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Dispersive properties of the Dirac equation

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Contents

CHAPTER 1

Introduction

1. Dispersive equations: Strichartz and smoothing estimates

With the term *dispersion* we mean, roughly speaking, the property of each elementary component of a wave packet to travel with a speed depending on the frequency. Many fundamental partial differential equations in quantum mechanics are dispersive: to name the most important ones in fact

$$\textbf{Schrödinger:}\qquad \begin{cases} iu_t+\Delta_x u(t,x)=0\\ u(0,x)=f(x), \end{cases}\qquad (t,x)\in\mathbb{R}\times\mathbb{R}^n, \tag{1.1}$$

$$\textbf{Wave}:\qquad \begin{cases} u_{tt}-\Delta_x u(t,x)=0\\ u(0,x)=f(x)\\ u_t(0,x)=g(x) \end{cases}\qquad (t,x)\in\mathbb{R}\times\mathbb{R}^n, \tag{1.2}$$

$$\textbf{Klein}-\textbf{Gordon}:\qquad \begin{cases} u_{tt}-\Delta_x u(t,x)+u(t,x)=0\\ u(0,x)=f(x)\\ u_t(0,x)=g(x) \end{cases}\qquad (t,x)\in\mathbb{R}\times\mathbb{R}^n, \tag{1.3}$$

$$\textbf{Dirac}:\qquad \begin{cases} iu_t-\mathcal{D}u(t,x)-\beta m u(t,x)=0\\ u(0,x)=f(x) \end{cases}\qquad (t,x)\in\mathbb{R}\times\mathbb{R}^3, \tag{1.4}$$

are all *dispersive equations*. The existence of some connections between these equations must not be completely surprising: although deeply different from many points of view, they in fact all can be written as

$$u_t+ih(D)u=0,\qquad h(D)=\mathcal{F}^{-1}(h(\xi)\mathcal{F})$$

where $\mathcal{F}$ is the Fourier transform with respect to x. As a consequence, the solutions can be defined by $u=e^{ith(D)}f$ once we impose the initial conditions, so it is not unnatural to suppose that they all show some common features, which have to be related to the structure of the propagator $e^{ith(D)}$ and so of the operator $h(D)$.
In this first introduction we mean to give an outline of some basic tools of dispersion theory (as a standard reference see [**73**]), mainly focusing on the aspects we shall deal with in the other chapters; we here begin by analizing the cases of the Schrödinger (1.1) and the wave equations (1.2) as main examples since the Dirac one, the main topic of this thesis, will be extensively introduced in the next section.
The Schrödinger equation can be by many meanings considered the clearest example: its propagator is indeed immediately determined to be $S(t)=e^{it\Delta}$ which is a unitary group of operators on L^2, and so it conserves the mass (the

L^2 spacial norm). The solution can be directly represented by its convolution kernel, or by Fourier transforming with respect to the space variable:

$$u(t,x) \cong \int_{\mathbb{R}^n} e^{i(t|\xi|^2+x\cdot\xi)}\hat{f}(\xi)d\xi = \frac{e^{i\frac{|x|^2}{4t}}}{(4\pi it)^{n/2}}\int_{\mathbb{R}^n} e^{-i\frac{x\cdot y}{2t}}e^{i\frac{|y|^2}{4t}}f(y)dy$$

which immediately yields the estimate

$$\|e^{it\Delta}f\|_{L^\infty} \lesssim \frac{1}{t^{n/2}}\|f\|_{L^1}. \qquad (1.5)$$

Interpolating with the conservation of energy

$$\|e^{it\Delta}f\|_{L^2} = \|f\|_{L^2}$$

we obtain the *dispersive estimates* (or *time-decay estimates*) for the Schrödinger flow:

$$\|e^{it\Delta}f\|_{L^p} \lesssim t^{-\frac{n}{2}+\frac{n}{p}}\|f\|_{L^{p'}}, \qquad p>2. \qquad (1.6)$$

