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On Stochastic, Irreversible Investment Problems in Continuous Time: a New Approach Based on First Order Conditions

Thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics for Economic-Financial Applications

by

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Le savant n'étudie pas la nature parce que cela est utile ;
il l'étudie parce qu'il y prend plaisir
et il y prend plaisir parce qu'elle est belle.

Jules Henri Poincaré

Ai miei genitori

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Introduction

In the last years the theory of irreversible investment under uncertainty has received much attention in Economics as well as in Mathematics (see, for example, the extensive review in Dixit and Pindyck [28]). From the mathematical point of view optimal irreversible investment problems under uncertainty are singular stochastic control problems. In fact the economic constraint that does not allow disinvestment may be modeled as a 'monotone follower' problem; that is a problem in which investment strategies are given by nondecreasing stochastic processes, not necessarily absolutely continuous with respect to the Lebesgue measure as functions of time. Work on 'monotone follower' problems and their application to Economics started with the pioneering papers by Karatzas, Karatzas and Shreve, El Karoui and Karatzas (cf. [40], [42] and [32]), among others. These Authors studied the problem of optimally minimizing a convex cost (or of optimally maximizing a concave profit) functional when the production capacity is a Brownian motion tracked by a nondecreasing process, i.e. the monotone follower. They showed that any such control problem is connected to a suitable optimal stopping problem whose value function v is the derivative of the value function V of the control problem; moreover, the optimal control ν_* defines an optimal stopping time τ^* in a very simple way through the formula $\tau^* := \inf\{t \in [0,T] : \nu_*(t) > 0\} \wedge T$. Later on, this kind of link has been established also for more complicated dynamics of the controlled diffusion; that is the case, for example, of a Geometric Brownian motion [2], or of a quite general controlled Ito diffusion (see [13] and [20], among others). More recently, Boetius [14], and Karatzas and Wang [47] showed that such connection holds in the case of bounded variation singular stochastic control problems as well; the value function of the control problem V satisfies $\frac{\partial}{\partial x}V = v$, where v is the saddle point of a suitable Dynkin game, that is a zero-sum optimal stopping game.

The link between irreversible investment problems and optimal stopping is also relevant in Economics. In fact a firm operating in a market with uncertainty not only has to decide how to invest but also when to invest. The optimal timing problem is then related to option theory, since it may be viewed as a 'real option', an option whose strike price is the cost of investment. It follows that exercising a real option means to invest properly at an optimal time.

Usually (see Kobila [48], Chiarolla and Haussmann [18] and [20], Riedel and Su [59], Oksendal [53] and Pham [55] among others) the optimal investment policy consists in waiting until the marginal expected future profit is below the marginal cost of investment; on the other hand, the times at which the future marginal expected profit equals the marginal cost of investment are optimal times to invest. Such simple policy is traditionally known in the economic literature as the 'Net Present Value' method. It follows that from the mathematical point of view one must find the region in which it is profitable to invest immediately (the so called 'action region') and the region in which it is optimal to wait (the so called 'no action region'). The boundary between these two regions is the free boundary of the optimal stopping problem naturally associated to the singular control one. The optimal investment is then the least effort to keep the controlled process inside the closure of the 'no-action' region, i.e. it is the local time of the optimal controlled diffusion at the free boundary.

The investment problem becomes even harder if one takes into account the fact that the available resources in which to invest may be limited. The problem turns into a 'finite fuel' singular stochastic control problem: the total amount of effort (fuel) available to the controller (for example the firm's manager) is limited. The mathematical literature on this field started in 1966 with Bather and Chernoff ([8] and [9]) in the context of controlling the motion of a spaceship. Finite fuel monotone follower problems were then studied by Benes, Shepp and Witsenhausen in 1980 [10]. In 1985 Chow, Menaldi and Robin [26], by a PDE approach, and Karatzas [43], by purely probabilistic arguments, showed that the optimal policy of a 'monotone follower' problem with constant finite fuel consists in following the unconstrained optimal policy until there is some fuel to spend. Much more difficult is the case of a finite fuel specified by a deterministic or stochastic time-dependent process. In 2005 Bank [7], without relying on any Markovian assumption, proved a suitable generalization of the optimal policy proposed by Karatzas [43] when the finite fuel is a stochastic, increasing, adapted process $\theta(t)$. The Author characterized the optimal policy of a cost minimization problem as the unique process satisfying some first order conditions for optimality ([7], Theorem 1): *the optimal control should be exercised only when its impact on future costs is maximal; on the other hand, when the cost functional's subgradient tends to decrease, then all the available fuel must be used.* More in detail, if $\mathbb{S}(\nu)$ is the Snell envelope of the total cost functional's subgradient $\nabla_\nu \mathcal{C}(\nu)$ (i.e. $\mathbb{S}(\nu)(t) := \operatorname{ess\,inf}_{t\leq\tau\leq T}\mathbb{E}\{\nabla_\nu\mathcal{C}(\nu)(\tau)|\mathcal{F}_t\}$), and $\mathcal{M}(\nu)+A(\nu)$ is its Doob-Meyer decomposition, then Bank [7] proved that ν^* is optimal if and only if

$$\begin{aligned} &\text{(i)} \quad \nu_* \text{ is flat off } \{\nabla_\nu\mathcal{C}(\nu_*) = \mathbb{S}(\nu_*)\} \\ &\text{(ii)} \quad A(\nu_*) \text{ is flat off } \{\nu_* = \theta\}. \end{aligned} \tag{1}$$

Moreover the Author constructed the optimal control ν_* in terms of the 'base capacity' process, a desirable value of capacity. Mathematically such process is the unique optional solution of the Bank-El Karoui representation problem [6].

The Bank-El Karoui Representation Theorem allows to write an optional process $Y = \{Y(t), t \in [0,T]\}$ as an optional projection of the form

$$Y(t) = \mathbb{E}\left\{\int_t^T f(s, \sup_{t\leq v\leq s}\xi(v))\,ds\,\Big|\mathcal{F}_t\right\}, \qquad t\in[0,T], \tag{2}$$

where $f = f(t,\xi)$ is a prescribed function, strictly decreasing in ξ, and $\{\xi(t), t\in[0,T]\}$ is a progressively measurable process to be found. It was shown in [6] that the representation problem (2) is closely linked to the solution of stochastic optimization problems as continuous time dynamic allocation problems with a limited amount of effort to spend on a fixed number of projects (e.g., cf. [31]), or the optimal consumption choice problem in a general semimartingale setting with Hindy-Huang-Kreps utility functional (cf. [3]).

The optimal stochastic, irreversible investment problem of a firm can also be involved into more complex problems as, for example, the existence of the General Equilibrium in a given economy. Roughly speaking, a market is in (intertemporal) equilibrium if prices (of labour, money, goods and financial instruments) vary over the time so that the firm's manager can maximize the profits of his company, the agents can optimize their utilities, and still 'market-clearing' conditions hold, i.e. there is a 'balance' between supply forces and demand forces. The mathematical treatment of such problem has been widely tackled in several classic papers like [27], [29], [35], [45], [51]. Also the economic literature is quite rich of models (usually in discrete time) which study the equilibrium problem for open and closed economies, cf. [49] and [52] among others. In [17] Chiarolla and Haussmann studied the equilibrium problem for a stochastic economy with consumption, wages and irreversible investment, whereas in [22] money, supplied by the government, was also considered. For an equilibrium model with irreversible investment simpler then that of [22], Paulsen [54] analyzed money market returns and reached some interesting economic conclusions.

In this Thesis we treat continuous time, stochastic, irreversible investment problems with both limited and unlimited resources. We develop a new approach based on first order conditions for optimality which correspond to a stochastic, infinite-dimensional analogue of the Kuhn-Tucker conditions of real analysis. Our approach is based on the identification of investment plans with the cumulative distribution of optional random measures on $[0, T]$. It does apply to very general semimartingale settings and not only to Markovian models; therefore it may be seen as a non-Markovian substitute of the dynamic programming approach. Moreover, as we show in Chapter 2, when the state process is a diffusion, then the dynamic programming method applies and our approach allows to obtain further regularity of the free boundary and a new characterization of it.

In Chapter 1 we study the Social Planner problem for a market with N firms in which the total investment is bounded by a stochastic, time-dependent finite fuel $\theta(t)$, that is $\sum_{i=1}^{N} \nu^{(i)}(t) \leq \theta(t)$ $\mathbb{P}$-a.s. for all $t \in [0,T]$. The Social Planner's objective is to pursue a vector of efficient irreversible investment processes $\underline{\nu}_* \in \mathbb{R}_+^N$ that maximizes the aggregate expected profit, net of investment cost, i.e.

$$\sup_{\sum_{i=1}^{N} \nu^{(i)} \leq \theta} \sum_{i=1}^{N} \mathbb{E}\bigg\{ \int_0^T e^{-\delta(t)} \; R^{(i)}(X(t), \nu^{(i)}(t))dt - \int_{[0,T)} e^{-\delta(t)} d\nu^{(i)}(t) \bigg\}.$$

Notice that the production function $R^{(i)}$ of firm i, $i = 1, 2, ..., N$, depends directly on the cumulative control exercised since in this problem we do not allow dynamics for the production capacity. As in [48] and [59], the uncertain status of the economy is modeled by an exogeneous economic shock $\{X(t), t \in [0,T]\}$. The application of a version of Komlòs' theorem for optional random measures (cf. [39]) allows us to prove existence and uniqueness of optimal irreversible investement policies. Then we use the concavity of the profit functional to characterize the optimal Social Planner policy as the unique solution of stochastic Kuhn-Tucker conditions. The Lagrange multiplier takes the form of a nonnegative optional random measure on $[0,T]$ whose support is the set of times for which the constraint is binding, i.e. all the fuel is spent. Hence, as a subproduct we obtain an enlightening interpretation of the first order conditions that Bank in [7] proved for a single firm optimal investment problem. Infact, condition (1)-(ii) may be interpreted as the Lagrange multiplier acting only when the constraint is binding; this is due to the identification of the Lagrange multiplier optional measure with the increment of the compensator in the Doob-Meyer decomposition of the net profit's supergradient at optimum.

Moreover, our generalized stochastic Kuhn-Tucker approach allows the explicit calculation of the Social Planner optimal investment strategy when the N firms have the same instantaneous production function (symmetric case) and, more interesting, in the case of Cobb-Douglas production functions with a different parameter for each firm. The Social Planner optimal policy is given in terms of the 'base capacity' process, i.e. the unique solution of Bank-El Karoui's Representation Problem [6].

Chapter 1 is organized as follows: in Section 1.1 we present the model. The generalized stochastic Kuhn-Tucker conditions for the Social Planner problem are introduced in Section 1.2. In Section 1.3 we test our approach on some 'finite-fuel' problems from the literature (cf. [7], [10], [48]) and we solve some N-firms Social Planner optimization problems.

In Chapter 2 we assume that the capacity is a diffusion process controlled by a nondecreasing process $\nu(t)$ representing the cumulative investment (as in Chiarolla and Haussmann [20] but without leisure, wages and scrap value), i.e.

$$\begin{cases} dC^{y,\nu}(t) = C^{y,\nu}(t)[-\mu_C(t)dt + \sigma_C(t)dW(t)] + f_C(t)d\nu(t), & t \in [0,T), \\ C^{y,\nu}(0) = y > 0. \end{cases}$$

Here we allow for unlimited resources, i.e. $\theta(t) = +\infty$. The firm's optimal investment problem is

$$\sup_{\nu} \mathbb{E}\bigg\{ \int_0^T e^{-\int_0^t \mu_F(s)ds} R(C^{y,\nu}(t))dt - \int_{[0,T)} e^{-\int_0^t \mu_F(s)ds} d\nu(t) \bigg\}.$$

In [20] the Authors proved the existence of the optimal investment process $\hat{\nu}$. As expected, the optimal time to invest τ^* was the solution of the associated optimal stopping problem. In particular, under constant coefficients and a Cobb-Douglas production function, they obtained a variational formulation for the optimal stopping problem, i.e. a free boundary problem. In order to characterize the moving boundary $\hat{y}(t)$ through an integral equation, the Authors proved the left-continuity of $\hat{y}(t)$ and assumed its right-continuity (cf. [20], Assumption-[Cfb]) since continuity of the free boundary was needed to prove the smooth fit property. Rather than trying to generalize the variational approach to the case of time-dependent coefficients, we characterize the free boundary by exploiting the Bank-El Karoui Representation Theorem (cf. [6]). In fact, by using the results in [6], Riedel and Su [59] in their irreversible investment problem with deterministic capacity and profit rate influenced by a stochastic parameter process, proved that *invest just enough to keep the production capacity above a certain lower bound* (their 'base capacity') is the optimal investment strategy. That means the optimal policy acts like the optimal control of the original monotone follower problem (e.g., cf. [40] and [42]) or, more generally, irreversible investment problems (cf. [2] and [20]). Hence in Chapter 2 we prove that the 'base capacity' and the free boundary arising in singular stochastic control problems are linked. That is done by identifying the 'base capacity' $l^*(t)$ of our irreversible investment problem with $\hat{y}(t)$. As a subproduct, in the case of constant coefficients and of a Cobb-Douglas production function, we obtain the right-continuity of the free boundary, whereas the variational approach did not lead to it in [20]. We start by proving some first order conditions for optimality. Then we obtain $l^*(t)$ as the unique solution of a representation problem in the spirit of Bank and El Karoui [6]. Hence we characterize the optimal solution of the investment problem in terms of $l^*(t)$ by the first order conditions for optimality. In particular, we prove that the 'base capacity' $l^*(t)$ is deterministic and coincides with the free boundary $\hat{y}(t)$ of the original irreversible investment problem when the coefficients of the controlled diffusion and the manager's discount factor are deterministic. It turns out that the representation problem for $l^*(t)$ provides an integral equation for the free boundary which might be solved numerically by backward induction.

Notice that when $T = +\infty$ we are able to find the explicit form of the free boundary which we show to coincide with that obtained in [55] by H. Pham via a viscosity solution approach.

Chapter 2 is organized as follows. In Section 2.1 we introduce the optimal investment problem, whereas in Section 2.2 we derive the first order conditions for optimality. In Section 2.3 we obtain the optimal production capacity. Under Markovian assumptions, in Section 2.4 we show that $l^*(t)$ is deterministic and coincides with $\hat{y}(t)$. Section 2.5 is devoted to the analysis of the Cobb-Douglas case with infinite time horizon. In Section 2.6 we recall the variational approach of Chiarolla and Haussmann [20] and we generalize some of their results to the case of deterministic, time-dependent coefficients. Such results are needed in Section 2.4.

In Chapter 3 we embed the firm's optimal irreversible investment problem into a stochastic continuous time economy on a finite time interval as it was done in [22]. The economy of [22] consists of a single perishable good producing firm which has to decide on cash holdings, levels of employment and investment for capacity expansion; rational agents that maximize their total expected utility of consumption, money holding and leisure, some of them are employed by the firm to facilitate capacity expansion and some who are retired or on welfare. Moreover, all the agents partecipate in a financial market consisting of a nominal bond, a real bond (i.e. valued in real terms), another type of contract, called derivative, and stocks of the firm. The shares' owner receives dividends. The firm produces a single kind of perishable consumption good and to do that it employs labour, borrows capital for its daily business, and sells shares to raise capital for capacity expansion. The agents and the firm's manager take the market parameters (e.g. the real interest rate, the wage process, the nominal interest rate, the real dividend process...) as given, but their 'optimal' value has to be characterized at equilibrium by some stochastic first order conditions ('market-clearing' conditions).

Rather than taking an exogeneous discount factor of the firm's manager we assume that it is the deflator of the economy. It was shown in [22] that this leads to a very difficult random fixed point for the deflator, the nominal interest rate and the wage process. We study the existence of a solution to such fixed point problem. In the simpler case of no leisure and no money, the deflator may be thought as an element of the Skorohod space of càdlàg processes endowed with the Meyer-Zheng topology (cf. [50] and Appendix A for a brief introduction on such topology). Then, under some reasonable assumptions, we are able to prove the continuity of the random operator arising in the fixed point problem and the compactness of its domain. Hence an application of Schauder Theorem (see, for example, [61]) guarantees the existence of an equilibrium deflator.

Finally, in the Conclusions we discuss some possible developments of the research and open problems.

Chapter 1

Generalized Kuhn-Tucker Conditions for N-Firms Stochastic Irreversible Investment under Limited Resources

In this Chapter we study a continuous time, optimal stochastic investment problem with limited resources in a market with N firms. The investment processes are subject to a time-dependent stochastic constraint. Rather than using a dynamic programming approach, we exploit the concavity of the profit functional to derive some necessary and sufficient first order conditions for the corresponding *Social Planner* optimal policy. Our conditions are a stochastic infinite-dimensional generalization of the Kuhn-Tucker Theorem. As a subproduct we obtain an enlightening interpretation of the first order conditions in Bank [7] for a single firm.

In the infinite-horizon case with Cobb-Douglas production functions our method allows the explicit calculation of the optimal policy in terms of the 'base capacity' process, i.e. the unique solution of the Bank and El Karoui representation problem [6].

1.1 The Model

We consider a market with N firms on a time horizon $T \leq +\infty$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a complete filtered probability space with the filtration $\{\mathcal{F}_t, t \in [0,T]\}$ satisfying the usual conditions. The cumulative irreversible investment of firm i, $i = 1, 2, ..., N$, denoted by $\nu^{(i)}(t)$, is an adapted process, nondecreasing, left-continuous, finite a.s. s.t. $\nu^{(i)}(0) = y^{(i)} > 0$.

The firms are financed entirely by equities but we focus primarily on the irreversibility of investments and do not model precisely the rest of the economy. It is reasonable to assume that the firms cannot invest in technologies or primary resources as much as they like. In fact, we assume that the total amount of technologies and available primary resources in the market is a finite quantity $\theta(t)$, at each time t, depending on the status of the economy. That is,

$$\sum_{i=1}^{N} \nu^{(i)}(t) \leq \theta(t), \quad \mathbb{P}-\text{a.s.}, \quad \text{for} \quad t \in [0, T]. \tag{1.1}$$

The stochastic time-dependent constraint $\{\theta(t), t \in [0, T]\}$ is the cumulative amount of resources extracted or technologies produced up to time t. It is a nonnegative and increasing adapted process with left-continuous paths, which starts at time zero from $\theta(0) = \theta_o > 0$. We assume

$$\mathbb{E}\{\theta(T)\} < +\infty. \tag{1.2}$$

We denote by $\mathcal{S}_\theta$ the nonempty set of admissible investment plans, i.e.

$$\begin{aligned} \mathcal{S}_\theta \;\; := \;\; & \{\underline{\nu} : \Omega \times [0, T] \to \mathbb{R}^N_+, \text{ nondecreasing, left-continuous, adapted process s.t.} \\ & \nu^{(i)}(0) = y^{(i)}, \; \mathbb{P}-\text{a.s.}, \; i = 1, 2, ..., N, \text{ and } \sum_{i=1}^{N} \nu^{(i)}(t) \leq \theta(t), \; \mathbb{P}-\text{a.s. } \forall t \in [0, T]\}. \end{aligned}$$

Let $\{X(t), t \in [0, T]\}$ be some exogenous real-valued state variable progressively measurable with respect to $\mathcal{F}_t$. It may be regarded as an economic shock, reflecting the changes in technological ouput, demand and macroeconomic conditions which have direct or indirect effect on the firm's profit. At the moment we do not make any Markovian assumption.

We work in a moneyless world and so all the quantities are measured in units of capital goods. That implies that the unitary price of the investment is set identically equal to one. We take the point of view of a fictitious *Social Planner* aiming to maximize the aggregate expected profit, net of investment costs, $\mathcal{J}_{SP}(\underline{\nu})$ (see equation (1.5) below). We denote by

$\delta(t)$ the Social Planner discount factor. $\delta(t)$ is a nonnegative, optional process, bounded uniformly in $(\omega, t) \in \Omega \times [0, T]$. Assumption (1.2) ensures

$$\mathbb{E}\left\{ \int_{[0,T)} e^{-\delta(t)} \, d\nu^{(i)}(t) \right\} < +\infty, \qquad i = 1, 2, ..., N, \tag{1.3}$$

i.e. the investment plan's expected net present value of firm i is finite.

The production function of firm i, $i = 1, 2, ..., N$, is $R^{(i)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$. At time t, when firm i investment is $\nu^{(i)}(t)$, $R^{(i)}\left(X(t), \nu^{(i)}(t)\right)$ represents the amount of goods produced by firm i under the shock process $X(t)$.

The Social Planner problem is

$$V_{SP} := \sup_{\underline{\nu} \in \mathcal{S}_\theta} \mathcal{J}_{SP}(\underline{\nu}), \tag{1.4}$$

where

$$\mathcal{J}_{SP}(\underline{\nu}) := \sum_{i=1}^{N} \mathcal{J}_i(\nu^{(i)}) \tag{1.5}$$

and, for $i = 1, 2, ..., N$,

$$\mathcal{J}_i(\nu^{(i)}) = \mathbb{E}\left\{ \int_0^T e^{-\delta(t)} \, R^{(i)}(X(t), \nu^{(i)}(t))dt - \int_{[0,T)} e^{-\delta(t)} d\nu^{(i)}(t) \right\}. \tag{1.6}$$

Notice that $\mathcal{J}_i(\nu^{(i)})$ is the expected total profit, net of investment costs, of firm i when the Social Planner picks $\underline{\nu} \in \mathcal{S}_\theta$.

The production functions satisfy the following concavity and regularity assumptions.

Assumption 1.1.1.

1. *For every $x \in \mathbb{R}$ and $i = 1, 2, ..., N$, the mapping $y \to R^{(i)}(x, y)$ is increasing, strictly concave, with continuous decreasing partial derivative $R^{(i)}_y(x, y)$ satisfying the Inada conditions*

$$\lim_{y \to 0} R^{(i)}_y(x, y) = \infty, \qquad \lim_{y \to \infty} R^{(i)}_y(x, y) = 0.$$

2. $R^{(i)}\left(X(\omega,t),\nu^{(i)}(\omega,t)\right)$ *is* $d\mathbb{P}\otimes dt$*-integrable, for* $i=1,2,...,N$.

3. *The process*

$$(\omega,t)\longrightarrow \sup_{\nu^{(i)}(\omega,t)\,:\,\underline{\nu}\in\mathcal{S}_\theta} R^{(i)}(X(\omega,t),\nu^{(i)}(\omega,t))$$

is $d\mathbb{P}\otimes dt$*-integrable, for* $i=1,2,...,N$.

Under (1.2) and Assumption 1.1.1 the net profit $\mathcal{J}_i(\nu^{(i)})$ is well definite and finite for all admissible plans.

In the next Section we show how to handle constraint (1.1) in order to find the solution to Social Planner problem (1.4).

1.2 A Stochastic Kuhn-Tucker Approach

In this Section we aim to find an optimal investment plan by means of a gradient approach. As in [59], proof of Theorem 2.6, by applying a suitable version of Komlòs' Theorem for optional random measures (cf. [39], Lemma 3.5), we obtain existence and uniqueness of a solution to problem (1.4). Komlòs' Theorem states that a sequence of random variables $(Z_n)_{n\in\mathbb{N}}$ upper-bounded in expectation, has a subsequence $(Z_{n_k})_{k\in\mathbb{N}}$ which converges in the Cesàro sense to some random variable Z. The limit identified by Komlòs' Theorem turns out to be the optimal investment strategy.

Theorem 1.2.1. *Under* (1.2) *and Assumption* 1.1.1*, there exists a unique optimal vector of irreversible investment plans* $\underline{\nu}_*\in\mathcal{S}_\theta$ *for problem* (1.4).

Proof. Let $\underline{\nu}\in\mathcal{S}_\theta$ and denote by $\mathcal{H}$ the space of optional measures on $[0,T]$. Then, the investment strategies $\nu^{(i)}$ may be regarded as elements of $\mathcal{H}$, hence $\mathcal{S}_\theta\subset\mathcal{H}^N$.

Let $(\underline{\nu}_n)_{n\in\mathbb{N}}$ be a maximizing sequence of investment plans in $\mathcal{S}_\theta$, i.e. a sequence such that $\lim_{n\to\infty}\mathcal{J}_{SP}(\underline{\nu}_n)=V_{SP}$. By (1.2) we have that the sequence $(\mathbb{E}\{\nu_n^{(i)}(T)\})_{n\in\mathbb{N}}$ is bounded for $i=1,2,...,N$; in fact, $\mathbb{E}\{\nu_n^{(i)}(T)\}\leq\mathbb{E}\left\{\theta(T)\right\}<\infty$. By a version of Komlòs' Theorem for optional measures, there exists a subsequence $(\hat{\underline{\nu}}_n)_{n\in\mathbb{N}}$ that converges weakly a.s. in the

Cesàro sense to some random vector $\underline{\nu}_* \in \mathcal{H}^N$. That is, for $i = 1, 2, ..., N$, we have, almost surely,

$$\hat{I}_n^{(i)}(t) := \frac{1}{n}\sum_{j=0}^{n} \hat{\nu}_j^{(i)}(t) \to \nu_*^{(i)}(t), \quad \text{as} \quad n \to \infty. \tag{1.7}$$

Notice that $\underline{\hat{\nu}}_n \in \mathcal{S}_\theta$ for all n implies that also the Cesàro sequence $\underline{\hat{I}}_n$ belongs to $\mathcal{S}_\theta$ due to the convexity of $\mathcal{S}_\theta$, hence $\sum_{i=1}^N \hat{I}_n^{(i)}(t) \leq \theta(t)$, for $n \in \mathbb{N}$. It follows that, almost surely,

$$\sum_{i=1}^{N} \nu_*^{(i)}(t) \leq \theta(t), \tag{1.8}$$

which means $\underline{\nu}_* \in \mathcal{S}_\theta$.

Since $(\nu_n^{(i)})_{n\in\mathbb{N}}$ is a maximizing sequence so is $(\hat{I}_n^{(i)})_{n\in\mathbb{N}}$ by concavity of the profit functional. Then Jensen inequality and the dominated convergence theorem yield

$$\mathcal{J}_{SP}(\underline{\nu}_*) \geq \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n} \mathcal{J}_{SP}(\underline{\hat{\nu}}_n) = V_{SP}. \tag{1.9}$$

Finally, uniqueness follows from the strict concavity of the Social Planner profit functional.

□

We now aim to characterize the Social Planner optimal policy as the unique solution of a set of first order generalized stochastic Kuhn-Tucker conditions. Notice that the strict concave functionals $\mathcal{J}_i$, $i = 1, 2, ..., N$, admit the supergradient

$$\nabla_y \mathcal{J}_i(\nu^{(i)})(t) := \mathbb{E}\left\{ \int_t^T e^{-\delta(s)}\, R_y^{(i)}(X(s), \nu^{(i)}(s))\, ds \,\middle|\, \mathcal{F}_t \right\} - e^{-\delta(t)}\mathbb{1}_{\{t<T\}} \tag{1.10}$$

for $t \in [0, T]$.

Remark 1.2.2. *The quantity $\nabla_y \mathcal{J}_i(\nu^{(i)})(t)$, $i = 1, 2, ..., N$, may be interpreted as the marginal expected profit resulting from an additional infinitesimal investment at time t when the investment plan is $\nu^{(i)}$. Mathematically, $\nabla_y \mathcal{J}_i(\nu^{(i)})$ is the Riesz representation of the profit gradient at $\nu^{(i)}$. More precisely, define $\nabla_y \mathcal{J}_i(\nu^{(i)})$ as the optional projection of the progressively measurable process*

$$\Phi_i(\omega, t) := \int_t^T e^{-\delta(\omega,s)}\, R_y^{(i)}(X(\omega, s), \nu^{(i)}(\omega, s))\, ds \,-\, e^{-\delta(\omega,t)}\mathbb{1}_{\{t<T\}}, \tag{1.11}$$

for $\omega \in \Omega$, *and* $t \in [0,T]$. *Hence* $\nabla_y \mathcal{J}_i(\nu^{(i)})$ *is uniquely determined up to* $\mathbb{P}$*-indistinguishability and it holds*

$$\mathbb{E}\Big\{ \int_{[0,T)} \nabla_y \mathcal{J}_i(\nu^{(i)})(t) d\nu^{(i)}(t) \Big\} = \mathbb{E}\Big\{ \int_{[0,T)} \Phi_i(t) d\nu^{(i)}(t) \Big\}$$

for all admissible $\nu^{(i)}(t)$ *(cf. Theorem* 1.33 *in [36]).*

1.2.1 Generalized Stochastic Kuhn-Tucker Conditions

Let $\mathcal{B}[0,T]$ denote the Borel σ-algebra on $[0,T]$. Recall that if $\beta(t)$ is a right-continuous, adapted and nondecreasing process, then the bracket operator

$$\langle \alpha, \beta \rangle = \mathbb{E}\Big\{ \int_{[0,T)} \alpha(t)\, d\beta(t) \Big\} \tag{1.12}$$

is well defined (possibly infinite) for all processes $\alpha(t)$ which are nonnegative and $\mathcal{F}_T \otimes \mathcal{B}[0,T]$-measurable. Notice that the bracket is preserved when we pass from α to its optional projection $\alpha^{(o)}$ (cf. [36], Theorem 1.33); that is

$$\langle \alpha, \beta \rangle = \langle \alpha^{(o)}, \beta \rangle. \tag{1.13}$$

Since the constraint is $\theta(t) - \sum_{i=1}^N \nu^{(i)}(t) \geq 0$, $\mathbb{P}$-a.s. for all $t \in [0,T]$ (cf. (1.1)), we define the *Lagrangian functional* of problem (1.4) as

$$\begin{aligned}
\mathcal{L}^{\theta}(\underline{\nu}, \lambda) &= \mathcal{J}_{SP}(\underline{\nu}) + \langle \theta - \sum_{i=1}^N \nu^{(i)}, \lambda \rangle \\
&= \sum_{i=1}^N \mathbb{E}\Big\{ \int_0^T e^{-\delta(t)}\ R^{(i)}(X(t), \nu^{(i)}(t))dt - \int_{[0,T)} e^{-\delta(t)} d\nu^{(i)}(t) \Big\} \\
&\quad + \mathbb{E}\Big\{ \int_{[0,T)} [\theta(t) - \sum_{i=1}^N \nu^{(i)}(t)] d\lambda(t) \Big\},
\end{aligned} \tag{1.14}$$

where $d\lambda(\omega, t)$ is a nonnegative optional measure, which may be interpreted as the Lagrange multiplier of Social Planner problem (1.4). By using Fubini's Theorem we write the bracket $\langle \theta - \sum_{i=1}^N \nu^{(i)}, \lambda \rangle$ in a more convenient form, that is

$$\begin{aligned}
\langle \theta - \sum_{i=1}^{N} \nu^{(i)}, \lambda \rangle &= \mathbb{E}\Big\{ \int_{[0,T)} [\theta(t) - \sum_{i=1}^{N} \nu^{(i)}(t)] d\lambda(t) \Big\} \\
&= \mathbb{E}\Big\{ \int_{[0,T)} \Big[\int_{[0,t)} (d\theta(s) - \sum_{i=1}^{N} d\nu^{(i)}(s)) \Big] d\lambda(t) \Big\} + K\,\mathbb{E}\Big\{ \int_{[0,T)} d\lambda(t) \Big\} \\
&= \mathbb{E}\Big\{ \int_{[0,T)} \Big[\int_{[t,T)} d\lambda(s) \Big] (d\theta(t) - \sum_{i=1}^{N} d\nu^{(i)}(t)) \Big\} + K\,\mathbb{E}\Big\{ \int_{[0,T)} d\lambda(t) \Big\},
\end{aligned}$$

where $K := \theta_o - \sum_{i=1}^{N} y^{(i)} = \theta(0) - \sum_{i=1}^{N} \nu^{(i)}(0)$. Hence

$$\begin{aligned}
\mathcal{L}^{\theta}(\underline{\nu}, \lambda) &= \mathcal{J}_{SP}(\underline{\nu}) + \langle \theta - \sum_{i=1}^{N} \nu^{(i)}, \lambda \rangle \\
&= \sum_{i=1}^{N} \mathbb{E}\Big\{ \int_0^T e^{-\delta(t)} \, R^{(i)}(X(t), \nu^{(i)}(t)) dt - \int_{[0,T)} e^{-\delta(t)} d\nu^{(i)}(t) \Big\} \\
&\quad + \mathbb{E}\Big\{ \int_{[0,T)} \Big[\int_{[t,T)} d\lambda(s) \Big] (d\theta(t) - \sum_{i=1}^{N} d\nu^{(i)}(t)) \Big\} + K\,\mathbb{E}\Big\{ \int_{[0,T)} d\lambda(t) \Big\}.
\end{aligned}$$

As done in [3] for an intertemporal utility maximization problem under uncertainty with Hindy, Huang and Kreps preferences, we now obtain stochastic Kuhn-Tucker conditions for optimality.

