{3.63}$$

in such a way the vertices are the four points in $\bigcap_{m\in M^c} V_m$ and all the points on the tetrahedron T_M belong to the vanishing set:

$$\begin{aligned} &\{\,[X_1,X_2,X_3,X_4] \in P^3 \mid \tilde{F}_M(X_1,X_2,X_3,X_4) = 0\,\} \\ &\text{where} \quad \tilde{F}_M := \textstyle\prod_{i=1}^4 \psi_i^M \end{aligned} \tag{3.64}$$

By a straightforward calculation for each $M \in C_4^+$, one is easily supplied with the linear forms appearing in each configuration:

$$T_1 : \begin{cases} \psi_1^1 = X_1 \\ \psi_2^1 = X_2 \\ \psi_3^1 = X_3 \\ \psi_4^1 = X_4 \end{cases}$$

$$T_2 : \begin{cases} \psi_1^1 = X_1 - X_3 \\ \psi_2^1 = X_1 + X_3 \\ \psi_3^1 = X_2 - X_4 \\ \psi_4^1 = X_2 + X_4 \end{cases} \qquad T_3 : \begin{cases} \psi_1^1 = X_1 - X_2 \\ \psi_2^1 = X_1 + X_2 \\ \psi_3^1 = X_3 - X_4 \\ \psi_4^1 = X_3 + X_4 \end{cases} \qquad T_4 : \begin{cases} \psi_1^4 = X_1 - X_4 \\ \psi_2^4 = X_1 + X_4 \\ \psi_3^4 = X_2 - X_3 \\ \psi_4^4 = X_2 + X_3 \end{cases}$$

$$T_5 : \begin{cases} \psi_1^5 = X_1 + iX_4 \\ \psi_2^5 = X_1 - iX_4 \\ \psi_3^5 = X_2 + iX_3 \\ \psi_4^5 = X_2 - iX_3 \end{cases} \qquad T_6 : \begin{cases} \psi_1^6 = X_1 + iX_2 \\ \psi_2^6 = X_1 - iX_2 \\ \psi_3^6 = X_3 + iX_4 \\ \psi_4^6 = X_3 - iX_4 \end{cases} \qquad T_7 : \begin{cases} \psi_1^7 = X_1 - iX_3 \\ \psi_2^7 = X_1 + iX_3 \\ \psi_3^7 = X_2 + iX_4 \\ \psi_4^7 = X_2 - iX_4 \end{cases}$$

$$T_8 : \begin{cases} \psi_1^8 = X_1 - X_2 + X_3 - X_4 \\ \psi_2^8 = X_1 + X_2 - X_3 - X_4 \\ \psi_3^8 = X_1 - X_2 - X_3 + X_4 \\ \psi_4^8 = X_1 + X_2 - X_3 + X_4 \end{cases} \qquad T_9 : \begin{cases} \psi_1^9 = X_1 - X_2 - X_3 - X_4 \\ \psi_2^9 = X_1 - X_2 + X_3 + X_4 \\ \psi_3^9 = X_1 + X_2 + X_3 - X_4 \\ \psi_4^9 = X_1 + X_2 - X_3 + X_4 \end{cases}$$

$$T_{10} : \begin{cases} \psi_1^{10} = X_1 + iX_2 - X_3 - iX_4 \\ \psi_2^{10} = X_1 - iX_2 - X_3 + iX_4 \\ \psi_3^{10} = X_1 + iX_2 + X_3 + iX_4 \\ \psi_4^{10} = X_1 - iX_2 + X_3 - iX_4 \end{cases} \qquad T_{11} : \begin{cases} \psi_1^{11} = X_1 + iX_2 - X_3 + iX_4 \\ \psi_2^{11} = X_1 - iX_2 + X_3 + iX_4 \\ \psi_3^{11} = X_1 + iX_2 + X_3 - iX_4 \\ \psi_4^{11} = X_1 - iX_2 - X_3 - iX_4 \end{cases}$$

$$T_{12} : \begin{cases} \psi_1^{12} = X_1 + iX_2 + iX_3 + X_4 \\ \psi_2^{12} = X_1 + iX_2 - iX_3 - X_4 \\ \psi_3^{12} = X_1 - iX_2 + iX_3 - X_4 \\ \psi_4^{12} = X_1 - iX_2 - iX_3 + X_4 \end{cases} \qquad T_{13} : \begin{cases} \psi_1^{13} = X_1 + iX_2 - iX_3 + X_4 \\ \psi_2^{13} = X_1 + iX_2 + iX_3 - X_4 \\ \psi_3^{13} = X_1 - iX_2 + iX_3 + X_4 \\ \psi_4^{13} = X_1 - iX_2 - iX_3 - X_4 \end{cases}$$

$$T_{14} : \begin{cases} \psi_1^{14} = X_1 - X_2 + iX_3 + iX_4 \\ \psi_2^{14} = X_1 - X_2 - iX_3 - iX_4 \\ \psi_3^{14} = X_1 + X_2 + iX_3 - iX_4 \\ \psi_4^{14} = X_1 + X_2 - iX_3 + iX_4 \end{cases} \qquad T_{15} : \begin{cases} \psi_1^{15} = X_1 - X_2 + iX_3 - iX_4 \\ \psi_2^{15} = X_1 - X_2 - iX_3 + iX_4 \\ \psi_3^{15} = X_1 + X_2 - iX_3 - iX_4 \\ \psi_4^{15} = X_1 + X_2 + iX_3 + iX_4 \end{cases}$$

These 15 configurations, describing the tetrahedrons T_M, can be used to build a modular form with respect to the Siegel modular group Γ_2. With reference to the notation introduced in (3.57), a holomorphic function can be associated to each $\tilde{F}_M$ (cf. (3.64)):

$$F_M(\tau) := \tilde{F}_M(\Theta_1(\tau), \Theta_2(\tau), \Theta_3(\tau), \Theta_4(\tau))$$

In particular:

$$F_1(\tau) := F_{M_1}(\tau) = \Theta_1(\tau)\Theta_2(\tau)\Theta_3(\tau)\Theta_4(\tau) = F^{(2)}(\tau) \tag{3.65}$$

since T_1 is the tetrahedron associated to the 4-plet M_1 in (3.61). F_1 is, therefore, the modular form of weight 2 with respect to $\Gamma_2(2)$ introduced in (3.48).
The product of all the 15 modular forms associated to the configurations is thus a good candidate to be investigated.

One has to remark that $\Gamma_{2,0}(2)$, as defined in (1.5), is the stabilizer St_{M_1} of M_1. $\Gamma_{2,0}(2) \subset St_{M_1}$ by (3.2), while, on the other hand, $\gamma \in St_{M_1}$ implies:

$$diag({}^t c_\gamma d_\gamma) - c_\gamma m'' \equiv 0 \bmod 2 \qquad \forall m'' \in \mathbb{Z}_2^2$$

which yields $c_\gamma \equiv 0 \bmod 2$, hence $\gamma \in \Gamma_{2,0}(2)$.
Since C_4^+ is an orbit, then $[\Gamma_2 : \Gamma_{2,0}(2)] = |C_4^+| = 15$; hence, a bijection exists between $\Gamma_2/\Gamma_{2,0}(2)$ and the tetrahedrons, being indeed provided by the map:

$$[\gamma] \mapsto T_{\gamma M_1}$$

Now, for the element:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_{2,0}(2)$$

one has, by (3.45), (3.20) and Lemma 3.11:

$$F_1(\gamma_0\tau) = -F_1(\tau)$$

Then F_1 admits a non trivial character under the action of $\Gamma_{2,0}(2)$; such a character is of course trivial on its kernel $\Gamma^+_{2,0}$ and extends to the non trivial character χ_P of Γ_2 described in (3.56), whose kernel is Γ^+_2 (cf. [Ib]); hence χ extends to χ_P, this one being the sole non-trivial character of $\Gamma_2/\Gamma_2(2)$, and:

$$F_1(\eta\tau) = \chi_P(\eta)\, det(c_\eta\tau + d_\eta)^2\, F_1(\tau) \qquad \forall \quad \eta \in \Gamma_{2,0}(2)$$

A holomorphic function can be defined now for any fixed $\gamma \in \Gamma_2$:

$$\varphi_\gamma(\tau) := \chi_P(\gamma)^{-1}(\gamma^{-1}|_2F_1)(\tau)$$

In particular, for any hoice $\gamma_1, \cdots \gamma_{15}$ of the representatives of the fifteen cosets of $\Gamma_{2,0}(2)$ in Γ_2 a notable holomorphic function can be defined:

$$\varphi(\tau) := \prod_{i=1}^{15} \varphi_{\gamma_i}(\tau)$$

which actually does not depend on the choice of the coset representatives; in fact, if $\gamma_1, \ldots, \gamma_{15}$ and $\gamma'_1, \ldots, \gamma'_{15}$ are two different choices for the fifteen representatives, then there exists a permutation j of the indices such that for each $i = 1, \ldots, 15$ one has $\gamma'_{j(i)} = \eta_i\gamma_i$ with $\eta_i \in \Gamma_{2,0}(2)$. Hence:

$$\begin{aligned}\prod_{j=1}^{15} \varphi_{\gamma'_j}(\tau) = \prod_{i=1}^{15} \varphi_{\gamma'_{j(i)}}(\tau) = \prod_{i=1}^{15} \varphi_{\eta_i\gamma_i}(\tau) = \prod_{i=1}^{15} \chi_P(\eta_i\gamma_i)^{-1}((\eta_i\gamma_i)^{-1}|_2F_1)(\tau) = \\ = \prod_{i=1}^{15} \chi_P(\eta_i\gamma_i)^{-1}\, det(c_{\eta_i\gamma_i}\tau + d_{\eta_i\gamma_i})^{-2}\, \chi_P(\eta_i)\, det(c_{\eta_i}\gamma_i\tau + d_{\eta_i})^2\, F_1(\gamma_i\tau) = \\ = \prod_{i=1}^{15} \chi_P(\gamma_i)^{-1}\, det(c_{\gamma_i}\tau + d_{\gamma_i})^{-2}\, F_1(\gamma_i\tau) = \prod_{i=1}^{15} \varphi_{\gamma_i}(\tau)\end{aligned}$$

Now the main theorem of the section can be stated:

Theorem 3.8. *There exists $\lambda \in \mathbb{C}^*$ such that $\varphi = \lambda\, \chi_{30}$.*

Proof. The transformation formula $\varphi(\gamma\tau) = \chi_P(\gamma)\, det(c_\gamma\tau + d_\gamma)^{30}\varphi(\tau)$ has to be proved first, whenever $\gamma \in \Gamma_2$. Let thus $\gamma_1, \ldots, \gamma_{15}$ be a fixed collection of coset representatives of $\Gamma_{2,0}(2)$ in Γ_2; then, for each $\gamma \in \Gamma_2$ one has:

$$\begin{aligned}
\varphi(\gamma\tau) &= \prod_{i=1}^{15} \varphi_{\gamma_i}(\gamma\tau) = \prod_{i=1}^{15} \chi_P(\gamma_i)^{-1} det(c_{\gamma_i}\gamma\tau + d_{\gamma_i})^{-2}\, F_1(\gamma_i\gamma\tau) = \\
&= \prod_{i=1}^{15} \chi_P(\gamma_i)^{-1}\, det(c_{\gamma_i\gamma}\tau + d_{\gamma_i\gamma})^{-2}\, det(c_\gamma\tau + d_\gamma)^2\, F_1(\gamma_i\gamma\tau) = \\
&= det(c_\gamma\tau + d_\gamma)^{30} \prod_{i=1}^{15} \chi_P(\gamma_i)^{-1}\, det(c_{\gamma_i\gamma}\tau + d_{\gamma_i\gamma})^{-2}\, F_1(\gamma_i\gamma\tau) = \\
&= det(c_\gamma\tau + d_\gamma)^{30} \prod_{i=1}^{15} \chi_P(\gamma_i)^{-1}\, \chi_P(\gamma_i\gamma)\, \varphi_{\gamma_i\gamma}(\tau) = \\
&= \chi_P(\gamma)^{15} det(c_\gamma\tau + d_\gamma)^{30} \prod_{i=1}^{15} \varphi_{\gamma_i\gamma}(\tau) = \ \chi_P(\gamma)\, det(c_\gamma\tau + d_\gamma)^{30}\varphi(\tau)
\end{aligned}$$

where the last equality holds as φ does not depend on the choice of the fifteen coset representetives. Hence φ is a modular form of weight 30 with respect to Γ_2^+ with character χ_P under the action of Γ_2; hence, the thesis follows by Theorem 3.7, since χ_{30} is the only modular form of weight 30 with respect to Γ_2^+ with a non-trivial character under the action of the full modular group. □

Such an expression for χ_{30} has been also found by Aloys Krieg and Dominic Gehre in a completely different way, by means of quaternionic Theta constants (cf. [Kr]).

3.7 Rings of Modular Forms with Levels

This section is devoted to recall important structure theorems for the rings of modular forms with respect to some of the congruence subgroups described in the Section 1.2.