As a direct consequence of dispersive estimates we obtain *Strichartz estimates* which are, as we shall see in details in the next pages, a fundamental tool in the analysis of nonlinear problems; first introduced as a restriction result, their application to questions of well posedness for dispersive equations turned to be remarkably useful since they provide suitable spaces on which develop the contraction method (as standard and extensive references see [**75**], [**64**], [**72**], [**59**], [**38**], [**47**]).
For the free Schrödinger operator, Strichartz estimates are given by

$$\|e^{it\Delta}f\|_{L^p_tL^q_x} \lesssim \|f\|_{L^2}, \qquad (1.7)$$

where the mixed spaces $L^p_tL^q_x = L^p(\mathbb{R}_t; L^q(\mathbb{R}^n_x))$ are called *Strichartz spaces*, and the exponents (p,q) must be *Schrödinger admissibile*, i.e. they have to satisfy the relation

$$\frac{2}{p}+\frac{n}{q}=\frac{n}{2}, \qquad 2\le p\le\infty, \qquad \frac{2n}{n-2}\ge q\ge 2, \qquad q\neq\infty.$$

The couple $(p,q)=(2,2n/(n-2)$ is called the *endpdoint* and has been separately proved to be true in [**47**] for $n\ge 2$.
With all of this in mind we turn to the wave equation, which is a bit more tricky, since it involves loss of derivatives. The Strichartz estimates for the wave flow

$$e^{it|D|}f = \cos(t|D|)f + \frac{\sin(t|D|)}{|D|}f$$

turn to be

$$\|e^{it|D|}f\|_{L^p\dot{H}_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2}, \qquad (1.8)$$

(here and in the following $\|f\|_{\dot{H}^s_q} = \||D|^sf\|_{L^q}$), provided the exponents (p,q) are *wave admissible*, i.e.

$$\frac{2}{p}+\frac{n-1}{q}=\frac{n-1}{2}, \qquad 2\le p\le\infty, \qquad \frac{2(n-1)}{n-3}\ge q\ge 2, \qquad q\neq\infty.$$

The wave equation endpoint is the couple $(p,q)=(2,2(n-1)/(n-3))$ and is allowed in dimension $n\ge 4$, while it is proved to be false for $n=3$.

Finally, we also recall the Strichartz estimates for the free Klein-Gordon equation, which turn to be

$$\|e^{it\langle D\rangle}f\|_{L^pH_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2}, \tag{1.9}$$

with the same admissibility condition of the Schrödinger equation.
The importance of Strichartz estimates is mainly connected, as already mentioned, to questions of existence of solutions for nonlinear problems with low regularity. When dealing with nonlinear perturbations it is indeed crucial to have some efficient ways to control the "size" of solutions to the linear problem in term of the size of the initial datum, and to understand which are the suitable function spaces in which the PDE is naturally wellposed. In fact, Strichartz estimates can be viewed in two ways.

Locally in time they describe a type of smoothing effect, reflected in a gain of integrability and/or of regularity, in a L^p time-averaged sense. For the Schrodinger case for instance we get that if the initial datum is in L^2 then the solution is in L^q with $q > 2$ for most of the time.

Globally in time they describe a decay effect, meaning that some spacial norm of a solution must decay to zero as $t \to \infty$, at least in some L^p time-averaged sense.

Another set of important estimates due to the dispersive property are the so called *Kato-smoothing estimates*, also known as weak dispersive estimates. First discovered by Kato for the KdV, it is not a completely surprising fact for equations with infinite speed of propagation that the solutions are more regular than the initial data; the gain of derivatives is a very interesting fact and often turns to be a crucial tool in the analysis of nonlinear problems. In the case of Schrödinger equation (see [**18**], [**68**], [**77**], [**76**] as extensive references) these estimates are given by

$$\|\langle x\rangle^{-\frac{1}{2}-}|D|^{\frac{1}{2}}e^{it\Delta}f\|_{L^2L^2} \lesssim \|f\|_{L^2}; \tag{1.10}$$

a stronger local version of this inequality is the following

$$\sup_{R>0}\frac{1}{R}\int_{-\infty}^{+\infty}\int_{|x|\leq R}|\nabla(e^{it\Delta}f)|^2dxdt \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}}. \tag{1.11}$$

Both of these show that the unique solution of the free Schrödinger equation with initial datum f gains half derivative in L^2 with respect to f, if we look to a weighted $L_t^2L_x^2$ norm. Analogous results are available for the wave equation, though no gain of regularity can be expected because of the finite speed of propagation of the semigroup $e^{it|D|}f$.