Theorem 1.2.3. *Under (1.2) and Assumption 1.1.1, an admissible investment vector* $\underline{\nu}_*$ *is the unique solution of the Social Planner problem (1.4) if there exists a nonnegative Lagrange multiplier measure* $d\lambda(\omega, t)$ *such that* $\mathbb{E}\{ \int_{[0,T)} d\lambda(t)\} < \infty$*, and the following generalized stochastic Kuhn-Tucker conditions hold true for* $i = 1, 2, ..., N$

$$\begin{cases}
\nabla_y \mathcal{J}_i(\nu_*^{(i)})(t) \leq \mathbb{E}\Big\{ \int_{[t,T)} d\lambda(s) \Big| \mathcal{F}_t \Big\}, \qquad \mathbb{P}-a.s., \ \forall t \in [0,T), \\
\int_{[0,T)} \Big[\nabla_y \mathcal{J}_i(\nu_*^{(i)})(t) - \mathbb{E}\Big\{ \int_{[t,T)} d\lambda(s) \Big| \mathcal{F}_t \Big\} \Big] d\nu_*^{(i)}(t) = 0, \qquad \mathbb{P}-a.s., \\
\mathbb{E}\Big\{ \int_{[0,T)} [\theta(t) - \sum_{i=1}^{N} \nu_*^{(i)}(t)] d\lambda(t) \Big\} = 0.
\end{cases} \tag{1.15}$$

Proof. Let $\underline{\nu}_*$ satisfy the first order Kuhn-Tucker conditions (1.15) and let $\underline{\nu}$ be an arbitrary admissible plan. By concavity of $R^{(i)}(x,\cdot)$, $i=1,2,...,N$, and Fubini's Theorem we have

$$\begin{aligned}
\mathcal{J}_{SP}(\underline{\nu}_*) - \mathcal{J}_{SP}(\underline{\nu}) &= \sum_{i=1}^{N} \mathbb{E}\bigg\{ \int_0^T e^{-\delta(t)}[R^{(i)}(X(t),\nu_*^{(i)}(t)) - R^{(i)}(X(t),\nu^{(i)}(t))\,]dt \\
&\qquad - \int_{[0,T)} e^{-\delta(t)} d(\nu_*^{(i)}(t) - \nu^{(i)}(t)) \bigg\} \\
&\geq \sum_{i=1}^{N} \mathbb{E}\bigg\{ \int_0^T e^{-\delta t} R_y^{(i)}(X(t),\nu_*^{(i)}(t))\,(\nu_*^{(i)}(t) - \nu^{(i)}(t))\,dt \qquad (1.16) \\
&\qquad - \int_{[0,T)} e^{-\delta(t)} d(\nu_*^{(i)}(t) - \nu^{(i)}(t)) \bigg\} \\
&= \sum_{i=1}^{N} \mathbb{E}\bigg\{ \int_{[0,T)} \int_s^T e^{-\delta(t)} R_y^{(i)}(X(t),\nu_*^{(i)}(t))\,dt\,d(\nu_*^{(i)}(s) - \nu^{(i)}(s)) \\
&\qquad - \int_{[0,T)} e^{-\delta(s)} d(\nu_*^{(i)}(s) - \nu^{(i)}(s)) \bigg\} \\
&= \sum_{i=1}^{N} \mathbb{E}\bigg\{ \int_{[0,T)} \nabla_y \mathcal{J}_i(\nu_*^{(i)})(t)\, d(\nu_*^{(i)}(t) - \nu^{(i)}(t)) \bigg\}.
\end{aligned}$$

Now (1.15) implies

$$\begin{aligned}
\mathcal{J}_{SP}(\underline{\nu}_*) - \mathcal{J}_{SP}(\underline{\nu}) &\geq \sum_{i=1}^{N} \mathbb{E}\bigg\{ \int_{[0,T)} \nabla_y \mathcal{J}_i(\nu_*^{(i)})(t)\, d(\nu_*^{(i)}(t) - \nu^{(i)}(t)) \bigg\} \\
&\geq \sum_{i=1}^{N} \mathbb{E}\bigg\{ \int_{[0,T)} \mathbb{E}\bigg\{ \int_{[t,T)} d\lambda(s) \Big| \mathcal{F}_t \bigg\} d(\nu_*^{(i)}(t) - \nu^{(i)}(t)) \bigg\} \qquad (1.17) \\
&= \sum_{i=1}^{N} \mathbb{E}\bigg\{ \int_{[0,T)} \Big[\int_{[t,T)} d\lambda(s) \Big] d(\nu_*^{(i)}(t) - \nu^{(i)}(t)) \bigg\}
\end{aligned}$$

and the nonnegativity of $d\lambda(t)$, the admissibility of $\underline{\nu}$, and another application of Fubini's Theorem give

$$\begin{aligned}
\mathcal{J}_{SP}(\underline{\nu}_*) - \mathcal{J}_{SP}(\underline{\nu}) &\geq \sum_{i=1}^{N} \mathbb{E}\bigg\{ \int_{[0,T)} \Big[\int_{[t,T)} d\lambda(s) \Big] d(\nu_*^{(i)}(t) - \nu^{(i)}(t)) \bigg\} \\
&= \mathbb{E}\bigg\{ \int_{[0,T)} \sum_{i=1}^{N} [\nu_*^{(i)}(t) - \nu^{(i)}(t)] d\lambda(t) \bigg\} \\
&= \mathbb{E}\bigg\{ \int_{[0,T)} [\theta(t) - \sum_{i=1}^{N} \nu^{(i)}(t)] d\lambda(t) \bigg\} \geq 0,
\end{aligned}$$

where the last line follows from (1.15), third condition. □

Conditions (1.15) are also necessary for optimality under the assumption that

$$\omega \to \theta(\omega, T) \int_0^T R^{(i)}(X(\omega, t), \theta(\omega, T))dt \quad \text{is} \quad d\mathbb{P} - \text{integrable}, \quad i = 1, 2, ..., N. \tag{1.18}$$

The proof is based on arguments similar to those used in the finite-dimensional Kuhn-Tucker Theorem. Denote by $\mathcal{T}$ the set of all stopping times in $[0, T]$, $\mathbb{P}$-a.s., and notice that

$$\nabla_y \mathcal{J}_i(\nu_*^{(i)})(\tau) \leq \mathbb{E}\Big\{ \int_{[\tau, T)} d\lambda(s) \Big| \mathcal{F}_\tau \Big\},$$

for every $i = 1, 2, ..., N$ and for all $\tau \in \mathcal{T}$. In fact, if not, then there would exist some $\overline{\tau} \in \mathcal{T}$ such that $\nabla_y \mathcal{J}_i(\nu_*^{(i)})(\overline{\tau}) > \mathbb{E}\{\int_{[\overline{\tau}, T)} d\lambda(s) | \mathcal{F}_{\overline{\tau}}\}$ which, togheter with the continuity of $R_y^{(i)}$ and the linearity of investment costs, would imply that a sufficiently small extra investment at $\overline{\tau}$ is profitable and hence contradict the optimality of $\nu_*^{(i)}$, $i = 1, 2, ..., N$.

In the next Lemma we show that under (1.18) the optimal policy $\underline{\nu}_*$ solves the linearized problem

$$\sup_{\underline{\nu} \in \mathcal{S}_\theta} \sum_{i=1}^N \mathbb{E}\Big\{ \int_{[0,T)} \Phi_i^*(s) d\nu^{(i)}(s) \Big\} \tag{1.19}$$

where Φ_i^* is the progressively measurable process associated to $\nabla_y \mathcal{J}_i(\nu_*^{(i)})$, $i = 1, 2, ..., N$, and defined in (1.11). Solutions of the linear problem will then be characterized by some 'flat-off conditions' in the second Lemma.

Lemma 1.2.4. *Let $\underline{\nu}_*$ be optimal for problem (1.4) and assume (1.18). Then it solves (1.19).*

Proof. Let $\underline{\nu}$ be an admissible plan. For $i = 1, 2, ..., N$ and $\epsilon \in (0, 1)$, set $\nu_\epsilon^{(i)} = \epsilon \nu^{(i)} + (1 - \epsilon)\nu_*^{(i)}$ and let Φ_i^ϵ be the progressively measurable process defined in (1.11) associated to $\nabla_{\nu_i} \mathcal{J}_i(\nu_\epsilon^{(i)})$. Then $\lim_{\epsilon \to 0} \nu_\epsilon^{(i)}(t) = \nu_*^{(i)}(t)$, $\mathbb{P}$-a.s., as well as $\lim_{\epsilon \to 0} \Phi_i^\epsilon(t) = \Phi_i^*(t)$, $\mathbb{P}$-a.s., by continuity of $R_y^{(i)}$. Optimality of $\underline{\nu}_*$, concavity of $y \to R^{(i)}(X(t), y)$ and Fubini's Theorem, imply

$$
\begin{aligned}
0 \;&\geq\; \frac{1}{\epsilon}\left[\mathcal{J}_{SP}(\underline{\nu}_{\epsilon}) - \mathcal{J}_{SP}(\underline{\nu}_{*})\right] \\
&= \frac{1}{\epsilon}\sum_{i=1}^{N}\mathbb{E}\bigg\{\int_0^T e^{-\delta(t)}[\,R^{(i)}(X(t),\nu_{\epsilon}^{(i)}(t)) - R^{(i)}(X(t),\nu_{*}^{(i)}(t))\,]dt \\
&\qquad\qquad -\epsilon\int_{[0,T)} e^{-\delta(t)}d(\nu^{(i)}(t) - \nu_{*}^{(i)}(t))\bigg\} \qquad (1.20) \\
&\geq \sum_{i=1}^{N}\mathbb{E}\bigg\{\int_{[0,T)}\Phi_i^{\epsilon}(t)d(\nu^{(i)}(t) - \nu_{*}^{(i)}(t))\bigg\},
\end{aligned}
$$

since $\epsilon(\nu^{(i)} - \nu_{*}^{(i)}) = \nu_{\epsilon}^{(i)} - \nu_{*}^{(i)}$.

In order to prove that

$$\sum_{i=1}^{N}\mathbb{E}\bigg\{\int_{[0,T)}\Phi_i^{*}(t)\,d(\nu^{(i)}(t) - \nu_{*}^{(i)}(t))\bigg\} \leq 0$$

we need to apply Fatou's Lemma to conclude (by (1.20))

$$\sum_{i=1}^{N}\mathbb{E}\bigg\{\int_{[0,T)}\Phi_i^{*}(t)\,d(\nu^{(i)}(t) - \nu_{*}^{(i)}(t))\bigg\} \leq \liminf_{\epsilon\to 0}\sum_{i=1}^{N}\mathbb{E}\bigg\{\int_{[0,T)}\Phi_i^{\epsilon}(t)\,d(\nu^{(i)}(t) - \nu_{*}^{(i)}(t))\bigg\} \leq 0.$$

To check the hypothesis of Fatou's Lemma, we must find $d\mathbb{P}$-integrable random variables, $G_i(\omega)$, $i = 1, 2, ..., N$, such that

$$I_i^{\epsilon}(\omega) := \int_{[0,T)}\Phi_i^{\epsilon}(\omega,t)\,d(\nu^{(i)}(\omega,t) - \nu_{*}^{(i)}(\omega,t)) \geq G_i(\omega), \quad \epsilon \in (0,1).$$

We write I_i^{ϵ} as

$$I_i^{\epsilon} = \int_0^T e^{-\delta(t)}R_y^{(i)}(X(t),\nu_{\epsilon}^{(i)}(t))(\nu^{(i)}(t) - \nu_{*}^{(i)}(t))dt - \int_{[0,T)} e^{-\delta(t)}d(\nu^{(i)}(t) - \nu_{*}^{(i)}(t)) \qquad (1.21)$$

by Fubini's Theorem. Then, from concavity of $R^{(i)}(x,\cdot)$ and

$$\nu_{\epsilon}^{(i)}(t)\begin{cases} \leq \nu^{(i)}(t), & \text{on } \{t : \nu^{(i)}(t) - \nu_{*}^{(i)}(t) \geq 0\}, \\ \\ > \nu^{(i)}(t), & \text{on } \{t : \nu^{(i)}(t) - \nu_{*}^{(i)}(t) < 0\}. \end{cases} \qquad (1.22)$$

we obtain

$$\begin{aligned} I_i^\epsilon \;\geq\; & \int_0^T e^{-\delta(t)} R_y^{(i)}(X(t), \nu^{(i)}(t))\,(\nu^{(i)}(t) - \nu_*^{(i)}(t))\,\mathbb{1}_{\{\nu^{(i)}(\cdot) \geq \nu_*^{(i)}(\cdot)\}}(t)dt \\ & + \int_0^T e^{-\delta(t)} R_y^{(i)}(X(t), \nu^{(i)}(t))\,(\nu^{(i)}(t) - \nu_*^{(i)}(t))\,\mathbb{1}_{\{\nu^{(i)}(\cdot) < \nu_*^{(i)}(\cdot)\}}(t)dt \\ & - \int_{[0,T)} e^{-\delta(t)} d(\nu^{(i)}(t) - \nu_*^{(i)}(t)) \\ = \; & \int_0^T e^{-\delta(t)} R_y^{(i)}(X(t), \nu^{(i)}(t))\,(\nu^{(i)}(t) - \nu_*^{(i)}(t))dt \\ & - \int_{[0,T)} e^{-\delta(t)} d(\nu^{(i)}(t) - \nu_*^{(i)}(t)) \\ = \; & \int_{[0,T)} \nabla_y \mathcal{J}_i(\nu^{(i)})(t)\, d(\nu^{(i)}(t) - \nu_*^{(i)}(t)). \end{aligned}$$

Hence we define

$$G_i(\omega) := \int_{[0,T)} \nabla_y \mathcal{J}_i(\nu^{(i)})(\omega, t) d(\nu^{(i)}(\omega, t) - \nu_*^{(i)}(\omega, t)). \tag{1.23}$$

Now (1.2), Assumption 1.1.1 and condition (1.18), imply the integrability of $G_i(\omega)$ since $|G_i(\omega)| \leq C[\theta(\omega, T) + (1 + \theta(\omega, T)) \int_0^T R^{(i)}\,(X(\omega, t), \theta(\omega, T))\,dt]$ with C a constant. □

Lemma 1.2.5. *Let f_i, $i = 1, 2, ..., N$, be optional processes and define*

$$\mu(s) := \max\left\{f_1^+(s), f_2^+(s), ..., f_N^+(s)\right\}. \tag{1.24}$$

Then every solution $\hat{\underline{\nu}}$ to the linear optimization problem

$$\sup_{\underline{\nu} \in \mathcal{S}_\theta} \sum_{i=1}^N \mathbb{E}\bigg\{ \int_{[0,T)} f_i(s)\, d\nu^{(i)}(s) \bigg\} \tag{1.25}$$

satisfies the 'flat-off conditions'

$$\mathbb{E}\bigg\{ \int_{[0,T)} (f_i(s) - \mu(s))\; d\hat{\nu}^{(i)}(s) \bigg\} = 0, \qquad i = 1, 2, ..., N. \tag{1.26}$$

Proof. Obviously

$$\sum_{i=1}^N \mathbb{E}\bigg\{ \int_{[0,T)} f_i(s)\, d\nu^{(i)}(s) \bigg\} \leq \sum_{i=1}^N \mathbb{E}\bigg\{ \int_{[0,T)} \mu(s)\, d\nu^{(i)}(s) \bigg\}. \tag{1.27}$$

The equality holds if and only if $\underline{\nu}$ satisfies (1.26). In fact (1.26) implies the equality. Conversely, if equality holds in (1.27), then $\sum_{i=1}^{N} \mathbb{E}\{ \int_{[0,T)} (f_i(s) - \mu(s))\, d\nu^{(i)}(s)\} = 0$. Hence (1.26) follows from the fact that the integrands are nonpositive. □

Remark 1.2.6. *We point out that our stochastic Kuhn-Tucker approach may be generalized to the case of investment processes also bounded from below by a stochastic process. In that case the Lagrangian functional is defined in terms of two Lagrange multipliers, $d\lambda_1(\omega, t)$ and $d\lambda_2(\omega, t)$.*

1.3 Applications of the Kuhn-Tucker Conditions

In this Section we test our approach on some 'finite-fuel' problems from the literature (cf. [7] and [43], among others) and we solve a N-firms Social Planner optimization problem. In the following examples we assume $\delta(t) = \delta t$, with $\delta > 0$, and $T = +\infty$.

1.3.1 The Finite Fuel Monotone Follower of Bank [7]

In the setting of Section 1.1, under (1.2) and Assumption 1.1.1, we take $N = 1$ and $T = +\infty$. We set $\nu := \nu^{(1)}$, $y := y^{(1)}$, $R := R^{(1)}$ and $\mathcal{J} := \mathcal{J}_1$. Notice that with

$$c(\omega, t, \nu(\omega, t)) := -e^{-\delta t} R(X(\omega, t), \nu(\omega, t)),$$

and instantaneous cost of investment

$$k(\omega, t) := -e^{-\delta t},$$

we fit into Bank's model [7]. Recall that Bank's optimal investment (cf. [7], Theorem 2) was given by

$$\nu_*(t) := \sup_{0 \le s < t} (l(s) \wedge \theta(s)) \vee y \tag{1.28}$$

in terms of the 'base capacity' process $l(t)$ (cf. [59] for this definition) which solves uniquely the stochastic backward equation (cf. [6], Theorem 3)

$$\mathbb{E}\bigg\{ \int_\tau^\infty e^{-\delta s} R_y(X(s), \sup_{\tau \le u < s} l(u))\, ds \,\bigg|\, \mathcal{F}_\tau \bigg\} = e^{-\delta \tau}, \quad \forall \tau \in \mathcal{T}. \tag{1.29}$$

When $l(t)$ is a continuous process we show the optimality of $\nu_*(t)$ by means of our Generalized Kuhn-Tucker conditions; as a subproduct we obtain an enlightening interpretation of the first order conditions stated in [7], Theorem 1, for a single firm optimal investment problem. Notice that continuity of $l(t)$ is guaranteed when the shock process $X(t)$ is continuous as well, as in the case of a diffusion (cf. [59], Theorem 6.5).

Recall that the supergradient of the net profit functional is the unique optional process given by

$$\nabla_y \mathcal{J}(\nu)(t) := \mathbb{E}\bigg\{ \int_t^{\infty} e^{-\delta s} R_y\left(X(s), \nu(s)\right) ds \,\bigg|\mathcal{F}_t\bigg\} - e^{-\delta t}. \tag{1.30}$$

By Theorem 1.2.3 an investment plan $\nu_*(t)$ is optimal if

$$\nabla_y \mathcal{J}(\nu_*)(t) \leq \mathbb{E}\bigg\{ \int_t^{\infty} d\lambda(s)\bigg|\mathcal{F}_t\bigg\}, \ \ \mathbb{P}-\text{a.s.}, \ t \geq 0, \tag{1.31}$$

$$\int_0^{\infty} \left[\nabla_y \mathcal{J}(\nu_*)(t) - \mathbb{E}\bigg\{ \int_t^{\infty} d\lambda(s)\bigg|\mathcal{F}_t\bigg\}\right] d\nu_*(t) = 0, \ \ \mathbb{P}-\text{a.s.}, \tag{1.32}$$

$$\nu_*(t) \leq \theta(t), \ \ \mathbb{P}-\text{a.s.}, \ \forall t \geq 0, \tag{1.33}$$

$$\mathbb{E}\bigg\{ \int_0^{\infty} \left(\theta(t) - \nu_*(t)\right) d\lambda(t)\bigg\} = 0, \tag{1.34}$$

for some nonnegative optional random measure $d\lambda(\omega, t)$ such that $\mathbb{E}\{\int_0^{\infty} d\lambda(s)\} < +\infty$.

Lemma 1.3.1. *For all $t \geq 0$ such that $\nu_*(t) = \theta(t)$ a.s., one has $R_y(X(t), \theta(t)) \geq \delta$ a.s.*

Proof. Let $t \geq 0$ such that $\nu_*(t) = \theta(t)$ a.s. Hence $l(t) \geq \theta(t)$ a.s. and therefore $R_y(X(t), \theta(t)) \geq R_y(X(t), l(t))$ a.s. due to the decreasing property of the mapping $y \to R_y(X(t), y)$. Recall that the base capacity process $l(t)$ was defined in [6], Theorem 1, as

$$l(t) = \operatorname*{ess\,inf}_{s \geq t} l_{s,t}, \tag{1.35}$$

where the $\mathcal{F}_t$-measurable random variable $l_{s,t}$ is the unique solution of the equation

$$\mathbb{E}\bigg\{ \int_t^{s} e^{-\delta u} R_y\left(X(u), l_{s,t}\right) du \,\bigg|\mathcal{F}_t\bigg\} = \mathbb{E}\bigg\{ e^{-\delta t} - e^{-\delta s} \,\bigg|\mathcal{F}_t\bigg\}. \tag{1.36}$$

Hence for each stopping time $\tau \geq t$ by (1.36) we have

$$\mathbb{E}\Big\{ \int_t^{\tau} e^{-\delta u} R_y\left(X(u), l(t)\right)\, du \Big| \mathcal{F}_t \Big\} \geq \mathbb{E}\Big\{ e^{-\delta t} - e^{-\delta \tau} \Big| \mathcal{F}_t \Big\}, \tag{1.37}$$

since $y \to R_y(X(t), y)$ is decreasing.

Therefore if $\epsilon > 0$ and $\tau_\epsilon(t) := \inf\{u \geq t : R_y\left(X(u), l(t)\right) > R_y\left(X(t), l(t)\right) + \epsilon\}$ we have

$$\mathbb{E}\Big\{ \int_t^{\tau_\epsilon(t)} e^{-\delta u} R_y\left(X(u), l(t)\right)\, du \Big| \mathcal{F}_t \Big\} \geq \mathbb{E}\Big\{ e^{-\delta t} - e^{-\delta \tau_\epsilon(t)} \Big| \mathcal{F}_t \Big\}.$$

On the other hand, the definition of $\tau_\epsilon(t)$ implies

$$\mathbb{E}\Big\{ \int_t^{\tau_\epsilon(t)} e^{-\delta u} R_y\left(X(u), l(t)\right)\, du \Big| \mathcal{F}_t \Big\} \leq \frac{1}{\delta}\left(R_y\left(X(t), l(t)\right) + \epsilon\right) \mathbb{E}\Big\{ e^{-\delta t} - e^{-\delta \tau_\epsilon(t)} \Big| \mathcal{F}_t \Big\}.$$

Therefore $R_y\left(X(t), l(t)\right) + \epsilon \geq \delta$. By taking $\epsilon \to 0$ we obtain $R_y\left(X(t), l(t)\right) \geq \delta$, and hence also $R_y\left(X(t), \theta(t)\right) \geq \delta$ a.s. □

Theorem 1.3.2. *If the base capacity process $l(t)$ has continuous paths, then $\nu_*(t)$ (cf. (1.28)) is optimal and the Lagrange multiplier $d\lambda(t)$ is absolutely continuous with respect to the Lebesgue measure.*

Proof. It sufficies to check the Generalized Kuhn-Tucker conditions (1.31) - (1.34) for $\nu_*(t)$. Obviously $\nu_*(t)$ satisfies (1.33). Recall that the available resources process $\theta(t)$ is increasing and left-continuous. To show (1.31) and (1.32), fix $\tau \in \mathcal{T}$, set $\tau_0 := \tau$, and recursively define

$$\begin{cases} \tau_{2n} := \inf\{s > \tau_{2n-1} : l(s) \leq \theta(s+)\} \\ \tau_{2n+1} := \inf\{s > \tau_{2n} : l(s) > \theta(s)\} \end{cases} \tag{1.38}$$

with the convention $\inf\{\emptyset\} = +\infty$. Notice that time τ_{2n+1}, $n \geq 0$, is a time of increase for $l(t)$. Then

$$\nu_*(s) = \theta(s) \qquad \text{for } s \in (\tau_{2n+1}, \tau_{2n+2}],$$

and

$$\nu_*(s) = \sup_{\tau_{2n} \leq u < s} l(u) \qquad \text{for } s \in (\tau_{2n}, \tau_{2n+1}],$$

by the continuity of $l(t)$. Moreover we have $l(s) \leq \theta(s)$ for $s \in (\tau, \tau_1]$, hence $\sup_{\tau \leq u < s} (l(u) \wedge \theta(u)) = \sup_{\tau \leq u < s} l(u)$.

Recalling (1.28) and the previous considerations we have

$$\begin{aligned}
&\mathbb{E}\Big\{ \int_\tau^\infty e^{-\delta s} R_y \left(X(s), \nu_*(s)\right) ds \Big| \mathcal{F}_\tau \Big\} \\
&\quad = \mathbb{E}\Big\{ \int_\tau^{\tau_1} e^{-\delta s} R_y(X(s), \nu_*(s)) ds \Big| \mathcal{F}_\tau \Big\} \\
&\qquad + \sum_{n=1}^\infty \mathbb{E}\Big\{ \int_{\tau_n}^{\tau_{n+1}} e^{-\delta s} R_y(X(s), \nu_*(s)) ds \Big| \mathcal{F}_\tau \Big\} \\
&\quad \leq \mathbb{E}\Big\{ \int_\tau^{\tau_1} e^{-\delta s} R_y(X(s), \sup_{\tau \leq u < s} l(u) \wedge \theta(u)) ds \Big| \mathcal{F}_\tau \Big\} \\
&\qquad + \sum_{n=1}^\infty \mathbb{E}\Big\{ \int_{\tau_{2n-1}}^{\tau_{2n}} e^{-\delta s} R_y(X(s), \theta(s)) ds \Big| \mathcal{F}_\tau \Big\} \\
&\qquad + \sum_{n=1}^\infty \mathbb{E}\Big\{ \int_{\tau_{2n}}^{\tau_{2n+1}} e^{-\delta s} R_y(X(s), \sup_{\tau_{2n} \leq u < s} l(u)) ds \Big| \mathcal{F}_\tau \Big\},
\end{aligned} \tag{1.39}$$

where the equality holds if and only if τ is a point of increase for ν_*. By definition of τ_1, from (1.39) we get

$$\begin{aligned}
&\mathbb{E}\Big\{ \int_\tau^\infty e^{-\delta s} R_y(X(s), \nu_*(s)) ds \Big| \mathcal{F}_\tau \Big\} \\
&\quad \leq \mathbb{E}\Big\{ \int_\tau^{\tau_1} e^{-\delta s} R_y(X(s), \sup_{\tau \leq u < s} l(u)) ds \Big| \mathcal{F}_\tau \Big\} \\
&\qquad + \sum_{n=1}^\infty \mathbb{E}\Big\{ \int_{\tau_{2n-1}}^{\tau_{2n}} e^{-\delta s} R_y(X(s), \theta(s)) ds \Big| \mathcal{F}_\tau \Big\} \\
&\qquad + \sum_{n=1}^\infty \mathbb{E}\Big\{ \int_{\tau_{2n}}^{\tau_{2n+1}} e^{-\delta s} R_y(X(s), \sup_{\tau_{2n} \leq u < s} l(u)) ds \Big| \mathcal{F}_\tau \Big\}.
\end{aligned} \tag{1.40}$$

Since τ_1 and all odd indexed stopping times are times of increase for the process $l(t)$, hence $\sup_{\tau \leq u < s} l(u) = \sup_{\tau_1 \leq u < s} l(u)$ for $s > \tau_1$, and $\sup_{\tau_{2n} \leq u < s} l(u) = \sup_{\tau_{2n+1} \leq u < s} l(u)$ for $s > \tau_{2n+1}$. Therefore, from (1.40) the stochastic backward equation (1.29) implies

$$\begin{aligned}
\mathbb{E}\Big\{ \int_\tau^\infty e^{-\delta s} R_y(X(s), \nu_*(s)) ds \Big| \mathcal{F}_\tau \Big\} &= \mathbb{E}\Big\{ e^{-\delta \tau} - e^{-\delta \tau_1} \Big| \mathcal{F}_\tau \Big\} \\
&+ \sum_{n=1}^\infty \mathbb{E}\Big\{ \int_{\tau_{2n-1}}^{\tau_{2n}} e^{-\delta s} R_y(X(s), \theta(s)) ds \Big| \mathcal{F}_\tau \Big\}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}\mathbb{E}\Big\{e^{-\delta\tau_{2n}}-e^{-\delta\tau_{2n+1}}\Big|\mathcal{F}_\tau\Big\}\\
&=e^{-\delta\tau}+\sum_{n=1}^{\infty}\mathbb{E}\Big\{\int_{\tau_{2n-1}}^{\tau_{2n}}e^{-\delta s}[R_y(X(s),\theta(s))-\delta]ds\Big|\mathcal{F}_\tau\Big\}\\
&=e^{-\delta\tau}+\mathbb{E}\left\{\int_{\tau}^{\infty}e^{-\delta s}[R_y(X(s),\theta(s))-\delta]\mathbb{1}(s)_{\{\nu_*=\theta\}}ds\,\Big|\mathcal{F}_\tau\right\}.
\end{aligned}
$$

Notice that the process $e^{-\delta t}[R_y(X(t),\theta(t))-\delta]\mathbb{1}_{\{\nu_*=\theta\}}(t)$ is nonnegative by Lemma 1.3.1 and it is $\mathcal{F}_t\otimes\mathcal{B}([0,t])-$measurable. Hence, we set

$$
d\lambda(t):=e^{-\delta t}[R_y(X(t),\theta(t))-\delta]\mathbb{1}_{\{\nu_*=\theta\}}(t)dt \tag{1.41}
$$

and we show that it is the optional measure Lagrange multiplier. Let us start by showing that $d\lambda(t)$ is an optional random measure on $\mathbb{R}_+$. That is, the continuous, increasing process

$$
\Lambda(t):=\int_{[0,t)}d\lambda(s) \tag{1.42}
$$

is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Assumption 1.1.1 and concavity of R in the second argument, imply that

$$
\begin{aligned}
\mathbb{E}\{\Lambda(t)\} &= \mathbb{E}\Big\{\int_0^t e^{-\delta s}[R_y(X(s),\theta(s))-\delta]\mathbb{1}_{\{\nu_*=\theta\}}(s)\,ds\Big\}\\
&\leq \mathbb{E}\Big\{\int_0^t e^{-\delta s}R_y(X(s),\theta_o)\mathbb{1}_{\{\nu_*=\theta\}}(s)\,ds\Big\}\\
&\leq \mathbb{E}\Big\{\int_0^\infty e^{-\delta s}R_y(X(s),\theta_o)\,ds\Big\}\\
&\leq \frac{1}{\theta_o}\mathbb{E}\Big\{\int_0^\infty \sup_{\nu(s)\in\mathcal{S}_\theta}R(X(s),\nu(s))\,ds\Big\}<+\infty.
\end{aligned} \tag{1.43}
$$

Hence $\Lambda(t)$ is $d\mathbb{P}$-integrable and $e^{-\delta t}[R_y(X(t),\theta(t))-\delta]\mathbb{1}_{\{\nu_*=\theta\}}(t)$ is $d\mathbb{P}\otimes dt$-integrable on $\Omega\times\mathbb{R}_+$. Therefore, by Fubini's Theorem, the application $\omega\to\Lambda(\omega,t)$ is $\mathcal{F}_t$-measurable and hence Λ is adapted. Then it is predictable since it is continuous.

It follows that (1.31) and (1.32) hold and hence the process (1.28) is optimal by Theorem 1.2.3. □

Remark 1.3.3. *The usual interpretation of the Lagrange multiplier as the shadow price of the value function may be heuristically shown as follows. After an integration by parts on the cost term, we may write the value function as*

$$V(\theta) = \mathbb{E}\left\{ \int_0^{\infty} e^{-\delta t}\Big[R(X(t), \sup_{0\le s<t}(l(s)\wedge\theta(s))) - \delta \sup_{0\le s<t}(l(s)\wedge\theta(s))\Big] dt \right\}.$$

Now, if $\nu_*(t) = \sup_{0\le s<t}(l(s)\wedge\theta(s))$*, then* $\mathbb{1}_{\{\nu_*=\theta\}}$ *is the derivative (in some sense) of* ν_* *with respect to* θ*. We thus expect that the 'derivative' of* V *with respect to the constraint* θ *is* $e^{-\delta t}[R_y(X(t),\theta(t)) - \delta]\mathbb{1}_{\{\nu_*=\theta\}}(t)$*, which is exactly the density of the Lagrange multiplier in the case of a continuous 'base capacity'* $l(t)$*.*

Proposition 1.3.4. *The process* $\mathcal{G}(t) := \mathbb{E}\{\Lambda(\infty)\,|\mathcal{F}_t\}$ *is a uniformly integrable martingale.*

Proof. By Assumption 1.1.1 the random variable $\Lambda(\infty)$ (cf. (1.42)) is $d\mathbb{P}$-integrable. Hence the process $\mathcal{G}(t)$ is a uniformly integrable martingale. □

Proposition 1.3.5. *The process*

$$\mathcal{U}(t) := \mathbb{E}\left\{ \int_t^{\infty} d\lambda(s)\,\Big|\mathcal{F}_t \right\} = \mathbb{E}\{\Lambda(\infty)\,|\mathcal{F}_t\} - \Lambda(t) = \mathcal{G}(t) - \Lambda(t) \tag{1.44}$$

is a supermartingale of class (D) and $\mathcal{G}(t) - \Lambda(t)$ *is its unique Doob-Meyer decomposition.*

Proof. Recall (cf. proof of Theorem 1.3.2) that the process $\Lambda(t)$ of (1.42) is increasing, adapted, continuous and integrable. Then $\mathcal{U}(t)$ is $d\mathbb{P}$-integrable. Moreover, being $d\lambda$ non-negative, for $s \le t$ we have $\mathbb{E}\{\mathcal{U}(t)\,|\mathcal{F}_s\} \le \mathcal{U}(s)$, i.e. $\mathcal{U}(t)$ is a supermartingale. Assumption 1.1.1 guarantees that it belongs to class (D). Hence $\mathcal{G}(t) - \Lambda(t)$ is the unique Doob-Meyer decomposition of the supermartingale $\mathcal{U}(t)$ and therefore the process $\Lambda(t)$ is the compensator of $\mathcal{U}(t)$. □

If $\mathbb{S}(\nu)$ is the Snell envelope of the supergradient $\nabla_y\mathcal{J}(\nu)$, i.e.

$$\mathbb{S}(\nu)(t) = \operatorname*{ess\,sup}_{t\le\tau\le+\infty} \mathbb{E}\left\{\nabla_y\mathcal{J}(\nu)(\tau)|\mathcal{F}_t\right\}, \tag{1.45}$$

then [7], Theorem 1, claims that the optimal investment plan ν_* is characterized by the following conditions

$$\begin{cases} \nu_* \text{ is flat off } \{\nabla_y \mathcal{J}(\nu_*) = \mathbb{S}(\nu_*)\} \\ A(\nu_*) \text{ is flat off } \{\nu_* = \theta\}, \end{cases} \tag{1.46}$$

where $A(\nu_*)$ is the predictable increasing process in the Doob-Meyer decomposition of the supermartingale $\mathbb{S}(\nu_*)$. Moreover $\mathbb{S}(\nu_*)(t) = \mathbb{E}\{A(\infty) - A(t)\,|\mathcal{F}_t\}$ since $\nabla_y \mathcal{J}(\nu_*)(\infty) = 0$. If (1.18) holds, then (1.46), (1.31) and (1.32) imply that

$$\mathcal{U}(t) \equiv \mathbb{S}(\nu_*)(t) \tag{1.47}$$

at times of investment (when $A(\nu_*)$ and $d\lambda$ are not flat). This argument allows an enlightening interpretation of the increasing, predictable, integrable process $\Lambda(t)$. In fact at times of investment

$$\mathcal{G}(t) - \Lambda(t) = \mathbb{E}\bigg\{ \int_t^\infty d\lambda(s) \,\bigg|\mathcal{F}_t \bigg\} \equiv \mathbb{S}(\nu_*)(t) = \mathcal{M}(\nu_*)(t) - A(\nu_*)(t), \tag{1.48}$$

where $\mathcal{M}(\nu_*)$ is the martingale process in the unique Doob-Meyer decomposition of $\mathbb{S}(\nu_*)$. By uniqueness

$$\mathbb{E}\bigg\{ \int_0^\infty d\lambda(s) \,\bigg|\mathcal{F}_t \bigg\} \equiv \mathcal{M}(\nu_*)(t) \quad \text{and} \quad \Lambda(t) \equiv A(\nu_*)(t), \tag{1.49}$$

hence

$$dA(\nu_*)(t) \equiv d\lambda(t), \;\; \mathbb{P} - \text{a.s.}, \;\; \forall t \geq 0. \tag{1.50}$$

Therefore the second first order condition of (1.46) coincides with the Kuhn-Tucker condition (1.34); that is the Lagrange multiplier acts only when the constraint is binding.