As concerns modular forms with respect to the congruence subgroup $\Gamma_g(4, 8)$, Theta constants with characteristics have already been described as a chief example; indeed their fundamental importance is soon revealed.
It is a remarkable fact that, for $g = 1, 2$, the only independent relations among the Theta constants θ_m are the so called Riemann's relations, which can be derived from (3.49) in a simple way.
For $g = 1$, (3.49) implies for the three non trivial Theta constants:

$$\theta^2_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}(\tau,0) = \theta^2_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}(2\tau,0) + \theta^2_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}(2\tau,0)$$

$$\theta^2_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}(\tau,0) = 2\theta_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}(2\tau,0)\, \theta_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}(2\tau,0)$$

$$\theta^2_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}(\tau,0) = \theta^2_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}(2\tau,0) - \theta^2_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}(2\tau,0)$$

Therefore, one obtains the only non trivial Riemann's relation for $g = 1$:

$$\theta^4_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]} - \theta^4_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} - \theta^4_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} = 0$$

Two kinds of Riemann's relations arise when $g = 2$, which can be likewise derived from (3.49); with reference to the notation introduced in Example 3.2 for even characteristics, one has 15 biquadratic Riemann's relations:

$$\begin{array}{lll}
\theta_2^2\theta_3^2 = \theta_1^2\theta_4^2 - \theta_7^2\theta_{10}^2; & \theta_2^2\theta_5^2 = \theta_7^2\theta_9^2 + \theta_4^2\theta_8^2; & \theta_3^2\theta_5^2 = \theta_9^2\theta_{10}^2 + \theta_1^2\theta_8^2; \\
\theta_2^2\theta_6^2 = \theta_1^2\theta_9^2 + \theta_8^2\theta_{10}^2; & \theta_3^2\theta_6^2 = \theta_4^2\theta_9^2 + \theta_7^2\theta_8^2; & \theta_6^2\theta_5^2 = \theta_1^2\theta_7^2 - \theta_4^2\theta_{10}^2; \\
\theta_6^2\theta_7^2 = \theta_3^2\theta_8^2 - \theta_1^2\theta_5^2; & \theta_6^2\theta_{10}^2 = \theta_4^2\theta_5^2 - \theta_2^2\theta_8^2; & \theta_6^2\theta_9^2 = \theta_1^2\theta_2^2 - \theta_3^2\theta_4^2; \\
\theta_5^2\theta_9^2 = \theta_2^2\theta_7^2 - \theta_3^2\theta_{10}^2; & \theta_4^2\theta_6^2 = \theta_5^2\theta_{10}^2 + \theta_3^2\theta_9^2; & \theta_1^2\theta_6^2 = \theta_5^2\theta_7^2 - \theta_2^2\theta_9^2; \\
\theta_6^2\theta_8^2 = \theta_3^2\theta_7^2 - \theta_2^2\theta_{10}^2; & \theta_5^2\theta_8^2 = \theta_1^2\theta_3^2 - \theta_2^2\theta_4^2; & \theta_8^2\theta_9^2 = \theta_4^2\theta_7^2 - \theta_1^2\theta_{10}^2;
\end{array}$$

and 15 quartic Riemann's relations, amongst which there are only 5 independent relations:

$$\begin{array}{lll}
\theta_1^4 - \theta_4^4 - \theta_5^4 - \theta_9^4 = 0; & \theta_2^4 - \theta_3^4 + \theta_5^4 - \theta_6^4 = 0; & \theta_2^4 - \theta_3^4 + \theta_8^4 - \theta_9^4 = 0; \\
\theta_1^4 - \theta_3^4 - \theta_6^4 - \theta_{10}^4 = 0; & \theta_1^4 - \theta_2^4 - \theta_7^4 - \theta_8^4 = 0; &
\end{array}$$

The 15 biquadratic Riemann's relations correspond to the elements of C_6^+ by the bijective map:

$$M = (m_1, \dots m_6) \quad \longmapsto \quad R_2(M) \quad : \theta^2_{m_1}\theta^2_{m_2} \pm \theta^2_{m_3}\theta^2_{m_4} \pm \theta^2_{m_5}\theta^2_{m_6} \tag{3.66}$$

while the 15 quartic Riemann's relations correspond to the elements of C_4^- by the bijective map:

$$M = (m_1, \dots m_4) \quad \longmapsto \quad R_4(M) \quad : \theta^4_{m_1} \pm \theta^4_{m_2} \pm \theta^4_{m_3} \pm \theta^4_{m_4} \tag{3.67}$$

Let $C[\theta_m]$ be the $\mathbb{C}$-algebra generated by the even Theta constants. Denoting by I_R the ideal generated by the Riemann's Relations one has, as already stated:

$$\mathbb{C}[\theta_m] = \frac{\mathbb{C}[X_m]}{I_R} \qquad (g = 1, 2) \tag{3.68}$$

For $g \geq 3$ Riemann's relations still provide independent relations between the generators θ_m; it is still not known, however, whether they are the only independent relations between generators for the ring $\mathbb{C}[\theta_m]$ or not. As for the ring in (3.68), Riemann's relations can be used to prove the following:

Theorem 3.9. *The graded ring $\mathbb{C}[\theta_m]$ is normal when $g = 1, 2$.*

Proof. The proof can be found in [I3] for the case $g = 1$ and in [I2] for the case $g = 2$. □

Corollary 3.10. *The graded ring $\mathbb{C}[\theta_m\theta_n] = \mathbb{C}[\theta_m]^{(e)}$ is normal.*

The ring $\mathbb{C}[\theta_m\theta_n]$ reveals itself to be strictly connected with the modular forms with respect to the congruence subgroup $\Gamma_g(4, 8)$, as a classical result proved by Igusa in [I3] states:

Theorem 3.10. (Igusa) *$A(\Gamma_g(4, 8))$ is the normalization of the ring $\mathbb{C}[\theta_m\theta_n]$.*

As a straight consequence of Corollary (3.10), the prominent role of Theta constants as generators of modular forms is clear:

Corollary 3.11. *$A(\Gamma_g(4, 8)) = \mathbb{C}[\theta_m\theta_n]$ when $g = 1, 2$.*

The section ends by recalling a structure theorem for cusp forms with respect to the subgroup $\Gamma_2(2, 4, 8)$ defined in (1.8), which will be used in the next chapter to prove new results; this structure theorem was proved by Bert van Geemen and Duco van Straten in [VGVS], also with reference to the geometrical construction described in Section 3.6.

Theorem 3.11. (van Geemen, van Straten) *A set of generators for the ideal $S(\Gamma_2(2, 4, 8))$ of the cusp forms with respect to $\Gamma_2(2, 4, 8)$ is given by:*

1. $D(M) \qquad \forall\, M \in C_4^-$
2. $\theta_{m_1}\theta_{m_2}\theta_{m_3}\theta_{m_4}\theta_{m_5} \qquad \forall\, \{m_1 \dots, m_5\} \in C_5^*$
3. $\theta_{m_1}\theta_{m_2}\theta_{m_3}\psi_i^M, \psi_j^M, \psi_k^M \qquad \forall\, \{m_1, m_2, m_3\} \in C_3^+ \qquad \forall\, i, j, k$

where, for each $\{m_1, m_2, m_3\} \in C_3^+$, $M \in C_4^+$ is the only element such that $\{m_1, m_2, m_3\} \subset M$ and $\psi_i^M, \psi_j^M, \psi_k^M$ are three of the four linear forms associated to T_M as in (3.62).

Chapter 4

The Group Γ and the structure of $A(\Gamma)$

This chapter aims to present the new results found, concerning with a remarkable map made by the gradients of odd Theta functions in genus 2. The first section will be devoted to outline the general description of this map, thus pointing out the exceptional features pertaining to the case $g = 2$, on which the other sections will be focused.
Since every statement will pertain to the case $g = 2$, the obvious indication of the grade will be conventionally omitted throughout the chapter.
For the sake of simplicity, the group $\Gamma(2,4)/\{\pm\Gamma(4,8)\}$, which will be involved in the discussion, will be denoted by the symbol G, while the symbol $\hat{G}$ will stand for the group of characters of G.

4.1 The Theta Gradients Map

By the transformation formula (3.34) one is allowed to define a map on the quotient space $A_g^{4,8} := \mathfrak{S}_g/\Gamma_g(4,8)$:

$$\mathbb{P}grTh : A_g^{4,8} \longrightarrow \overbrace{\mathbb{C}^g \times \cdots \times \mathbb{C}^g}^{2^{g-1}(2^g-1)\text{times}} / T_0(Gl(g,\mathbb{C}))$$

$$\tau \longrightarrow \left\{\operatorname{grad}_z \theta_n|_{z=0}\right\}_{n\,\mathrm{odd}}$$

where T_0 is the representation defined in (3.35).

Due to Lefschetz's theorem (cf. [GH]) the range of this map lies in the Grassmannian $Gr_{\mathbb{C}}(g, 2^{g-1}(2^g-1))$, as proved by Riccardo Salvati Manni in [SM1]; thanks to (3.34) the Plücker coordinates of this map are modular forms with respect to $\Gamma_g(4,8)$.

Studying the injectivity of such a map is strictly related to the problem of recovering plane curves form their tangent hyperplanes. In fact, denoting by:

$$\phi : C \to \mathbb{P}^{g-1}$$

the canonical map of a smooth curve C of genus g, and by $\tau_C = [J(C)] \in J_g \subset A_g$ the correspondent point in the locus of Jacobians J_g, a hyperplane $H \subset \mathbb{P}^{g-1}$, tangent to the canonical curve $\phi(C)$ in $g-1$ points, is known to cut on $\phi(C)$ a divisor, which is the zero locus of one of the $2^{g-1}(2^g - 1)$ Riemann Theta functions with odd characteristics $\theta_n(z) = \theta_n(\tau_C, z)$; on the converse, each Riemann Theta function with odd characteristic related to the curve C determines such a hyperplane $H \subset \mathbb{P}^{g-1}$, whose direction is given by the gradient of the correspondent Riemann Theta function in zero (see [GH] and [ACGH] for details). In particular, the map $\mathbb{P}grTh$ sends an element $\tau \in J_g^{4,8} \subset A_g^{4,8}$ to an ordered set of all the hyperplanes tangent in $g-1$ points; it is, therefore, related to the map which sends $\tau \in J_g$ to the sets of all the hyperplanes tangent in $g-1$ points, whose injectivity has been already investigated by Lucia Caporaso and Edoardo Sernesi in [CS1] and [CS2]. Indeed, such a map factors through $\mathbb{P}grTh$ where both are defined (see [GSM] for details).

In their work [GSM], Samuel Grushevsky and Riccardo Salvati Manni proved that $\mathbb{P}grTh$ is generically injective on $A_g^{4,8}$ when $g \geq 3$ and injective on the tangent spaces when $g \geq 2$.
The map was also conjectured to be injective whenever $g \geq 3$, albeit it has not been proved yet.

The case $g = 2$ is a case of special interest. The six odd 2-characteristics originate fifteen Plücker coordinates, which are the Jacobian determinants enumerated in Example 3.2; such coordinates satisfy by (3.36) the transformation law:

$$\begin{aligned} & D(N)(\gamma\tau) = \kappa(\gamma)^2 \chi_N(\gamma) \det(c_\gamma \tau + d_\gamma)^2 D(N)(\tau) \\ & \forall\, \tau \in \mathfrak{S}_2 \qquad \forall\, \gamma \in \Gamma_2(2,4) \qquad \forall\, N = \{n_1, n_2\} \in \tilde{C}_2 \end{aligned} \tag{4.1}$$

where $\chi_N = \chi_{n_1}\chi_{n_2}$.

In particular, by (1.7), there are 2^{10} distinct cosets in $\Gamma_g(2,4)/\Gamma_g(4,8)$; however, since $\kappa(\gamma)^2\chi_N(\gamma)$ is a sign whenever $\gamma \in \Gamma_g(2,4)$, the 6 odd characteristics yield $\sum_{k=0}^{6}\binom{6}{k} = 64$ possible image points for each set $\{\gamma\tau_0\}_{[\gamma]\in\Gamma_g(2,4)/\Gamma_g(4,8)}$ through the map $\mathbb{P}grTh_2$:

$$\begin{aligned} \mathbb{P}grTh_2 : A^{4,8} &\longrightarrow \mathbb{P}^{14} \\ \tau &\longrightarrow [\, D(N_1)(\tau), \cdots D(N_{15})(\tau)\,] \end{aligned}$$

Therefore, $\mathbb{P}grTh_2$ can not be injective, although it is seen to be finite of degree 16 (cf. [GSM]).

There exists, indeed, a suitable intermediate congruence subgroup $\Gamma(4,8) \subset \Gamma \subset \Gamma(2,4)$ such that $\mathbb{P}grTh_2$ factors on the correspondent level moduli space $A_\Gamma := \mathfrak{S}_2/\Gamma$ as:

$$\mathbb{P}grTh^* : A_2^\Gamma \to \mathbb{P}^{14} \qquad A_2^\Gamma := \mathfrak{S}_2/\Gamma$$

and the new map $\mathbb{P}grTh^*$ turns out to be injective.