2. The Dirac equation

In relativistic quantum mechanics the state of a free electron is represented by a wave function $\Psi(t,x)$ with $\Psi(t,\cdot) \in L^2(\mathbb{R}^3,\mathbb{C}^4)$ for any t. This wave satisfies the *free Dirac equation*:

$$i\partial_t\Psi = \mathcal{D}\Psi + m\beta\Psi \tag{1.12}$$

where $m \geq 0$ is the mass of the electron and $\mathcal{D}$, the Dirac operator, is the first order operator defined as

$$\mathcal{D} = -i \sum_{k=1}^{3} \alpha_k \partial_k = -i(\alpha \cdot \nabla)$$

where α_1, α_2, α_3 and β are 4×4 complex matrices, that in the standard representation can be written in terms of the Pauli matrices as

$$\beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} \qquad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3 \tag{1.13}$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.14}$$

To briefly retrace the original argument followed by Dirac that led to equation (1.12), we recall that formally the transition from classical to quantum mechanics can be accomplished by substituting appropriate operators, usually differential or multiplication operators, for the classical quantities, which act on suitable wavefunctions. In particular, for the energy E and the momentum p of a free particle the substitution

$$E \to ih\frac{\partial}{\partial t}, \qquad p \to -ih\nabla \tag{1.15}$$

is familiar from the nonrelativistic theory. If applied to the classical relativistic energy-momentum relation,

$$E = \sqrt{c^2p^2 + m^2c^4} \tag{1.16}$$

gives the square-root Klein-Gordon equation

$$ih\frac{\partial}{\partial t}\Psi(t,x) = \sqrt{-c^2h^2\Delta + m^2c^4}\,\Psi(t,x), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^3.$$

Due to the asymmetry of space and time derivatives, Dirac found it impossible to include external electromagnetic fields in a relativistically invariant way; so he looked for another equation which can be modified in order to describe the action of electromagnetic forces. This new equation should also describe the internal structure of the electrons, the *spin*. The *Klein-Gordon equation*

$$-h^2\frac{\partial^2}{\partial t^2}\Psi(t,x) = (-c^2h^2\Delta + m^2c^4)\Psi(t,x) \tag{1.17}$$

with a scalar wavefunction Ψ was not able to do so. Moreover, a quantum mechanical evolution equation should be of first order in the time derivative. So Dirac reconsidered the energy-momentum relation (1.16) and before translating it to quantum mechanics with the help of (1.15), he linearized it by writing

$$E = c\sum_{j=1}^{3} \alpha_j p_j + \beta mc^2 = c\alpha \cdot p + \beta mc^2 \tag{1.18}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β have to be determined by (1.16). Indeed, (1.16) can be satisfied if one assumes that α and β are anticommuting quantities, which are most naturally represented by $n \times n$ matrices (the "Dirac matrices"). Comparing E^2 according to equations (1.16) and (1.18) we find that the following relations must hold

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}\mathbf{1}, \quad j,k = 1,2,3; \tag{1.19}$$

$$\alpha_j\beta + \beta\alpha_j = \mathbf{0}, \quad j = 1,2,3; \tag{1.20}$$

$$\beta^2 = \mathbf{1} \tag{1.21}$$

where δ_{jk} denotes the Kronecker symbol and $\mathbf{1}$,$\mathbf{0}$ are the n-th dimensional unit and zero matrices. The $n \times n$ matrices α and β should be Hermitian so that (1.18) can lead to a self-adjoint expression, that is a necessary tool for a quantum mechanical interpretation. For what concerns the dimension n of such matrices, it can be determined as follows. From (1.19)-(1.21) we have

$$\mathrm{Tr}(\alpha_{\mathrm{i}}) = \mathrm{Tr}(\beta^2\alpha_{\mathrm{i}}) = -\mathrm{Tr}(\beta\alpha_{\mathrm{i}}\beta) = -\mathrm{Tr}(\alpha_{\mathrm{i}}\beta\beta) = -\mathrm{Tr}(\alpha_{\mathrm{i}}) = 0. \tag{1.22}$$