When $l(t)$ is continuous, the explicit form of the Lagrange multiplier is known (cf. (1.41)), hence the compensator $A(\nu_*)(t)$ is known as well. It follows that its paths are absolutely continuous with respect to the Lebesgue measure and the Radon-Nykodym derivative of $dA(\nu_*)(t)$ is $e^{-\delta t}[R_y(X(t), \theta(t)) - \delta]\mathbb{1}_{\{\nu_* = \theta\}}(t)$.

1.3.2 N Firms with Finite Fuel: the Symmetric Case

In the same setting of Section 1.1 with $T = +\infty$, we may start with studying the symmetric Social Planner problem, i.e. problem (1.4) when $R^{(i)}(x, y) := R(x, y)$, $i = 1, 2, ..., N$. Moreover, suppose that the investment processes of the N firms have the same initial conditions, i.e. $\nu^{(i)}(0) = y$, $i = 1, 2, ..., N$. Notice that in such case the supergradient processes coincide for $i = 1, 2, ..., N$ and are given by

$$\nabla_y \mathcal{J}_i(\nu^{(i)})(t) = \mathbb{E}\bigg\{ \int_t^{\infty} e^{-\delta s} R_y(X(s), \nu^{(i)}(s))\, ds \,\Big|\mathcal{F}_t \bigg\} - e^{-\delta t}, \tag{1.51}$$

Moreover, recall that the 'base capacity' is defined as the optional process that uniquely solves the backward stochastic differential equation (cf. [6])

$$\mathbb{E}\bigg\{ \int_\tau^{\infty} e^{-\delta s} R_y(X(s), \sup_{\tau \leq u < s} l(u))\, ds \,\Big|\mathcal{F}_\tau \bigg\} = e^{-\delta \tau}, \;\; \forall \tau \in \mathcal{T}. \tag{1.52}$$

By Theorem (1.2.3) we may prove the following Proposition.

Proposition 1.3.6. *Let the production functions be identical, the investment processes start from the same level and the base capacity $l(t)$ have continuous paths. Then, the unique optimal solution for the Social Planner problem (1.4) is given by*

$$\nu_*^{(i)}(t) = \sup_{0 \leq u < t} (l(u) \wedge \frac{\theta(u)}{N}) \vee y, \quad i = 1, 2, ..., N.$$

Moreover the Lagrange multiplier associated to problem (1.4) is absolutely continuous with respect to the Lebesgue measure.

Proof. We are going only to sketch the proof, since it is very similar to that of Theorem (1.3.2).

Obviously $\sum_{i=1}^{N} \nu_*^{(i)}(t) \leq \theta(t)$, a.s. for all $t \geq 0$. To prove the optimality of $\sup_{0 \leq u < t}(l(u) \wedge \frac{\theta(u)}{N}) \vee y$ we proceed as in the case of only one firm (see Section 1.3.1, proof of Theorem (1.2.3)) but with a base capacity given by $Nl(t)$. For $\tau \geq 0$ fixed, and $n \geq 0$, we may introduce the random times

$$\begin{cases} \tau_0 \equiv \tau \\ . \\ . \\ \tau_{2n+1} := \inf\{s > \tau_{2n} : Nl(s) > \theta(s)\} \\ \tau_{2n+2} := \inf\{s > \tau_{2n+1} : Nl(s) \leq \theta(s+)\} \\ . \\ . \end{cases} \tag{1.53}$$

with the convention $\inf\{\emptyset\} = +\infty$. Notice that time τ_{2n+1}, $n \geq 0$, is a time of increase for the process $Nl(s)$. Following from now on the same considerations as in the proof of Theorem 1.3.2 (obviously by substituting the process $l(t)$ with $Nl(t)$, or, equivalentely, $\theta(t)$ with $\frac{\theta(t)}{N}$), we may conclude that (1.15) are actually satisfied for a Lagrange multiplier optional measure given by

$$d\lambda(t) = e^{-\delta t}[R_y(X(t), N^{-1}\theta(t)) - \delta]\mathbb{1}_{\{\sum_{i=1}^N \nu_*^{(i)} = \theta\}}(s)ds. \tag{1.54}$$

□

Asymmetric Capital Levels

Since the results in this Section can be easily generalized, set $N = 2$ for sake of semplicity. We now allow for some heterogeneity by considering that the two firms have the same production functions but investment processes with different capital installed before the starting time. We aim to prove that, as long as the levels of installed capital are not all the same, only the 'smallest firm' (i.e. the firm with the smallest value of installed capital at the beginning) will invest. This particular behavior of the investment policies has been already discussed in [60].

Interpret the investment processes as including the respective initial capital; without loss of generality we may set: $\nu^{(1)}(0) = y_1$, $\nu^{(2)}(0) = y_2$ and $y_1 > y_2$. Hence we shall refer to firm 2 as the smallest one.

Proposition 1.3.7. *Suppose* $R^{(1)}(x,y) = R^{(2)}(x,y)$, $\nu^{(1)}(0) = y_1$, $\nu^{(2)}(0) = y_2$ *and* $y_1 > y_2$. *Then, we have* $d\nu_*^{(1)}(t) = 0$ *as long as* $y_1 > y_2 + \int_0^t d\nu_*^{(2)}(s)$.

Proof. Let σ_1 be the first time at which firm 1 invests, and σ_2 the first time of investment for firm 2. Suppose, by *absurdum*, that $\sigma_1 < \sigma_2$ and let $(\nu_*^{(1)}, \nu_*^{(2)})$ be the optimal solution for the Social Planner problem (1.4). From the generalized stochastic Kuhn-Tucker first order conditions in Theorem 1.3.2 we have at σ_1

$$\mathbb{E}\Big\{ \int_{\sigma_1}^{\sigma_2} e^{-\delta s} R_y(X(s), \nu_*^{(1)}(s))\, ds \,\Big|\mathcal{F}_{\sigma_1}\Big\} + \mathbb{E}\Big\{ \int_{\sigma_2}^{\infty} e^{-\delta s} R_y(X(s), \nu_*^{(1)}(s))\, ds \,\Big|\mathcal{F}_{\sigma_1}\Big\} = e^{-\delta\sigma_1} + \mathbb{E}\Big\{ \int_{\sigma_1}^{\infty} d\lambda(s) \,\Big|\mathcal{F}_{\sigma_1}\Big\} \tag{1.55}$$

for firm 1 and

$$\mathbb{E}\Big\{ \int_{\sigma_1}^{\sigma_2} e^{-\delta s} R_y(X(s), y_2)\, ds \,\Big|\mathcal{F}_{\sigma_1}\Big\} + \mathbb{E}\Big\{ \int_{\sigma_2}^{\infty} e^{-\delta s} R_y(X(s), \nu_*^{(2)}(s))\, ds \,\Big|\mathcal{F}_{\sigma_1}\Big\} \leq e^{-\delta\sigma_1} + \mathbb{E}\Big\{ \int_{\sigma_1}^{\infty} d\lambda(s) \,\Big|\mathcal{F}_{\sigma_1}\Big\} \tag{1.56}$$

since σ_1 is not optimal for firm 2.

Notice that σ_2 is an optimal time for firm 2. Hence

$$\mathbb{E}\Big\{ \int_{\sigma_2}^{\infty} e^{-\delta s} R_y(X(s), \nu_*^{(2)}(s))\, ds \,\Big|\mathcal{F}_{\sigma_1}\Big\} = \mathbb{E}\Big\{ e^{-\delta\sigma_2} + \int_{\sigma_2}^{\infty} d\lambda(s) \,\Big|\mathcal{F}_{\sigma_1}\Big\}. \tag{1.57}$$

On the other hand, time σ_2 is not optimal for firm 1

$$\begin{aligned} &\mathbb{E}\Big\{ \int_{\sigma_2}^{\infty} e^{-\delta s} R_y(X(s), \nu_*^{(1)}(s))\, ds \,\Big|\mathcal{F}_{\sigma_1}\Big\} \\ &\quad\leq \mathbb{E}\Big\{ e^{-\delta\sigma_2} + \int_{\sigma_2}^{\infty} d\lambda(s) \,\Big|\mathcal{F}_{\sigma_1}\Big\} \\ &\quad= \mathbb{E}\Big\{ \int_{\sigma_2}^{\infty} e^{-\delta s} R_y(X(s), \nu_*^{(2)}(s))\, ds \,\Big|\mathcal{F}_{\sigma_1}\Big\}, \end{aligned} \tag{1.58}$$

where in the last equality we have used (1.57). Moreover, by hypothesis $\nu_*^{(1)}(t) \geq y_1 > y_2$, on $\sigma_1 \leq t < \sigma_2$; hence concavity of $R_y(X(t), \cdot)$ implies

$$\begin{aligned} e^{-\delta\sigma_1} + \mathbb{E}\Big\{ \int_{\sigma_1}^{\infty} d\lambda(s) \,\Big|\mathcal{F}_{\sigma_1}\Big\} &= \mathbb{E}\Big\{ \int_{\sigma_1}^{\sigma_2} e^{-\delta s} R_y\left(X(s), \nu_*^{(1)}(s)\right)\, ds \,\Big|\mathcal{F}_{\sigma_1}\Big\} \\ &\quad + \mathbb{E}\Big\{ \int_{\sigma_2}^{\infty} e^{-\delta s} R_y(X(s), \nu_*^{(1)}(s))\, ds \,\Big|\mathcal{F}_{\sigma_1}\Big\} \\ &\leq e^{-\delta\sigma_1} + \mathbb{E}\Big\{ \int_{\sigma_1}^{\infty} d\lambda(s) \,\Big|\mathcal{F}_{\sigma_1}\Big\} \end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{E}\Big\{ \int_{\sigma_1}^{\sigma_2} e^{-\delta s} R_y(X(s), y_2)\, ds \,\Big|\mathcal{F}_{\sigma_1}\Big\} \\
&\qquad + \mathbb{E}\Big\{ \int_{\sigma_2}^{\infty} e^{-\delta s} R_y(X(s), \nu_*^{(2)}(s))\, ds \,\Big|\mathcal{F}_{\sigma_1}\Big\} \\
&\leq e^{-\delta\sigma_1} + \mathbb{E}\Big\{ \int_{\sigma_1}^{\infty} d\lambda(s) \,\Big|\mathcal{F}_{\sigma_1}\Big\}.
\end{aligned}
$$

It means that

$$
\begin{aligned}
&\mathbb{E}\Big\{ \int_{\sigma_1}^{\sigma_2} e^{-\delta s} R_y(X(s), y_2)\, ds \,\Big|\mathcal{F}_{\sigma_1}\Big\} + \mathbb{E}\Big\{ \int_{\sigma_2}^{\infty} e^{-\delta s} R_y(X(s), \nu_*^{(2)}(s))\, ds \,\Big|\mathcal{F}_{\sigma_1}\Big\} \\
&= e^{-\delta\sigma_1} + \mathbb{E}\Big\{ \int_{\sigma_1}^{\infty} d\lambda(s) \,\Big|\mathcal{F}_{\sigma_1}\Big\}
\end{aligned}
$$

which is, obviously, a contraddiction since σ_1 is not an optimal time for firm 2. □

Proposition (1.3.7) states that the smallest firm will catch up before any other invests. Once all firms are equally sized, they act identically as suggested by Proposition (1.3.6). If we define

$$\tau_{y_1} := \inf\{t \geq 0 : y_1 = y_2 + \int_0^t d\nu_*^{(2)}(u)\},$$

then the optimal investment couple $(\nu_*^{(1)}(t), \nu_*^{(2)}(t))$ is given by

$$
\begin{cases}
\nu_*^{(1)}(t) = y_1, \quad t \in [0, \tau_{y_1}) \\
\\
\nu_*^{(2)}(t) = \sup_{0 \leq u < t}(l(u) \wedge \theta(u)), \quad t \in [0, \tau_{y_1})
\end{cases}
\tag{1.59}
$$

and

$$
\begin{cases}
\nu_*^{(1)}(t) = y_1 \vee \sup_{\tau_{y_1} \leq u < t}(l(u) \wedge \frac{\theta(u)}{2}), \quad t \geq \tau_{y_1} \\
\\
\nu_*^{(2)}(t) = y_1 \vee \sup_{\tau_{y_1} \leq u < t}(l(u) \wedge \frac{\theta(u)}{2}), \quad t \geq \tau_{y_1}
\end{cases}
\tag{1.60}
$$

The Social Planner leads only to the smallest firm to invest until its capital reaches the value of the initial capital of the other firm at time τ_{y_1}. This means that until that time firm 2 behaves as a monopoly in a market with only one firm; after τ_{y_1} the two firms behave in a symmetric way (see Proposition 1.3.6 for further details). Notice that (1.60) is the version of the 'dynamic programming' principle formulated in [7], Corollary 4.2: the base capacity process $l(t)$ may be used to describe optimal solutions not only when starting at time zero, but actually for an arbitrary initial stopping time.

1.3.3 N Firms: Finite Fuel and Cobb-Douglas Production

In the setting of Section 1.1 with $T = +\infty$ we consider the Social Planner optimal investment problem (1.4) for a market with N firms endowed with Cobb-Douglas production functions, i.e. $R^{(i)}(x, y) = \frac{x^{\alpha_i} y^{1-\alpha_i}}{1-\alpha_i}$ with $\alpha_i \in (0, 1)$, $i = 1, 2, ..., N$.

Suppose that the economic shock process $X(t)$ is given by $X(t) = \exp\{Y(t)\}$ for some Levy process $Y(t)$ such that $Y(0) = 0$ and with finite Laplace transform. Then (cf. [59], Proposition 7.1)

$$l^{(i)}(t) = k_i X(t), \qquad i = 1, 2, ..., N, \tag{1.61}$$

with

$$k_i = \left(\mathbb{E}\left\{\int_0^{+\infty} e^{-\delta t} e^{\alpha_i \inf_{0\leq u<t} Y(u)} dt\right\}\right)^{\frac{1}{\alpha_i}}, \qquad i = 1, 2, ..., N,$$

is the unique optional solution of the stochastic backward equation

$$\mathbb{E}\left\{\int_\tau^\infty e^{-\delta s} R_y^{(i)}(X(s), \sup_{\tau\leq u<s} l^{(i)}(u))\, ds \,\Big|\, \mathcal{F}_\tau\right\} = e^{-\delta\tau}, \qquad \forall \tau \in \mathcal{T}. \tag{1.62}$$

Define the optional process

$$\beta_i(t) := \frac{l^{(i)}(t)}{\sum_{j=1}^N l^{(j)}(t)}. \tag{1.63}$$

Here $\beta_i(t)$ may be thought as the fraction of desirable investment of the i-th firm. By (1.61), for $t \geq 0$ and $i = 1, 2, ..., N$, we have that $\beta_i(t)$ is constant in time; in fact $\beta_i(t) = \frac{k_i}{\sum_{j=1}^N k_j} =: \beta_i$.

Fix $\tau \in \mathcal{T}$ and introduce the random times

$$\begin{cases} \sigma_1(\tau) = \inf\{s \geq \tau : \sum_{i=1}^N l^{(i)}(s) > \theta(s)\} \\ \sigma_2(\tau) = \inf\{s \geq \tau : l^{(i)}(s) > \beta_i\theta(s), \ \forall i = 1, 2, ..., N\}. \end{cases} \tag{1.64}$$

Lemma 1.3.8. *For all $\tau \in \mathcal{T}$ we have $\sigma_1(\tau) = \sigma_2(\tau)$ $\mathbb{P}$-almost surely.*

Proof. Notice that (1.61) implies $\sigma_1(\tau) = \inf\{s \geq \tau : X(s) > \frac{\theta(s)}{\sum_{i=1}^N k_i}\} = \inf\{s \geq \tau : k_i X(s) > \beta_i\theta(s), \ \forall i = 1, 2, ..., N\} = \sigma_2(\tau)$. □

Remark 1.3.9. *If $\tau \in \mathcal{T}$ is a time of investment for all firms, that is $d\nu_*^{(i)}(\tau) > 0$ for all i, then the first Kuhn-Tucker condition in (1.15) guarantees that*

$$\mathbb{E}\bigg\{ \int_\tau^{+\infty} e^{-\delta s} R_y^{(i)}(X(s), \nu_*^{(i)}(s))\, ds \,\bigg|\, \mathcal{F}_\tau \bigg\} = \mathbb{E}\bigg\{ \int_\tau^{+\infty} e^{-\delta s} R_y^{(j)}(X(s), \nu_*^{(j)}(s))\, ds \,\bigg|\, \mathcal{F}_\tau \bigg\}.$$

Notice that if X is continuous, then $l^{(i)}$ is continuous too due to (1.61).

Theorem 1.3.10. *If the shock process $X(t)$ is continuous then the process $\underline{\nu}_*$ with components*

$$\nu_*^{(i)}(t) = \sup_{0 \le u < t} (l^{(i)}(u) \wedge \beta_i \theta(u)) \vee y^{(i)}, \qquad i = 1, 2, ..., N, \tag{1.65}$$

is optimal for problem (1.4). Moreover, the Lagrange multiplier $d\lambda(t)$ associated to (1.4) is absolutely continuous with respect to the Lebesgue measure.

Proof. Let us check that $\nu_*^{(i)}(t)$ satisfies the first order conditions of Theorem 1.2.3. Obviously $\sum_{i=1}^N \nu_*^{(i)}(t) \le \theta(t)$ a.s. for all $t \ge 0$.

The arguments of the proof are similar to those in the proof of Theorem 1.3.2. Fix $\tau \in \mathcal{T}$, set $\tau_0 := \tau$ and define the sequence of stopping times τ_n as in (1.38) but with $\sum_{i=1}^N l^{(i)}$ instead of l; that is,

$$\begin{cases} \tau_{2n+1} := \inf\{s > \tau_{2n} : \sum_{i=1}^N l^{(i)}(s) > \theta(s)\} \\ \tau_{2n+2} := \inf\{s > \tau_{2n+1} : \sum_{i=1}^N l^{(i)}(s) \le \theta(s+)\}. \end{cases} \tag{1.66}$$

Notice that the continuity of $l^{(i)}$ implies

$$\nu_*^{(i)}(s) = \sup_{\tau_{2n} \le u < s} l^{(i)}(u) \qquad \text{for } s \in (\tau_{2n}, \tau_{2n+1}].$$

Also $\tau_{2n+1} = \sigma_1(\tau_{2n}) = \sigma_2(\tau_{2n})$ by Lemma 1.3.8, hence τ_{2n+1} is a time of increase for all $l^{(i)}$. It follows

$$\nu_*^{(i)}(s) = \beta_i \theta(s) \qquad \text{for } s \in (\tau_{2n+1}, \tau_{2n+2}].$$

Fix $i = 1, 2, ..., N$, and consider $\mathbb{E}\{ \int_\tau^\infty e^{-\delta s} R_y^{(i)}(X(s), \nu_*^{(i)}(s)) ds \,|\mathcal{F}_\tau\}$. Split the integral into two integrals $\int_\tau^{\tau_1}$ and $\int_{\tau_1}^\infty$. Since τ_1 is a time of increase for every $l^{(i)}$, Remark (1.3.9) holds and we may write

$$\mathbb{E}\Big\{\int_{\tau}^{\infty} e^{-\delta s} R^{(i)}_y(X(s),\nu^{(i)}_*(s))ds\,\Big|\mathcal{F}_\tau\Big\} = \mathbb{E}\Big\{\int_{\tau}^{\tau_1} e^{-\delta s} R^{(i)}_y(X(s),\nu^{(i)}_*(s))ds\,\Big|\mathcal{F}_\tau\Big\} \tag{1.67}$$
$$+\mathbb{E}\Big\{\int_{\tau_1}^{\infty} e^{-\delta s}\beta_i R^{(i)}_y(X(s),\nu^{(i)}_*(s))ds\,\Big|\mathcal{F}_\tau\Big\} + \mathbb{E}\Big\{\int_{\tau_1}^{\infty} e^{-\delta s}\sum_{j\neq i}\beta_j R^{(j)}_y(X(s),\nu^{(i)}_*(s))ds\,\Big|\mathcal{F}_\tau\Big\}$$

since $\mathcal{F}_\tau \subseteq \mathcal{F}_{\tau_1}$. Now, as in the proof of Theorem 1.3.2, we use the stopping times τ_n to split the last two integrals above and by the backward equation (1.62) corresponding to $l^{(i)}(t)$ we may write

$$\mathbb{E}\Big\{\int_{\tau}^{\infty} e^{-\delta s} R^{(i)}_y(X(s),\nu^{(i)}_*(s))ds\,\Big|\mathcal{F}_\tau\Big\} \tag{1.68}$$
$$\leq e^{-\delta\tau} + \mathbb{E}\Big\{\int_{\tau}^{\infty} e^{-\delta s}\Big[\sum_{i=1}^{N}\beta_i R^{(i)}_y(X(s),\beta_i\theta(s)) - \delta\Big]\mathbb{1}_{\{\sum_{i=1}^N \nu^{(i)}_*=\theta\}}(s)ds\,\Big|\mathcal{F}_\tau\Big\},$$

with equality if and only if $d\nu^{(i)}_*(\tau) > 0$. Hence

$$\rho(t) := e^{-\delta t}\Big[\sum_{i=1}^{N}\beta_i(R^{(i)}_y(X(t),\beta_i\theta(t)) - \delta)\Big]\mathbb{1}_{\{\sum_{i=1}^N \nu^{(i)}_*=\theta\}}(t)$$

is nonnegative by Lemma 1.3.1. We may now define the Lagrange multiplier for the N-firms Social Planner problem by $d\lambda(t) := \rho(t)dt$ since such $d\lambda$ is a nonnegative optional measure as in the proof of Theorem 1.3.2. □

Remark 1.3.11. *For general production functions satisfying Assumption 1.1.1, we expect the solution for the Social Planner problem (1.4) to be*

$$\nu^{(i)}_*(t) = \sup_{0\leq u<t}(l^{(i)}(u)\wedge\beta_i(u)\theta(u))\vee y^{(i)}, \qquad i = 1,2,...,N,$$

with

$$\beta_i(t) := \frac{l^{(i)}(t)}{\sum_{j=1}^N l^{(j)}(t)}.$$

1.3.4 Constant Finite Fuel and Quadratic Cost

Here we consider a monotone follower problem with constant finite fuel similar to those studied by Karatzas ([40], [43]), and Karatzas and Shreve [42] (among others). In particular we discuss the example (cf. [7]) of optimal cost minimization for a firm that does not incur into investment's costs and has a running cost flow given by the convex function $c(x, y) = \frac{1}{2}(x - y)^2$ of the economic shock x and the investment y. That is, we study the constrained convex minimization problem

$$\inf_{\nu \in \mathcal{S}_{\theta_o}} \mathcal{C}(\nu) := \inf_{\nu \in \mathcal{S}_{\theta_o}} \mathbb{E}\left\{ \int_0^{\infty} \delta e^{-\delta s} \frac{1}{2} (W(t) - \nu(t))^2 \, dt \right\} \tag{1.69}$$

where $W(t)$ is a standard Brownian motion and θ_o is the positive constant finite fuel such that $\nu(t) \leq \theta_o$, $\mathbb{P}$-a.s. for all $t \geq 0$.

We expect to find a nonpositive Lagrange multiplier. Notice that

$$\nabla_y \mathcal{C}(\nu)(t) = \mathbb{E}\left\{ \int_t^{\infty} \delta e^{-\delta s} (\nu(s) - W(s)) \, ds \,\Big|\, \mathcal{F}_t \right\}. \tag{1.70}$$

Moreover, the backward equation

$$\mathbb{E}\left\{ \int_{\tau}^{\infty} \delta e^{-\delta s} \sup_{\tau \leq u < s} l(u) \, ds \,\Big|\, \mathcal{F}_{\tau} \right\} = e^{-\delta \tau} W(\tau), \quad \forall \tau \in \mathcal{T}, \tag{1.71}$$

is uniquely solved by

$$l(s) = W(s) - c, \tag{1.72}$$

where c is the positive constant $c := \mathbb{E}\{ \int_0^{\infty} \delta e^{-\delta s} \sup_{0 \leq u < s} W(u) \, ds \}$, by independence and time-homogeneity of Brownian increments.

From [7] we know that the optimal investment policy is

$$\nu_*(t) = \sup_{0 \leq s < t} ((W(s) - c) \wedge \theta_o) \vee \nu(0), \tag{1.73}$$

which is the well known strategy of reflecting the Brownian motion at the threshold c until all the fuel is spent (cf. [43]).

We may write the subgradient (1.70) at ν_* as

$$\begin{aligned}\nabla_y \mathcal{C}(\nu_*)(t) &= \mathbb{E}\Big\{ \int_t^\infty \delta e^{-\delta s} \left(\nu_*(s) - W(s)\right) ds \Big| \mathcal{F}_t \Big\} - 0 \\ &= \mathbb{E}\Big\{ \int_t^\infty \delta e^{-\delta s} \left(\nu_*(s) - W(s)\right) ds \Big| \mathcal{F}_t \Big\} \\ &\qquad -\mathbb{E}\Big\{ \int_t^\infty \delta e^{-\delta s} \sup_{t \le u < s} \left(W(u) - c\right) ds \Big| \mathcal{F}_t \Big\} + e^{-\delta t} W(t) \\ &= \mathbb{E}\Big\{ \int_t^\infty \delta e^{-\delta s} \Big[\nu_*(s) - \sup_{t \le u < s} (W(u) - c)\Big] ds \Big| \mathcal{F}_t \Big\}\end{aligned}$$

where we have used (1.71) in the second equality with l given by (1.72). With this trivial trick we are in the same setting as [7], proof of Theorem 2. Hence we have that the Snell envelope of the subgradient evaluated at the optimum ν_* (cf. (1.73)) is

$$\mathbb{S}(\nu_*)(t) = \mathbb{E}\Big\{ \int_t^\infty \delta e^{-\delta s} \Big[\nu_*(s) - \sup_{t \le u < s} (W(u) - c)\Big] \wedge 0 \, ds \Big| \mathcal{F}_t \Big\} \tag{1.74}$$

or, equivalently,

$$\mathbb{S}(\nu_*)(t) = \mathbb{E}\Big\{ \int_{\tau_{\theta_o}(t)}^\infty \delta e^{-\delta s} \Big[\theta_o - \sup_{t \le u < s} (W(u) - c)\Big] ds \Big| \mathcal{F}_t \Big\}$$

with

$$\tau_{\theta_o}(t) := \inf\{s \ge t : W(s) - c > \theta_o\}, \tag{1.75}$$

by means of (1.73). Notice that $\tau_{\theta_o}(t)$ is a time of increase for $W(t) - c$. Hence we have $\sup_{t \le u < s}(W(u) - c) = \sup_{\tau_{\theta_o}(t) \le u < s}(W(u) - c)$ for $s \in (\tau_{\theta_o}(t), +\infty]$. Therefore (1.71) implies

$$\mathbb{S}(\nu_*)(t) = \mathbb{E}\Big\{ \int_{\tau_{\theta_o}(t)}^\infty \delta e^{-\delta s} \, \theta_o \, ds \Big| \mathcal{F}_t \Big\} - \mathbb{E}\Big\{ e^{-\delta \tau_{\theta_o}(t)} W(\tau_{\theta_o}(t)) \Big| \mathcal{F}_t \Big\};$$

that is,

$$\mathbb{S}(\nu_*)(t) = \mathbb{E}\Big\{ e^{-\delta \tau_{\theta_o}(t)} \Big[\theta_o - W(\tau_{\theta_o}(t))\Big] \Big| \mathcal{F}_t \Big\}. \tag{1.76}$$

We now find the explicit form of the Snell envelope $\mathbb{S}(\nu_*)(t)$ and then we use it to identify the compensator part of its Doob-Meyer decomposition; that is the Lagrange multiplier of problem (1.69) (cf. (1.50)). Notice that $\tau_{\theta_o}(t) \ge t$ a.s. by definition (1.75). Hence we have two cases.

- If t is such that $W(t) < \theta_o + c$, then $\tau_{\theta_o}(t) > t$, a.s. Since $\tau_{\theta_o}(u) = \tau_{\theta_o}(t)$ a.s. for all $u \in [t, \tau_{\theta_o}(t))$, then for $u_1 < u_2$ in $[t, \tau_{\theta_o}(t))$ we have $\mathbb{E}\{S(\nu_*)(u_2)|\mathcal{F}_{u_1}\} = S(\nu_*)(u_1)$, hence $(S(\nu_*)(u))_{u\in[t,\tau_{\theta_o}(t))}$ is a $\mathcal{F}_u$-martingale. The Markov property, the continuity of Brownian paths, and the Laplace transform formula for the hitting time of a standard Brownian motion (see, for example, [44]) imply that $S(\nu_*)(u) = Ke^{\sqrt{2\delta}W(u)-\delta u}$ for $u \in [t, \tau_{\theta_o}(t))$, with $K = K(\delta, \theta_o, c)$ a constant.

- If t is such that $W(t) > \theta_o + c$, then define the stopping time

  $$\sigma(t) := \inf\{s > t : W(s) \leq \theta_o + c\}.$$

  Notice that $t < \sigma(t)$ a.s. and $\tau_{\theta_o}(u) = u$ a.s., for every $u \in [t, \sigma(t))$. Hence

  $$S(\nu_*)(u) = e^{-\delta u}(\theta_o - W(u)).$$

  Fix now $u_1 < u_2$ in $[t, \sigma(t))$ and notice that $\mathbb{E}\{S(\nu_*)(u_2)|\mathcal{F}_{u_1}\} \geq e^{-\delta u_1}(\theta_o - W(u_1)) = S(\nu_*)(u_1)$. It follows that $(S(\nu_*)(u))_{u\in[t,\sigma(t))}$ is a $\mathcal{F}_u$-submartingale with

  $$d\mathbb{S}(\nu_*)(u) = -e^{-\delta u}dW(u) + [-\delta e^{-\delta u}(\theta_o - W(u))]du, \tag{1.77}$$

  i.e. with absolutely continuous compensator $A(\nu_*)(u)$ such that

  $$dA(\nu_*)(u) := -\delta e^{-\delta u}\left(\theta_o - W(u)\right)du. \tag{1.78}$$

Recall that the Lagrange multiplier (1.41) acts only when $\nu_*(t) = \theta(t)$, i.e. only when $l(t) > \theta(t)$. Therefore, the Lagrange multiplier of problem (1.69) must be

$$d\lambda(t) = \delta e^{-\delta t}\left[\theta_o - W(t)\right] \mathbb{1}_{\{W(t)>\theta_o+c\}}\, dt, \tag{1.79}$$

which, as expected, is negative and coincides with the opposite of the optional measure $dA(\nu_*)(t)$ (cf. (1.78)).

Remark 1.3.12. *In [10] Benes, Shepp and Witsenhausen considered a problem with the same cost functional but they allowed controls of bounded variation.*

1.3.5 Constant Finite Fuel and Cobb-Douglas Production

We consider the maximization problem of profit, net of investment costs,

$$\sup_{\nu \in \mathcal{S}_{\theta_o}} \mathcal{J}(\nu) := \sup_{\nu \in \mathcal{S}_{\theta_o}} \mathbb{E}\bigg\{ \int_0^\infty e^{-\delta s} R\left(X(s), \nu(s)\right) ds - \int_0^\infty e^{-\delta s} d\nu(s) \bigg\}. \tag{1.80}$$

The finite fuel is given by the positive constant θ_o, hence the controls satisfy $0 \leq \nu(t) \leq \theta_o$ $\mathbb{P}$-a.s., for all $t \geq 0$. The economic shock process $X(t)$ is modeled by a Geometric Brownian motion

$$X(t) = x_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)} \qquad \text{with} \quad x_0 > 0. \tag{1.81}$$

The firm's production function is of the Cobb-Douglas type and depends on the economic shock x and the investment policy y; i.e., $R\left(x, y\right) = \frac{1}{1-\alpha} x^\alpha y^{1-\alpha}$ with $0 < \alpha < 1$. As pointed out in [59] this construction is consistent with a competitive firm which produces at decreasing returns to scale or with a monopolist firm facing a constant elasticity demand function and constant returns to scale production. Notice that problem (1.80) has been studied in detail in [48] in the case of $\theta_o = +\infty$ by a dynamic programming approach.