In the following section such a remarkable group will be described.

4.2 The Congruence Subgroup Γ

As explained above, the chief interest of this section is to locate between $\Gamma(4,8)$ and $\Gamma(2,4)$ the proper congruence subgroup Γ, whose correspondent level moduli space $A_\Gamma := \mathfrak{S}_2/\Gamma$ is such that the Theta gradients map $\mathbb{P}grTh_2$ is still well defined on it and injective. The following description is provided for such a group:

Proposition 4.1.

$$\Gamma = \bigcap_{i=1}^{15} Ker\chi_{N_i} = \{\gamma \in \Gamma(2,4) \mid \kappa(\gamma)^2\chi_{N_i}(\gamma) = 1 \quad \forall i = 1, \ldots, 15\} \tag{4.2}$$

where χ_{N_i} are the fifteen characters involved in the transformation law (4.1) pertaining to the fifteen Jacobian determinants.

Proof. As far as one knows by (4.1), an *a priori* description for the subgroup Γ is:

$$\Gamma := \Gamma^{(1)} \cup \Gamma^{(-1)} \tag{4.3}$$

where:

$$\begin{aligned} \Gamma^{(1)} &:= \{\gamma \in \Gamma(2,4) \mid \kappa(\gamma)^2\chi_{N_i}(\gamma) = 1 \quad \forall i = 1, \ldots, 15\} \\ \Gamma^{(-1)} &:= \{\gamma \in \Gamma(2,4) \mid \kappa(\gamma)^2\chi_{N_i}(\gamma) = -1 \quad \forall i = 1, \ldots, 15\} \end{aligned}$$

Γ, as defined in (4.3), is plainly checked to be a subgroup of the Siegel modular group Γ_2, for Corollary 3.7 and Lemma 3.15 imply for each $i = 1, \ldots, 15$:

$$\left.\begin{aligned} &\kappa^2(\gamma\gamma')\chi_{N_i}(\gamma\gamma') = \kappa^2(\gamma)\chi_{N_i}(\gamma)\kappa^2(\gamma')\chi_{N_i}(\gamma') = 1 \\ &\kappa^2(\gamma^{-1})\chi_{N_i}(\gamma^{-1}) = [\,\kappa^2(\gamma)\chi_{N_i}(\gamma)\,]^{-1} = \kappa^2(\gamma)\chi_{N_i}(\gamma) \end{aligned}\right. \qquad \forall \gamma, \gamma' \in \Gamma$$

Moreover, by (3.22) and Lemma 3.16, one has $\Gamma(4,8) \subset \Gamma$; hence the subgroup Γ in (4.3) is indeed a congruence subgroup such that $\Gamma(4,8) \subset \Gamma \subset \Gamma(2,4)$.
The next step is to refine the definition of Γ, by detecting which elements of $\Gamma(2,4)$ belong to this subgroup.
Clearly $\bigcap_{i=1}^{15} \mathrm{Ker}\chi_{N_i} \subset \Gamma$. Therefore, only the reverse inclusion has to be shown to prove the first part of the statement.

Let thus $\gamma \in \Gamma$. Due to the definition in (4.3), either $\chi_{N_i}(\gamma) = 1$ for each $i = 1, \dots, 15$ or $\chi_{N_i}(\gamma) = -1$ for each $i = 1, \dots, 15$. However, if $\chi_{N_i}(\gamma) = -1$, for each i, an absurd statement turns up:

$$-1 = \chi_{(n,n_i)}(\gamma)\,\chi_{(n,n_j)}(\gamma)\,\chi_{(n,n_k)}(\gamma) = \chi_{(n_i,n_j)}(\gamma)\,\chi_{(n,n_k)}\,(\gamma) = 1$$

Hence, the only possible case is $\gamma \in \bigcap_{i=1}^{15} \mathrm{Ker}\chi_{N_i}$. Consequently, one has:

$$\Gamma = \bigcap_{i=1}^{15} \mathrm{Ker}\chi_{N_i} \tag{4.4}$$

and this part of the statement is proved. In order to gain the second identity in the statement, one needs to prove that $k^2(\gamma) = 1$ whenever $\gamma \in \Gamma$. By the criterion described in Proposition 3.19, the products:

$$D := D(n_1, n_2)D(n_3, n_4)D(n_5, n_6)$$

with $n_1, \dots n_6$ all distinct, are plainly checked to be modular forms with respect to $\Gamma(2, 4)$. Moreover, by (4.1) the following transformation law holds for each $\gamma \in \Gamma(2, 4)$:

$$D(\gamma\tau) = k^2(\gamma)\chi_{n_1} \cdots \chi_{n_6} det(c_\gamma \tau + d_\gamma)^6 D(\tau)$$

Hence, by Lemma (3.14) one has for each $\gamma \in \Gamma(2, 4)$:

$$k^2(\gamma) = \prod_{i=1}^{6} \chi_{n_i}(\gamma) = \chi_{(n_1,n_2)}(\gamma)\,\chi_{(n_3,n_4)}(\gamma)\,\chi_{(n_5,n_6)}(\gamma)$$

Therefore, whenever $\gamma \in \Gamma$, $k^2(\gamma) = 1$ by the characterization in (4.4). This shows $\Gamma^{(-1)}$ in (4.3) is indeed an empty set, and $\Gamma = \Gamma^{(1)}$. The proof is thus concluded.

□

Thanks to Proposition 4.1, an important property can be immediately stated:

Proposition 4.2. Γ *is normal in* Γ_2.

Proof. One has to prove that:

$$\chi_{N_i}(\gamma^{-1}\eta\gamma) = 1 \qquad \forall \gamma \in \Gamma_2 \quad , \quad \forall \eta \in \Gamma \quad , \quad \forall i = 1, \dots, 15$$

By setting $N_i = (n_{1\,i}, n_{2\,i})$ for each $i = 1, \dots, 15$, one has:

$$\chi_{N_i}(\gamma^{-1}\eta\gamma) = \chi_{n_{1\,i}}(\gamma^{-1}\eta\gamma)\chi_{n_{2\,i}}(\gamma^{-1}\eta\gamma) = \gamma^{-1}(\chi_{n_{1\,i}}, \chi_{n_{2\,i}})(\eta) = \chi_{\gamma n_{1\,i}}(\eta)\chi_{\gamma n_{2\,i}}(\eta)$$

Since the action in (3.2) preserves the parity, for each $i = 1, \dots, 15$ there exists a j, depending on i and γ, such that $(\gamma n_{1\,i}, \gamma n_{2\,i}) = N_j$. Therefore:

$$\chi_{N_j}(\gamma^{-1}\eta\gamma) = \chi_{\gamma n_{1\,i}}(\eta)\chi_{\gamma n_{2\,i}}(\eta) = \chi_{N_j}(\eta) = 1$$

where the last equality on the right holds since $\eta \in \Gamma$. □

A concrete description for the congruence subgroup Γ in terms of generators can be also a useful tool to work with. Corollary 3.9 suggests how to find such a description. Since the functions χ_{N_i} are characters of $G = \Gamma(2,4)/\{\pm\Gamma(4,8)\}$, the elements in $\Gamma(2,4)$ belonging to $\bigcap_{i=1}^{15} \mathrm{Ker}\chi_{N_i}$ can be found, by checking the representative elements for the cosets of $\Gamma(4,8)$ in $\Gamma(2,4)$. For such a purpose, Proposition 1.3 and Corollary 1.2 can be applied, to obtain:

Proposition 4.3. *G is a 9-dimensional vector space on $\mathbb{Z}_2$. A basis is given by:*

$$A_{11} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_{12} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix} \quad A_{21} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_{11}^2 = \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad B_{22}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad B_{12} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C_{11}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad C_{22}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{pmatrix} \quad C_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

Then, with reference to the notation introduced in Example 3.2, a table can be written down by means of the values in Lemma 3.13.

$\chi_{i,j} := \chi_{(n^{(i)},n^{(j)})}$	A_{11}	A_{12}	A_{21}	B_{12}	B_{11}^2	B_{22}^2	C_{12}	C_{11}^2	C_{22}^2
χ_{12}	-1	1	1	1	-1	-1	1	-1	-1
χ_{13}	1	1	-1	1	1	1	-1	-1	1
χ_{14}	-1	-1	1	1	-1	-1	-1	-1	1
χ_{15}	1	-1	1	-1	-1	1	1	1	1
χ_{16}	-1	1	-1	-1	-1	1	1	-1	-1
χ_{23}	-1	1	-1	1	-1	-1	-1	1	-1
χ_{24}	1	-1	1	1	1	1	-1	1	-1
χ_{25}	-1	-1	1	-1	1	-1	1	-1	-1
χ_{26}	1	1	-1	-1	1	-1	1	1	1
χ_{34}	-1	-1	-1	1	-1	-1	1	1	1
χ_{35}	1	-1	-1	-1	-1	1	-1	-1	1
χ_{36}	-1	1	1	-1	-1	1	-1	1	-1
χ_{45}	-1	1	1	-1	1	-1	-1	-1	1
χ_{46}	1	-1	-1	-1	1	-1	-1	1	-1
χ_{56}	-1	-1	-1	1	1	1	1	-1	-1

Table 4.1: Values of χ_{N_i} on a basis of G

Proposition 4.4. *The group Γ is generated by $\Gamma(4,8)$ and the elements:*

$$A_{12}B_{11}^2C_{22}^2 = \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 4 & -2 & 1 \end{pmatrix}$$

$${}^t(A_{12}B_{11}^2C_{22}^2) = A_{21}B_{22}^2C_{11}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 4 \\ 4 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_{12}B_{11}^2B_{22}^2 = \begin{pmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^t(B_{12}B_{11}^2B_{22}^2) = C_{12}C_{11}^2C_{22}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{pmatrix}$$

Proof. The independent elements satisfying the desired properties can be easily detected by the table. □

Corollary 4.1. *$\mathfrak{S}_2$ does not admit any fixed point for the action of Γ. In particular, the space $\mathfrak{S}_2/\Gamma$ is smooth.*

Proof. Since $\Gamma \subset \Gamma(2,4)$, the thesis follows from Corollary 1.5 and Proposition 4.4. □

4.3 Structure of $A(\Gamma)$: generators

Since the Plücker coordinates $D(N)$ are known to be cusp forms, the Theta gradients map does not extend to the boundary of the Satake's compactification [1] of the level moduli space $A^{4,8}$. The graded ring $A(\Gamma)$ of modular forms with respect to the congruence subgroup Γ, as well as the ideal $S(\Gamma) \subset A(\Gamma)$ of the cusp forms, are needed to describe Satake's compactification Proj$A(\Gamma)$ and the desingularization Proj$S(\Gamma)$. Only the even parts $A(\Gamma)^e$ and $S(\Gamma)^e$ are, indeed, relevant in describing respectively the Proj scheme [2]. This work will focus, therefore, on the structure of $A(\Gamma)^e$ and $S(\Gamma)^e$, in order to decrease the number of the generators involved; as a first step, this section will aim to find the generators of $A(\Gamma)^e$. A structure theorem have to be proved first, in order to describe $A(\Gamma)^e$:

[1] See Appendix B

[2] A detailed exposition of these topics can be found, for instance, in the classic book [Ht].

Proposition 4.5. *The following decomposition holds for* $\Gamma(4,8)$

$$A(\Gamma(4,8)) = \left(\bigoplus_{d=0,2,4} \mathbb{C}[\theta_m^2\theta_n^2]\theta_{m_1}\cdots\theta_{m_{2d}}\right)\bigoplus\left(\bigoplus_{d=1,3,5} \mathbb{C}[\theta_m^2\theta_n^2]\theta_{m_1}\cdots\theta_{m_{2d}}\right)$$

where:

$$A(\Gamma(4,8))^e = \bigoplus_{d=0,2,4} \mathbb{C}[\theta_m^2\theta_n^2]\theta_{m_1}\cdots\theta_{m_{2d}}$$

is the even part of the graded ring, and

$$A(\Gamma(4,8))^o = \bigoplus_{d=1,3,5} \mathbb{C}[\theta_m^2\theta_n^2]\theta_{m_1}\cdots\theta_{m_{2d}}$$

is the odd part.

Proof. By Igusa's Theorem, $A(\Gamma(4,8) = \mathbb{C}[\theta_m\theta_n]$ (Corollary 3.11); the result of the decomposition of the ring under the action of $\Gamma(2,4)$, according to the general procedure described in (2.2), is then:

$$A(\Gamma(4,8)) = \bigoplus_{\chi\in\hat{G}} \mathbb{C}[\theta_m\theta_n,\chi]$$

Since monomials in Theta constants transforms into monomials under the action of $\Gamma(2,4)$ (cf. (3.36)), one only needs to focus on monomials in $\theta_m\theta_n$, in order to study the transformation law for elements of $\mathbb{C}[\theta_m\theta_n]$.