On the other hand, in view of $\alpha_1^2 = 1$, the eigenvalues of α_i must be ± 1. This together with (1.22) shows that the dimension n of the matrices has to be an even number. For $n = 2$ there are at most three linearly independent anticommuting matrices: for example the Pauli matrices (1.14) together with the unit matrix $\mathbf{1}_n$ form a basis in the space of Hermitian 2×2 matrices. Hence there is no room for a "rest energy" matrix β in two dimensions. In four dimensions all the properties (1.19), (1.20), (1.21) can be satisfied with the choice (1.13), that is the so called standard representation. Notice that "translating" with this choice equation (1.18) to quantum mechanics (and settig for convenience $h = 1$) one immediately obtains equation (1.23).

Our attention will be mostly devoted to the 3-dimensional *massless Dirac equation*

$$\begin{cases} iu_t - \mathcal{D}u(t,x) = 0 \\ u(0,x) = f(x) \end{cases} \tag{1.23}$$

(we shall follow the mathematical notation indicating with u the unknown function). The first thing to be observed is that massless Dirac equation is strictly connected to to the wave one; by properties (1.19)-(1.21) we have indeed

$$(i\partial_t + \mathcal{D})(i\partial_t - \mathcal{D})u = u_{tt} - \Delta u, \tag{1.24}$$

so that every component of the solution of (1.23) satisfies a wave equation. As a consequence, the Strichartz estimates that the free Dirac flow

$$e^{it\mathcal{D}} = \cos(t|D|) + \frac{\sin(t|D|}{|D|}\mathcal{D} \tag{1.25}$$

satisfies are the same as the wave ones for $n = 3$, so that

$$\|e^{it\mathcal{D}}f\|_{L^p\dot{H}_q^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}} \lesssim \|f\|_{L^2} \tag{1.26}$$

with the admissibility condition

$$\frac{2}{p} + \frac{2}{q} = 1, \quad 2 < p \leq \infty, \quad 2 \leq q < \infty.$$

As for the wave equation, the 3D endpoint estimate

$$\|e^{it\mathcal{D}}f\|_{L_t^2 L_x^\infty} \lesssim \|f\|_{\dot{H}^1} \tag{1.27}$$

is generally known to be false.
While thus the linear homogeneous case turns to be quietly straightforward analogous to the wave equation, when we deal with nonlinear or potential perturbations (or both of them) things get of course much more messy, and a deeper insight of the Dirac structure becomes often necessary.

3. Main results

Each chapter of this thesis is almost completely self-contained, and consists of a different and independent paper: in this section we give a rapid outline of the results we have proved, referring to the single introductions for greater details.
The initial aim of this thesis was the study of the well-posedness for the 3D $\dot{H}^1$-critical nonlinear Dirac equation perturbed by a suitable electric potential, i.e. the study of the Cauchy problem

$$\begin{cases} iu_t - \mathcal{D}u(t,x) + V(x)u = P_3(u) \\ u(0,x) = f(x) \in \dot{H}^1 \end{cases} \tag{1.28}$$

where the function

$$P_3(u) = \langle \beta u, u \rangle u \tag{1.29}$$

is the *cubic nonlinearity.*
The unperturbed nonlinear Dirac equation (that is. eq. (1.28) with $V = 0$) is important in relativistic quantum mechanics, and was studied in a number of works (see e.g. [**62**], [**31**], [**57**], [**55**], [**33**], [**52**], [**51**] and for the more general Dirac-Klein-Gordon system see [**25**], [**24**]). In particular, it is well known that the nonlinearity (1.29) is critical for solvability in the energy space $\dot{H}^1$; global existence in $\dot{H}^1$ is still an open problem even for small initial data, while the case of subcritical spaces H^s, $s > 1$ was settled in the positive in [**33**], [**52**]. The major difficulty to overcome here is the lack of the endpoint Strichartz estimate (1.27), since it prevents the standard application of the contraction method (see the beginning of next chapter for further details), and thus a different approach is required. Though nor a positive or negative answer has been obtained, we have proved several interesting improvements in this direction, most of all in the potential-perturbed case, whose interest has rapidly grown in recent years.
In chapter 2 (see [**13**]) we first prove (Theorem 2.1) the endpoint estimate for the perturbed Dirac flow