It is known (cf. [7]) that the unique optimal solution for problem (1.80) is given by

$$\nu_*(t) = \sup_{0 \leq s < t} \left(l(s) \wedge \theta_o\right) \vee \nu(0), \tag{1.82}$$

where the optional process $l(t)$ uniquely solves the stochastic backward equation (cf. [6])

$$\mathbb{E}\bigg\{ \int_\tau^\infty e^{-\delta s} X^\alpha(s) \big(\sup_{\tau \leq u < s} l(u) \big)^{-\alpha} ds \,\Big|\, \mathcal{F}_\tau \bigg\} = e^{-\delta \tau}, \quad \forall \tau \in \mathcal{T}. \tag{1.83}$$

As shown in [59], Proposition 7.1, when the shock process is of exponential Levy type, i.e. $X(t) = x_0 e^{Y(t)}$, with $Y(t)$ a Levy process such that $Y(0) = 0$, then the solution of (1.83) is given by

$$l(t) = kX(t), \tag{1.84}$$

where $k = (\frac{1}{\delta}\mathbb{E}\{e^{\alpha \underline{Y}(\tau(\delta))}\})^{\frac{1}{\alpha}}$, $\underline{Y}(t) := \inf_{0 \leq u \leq t} Y(u)$ and $\tau(\delta)$ is an independent exponentially distributed time with parameter δ.

From (1.80) we have

$$\nabla_y \mathcal{J}(\nu)(t) = \mathbb{E}\bigg\{ \int_t^{\infty} e^{-\delta s} X^{\alpha}(s) \nu^{-\alpha}(s) ds \,\bigg|\mathcal{F}_t \bigg\} - e^{-\delta t}. \tag{1.85}$$

Following [7], proof of Theorem 2, we know that the Snell envelope of supergradient (1.85) evaluated at the optimal control policy (1.82) is

$$\mathbb{S}(\nu_*)(t) \tag{1.86}$$
$$= \mathbb{E}\bigg\{ \int_t^{\infty} e^{-\delta s} \Big[X^{\alpha}(s) \Big(\Big(\sup_{0 \le u < s} (kX(u) \wedge \theta_o) \vee \nu(0) \Big)^{-\alpha} - \Big(\sup_{t \le u < s} kX(u) \Big)^{-\alpha} \Big) \Big]^{+} ds \,\bigg|\mathcal{F}_t \bigg\}.$$

Fix $t \geq 0$ and define the stopping time

$$\tau_{\theta_o}(t) := \inf\{s \geq t : kX(s) > \theta_o\}. \tag{1.87}$$

It is a time of increase for $X(t)$. Now we split the integral into $\int_t^{\tau_{\theta_o}(t)} + \int_{\tau_{\theta_o}(t)}^{\infty}$, then the first one vanishes due to (1.87) and we are left with

$$\begin{aligned} \mathbb{S}(\nu_*)(t) &= \mathbb{E}\bigg\{ \int_{\tau_{\theta_o}(t)}^{\infty} e^{-\delta s} \Big[X^{\alpha}(s) \Big((\theta_o)^{-\alpha} - (\sup_{t \le u < s} kX(u))^{-\alpha} \Big) \Big] \, ds \,\bigg|\mathcal{F}_t \bigg\} \\ &= (\theta_o)^{-\alpha} \mathbb{E}\bigg\{ \int_{\tau_{\theta_o}(t)}^{\infty} e^{-\delta s} X^{\alpha}(s) \, ds \,\bigg|\mathcal{F}_t \bigg\} - \mathbb{E}\Big\{ e^{-\delta \tau_{\theta_o}(t)} \,\Big|\mathcal{F}_t \Big\} \end{aligned}$$

where we have used (1.83) to obtain the second equality.

Lemma 1.3.13. *Assume $\delta > \mu + \sigma^2$. Then for every $t \geq 0$, one has*

$$\mathbb{E}\bigg\{ \int_{\tau_{\theta_o}(t)}^{\infty} e^{-\delta s} X^{\alpha}(s) \, ds \,\bigg|\mathcal{F}_t \bigg\} = \frac{1}{(\delta - \mu\alpha) + \frac{1}{2}\sigma^2 \alpha (1-\alpha)} \mathbb{E}\Big\{ e^{-\delta \tau_{\theta_o}(t)} X^{\alpha}(\tau_{\theta_o}(t)) \,\Big|\mathcal{F}_t \Big\}. \tag{1.88}$$

Proof. The proof follows from the Markov property and the Laplace transform of a Gaussian process. Independence of Brownian increments, together with $W(u + \tau_{\theta_o}(t)) - W(\tau_{\theta_o}(t)) \sim W(u)$, allow us to write

$$\begin{aligned} &\mathbb{E}\bigg\{ \int_{\tau_{\theta_o}(t)}^{\infty} e^{-\delta s} X^{\alpha}(s) \, ds \,\bigg|\mathcal{F}_t \bigg\} = \mathbb{E}\bigg\{ \mathbb{E}\bigg\{ \int_{\tau_{\theta_o}(t)}^{\infty} e^{-\delta s} X^{\alpha}(s) \, ds \,\bigg|\mathcal{F}_{\tau_{\theta_o}(t)} \bigg\} \,\bigg|\mathcal{F}_t \bigg\} \\ &= \mathbb{E}\bigg\{ e^{-\delta \tau_{\theta_o}(t)} X^{\alpha}(\tau_{\theta_o}(t)) \mathbb{E}\bigg\{ \int_0^{\infty} e^{-\delta u} e^{\alpha(\mu - \frac{1}{2}\sigma^2)u + \alpha\sigma(W(u + \tau_{\theta_o}(t)) - W(\tau_{\theta_o}(t)))} \, ds \bigg\} \,\bigg|\mathcal{F}_t \bigg\} \\ &= \mathbb{E}\bigg\{ e^{-\delta \tau_{\theta_o}(t)} X^{\alpha}(\tau_{\theta_o}(t)) \mathbb{E}\bigg\{ \int_0^{\infty} e^{-\delta u} e^{\alpha(\mu - \frac{1}{2}\sigma^2)u + \alpha\sigma W(u)} \, ds \bigg\} \,\bigg|\mathcal{F}_t \bigg\} \\ &= \mathbb{E}\bigg\{ e^{-\delta \tau_{\theta_o}(t)} X^{\alpha}(\tau_{\theta_o}(t)) \int_0^{\infty} e^{-(\delta - \mu\alpha)u - \frac{1}{2}\sigma^2 \alpha(1-\alpha)u} du \,\bigg|\mathcal{F}_t \bigg\}. \end{aligned}$$

Notice that $(\delta - \mu\alpha) + \frac{1}{2}\sigma^2\alpha(1-\alpha) > 0$ by the assumption, hence (1.88) follows. $\square$

Now Lemma 1.3.13 and (1.88) imply

$$\mathbb{S}(\nu_*)(t) = \frac{(\theta_o)^{-\alpha}}{(\delta - \mu\alpha) + \frac{1}{2}\sigma^2\alpha(1-\alpha)}\mathbb{E}\Big\{e^{-\delta\tau_{\theta_o}(t)}X^{\alpha}(\tau_{\theta_o}(t))\Big|\mathcal{F}_t\Big\} - \mathbb{E}\Big\{e^{-\delta\tau_{\theta_o}(t)}\Big|\mathcal{F}_t\Big\}. \quad (1.89)$$

As in Subsection 1.3.4 we now find the explicit form of the compensator of the Snell envelope $\mathbb{S}(\nu_*)(t)$ and hence we identify the compensator part of its Doob-Meyer decomposition, which is the Lagrange multiplier of problem (1.80). By definition we have $\tau_{\theta_o}(t) \geq t$ a.s., hence we consider two cases.

- If t is such that $kX(t) < \theta_o$, then $\tau_{\theta_o}(t) > t$ a.s. Since $\tau_{\theta_o}(u) = \tau_{\theta_o}(t)$ a.s. for all $u \in [t, \tau_{\theta_o}(t))$, then for $u_1 < u_2$ in $[t, \tau_{\theta_o}(t))$ we have $\mathbb{E}\{S(\nu_*)(u_2)|\mathcal{F}_{u_1}\} = S(\nu_*)(u_1)$, hence $(S(\nu_*)(u))_{u\in[t,\tau_{\theta_o}(t))}$ is a $\mathcal{F}_u$-martingale. By using the continuity of Brownian paths, the Laplace transform formula for the hitting time of a Brownian motion with drift (cf. [44]) and the Markov property, we may write

$$S(\nu_*)(u) = Ke^{W(u)(\sqrt{\gamma^2+2\delta}-\gamma)-u(\delta+\gamma^2-\gamma\sqrt{\gamma^2+2\delta})} \qquad \text{for } u \in [t, \tau_{\theta_o}(t)),$$

  with $\gamma := \frac{1}{\sigma}(\mu - \frac{1}{2}\sigma^2)$ and $K = K(\sigma, \theta_o, k)$ a constant.

- If t is such that $kX(t) > \theta_o$, then define the stopping time

$$\sigma(t) := \inf\{s > t : kX(s) \leq \theta_o\}.$$

  Notice that $t < \sigma(t)$ a.s. and we have $\tau_{\theta_o}(u) = u$ a.s., for every $u \in [t, \sigma(t))$. Hence

$$S(\nu_*)(u) = \frac{(\theta_o)^{-\alpha}}{(\delta - \mu\alpha) + \frac{1}{2}\sigma^2\alpha(1-\alpha)}\, e^{-\delta u}X^{\alpha}(u) - e^{-\delta u}, \qquad \text{for } u \in [t, \sigma(t)).$$

  Moreover $X^{\alpha}(u)(\theta_o)^{-\alpha} > \delta$ for all $u \in [t, \sigma(t))$. In fact, with k as in (1.84) and $\underline{Y}(u) := \inf_{0\leq s\leq u}\left[\left(\mu - \frac{1}{2}\sigma^2\right)s + \sigma W(s)\right]$, we have $X^{\alpha}(u)(\theta_o)^{-\alpha} > k^{-\alpha}$ for $u \in [t, \sigma(t))$ because $\mathbb{E}\{e^{\alpha\underline{Y}(\tau(\delta))}\} = \beta_-(\beta_- - \alpha)^{-1} < 1$, being β_- the negative root of $\frac{1}{2}\sigma^2x^2 + \left(\mu - \frac{1}{2}\sigma^2\right)x - \delta = 0$ and $\alpha > 0$.

Fix now $u_1 < u_2$ in $[t, \sigma(t))$ and apply Ito's Lemma,

$$\begin{aligned}\mathbb{E}\Big\{\mathbb{S}(\nu_*)(u_2)\Big|\mathcal{F}_{u_1}\Big\} &= \mathbb{S}(\nu_*)(u_1) + \frac{\sigma\alpha(\theta_o)^{-\alpha}}{(\delta-\mu\alpha)+\frac{1}{2}\sigma^2\alpha(1-\alpha)}\mathbb{E}\Big\{\int_{u_1}^{u_2} e^{-\delta t}X^\alpha(s)dW(s)\,\Big|\mathcal{F}_{u_1}\Big\} \\ &\quad -\mathbb{E}\Big\{\int_{u_1}^{u_2} e^{-\delta s}\left(X^\alpha(s)(\theta_o)^{-\alpha}-\delta\right)ds\,\Big|\mathcal{F}_{u_1}\Big\} \leq \mathbb{S}(\nu_*)(u_1).\end{aligned}$$

Hence $(S(\nu_*)(u))_{u\in[t,\sigma(t))}$ is a $\mathcal{F}_u$-supermartingale whose compensator is the absolutely continuous process

$$dA(\nu_*)(u) := e^{-\delta u}\left(X^\alpha(u)(\theta_o)^{-\alpha}-\delta\right)du. \tag{1.90}$$

Recall that the Lagrange multiplier optional measure $d\lambda$ (cf. (1.41)) acts only at times such that $\nu_*(t) = \theta(t)$ (i.e., only when $l(t) > \theta(t)$). Therefore, for problem (1.80), $d\lambda$ must be given by

$$d\lambda(t) = e^{-r\delta}\left(X^\alpha(t)(\theta_o)^{-\alpha}-\delta\right)\mathbb{1}_{\{kX(t)>\theta_o\}}\,dt, \tag{1.91}$$

which coincides with the random measure $dA(\nu_*)(t)$ (cf. (1.90)).

Chapter 2

Identifying the Free Boundary of a Stochastic, Irreversible Investment Problem via the Bank-El Karoui Representation Theorem

In this Chapter we study a stochastic, continuous time model on a finite horizon for a firm that produces a single good. In contrast with Chapter 1 in which we did not make any Markovian assumption and in which there was not a production capacity dynamics, here we model the capacity as an Ito diffusion controlled by a nondecreasing process representing the cumulative investment. We suppose now that the resources are unlimited and the firm's manager aims to maximize its expected total net profit by choosing the optimal investment process. That is a singular stochastic control problem. The aim of this Chapter is to understand the significance of the *base capacity* process in such a diffusion framework.

We derive some first order conditions for optimality and we characterize the optimal solution in terms of the *base capacity* process $l^*(t)$, i.e. the unique solution of a representation problem in the spirit of Bank and El Karoui [6].

Under further assumptions we show that the base capacity is in fact deterministic and coincides with the free boundary $\hat{y}(t)$ of the optimal stopping problem naturally associated to the original singular control problem. This result allows us to obtain the continuity of the free boundary $\hat{y}(t)$ in the case of a Cobb-Douglas production function and of constant coefficients in the controlled capacity process.

2.1 The Firm's Investment Problem

The setting is as in Chiarolla and Haussmann [20] but without leisure, wages and scrap value. We briefly recall their notation. An economy with finite horizon T and productive sector represented by a firm is considered on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t, t \in [0,T]\}$. Such filtration is the usual augmentation of the filtration generated by an exogeneous Brownian motion $\{W(t) : t \in [0,T]\}$ and augmented by $\mathbb{P}$-null sets. The firm produces at rate $R(C)$ when its capacity is C. The cumulative, irreversible investment is denoted by $\nu(t)$. It is an a.s. finite, left-continuous with right-limits, nondecreasing, and adapted process. The irreversibility of investment is expressed by the nondecreasing nature of ν. The production capacity $C^{y,\nu}$ associated to the investment strategy ν satisfies

$$\begin{cases} dC^{y,\nu}(t) = C^{y,\nu}(t)[-\mu_C(t)dt + \sigma_C(t)dW(t)] + f_C(t)d\nu(t), \quad t \in [0,T), \\ \\ C^{y,\nu}(0) = y > 0, \end{cases} \tag{2.1}$$

where μ_C, σ_C and f_C are given measurable, uniformly bounded in (ω, t) adapted processes. Moreover f_C is continuous with $0 < k_f \leq f_C(t) \leq \kappa_f$ and $\mu_C \geq 0$. Here f_C is a conversion factor since any unit of investment is converted into f_C units of production capacity.

By setting

$$C^0(t) := C^{1,0}(t), \qquad \overline{\nu}(t) := \int_{[0,t)} \frac{f_C(s)}{C^0(s)} d\nu(s), \tag{2.2}$$

then we may write

$$C^0(t) = e^{-\int_0^t \mu_C(s)ds} \mathcal{M}_0(t), \tag{2.3}$$

where the exponential martingale

$$\mathcal{M}_s(t) := e^{-\int_s^t \frac{1}{2}\sigma_C^2(u)du + \int_s^t \sigma_C(u)dW(u)}, \quad t \in [s,T], \tag{2.4}$$

is defined for $s \in [0,T]$. Without investment, C^0 represents the decay of a unit of initial capital and we have

$$C^{y,\nu}(t) = C^0(t)[y + \overline{\nu}(t)]. \tag{2.5}$$

The production function of the firm is a nonnegative, measurable function $R(C)$. We make the following

Assumption 2.1.1.

1. *the mapping $C \to R(C)$ is strictly increasing and strictly concave with continuous derivative $R_c(C) := \frac{\partial}{\partial C} R(C)$ satisfying the Inada conditions*

$$\lim_{C \to 0} R_c(C) = \infty, \qquad \lim_{C \to \infty} R_c(C) = 0.$$

2. $\forall \eta > 0 : \quad \sup_{C \geq 0} \{R(C) - \eta C\} < \infty$.

Our Assumption 2.1.1 is not as general as the Assumption in [20] but it is needed to apply the Bank-El Karoui Representation Theorem [6]. Assumption 2.1.1, part 2 is satisfied by any production function which grows at infinity less than linearly as, for example, the Cobb-Douglas one, i.e. $R(C) = C^\alpha / \alpha$ with $\alpha \in (0,1)$.

Each investment plan $\nu \in \mathcal{S}_0$ leads to the expected total profit net of investment

$$\mathcal{J}_{0,y}(\nu) = \mathbb{E}\left\{ \int_0^T e^{-\int_0^t \mu_F(s)ds} R(C^{y,\nu}(t))dt - \int_{[0,T)} e^{-\int_0^t \mu_F(s)ds} d\nu(t) \right\} \tag{2.6}$$

where

$$\mathcal{S}_0 \;:=\; \{\nu : \Omega \times [0,T] \to \mathbb{R}_+, \text{ nondecreasing, left-continuous, adapted s.t. } \nu(0) = 0, \ \mathbb{P} - \text{a.s.}\}$$

is the convex set of irreversible investment processes. Here μ_F is the firm's manager discount factor; it is a nonnegative, measurable, uniformly bounded in (ω, t) adapted process. Of course $\mathcal{S}_0 \neq \emptyset$ because $\nu(t) = 0$ for all $t \in [0,T]$ belongs to $\mathcal{S}_0$.

The firm's problem is

$$V(0,y) := \sup_{\nu \in \mathcal{S}_0} \mathcal{J}_{0,y}(\nu). \tag{2.7}$$

V is finite thanks to Assumption 2.1.1, part 2 (cf. [20], Proposition 2.1). Moreover, the concavity of R and the affine nature of $C^{y,\nu}$ in ν imply that $\mathcal{J}_{0,y}(\nu)$ is strictly concave on $\mathcal{S}_0$. Hence if a solution $\hat{\nu}$ of (2.7) exists, it is unique. The existence of the solution has been proved in [20], Theorem 3.1. We provide a new characterization of it in Theorem 2.3.1.

2.2 First Order Conditions for Optimality

As in [3] and [24], we aim to characterize the optimal solution of (2.7) by some first order conditions for optimality.

Let $\mathcal{T}$ denote the set of all stopping times with value in $[0,T]$, $\mathbb{P}$-a.s. Note that the strict concave functional $\mathcal{J}_{0,y}(\nu)$ admits the supergradient

$$\begin{aligned}\nabla_\nu \mathcal{J}_{0,y}(\nu)(\tau) \;:=\; &\mathbb{E}\bigg\{ \int_\tau^T e^{-\int_0^s \mu_F(u)du} C^0(s) \frac{f_C(\tau)}{C^0(\tau)}\, R_c(C^{y,\nu}(s))\, ds \,\bigg|\, \mathcal{F}_\tau \bigg\} \\ &- e^{-\int_0^\tau \mu_F(u)du}\, \mathbb{1}_{\{\tau<T\}},\end{aligned} \tag{2.8}$$

for $\tau \in \mathcal{T}$.

Remark 2.2.1. *The quantity $\nabla_\nu \mathcal{J}_{0,y}(\nu)(t)$ may be interpreted as the marginal expected future profit resulting from an additional infinitesimal investment at time t. Mathematically, $\nabla_\nu \mathcal{J}_{0,y}(\nu)$ can be viewed as the Riesz representation of the profit's gradient at ν. More precisely, we may define $\nabla_\nu \mathcal{J}_{0,y}(\nu)$ as the optional projection of the progressively measurable process*

$$\phi(\omega,t) := \int_t^T e^{-\int_0^s \mu_F(\omega,u)du} C^0(\omega,s) \frac{f_C(\omega,t)}{C^0(\omega,t)}\, R_c(C^{y,\nu}(\omega,s))\, ds - e^{-\int_0^t \mu_F(\omega,u)du} \mathbb{1}_{\{t<T\}}, \tag{2.9}$$

for $\omega \in \Omega$, $t \in [0,T]$. Hence $\nabla_\nu \mathcal{J}_{0,y}(\nu)$ is uniquely determined up to $\mathbb{P}$-indistinguishability and it holds

$$\mathbb{E}\bigg\{ \int_{[0,T)} \nabla_\nu \mathcal{J}_{0,y}(\nu)(t) d\nu(t) \bigg\} = \mathbb{E}\bigg\{ \int_{[0,T)} \phi(t) d\nu(t) \bigg\} \tag{2.10}$$

for all $\nu \in \mathcal{S}_0$ (cf. Theorem 1.33 in [36]).

We shall prove that

Theorem 2.2.1. *Given problem (2.7), the following first-order conditions*

$$\nabla_\nu \mathcal{J}_{0,y}(\hat{\nu})(\tau) \leq 0, \quad \forall \tau \in \mathcal{T}, \ \ \mathbb{P} - a.s., \tag{2.11}$$

$$\mathbb{E}\bigg\{ \int_{[0,T)} \nabla_\nu \mathcal{J}_{0,y}(\hat{\nu})(\tau) \, d\hat{\nu}(\tau) \bigg\} = 0, \tag{2.12}$$

are sufficient for the optimality of $\hat{\nu}(t)$.

Conversely, if $\mathcal{J}_{0,y}(\nu) \geq 0$ *for all* $\nu \in \mathcal{S}_0$ *and* $\overline{\nu}^y(T) := \int_{[0,T)} \frac{f_C(s)}{C_0(s)} d\hat{\nu}(s)$ *is* $\mathbb{P}$*-integrable, then (2.11) and (2.12) are also necessary for optimality.*

The proof of Theorem 2.2.1 relies on the following Lemma. The idea is to use arguments similar to those in the proof of the finite dimensional Kuhn-Tucker Theorem. First of all, notice that the supergradient (2.8) evaluated at the optimal investment plan cannot be positive. In fact there cannot exist a stopping time $\overline{\tau} \in \mathcal{T}$ such that $\nabla_\nu \mathcal{J}_{0,y}(\hat{\nu})(\overline{\tau}) > 0$, since the continuity of R_c and the linearity of the investment cost imply that a sufficiently small extra investment at $\overline{\tau}$ would be profitable, and hence $\hat{\nu}$ would not be optimal. Therefore $\nabla_\nu \mathcal{J}_{0,y}(\hat{\nu})(\tau) \leq 0$ for all $\tau \in \mathcal{T}$.

In the next Lemma we show that if $\mathcal{J}_{0,y}(\nu) \geq 0$ for all $\nu \in \mathcal{S}_0$ and $\mathbb{E}\{\overline{\nu}^y(T)\} < +\infty$, then the optimal policy $\hat{\nu}$ solves the problem linearized near $\hat{\nu}$. The solutions of the linear problem are characterized by a *flat-off condition*.

Lemma 2.2.2. *Let* $\hat{\nu}$ *be optimal for problem (2.7) and let* $\hat{\phi}$ *be the progressively measurable process given by (2.9) and corresponding to* $\nabla_\nu \mathcal{J}_{0,y}(\hat{\nu})$*. If* $\mathcal{J}_{0,y}(\nu) \geq 0$ *for all* $\nu \in \mathcal{S}_0$ *and* $\mathbb{E}\{\overline{\nu}^y(T)\} < +\infty$*, then* $\hat{\nu}$ *solves the linear problem*

$$\sup_{\nu \in \mathcal{S}_0} \mathbb{E}\bigg\{ \int_{[0,T)} \hat{\phi}(s) d\nu(s) \bigg\}. \tag{2.13}$$

Proof. Let $\nu \in \mathcal{S}_0$. We set $\nu^\epsilon(t) = \epsilon\nu(t) + (1-\epsilon)\hat{\nu}(t)$, for $\epsilon \in (0,1)$, and define ϕ^ϵ to be the progressively measurable process given by (2.9) and corresponding to $\nabla_\nu \mathcal{J}_{0,y}(\nu^\epsilon)$. Of course $\lim_{\epsilon \to 0} \nu^\epsilon(t) = \hat{\nu}(t)$ for all $t \leq T$, $\mathbb{P}$-a.s., as well as $\lim_{\epsilon \to 0} \phi^\epsilon(t) = \hat{\phi}(t)$ by continuity of R_c.

By optimality of $\hat{\nu}$ and concavity of R we have

$$
\begin{aligned}
0 \;\geq\;& \frac{\mathcal{J}_{0,y}(\nu^{\epsilon}) - \mathcal{J}_{0,y}(\hat{\nu})}{\epsilon} \\
=\;& \frac{1}{\epsilon}\mathbb{E}\bigg\{ \int_0^T e^{-\int_0^t \mu_F(u)du} \left(R(C^{y,\nu^{\epsilon}}(t)) - R(C^{y,\hat{\nu}}(t))\right)dt \bigg\} \\
& -\frac{1}{\epsilon}\mathbb{E}\bigg\{ \int_{[0,T)} e^{-\int_0^t \mu_F(u)du} \left(d\nu^{\epsilon}(t) - d\hat{\nu}(t)\right) \bigg\} \\
\geq\;& \frac{1}{\epsilon}\mathbb{E}\bigg\{ \int_0^T e^{-\int_0^t \mu_F(u)du} R_c(C^{y,\nu^{\epsilon}}(t))(C^{y,\nu^{\epsilon}}(t) - C^{y,\hat{\nu}}(t)) \bigg\} dt \\
& - \mathbb{E}\bigg\{ \int_{[0,T)} e^{-\int_0^t \mu_F(u)du} \left(d\nu(t) - d\hat{\nu}(t)\right) \bigg\} \qquad (2.14) \\
=\;& \mathbb{E}\bigg\{ \int_0^T e^{-\int_0^t \mu_F(u)du} R_c(C^{y,\nu^{\epsilon}}(t)) \Big[\int_{[0,t)} C^0(t)\frac{f_C(u)}{C^0(u)} \left(d\nu(u) - d\hat{\nu}(u)\right) \Big] \bigg\} dt \\
& - \mathbb{E}\bigg\{ \int_{[0,T)} e^{-\int_0^t \mu_F(u)du} \left(d\nu(t) - d\hat{\nu}(t)\right) \bigg\} \\
=\;& \mathbb{E}\bigg\{ \int_{[0,T)} \Big[\int_t^T e^{-\int_0^s \mu_F(u)du} R_c(C^{y,\nu^{\epsilon}}(s)) C^0(s)\frac{f_C(t)}{C^0(t)} ds - e^{-\int_0^t \mu_F(u)du} \Big] \left(d\nu(t) - d\hat{\nu}(t)\right) \bigg\} \\
=\;& \mathbb{E}\bigg\{ \int_{[0,T)} \phi^{\epsilon}(t) \left(d\nu(t) - d\hat{\nu}(t)\right) \bigg\}
\end{aligned}
$$

where in the third equality we have used Fubini's Theorem.

We would like to prove that

$$
\mathbb{E}\bigg\{ \int_{[0,T)} \hat{\phi}(t) \left(d\nu(t) - d\hat{\nu}(t)\right) \bigg\} \leq 0.
$$

Consider

$$
\begin{aligned}
I^{\epsilon} \;:=\;& \int_{[0,T)} \phi^{\epsilon}(t) \left(d\nu(t) - d\hat{\nu}(t)\right) \\
=\;& \int_0^T e^{-\int_0^t \mu_F(u)du} R_c(C^{y,\nu^{\epsilon}}(t)) \Big[\int_{[0,t)} C^0(t)\frac{f_C(s)}{C^0(s)} \left(d\nu(s) - d\hat{\nu}(s)\right) \Big] dt \\
& - \int_{[0,T)} e^{-\int_0^t \mu_F(u)du} \left(d\nu(t) - d\hat{\nu}(t)\right) \qquad (2.15) \\
=\;& \int_0^T e^{-\int_0^t \mu_F(u)du} R_c(C^{y,\nu^{\epsilon}}(t))(C^{y,\nu}(t) - C^{y,\hat{\nu}}(t))dt \\
& - \int_{[0,T)} e^{-\int_0^t \mu_F(u)du} \left(d\nu(t) - d\hat{\nu}(t)\right),
\end{aligned}
$$

where in the first equality we have used Fubini's Theorem. Notice that

$$C^{y,\nu^{\epsilon}}(t) = \epsilon C^{y,\nu}(t) + (1-\epsilon)C^{y,\hat{\nu}}(t) \begin{cases} \leq C^{y,\nu}(t) & \text{on} \quad \{C^{y,\nu}(t) - C^{y,\hat{\nu}}(t) \geq 0\}, \\ \\ > C^{y,\nu}(t) & \text{on} \quad \{C^{y,\nu}(t) - C^{y,\hat{\nu}}(t) < 0\}. \end{cases} \tag{2.16}$$

Since the production function is concave, then for $\epsilon \in (0,1)$ the decreasing property of R_c and (2.16) give

$$\begin{aligned} I^{\epsilon} &= \int_{[0,T)} \phi^{\epsilon}(t)(d\nu(t) - d\hat{\nu}(t)) \\ &\geq \int_0^T e^{-\int_0^t \mu_F(u)du} R_c(C^{y,\nu}(t))(C^{y,\nu}(t) - C^{y,\hat{\nu}}(t))\, \mathbb{1}_{\{C^{y,\nu}(t) \geq C^{y,\hat{\nu}}(t)\}} dt \\ &\quad + \int_0^T e^{-\int_0^t \mu_F(u)du} R_c(C^{y,\nu}(t))(C^{y,\nu}(t) - C^{y,\hat{\nu}}(t))\, \mathbb{1}_{\{C^{y,\nu}(t) < C^{y,\hat{\nu}}(t)\}} dt \\ &\quad - \int_{[0,T)} e^{-\int_0^t \mu_F(u)du}(d\nu(t) - d\hat{\nu}(t)) \\ &= \int_0^T e^{-\int_0^t \mu_F(u)du} R_c(C^{y,\nu}(t))(C^{y,\nu}(t) - C^{y,\hat{\nu}}(t)) dt \\ &\quad - \int_{[0,T)} e^{-\int_0^t \mu_F(u)du}(d\nu(t) - d\hat{\nu}(t)) \qquad (2.17) \\ &= \int_{[0,T)} \nabla_{\nu} \mathcal{J}_{0,y}(\nu)(t)(d\nu(t) - d\hat{\nu}(t)) =: G(\omega). \end{aligned}$$

If $\mathbb{E}\{|G(\omega)|\} < +\infty$, we may apply Fatou's Lemma to $\lim_{\epsilon \to 0} \mathbb{E}\{I^{\epsilon}(\omega)\}$ and obtain

$$\begin{aligned} &\mathbb{E}\left\{ \int_{[0,T)} \hat{\phi}(s)(d\nu(s) - d\hat{\nu}(s)) \right\} \\ &\leq \liminf_{\epsilon \to 0} \mathbb{E}\left\{ \int_{[0,T)} \phi^{\epsilon}(s)(d\nu(s) - d\hat{\nu}(s)) \right\} \leq 0 \end{aligned} \tag{2.18}$$

by (2.14).

It remains to show that $\mathbb{E}\{|G(\omega)|\} < +\infty$. The growth assumption on R (cf. Assumption 2.1.1, part 2) implies that for every $\eta > 0$ there exists κ_η such that $R(C) \leq \kappa_\eta + \eta C$. As in [20] we define $\overline{\nu}^y(t) := \int_{[0,t)} \frac{f_C(s)}{C^0(s)} d\hat{\nu}(s)$. By concavity of R we have

$$|G(\omega)| \leq \int_0^T e^{-\int_0^t \mu_F(u)du} \frac{R\left(C^{y,\nu}(t)\right)}{C^{y,\nu}(t)} \left|C^{y,\nu}(t) - C^{y,\hat{\nu}}(t)\right| dt$$

$$
\begin{aligned}
&+\int_{[0,T)} e^{-\int_0^t \mu_F(u)du} \; |d\nu(t) - d\hat{\nu}(t)| \\
\leq\; & \int_0^T e^{-\int_0^t \mu_F(u)du} \, \frac{\kappa_\eta + \eta C^{y,\nu}(t)}{C^{y,\nu}(t)} \left| C^{y,\nu}(t) - C^{y,\hat{\nu}}(t) \right| dt \\
&+\int_{[0,T)} e^{-\int_0^t \mu_F(u)du} \, (d\nu(t) + d\hat{\nu}(t)) \qquad (2.19) \\
\leq\; & \kappa_\eta T + \int_0^T e^{-\int_0^t \mu_F(u)du} \, \kappa_\eta \left(\frac{y + \overline{\nu}^y(t)}{y} \right) dt + \\
&+\eta \int_0^T e^{-\int_0^t \mu_F(u)du} C^0(t) \, (\overline{\nu}(t) + \overline{\nu}^y(t)) dt \\
&+\int_{[0,T)} e^{-\int_0^t \mu_F(u)du} \, (d\nu(t) + d\hat{\nu}(t)) \\
\leq\; & 2\kappa_\eta T + \frac{\kappa_\eta}{y} \int_0^T e^{-\int_0^t \mu_F(u)du} \, \overline{\nu}^y(t) dt + \eta \int_0^T e^{-\int_0^t \mu_F(u)du} \, C^0(t)(\overline{\nu}(t) + \overline{\nu}^y(t)) dt \\
&+\int_{[0,T)} e^{-\int_0^t \mu_F(u)du} \, (d\nu(t) + d\hat{\nu}(t)).
\end{aligned}
$$

If $\mathcal{J}_{0,y}(\nu) \geq 0$, then [20], Proposition 2.1 part (b), implies $\mathbb{E}\{\int_{[0,T)} e^{-\int_0^t \mu_F(u)du} \, d\nu(t)\} < +\infty$ for all $\nu \in \mathcal{S}_0$. Thus we have only to show that

$$
\mathbb{E}\left\{ \int_0^T e^{-\int_0^t \mu_F(u)du} \, \overline{\nu}^y(t) dt \right\} < \infty \quad \text{and} \quad \mathbb{E}\left\{ \int_0^T e^{-\int_0^t \mu_F(u)du} \, C^0(t)(\overline{\nu}(t) + \overline{\nu}^y(t)) dt \right\} < +\infty.
$$

Clearly, if $\overline{\nu}^y(T)$ is $\mathbb{P}$-integrable then $\mathbb{E}\left\{ \int_0^T e^{-\int_0^t \mu_F(u)du} \overline{\nu}^y(t) dt \right\} < +\infty$.