In particular, if $P_d = \theta_{m_1}\cdots\theta_{m_{2d}} \in \mathbb{C}[\theta_m\theta_n]_d$ is a monomial of degree d in the variables $\theta_m\theta_n$, (3.36) implies the following transformation law:

$$P_d(\gamma\tau) = \kappa^{2d}(\gamma)\chi_{m_1}\cdots\chi_{m_{2d}}\det(c_\gamma\tau+d_\gamma)^d P_d(\tau) \qquad \forall\gamma\in\Gamma(2,4)$$

If $d=2l$, $P_d \in \mathbb{C}[\theta_m\theta_n,\chi_{m_1}\cdots\chi_{m_{2d}}]$, because $\kappa^4(\gamma)=1$ for each $\gamma\in\Gamma(2,4)$ by (3.22). Moreover, Corollary 3.9 and Lemma 3.14 imply that for each couple of characteristics m,n, the product $\chi_m^2\chi_n^2$ is a trivial character of G; the following decomposition arises, therefore, for the even part $A(\Gamma(4,8))^e$ of the ring:

$$A(\Gamma(4,8))^e = \bigoplus_{d\,\text{even}} \mathbb{C}[\theta_m^2\theta_n^2]\theta_{m_1}\cdots\theta_{m_{2d}} \tag{4.5}$$

On the other hand, if $d=2l+1$, $P_d\in\mathbb{C}[\theta_m\theta_n,\kappa^2\chi_{m_1}\cdots\chi_{m_{2d}}]$. Since κ^2 is a character of G, as stated in Corollary 3.7, it is indeed a product of the functions χ_m by Proposition 3.11; the following decomposition arises, therefore, for the odd part $A(\Gamma(4,8))^o$ of the ring:

$$A(\Gamma(4,8))^o = \bigoplus_{d=1,3,5} \mathbb{C}[\theta_m^2\theta_n^2]\theta_{m_1}\cdots\theta_{m_{2d}} \tag{4.6}$$

This concludes the proof.

□

Thanks to Proposition 4.5 a structure theorem can be finally stated for $A(\Gamma)^e$:

Theorem 4.1. $A(\Gamma)^e = \mathbb{C}[\theta_m^2, D(N)]^{(e)}$.

Proof. By Proposition 4.1, $\Gamma/\{\pm\Gamma(4,8)\} \subset G$ is the dual subgroup corresponding to the subgroup $<\chi_{N_i}>\subset \hat{G}$ generated by the fifteen characters χ_{N_i} related to the Jacobian determinants $D(N_i)$. One has, therefore:

$$A(\Gamma) = \bigoplus_{\chi\in<\chi_{N_i}>} A(\Gamma(4,8),\chi)$$

Hence, the thesis follows from (4.5). □

4.4 Structure of $A(\Gamma)$: Relations

The foregoing section has been devoted to the detection of the generators of $A(\Gamma)^e$, as stated in Theorem 4.1. Some relations exist amongst these generators, most of which are induced by Riemann's relations. This section aims to provide them through a combinatorial description; for this purpose a threefold investigation will be needed, in order to describe the relations involving only the θ_m^2, the relations involving only the $D(N)$, and finally the ones between the θ_m^2 and the $D(N)$.

4.4.1 Relations among θ_m^2

The relations among θ_m^2 are completely described by Riemann's relations (see Section 3.7). Therefore, with reference to the notation introduced in (3.66) and (3.67), there are 15 independent biquadratic relations:

$$R_2(M) = 0 \qquad \forall M \in C_6^+ \tag{4.7a}$$

and 5 independent quartic relations:

$$R_4(M_i) = 0 \qquad i = 1,\ldots 5 \tag{4.7b}$$

4.4.2 Relations among $D(N)$

The 15 relations of the kind:

$$D(M)^2 = \theta_{m_1}^2\theta_{m_2}^2\theta_{m_3}^2\theta_{m_4}^2 \qquad \forall\, M = \{m_1,m_2,m_3,m_4\} \in C_4^-$$

induced by Jacobi's formula and described in Example 3.2, have to be used together with the ones in (4.7a) and (4.7b), to find all the desired relations amongst the fifteen Jacobian determinants $\{D(M)\}_{M\in C_4^-}$. Here a combinatorial explanation for all these relations follows.

1. For each even characteristic m, one can enumerate the six 4-plets $\{M_i^m\}_{i=1,\dots 6}$ in C_4^- containing m, in such a way that:

$$M_1^m \cap M_2^m \cap M_3^m = \{m\} = M_4^m \cap M_5^m \cap M_6^m$$

Then, one has:

$$D(M_1^m)D(M_2^m)D(M_3^m) = \chi_5\theta_m^2 = D(M_4^m)D(M_5^m)D(M_6^m) \tag{4.8a}$$

which are obviously 10 relations, namely one for each choice of m.

2. For each $M = \{m_1, \dots m_6\} \in C_6^+$ there are eight 4-plets of C_4^-, containing exactly a triplet $\{m_i, m_j, m_k\} \subset M$; these 4-plets can be enumerated in such a way that:

$$\begin{aligned} D(\tilde{M}_1)D(\tilde{M}_2)D(\tilde{M}_3)D(\tilde{M}_4) = \chi_5 \textstyle\prod_{i=1}^6 \theta_{m_i}^2 = \\ = D(\tilde{M}_5)D(\tilde{M}_6)D(\tilde{M}_7)D(\tilde{M}_8) \end{aligned} \tag{4.8b}$$

These are 15 relations, namely one for each choice of $M \in C_6^+$.

3. Let $M = \{m_1, \dots m_6\} \in C_6^+$ and let

$$R_2(M) = \theta_{m_1}^2\theta_{m_2}^2 \pm \theta_{m_3}^2\theta_{m_4}^2 \pm \theta_{m_5}^2\theta_{m_6}^2 = 0$$

be the associated biquadratic Riemann's relation as in (3.66).
For each couple of even characteristics $\{m_i, m_j\}$, only two 4-plets $M_1^{i,j}$ and $M_2^{i,j}$ belonging to C_4^- contain $\{m_i, m_j\}$; then, one has:

$$\begin{aligned} D(M_1^{1,2})D(M_2^{1,2}) \pm D(M_1^{3,4})D(M_2^{3,4}) \pm D(M_1^{5,6})D(M_2^{5,6}) = \\ = \pm D(M')R_2(M) = 0 \end{aligned} \tag{4.8c}$$

These are 15 relations, since they correspond to the elements of C_6^+.

Otherwise, by choosing for each couple m_i, m_j only one of the two 4-plets $M_1^{i,j}$ e $M_2^{i,j}$, one has the following general identity:

$$\begin{aligned} D^2(M_\alpha^{1,2}) \pm D^2(M_\beta^{3,4}) \pm D^2(M_\epsilon^{5,6}) = \\ = \theta_{\alpha_1}^2\theta_{\alpha_2}^2\theta_{m_1}^2\theta_{m_2}^2 \pm \theta_{\beta_1}^2\theta_{\beta_2}^2\theta_{m_3}^2\theta_{m_4}^2 \pm \theta_{\epsilon_1}^2\theta_{\epsilon_2}^2\theta_{m_5}^2\theta_{m_6}^2 \end{aligned}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \epsilon_1, \epsilon_2$ are characteristics in $M^c \in C_4^-$.

In particular, each triplet of determinants can be chosen in such a way that only three of the four characteristics in M^c appear; for combinatorial reasons, there is a unique way to build a relation among $D(M)$'s by multiplying each determinant of such a triplet by two other distinct determinants:

$$D_h\, D_k\, D^2(M_\alpha^{1,2}) \pm D_l\, D_r\, D^2(M_\beta^{3,4}) \pm D_s\, D_t\, D^2(M_\epsilon^{5,6}) = 0 \tag{4.8d}$$

One can observe that, for each choice of $M \in C_6^+$, there exist four distinct triplets of determinants $D(M_\alpha^{1,2}), D^2(M_\beta^{3,4}), D^2(M_\epsilon^{5,6})$ satisfying the desired condition; more precisely, each triplet corresponds to a choice for the characteristic in M^c which does not appear in the identity, and obviously there are four possible choices for such a characteristic. Therefore, the relations (4.8d) are in number of $15 \cdot 4 = 60$.

4. Let $M = \{m_1, \ldots m_4\} \in C_4^-$ an let

$$R_4(M) = \theta_{m_1}^4 \pm \theta_{m_2}^4 \pm \theta_{m_3}^4 \pm \theta_{m_4}^4 = 0$$

be the associated quartic Riemann's relation as in (3.67). For each $m_i \in M$ there exist only 2 elements $M_1^i, M_2^i \in C_4^-$ containing m_i and such that:

$$M_1^i \vartriangle M_2^i = M^c = \{m_5, \ldots m_{10}\}$$

One has, therefore:

$$\sum_{i=1}^{4} \pm D(M_1^i)^2\, D(M_2^i)^2 = \theta_{m_5} \cdots \theta_{m_{10}} R_4(M) = 0 \tag{4.8e}$$

All the 15 quartic Riemann relations yield independent relations on the $D(M)$, even though they are not independent themselves; the relations in (4.8e) are, therefore, in number of 15.

5. Let $M = \{m_1, m_2, m_3\} \in C_3^-$ be fixed.
There exist only two distinct 6-plets in C_6^+, containing M. Then let $\tilde{M} = \{m_1, m_2, m_3, m_1', m_2', m_3'\}$ be such a 6-plet; the corresponding biquadratic Riemann's relation is easily seen to be such that:

$$R_2(\tilde{M}) = \theta_{m_1}^2 \theta_{m_1'}^2 \pm \theta_{m_2}^2 \theta_{m_2'}^2 \pm \theta_{m_3}^2 \theta_{m_3'}^2 = 0$$

with $\{m_1', m_2', m_3'\} \in C_3^-$.
Moreover, for each couple of characteristics $\{m_i, m_j\} \subset M$, there exists a unique $M_{i,j} \in C_6^+$, containing $\{m_i, m_j\}$ and satisfying:

$$R_2(M_{i,j}) = \pm\theta^2_{m_i}\theta^2_{m_j} + P_{ij} = 0$$

For combinatorial reasons, all the terms $\theta^2_{m'_i}P_{jk}$ share the common addend $\theta^2_{m'_1}\theta^2_{m'_2}\theta^2_{m'_3}$; therefore, one has:

$$0 = \theta^2_{m_1}\theta^2_{m_2}\theta^2_{m_3}R_2(\tilde{M}) = \pm\theta^2_{m_1}\theta^2_{m'_1}P_{2,3} \pm \theta^2_{m_2}\theta^2_{m'_2}P_{1,3} \pm \theta^2_{m_3}\theta^2_{m'_3}P_{1,2} =$$

$$= \pm\theta^2_{m_4}\theta^2_{m'_1}\theta^2_{m'_2}\theta^2_{m'_3} \pm \theta^2_{m_1}\theta^2_{m'_1}\theta^2_{m_\alpha}\theta^2_{m_\beta} \pm \theta^2_{m_2}\theta^2_{m'_2}\theta^2_{m_\alpha}\theta^2_{m_\epsilon} \pm \theta^2_{m_3}\theta^2_{m'_3}\theta^2_{m_\beta}\theta^2_{m_\epsilon}$$

where m_4 is the unique even characteristic which completes $M = \{m_1, m_2, m_3\}$ to a 4-plet in C_4^- (to which a quartic Riemann's relations correspond, as in (3.67)), and $\{m_\alpha, m_\beta, m_\epsilon\} = \{m_1, m_2, m_3, m_4, m'_1, m'_2, m'_3\}^c$.
Then it is easily checked that:

$$0 = \left(\prod_{m \notin \{m_1,m_2,m_3,m_4\}} \theta_m\right)\theta^2_{m_1}\theta^2_{m_2}\theta^2_{m_3}R_2(\tilde{M}) = \sum_{i=1}^{4} \pm D(M_1^i)\, D(M_2^i)^3 \tag{4.8f}$$

where M_1^i and M_2^i are the 4-plets in C_4^- containing m_i and such that $M_1^i \vartriangle M_2^i = \{m_1, m_2, m_3, m_4\}^c$, as in (4.8e).
By choosing the other 6-plet $\tilde{M} = \{m_1, m_2, m_3, m_\alpha, m_\beta, m_\epsilon\}$ containing $M = \{m_1, m_2, m_3\}$, one obtains the same relations with interchanged exponents:

$$0 = \sum_{i=1}^{4} \pm D(M_1^i)^3\, D(M_2^i) \tag{4.8g}$$

Triplets $M = \{m_1, m_2, m_3\}$ contained in the same 4-plet in C_4^- yield exactly the same relation (essentially because the same related quartic Riemann relation turns out to be replaced in the foregoing null expression). Therefore, these relations are parameterized by the elements in C_4^-; hence, there are 15 relations of the type (4.8f) and 15 of the type (4.8g).