Recall that $C^0(t) = e^{-\int_0^t \mu_C(u)du} \mathcal{M}_0(t)$ and apply Fubini's Theorem to obtain

$$
\begin{aligned}
\mathbb{E}\left\{ \int_0^T e^{-\int_0^t \mu_F(u)du} \, C^0(t) \overline{\nu}(t) dt \right\} &= \mathbb{E}\left\{ \int_{[0,T)} f_C(s) \, \frac{d\nu(s)}{C^0(s)} \int_s^T e^{-\int_0^t \mu_F(u)du} \, C^0(t) dt \right\} \\
&\leq \mathbb{E}\left\{ \int_{[0,T)} e^{\int_0^s \mu_C(u)} \frac{f_C(s)}{\mathcal{M}_0(s)} d\nu(s) \, \mathbb{E}\left\{ \int_s^T e^{-\int_0^t (\mu_C(u) + \mu_F(u))du} \mathcal{M}_0(t) \, dt \, | \mathcal{F}_s \right\} \right\} \qquad (2.20) \\
&\leq \text{const} \, \mathbb{E}\{\nu(T)\},
\end{aligned}
$$

since f_C, μ_C and μ_F are uniformly bounded in (t, ω). Again by [20], Proposition 2.1 part (b), if $\mathcal{J}_{0,y}(\nu) \geq 0$ then $\mathbb{E}\{\nu(T)\} < \infty$ for all $\nu \in \mathcal{S}_0$. Hence $\mathbb{E}\left\{ \int_0^T e^{-\int_0^t \mu_F(u)du} \, C^0(t) \, \overline{\nu}(t) dt \right\} < +\infty$ and similarly for $\mathbb{E}\left\{ \int_0^T e^{-\int_0^t \mu_F(u)du} \, C^0(t) \, \overline{\nu}^y(t) dt \right\}$. It means that $G(\omega)$ is $\mathbb{P}$-integrable and this concludes the proof. □

Notice that for every nonpositive, optional, and adapted process $f(t)$, the linear optimization problem

$$\sup_{\nu\in\mathcal{S}_0} \mathbb{E}\Big\{ \int_{[0,T)} f(s)d\nu(s)\Big\} \tag{2.21}$$

has value zero since $\nu \in \mathcal{S}_0$ has nondecreasing paths.

Proof of Theorem 2.2.1. Let $\hat{\nu}$ satisfy the first-order conditions (2.11) and (2.12) and let $\nu \in \mathcal{S}_0$. Then it follows from (2.5) that

$$C^{y,\hat{\nu}}(t) - C^{y,\nu}(t) = \int_{[0,t)} C^0(t)\frac{f_C(s)}{C^0(s)}(d\hat{\nu}(s) - d\nu(s)).$$

Hence the strict concavity of R implies

$$\begin{aligned}
&\mathcal{J}_{0,y}(\hat{\nu}) - \mathcal{J}_{0,y}(\nu)\\
&= \mathbb{E}\Big\{ \int_0^T e^{-\int_0^t \mu_F(u)du}(R(C^{y,\hat{\nu}}(t)) - R(C^{y,\nu}(t)))dt - \int_{[0,T)} e^{-\int_0^t \mu_F(u)du}(d\hat{\nu}(t) - d\nu(t))\Big\}\\
&> \mathbb{E}\Big\{ \int_0^T e^{-\int_0^t \mu_F(u)du}\, R_c(C^{y,\hat{\nu}}(t))(C^{y,\hat{\nu}}(t) - C^{y,\nu}(t))dt - \int_{[0,T)} e^{-\int_0^t \mu_F(u)du}(d\hat{\nu}(t) - d\nu(t))\Big\}\\
&= \mathbb{E}\Big\{ \int_0^T e^{-\int_0^t \mu_F(u)du}\, R_c(C^{y,\hat{\nu}}(t)) \int_{[0,t)} C^0(t)\frac{f_C(s)}{C^0(s)}(d\hat{\nu}(s) - d\nu(s))dt\\
&\qquad - \int_{[0,T)} e^{-\int_0^t \mu_F(u)du}(d\hat{\nu}(t) - d\nu(t))\Big\} \qquad (2.22)\\
&= \mathbb{E}\Big\{ \int_{[0,T)} \Big[\int_t^T e^{-\int_0^s \mu_F(u)du}\, R_c(C^{y,\hat{\nu}}(s))\, C^0(s)\frac{f_C(t)}{C^0(t)}ds - e^{-\int_0^t \mu_F(u)du}\Big](d\hat{\nu}(t) - d\nu(t))\Big\}\\
&= \mathbb{E}\Big\{ \int_{[0,T)} \nabla_\nu \mathcal{J}_{0,y}(\hat{\nu})(t)\,(d\hat{\nu}(t) - d\nu(t))\Big\} \geq 0
\end{aligned}$$

where we have used Fubini's theorem in the third equality, and (2.11) and (2.12) in the last one. It follows that $\hat{\nu}$ is optimal for problem (2.7).

On the other hand, that (2.11) and (2.12) are necessary for optimality follows from Lemma 2.2.2. □

Theorem (2.2.1) characterizes the optimal investment plan but it might not be useful if one aims to find the explicit solution, since the first order conditions are not always binding.

In what follows we construct the optimal capacity in terms of the '*base capacity*' $\{l(t), t \in [0,T]\}$ (cf. [59]) which represents the capacity level that is optimal for a firm starting at time t without any knowledge of the past capacity, and we show that it is optimal for (2.7) to invest up to the base capacity level if the current capacity level is below it; otherwise no investment is optimal.

2.3 Finding the Optimal Capacity Process

Recall the Bank-El Karoui Representation Theorem (cf. [6], Theorem 3); that is, given

- an optional process $X = \{X(t), t \in [0,T]\}$ of class (D), lower-semicontinuous in expectation with $X(T) = 0$,
- a nonnegative optional random Borel measure $\mu(\omega, dt)$,
- $f(\omega, t, x) : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ such that $f(\omega, t, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous in x, strictly decreasing from $+\infty$ to $-\infty$, and the stochastic process $f(\cdot, \cdot, x) : \Omega \times [0,T] \to \mathbb{R}$ is progressively measurable and integrable with respect to $d\mathbb{P} \otimes \mu(\omega, dt)$,

then there exists a unique optional process $\xi = \{\xi(t), t \in [0,T]\}$ such that for all $\tau \in \mathcal{T}$

$$f(t, \sup_{\tau \le u \le t} \xi(u)) \mathbb{1}_{[\tau,T)}(t) \in \mathbf{L}^1 \left(d\mathbb{P} \otimes \mu(\omega, dt) \right)$$

and

$$\mathbb{E}\left\{ \int_\tau^T f(s, \sup_{\tau \le u \le s} \xi(u))\, \mu(\omega, ds) \,\Big|\, \mathcal{F}_\tau \right\} = X(\tau).$$

Note that ξ may be taken to be upper right-continuous a.s. (cf. [6], Lemma 4.1).

Lemma 2.3.1. *There exists a unique upper right-continuous process $\xi^*(t)$ that solves*

$$\begin{aligned} &\mathbb{E}\left\{ \int_\tau^T e^{-\int_0^s \mu_F(u)du} C^0(s)\, R_c\left(-\frac{C^0(s)}{\sup_{\tau \le u \le s} \xi^*(u)} \right) ds \,\Big|\, \mathcal{F}_\tau \right\} \\ &= e^{-\int_0^\tau \mu_F(u)du} \frac{C^0(\tau)}{f_C(\tau)} \mathbb{1}_{\{\tau < T\}} \end{aligned} \tag{2.23}$$

for all $\tau \in \mathcal{T}$*. Moreover* $\xi^*(t) < 0$ *for all* $t \in [0,T)$ *a.s.*

Proof. We apply the Bank-El Karoui Representation Theorem to

$$X(\omega,t) := e^{-\int_0^t \mu_F(\omega,u)du} \frac{C^0(\omega,t)}{f_C(\omega,t)} \mathbb{1}_{[0,T)}(t), \qquad \mu(\omega,dt) := e^{-\int_0^t \mu_F(\omega,u)du} C^0(\omega,t)dt \quad (2.24)$$

and

$$f(\omega,t,x) := \begin{cases} R_c\left(-\frac{C^0(\omega,t)}{x}\right), & \text{for } x<0, \\ \\ -x\,, & \text{for } x \geq 0. \end{cases} \quad (2.25)$$

Then there exists a unique upper right-continuous process ξ^* such that, for all $\tau \in \mathcal{T}$

$$e^{-\int_0^\tau \mu_F(u)du} \frac{C^0(\tau)}{f_C(\tau)} \mathbb{1}_{\{\tau<T\}} = \mathbb{E}\bigg\{ \int_\tau^T f(s, \sup_{\tau \leq u \leq s} \xi^*(u))\, \mu(ds) \,\bigg|\, \mathcal{F}_\tau \bigg\}. \quad (2.26)$$

It now sufficies to show that $\xi^* < 0$ on $[0,T)$ a.s. Define

$$\sigma := \inf\{t \in [0,T) : \xi^*(t) \geq 0\} \wedge T,$$

then for $\omega \in \{\sigma < T\}$, the upper right-continuity of ξ^* implies $\xi^*(\sigma) \geq 0$ and therefore $\sup_{\sigma \leq u \leq s} \xi^*(u) \geq 0$ for all $s \in [\sigma, T]$. Therefore, (2.26) with $\tau = \sigma$, i.e.

$$e^{-\int_0^\sigma \mu_F(u)du} \frac{C^0(\sigma)}{f_C(\sigma)} \mathbb{1}_{\{\sigma<T\}} = -\mathbb{E}\bigg\{ \int_\sigma^T e^{-\int_0^s \mu_F(u)du} C^0(s) \sup_{\sigma \leq u \leq s} \xi^*(u)\, ds \,\bigg|\, \mathcal{F}_\sigma \bigg\}, \quad (2.27)$$

is not possible for $\omega \in \{\sigma < T\}$ since the right-hand side of (2.27) is nonpositive, whereas the left-hand side is always strictly positive. It follows that $\sigma = T$ a.s. and hence $\xi^*(t) < 0$ for all $t \in [0,T)$ a.s. □

Proposition 2.3.2. *There exists a unique upper right-continuous solution* $l^*(t)$ *of*

$$\mathbb{E}\bigg\{ \int_\tau^T e^{-\int_0^s \mu_F(u)du} C^0(s)\, R_c\bigg(C^0(s) \sup_{\tau \leq u \leq s} \bigg(\frac{l^*(u)}{C^0(u)}\bigg)\bigg)\, ds \,\bigg|\, \mathcal{F}_\tau \bigg\}$$
$$= e^{-\int_0^\tau \mu_F(u)du} \frac{C^0(\tau)}{f_C(\tau)} \mathbb{1}_{\{\tau<T\}} \quad (2.28)$$

for $\tau \in \mathcal{T}$*, and it is given by*

$$l^*(t) := -\frac{C^0(t)}{\xi^*(t)}. \quad (2.29)$$

Proof. With $\xi^*(t) = -\frac{C^0(t)}{l^*(t)}$ as in (2.29), it follows from (2.23) that

$$\begin{aligned} e^{-\int_0^\tau \mu_F(u)du}\frac{C^0(\tau)}{f_C(\tau)}\mathbb{1}_{\{\tau<T\}} &= \mathbb{E}\Big\{ \int_\tau^T e^{-\int_0^s \mu_F(u)du} C^0(s)\, R_c\Big(\frac{C^0(s)}{-\sup_{\tau\le u\le s}(-\frac{C^0(u)}{l^*(u)})}\Big)\, ds \,\Big|\, \mathcal{F}_\tau \Big\} \\ &= \mathbb{E}\Big\{ \int_\tau^T e^{-\int_0^s \mu_F(u)du} C^0(s)\, R_c\Big(\frac{C^0(s)}{\inf_{\tau\le u\le s}(\frac{C^0(u)}{l^*(u)})}\Big)\, ds \,\Big|\, \mathcal{F}_\tau \Big\} \quad (2.30) \\ &= \mathbb{E}\Big\{ \int_\tau^T e^{-\int_0^s \mu_F(u)du} C^0(s)\, R_c\Big(C^0(s) \sup_{\tau\le u\le s}\Big(\frac{l^*(u)}{C^0(u)}\Big)\Big)\, ds \,\Big|\, \mathcal{F}_\tau \Big\}. \end{aligned}$$

Finally, the upper right-continuity of $l^*(t)$ follows from that of $\xi^*(t)$ and from the continuity of $C^0(t)$. □

Notice that $l^*(t)$ may be found numerically from (2.28) by backward induction. In some cases, when $T = +\infty$, (2.28) has a closed form solution as in the case of a Cobb-Douglas production function.

We are now able to find the unique optimally controlled capacity plan for problem (2.7).

Definition 2.3.3. *For a given optional process $l(t)$, the capacity process that tracks l is defined as*

$$C^{(l)}(t) := C^0(t)\Big(y \vee \sup_{0\le u\le t}\Big(\frac{l(u)}{C^0(u)}\Big)\Big). \quad (2.31)$$

Theorem 2.3.1. *Let $l^*(t)$ be the unique solution of (2.28) and let $C^{(l^*)}$ be the capacity process that tracks l^*. Then the investment plan $\nu^{(l^*)}(t)$ that finances $C^{(l^*)}$, i.e.*

$$d\nu^{(l^*)}(t) = C^{(l^*)}(t)[\mu(t)dt - \sigma(t)dW(t)] + dC^{(l^*)}(t), \quad \text{with} \quad \nu^{(l^*)}(0) = 0,$$

is optimal for the firm's problem (2.7).

Proof. In order to prove that $C^{(l^*)}(t)$ is the optimal capacity, we only have to show that $C^{(l^*)}(t)$ solves the two first-order conditions of Theorem 2.2.1. In fact, for all $\tau \in \mathcal{T}$

$$\begin{aligned} &\mathbb{E}\Big\{ \int_\tau^T e^{-\int_0^s \mu_F(u)du} C^0(s)\, R_c\big(C^{(l^*)}(s)\big)\, ds \,\Big|\, \mathcal{F}_\tau \Big\} \\ &= \mathbb{E}\Big\{ \int_\tau^T e^{-\int_0^s \mu_F(u)du} C^0(s)\, R_c\Big(C^0(s)\Big(y \vee \sup_{0\le u\le s}\Big(\frac{l^*(u)}{C^0(u)}\Big)\Big)\Big)\, ds \,\Big|\, \mathcal{F}_\tau \Big\} \quad (2.32) \end{aligned}$$

$$\leq \mathbb{E}\bigg\{ \int_\tau^T e^{-\int_0^s \mu_F(u)du} C^0(s)\, R_c\bigg(C^0(s) \sup_{\tau \leq u \leq s} \bigg(\frac{l^*(u)}{C^0(u)}\bigg)\bigg)\, ds \,\bigg|\, \mathcal{F}_\tau \bigg\}$$
$$= e^{-\int_0^\tau \mu_F(u)du} \frac{C^0(\tau)}{f_C(\tau)} \mathbb{1}_{\{\tau < T\}}$$

where in the last step we have used (2.28). Notice that in (2.32) we have equality if and only if τ is a time of investment; that is a time of strict increase for $C^{(l^*)}$, i.e. $dC^{(l^*)}(\tau) > 0$. In fact, at such time, we have $C^{(l^*)}(t) = C^0(t) \sup_{\tau \leq u \leq t} \left(\frac{l^*(u)}{C^0(u)}\right)$ for $t \geq \tau$. Hence (2.11) and (2.12) hold (see also (2.8)) and so $\nu^{(l^*)}(t) \equiv \hat{\nu}(t)$. □

Remark 2.3.4. *Recall that $C^{y,\hat{\nu}}(t) = C^0(t)[y + \overline{\nu}^y(t)]$ (cf. 2.5) where $\overline{\nu}^y(t) = \int_{[0,t)} \frac{f_C(s)}{C^0(s)} d\hat{\nu}(s)$. Hence it follows from (2.31) with $l = l^*$ that*

$$\overline{\nu}^y(t) = \sup_{0 \leq u \leq t} \left(y \vee \frac{l^*(u)}{C^0(u)} \right) - y. \tag{2.33}$$

Therefore

$$\overline{\nu}^y(t) = \sup_{0 \leq u \leq t} \left(\frac{l^*(u) - yC^0(u)}{C^0(u)} \right) \vee 0. \tag{2.34}$$

2.4 Identifying the Base Capacity Process

In this Section we find the explicit link between our 'base capacity' approach and the variational approach in Chiarolla and Haussmann [20] based on the shadow value of installed capital, $v := \frac{\partial}{\partial y} V$ (see Section 2.6 for a generalization of [20] in the case of deterministic, time-dependent coefficients).

We make the following Markovian Assumption

Assumption 2.4.1. *$\mu_C(t)$, $\sigma_C(t)$, $f_C(t)$ and $\mu_F(t)$ are deterministic functions of $t \in [0,T]$.*

Define

$$\Gamma^\xi(t) \;:=\; \operatorname*{ess\,inf}_{t \leq \tau \leq T} \mathbb{E}\bigg\{ \int_t^\tau e^{-\int_0^u \mu_F(r)dr} C^0(u)\, R_c\bigg(-\frac{1}{\xi} C^0(u) \bigg)\, du + e^{-\int_0^\tau \mu_F(r)dr} C_0(\tau) \frac{1}{f_C(\tau)} \mathbb{1}_{\{\tau < T\}} \,\bigg|\, \mathcal{F}_t \bigg\}, \tag{2.35}$$

for $\xi \in \mathbb{R}$ and $t \in [0,T]$. Then [6], Lemma 4.12 and Lemma 4.13 guarantee that

- the stopping time

$$\tau_t^{\xi} := \inf\left\{s \in [t,T) : \Gamma^{\xi}(s) = e^{-\int_0^s \mu_F(r)dr} C^0(s)\frac{1}{f_C(s)}\right\} \wedge T \tag{2.36}$$

  is optimal for (2.35);

- the optional process

$$\xi^*(t) := \sup\left\{\xi \in \mathbb{R} : \Gamma^{\xi}(t) = e^{-\int_0^t \mu_F(r)dr} C^0(t)\frac{1}{f_C(t)}\right\}, \quad t \in [0,T), \tag{2.37}$$

  is the unique solution of the representation problem (2.23).

We now make an absolutely continuous change of probability measure. In fact, consider the exponential martingale $\mathcal{M}_t(s) := e^{\int_t^s \sigma(u)dW(u) - \frac{1}{2}\int_t^s \sigma^2(u)du}$, $t \in [0,T]$ and $t \le s \le T$, and define the probability measure $\widetilde{\mathbb{P}}_t$ by $\widetilde{\mathbb{P}}_t(A) := \mathbb{E}\left\{\mathcal{M}_t(T)\mathbb{1}_A\right\}$, for $A \in \widetilde{\mathcal{F}}_{t,T} := \sigma\{W(u) - W(t), t \le u \le T\}$. Then the Radon-Nikodym derivative is

$$\frac{d\widetilde{\mathbb{P}}_t}{d\mathbb{P}}\Big|_{\widetilde{\mathcal{F}}_{t,s}} = \mathcal{M}_t(s), \qquad s \in [0,T], \tag{2.38}$$

and the process $\widetilde{W}^t(s) := W(s) - W(t) - \int_t^s \sigma(u)du$ is a standard Brownian motion under $\widetilde{\mathbb{P}}_t$. We denote by $\widetilde{\mathbb{E}}_t\left\{\cdot\right\}$ the expectation w.r.t. $\widetilde{\mathbb{P}}_t$.

Hence under $\widetilde{\mathbb{P}} := \widetilde{\mathbb{P}}_0$ the process $e^{\int_0^t \mu_F(r)dr}\frac{\Gamma^{\xi}(t)}{C^0(t)}$ becomes

$$\widetilde{\Gamma}^{\xi}(t) := \operatorname*{ess\,inf}_{t\le\tau\le T} \widetilde{\mathbb{E}}\left\{\int_t^{\tau} e^{-\int_t^u \overline{\mu}(r)dr} R_c\left(-\frac{1}{\xi}C^0(u)\right)du + e^{-\int_t^{\tau}\overline{\mu}(r)dr}\frac{1}{f_C(\tau)}\mathbb{1}_{\{\tau<T\}}\Big| \mathcal{F}_t\right\}, \tag{2.39}$$

and so the optional process $\xi^*(t)$ (cf. 2.37) may be written as

$$\xi^*(t) = \sup\left\{\xi \in \mathbb{R} : \widetilde{\Gamma}^{\xi}(t) = \frac{1}{f_C(t)}\right\}. \tag{2.40}$$

In fact,

$$\xi^*(t) = \sup\Bigg\{\xi \in \mathbb{R} : \operatorname*{ess\,inf}_{t\le\tau\le T} \mathbb{E}\bigg\{\int_t^{\tau} e^{-\int_t^u \mu_F(r)dr}\frac{C^0(u)}{C^0(t)} R_c\left(-\frac{1}{\xi}C^0(u)\right)du$$
$$+ e^{-\int_t^{\tau}\mu_F(r)dr}\frac{C_0(\tau)}{C^0(t)}\frac{1}{f_C(\tau)}\mathbb{1}_{\{\tau<T\}}\Big| \mathcal{F}_t\bigg\} = \frac{1}{f_C(t)}\Bigg\},$$

and by the continuous time Bayes' Rule (see e.g. [44]), we obtain

$$\begin{aligned}&\mathbb{E}\Big\{ \int_t^{\tau} e^{-\int_t^u \mu_F(r)dr} \frac{C^0(u)}{C^0(t)} R_c\Big(-\frac{1}{\xi}C^0(u)\Big)\, du \, + e^{-\int_t^{\tau} \mu_F(r)dr} \frac{C_0(\tau)}{C^0(t)} \frac{1}{f_C(\tau)} \mathbb{1}_{\{\tau<T\}}\Big| \mathcal{F}_t \Big\} \\ &\quad = \widetilde{\mathbb{E}}\Big\{ \int_t^{\tau} e^{-\int_t^u \overline{\mu}(r)dr} R_c\Big(-\frac{1}{\xi}C^0(u)\Big)\, du \, + e^{-\int_t^{\tau} \overline{\mu}(r)dr} \frac{1}{f_C(\tau)} \mathbb{1}_{\{\tau<T\}}\Big| \mathcal{F}_t \Big\}\end{aligned}$$

with $\overline{\mu}(t) := \mu_C(t) + \mu_F(t)$. Now (2.40) follows from (2.39).

For an appropriate value of ξ, we are now able to link $\widetilde{\Gamma}^{\xi}(t)$ to $v(t,y)$, the shadow value of installed capital (cf. (2.81)).

Proposition 2.4.2. *With $\widetilde{\Gamma}^{\xi}(t)$ as in (2.39) and $v(t,y)$ as in (2.81), that is*

$$v(t,y) = \inf_{t\le\tau\le T} \widetilde{\mathbb{E}}_t\Big\{ \int_t^{\tau} e^{-\int_t^u \overline{\mu}(r)dr} R_c\left(Y^{t,y}(u)\right) du + e^{-\int_t^{\tau} \overline{\mu}(r)dr} \frac{1}{f_C(\tau)} \mathbb{1}_{\{\tau<T\}} \Big\}, \tag{2.41}$$

we have

$$\widetilde{\Gamma}^{-\frac{1}{y}}(t) = v\left(t, yC^0(t)\right). \tag{2.42}$$

Proof. The proof borrows arguments from [18], Theorem 4.1. As in Section 2.6 we set $Y^{t,y}(s) := y\widetilde{C}^t(s) = yC^{t,1,0}(s)$ for $s \ge t$ (cf. (2.69)). Then, for $t \in [0,T)$ and $\tau \in [t,T]$, notice that

$$\begin{aligned}&\widetilde{\mathbb{E}}\Big\{ \int_t^{\tau} e^{-\int_t^u \overline{\mu}(r)dr} R_c\left(yC^0(u)\right)\, du \, + e^{-\int_t^{\tau} \overline{\mu}(r)dr} \frac{1}{f_C(\tau)} \mathbb{1}_{\{\tau<T\}}\Big| \mathcal{F}_t \Big\} \\ &\quad = \widetilde{\mathbb{E}}\Big\{ \int_t^{\tau} e^{-\int_t^u \overline{\mu}(r)dr} R_c\left(Y^{t,yC^0(t)}(u)\right)\, du \, + e^{-\int_t^{\tau} \overline{\mu}(r)dr} \frac{1}{f_C(\tau)} \mathbb{1}_{\{\tau<T\}}\Big| \mathcal{F}_t \Big\} \\ &\quad = \widetilde{\mathbb{E}}\Big\{ \int_t^{\tau} e^{-\int_t^u \overline{\mu}(r)dr} R_c\left(yC^0(t)\widetilde{C}^t(u)\right)\, du \, + e^{-\int_t^{\tau} \overline{\mu}(r)dr} \frac{1}{f_C(\tau)} \mathbb{1}_{\{\tau<T\}}\Big| \mathcal{F}_t \Big\}.\end{aligned} \tag{2.43}$$

In order to take care of the conditioning, it is convenient to work on the canonical probability space $\left(\overline{\Omega}, \overline{\mathbb{P}}\right)$, where $\overline{\mathbb{P}}$ is the Wiener measure on $\overline{\Omega} := \mathcal{C}_0\left([0,T]\right)$, the space of all continuous functions on $[0,T]$ which are zero at $t=0$. In fact, we may take $\widetilde{W}^0(\cdot) = \overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2)$ where $\overline{\omega}_1 = \left\{\widetilde{W}^0(v), 0 \le v \le t\right\}$ and $\overline{\omega}_2 = \left\{\widetilde{W}^0(v) - \widetilde{W}^0(t), t \le v \le T\right\} = \left\{\widetilde{W}'(v), 0 \le v \le T-t\right\}$. Since Brownian increments are independent then $\overline{\mathbb{P}}$ is a product-measure on $\mathcal{C}_0\left([0,T]\right) = \mathcal{C}_0\left([0,t]\right) \times \mathcal{C}_0\left([0,T-t]\right)$ and $\tau \ge t$ $\widetilde{\mathbb{P}}$-a.s. may be written in the form $\tau(\overline{\omega}_1,\overline{\omega}_2) = t + \tau'_{\overline{\omega}_1}(\overline{\omega}_2)$ with $\tau'_{\overline{\omega}_1}(\cdot)$ a $\left\{\mathcal{F}_v^{\widetilde{W}'}\right\}_{0\leq v\leq T-t}$-stopping time for every $\overline{\omega}_1$. Then, since $\widetilde{C}^t(\cdot)$ is independent of $\mathcal{F}_t$, the last conditional expectation in (2.43) is equal to

$$\begin{aligned}
&\widetilde{\mathbb{E}}_{\overline{\omega}_2}\left\{ \int_t^{t+\tau'_{\overline{\omega}_1}} e^{-\int_t^u \overline{\mu}(r)dr} R_c\left(yC^0(t)\widetilde{C}^t(u)\right) du + e^{-\int_t^{t+\tau'_{\overline{\omega}_1}} \overline{\mu}(r)dr} \frac{1}{f_C(t+\tau'_{\overline{\omega}_1})} \mathbb{1}_{\{\tau'_{\overline{\omega}_1}+t<T\}} \right\} \\
&= \widetilde{\mathbb{E}}_t\left\{ \int_t^{\tau} e^{-\int_t^u \overline{\mu}(r)dr} R_c\left(Y^{t,yC^0(t)}(u)\right) du + e^{-\int_t^{\tau} \overline{\mu}(r)dr} \frac{1}{f_C(\tau)} \mathbb{1}_{\{\tau<T\}} \right\}
\end{aligned} \tag{2.44}$$

where $\widetilde{\mathbb{E}}_{\overline{\omega}_2}\{\cdot\}$ denotes expectation over $\overline{\omega}_2$ or $\widetilde{W}'$. Hence (2.42) follows from (2.39) and (2.41). □

Notice that the process $\xi^*(t)$ is negative for all $t \in [0,T)$ a.s. (cf. Lemma 2.23), then for $t \in [0,T)$ it must be $\xi^*(t) = \sup\left\{\xi < 0 : \Gamma^{\xi}(t) = e^{-rt}C^0(t)\frac{1}{f_C(t)}\right\}$ or equivalently $\xi^*(t) = \sup\left\{\xi < 0 : \widetilde{\Gamma}^{\xi}(t) = \frac{1}{f_C(t)}\right\}$.

The following Proposition provides another representation of the base capacity $l^*(t) := -\frac{C^0(t)}{\xi^*(t)}$ (cf. (2.29)).

Proposition 2.4.3. *The base capacity $l^*(t)$, unique solution of (2.28), admits the representation*

$$l^*(t) = \sup\left\{y\,C^0(t) > 0 : v\left(t, y\,C^0(t)\right) = \frac{1}{f_C(t)}\right\}. \tag{2.45}$$

Proof. For $t \in [0,T)$ and $y > 0$ we have

$$\begin{aligned}
l^*(t) &:= -\frac{C^0(t)}{\xi^*(t)} = -\frac{C^0(t)}{\sup\left\{\xi < 0 : \widetilde{\Gamma}^{\xi}(t) = \frac{1}{f_C(t)}\right\}} = -\frac{C^0(t)}{\sup\left\{-\frac{1}{y} < 0 : \widetilde{\Gamma}^{-\frac{1}{y}}(t) = \frac{1}{f_C(t)}\right\}} \\
&= \frac{C^0(t)}{-\sup\left\{-\frac{1}{y} < 0 : \widetilde{\Gamma}^{-\frac{1}{y}}(t) = \frac{1}{f_C(t)}\right\}} = \frac{C^0(t)}{\inf\left\{\frac{1}{y} > 0 : \widetilde{\Gamma}^{-\frac{1}{y}}(t) = \frac{1}{f_C(t)}\right\}} \\
&= C^0(t)\sup\left\{y > 0 : \widetilde{\Gamma}^{-\frac{1}{y}}(t) = \frac{1}{f_C(t)}\right\} = \sup\left\{yC^0(t) > 0 : \widetilde{\Gamma}^{-\frac{1}{y}}(t) = \frac{1}{f_C(t)}\right\} \\
&= \sup\left\{y\,C^0(t) > 0 : v\left(t, y\,C^0(t)\right) = \frac{1}{f_C(t)}\right\}
\end{aligned}$$

where the last equality follows from Proposition 2.4.2. □

Notice that $v(t,y) \le \frac{1}{f_C(t)}$ for all $t \in [0,T)$ and $y > 0$. As in [20], (3.19) introduce the *Continuation Region* (or 'no-action region') of problem (2.41)

$$D := \left\{ (t,y) \in [0,T) \times (0,\infty) : v(t,y) < \frac{1}{f_C(t)} \right\}. \tag{2.46}$$

This is a Borel set and, roughly speaking, D is the region where it is not profitable to invest, since the expected marginal profit is strictly less than the capital's replacement cost. Similarly its complement is the *Stopping Region* (or 'action region'), i.e.

$$D^c := \left\{ (t,y) \in [0,T) \times (0,\infty) : v(t,y) = \frac{1}{f_C(t)} \right\}, \tag{2.47}$$

is the region where it is profitable to invest immediately.

The boundary between these two regions is the free boundary $\hat{y}(t)$ of the optimal stopping problem (2.41).

Theorem 2.4.1. *The base capacity process $l^*(t)$, unique solution of (2.28), is deterministic and coincides with the free boundary $\hat{y}(t)$ associated to the optimal stopping problem (2.41). Hence*

$$l^*(t) = \sup\left\{ z > 0 : v(t,z) = \frac{1}{f_C(t)} \right\} \qquad \textit{for } t \in [0,T). \tag{2.48}$$

Proof. Recall (2.45). Fix $t \in [0,T)$ and set

$$z(\omega,y) := yC^0(\omega,t).$$

It follows that

$$\begin{aligned} \left\{ yC^0(\omega,t) > 0 : v\left(t, yC^0(\omega,t)\right) = \frac{1}{f_C(t)} \right\} &= \left\{ z(\omega,y) > 0 : v(t,z(\omega,y)) = \frac{1}{f_C(t)} \right\} \\ &\subseteq \left\{ z > 0 : v(t,z) = \frac{1}{f_C(t)} \right\} \end{aligned}$$

for a.e. $\omega \in \Omega$ and $y > 0$, hence the inclusion holds a.s. for all $y > 0$.

To show the reverse inclusion, fix $\omega \in \Omega$ and $t \in [0,T)$, then for $z > 0$ define

$$y(\omega,z) := \frac{z}{C^0(\omega,t)}$$

so that that every $z > 0$ may be written as $z = y(z,\omega)C^0(t,\omega)$. Then

$$\begin{aligned}\left\{ z > 0 : v(t,z) = \frac{1}{f_C(t)} \right\} &= \left\{ y(\omega,z)C^0(\omega,t) > 0 : v\left(t, y(\omega,z)C^0(\omega,t)\right) = \frac{1}{f_C(t)} \right\} \\ &\subseteq \left\{ yC^0(\omega,t) > 0 : v\left(t, yC^0(\omega,t)\right) = \frac{1}{f_C(t)} \right\}.\end{aligned}$$

This inclusion holds for a.e. $\omega \in \Omega$, hence a.s. Hence, it holds $\widetilde{\mathbb{P}}$-a.s. that

$$\sup\left\{ yC^0(\omega,t) > 0 : v\left(t, yC^0(\omega,t)\right) = \frac{1}{f_C(t)} \right\} = \sup\left\{ z > 0 : v(t,z) = \frac{1}{f_C(t)} \right\} \tag{2.49}$$

and $l^*(t)$ is deterministic (cf. (2.45)). Now the right-hand side of (2.49) (cf. [20], (3.13)) identifies $l^*(t)$ with the free boundary $\hat{y}(t)$ of problem (2.41). □

Since $\hat{y}(t)$ coincides with $l^*(t)$, equation (2.28) provides an integral equation for the free boundary $\hat{y}(t)$.

Theorem 2.4.2. *The free boundary $\hat{y}(t)$ of problem (2.41) is the unique upper right-continuous solution of the integral equation*

$$\widetilde{\mathbb{E}}\left\{ \int_0^{T-t} e^{-\int_t^{v+t} \overline{\mu}(r)dr} R_c\left(\sup_{0 \le u' \le v} \left(\hat{y}(u'+t) \frac{C^0(v+t)}{C^0(u'+t)} \right) \right) dv \right\} = \frac{1}{f_C(t)}, \quad t \in [0,T). \tag{2.50}$$

Proof. Fix $t \in [0,T)$. Set $\tau = t$ and recall that $l^*(t) = \hat{y}(t)$. Then write (2.28) under $\widetilde{\mathbb{P}}$ and apply the continuous time Bayes' Rule to obtain

$$\widetilde{\mathbb{E}}\left\{ \int_0^{T-t} e^{-\int_t^{v+t} \overline{\mu}(r)dr} R_c\left(\sup_{0 \le u' \le v} \left(\hat{y}(u'+t) \frac{C^0(v+t)}{C^0(u'+t)} \right) \right) dv \,\Big|\, \mathcal{F}_t \right\} = \frac{1}{f_C(t)}.$$

Now (2.50) follows since $\frac{C^0(v+t)}{C^0(u'+t)}$, $v > u' \ge 0$, is independent of $\mathcal{F}_t$. □

We now aim to obtain paths' regularity of the free boundary $\hat{y}(t)$ of problem (2.41) from the fact that it coincides with the base capacity process $l^*(t)$. Recall that $l^*(t)$ has upper right-continuous paths on $[0,T)$ and satisfies $l^*(t) > 0$ on $[0,T)$ a.s. (cf. Lemma 2.3.1).

As in [20], Section 4, we make the following assumptions

Assumption 2.4.4.

1. $R(C) = \frac{1}{\alpha}C^{\alpha}$ *with* $\alpha \in (0,1)$ *(i.e. Cobb-Douglas production function);*

2. $\mu_C(t) \equiv \mu_C, \quad \sigma_C(t) \equiv \sigma_C, \quad \mu_F(t) \equiv \mu_F, \quad f_C(t) \equiv f_C.$

Remark 2.4.5. *Notice that under Assumption 2.4.4, part 2, the process* $\frac{C^0(v+t)}{C^0(u'+t)}$ *has the same law as* $\frac{C^0(v)}{C^0(u')}$*. Hence, the integral equation (2.50) takes the form*

$$\widetilde{\mathbb{E}}\left\{ \int_0^{T-t} e^{-\bar{\mu}v} R_c\left(\sup_{0\le u'\le v} \left(\hat{y}(u'+t)\frac{C^0(v)}{C^0(u')}\right)\right) dv \right\} = \frac{1}{f_C(t)}. \tag{2.51}$$

Under Assumption 2.4.4, the properties of the free boundary obtained in [20] hold. The novelty is the continuity of $\hat{y}(t)$ which we prove thanks to its identification with $l^*(t)$.

Theorem 2.4.3. *Let Assumptions 2.4.4 hold and recall that* $\hat{y}(t)$ *is the function representing the free boundary between the continuation region* D *and the stopping region* D^c*. Then we have*

1. $\hat{y}(t) > 0$ *on* $t \in [0,T)$*;*

2. $\hat{y}(T-) = 0$*;*

3. $\hat{y}(t)$ *is nonincreasing for* $t \in [0,T)$*;*

4. $\hat{y}(t)$ *is left-continuous for* $t \in [0,T)$*;*

5. $\hat{y}(t)$ *is continuous on* $t \in [0,T)$*.*

Proof. Property 1 follows from the analogous property of $l^*(t)$ (see Lemma 2.3.1). For the proof of properties 2, 3 and 4 see [20], Proposition 4.3. To prove property 5 recall that $l^*(t)$ has upper right-continuous paths (see Lemma 2.3.1), but $l^*(t) = \hat{y}(t)$ admits right-hand limits thanks to property 3, then it is right-continuous, i.e.

$$l^*(t) = \limsup_{s\downarrow t} l^*(s) = \lim_{s\downarrow t} l^*(s).$$

Hence the continuity of $\hat{y}(t)$ follows from property 4. □

Notice that property 5 was assumed in [20] (see [20], Assumption-[Cfb]). Chiarolla and Haussmann in [20] stressed that the data regularity of problem (2.41) gave no indication that continuity of the free boundary should fail, but they were unable to prove it, as they could not show its right-continuity. In fact arguments similar to those used in [38] for the free boundary of the American put did not apply being the value function of their stopping problem an inf rather than a sup as in the option problem.

In this Section we have linked the Bank-El Karoui's probabilistic approach to the variational approach followed by Chiarolla and Haussmann in [18] and [20] for an irreversible investment problem similar to (2.7). Under Markovian assumptions we have proved that the base capacity process $l^*(t)$ is a deterministic process and it coincides with the free-boundary of the optimal stopping problem (2.41). Moreover, in the Cobb-Douglas case, we have obtained its continuity so to remove Assumption-[Cfb] in [20]. We have characterized the free boundary as the unique solution of an integral equation based on the stochastic Representation Theorem of [6]. Even under Assumption 2.4.4, the integral equation for the free boundary (2.50) cannot be analitically solved when the time horizon is finite. However it is possible to find a curve bounding the free boundary from above. In Section 2.5 we shall see that, instead, when $T = +\infty$ (as in H. Pham [55]) the free boundary is a constant whose value we find explicitly by applying Proposition 2.5.1.

Recall that $T < +\infty$.

Proposition 2.4.6. *Under Assumption 2.4.4 the boundary $\hat{y}(t)$ of the continuation region D satisfies*

$$\hat{y}(t) \leq \left[f_C \left(\frac{1 - e^{-(\mu_F + \alpha\mu_C + \frac{1}{2}\alpha(1-\alpha)\sigma_C^2)(T-t)}}{\mu_F + \alpha\mu_C + \frac{1}{2}\alpha(1-\alpha)\sigma_C^2} \right) \right]^{\frac{1}{1-\alpha}} =: y^*(t), \tag{2.52}$$

for every $t \in [0, T)$.

Proof. Fix $t \in [0, T)$. The representation formula (2.28) for $\tau = t$ and in the Cobb-Douglas case becomes

$$e^{-\mu_F t} \frac{1}{f_C} = \mathbb{E}\bigg\{ \int_t^T e^{-\mu_F s} \frac{C^0(s)}{C^0(t)} \left(\sup_{t \leq u \leq s} \left(C^0(s) \frac{l^*(u)}{C^0(u)} \right) \right)^{\alpha - 1} ds \, \bigg| \, \mathcal{F}_t \bigg\}. \tag{2.53}$$

Set $\widetilde{\mu}_C := \mu_C + \frac{1}{2}\sigma_C^2$, then the right-hand side of (2.53) gives

$$\begin{aligned}
&\mathbb{E}\Big\{ \int_t^T e^{-\mu_F s}\frac{C^0(s)}{C^0(t)} \left(\sup_{t\le u\le s} \left(C^0(s)\frac{l^*(u)}{C^0(u)}\right)\right)^{\alpha-1} ds \,\Big|\, \mathcal{F}_t \Big\} \qquad (2.54)\\
&= \mathbb{E}\Big\{ \int_t^T e^{-\mu_F s}\, e^{-\widetilde{\mu}_C(s-t)+\sigma_C(W(s)-W(t))} \\
&\qquad\qquad \times \inf_{t\le u\le s}\left([l^*(u)]^{\alpha-1}\, e^{(\alpha-1)(-\widetilde{\mu}_C(s-u)+\sigma_C(W(s)-W(u)))}\right) ds \Big|\, \mathcal{F}_t \Big\}\\
&\le \mathbb{E}\Big\{ \int_t^T e^{-\mu_F s}\, e^{-\widetilde{\mu}_C(s-t)+\sigma_C(W(s)-W(t))}\,[l^*(t)]^{\alpha-1}\, e^{(\alpha-1)(-\widetilde{\mu}_C(s-t)+\sigma_C(W(s)-W(t)))}\, ds \Big|\, \mathcal{F}_t \Big\}\\
&= e^{-\mu_F t}\,[l^*(t)]^{\alpha-1}\, \mathbb{E}\Big\{ \int_0^{T-t} e^{-\mu_F v}\, e^{-\widetilde{\mu}_C v+\sigma_C(W(v+t)-W(t))}\\
&\qquad\qquad \times e^{(\alpha-1)(-\widetilde{\mu}_C v+\sigma_C(W(v+t)-W(t)))}\, dv \Big|\, \mathcal{F}_t \Big\}
\end{aligned}$$

Since the Brownian increments in the integral above are independent of $\mathcal{F}_t$, we obtain

$$\begin{aligned}
&\mathbb{E}\Big\{ \int_t^T e^{-\mu_F s}\frac{C^0(s)}{C^0(t)} \left(\sup_{t\le u\le s} \left(C^0(s)\frac{l^*(u)}{C^0(u)}\right)\right)^{\alpha-1} ds \,\Big|\, \mathcal{F}_t \Big\}\\
&\le e^{-\mu_F t}\,[l^*(t)]^{\alpha-1}\, \mathbb{E}\Big\{ \int_0^{T-t} e^{-\mu_F v}\, e^{-\widetilde{\mu}_C v+\sigma_C(W(v+t)-W(t))}\\
&\qquad\qquad \times e^{(\alpha-1)(-\widetilde{\mu}_C v+\sigma_C(W(v+t)-W(t)))} \Big\}\, dv \qquad (2.55)\\
&= e^{-\mu_F t}\,[l^*(t)]^{\alpha-1} \int_0^{T-t} e^{-\mu_F v}\, \mathbb{E}\left\{ e^{\alpha(-\widetilde{\mu}_C v+\sigma_C(W(v+t)-W(t)))}\right\} dv\\
&= e^{-\mu_F t}\,[l^*(t)]^{\alpha-1} \int_0^{T-t} e^{-\mu_F v} e^{-\alpha\widetilde{\mu}_C v}\, \mathbb{E}\left\{ e^{\alpha\sigma_C W(v)}\right\} dv\\
&= e^{-\mu_F t}\,[l^*(t)]^{\alpha-1} \int_0^{T-t} e^{-\mu_F v} e^{-\alpha\widetilde{\mu}_C v}\, e^{\frac{1}{2}\alpha^2\sigma_C^2\, v}\, dv.
\end{aligned}$$

Notice that

$$\mu_F + \alpha\widetilde{\mu}_C - \frac{1}{2}\alpha^2\sigma_C^2 = \mu_F + \alpha\mu_C + \frac{1}{2}\alpha(1-\alpha)\sigma_C^2 > 0,$$

hence (2.53) and (2.55) imply that

$$e^{-\mu_F t}\frac{1}{f_C} \;\le\; e^{-\mu_F t}\,[l^*(t)]^{\alpha-1} \int_0^{T-t} e^{-\left(\mu_F+\alpha\mu_C+\frac{1}{2}\alpha(1-\alpha)\sigma_C^2\right)v}\, dv \qquad (2.56)$$

$$= \quad e^{-\mu_F t}\,[l^*(t)]^{\alpha-1}\left(\frac{1-e^{-\left(\mu_F+\alpha\mu_C+\frac{1}{2}\alpha(1-\alpha)\sigma_C^2\right)(T-t)}}{\mu_F+\alpha\mu_C+\frac{1}{2}\alpha(1-\alpha)\sigma_C^2}\right).$$

Now 2.56 gives

$$[l^*(t)]^{1-\alpha} \le f_C\left(\frac{1-e^{-\left(\mu_F+\alpha\mu_C+\frac{1}{2}\alpha(1-\alpha)\sigma_C^2\right)(T-t)}}{\mu_F+\alpha\mu_C+\frac{1}{2}\alpha(1-\alpha)\sigma_C^2}\right) =: y^*(t)^{1-\alpha}, \tag{2.57}$$

and (2.52) follows from the identification of $l^*(\cdot)$ with $\hat{y}(\cdot)$ (cf. Theorem 2.4.1). □

Remark 2.4.7. *Notice that the curve $y^*(t)$ is exactly what in [18] was incorrectly identified as the free boundary between the 'action' and the 'no-action' regions. In [20] the authors characterized the free boundary $\widehat{y}(t)$ as the unique solution of a nonlinear integral equation (see [20], Theorem 4.8). Then, by using a discrete approximation of such integral equation, they showed that $\widehat{y}(t) \le y^*(t)$, for $t \le T$. That is exactly what we prove here in Proposition 2.4.6.*

Remark 2.4.8. *Notice that the arguments in the proof of Proposition 2.4.6 apply even under the more general conditions of Assumption 2.4.1. That is, under deterministic, time-dependent coefficients we have*

$$\hat{y}(t) \le \left[f_C(t)\int_0^{T-t} e^{-\int_t^{v+t}\left(\mu_F(s)+\alpha\mu_C(s)+\frac{1}{2}\alpha(1-\alpha)\sigma_C^2(s)\right)ds}dv\right]^{\frac{1}{1-\alpha}}, \quad \forall t\in[0,T).$$

2.5 Explicit Results when $T = +\infty$

In this Section, with $T = +\infty$ and under Assumption 2.4.4, we set $f_C = 1$ in order to compare our finding with the results in H. Pham [55]. As one would expect, when the time horizon is infinite, the free boundary is a point. That is what we show below.

Proposition 2.5.1. *The unique solution of the representation problem (2.28) is given by*

$$l^*(t) = \left[\frac{2}{2\mu_F-\sigma_C^2\beta_- - \alpha\sigma_C^2(1+\beta_+)}\right]^{\frac{1}{1-\alpha}} =: a \tag{2.58}$$

where $\beta_\pm$ are, respectively, the positive and negative roots of $\frac{1}{2}\sigma_C^2x^2+\widetilde{\mu}_Cx-\mu_F=0$ with $\widetilde{\mu}_C := \mu_C+\frac{1}{2}\sigma_C^2$.

Hence (cf. Definition 2.3.3 and Theorem 2.3.1) the optimal capacity is given by

$$C^{y,\hat{\nu}}(t) = C^{(a)}(t) \equiv C^0(t)\Big(y \vee \sup_{0\le u\le t}\Big(\frac{a}{C^0(u)}\Big)\Big). \tag{2.59}$$

Proof. We make the *ansatz* that $l^*(t) \equiv a$ for all $t \in [0,\infty)$ and we plug it into the left-hand side of (2.28) to obtain

$$\begin{aligned}
&a^{\alpha-1}\,\mathbb{E}\bigg\{\int_\tau^\infty e^{-\mu_F s}\frac{C^0(s)}{C^0(\tau)}\left[\sup_{\tau\le u\le s}\left(\frac{C^0(s)}{C^0(u)}\right)\right]^{\alpha-1} ds\,\bigg|\,\mathcal{F}_\tau\bigg\}\\
&= a^{\alpha-1}\,\mathbb{E}\bigg\{\int_\tau^\infty e^{-\mu_F s}\frac{C^0(s)}{C^0(\tau)}\inf_{\tau\le u\le s}\left(\left[\frac{C^0(s)}{C^0(u)}\right]^{\alpha-1}\right) ds\,\bigg|\,\mathcal{F}_\tau\bigg\}\\
&= a^{\alpha-1}\,\mathbb{E}\bigg\{\int_\tau^\infty e^{-\mu_F s}e^{\sigma_C(W(s)-W(\tau))-\widetilde{\mu}_C(s-\tau)}\\
&\qquad\times\inf_{0\le u'\le s-\tau}\left[e^{\sigma_C(W(s)-W(u'+\tau))-\widetilde{\mu}_C(s-u'-\tau)}\right]^{(\alpha-1)} ds\,\bigg|\,\mathcal{F}_\tau\bigg\}\\
&= a^{\alpha-1}e^{-\mu_F\tau}\,\mathbb{E}\bigg\{\int_0^\infty e^{-\mu_F v}e^{\sigma_C W(v)-\widetilde{\mu}_C v}\inf_{0\le u'\le v}\left(e^{(\alpha-1)(\sigma_C(W(v)-W(u'))-\widetilde{\mu}_C(v-u'))}\right) dv\bigg\}
\end{aligned} \tag{2.60}$$

since the Brownian increments are independent of $\mathcal{F}_\tau$.

Define now $Y(v) := \widetilde{\mu}_C v - \sigma_C W(v)$, $\underline{Y}(v) := \inf_{0\le u'<v} Y(u')$ and $\overline{Y}(v) := \sup_{0\le u'<v} Y(u')$, then

$$\begin{aligned}
&a^{\alpha-1}e^{-\mu_F\tau}\,\mathbb{E}\bigg\{\int_0^\infty e^{-\mu_F v}e^{\sigma_C W(v)-\widetilde{\mu}_C v}\inf_{0\le u'\le v}\left(e^{(\alpha-1)(\sigma_C(W(v)-W(u'))-\widetilde{\mu}_C(v-u'))}\right) dv\bigg\}\\
&= a^{\alpha-1}e^{-\mu_F\tau}\,\mathbb{E}\bigg\{\int_0^\infty e^{-\mu_F v}e^{-\alpha(\widetilde{\mu}_C v-\sigma_C W(v))}\,e^{\sup_{0\le u'\le v}[(\alpha-1)(\widetilde{\mu}_C u'-\sigma_C W(u'))]}\,dv\bigg\}\\
&= a^{\alpha-1}e^{-\mu_F\tau}\,\mathbb{E}\bigg\{\int_0^\infty e^{-\mu_F v}e^{-\alpha Y(v)}e^{(\alpha-1)\overline{Y}(v)}dv\bigg\}\\
&= \frac{1}{\mu_F}a^{\alpha-1}e^{-\mu_F\tau}\,\mathbb{E}\bigg\{\int_0^\infty \mu_F\,e^{-\mu_F v}e^{-\alpha\left(Y(v)-\overline{Y}(v)\right)}e^{-\overline{Y}(v)}\,dv\bigg\}\\
&= \frac{1}{\mu_F}a^{\alpha-1}e^{-\mu_F\tau}\,\mathbb{E}\left\{e^{-\alpha\left(Y(\tau(\mu_F))-\overline{Y}(\tau(\mu_F))\right)}e^{-\overline{Y}(\tau(\mu_F))}\right\}
\end{aligned} \tag{2.61}$$

where $\tau(\mu_F)$ denotes an independent exponential distributed random time.

Using the Excursion Theory for Levy processes (cf. [11]), $Y - \overline{Y}$ is independent of $\overline{Y}$, and by the Duality Theorem, $Y - \overline{Y}$ has the same distribution of $\underline{Y}$. Hence from (2.61) we obtain

$$\begin{aligned}&\frac{1}{\mu_F} a^{\alpha-1} e^{-\mu_F \tau}\, \mathbb{E}\left\{ e^{-\alpha\left(Y(\tau(\mu_F)) - \overline{Y}(\tau(\mu_F))\right)} e^{-\overline{Y}(\tau(\mu_F))} \right\} \\ &\quad = \frac{1}{\mu_F} a^{\alpha-1} e^{-\mu_F \tau}\, \mathbb{E}\left\{ e^{-\alpha \underline{Y}(\tau(\mu_F))} \right\} \mathbb{E}\left\{ e^{-\overline{Y}(\tau(\mu_F))} \right\}. \end{aligned} \tag{2.62}$$

It is well known that for a Brownian motion with drift

$$\mathbb{E}\left\{ e^{z\overline{Y}(\tau(\mu_F))} \right\} = \frac{\beta_+}{\beta_+ - z} \quad \text{and} \quad \mathbb{E}\left\{ e^{z\underline{Y}(\tau(\mu_F))} \right\} = \frac{\beta_-}{\beta_- - z},$$

if β_+ and β_- are, respectively, the positive and negative roots of $\frac{1}{2}\sigma_C^2 x^2 + \widetilde{\mu}_C x - \mu_F = 0$, i.e.

$$\beta_\pm = -\frac{\widetilde{\mu}_C}{\sigma_C^2} \pm \sqrt{\left(\frac{\widetilde{\mu}_C}{\sigma_C^2}\right)^2 + \frac{2\mu_F}{\sigma_C^2}}.$$

Hence (cf. (2.28))

$$\begin{aligned} e^{-\mu_F \tau} &= \mathbb{E}\left\{ \int_\tau^\infty e^{-\mu_F s} \frac{C^0(s)}{C^0(\tau)} \left[C^0(s) \sup_{\tau \le u \le s} \left(\frac{l^*(u)}{C^0(u)} \right) \right]^{\alpha-1} ds \,\Big|\, \mathcal{F}_\tau \right\} \\ &= \frac{1}{\mu_F} a^{\alpha-1} e^{-\mu_F \tau} \mathbb{E}\left\{ e^{-\alpha \underline{Y}(\tau(\mu_F))} \right\} \mathbb{E}\left\{ e^{-\overline{Y}(\tau(\mu_F))} \right\} \\ &= \frac{1}{\mu_F} a^{\alpha-1} e^{-\mu_F \tau} \frac{\beta_+ \beta_-}{(1+\beta_+)(\alpha+\beta_-)}. \end{aligned} \tag{2.63}$$

Then, we solve for a and we obtain

$$a^{\alpha-1} = \left(\frac{\mu_F (1+\beta_+)(\alpha+\beta_-)}{\beta_+ \beta_-} \right),$$

which may also be written as

$$a = \left(\frac{2}{2\mu_F - \sigma_C^2 \beta_- - \alpha \sigma_C^2 (1+\beta_+)} \right)^{\frac{1}{1-\alpha}}$$

being $\beta_+ \beta_- = -\frac{2\mu_F}{\sigma_C^2}$.

Hence (cf. Theorem 2.3.1) the optimal capacity is

$$C^{y,\hat{\nu}}(t) = C^{(a)}(t) = C^0(t) \left(y \vee \sup_{0 \le u \le t} \left(\frac{a}{C^0(u)} \right) \right). \tag{2.64}$$

□

From Remark 2.3.4 we have

$$\overline{\nu}^y(t) = \sup_{0 \le u \le t} \left(\frac{a - yC^0(u)}{C^0(u)} \right) \vee 0, \tag{2.65}$$

and the corresponding control $\hat{\nu}(t)$ (cf. (2.2)) makes the diffusion reflect at the boundary a, it is the local time of $C^{y,\hat{\nu}}(t)$ at a.

Notice that the boundary a in (2.58) coincides with the free boundary k_b obtained by H. Pham in [55] for a unit cost of investment p. In fact from [55], Example 1.5.1

$$k_b^{\alpha-1} = \frac{1-m}{C(\alpha - m)},$$

with

$$C = \frac{1}{\mu_F + \alpha\widetilde{\mu}_C - \frac{\alpha^2\sigma_C^2}{2}} \qquad \text{and} \qquad m = -\beta_+.$$

It is easy to see that

$$a^{\alpha-1} = \frac{\mu_F(1+\beta_+)(\alpha+\beta_-)}{\beta_+\beta_-} = \frac{1-m}{C(\alpha-m)} = k_b^{\alpha-1}, \tag{2.66}$$

hence $a = k_b$.

The following Proposition permits to have some comparative statics results.

Proposition 2.5.2. *The free boundary of problem (2.7) is a positive decreasing function of the diffusion coefficient σ_C and satisfies*

$$\lim_{\sigma_C \to 0} a = \left(\frac{1}{\mu_F + \alpha\mu_C} \right)^{\frac{1}{1-\alpha}}$$

$$\lim_{\sigma_C \to +\infty} a = 0.$$

Proof. Notice that $k_b > 0$ since β_+ and α are positive as well as C. By means of (2.66) it follows $a > 0$.

Straighforward calculations give

$$\frac{\partial \beta_+}{\partial \sigma_C} = -\frac{\beta_+}{\sigma_C}\left(1 + \frac{\left(\frac{\widetilde{\mu}_C}{\sigma_C^2}\right)}{\sqrt{\left(\frac{\widetilde{\mu}_C}{\sigma_C^2}\right)^2 + \frac{2\mu_F}{\sigma_C^2}}}\right) = -\frac{2\mu_F}{\sigma_C^3\sqrt{\left(\frac{\widetilde{\mu}_C}{\sigma_C^2}\right)^2 + \frac{2\mu_F}{\sigma_C^2}}} < 0$$

where we have used the definition of β_- and the relation $\beta_+\beta_- = -\frac{2\mu_F}{\sigma_C^2}$. Thanks to (2.66) we get

$$\frac{\partial a^{1-\alpha}}{\partial \sigma_C} = \frac{C^2(1-\alpha)[\frac{1}{C}\frac{\partial \beta_+}{\partial \sigma_C} - \sigma_C\alpha(1+\beta_+)(\alpha+\beta_+)]}{(1+\beta_+)^2} < 0$$

from which it is clear that $\sigma_C \to a(\sigma_C)$ is a decreasing mapping.

Moreover, we have

$$\lim_{\sigma_C \to 0} k_b^{\alpha-1} = \mu_F + \alpha\mu_C$$

and

$$\lim_{\sigma_C \to +\infty} k_b^{\alpha-1} = +\infty.$$

Hence, thanks to (2.66) we get

$$\lim_{\sigma_C \to 0} a = \left(\frac{1}{\mu_F + \alpha\mu_C}\right)^{\frac{1}{1-\alpha}}$$

and

$$\lim_{\sigma_C \to +\infty} a = 0.$$

□

We notice that when σ_C and μ_C are zero (i.e. in the deterministic setting with zero depreciation) $R_c(l^*(t)) = R_c(a) = \mu_F$ as expected from economic theory.

Remark 2.5.3. *For a general production function $R(\cdot)$ satisfying Assumption 2.1.1, to find the free boundary a one should solve the analogue of (2.62), i.e.*

$$\frac{1}{\mu_F}\mathbb{E}\left\{e^{-\underline{Y}(\tau(\mu_F))}R_c\left(a\, e^{-\underline{Y}(\tau(\mu_F))}\right)\right\}\mathbb{E}\left\{e^{-\overline{Y}(\tau(\mu_F))}\right\} = 1,$$

or equivalently

$$\frac{1}{\mu_F}\mathbb{E}\left\{e^{-\underline{Y}(\tau(\mu_F))}R_c\left(a\, e^{-\underline{Y}(\tau(\mu_F))}\right)\right\}\frac{\beta_+}{1+\beta_+} = 1.$$

That is, a is the unique solution of

$$\mathbb{E}\left\{e^{-\underline{Y}(\tau(\mu_F))}R_c\left(a\, e^{-\underline{Y}(\tau(\mu_F))}\right)\right\} = \frac{\mu_F(1+\beta_+)}{\beta_+}. \tag{2.67}$$

2.6 The Variational Approach in the Case of Time-Dependent Coefficients

In this Section we recall the solution of problem (2.7) obtained in Chiarolla and Haussmann [20] by a variational approach and we generalize some of their results to the case of deterministic, time-dependent coefficients of the controlled diffusion (cf. Assumption 2.4.1).

Let $C^{s,y,\nu}(t)$ be the capacity process starting at time $s \in [0,T)$ from y, controlled by ν, then

$$\begin{cases} dC^{s,y,\nu}(t) = C^{s,y,\nu}(t)[-\mu_C(t)dt + \sigma_C(t)dW(t)] + f_C(t)d\nu(t), \;\; t \in [s,T), \\ C^{s,y,\nu}(s) = y > 0, \end{cases} \tag{2.68}$$

hence

$$C^{s,y,\nu}(t) = \frac{C^0(t)}{C^0(s)}\left\{y + \int_{[s,t)} \frac{C^0(s)}{C^0(u)} f_C(u)d\nu(u)\right\}$$

with $C^0(t)$ as defined in (2.3).

To semplify notation write

$$\widetilde{C}^s(t) := C^{s,1,0}(t) = \frac{C^0(t)}{C^0(s)} = e^{-\int_s^t(\mu_C(u)+\frac{1}{2}\sigma_C^2(u))du + \int_s^t \sigma_C(u)dW(u)}, \tag{2.69}$$

this process is $\widetilde{\mathcal{F}}_{s,t} := \sigma\{W(u) - W(s), s \leq u \leq t\}$-measurable.

To $C^{s,y,\nu}$ we associate the expected total profit net of investment given by

$$J_{s,y}(\nu) = \mathbb{E}\left\{\int_s^T e^{-\int_s^t \mu_F(u)du} R(C^{s,y,\nu}(t))dt - \int_{[s,T)} e^{-\int_s^t \mu_F(u)du} d\nu(t)\right\}. \tag{2.70}$$

The corresponding optimal investment problem is

$$V(s,y) := \sup_{\nu \in \mathcal{S}_s} J_{s,y}(\nu), \tag{2.71}$$

where

$$\mathcal{S}_s \;\; := \;\; \{\nu : \Omega \times [s,T] \to \mathbb{R}_+, \text{ nondecreasing, left-continuous, adapted s.t. } \nu(s) = 0, \; \mathbb{P} - \text{a.s.}\}$$

is the convex set of irreversible investments.