6. For each even characteristic $m \in C_1$, there are exactly six determinants $\{D_i^m\}_{i=1,\ldots 6}$ such that:

$$D_i^m = D(M) = \pm\theta_{m_1}\theta_{m_2}\theta_{m_3}\theta_{m_4} \qquad \text{with } m \in M = \{m_1, m_2, m_3, m_4\} \in C_4^-$$

Then, one has:

$$\sum_{i=1}^{6}(D_i^m)^4 = \theta^4_m[\theta^4_{n_1}(\theta^4_{\alpha_1}\theta^4_{\alpha_3} \pm \theta^4_{\alpha_2}\theta^4_{\alpha_4})\pm$$
$$\pm\, \theta^4_{n_2}(\theta^4_{\alpha_1}\theta^4_{\alpha_5} \pm \theta^4_{\alpha_2}\theta^4_{\alpha_6}) \pm \theta^4_{n_3}(\theta^4_{\alpha_3}\theta^4_{\alpha_5} \pm \theta^4_{\alpha_4}\theta^4_{\alpha_6})]$$

where $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \{\alpha_1, \alpha_2, \alpha_5, \alpha_6\} \in C_4^-$.
Then, for each $i, j, k = 1, \ldots 6$ such that $\{\alpha_i, \alpha_j, \alpha_k\} \in C_3^-$, one can denote by the symbol M_{ij}^k the only 4-plet in C_4^- containing α_i, α_j, and not α_k, and by the symbol $P(M_{ij}^k)$ the polynomial:

$$P(M_{ij}^k) := R_4(M_{ij}^k) - \theta_{\alpha_j}^4$$

where $R_4(M_{ij}^k)$ is the quartic Riemann relation associated to M_{ij}^k as in (3.67). Then, in particular:

$$\sum_{i=1}^{6} \pm (D_i^m)^4 = \theta_m^4 \{ \theta_{n_1}^4 [\theta_{\alpha_1}^4 P(M_{23}^1) \pm \theta_{\alpha_2}^4 P(M_{14}^2)] \pm \\ \pm \theta_{n_2}^4 [\theta_{\alpha_1}^4 P(M_{25}^1) \pm \theta_{\alpha_2}^4 P(M_{16}^2)] \pm \theta_{n_3}^4 [\theta_{\alpha_3}^4 P(M_{45}^3) \pm \theta_{\alpha_4}^4 P(M_{36}^4)] \}$$

where:

$$\begin{cases} M_{23}^1 \cap M_{14}^2 = \{n_2, n_3\} \\ M_{25}^1 \cap M_{16}^2 = \{n_1, n_3\} \\ M_{45}^3 \cap M_{36}^4 = \{n_1, n_2\} \end{cases}$$

Therefore, for a suitable choice of the signs, one has:

$$\sum_{i=1}^{6} \pm (D_i^m)^4 = \theta_m^4 [\theta_{n_1}^4 (\theta_{\alpha_1}^4 \pm \theta_{\alpha_2}^4)(\pm \theta_{n_2}^4 \pm \theta_{n_3}^4) \pm \\ \theta_{n_2}^4 (\theta_{\alpha_1}^4 \pm \theta_{\alpha_2}^4)(\pm \theta_{n_1}^4 \pm \theta_{n_3}^4) \pm \theta_{n_3}^4 (\theta_{\alpha_1}^4 \pm \theta_{\alpha_2}^4)(\pm \theta_{n_1}^4 \pm \theta_{n_2}^4)] = \\ = \theta_m^4 (\theta_{\alpha_1}^4 \pm \theta_{\alpha_2}^4)[\theta_{n_1}^4 (\pm \theta_{n_2}^4 \pm \theta_{n_3}^4) \pm \theta_{n_2}^4 (\pm \theta_{n_1}^4 \pm \theta_{n_3}^4) \pm \theta_{n_3}^4 (\pm \theta_{n_1}^4 \pm \theta_{n_2}^4)]$$

which is made null by properly selecting the remaining signs. To sum up, therefore, for each $m \in C_1$ one has the relation:

$$\sum_{i=1}^{6} \pm (D_i^m)^4 = 0 \tag{4.8h}$$

where the choice of the signs is uniquely determined as explained. Only 6 of this ten relations are easily seen to be independent.

A final Proposition can thus be stated about the relations among the $D(N)$:

Proposition 4.6. *All the relations amongst the $D(N)$ are generated by:*

1. *The 10 relations in (4.8a);*
2. *The 15 relations in (4.8b);*

3. *The* 15 *relations in* (4.8c)*;*
4. *The* 60 *relations in* (4.8d)*;*
5. *The* 15 *relations in* (4.8e)*;*
6. *The* 15 *relations in* (4.8f)*;*
7. *The* 15 *relations in* (4.8g)*;*
8. *The* 6 *relations in* (4.8h)*;*

Proof. The statement has been proved by elimination, thanks to a Singular program created by Professor Eberhard Freitag; working, in particular, with the ideal generated by the one given by the Riemann relations between the θ_m and the one given by the expressions of the $D(N)$ as products of θ_m (see Example 3.2), the desired relations have been detected by eliminating the variables θ_m by means of a Gröebner basis [3]. □

4.4.3 Relations among $D(N)$ and θ_m^2

Since the generators $D(N)$ are modular forms of even weight, the relations among $D(N)$ and θ_m^2 are indeed among $D(N)$ and $\theta_m^2\theta_n^2$.

There are of course the 15 relations induced by Jacobi's formula:

$$D(M)^2 = \theta_{m_1}^2\theta_{m_2}^2\theta_{m_3}^2\theta_{m_4}^2 \qquad \forall\, M = \{m_1, m_2, m_3, m_4\} \in C_4^- \tag{4.9a}$$

as described in Example 3.2. Any other relation is clearly generated by the ones in (4.9a) and by all the relations of the kind:

$$\prod_{i=1}^{h} D(N_i) = P(\theta_m^2\theta_n^2)$$

where $P(\theta_m^2\theta_n^2)$ is a polynomial in $\theta_m^2\theta_n^2$ and the determinants $D(N_i)$ are all distinct.
Since by (3.36) and (3.23) $\theta_m^2\theta_n^2$ is a modular form with respect to $\Gamma(2,4)$ for each couple of characteristics m, n, such a relation holds if and only if $\prod_i D(N_i)$ itself is a modular form with respect to $\Gamma(2,4)$.
By Proposition 3.8 such a condition is equivalent to $M^tM \equiv 0\, mod 2$ for the $4\times4h$ matrix $M = (M_1 \dots M_h)$ of even characteristics, associated to $\prod_i D(M_i)$, and also, by Proposition 3.19, to $N^tN \equiv 0\, mod 2$ for the $4 \times 2h$ matrix $N = (N_1 \dots N_h)$ of odd characteristics, associated to $\prod_i D(N_i)$. A necessary condition is therefore given by:

$$diag(M^tM) \equiv 0 \bmod 2$$

or, likewise, concerning with odd characteristics, by:

$$diag(N^tN) \equiv 0 \bmod 2$$

The investigation will be therefore focused on products $\prod_i D(N_i)$ of distinct Jacobian determinants, fulfilling this condition. For such a purpose the following technical definition will be needed throughout this section:

[3]For the definition of a Gröebner basis see, for instance, [E].

Definition 4.1. *Let $1 \leq h \leq 15$, and let $D(N_1)\cdots D(N_h) = D(M_1)\cdots D(M_h)$ be a product of distinct Jacobian determinants. If the sum of all the even characteristics m appearing in the 4plets $M_i \in C_4^-$ (or, equivalently, the sum of all the odd characteristics n appearing in the couples $N \in \tilde{C}_2$), each counted with its multiplicity, is congruent to 0 mod 2, such a product will be called a* **remarkable factor** *of degree h. A remarkable factor which is product of remarkable factors will be named* **reducible**, *otherwise it will be called* **non-reducible**.

As already seen, if the product $D(N_1)\cdots D(N_h)$ of distinct determinants is a modular form with respect to $\Gamma(2,4)$, it is a remarkable factor, while the converse statement is not necessarily true.

Remarkable factors can be easily characterized:

Proposition 4.7. *P is a remarkable factor if and only if P is a monomial in the variables θ_m^2 and χ_5 (cf. (3.40)). More precisely:*

$$P = {\chi_5}^h \prod_m {\theta_m}^2 \qquad h = 0, 1$$

Proof. If P is a monomial in the variables θ_m^2 and χ_5, then P is clearly a remarkable factor; therefore, only the converse statement has to be proved. To aim at this, one can use the function, defined in (3.52):

$$F : C \longrightarrow \mathbb{C}[\theta_m]$$

$$\{m_1, m_2, \ldots, m_h\} \longrightarrow \theta_{m_1}\theta_{m_2}\cdots\theta_{m_h}$$

$$F(\emptyset) := 1$$

Then, in particular:

$$F(\{m\}) = \theta_m; \qquad F(C_1) = \chi_5; \qquad F(M) = D(M) \quad \forall M \in C_4^-;$$

and moreover one has:

$$F(M_i)\,F(M_j) = F(M_1 \vartriangle M_j) \prod_{m \in M_i \cap M_j} {\theta_m}^2$$

If $P = F(M_1)\cdots F(M_h)$ is a remarkable factor, by this identity and Propositions 3.4, 3.5 and 3.6, it necessarily follows that:

$$P = {\chi_5}^h \prod_m {\theta_m}^2$$

with $h = 0, 1$. □

In order to classify remarkable factors in terms of the Jacobian determinants appearing in the product, the law associating to each couple of odd characteristics $N \in \tilde{C}_2$ their sum $S(N) \in \mathbb{Z}_2^4$ will be used:

$$N = \{n_1, n_2\} \quad \longrightarrow \quad S(N) := n_1 + n_2$$

In fact, a product $P = \prod_i D(N_i)$ of distinct Jacobian determinants is clearly a remarkable factor if and only if $\sum_i S(N_i) = 0$.

Lemma 4.1. *Remarkable factors of degree greater than* 5 *are reducible.*

Proof. Let $P = \prod_{i=1}^{h} D(N_i)$ be a remarkable factor with $h > 5$. Then, the set $\{S(N_i)\}_{i=1,\dots h} \subset \mathbb{Z}_2^4$ necessarily contains at least two elements being linearly dependent from the others. Since $S(N) \neq 0$ for each $N \in \tilde{C}_2$, the thesis follows. □

By Lemma 4.1, remarkable factors of degree at most 5 are the only ones to be checked, in order to find the non-reducible remarkable factors.

Proposition 4.8. *The non-reducible remarkable factors are:*

1. $D(n_i, n_j)D(n_j, n_k)D(n_k, n_i)$;
2. $D(n_i, n_j)D(n_k, n_l)D(n_s, n_t)$;
3. $D(n_i, n_j)D(n_j, n_k)D(n_k, n_l)D(n_l, n_i)$;
4. $D(n_i, n_j)D(n_i, n_k)D(n_i, n_l)D(n_s, n_t)$;
5. $D(n_i, n_j)D(n_j, n_k)D(n_k, n_l)D(n_l, n_r)D(n_r, n_i)$;
6. $D(n, n_j)D(n, n_k)D(n, n_l)D(m, n_r)D(m, n_s)$;

Proof. They can be plainly detected by the following table, which is redacted with reference to the notation introduced in the Example 3.2 for odd characteristics:

$D_{ij} := D(n^{(i)}, n^{(n_j)})$	${}^tS(N)$
D_{12}	(1111)
D_{13}	(0010)
D_{23}	(1101)
D_{14}	(1110)
D_{24}	(0001)
D_{15}	(1000)
D_{25}	(0111)
D_{16}	(1011)
D_{26}	(0100)
D_{34}	(1100)
D_{35}	(1010)
D_{36}	(1001)
D_{45}	(0110)
D_{46}	(0101)
D_{56}	(0011)

Table 4.2: Values of S(N)

□

By (3.19) one notes only factors of the type 2., 3. and 6. are modular forms with respect to $\Gamma(2,4)$. Due to Proposition 4.7, no Theta constant appears in such factors with even multiplicity, while χ_5 appears in factors of type 1., 4. and 5. χ_5 with odd multiplicity.

The products $\prod_i D(N_i)$ of distinct determinants which are functions of $\theta_m^2\theta_n^2$ are, therefore, products of the factors listed in Proposition 4.8.