We define the opportunity cost of not investing until time t as (compare with [20], Section 3)

$$\zeta^{s,y,T}(t) := \int_s^t e^{-\int_s^u \mu_F(r)dr}\, \widetilde{C}^s(u) R_c(y\widetilde{C}^s(u))du + e^{-\int_s^t \mu_F(r)dr} \widetilde{C}^s(t)\frac{1}{f_C(t)}\mathbb{1}_{\{t<T\}}, \tag{2.72}$$

and the optimal stopping problem (compare with [20], (3.1))

$$Z^{s,y,T}(t) := \operatorname*{ess\,inf}_{t\le\tau\le T} \mathbb{E}\left\{\zeta^{s,y,T}(\tau)\middle|\widetilde{\mathcal{F}}_{s,t}\right\}. \tag{2.73}$$

Denoting by $\mathcal{Z}^{s,y,T}(\cdot)$ the right-continuous with left-limits modification of $Z^{s,y,T}(\cdot)$, for $s=t$ we set $v(s,y) := \mathcal{Z}^{s,y,T}(s)$, so that up to a null set,

$$\begin{aligned} v(s,y) &= \operatorname*{ess\,inf}_{s\le\tau\le T} \mathbb{E}\bigg\{ \int_s^\tau e^{-\int_s^u \mu_F(r)dr}\, \widetilde{C}^s(u) R_c\left(y\widetilde{C}^s(u)\right) du \\ &\qquad + e^{-\int_s^\tau \mu_F(r)dr} \widetilde{C}^s(\tau)\frac{1}{f_C(\tau)}\mathbb{1}_{\{\tau<T\}}\bigg\}. \end{aligned} \tag{2.74}$$

Now, the results in [2], Proposition 2 and Theorem 3, guarantee that for $s \in [0,T)$ the stopping time

$$\tau^*(s,y) = \inf\left\{t\in[s,T) : \mathcal{Z}^{s,y,T}(t) = \zeta^{s,y,T}(t)\right\}\wedge T \tag{2.75}$$

is optimal for (2.73) and the function $v(s,y)$ is the shadow value of installed capital, i.e.

$$v(s,y) = \frac{\partial}{\partial y}V(s,y).$$

Theorem 2.6.1. *Under Assumption 2.4.1, for every (s,y) in $[0,T)\times(0,\infty)$ the optimal stopping time (2.75) may be written as*

$$\tau^*(s,y) = \inf\left\{t\in[s,T) : v(t, Y^{s,y}(t)) = \frac{1}{f_C(t)}\right\}\wedge T. \tag{2.76}$$

Proof. Recall that $Y^{s,y}(t) := y\widetilde{C}^s(t)$. From (2.73) we may write

$$\begin{aligned} \mathcal{Z}^{s,y,T}(t) &= \operatorname*{ess\,inf}_{t\le\tau\le T} \mathbb{E}\bigg\{ \int_s^t e^{-\int_s^u \mu_F(r)dr}\, \widetilde{C}^s(u) R_c\left(Y^{s,y}(u)\right) du \\ &\qquad + \int_t^\tau e^{-\int_s^u \mu_F(r)dr}\, \widetilde{C}^s(u) R_c\left(Y^{s,y}(u)\right) du \end{aligned}$$

$$
\begin{aligned}
&\qquad + e^{-\int_s^t \mu_F(r)dr} e^{-\int_t^\tau \mu_F(r)dr} \widetilde{C}^s(\tau) \frac{1}{f_C(\tau)} \mathbb{1}_{\{\tau<T\}} \Big| \widetilde{\mathcal{F}}_{s,t} \Big\} \qquad (2.77)\\
&= \zeta^{s,y,T}(t) + \operatorname*{ess\,inf}_{t\le\tau\le T} \mathbb{E}\Big\{ \int_t^\tau e^{-\int_s^u \mu_F(r)dr}\, \widetilde{C}^s(u) R_c\left(Y^{s,y}(u)\right) du \\
&\qquad + e^{-\int_s^t \mu_F(r)dr} \Big(e^{-\int_t^\tau \mu_F(r)dr} \widetilde{C}^s(\tau) \frac{1}{f_C(\tau)} \mathbb{1}_{\{\tau<T\}} - \widetilde{C}^s(t) \frac{1}{f_C(t)} \mathbb{1}_{\{t<T\}} \Big) \Big| \widetilde{\mathcal{F}}_{s,t} \Big\}.
\end{aligned}
$$

Notice that

$$
\widetilde{C}^s(u) = \widetilde{C}^s(t)\widetilde{C}^t(u), \ \forall u \ge t, \quad \text{and} \quad e^{-\int_s^\tau \mu_F(r)dr} \widetilde{C}^s(\tau) = e^{-\int_s^t \mu_F(r)dr} e^{-\int_t^\tau \mu_F(r)dr} \widetilde{C}^s(t)\widetilde{C}^t(\tau).
$$

Hence for $t < T$ we have

$$
\begin{aligned}
\mathcal{Z}^{s,y,T}(t) &= \zeta^{s,y,T}(t) \qquad (2.78)\\
&\quad + e^{-\int_s^t \mu_F(r)dr} \widetilde{C}^s(t) \operatorname*{ess\,inf}_{t\le\tau\le T} \mathbb{E}\Big\{ \int_t^\tau e^{-\int_t^u \mu_F(r)dr}\, \widetilde{C}^t(u) R_c\Big(Y^{s,y}(t)\widetilde{C}^t(u)\Big) du \\
&\qquad + e^{-\int_t^\tau \mu_F(r)dr} \widetilde{C}^s(\tau) \frac{1}{f_C(\tau)} \mathbb{1}_{\{\tau<T\}} - \frac{1}{f_C(t)} \Big| \widetilde{\mathcal{F}}_{s,t} \Big\}.
\end{aligned}
$$

In order to take care of the conditioning in (2.78) we proceed exactly as in the proof of Proposition 2.4.2 by working on the canonical probability space $(\overline{\Omega}, \overline{\mathbb{P}})$, where $\overline{\mathbb{P}}$ is the Wiener measure on $\overline{\Omega} := \mathcal{C}_0([0,T])$, the space of continuous functions on $[0,T]$ which are zero at time zero. Since $\widetilde{C}^t(\cdot)$ is independent of $\widetilde{\mathcal{F}}_{s,t}$ and $Y^{s,y}(t)$ is $\widetilde{\mathcal{F}}_{s,t}$-adapted, recalling (2.74), from (2.78) we get

$$
\mathcal{Z}^{s,y,T}(t) = \zeta^{s,y,T}(t) + e^{-\int_s^t \mu_F(r)dr} \widetilde{C}^s(t) \Big(v\left(t, Y^{s,y}(t)\right) - \frac{1}{f_C(t)} \Big). \qquad (2.79)
$$

Finally, (2.75) and (2.79) imply

$$
\begin{aligned}
\tau^*(s,y) &= \inf\{t \in [s,T) : \mathcal{Z}^{s,y,T}(t) = \zeta^{s,y,T}(t)\} \wedge T \\
&= \inf\Big\{ t \in [s,T) : e^{-\int_s^t \mu_F(r)dr} \widetilde{C}^s(t) \Big(v\left(t, Y^{s,y}(t)\right) - \frac{1}{f_C(t)} \Big) = 0 \Big\} \wedge T \\
&= \inf\Big\{ t \in [s,T) : v(t, Y^{s,y}(t)) = \frac{1}{f_C(t)} \Big\} \wedge T. \qquad (2.80)
\end{aligned}
$$

□

Notice that if $\widetilde{\mathbb{E}}_s\{\cdot\}$ is the expectation w.r.t. $\widetilde{\mathbb{P}}_s$ (cf. (2.38) for its definition), then (2.74) may also be written as

$$v(s,y) = \inf_{s\le\tau\le T} \widetilde{\mathbb{E}}_s\bigg\{ \int_s^\tau e^{-\int_s^u \overline{\mu}(r)dr} R_c\left(Y^{s,y}(t)\right) du + e^{-\int_s^\tau \overline{\mu}(r)dr} \frac{1}{f_C(\tau)} \mathbb{1}_{\{\tau<T\}} \bigg\} \tag{2.81}$$

with $\overline{\mu}(t) = \mu_F(t) + \mu_C(t)$, for $t \in [0,T]$.

The value function $v(s,y)$ is expected to be the solution of a variational inequality similar to that obtained in Chiarolla and Haussmann [20] under Markovian restrictions (cf. [20], Assumption-[M]) and with a Cobb-Douglas production function (cf. [20], (4.5) and Theorem 4.4).

In order to show that $v(s,y)$ is directly related to the solution of a suitable variational inequality, we introduce for $s \in [0,T]$ and $t \ge s$ the diffusion

$$X^{s,x}(t) = x + \int_s^t (\mu_C(r) - \frac{1}{2}\sigma_C^2(r))dr + \int_s^t \sigma_C(r) d\widetilde{W}^s(r), \quad X^{s,x}(s) = x. \tag{2.82}$$

Clearly, $X^{s,\ln(y)}(t) = \ln(Y^{s,y}(t))$. Moreover, we may define the second order differential operator

$$\mathcal{L} := \frac{1}{2}\sigma^2(s)\partial_{xx} + (\frac{1}{2}\sigma^2(s) - \mu(s))\partial_x. \tag{2.83}$$

Definition 2.6.1. *Let $1 \le p \le \infty$ and let $\mathcal{O}$ be an open set of $\mathbb{R}$. For any $m \in \mathbb{N}$ and $\lambda > 0$ we denote by $\boldsymbol{W}^{m,p,\lambda}(\mathcal{O})$ the space of all functions u such that*

$$||u||_{m,p,\lambda} := \left(\sum_{k\le m} \int_{\mathcal{O}} e^{-\lambda|x|} |u^{(k)}(x)|^p dx \right)^{\frac{1}{p}} < +\infty,$$

where $u^{(k)}(x)$ denotes the k-th derivate of u.
Moreover, we write $u \in \boldsymbol{L}^{p,\lambda}(\mathcal{O})$ if u is such that

$$\left(\int_{\mathcal{O}} e^{-\lambda|x|} |u(x)|^p dx \right)^{\frac{1}{p}} < +\infty.$$

Theorem 2.6.2. *Assume that μ_C, σ_C, f_C and μ_F have bounded first order derivatives and fix $\lambda > 2$. Then, there exists a unique solution $\Phi(s,x) \in \boldsymbol{L}^2\left(0,T; \boldsymbol{W}^{1,2,\lambda}(\mathbb{R})\right) \cap$*

$\boldsymbol{L}^2\left(0,T;\,\boldsymbol{W}^{2,2,\lambda}(\mathbb{R})\right)$ *such that* $\frac{\partial\Phi}{\partial s}\in\boldsymbol{L}^2\left(0,T;\boldsymbol{L}^{2,\lambda}(\mathbb{R})\right)\cap\boldsymbol{L}^2\left(0,T;\boldsymbol{L}^{2,\lambda}(\mathbb{R})\right)$ *and*

$$\begin{cases}(-\partial_s-\mathcal{L}-\overline{\mu}(s))\,\Phi(s,x)\leq R_c\left(e^x\right), & (s,x)\in(0,T)\times\mathbb{R},\\[2ex] (-(\partial_s+\mathcal{L}+\overline{\mu}(s))\Phi(s,x)-R_c\left(e^x\right))\left(\Phi(s,x)-\dfrac{1}{f_C(s)}\right)=0, & (s,x)\in(0,T)\times\mathbb{R},\\[2ex] \Phi(s,x)\leq\dfrac{1}{f_C(s)}, & (s,x)\in(0,T)\times\mathbb{R},\\[2ex] \Phi(T,x)=0, & x\in\mathbb{R}.\end{cases}\tag{2.84}$$

Moreover, the unique solution $\Phi(s,x)$ *admits the representation*

$$\Phi(s,x)=\inf_{s\leq\tau\leq T}\widetilde{\mathbb{E}}_s\left\{\int_s^\tau e^{-\int_s^u\overline{\mu}(r)dr}\,R_c(e^{X^{s,x}(u)})du+e^{-\int_s^\tau\overline{\mu}(r)dr}\frac{1}{f_C(\tau)}\mathbb{1}_{\{\tau<T\}}\right\}$$

with the diffusion $X^{s,x}(t)$ *given by (2.82).*

Proof. The proof is an application of the results of Bensoussan and Lions in [12] (see [12], Chapter 3, Section 4.9, p. 442). With respect to the notation in [12] we may write system (2.84) as

$$\begin{cases}(-\partial_s+A(s))\,\Phi(s,x)\leq R_c\left(e^x\right), & (s,x)\in(0,T)\times\mathbb{R},\\[2ex] (-\partial_s\Phi(s,x)+A(s)\Phi(s,x)-R_c\left(e^x\right))\left(\Phi(s,x)-\dfrac{1}{f_C(s)}\right)=0, & (s,x)\in(0,T)\times\mathbb{R},\\[2ex] \Phi(s,x)\leq\dfrac{1}{f_C(s)}, & (s,x)\in(0,T)\times\mathbb{R},\\[2ex] \Phi(T,x)=0, & x\in\mathbb{R},\end{cases}$$

where we have introduced the differential operator $A(s):=-\left(\mathcal{L}-\overline{\mu}(s)\right).$

Notice that

- $R_c(e^x)\in\mathbf{L}^2\left(0,T;\mathbf{L}^{2,\lambda}(\mathbb{R})\right)$ for every $\lambda>2$. In fact, the strict concavity of the production function and the growth assumption on the production function (cf. the second part of Assumption 2.1.1) imply

$$\int_0^T\left(\int_{-\infty}^{\infty}e^{-\lambda|x|}|R_c\left(e^x\right)|^2dx\right)^{\frac{1}{2}}dt\leq T\left(\int_{-\infty}^{\infty}e^{-\lambda|x|}e^{-2x}\left(k_\eta+\eta e^x\right)^2dx\right)^{\frac{1}{2}}<\infty.$$

- $\dfrac{1}{f_C(s)}$ is bounded with bounded first order derivative and clearly it belongs to the space $\mathbf{L}^2\left(0,T;\mathbf{W}^{1,2,\lambda}(\mathbb{R})\right)$ for all $\lambda > 0$. Moreover $\frac{1}{f_C(s)} \in C^0([0,T]\times\mathbb{R})$ and
$$\left(-\frac{\partial}{\partial s} + A(s)\right)\frac{1}{f_C(s)} \in \mathbf{L}^2\left(0,T;\mathbf{L}^{2,\lambda}(\mathbb{R})\right), \quad \forall\lambda > 0$$
being $\overline{\mu}(s)$ bounded as well as $f_C(s)$ and its first order derivative.

- $\Phi(T,x) = 0$, hence it belongs to $\mathbf{W}^{2,2,\lambda}(\mathbb{R})$ for all $\lambda > 0$, and $\Phi(T,x) \leq \dfrac{1}{f_C(T)}$.

Moreover we have $\frac{1}{2}\sigma_C^2(s)$, $\mu_C(s) - \frac{1}{2}\sigma_C^2(s)$ and $\overline{\mu}(s)$ bounded with bounded first order derivatives. Finally, the differential operator $\mathcal{L}$ is uniformly parabolic.

Hence, thanks to the results in [12], Chapter 3, Section 4.9, p. 442, we have that there exists a unique Φ s.t.

$$\begin{cases}
\Phi(s,x) \in \mathbf{L}^2\left(0,T;\mathbf{W}^{1,2,\lambda}(\mathbb{R})\right) \cap \mathbf{L}^2\left(0,T;\mathbf{W}^{2,2,\lambda}(\mathbb{R})\right), \\
\dfrac{\partial\Phi}{\partial s} \in \mathbf{L}^2\left(0,T;\mathbf{L}^{2,\lambda}(\mathbb{R})\right) \cap \mathbf{L}^2\left(0,T;\mathbf{L}^{2,\lambda}(\mathbb{R})\right), \\
\left(-\partial_s + A(s)\right)\Phi(s,x) \leq R_c\left(e^x\right), \quad (s,x) \in (0,T)\times\mathbb{R}, \\
\left(-\partial_s\Phi(s,x) + A(s)\Phi(s,x) - R_c\left(e^x\right)\right)\left(\Phi(s,x) - \dfrac{1}{f_C(s)}\right) = 0, \quad (s,x) \in (0,T)\times\mathbb{R}, \\
\Phi(s,x) \leq \dfrac{1}{f_C(s)}, \quad (s,x) \in (0,T)\times\mathbb{R}, \\
\Phi(T,x) = 0, \quad x \in \mathbb{R}.
\end{cases}$$

In particular it follows that

$$\Phi(s,x) = \inf_{s\leq\tau\leq T}\widetilde{\mathbb{E}}_s\left\{\int_s^\tau e^{-\int_s^u\overline{\mu}(r)dr}R_c(e^{X^{s,x}(u)})du + e^{-\int_s^\tau\overline{\mu}(r)dr}\frac{1}{f_C(\tau)}\mathbb{1}_{\{\tau<T\}}\right\}, \tag{2.85}$$

where the diffusion $X^{s,x}(t)$ is given by (2.82). □

Corollary 2.6.2. *Let $\Phi(s,x)$ be the unique solution of (2.84). Then we have $\Phi(s,\ln(y)) = v(t,y)$.*

Proof. Since $Y^{s,y}(t) \equiv e^{X^{s,\ln(y)}}(t)$, then it is clear by (2.85) that $v(t,y) = \Phi(s,\ln(y))$. □

Chapter 3

A Stochastic Economy in Continuous Time: First Order Conditions and a Fixed Point Problem

In the previous two Chapters of this Thesis a firm has represented the productive sector of a market, but we have not modeled precisely the rest of the economy. In fact we focused only on the firm's manager problem, i.e. to choose an irreversible investment strategy that maximizes the company's expected total profit, net of investment costs. In this Chapter we consider the optimal irreversible investment problem for a firm embedded in a stochastic, continuous time economy on a finite time interval, as modeled in [22]. When the discount factor of the firm's manager coincides with the deflator, stochastic first order conditions for the general equilibrium of the economy lead to a very difficult random fixed point problem that the Authors in [22] have been unable to solve. We now aim to study the existence of a solution to such fixed point problem.

3.1 A Stochastic Economy with Irreversible Investment

We briefly recall the model of [22] for a continuous time, stochastic economy with irreversible investment and money. A stochastic, continuous time economy consists of a single perishable good producing firm which has to decide on cash holdings, levels of employment and how to invest for capacity expansion; rational agents that maximize their total expected utility of consumption, money holding and leisure, some of them are employed by the firm to facilitate capacity expansion, and some are retired or on welfare. Moreover, all the agents partecipate in a financial market consisting of a nominal bond, a real bond (i.e. valued in real terms), stocks of the firm and another type of contract, called derivative. The shares' owner receives dividends. For its production the firm employs labour, borrows capital for its daily business, and sells shares to raise capital for capacity expansion.

We may think of the agents partecipating in the economy as members of three different categories: the first kind provides labour to the firm (production sector), the second turns investment cash into capacity expansion (construction sector), and the third category provides no labour (the welfare and retired sector). All the agents own the firm's shares and the bonds, and consume the good produced by the firm. Moreover, in order to facilitate purchases of the goods and the other financial transactions, the government supplies money to the agents as an exogeneous process M. On the other hand, all the price processes are endogeneous and their optimal value is determined from equilibrium considerations.

We now start with briefly introducing the 'ingredients' we need. Let J be the total number of agents operating in the three sector of the economy. At equilibrium, market clearing conditions has to be solved, and, following the original approach of [45], the analysis becomes much easier if only one (representative) agent is present. For that reason the actions of the single agents are aggregated into the action of a single fictitious Social Planner. His utility function, U, is a suitable weight of the utility functions of the individual agents in the economy. The factor $\Lambda := (\lambda_1, ..., \lambda_J) \in \mathbb{R}^J_{++}$ accomplishes this. The suitable Λ is determined at equilibrium as solution of a fixed point problem in the spirit of [45], Theorem 11.1.

Denote by $\hat{C}(t)$ the optimal (at equilibrium) production capacity process, by $\hat{K}(t)$ the optimal real capital at time $t \in [0, T]$, and by $\hat{L}(t)$ the optimal labour level process. As already introduced, $M(t)$ represents the exogeneous money supply, whereas J^p is the total number of agents who supply labour to the firm (that is, $J^p - \hat{L}(t)$ is the equilibrium number of agents in the welfare and retired sector at time $t \in [0,T]$).

Regarding the Social Planner's utility function U, we assume all the properties of [22], Lemma 4.1; briefly, $U(t,\cdot;\Lambda)$ is a concave, increasing, continuous function on $\mathbb{R}^3$ with continuous partial first order derivatives U_c, U_l, U_m.

The process $\zeta(t)$ is the deflator (or real state-price density) and (cf. [22], (2.18)) it is assumed to be continuous, uniformly bounded in $(\omega,t) \in \Omega \times [0,T]$ (i.e. there exist finite constants k_ζ, κ_ζ such that a.s. a.e. $0 < k_\zeta \le \zeta(\omega,t) \le \kappa_\zeta$.), and to satisfy

$$\begin{cases} d\zeta(t) = \zeta(t)[-r(t)dt - \theta^T(t)dW(t) - d\beta(t)], \\ \zeta(0) = 1. \end{cases} \tag{3.1}$$

Here, $r(t)$ is the real interest rate (cf. [22], (2.1)), whereas $\beta(t)$ is the singular continuous part in the dynamics of the real bond (cf. [22], (2.1)). In [22], (2.2), it is assumed that

$$\int_0^T |r(t)|dt < +\infty \;\; a.s., \qquad ||\beta||_T < \infty \;\; a.s.,$$

where $||\beta||_T$ denotes the total variation of the process β on $[0,T]$. $W(t)$ is an exogeneous two-dimensional Brownian motion and $\theta(t)$ is the 'market price of risk' (cf. [22], (2.16), (2.17) and Remark 2.1) such that

$$\begin{cases} (i) \quad \int_0^T ||\theta(t)||^2 dt < \infty \;\; \text{a.s.} \\ \\ (ii) \quad \mathcal{E}(t) := \exp[-\int_0^t \theta^T(s)dW(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds] \quad \text{is a martingale.} \end{cases} \tag{3.2}$$

On the other hand, $\widetilde{r}(t)$ and $w'(t)$ are the nominal interest rate and the real cost of labour (the real wage process) respectively. They are uniformly bounded in $(\omega,t) \in \Omega \times [0,T]$ continuous processes (cf. [22], (2.4) and (3.5)).

The firm's production function is denoted by $R : \mathbb{R}^3 \to [-\infty,\infty)$, and, according to [22], Assumption R (3.4), it is a nondecreasing, upper semicontinuous, concave function with sub-linear growth, twice continuously differentiable on the interior of its domain. Since R is the rate of production of goods, the production profit per unit of time is given by

$R(C,K,L) - \widetilde{r}K - w'L$ which the manager aims to maximize at each moment in time by choosing capital and labour (K,L) at the current capacity C. Take $M \leq \kappa_M$ to be the current money supply and set $Q(M) := [0,M] \times [0,J^p]$. Then, (cf. [22], (3.11))

$$\widetilde{R}(C,\widetilde{r},w') := \max_{(K,M)\in Q(M)} [R(C,K,L) - \widetilde{r}K - w'L]$$

is the maximal profit rate. Notice that for $C \geq 0$ fixed, $-\widetilde{R}(C,\cdot,\cdot)$ is the concave conjugate of $R^{Q(M)}(C,\cdot,\cdot) = R(C,\cdot,\cdot) - \chi_{Q(M)}(\cdot,\cdot)$ where

$$\chi_{Q(M)} := \begin{cases} 0, & x \in Q(M) \\ +\infty, & \text{otherwise} \end{cases}$$

The firm's manager chooses the investment strategy $\nu(\omega,t)$ in order to maximize the total profit plus scrap value, net of investment costs, that is he maximizes

$$\begin{aligned} \mathcal{J}_{0,y}(\nu) \quad := \quad \mathbb{E}\bigg\{ \int_0^T & e^{-\mu_F(t)} \widetilde{R}(C^{y,\nu}(t), \widetilde{r}(t), w'(t))dt + e^{-\mu_F(T)} G(C^{y,\nu}(T)) \\ & - \int_{[0,T)} e^{-\mu_F(t)} d\nu(t) \bigg\} \end{aligned} \tag{3.3}$$

over the convex set

$$\mathcal{S} := \{\nu : \text{left-continuous, nondecreasing, adapted process, a.s. finite, s.t. } \nu(0) = 0 \text{ a.s.}\}.$$

Here G is the scrap-value function; it is strictly concave, non decreasing, continuously differentiable with sub-linear growth (cf. [22], Assumption G). Recall that the controlled production capacity $C^{y,\nu}(t)$, unique strong solution of

$$\begin{cases} dC^{y,\nu}(t) = C^{y,\nu}(t)[-\mu_C(t)dt + \sigma_C(t)dW(t)] + f_C(t)d\nu(t), \quad t \in [0,T), \\ \\ C^{y,\nu}(0) = y > 0, \end{cases}$$

is given by $C^{y,\nu}(t) = C^0(t)[y + \overline{\nu}(t)]$, with $C^0(t) := C^{1,0}(t)$ and $\overline{\nu}(t) := \int_{[0,t)} \frac{f_C(s)}{C^0(s)} d\nu(s)$.

At equilibrium all the agents act optimally: the firm's manager has to choose investment, labour and operating capital to maximize the expected total net profits; the profits have to be distributed as dividends; the investment capital has to be passed as wages to the non-production sector; the changes in the money supply have to be passed into the economy as 'welfare' and the market (goods, bonds, money, labour, derivative and stocks) have to be clear. In [22], Definition 6.1, the general equilibrium of the economy is characterized by stochastic first order conditions that must hold simultaneously.

3.1.1 First Order Conditions and a Random Fixed Point Problem

Notice that, as pointed out in [22], Remark 3.2, we may look at the manager's situation (cf. (3.3)) slightly different. In fact, we may think of the firm's net present value as a claim to be sold with no-arbitrage price given by $\mathcal{J}_{0,y}$ when the manager's discount factor coincides with the deflator of the economy (cf. (3.1)), i.e. $e^{-\mu_F(t)} \equiv \zeta(t)$. In such a case $\mathcal{J}_{0,y}(\nu)$ is the total number of firm's shares at time zero, i.e. $\mathcal{J}_{0,y} = N(0)S(0)$ (cf. [22], (3.13)). If $N(0)$ is given (exogeneous), on the other hand $S(0)$ depends on the future expected payments, hence the firm's manager attempt is to maximize the present share value. Notice that he may well be motivated to do so if he owns a significant number of stock options. It follows that, when $e^{-\mu_F(t)} \equiv \zeta(t)$, the market parameters $\zeta, \widetilde{r}$ and w' are expected to be at equilibrium solutions of a very difficult random fixed point problem. In fact, (cf. [22], (6.16) and (6.17)),

$$\begin{cases} \zeta(t) = U_c(t, R(\hat{C}(t), \hat{K}(t), \hat{L}(t)), M(t) - \hat{K}(t), J^p - \hat{L}(t); \Lambda), \\ \widetilde{r}(t) = \dfrac{U_m(t, R(\hat{C}(t), \hat{K}(t), \hat{L}(t)), M(t) - \hat{K}(t), J^p - \hat{L}(t); \Lambda)}{U_c(t, R(\hat{C}(t), \hat{K}(t), \hat{L}(t)), M(t) - \hat{K}(t), J^p - \hat{L}(t); \Lambda)}, \\ w'(t) = \dfrac{U_l(t, R(\hat{C}(t), \hat{K}(t), \hat{L}(t)), M(t) - \hat{K}(t), J^p - \hat{L}(t); \Lambda)}{U_c(t, R(\hat{C}(t), \hat{K}(t), \hat{L}(t)), M(t) - \hat{K}(t), J^p - \hat{L}(t); \Lambda)}. \end{cases} \tag{3.4}$$

Equations (3.4) state that, at equilibrium, the deflator ζ equals the marginal utility of consumption; the nominal interest rate $\widetilde{r}$ coincides with the marginal utility of money relative to the marginal utility of consumption; whereas the real cost of labour w' is the marginal utility of labour relative to the marginal utility of consumption. Notice that equations (3.4) are well posed since, as ζ, $\widetilde{r}$ and w', also U_c, U_l and U_m are bounded (cf. [22], proof of Theorem 7.1). We stress that in [22] the Authors did not work with the endogeneous discount factor $\zeta(t)$, but only with the exogeneous one $e^{-\mu_F(t)}$, since they were unable to solve (3.4). In this Section, we aim to prove the existence of a solution to the daunting fixed point problem (3.4).

We shall write $(\zeta(t), \widetilde{r}(t), w'(t)) \in \mathcal{A}$, where

$$\mathcal{A} := \{\underline{X} : \Omega \times [0,T] \to \mathbb{R}^3 \text{ continuous and s.t. } ||\underline{X}||_{\mathbb{R}^3} \leq K < +\infty, \forall t \in [0,T], \ \mathbb{P}-\text{a.s.}\}.$$

Notice that the right-hand sides of (3.4) are highly nonlinear functionals of the processes $\zeta(t)$, $\widetilde{r}(t)$ and $w'(t)$. The marginal utilities U_l, U_m and U_c depend on $(\hat{C}(t), \hat{K}(t), \hat{L}(t))$ which in turn are functions of $\zeta(t)$, $\widetilde{r}(t)$ and $w'(t)$. In fact

$$(\hat{K}(t), \hat{L}(t)) = I^{R^{Q(M)(\hat{C}(t),\cdot,\cdot)}}(\widetilde{r}(t), w'(t)),$$

where $I^{R^{Q(M)(C,\cdot,\cdot)}}$ is an extension of the inverse of $\nabla_{K,L} R^{Q(M)}(C,\cdot,\cdot)$, whereas the optimal productive capacity $\hat{C}(t) = C^0(t)[y + \overline{\nu}^y(t)]$ depends on $(\zeta(t), \widetilde{r}(t), w'(t))$ in the following way. Define

$$\begin{aligned}\mathbf{R}^{y,T}(t) := & \int_0^t \zeta(s) C^0(s) \widetilde{R}_c(yC^0(s), \widetilde{r}(s), w'(s)) ds \\ & + \zeta(t) \frac{C^0(t)}{f_C(t)} \mathbb{1}_{\{t<T\}} + \zeta(T) C^0(T) G'(yC^0(T)) \mathbb{1}_{\{t=T\}}\end{aligned}$$

and the Snell Envelope of $\mathbf{R}^{y,T}(t)$,

$$Z^{y,T} := \operatorname*{ess\,inf}_{t \leq \tau \leq T} \mathbb{E}\{\mathbf{R}^{y,T}(\tau) | \mathcal{F}_t\}.$$

Let $\mathcal{Z}^{y,T}(t)$ be a modification of $Z^{y,T}$ with right-continuous paths, then the stopping time

$$\tau^*(0,y) := \inf\{s \in [0,T) : \mathcal{Z}^{y,T}(t) = \mathbf{R}^{y,T}(t)\} \wedge T$$

is the optimal time to invest and its left-continuous inverse (modulo a shift)

$$\overline{\nu}^y(t) := [\sup\{z \geq y : \tau^*(0, z+) < t\} - y]^+ \text{ if } t > 0, \qquad \overline{\nu}^y(0) = 0,$$

is related to the optimal investment $\hat{\nu}(t)$ through

$$\hat{\nu}(t) := \int_{[0,t)} \frac{C^0(s)}{f_C(s)} d\overline{\nu}^y(s).$$

Unfortunately the right-hand sides of (3.4) are not monotone operators, hence fixed point theorems for monotone operators (e.g. Tarski Theorem) cannot be applied. Therefore the idea is to look for a topology under which the mapping

$$(\zeta(t), \widetilde{r}(t), w'(t)) \to (U_c, \frac{U_m}{U_c}, \frac{U_l}{U_c})(t, R(\hat{C}(t), \hat{K}(t), \hat{L}(t)), M(t) - \hat{K}(t), J^p - \hat{L}(t); \Lambda)$$

is continuous and under which the set $\mathcal{A}$ is compact. We may think of $\underline{X}(t) := (\zeta(t), \widetilde{r}(t), w'(t))$ as an element of $D([0,T];\mathbb{R}^3)$, the Skorohod space of càdlàg functions from $[0,T]$ into $\mathbb{R}^3$ endowed with the Meyer-Zheng (or pseudo-path) topology. As pointed out in [50], $D([0,T];\mathbb{R}^3)$ with the pseudo-path topology is a separable metric space and the Meyer-Zheng topology is equivalent to convergence in measure (see Appendix A for further details on the Meyer-Zheng topology). We aim to check if $\mathcal{A}$ is compact under such topology and the right-hand sides of (3.4) are continuous.

It has been proved in [57] that if Y is a stochastic process satisfying some conditions and such that $Y_n \overset{MZ}{\Rightarrow} Y$ (i.e. the probabilities $\mathbb{P}^{Y_n}$ converge to $\mathbb{P}^{Y}$ when the Skorohod sample space is endowed with Meyer-Zheng topology), then the Snell envelope Υ_n associated to Y_n is such that $\mathbb{P}^{\Upsilon_n} \to \mathbb{P}^{\Upsilon}$, where Υ is the Snell envelope of Y. That result motivates our attempt to look at our fixed point problem under the Meyer-Zheng topology, possibly under some additional assumptions. Our daunting program of action is described below.

Set $\underline{X}_n := (\zeta_n(t), \widetilde{r}_n(t), w_n'(t)) \in \mathcal{A}$ and define

$$\begin{aligned}\mathbf{R}_n^{y,T}(t) &:= \int_0^t \zeta_n(s) C^0(s) \widetilde{R}_c(yC^0(s), \widetilde{r}_n(s), w_n^{'}(s))ds \\ &+ \zeta_n(t)\frac{C^0(t)}{f_C(t)}\mathbb{1}_{\{t<T\}} + \zeta_n(T)C^0(T)G'(yC^0(T))\mathbb{1}_{\{t=T\}},\end{aligned}$$

$\mathcal{Z}_n^{y,T}(t)$ as the right-continuous modification of the Snell envelope of $\mathbf{R}_n^{y,T}(t)$,

$$\tau_n^*(0,y) := \inf\{s \in [0,T) : \mathcal{Z}_n^{y,T}(t) = \mathbf{R}_n^{y,T}(t)\} \wedge T,$$

$$\overline{\nu}_n^y(t) := [\sup\{z \geq y : \tau_n^*(0, z+) < t\} - y]^+ \text{ if } t > 0, \quad \overline{\nu}_n^y(0) = 0,$$

and

$$\hat{C}_n(t) = C^0(t)[y + \overline{\nu}_n^y(t)].$$

Suppose $\underline{X}_n \overset{MZ}{\Rightarrow} \underline{X}$, then we would like to prove the following steps.

$$\mathbf{R}_n^{y,T} \overset{MZ}{\Rightarrow} \mathbf{R}^{y,T} \tag{3.5}$$

$$\mathcal{Z}_n^{y,T} \overset{MZ}{\Rightarrow} \mathcal{Z}^{y,T} \tag{3.6}$$

$$\tau_n^*(0,y) \overset{a.s}{\to} \tau^*(0,y) \tag{3.7}$$

(recall that a stopping time, being a functional of a process, is continuous in the more usual Skorohod topology (cf. [37], Chapter VI))

$$\overline{\nu}_n^y \overset{a.s.}{\to} \overline{\nu}^y \tag{3.8}$$

$$\hat{C}_n \overset{a.s.}{\to} \hat{C}. \tag{3.9}$$

The continuity of $I^{R^{Q(M)(\hat{C},\cdot,\cdot)}}$ and of U_l, U_m, U_c (cf. [22] and [21]) might then enable us to obtain the continuity of the right-hand sides of (3.4).

3.1.2 The Case of no Leisure and no Money

We may start with studying the simpler case of no leisure and no money; hence throughout this Section we make use of the following Assumption.

Assumption 3.1.1.

1. $\widetilde{r}(t) \equiv 0 \equiv w'(t)$ *for* $t \in [0,T]$;
2. $\mathbb{E}\left\{\int_0^T r(u)du\right\} \leq C_1, \quad \mathbb{E}\left\{\int_0^T ||\theta(u)||^2 du\right\} \leq C_2$ *and* $\mathbb{E}\{||\beta||_T\} \leq C_3$.

Notice that the second part of Assumption 3.1.1 is stronger than [22], (2.2) and the first of (2.17), but, in any case, that will be verified in equilibrium (cf. [22], page 42). Under Assumption 3.1.1 the fixed point problem (3.4) becomes

$$\zeta(t) = U_c(R(\hat{C}(t))). \tag{3.10}$$

Introduce the set

$$\mathcal{A}' := \{X : \Omega \times [0,T] \to \mathbb{R} \text{ continuous, uniformly of class (D) and uniformly bounded in } (\omega, t) \text{ with bounded conditional variation}\},$$

and recall that (cf. (3.1))

$$\begin{cases} d\zeta(t) = \zeta(t)[-r(t)dt - \theta^T(t)dW(t) - d\beta(t)], \\ \zeta(0) = 1, \end{cases} \tag{3.11}$$

with $k_\zeta \leq \zeta(\omega, t) \leq \kappa_\zeta$, $\mathbb{P}$-a.s. for all $t \in [0,T]$.