Proposition 4.9. *The relations involving products of* 3 *determinants are:*

$$D(n_i,n_j)D(n_k,n_l)D(n_s,n_t) = \prod_{i=1}^{6} \theta_{m_i}{}^2 \tag{4.9b}$$

Proof. As already stated, factors of type 2. are the only ones involved.
In order to prove that the six Theta constants appearing in the expression (4.9b) are all distinct, let P be a product of determinants as in (4.9b), and let $M_1, M_2, M_3 \in C_4^-$ be the 4-plets satisfying:

$$P = D(M_1)D(M_2)D(M_3)$$

Since P is a remarkable factor, if $M_i \vartriangle M_j \in C_6^+$ for any couple of this 4-plets, it would follow that $(M_1 \vartriangle M_2)^c = M_3$; then one would have $P = \theta_{m_1}{}^2\theta_{m_2}{}^2\theta_{m_3}{}^2\chi_5$, which is an absurd statement, because P is also a modular form with respect to $\Gamma_2(2,4)$. Then, by Proposition 3.4, $M_i \vartriangle M_j \in C_4^-$ for each distinct couple M_i, M_j, and the only possibility is $M_1 \vartriangle M_2 = M_3 \in C_4^-$, namely:

$$M_1 = \{m_1,m_2,m_3,m_4\} \qquad M_2 = \{m_1,m_2,m_5,m_6\} \qquad M_3 = \{m_3,m_4,m_5,m_6\}$$

Therefore, the thesis follows. □

Proposition 4.10. *The relations involving products of* 4 *determinants are:*

$$D(n_i,n_j)D(n_j,n_k)D(n_k,n_l)D(n_l,n_i) = \prod_{i=1}^{8} \theta_{m_i}{}^2 \tag{4.9c}$$

Proof. Remarkable factors of type 3. are the only ones involved.
To prove the eight Theta constants appearing in the expression (4.9c) are all distinct, let P be a product of determinants of the type in (4.9c), and let $M_1, M_2, M_3, M_4 \in C_4^-$ the 4-plets satisfying:

$$P = D(M_1)D(M_2)D(M_3)D(M_4)$$

If $M_1 \vartriangle M_2 \in C_4^-$, then $M_3 \vartriangle M_4 = M_1 \vartriangle M_2 \in C_4^-$, since P is a modular form with respect to $\Gamma_2(2,4)$. Due to $|M_1 \vartriangle M_2| = 4$, there are at least six distinct characteristics among the ones associated to the Theta constants appearing in the expression (each of them appearing with multiplicity 2); however, $M_3 \vartriangle M_4 = M_1 \vartriangle M_2$ and $|M_3 \cap M_4| = 2$, hence the eight characteristics, appearing with multiplicity 2, are all distinct, the determinants involved $D(M_i)$ being all distinct.
If $M_1 \vartriangle M_2 \in C_6^+$, then $M_3 \vartriangle M_4 = M_1 \vartriangle M_2 \in C_6^+$. Since $|M_1 \cap M_2| = 1$, at

least seven distinct characteristics appear, each with multiplicity 2. As before, $M_3 \vartriangle M_4 = M_1 \vartriangle M_2$ with $|M_3 \cap M_4| = 1$ and the common characteristic in M_3 and M_4 must be different from the other seven, since the $D(M_i)$ are all distinct. □

Proposition 4.11. *The relations involving products of* 5 *determinants are:*

$$D(n,n_j)D(n,n_k)D(n,n_l)D(m,n_r)D(m,n_s) = \prod_m \theta_m^2 \tag{4.9d}$$

Proof. As seen, factors of type 6 are the only ones involved. By Proposition 4.7 χ_5 appears with even multiplicity. However, the ten Theta constants in the expression (4.9d) are plainly seen to be not necessarily all distinct in this case. □

As concerns the relations involving products of more than 5 determinants, one has to note that the product of two non-reducible remarkable factors of type 2, 4 and 5 is indeed a modular form with respect to $\Gamma(2,4)$; therefore, if it does not factorize into a product of the factors already discussed, it will yield new independent relations. One has, in particular, the following:

Proposition 4.12. *The relations involving products of* 6 *determinants are:*

$$D(n_i,n_j)D(n_j,n_k)D(n_k,n_i)D(n_l,n_r)D(n_r,n_s)D(n_s,n_l) = {\chi_5}^2{\theta_m}^2{\theta_n}^2 \tag{4.9e}$$

Proof. The only possible case rises from the product P of two distinct factors of type 1. :

$$Q_1 = D(n_i,n_j)D(n_j,n_k)D(n_k,n_i) = \chi_5{\theta_m}^2$$

$$Q_1' = D(n_i',n_j')D(n_j',n_k')D(n_k',n_i') = \chi_5\theta_n^2$$

Clearly $Q_1 \cdot Q_1'$ does not factorize into products of determinants which are in turn modular forms with respect to $\Gamma(2,4)$; therefore, the relations in (4.9e) are not generated by the previous ones. □

The following Proposition ends the investigation around these relations.

Proposition 4.13. *Let P be a product of more than* 6 *distinct determinants, which is a modular form with respect to* $\Gamma(2,4)$. *Then, each relation involving P is dependent from the ones in (4.9a), (4.9b), (4.9c), (4.9d) and (4.9e).*

Proof. The single cases have to be briefly discussed.

Let P be a product of 7 distinct determinants such that $P \in A(\Gamma(2,4))$. Then P is necessarily the product of a factor P_1 of type 1. and a factor P_4 of type 4. and the only possible cases are:

$$P_1 \cdot P_4 = [D(n_i,n_j)D(n_j,n_k)D(n_k,n_i)][D(n_l,n_i)D(n_l,n_j)D(n_l,n_k)D(n_r,n_s)]$$

$$P_1 \cdot P_4 = [D(n_i,n_j)D(n_j,n_k)D(n_k,n_i)][D(n_l,n_i)D(n_l,n_j)D(n_l,n_r)D(n_k,n_s)]$$

However, by using the relations (4.8a), one gains:

$$D(n_i, n_j)D(n_j, n_k)D(n_k, n_i) = D(n_l, n_r)D(n_r, n_s)D(n_s, n_l)$$

Therefore, at least a $D(N)^2$ appears in both cases, hence the relations involving P are dependent from the ones which have been already found (the relations in (4.9a) hold, in particular).

Regarding products of more than 7 determinants:

$$P = \prod_{N \in C} D(N) \qquad C \subset \tilde{C}_2 \quad \text{s. t. } |C| > 7$$

the product of the determinants associated to the complementary couples N:

$$P^c := \prod_{N \notin C} D(N)$$

can be plainly investigated. In fact, if $P \in A(\Gamma(2,4))$, then clearly $P^c \in A(\Gamma(2,4))$, hence the behaviour of P^c pertains to the previous cases.

Let P be, therefore, a product of 8 distinct determinants such that $P \in A(\Gamma(2,4))$. P^c has thus degree 7; then, either $P^c = Q_1 \cdot Q_4$ with Q_1 of type 1. and Q_4 of type 4, or $P^c = Q_2 \cdot Q_3$ with Q_2 of type 2. and Q_3 of type 3.
If $P^c = Q_1 \cdot Q_4$ the only two possible cases are the ones discussed above; then, P is plainly verified to admit a factor of type (4.9d):

$$D(n_i, n_k)D(n_i, n_r)D(n_i, n_s)D(n_s, n_j)D(n_s, n_l)$$

which does not appear in the product $Q_1 \cdot Q_4$. Then P factorizes into the product of two factors, which are modular forms with respect to $A(\Gamma(2,4))$ and thus have been already studied.
If $P^c = Q_2 \cdot Q_3$, then:

$$P^c = D(n_i, n_j)D(n_k, n_l)D(n_r, n_s)D(n'_i, n'_j)D(n'_j, n'_k)D(n'_k, n'_l)D(n'_l, n'_i)$$

Since four of the six odd characteristics appear with multiplicity 3 and the other two with multiplicity 1, P always contains a factor of type (4.9c):

$$D(n, n_\alpha)D(n_\alpha, n_\beta)D(n_\beta, n_\gamma)D(n_\gamma, n)$$

Therefore, P factorizes again into the product of two factors which are modular forms with respect to $A(\Gamma(2,4))$.

Let P be the product of 9 distinct determinants such that $P \in A(\Gamma(2,4))$. $P^c \in A(\Gamma(2,4))$ has, therefore, degree 6; hence, it is either the product $Q_2 \cdot Q'_2$ of two factors of type 2, or the product $Q_1 \cdot Q'_1$ of two factors of type 1.
In the first case:

$$P^c = Q_2 \cdot Q'_2 = D(n_i, n_j)D(n_k, n_l)D(n_r, n_s)D(n'_i, n'_j)D(n'_k, n'_l)D(n'_r, n'_s)$$

all the characteristics appear with multiplicity 2; therefore, P always contains a factor of type (4.9d).
In the second case P^c is of type (4.9e); then, at least five characteristics appear in P^c with multiplicity 2. Hence, P always contains a factor of type (4.9d).

If P is a product of 10 distinct determinants such that $P \in A(\Gamma(2,4))$, by Proposition 4.11 P^c is of type (4.9d). Then, P is easily checked to necessarily contain a factor P_3 of type (4.9c) both when $n \neq m$ and when $n = m$.

If P is a product of 11 distinct determinants such that $P \in A(\Gamma(2,4))$, by Proposition 4.10 P^c is of type (4.9c). Then, $P = P_6 \cdot P'$ where P_6 is of type (4.9d) and $P' = P_1 \cdot P_1$ is of type (4.9e).

Finally, if P is a product of 12 distinct determinants such that $P \in A(\Gamma(2,4))$, by Proposition 4.9 P^c is of type (4.9b). Then, $P = P_6 \cdot P_3 \cdot P_2$ where P_6 is of type (4.9d), P_3 is of type (4.9c) and P_2 is of type (4.9b).

Products of 13 or 14 Jacobian determinants can not be modular forms with respect to $\Gamma(2,4)$, while the product of all the 15 determinants trivially factorizes into factors already selected. The proof is thus concluded.

□

To sum up, one has:

Proposition 4.14. *A system of independent relations between the $D(M)$ and the $\theta_m^2\theta_n^2$ is given by (4.9a), (4.9b), (4.9c), (4.9d) and (4.9e).*

The results about the description of $A(\Gamma)^{(e)}$ in terms of relations can be now summarized as follows:

Theorem 4.2. *The ideal of the relations amongst $D(N)$ and θ_m^2 is generated by (4.7a), (4.7b), (4.8a), (4.8b), (4.8c), (4.8d), (4.8e), (4.8f), (4.8g), (4.8h), (4.9a), (4.9b), (4.9c), (4.9d) and (4.9e).*

4.5 The Ideal $S(\Gamma)$

The even part $S(\Gamma)^e$ of the ideal of cusp forms with respect to the subgroup Γ can be described thanks to the results gained by Van Geemen and Van Straten about the generators of the cusp forms with respect to $\Gamma(2,4,8)$:

Theorem 4.3. *A system of generators for $S(\Gamma)^e$ is given by:*

1. $D(M) \qquad \forall\, M \in C_4^-$
2. $\theta_{m_1}^4\theta_{m_2}^2\theta_{m_3}^2\theta_{m_4}^2\theta_{m_5}^2 \qquad \forall\,\{m_1, m_2, m_3, m_4, m_5\} \notin C_5^+ \cup C_5^-$

In particular, there are $15 + 5 \cdot 72 = 375$ generators for this ideal.

Proof. Since $\Gamma(2,4,8) \subset \Gamma(4,8) \subset \Gamma$, the inclusion $S(\Gamma) \subset S(\Gamma(2,4,8)$ holds; the generators of $S(\Gamma)^e$ can be therefore selected amongst the ones described in Theorem 3.11. Since, by Theorem 4.1, $S(\Gamma)^e \subset \mathbb{C}[\theta_m^2\theta_n^2, D(M)]$, only the types 1. and 2., enumerated in the statement of Theorem 3.11, generates $S(\Gamma)^e$; in fact, by using $\theta_m^2 = Q_m(\Theta_{m'})$, elements of type 3. are easily seen no to be in the ideal, being expressed as $P(\theta_m^2\theta_n^2)\Theta_{m'}$ where $\Theta_{m'}$ is a second order Theta constant. □

Appendix A

Elementary Results of Matrix Calculus

Proposition A.1. *Let $k = {}^t(k_1, \dots k_g) \in \mathbb{Z}^g$ be a column vector and $D := MCD(k_1, \dots k_g)$ the greatest common divisor of $k_1, \dots k_g$. There exists a matrix $M \in M(g, \mathbb{Z})$ swhose first column is k and* $\det M = D$.

Proof. The statement can be proved by induction on g. It is trivial indeed for $g = 1$. Then, one can suppose it holds for $g - 1$; let, now, $k = (k_1, \dots k_g) \in \mathbb{Z}^g$; by the inductive hypothesis, a matrix $M' \in M(g-1, \mathbb{Z})$ exists such that its first column is $k' = {}^t(k_1, \dots k_{g-1}) \in \mathbb{Z}^{g-1}$ and $det(M') = D' = MCD(k_1, \dots k_{g-1})$. Now, let $p, q \in \mathbb{Z}$ verifying the Bezout identity $pD' - qk_g = D$. Then the matrix M defined by:

$$M = \begin{pmatrix} & & & \frac{k_1 q}{D'} \\ & & & \frac{k_2 q}{D'} \\ & M' & & \vdots \\ & & & \frac{k_{g-1} q}{D'} \\ k_g & 0 & \cdots \; 0 & p \end{pmatrix}$$

fulfills the desired requirements [1]. ☐

The following is an immediate application of Proposition A.1:

Corollary A.1. *Let $k = {}^t(k_1, \dots k_g) \in \mathbb{Z}^g$ a primitive column vector. Then, a unimodular matrix $M \in GL(g, \mathbb{Z})$ exists, such that k is its first column.*

As concerns matrices with entries in a field, decompositions into triangular factors are often useful to be considered.