Proposition 3.1.2. *Let Assumption 3.1.1 hold, then $\zeta \in \mathcal{A}'$.*

Proof. Under Assumption 3.1.1 the solution of (3.11) $\zeta(t) = \mathcal{E}(t)\exp\{\int_0^t r(s)ds - \beta(t)\}$ (see (3.1) for the definition of the exponential martingale $\mathcal{E}(t)$) is continuous; moreover, being uniformly bounded in $(\omega, t) \in \Omega \times [0,T]$, it is uniformly of class (D) as well, i.e. the random variable $\zeta(\tau)$ is uniformly integrable for every stopping time $\tau \leq T$.

Let π be any partition of $[0,T]$. Denoting by $V_\pi(\zeta)$ the conditional variation of $\zeta(t)$, we have

$$\begin{aligned} V_\pi(\zeta) &= \mathbb{E}\{|\zeta(T)|\} \\ &\quad + \sum_{k=0}^{n-1} \mathbb{E}\Big\{\Big|\mathbb{E}\Big\{ - \int_{t_k}^{t_{k+1}} \zeta(u)r(u)du - \int_{t_k}^{t_{k+1}} \zeta(u)\theta^T(u)dW(u) - \int_{t_k}^{t_{k+1}} \zeta(u)d\beta(u)\Big|\mathcal{F}_{t_k}\Big\}\Big|\Big\} \\ &\leq C\Big(1 + \sum_{k=0}^{n-1} \mathbb{E}\Big\{\int_{t_k}^{t_{k+1}} r(u)du + \int_{t_k}^{t_{k+1}} |d\beta(u)|\Big\}\Big) \\ &= C\Big(1 + \mathbb{E}\Big\{\int_0^T r(u)du\Big\} + \mathbb{E}\{||\beta||_T\}\Big) \leq \widetilde{C} \end{aligned}$$

by Assumption 3.1.1. It follows that ζ belongs to $\mathcal{A}'$. □

Proposition 3.1.3. *The set $\mathcal{A}'$ is relatively compact (in the sense of convergence in distribution) when $D([0,T];\mathbb{R})$ is endowed with the Meyer-Zheng topology.*

Proof. Notice that every $X \in \mathcal{A}'$ is a uniformly bounded in (ω, t) quasimartingale. Hence [50], Theorem 4 implies the thesis. □

From now on we use the notation $X_n \overset{MZ}{\Rightarrow} X$ to indicate that the probabilities $\mathbb{P}^{X_n}$ converge to $\mathbb{P}^X$ when the sample space $D([0,T];\mathbb{R})$ is endowed with the Meyer-Zheng topology. Moreover $X_n \overset{MZ}{\rightarrow} X$ if the sample path X_n converges to X in the Meyer-Zheng topology. We may now start with studying the continuity of the right-hand side of (3.10). Hence, letting $\zeta_n \overset{MZ}{\Rightarrow} \zeta$, we aim to prove that $U_c(R(\hat{C}_n(t)))$ converges (in some sense) to $U_c(R(\hat{C}(t)))$.

Recall that F is a continuous functional on $D([0,T];\mathbb{R})$ endowed with the Meyer-Zheng topology if for all x_n and x in $D([0,T];\mathbb{R})$ such that $x_n \overset{MZ}{\rightarrow} x$ we have

$$\lim_{n\to\infty} \frac{1}{T}\int_0^T |F(t,x_n(t)) - F(t,x(t))|dt = 0.$$

Proposition 3.1.4. *The functional* $\boldsymbol{R}^{y,T}(\cdot) : D([0,T];\mathbb{R}) \to D([0,T];\mathbb{R})$ *is continuous under the Meyer-Zheng topology.*

Proof. Let ζ_n be a sequence in $\mathcal{A}'$ which converges to ζ in the Meyer-Zheng topology. We have

$$\begin{aligned} |\mathbf{R}_n^{y,T}(t) - \mathbf{R}^{y,T}(t)| &\leq \int_0^t |\zeta_n(s) - \zeta(s)| C^0(s) R_c(yC^0(s))ds \\ + & \quad |\zeta_n(t) - \zeta(t)| \frac{C^0(t)}{f_C(t)} \mathbb{1}_{\{t<T\}} + |\zeta_n(T) - \zeta(T)| C^0(T) G'(yC^0(T)) \mathbb{1}_{\{t=T\}}. \end{aligned} \tag{3.12}$$

Hence, concavity of R and G, and continuity of $C^0(t)$ imply

$$\int_0^T |\mathbf{R}^{y,T}(\zeta)(t) - \mathbf{R}^{y,T}(\zeta_n)(t)|dt \leq K \int_0^T |\zeta_n(t) - \zeta(t)|dt, \tag{3.13}$$

where K denotes a suitable constant depending on y, $\inf_{t\in[0,T]} f_C(t)$, $\sup_{t\in[0,T]} C^0(t)$ and on the constants that appear in the growth conditions on G and R. The thesis follows by recalling that convergence in the Meyer-Zheng topology is just convergence in Lebesgue measure (cf. Appendix A, Lemma A.0.10). □

By continuous mapping Theorem we have the following simple result.

Lemma 3.1.5. *If* $\zeta_n \overset{MZ}{\Rightarrow} \zeta$ *then* $\boldsymbol{R}_n^{y,T} \overset{MZ}{\Rightarrow} \boldsymbol{R}^{y,T}$.

Proposition 3.1.6. *Denote by* $\underline{Y}_n := (\boldsymbol{R}_n^{y,T}, \zeta_n, C^0)$ *and by* $\underline{\Upsilon}_n$ *the random vector whose components are the Snell envelopes of the elements of* $\underline{Y}$. *If* $(\zeta_n, C^0, \boldsymbol{R}_n^{y,T}) \overset{MZ}{\Rightarrow} (\zeta, C^0, \boldsymbol{R}^{y,T})$ *then* $(\underline{Y}_n, \underline{\Upsilon}_n) \overset{MZ}{\Rightarrow} (\underline{Y}, \underline{\Upsilon})$.

Proof. For this proof we aim to apply the results in [57] regarding the functional convergence of the Snell envelopes in American Options approximations. First of all, notice that the process $\mathbf{R}^{y,T}(t)$ does not satisfy hypothesis (**H**) in [57], Definition 2.4 since it is not Markovian depending on the path of processes C^0 and ζ up to time t. On the other hand, if we consider the Markovian three-dimensional process $(\mathbf{R}^{y,T}(t), C^0(t), \zeta(t)) \in D([0,T];\mathbb{R}^3)$, hypothesis (**H**) of [57] does apply.

Notice that [57], condition (3.1), is satisfied being $(\mathbf{R}_n^{y,T})_{n\in\mathbb{N}}$ of class (D), i.e. the r.v. $(\mathbf{R}_n^{y,T}(\tau))_{n\in\mathbb{N}}$ are uniformly integrable for every stopping time $\tau \le T$. In fact, since R and G are concave functions such that $R(x) \le \kappa_\eta + \eta x$ and $G(x) \le k_\epsilon + \epsilon x$ and f_C and ζ_n are bounded processes, we may write

$$|\mathbf{R}_n^{y,T}(\tau)| \le K(1 + \sup_{0\le t\le T} C^0(t)), \qquad \forall \tau \in [0,T],$$

where K denotes a suitable constant depending on y, κ_ϵ, ϵ, κ_η, η, T and the bounds on f_C and ζ_n. Moreover, notice that the process $\mathbf{R}_n^{y,T}(t)$ has right-continuous paths, hence

$$\lim_{\delta\to 0^+} \frac{1}{\delta} \int_t^{t+\delta} \mathbf{R}_n^{y,T}(s)ds = \mathbf{R}_n^{y,T}(t).$$

Denote by $\mathbb{E}_n\{\cdot\}$ the expectation under the distribution $\mathbb{P}_n$ of $\mathbf{R}_n^{y,T}$ on the canonical space $D([0,T];\mathbb{R})$. By Lebesgue Theorem, there exists $\delta(\epsilon;n)$ such that

$$\mathbb{E}_n\left\{|\frac{1}{\delta}\int_t^{t+\delta} \mathbf{R}_n^{y,T}(s)ds - \mathbf{R}_n^{y,T}(t)|\right\} < \epsilon, \qquad \delta < \delta(\epsilon;n).$$

Set now $\gamma := \inf_{n\in\mathbb{N}} \delta(\epsilon;n)$. Then, if $\delta < \gamma$ we have

$$\mathbb{E}_n\left\{|\frac{1}{\delta}\int_t^{t+\delta} \mathbf{R}_n^{y,T}(s)ds - \mathbf{R}_n^{y,T}(t)|\right\} < \epsilon, \qquad \forall n \in \mathbb{N}.$$

It follows that the assertion of Lemma 3.3 in [57] is also fulfilled.

Moreover, a sequence $(C^0(t), \zeta_n(t))_{n\in\mathbb{N}} \in D([0,T];\mathbb{R}^2)$, with $(\zeta_n(t))_{n\in\mathbb{N}} \in \mathcal{A}'$ and converging in the Meyer-Zheng topology to $(C^0(t), \zeta(t)) \in D([0,T];\mathbb{R}^2)$, fulfill hypotheses 3.1 being $(\zeta_n(t))_{n\in\mathbb{N}}$ and $C^0(t)$ uniformly of class (D). By using the same arguments as for $\mathbf{R}^{y,T}_n(t)$, we may prove that $(\zeta_n(t))_{n\in\mathbb{N}}$ and $C^0(t)$ satisfy the assertion of Lemma 3.3 in [57].

Denoting by $\underline{Y}_n := (\mathbf{R}^{y,T}_n(t), \zeta_n(t), C^0(t))$ and by $\underline{\Upsilon}_n(t)$ the random vector whose components are the Snell envelopes of the elements of $\underline{Y}$, i.e.

$$\underline{\Upsilon}_n(t) := \left(\operatorname*{ess\,inf}_{t\le\tau\le T} \mathbb{E}_n\{\mathbf{R}^{y,T}_n(\tau)|\mathcal{F}_t\}, \operatorname*{ess\,inf}_{t\le\tau\le T} \mathbb{E}_n\{\zeta_n(\tau)|\mathcal{F}_t\}, \operatorname*{ess\,inf}_{t\le\tau\le T} \mathbb{E}\{C^0(\tau)|\mathcal{F}_t\} \right),$$

then by [57], Theorem 3.5 and Remark 3.6, we have $(\underline{Y}_n, \underline{\Upsilon}_n) \overset{MZ}{\Rightarrow} (\underline{Y}, \underline{\Upsilon})$ with

$$\underline{\Upsilon}(t) := \left(\operatorname*{ess\,inf}_{t\le\tau\le T} \mathbb{E}\{\mathbf{R}^{y,T}(\tau)|\mathcal{F}_t\}, \operatorname*{ess\,inf}_{t\le\tau\le T} \mathbb{E}\{\zeta(\tau)|\mathcal{F}_t\}, \operatorname*{ess\,inf}_{t\le\tau\le T} \mathbb{E}\{C^0(\tau)|\mathcal{F}_t\} \right).$$

Therefore the thesis follows. □

Notice that, under hypothesis (**H**) in [57], Definition 2.4, every limit law of $(\underline{Y}_n, \underline{\Upsilon}_n, \tau^*_n)$ on $D([0,T];\mathbb{R}^6) \times [0,T]$ is the law of $(\underline{Y}, \underline{\Upsilon}, \tau^*)$ on $D([0,T];\mathbb{R}^6) \times [0,T]$ where τ^* is an optimal stopping time for $(\mathbf{R}^{y,T}, \mathcal{Z}^{y,T})$ (cf. [57], Remark 3.8). Then, by the Skorohod representation theorem (see, e.g., [15]), we can assume without loss of generality that on a common probability space, still denoted by $(\Omega, \mathcal{F}, \mathbb{P})$, τ^*_n converges to τ^* almost surely. Hence the following proposition holds.

Proposition 3.1.7. *$\overline{\nu}^y_n(t) \to \overline{\nu}^y(t)$ a.s. for every $t \in [0,T]$.*

Proof. If $\tau^*_n(0,y) \to \tau^*(0,y)$ a.s. for all $y > 0$, then it is well known that its generalized inverse, i.e. $\overline{\nu}^y_n$, converges weakly to $\overline{\nu}^y$, that is $\overline{\nu}^y_n(t) \to \overline{\nu}^y(t)$ a.s. for all the times $t \in [0,T]$ of continuity of $\overline{\nu}^y(\cdot)$. Being $\hat{\nu}(t)$, with continuous paths and thus a.s. finite on $[0,T]$ (cf. [22], (3.18)), by Dominated Convergence Theorem we have that $\overline{\nu}^y(t) := \int_{[0,t)} \frac{C^0(s)}{f_C(s)} d\hat{\nu}(s)$ has continuous trajectories as well. Hence $\overline{\nu}^y_n(t)$ converges a.s. to $\overline{\nu}^y(t)$ for all $t \in [0,T]$. □

Corollary 3.1.8. *$\hat{C}_n(t)$ converges a.s. to $\hat{C}(t)$ for all $t \in [0,T]$.*

Proof. Since $\hat{C}_n(t) := C^0(t)[y + \overline{\nu}^y_n(t)]$, obviously $\lim_{n\to\infty} \hat{C}_n(t) = \hat{C}(t)$ a.s. for all $t \in [0,T]$. □

We may now prove the main Theorem.

Theorem 3.1.1. *There exists a solution to the fixed point problem (3.10).*

Proof. Proposition 3.1.4, Proposition 3.1.6, Proposition 3.1.7, Corollary 3.1.8 and, finally, the continuity of R and U_c (cf. [23], Lemma 4.4) imply the continuity of the right-hand side of (3.10). Hence an application of Schauder Fixed Point Theorem (e.g., cf. [61]) guarantees the existence of an equilibrium deflator, that is the existence of a solution ζ to (3.10). □

Remark 3.1.9. *A future attempt will be surely to remove Assumption 3.1.1. By assuming that the uniformly bounded continuous processes $\widetilde{r}$ and w' have uniformly bounded conditional variation, then ζ, $\widetilde{r}$ and w' are continuous quasimartingale that belong to the set*

$$\begin{aligned}\mathcal{D} \;:=\; & \{X : \Omega \times [0,T] \to \mathbb{R}^3 \text{ continuous, uniformly of class (D) and} \\ & \text{uniformly bounded in } (\omega, t) \text{ with bounded conditional variation}\}.\end{aligned}$$

By [50], Theorem 4, the set $\mathcal{D}$ is relatively compact (in the sense of convergence in distribution) if $D([0,T];\mathbb{R}^3)$ is endowed with the Meyer-Zheng topology. Then, arguments similar to those used in Proposition 3.1.4, Proposition 3.1.6, Proposition 3.1.7, Corollary 3.1.8, together with the continuity of R, U_c, U_l, U_m and $I^{R Q(M)(\hat{C}(t),\cdot,\cdot)}$ (cf. [23], Lemma 4.4, and [21], Proposition 3.2) allow to conclude that there exists a solution $(\zeta, \widetilde{r}, w')$ to (3.4).

Chapter 4

Concluding Remarks and Open Problems

In this Thesis we have studied stochastic, irreversible investment problems in continuous time. We have developed a new approach based on first order conditions for optimality which may be thought as generalized stochastic Kuhn-Tucker conditions.

In particular, in Chapter 1, we analyzed in a very general semimartingale setting the optimal investment problem for the Social Planner of a market with N firms and with limited resources. Our approach generalizes that of Peter Bank [7] for a single firm. The optimal solution has been given in terms of the base capacity process $l^*(t)$, a desirable value of production capacity, unique optional solution of a representation problem in the spirit of Bank and El Karoui [6].

Chapter 2 has tackled the problem of the meaning of $l^*(t)$ in a diffusion setting. We have studied a stochastic, continuous time model on a finite horizon for a firm that produces a single good and whose production capacity is a controlled Ito process. Under suitable assumptions on the controlled diffusion coefficients, we have showed that the base capacity process is actually deterministic and coincides with the free boundary $\hat{y}(t)$ of the optimal stopping problem naturally associated to the singular control one. It follows that the Bank-El Karoui representation problem gives rise to an integral equation for $\hat{y}(t)$ which might be solved numerically by backward induction.

Finally, in Chapter 3 we have considered the optimal irreversible investment problem for a firm embedded in a stochastic continuous time economy on a finite time interval, as modeled in [22]. Under Markovian assumptions, we have studied a very daunting random fixed point problem arising from stochastic first order conditions for the general equilibrium of the economy when the discount factor of the firm's manager coincides with the deflator.

The new approach for solving singular, stochastic control problems we have developed in this Thesis might apply to other some very interesting problems arising in Economics. In this Chapter we want to discuss some possible developments of the research.

First of all we may study the singular stochastic control problem of Chapter 2 when in the economy there are limited resources. In particular we may consider an economy on a finite time horizon T whose productive sector is represented by a firm with capacity dynamics

$$\begin{cases} dC^{y,\nu}(t) = C^{y,\nu}(t)[-\mu_C(t)dt + \sigma_C(t)dW(t)] + f_C(t)d\nu(t), \;\; t \in [0,T), \\ C^{y,\nu}(0) = y > 0. \end{cases} \tag{4.1}$$

We may assume that any investment strategy $\nu(t)$ is such that $\nu(t) \leq \theta(t)$, for some nondecreasing, integrable, left-continuous, adapted process θ. The firm's manager problem is

$$\sup_{\nu \leq \theta} \mathbb{E}\bigg\{ \int_0^T e^{-\int_0^t \mu_F(s)ds} R(C^{y,\nu}(t))dt - \int_{[0,T)} e^{-\int_0^t \mu_F(s)ds} d\nu(t) \bigg\}. \tag{4.2}$$

Recall that $C^0(t) := C^{1,0}(t)$, $\mathcal{T}$ denotes the set of all stopping times with values in $[0,T]$, and that the optional process

$$\begin{aligned} \nabla_\nu \mathcal{J}_{0,y}(\nu)(\tau) \;\; := \;\; & \mathbb{E}\bigg\{ \int_\tau^T e^{-\int_0^s \mu_F(u)du} C^0(s) \frac{f_C(\tau)}{C^0(\tau)} R_c(C^{y,\nu}(s))\, ds \,\bigg|\, \mathcal{F}_\tau \bigg\} \\ & - e^{-\int_0^\tau \mu_F(u)du} \mathbb{1}_{\{\tau < T\}} \end{aligned}$$

is the supergradient of $\mathcal{J}_{0,y}(\nu)$. At the moment we are able to prove that necessary and sufficient conditions for optimality of $\hat{\nu}(t)$ are

$$\nabla_\nu \mathcal{J}_{0,y}(\hat{\nu})(\tau) \leq 0, \;\; \forall \tau \in \mathcal{T}, \;\; \mathbb{P}-\text{a.s.},$$

$$\mathbb{E}\bigg\{ \int_{[0,T)} \nabla_\nu \mathcal{J}_{0,y}(\hat{\nu})(\tau)\, d\hat{\nu}(\tau) \bigg\} = 0,$$

$$\hat{\nu}(t) \leq \theta(t), \quad \forall t \in [0,T], \;\; \mathbb{P}-\text{a.s.},$$

$$\mathbb{E}\bigg\{ \int_{[0,T)} (\theta(t) - \hat{\nu}(t)) d\lambda(t) \bigg\} = 0.$$

As in Chapter 1 the Lagrange multiplier $d\lambda(\omega,t)$ is an optional random measure on $[0,T]$. We guess that, at least in the case of a constant finite fuel θ_0, the optimal investment strategy takes the form

$$\hat{\nu}(t) = \nu^*(t) \wedge \theta_0, \tag{4.3}$$

with $\nu^*(t)$ the optimal solution of the infinite fuel case discussed in Chapter 2, Theorem 2.3.1. Such policy represents a quite natural generalization of that in [43] for a Brownian Motion controlled by a nondecreasing process. Notice that even in this case the base capacity process $l^*(t)$, unique solution of a representation problem in the spirit of Bank-El Karoui (cf. (2.28)), should coincide with the free boundary of the optimal stopping problem associated to the singular control one. In fact, as stressed in [7], it does not depend on the fuel and, therefore, it may be viewed as a universal signal for a big class of finite fuel optimal stochastic control problem. It follows that it would be interesting to prove guess (4.3) and to study which is the optimal investment strategy in the case of a time-dependent, stochastic, increasing fuel.

It is reasonable to think that a firm might also disinvest. In that case the controls are stochastic processes of bounded variation with minimal decomposition

$$\nu(t) = \nu^+(t) - \nu^-(t),$$

for ν^+ and ν^- increasing and left-continuous. The total expected profit associated to the investment-disinvestment strategy ν might be

$$\mathcal{J}(\nu;x) = \mathbb{E}\bigg\{ \int_0^T R(t,X(t))\,dt - \int_{[0,T)} \gamma(t)\,d\nu^+(t) - \int_{[0,T)} \beta(t)\,d\nu^-(t) \bigg\},$$

hence the firm's optimal problem

$$\sup_{\nu} \mathcal{J}(\nu; x). \tag{4.4}$$

Here the process $\gamma(t)$ (resp. $\beta(t)$) is the running cost per unit of fuel spent to push in the positive (resp. negative) direction, whereas the process $X(t) = x + \nu(t)$ represents the state at time t, when starting at $X(0) = x$. Economically, $\gamma(t)$ is the running cost of investment and $\beta(t)$ the rebate for disinvestment. We may assume that γ and β are optional processes, of class (D), continuous in expectation, such that $\gamma \geq \beta$ and $\gamma(T) = \beta(T) = 0$.

Denoting by $\hat{\nu}$ the optimal solution of (4.4) and by $\hat{X}(t) = x + \hat{\nu}(t)$ the optimal controlled state, then the first order conditions for optimality for problem (4.4) would be

$$\mathbb{E}\Big\{ \int_{\tau}^{T} R_x(t, \hat{X}(t))\, dt \Big| \mathcal{F}_{\tau} \Big\} \leq \gamma(\tau), \quad \forall \tau \in \mathcal{T}, \ \mathbb{P} - \text{a.s.}, \tag{4.5}$$

$$\mathbb{E}\Big\{ \int_{\tau}^{T} R_x(t, \hat{X}(t))\, dt \Big| \mathcal{F}_{\tau} \Big\} \geq -\beta(\tau), \quad \forall \tau \in \mathcal{T}, \ \mathbb{P} - \text{a.s.}, \tag{4.6}$$

with equality whenever investment and disinvestment actually occur. We expect that the optimal control policy $(\hat{\nu}^{+}, \hat{\nu}^{-})$ might be expressed in terms of the solutions (l^*_+, l^*_-) of a coupled representation problem of the Bank-El Karoui's type arising from (4.5) and (4.6). Hence it is engaging to compare this kind of analysis with that one based on the connections between bounded variation control and Dynkin Games (e.g., cf. [47] and [14]). Moreover, when X is a controlled diffusion, it would be intriguing to understand if l^*_+ and l^*_- coincide, as in the irreversible investment case (cf. Chapter 2), with the free boundaries of the investment-disinvestment problem (4.4) and to analyze their path properties, e.g. if they cross each other, if they are monotone, continuous...

A daunting task is to study the problem of Chapter 2 when the production capacity is a very general controlled diffusion given by

$$\begin{cases} dC^{y,\nu}(t) = C^{y,\nu}(t)[-\mu_C(t, C^{y,\nu}(t))\, dt + \sigma_C(t, C^{y,\nu}(t))\, dW(t)] + f_C(t) d\nu(t), \ \ t \in [0, T) \\ C^{y,\nu}(0) = y > 0, \end{cases} \tag{4.7}$$

for some deterministic coefficients satisfying suitable assumptions. By introducing the Doléans-Dade exponential of $C^{y,\nu}$, i.e.

$$\mathcal{E}(C^{y,\nu}(t)) := \exp\bigg\{ - \int_0^t (\mu_C(s, C^{y,\nu}(s)) + \frac{1}{2}\sigma_C^2(s, C^{y,\nu}(s))\, ds + \int_0^t \sigma_C(s, C^{y,\nu}(s))\, dW(s)\bigg\},$$

and

$$\overline{\nu}(t) := \int_{[0,t)} \frac{f_C(s)}{\mathcal{E}(C^{y,\nu}(s))} d\nu(s),$$

then, by Ito's Lemma, one formally obtains

$$C^{y,\nu}(t) = \mathcal{E}(C^{y,\nu}(t))[y + \overline{\nu}(t)]. \tag{4.8}$$

Notice that, under some assumptions on the diffusion coefficients and on the production function, we are in the same setting of [34] with no absolutely continuous controls. Hence, the control problem

$$\sup_{\nu} \mathbb{E}\bigg\{ \int_0^T e^{-\int_0^t \mu_F(s)ds}\, R(C^{y,\nu}(t))dt - \int_{[0,T)} e^{-\int_0^t \mu_F(s)ds} d\nu(t) \bigg\} \tag{4.9}$$

has an optimal solution $\widehat{\nu}(t)$. However that solution is not unique; in fact, althought the production function is strictly concave, the capacity process $C^{y,\nu}(t)$ is not affine in ν. We are able to show that a process $\hat{\nu}$ is optimal for problem (4.9) if it satisfies the following first-order conditions

$$\nabla_\nu \mathcal{J}_{0,y}(\hat{\nu})(\tau) \le 0, \quad \forall \tau \in \mathcal{T}, \;\; \mathbb{P}-\text{a.s.}, \tag{4.10}$$

$$\mathbb{E}\bigg\{ \int_{[0,T)} \nabla_\nu \mathcal{J}_{0,y}(\hat{\nu})(\tau)\, d\hat{\nu}(\tau) \bigg\} = 0, \tag{4.11}$$

with $\nabla_\nu \mathcal{J}_{0,y}(\nu)$ the unique optional process given by

$$\begin{aligned} \nabla_\nu \mathcal{J}_{0,y}(\nu)(\tau) \;\; := \;\; & \mathbb{E}\bigg\{ \int_\tau^T e^{-\int_0^s \mu_F(u)du} \mathcal{E}(C^{y,\nu}(s)) \frac{f_C(\tau)}{\mathcal{E}(C^{y,\nu}(\tau))} R_c(C^{y,\nu}(s))\, ds \,\bigg|\, \mathcal{F}_\tau \bigg\} \\ & -e^{-\int_0^\tau \mu_F(u)du} \mathbb{1}_{\{\tau<T\}}. \end{aligned} \tag{4.12}$$

Notice that (4.10) and (4.11) are not necessary conditions as instead (2.11) and (2.12). In fact in this general case we are not able to apply Fatou's Lemma, a crucial tool for proving the necessity of the first-order conditions for optimality. We can show that an optimal capacity production can be represented as

$$C^{y,\hat{\nu}}(t) = C^0(t)\left(y \vee \sup_{0\le u\le t}\left(\frac{l^*(u)}{C^0(u)}\right)\right), \tag{4.13}$$

with $C^0(t) := C^{1,0}(t)$. The optional process $l^*(t)$ is expected to be the solution of a representation problem in the spirit of Bank-El Karoui, i.e.

$$\begin{aligned}\mathbb{E}\left\{\int_\tau^T e^{-\int_0^s \mu_F(u)du}\mathcal{E}(C^{y,\hat{\nu}}(s))\, R_c\left(C^0(s)\sup_{\tau\le u\le s}\frac{l^*(u)}{C^0(u)}\right)ds\,\middle|\,\mathcal{F}_\tau\right\}\\ = e^{-\int_0^\tau \mu_F(u)du}\,\frac{\mathcal{E}(C^{y,\hat{\nu}}(\tau))}{f_C(\tau)}\,\mathbb{1}_{\{\tau<T\}}.\end{aligned} \tag{4.14}$$

We stress that (4.13) is not an explicit solution for the optimal investment problem (4.9) depending $l^*(t)$ through (4.14) on $C^{y,\hat{\nu}}(t)$ itself. Hence (4.13) is actually a fixed point problem. However, formula (4.13) is an interesting characterization of an optimal production capacity $C^{y,\hat{\nu}}(t)$ in terms of the solution of representation problem (4.14). It would be then of some interests to understand which is in this setting the meaning of the the 'base capacity' $l^*(t)$, i.e. of the optional solution to (4.14).

Finally a further attempt is to study the fixed point problem of Chapter 3 removing Assumption 3.1.1.

Appendix A

The Meyer-Zheng Topology

In this Appendix we recall the main facts about the Meyer-Zheng (or pseudo-path) topology [50] for the Skorohod space $D([0,T];\mathbb{R}^N)$ of càdlàg processes with values in $\mathbb{R}^N$, $N \geq 1$. The Meyer-Zheng topology has been widely used concerning the existence of solutions to backward stochastic differential equations (see [1] and [16] among others) and for showing the existence of singular stochastic optimal controls as in [34].

For sake of semplicity set $N = 1$, then for every $X \in D([0,T];\mathbb{R})$ define the *pseudo-path* of X to be a probability measure on $[0,T] \times \overline{\mathbb{R}}$ as

$$\mathbb{P}^X(A) := \frac{1}{T}\int_0^T \mathbb{1}_A(t, X(t))dt, \qquad \forall A \in \mathcal{B}([0,T] \times \overline{\mathbb{R}}), \tag{A-1}$$

where $\mathcal{B}([0,T] \times \overline{\mathbb{R}})$ denotes the Borel σ-algebra and $\mathbb{1}_A(\cdot)$ the indicator function of a set A. By definition, the pseudo-path of X is the image measure of the Lebesgue measure over $[0,T]$, $\lambda(dt) := \frac{1}{T}dt$, under the mapping $t \to (t, X(t))$.

Denote by ψ the mapping which associates to a path X its pseudo-path; obviously ψ identifies two paths if and only if they are equal a.e. in the Lebesgue sense. In particular, ψ is 1-1 on $D([0,T];\mathbb{R})$, hence we can identify every $X \in D([0,T];\mathbb{R})$ with its pseudo-path. Moreover, ψ provide an imbedding of $D([0,T];\mathbb{R})$ into the compact space $\overline{\mathcal{P}}$ of all probability laws on the compact space $[0,T] \times \overline{\mathbb{R}}$. The topology induced on $D([0,T];\mathbb{R})$ by this embedding is the *pesudo-path* or *Meyer-Zheng topology*. It can also be introduced as the topology generated by the metric

$$d(x,y) = \frac{1}{T}\int_0^T (|x(s) - y(s)| \wedge 1)\, ds, \qquad x, y \in D([0,T];\mathbb{R}). \tag{A-2}$$

Convergence in the metric d is equivalent to convergence with respect to the Lebesgue measure.

Lemma A.0.10. *Let Ψ denote the set of all the pseudo-paths. Then, the pseudo-path topology on Ψ is just convergence in Lebesgue measure, that is $X_n \overset{MZ}{\to} X$ if and only if for every bounded continuous function $f(s,x)$ on $[0,T]\times\overline{\mathbb{R}}$ we have*

$$\lim_{n\to\infty} \frac{1}{T}\int_0^T f(s, X_n(s))ds = \frac{1}{T}\int_0^T f(s, X(s))ds.$$

For the proof see [50], Lemma 1. Notice that the Meyer-Zheng topology is much weaker than the usual Skorohod topology (see [37] for a good introduction on the usual Skorohod topology).

Let $X \in D([0,T];\mathbb{R})$ and define $X^* := \sup_{t\in[0,T]} |X(t)|$. For u and v in $\mathbb{R}$ such that $u < v$, denote by $N^{u,v}(X)$ the number of upcrossing of $X(\cdot)$ on $[0,T]$ between levels u and v. Then a subset $A \subset D([0,T];\mathbb{R})$ such that for every $u < v$

$$\sup_{X\in A} X^* < \infty, \qquad \sup_{X\in A} N^{u,v}(X) < \infty \tag{A-3}$$

is relatively compact in $D([0,T];\mathbb{R})$ endowed with the Meyer-Zheng topology. For details, see [50], Corollary of Theorem 2.

The most significant application of the Meyer-Zheng topology is a tightness criterion for *quasimartingales*. Let X be an adapted, càdlàg process defined on $[0,T]$, such that $\mathbb{E}\{|X(t)|\} < \infty$ for all $t \in [0,T]$. Let $\pi := \{t_0, t_1, ..., t_n : 0 = t_0 < t_1 < ... < t_n = T\}$ be a partition of the time interval $[0,T]$ and define

$$V_\pi(X) := \sum_{i=0}^{n-1} \mathbb{E}\{|\,\mathbb{E}\{X(t_{i+1}) - X(t_i)|\mathcal{F}_{t_i}\}\,|\} + \mathbb{E}\{|X(T)|\}.$$

If the *conditional variation* of X is finite, i.e. if $\sup_\pi V_\pi(X) < \infty$, then X is a *quasimartingale*.

In [50], Theorem 4, the following tightness result for quasimartingales is proved.

Theorem A.0.2. *Let* $(\mathbb{P}_n)_{n\in\mathbb{N}}$ *be a sequence of probability laws on* $D([0,T];\mathbb{R})$ *such that, under* $\mathbb{P}_n$*, the coordinate process* $X(t)$ *is a quasimartingale with variation* $V_n(X)$ *uniformly bounded in* n*. Then there exists a subsequence* $(\mathbb{P}_{n_k})_{k\in\mathbb{N}}$ *which converges weakly on* $D([0,T];\mathbb{R})$ *to a law* $\mathbb{P}$*, and* X *is a quasimartingale under* $\mathbb{P}$*.*

Finally notice that, contrary to the case of the usual Skorohod topology, the Meyer-Zheng topology on the product space $D([0,T];\mathbb{R}^N)$ is the product topology; then, if each component of the $\mathbb{R}^N$ valued random vector is tight, then the vector is tight as well.

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