Definition A.1. *Let K be a field and $M \in GL(n, K)$. The matrix M is said to admit a LU decomposition if $M = LU$, where $L \in GL(n, K)$ is a lower triangular matrix with entries* 1 *on the diagonal and $U \in GL(n, K)$ is an upper triangular matrix.*

[1] Since the main ingredient of the proof is the Bezout identity, the statement generally holds for principal ideal domains with no crucial differences in the proof. The existence of such a matrix M is likewise proved for each distinct greater common divisor D.

Proposition A.2. *The LU decomposition is unique.*

Proof. Let $M \in GL(n, K)$ be such that $LU = M = L'U'$, where $L, L' \in GL(n, K)$ are lower triangular matrices with entries 1 on the diagonal, and $U, U' \in GL(n, K)$ are upper triangular matrices. Then, $(L')^{-1}L = U'U^{-1}$. However, since $(L')^{-1}L$ is lower triangular and $U'U^{-1}$ is upper triangular, both must be diagonal. Moreover, the entries of $(L')^{-1}L$ on the diagonal are all 1; hence, $(L')^{-1}L = U'U^{-1} = 1_n$. Therefore, $L = L'$ and $U = U'$. □

Proposition A.3. *Let be K a field, and $M \in GL(n, K)$. Then M admits a LU decomposition if and only if all the leading principal minors are nonzero.*

Proof. Let $M \in GL(n, K)$ be such that $M = LU$ where $L \in GL(n, K)$ is a lower triangular matrix with entries 1 on the diagonal, and $U \in GL(n, K)$ is an upper triangular matrix. Then, for each $1 \leq h \leq n$ the $h \times h$ submatrix $M^{(h)}$ of M, corresponding to the $h \times h$ leading principal minor, clearly admits the decomposition $M^{(h)} = L^{(h)}U^{(h)}$, where $L^{(h)}$ and $U^{(h)}$ correspond to the $h \times h$ leading principal minor respectively of L and U; hence:

$$\det M^{(h)} = \det L^{(h)} \det U^{(h)} = \prod_{i=1}^{h} U_{ii} \neq 0$$

On the converse, let $M \in GL(n, K)$ be such that all the leading principal minors are nonzero; one can prove the LU decomposition for M by induction on n. For $n = 1$ the statement is trivial. Then, one can suppose it holds for $n - 1$; such a matrix $M \in GL(n, K)$ can be then described in a suitable block notation as:

$$M = \begin{pmatrix} M_{(n-1)} & a \\ {}^t b & M_{nn} \end{pmatrix}$$

where $M_{(n-1)}$ is a $(n-1) \times (n-1)$ matrix, whose leading principal minors are nonzero. By the inductive hypothesis, $M_{(n-1)} = L_{(n-1)}U_{(n-1)}$, where $L_{(n-1)} \in GL(n-1, K)$ is a lower triangular matrix with entries 1 on the diagonal, and $U_{(n-1)} \in GL(n-1, K)$ is an upper triangular matrix. Then, the matrices L and U are:

$$L = \begin{pmatrix} L_{(n-1)} & 0 \\ {}^t x & 1 \end{pmatrix} \qquad U = \begin{pmatrix} U_{(n-1)} & y \\ 0 & U_{nn} \end{pmatrix}$$

where x and y are the unique solutions of the systems:

$$L_{(n-1)}y = a \qquad {}^t U_{(n-1)}x = b$$

and $U_{nn} = M_{nn} - {}^t xy$.

□

As a corollary, one has the following:

Proposition A.4. **(Jacobi decomposition)** *Let $M \in Sym_n(\mathbb{R})$ be a definite positive matrix. Then, a unique decomposition $M = {}^t UDU$ exists, where $U \in GL(n, \mathbb{R})$ is an upper triangular matrix with entries* 1 *on the diagonal and D is a diagonal matrix with positive entries.*

Proof. Since M is a real symmetric definite positive matrix, all its leading principal minors are nonzero. By Proposition A.3, M admits a LU decomposition. Let be $M = L_0 U_0$ such a unique decomposition. Since $\det U_0 \neq 0$, each diagonal entry U_{ii} of the matrix U_0 is nonzero; then, one has:

$$U_0 = \begin{pmatrix} U_{11} & 0 & \dots & 0 \\ 0 & U_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & U_{nn} \end{pmatrix} \begin{pmatrix} 1 & \frac{U_{12}}{U_{11}} & \cdots & \frac{U_{1n}}{U_{11}} \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \frac{U_{n-1n}}{U_{n-1n-1}} \\ 0 & \cdots & 0 & 1 \end{pmatrix} = DU$$

By transposing, one also obtains $M = {}^tUD\,{}^tL_0$. Since the LU decomposition is unique, ${}^tL_0 = U$, and consequently $M = {}^tUDU$. One also has $D = {}^tU^{-1}MU^{-1}$, as U is invertible; hence, D is definite positive, which concludes the proof. □

A straightforward corollary is the following classical result:

Corollary A.2. **(Cholesky Decomposition)** *Let be $M \in Sym_n(\mathbb{R})$ a definite positive matrix. Then, a unique decomposition $M = L^tL$ exists, where $L \in GL(n, \mathbb{R})$ is a lower triangular matrix with positive entries on the diagonal.*

Proof. Since the diagonal matrix D appearing in the Jacobi decomposition $M = {}^tUDU$ is definite positive, $D^{1/2}$ exists. Then, the thesis follows by setting $L = {}^tUD^{1/2}$. □

Appendix B

Satake's Compactification

This appendix is designed to describe a singular compactification of the moduli space $A_g = \mathfrak{S}_g/\Gamma_g$, provided by Satake in [Sa], which is realized by adding cusps, namely orbits under the action of suitable subgroups of Γ_g; in this construction these cusps play indeed the role of different directions to infinity to be added to the space in order to make it compact. This is done by means of consequent steps.

B.1 Realization of $\mathfrak{S}_g$ as a Bounded Domain

To compactify $\mathfrak{S}_g/\Gamma_g$ in the way described above, one needs to find the right cusps to add; the first step consists of realizing $\mathfrak{S}_g$ as a bounded domain in $Sym_g(\mathbb{C})$. In the $g = 1$ case by the *Cayley transformation*:

$$C_1(\tau) := \frac{\tau - i}{\tau + i} \quad \forall \tau \in \mathfrak{S}_1 \tag{B.1}$$

one is provided with the Poincaré model, which realizes the complex upper half-plane $\mathbb{H}$ as the open unit disk $D_1 := \{z \in \mathbb{C} \mid |z| < 1\}$. The Cayley transformation admits a generalization to the upper half-space $\mathfrak{S}_g$:

$$\begin{aligned} &C_g : \mathfrak{S}_g \to D_g \\ &\tau \to (\tau - i1_g) \cdot (\tau + i1_g)^{-1} \end{aligned} \tag{B.2}$$

where $D_g = \{z \in Sym_g\mathbb{C} \mid z\bar{z} - 1 < 0\}$ is the natural generalization of the open unit disk D_1.

Proposition B.1. *The map C_g in (B.2) is an analytic isomorphism, whose inverse map is:*

$$C_g^{-1}(z) = i(1_g + z)(1_g - z)^{-1} \tag{B.3}$$

Proof. As proved in Proposition 1.5, $c_\gamma\tau + d_\gamma$ is invertible whenever $\tau \in \mathfrak{S}_g$ and $\gamma \in Sp(g,\mathbb{R})$. Hence, in particular $\det\tau \neq 0$ whenever $\tau \in \mathfrak{S}_g$; then, $\det\tau + i1_g \neq 0$ whenever $\tau \in \mathfrak{S}_g$ and the map C_g is consequently a well defined analytic map. Moreover, for each $\tau \in \mathfrak{S}_g$, one has:

$$\begin{aligned} C_g(\tau)\overline{C_g(\tau)} - 1_g &= (\tau - i1_g)(\tau + i1_g)^{-1}(\overline{\tau} + i1_g)(\overline{\tau} - i1_g)^{-1} = \\ &= {}^t\overline{(\overline{\tau} - i1_g)^{-1}}[(\tau - i1_g)(\overline{\tau} + i1_g) + (\tau + i1_g)(\overline{\tau} - i1_g)](\overline{\tau} - i1_g)^{-1} = \\ &= -4\,{}^t\overline{(\overline{\tau} - i1_g)^{-1}}Im\tau(\overline{\tau} - i1_g)^{-1} < 0 \end{aligned}$$

Therefore, C_g maps $\mathfrak{S}_g$ into D_g. Moreover, for each $z \in D_g$, $1_g - z$ is also invertible; in fact, if $w \in \mathbb{C}^g$ is such that $(1_g - z)w = 0$, then:

$${}^tw(1_g - z\overline{z})\overline{w} = 0$$

hence, $w = 0$ whenever $z \in D_g$; then, the map in (B.3) is also a well defined analytic map. Moreover, for each $z \in D_g$, one has:

$$\begin{aligned} ImC_g^{-1}(z) &= \frac{1}{2}[(1_g - z)^{-1}(1_g + z) + (1_g + \overline{z})(1_g - \overline{z})^{-1}] = \\ &= \frac{1}{2}{}^t\overline{(1_g - \overline{z})^{-1}}[(1_g + z)(1_g - \overline{z}) + (1_g - z)(1_g + \overline{z})](1_g - \overline{z})^{-1} = \\ &= {}^t\overline{(1_g - \overline{z})^{-1}}(1_g - z\overline{z})(1_g - \overline{z})^{-1} > 0 \end{aligned}$$

Then, C_g^{-1} maps D_g into $\mathfrak{S}_g$ and, as easily checked, is inverse to C_g. □

In general, an embedding theorem proved by Borel and Harish-Chandra (cf. [AMRT]) states that every symmetric domain can be realized as a bounded domain in a complex affine space of the same dimension if and only if it does not admit a direct factor, which is isomorphic to $\mathbb{C}^n$ modulus a discrete group of translation. Since, $Sp(g,\mathbb{R})$ is a simple Lie group, (1.14) implies $\mathfrak{S}_g$ does not admit such a factor and consequently the Borel and Harish-Chandra theorem applies. The Cayley transformation C_g is indeed seen to be the Harish-Chandra embedding.

Due to (1.10), an action of the symplectic group $Sp(g,\mathbb{R})$ is induced on the bounded domain D_g by the Cayley transform:

$$\begin{aligned} &Sp(g,\mathbb{R}) \times D_g \to D_g \\ &(\gamma, z) \to C_g\,\gamma\,C_g^{-1}z \end{aligned} \tag{B.4}$$

As easily seen, one has:

$$\gamma z := [\,(a_\gamma - ic_\gamma)(z+1_g) + i(b_\gamma - id_\gamma)(z-1_g)\,]\cdot[\,(a_\gamma + ic_\gamma)(z+1_g) + i(b_\gamma + id_\gamma)(z-1_g)\,]^{-1}$$

By virtue of the bounded realization of the Siegel upper half-space $\mathfrak{S}_g$, one is allowed to look for the cusps along the boundary. Indeed the symplectic group acts on the boundary of D_g as well, as the following Proposition states:

Proposition B.2. *Let* $\overline{D}_g := \{z \in Sym_g(\mathbb{C}|1_g - z\overline{z} \leq 0)\}$ *be the closure of* D_g *in* $Sym(g,\mathbb{C})$. *The action of* $Sp(g,\mathbb{R})$ *extends to* $\overline{D}_g$.

Proof. One has only to prove that the matrix

$$M^+_\gamma(z) := (a_\gamma + ic_\gamma)(z + 1_g) + i(b_\gamma + id_\gamma)(z - 1_g)$$

is invertible whenever $\gamma \in Sp(g, \mathbb{R})$ and $z \in \overline{D}_g$, namely $M^+_\gamma(z)$ is of maximum rank. By setting:

$$M^-_\gamma(z) := (a_\gamma - ic_\gamma)(z + 1_g) + i(b_\gamma - id_\gamma)(z - 1_g)$$

one has:

$$\begin{aligned} {}^t\overline{M^\pm_\gamma(z)}M^\pm_\gamma(z) = {} & (\overline{z} + 1_g)({}^ta_\gamma a_\gamma + {}^tc_\gamma c_\gamma)(z + 1_g) + (\overline{z} - 1_g)({}^tb_\gamma b_\gamma + {}^td_\gamma d_\gamma)(z - 1_g) + \\ & - i(\overline{z} - 1_g)({}^tb_\gamma a_\gamma + {}^td_\gamma c_\gamma)(z + 1_g) + i(\overline{z} + 1_g)({}^ta_\gamma b_\gamma + {}^tc_\gamma d_\gamma)(z - 1_g) + \\ & \pm 2(1_g - \overline{z}z) \end{aligned}$$

Hence:

$${}^t\overline{M^+_\gamma(z)}M^+_\gamma(z) = \frac{1}{2}({}^t\overline{M^+_\gamma(z)}M^+_\gamma(z) + {}^t\overline{M^-_\gamma(z)}M^-_\gamma(z)) + 2(1_g - \overline{z}z) \tag{B.5}$$

Now, since:

$$\begin{pmatrix} M^-_\gamma(z) \\ M^+_\gamma(z) \end{pmatrix} = \begin{pmatrix} 1_g & -i1_g \\ 1_g & i1_g \end{pmatrix} \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \begin{pmatrix} 1_g & 1_g \\ i1_g & -i1_g \end{pmatrix} \begin{pmatrix} z \\ 1_g \end{pmatrix}$$

one has:

$$\operatorname{rank} \begin{pmatrix} M^-_\gamma(z) \\ M^+_\gamma(z) \end{pmatrix} = \operatorname{rank} \begin{pmatrix} z \\ 1_g \end{pmatrix} = g$$

hence ${}^t\overline{M^+_\gamma(z)}M^+_\gamma(z) + {}^t\overline{M^-_\gamma(z)}M^-_\gamma(z) > 0$. Therefore, for each $z \in \overline{D}_g$, (B.5) implies:

$${}^t\overline{M^+_\gamma(z)}M^+_\gamma(z) > 0$$

and, therefore, $\operatorname{rank} M^+_\gamma(z) = g$. □

B.2 Boundary Components

The next step is to decompose $\overline{D}_g$ in such a way the decomposition is preserved by the action of $Sp(g, \mathbb{R})$; the group will thus operate on the components of $\overline{D}_g$. First of all, an equivalence relation can be introduced in $\overline{D}_g$, two points of $\overline{D}_g$ being declared equivalent if and only if they are connected by finitely many holomorphic arcs:

Definition B.1. *For $z, w \in \overline{D}_g$, one sets $z\ \rho\ w$, if and only if there exist finitely many holomorphic maps $f_1, \ldots, f_k : D_1 \to \overline{D}_g$, such that $f_1(0) = z$, $f_k(0) = w$, and $f_i(D_1) \cap f_{i+1}(D_1) \neq \emptyset$ for each $i = 1, \ldots, k$.*

The relation ρ is thus easily checked to be an equivalence relation on $\overline{D}_g$.

Definition B.2. *The equivalence classes of ρ are known as the* **boundary components** *of $\overline{D}_g$.*

In order to classify boundary components, a map $\psi_z : \mathbb{R}^{2g} \to \mathbb{C}^g$ can be defined for any fixed $z \in \overline{D}_g$:

$$\psi_z(x) := x \begin{pmatrix} i(1_g + z) \\ 1_g - z \end{pmatrix} \tag{B.6}$$

The importance of these maps is related to the subspaces $Ker\psi_z$, which are invariants of the boundary components.

Proposition B.3. *Let $z \in \overline{D}_g$ and let ψ_z be the correspondent map as in (B.6). Then:*

1. *$Ker\psi_z$ is an isotropic subspace of $\mathbb{R}^{2g}$, namely $xJ_g{}^t y = 0$ whenever $x, y \in Ker\psi_z$;*
2. *$Ker\psi_z \neq \{0\}$ if and only if $z \notin D_g$;*
3. *$Ker\psi_{\gamma z} = Ker\psi_z \gamma^{-1}$ for each $\gamma \in Sp(g, \mathbb{R})$;*

Proof. By identifying $\mathbb{R}^{2g}$ with $\mathbb{C}^g$ by means of the map:

$$(x_1, \dots x_{2g}) \xrightarrow{\phi} (x_1 + ix_{g+1}, x_2 + ix_{g+2}, \dots x_g + ix_{2g})$$

one has for each $z \in \overline{D}_g$:

$$\begin{aligned} \psi_z\phi^{-1}(w) &= \phi^{-1}(w) \begin{pmatrix} i1_g & i1_g \\ -1_g & -1_g \end{pmatrix} \begin{pmatrix} z \\ 1_g \end{pmatrix} = \\ &= (ix_1 - x_{g+1}, \dots ix_g - x_{2g}, ix_1 + x_{g+1}, \dots ix_g + x_{2g}) \begin{pmatrix} z \\ 1_g \end{pmatrix} = \\ &= i(wz + \overline{w}) \end{aligned}$$

Hence, $\overline{\phi(x)} = -i\phi(x)z$ and $\overline{\phi(y)} = -i\phi(y)z$ hold whenever $x, y \in Ker\psi_z$; then:

$$\begin{aligned} xJ_g{}^t y &= \sum_{i=1}^{g} x_i y_{g+i} - \sum_{i=1}^{g} x_{g+i} y_i = Im\left(\overline{\phi(x)}\,{}^t\phi(y)\right) = \\ &= \frac{1}{2}(\overline{\phi(x)}\,{}^t\phi(y) - \phi(x)^t\overline{\phi(y)}) = 0 \end{aligned}$$

and thus 1. is proved.
To prove 2, one has to note there exists $w \in \mathbb{C}^g$ such that $0 = wz + \overline{w}$ whenever $Ker\psi_z \neq 0$. Then, $w(1_g - z\overline{z}) = w + \overline{w}\,\overline{z} = w - w = 0$, hence $1_g - z\overline{z}$ is not positive definite; therefore, $z \in \overline{D}_g - D_g$. On the other hand, $z \notin \overline{D}_g$ implies there exists an eigenvector $w \in \mathbb{C}^{2g}$ such that $(z\overline{z})w = w$. If $w = -\overline{w}\,\overline{z}$, then w is a non null vector in $Ker\psi_z$; otherwise $(i\,w + i\,\overline{w}\,\overline{z})z = i(w\overline{z} + \overline{w}) = -(i\,w + i\,\overline{w}\,\overline{z})$ and $i\,w + i\,\overline{w}\,\overline{z}$ is thus a non null vector in $Ker\psi_z$.
Finally, 3. follows by a straightforward computation.

□

Corollary B.1. *Let $z_1, z_2 \in \overline{D}_g$. If $z_1 \,\rho\, z_2$, then $Ker\psi_{z_1} = Ker\psi_{z_2}$.*

Thanks to Corollary B.1, the following definition can be introduced:

Definition B.3. *Let F be a boundary component of D_g. The isotropic subspace associated to F is:*

$$U(F) := Ker\psi_z \qquad z \in F \tag{B.7}$$

By using the associated isotropic subspaces, the boundary components can be classified.

Proposition B.4. *The following subsets of $\overline{D}_g$:*

$$F_0 := \left\{1_g\right\} \tag{B.8a}$$

$$F_h := \left\{\begin{pmatrix}\tau & 0\\ 0 & 1_h\end{pmatrix} \;\middle|\; \tau \in D_h\right\} \cong D_h \quad \forall \quad 0 < h < g \tag{B.8b}$$

$$F_g := D_g \tag{B.8c}$$

are boundary components.

Proof. The following isotropic subspaces of $\mathbb{R}^{2g}$:

$$U^{(h)} := \sum_{i=0}^{h} \mathbb{R}e_{g+i}$$

are easily checked to be such that $U^h = U(F_h)$ for each $h = 1, \ldots g$. Therefore, each F_h must be a union of boundary components; F_h itself is a boundary component, being connected by holomorphic arcs. □

(B.7) defines a one-to-one correspondence between boundary components of $\overline{D}_g$ and isotropic subspaces of $\mathbb{R}^{2g}$ (cf. [HKW]). Therefore, $\overline{D}_g$ admits a decomposition into boundary components:

$$\overline{D}_g = \bigcup_{\substack{\gamma \in Sp(g,\mathbb{R})\\ 0 \le k \le g}} \gamma(F_k)$$

The boundary components, corresponding to isotropic spaces which are $\mathbb{Q}$-generated, are the only ones to be considered for adding the cusps:

Definition B.4. *The following subset of $\overline{D}_g$:*

$$\overline{D}_g^{rc} = \bigcup_{\substack{\gamma \in Sp(g,\mathbb{Q})\\ 0 \le k \le g}} \gamma(F_k)$$

is called the **rational closure** *of D_g.*

B.3 The Cylindrical Topology on D_g^{rc}

In this section, the so-called cylindrical topology on the rational closure D_g^{rc} will be briefly reviewed (details can be found in [Na], [HKW] and [F]) [1]. For the sake of simplicity, a suitable block notation can be introduced by means of the following maps:

$$\pi_{ji}: \quad \mathfrak{S}_j \to \mathfrak{S}_i \qquad\qquad \rho_{ji}: \quad \mathfrak{S}_j \to Sym^+_{j-i}(\mathbb{R})$$

$$\begin{pmatrix} \tau & {}^t w \\ w & \tau' \end{pmatrix} \to \tau' \qquad\qquad \begin{pmatrix} \tau & {}^t w \\ w & \tau' \end{pmatrix} \to Im\tau' - Imw\, Im\tau^{-1}\, Im^t w$$

Then, for each open set $U \subset \mathfrak{S}_i$ and $S_{j-i} \in Sym^+_{j-i}(\mathbb{R})$ $(j \geq i)$ a generalized open cylindrical neighbourhood is defined by:

$$N_{U,S_i} := \bigcup_{j \geq i} V_{U,S_{j-i}}$$

where:

$$V_{U,S_{j-i}} := \{\tau \in \mathfrak{S}_i \,|\, \pi_{ji}(\tau) \in U\,,\ \rho_{ji} - S_{J-i} > 0\}$$

A basis for a topology is then provided by the sets:

$$\tilde{N}_{U,S_i} := \bigcup_{\substack{\gamma \in Sp(g,\mathbb{Z}) \\ \gamma = \gamma_1\gamma_2\gamma_3}} \gamma N_{U,S_i}$$

and by their translates under the action of $Sp(g, \mathbb{Q})$, $\gamma_1, \gamma_2, \gamma_3$ being peculiar elements belonging to the stabilizer P_i of the rational boundary component F_i:

$$P_i = P(F_i) = \left\{ \begin{pmatrix} a & 0 & b & \alpha_1 \\ \alpha_2 & u & \alpha_3 & \alpha_4 \\ c & 0 & d & \alpha_5 \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} \in Sp(g, \mathbb{R}) \;\middle|\; \begin{matrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(i, \mathbb{R}) \\ \\ u \in GL(g-i, \mathbb{R}) \end{matrix} \right\}$$

More precisely:

$$\gamma_1 = \begin{pmatrix} 1_i & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & 1_i & 0 \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} \qquad u \in GL(g-i, \mathbb{R})$$

$$\gamma_2 = \begin{pmatrix} 1_i & 0 & 0 & {}^t n \\ m & 1_{g-i} & n & 0 \\ 0 & 0 & 1_i & -{}^t m \\ 0 & 0 & 0 & 1_{g-i} \end{pmatrix} \qquad m^t n \in Sym_{g-i}(\mathbb{R})$$

[1] The cylindrical topology, which have been introduced by Pjatečkii-Šapiro [PS], does not coincide with the one defined by Satake on the rational closure D_g^{rc}; however, they induce on the quotient D_g^{rc}/Γ the same topology.

$$\gamma_3 = \begin{pmatrix} 1_i & 0 & 0 & 0 \\ 0 & 1_{g-i} & 0 & S \\ 0 & 0 & 1_i & 0 \\ 0 & 0 & 0 & 1_{g-i} \end{pmatrix} \qquad S \in Sym_{g-i}(\mathbb{R})$$

The topology generated by $\tilde{N}_{U,S_i}$ and its translates under the action of $Sp(g,\mathbb{Q})$ is called the **cylindrical topology**.

The following statement holds:

Proposition B.5. *The cylindrical topology turns D_g^{rc}/Γ into a compact Hausdorff space containing D_g/Γ as a dense open subset.*

Proof. A proof can be found in [AMRT] [2]. □

As concerns the full modular group Γ_g, one can observe (cf. [Na]) that a sequence

$$\left\{\tau^{(n)} := \begin{pmatrix} \tau_1^{(n)} & \tau_2^{(n)} \\ \tau_2^{(n)} & \tau_3^{(n)} \end{pmatrix}\right\}_{n\in\mathbb{N}} \subset \mathfrak{S}_2$$

with $\{\tau_1^{(n)}\} \subset \mathfrak{S}_1$, converges in D_g^{rc} to an element $\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathfrak{S}_1 \cong F_1$ in the cylindrical topology if and only if

$$\tau_1^{(n)} \xrightarrow[n\to\infty]{} \tau_1 \qquad (\tau_3^{(n)} - Im\tau_2^{(n)} \left(Im\tau_1^{(n)}\right)^{-1} Im^t\tau_2^{(n)}) \xrightarrow[n\to\infty]{} \infty \tag{B.9}$$

If the sequence $\{\tau_2^{(n)}\} \subset \mathbb{C}$ is bounded, the convergence is thus characterized by the conditions:

$$\tau_1^{(n)} \xrightarrow[n\to\infty]{} \tau_1 \quad Im\tau_3^{(n)} \xrightarrow[n\to\infty]{} \infty \tag{B.10}$$

[2] The first part of the statement was proved by Satake in [Sa], while the proof of the second statement was provided by Baily in [Ba].

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