$$\|e^{it(\mathcal{D}+V)} f\|_{L^2_t L^\infty_x} \lesssim \|f\|_{\dot{H}^1}$$

for initial data f belonging to the class

$$\dot{\mathcal{H}}^1 = \{f_1 + \mathcal{D}f_2,\ f_1 \in \dot{H}^1,\ f_2 \in \dot{H}^2;\ f_1, f_2 \text{ radial}\}.$$

This result, already known in the free case, is still new in presence of a potential; as a consequence of this estimate and of the remarkable fact that a proper subset (see section 5) of $\dot{\mathcal{H}}^1$ is preserved by the cubic non linearity, the standard application of the fixed point theorem yields a result of global existence for problem (1.28) for $\dot{H}^1$-small initial data f with some additional algebraic structure (Theorem 2.2).
In chapter 3 (see [**16**]) we generalize to the potential perturbed case and to

higher dimensions $n \geq 4$ some very recent results involving angular integrability: in [**51**] the authors prove in fact the following endpoint estimate for the wave flow

$$\|e^{it|D|}f\|_{L^2L^\infty} \lesssim \|\Lambda_\omega^\epsilon e^{it|D|}f\|_{L^2L_r^\infty L_\omega^p} \lesssim \||D|\Lambda_\omega^\epsilon f\|_{L^2}, \qquad p > \frac{2}{\epsilon}$$

where the angular derivative operator Λ_ω^s is defined in terms of the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$ as

$$\Lambda_\omega^s = (1 - \Delta_{\mathbb{S}^{n-1}})^{s/2}.$$

Using new mixed Strichartz-smoothing estimates, we extend this result to the perturbed case both for the wave, proving

$$\|u\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \|f\|_{\dot{H}^1} + \|g\|_{L^2} + \|\langle x\rangle^{\frac{1}{2}+}F\|_{L_t^2 L_x^2}.$$

(here u solves $u_{tt} - \Delta u + V(x)u = F(t,x)$ with initial data f and g), and for the Dirac equation, proving

$$\|e^{it(\mathcal{D}+V)}f\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \|f\|_{H^1}.$$

(for the precise statements and all the hypothesis on the potentials see Theorems 3.1 and 3.2). As a consequence, we are able (see Theorem 3.3) to prove global well-posedness for problem (1.28) assuming suitable smallness of the norm $\|\Lambda_\omega^s f\|_{H^1}$, $s > 1$.
chapter 4 (see [**14**]) is devoted to the study of the dispersive properties of the magnetic-potential perturbation of the Dirac equation (this time with $m \geq 0$) for every dimension $n \geq 3$. In fact, generalizing the 3-dimensional results proved in [**11**], we use the classical multiplier method to prove a virial identity for the operator

$$\mathcal{H} = \mathcal{D}_A + m\beta$$

where $\mathcal{D}_A = i^{-1}\sum_{k=1}^{n}(\partial_k - iA^k)$, and $A(x) = (A^1(x), ..., A^n(x)) : \mathbb{R}^n \to \mathbb{R}^n$ is a static magnetic potential (see Theorem 4.2). As a standard consequence of this identity, we obtain the following smoothing estimates (Theorem 4.3)

$$\sup_{R>0}\int_{-\infty}^{+\infty}\int_{|x|\leq R}|e^{it\mathcal{H}}f|^2 dxdt \lesssim \|f\|_{L^2}^2, \qquad \forall f \in L^2$$

and Strichartz estimates as well (Theorem 4.4)

$$\||D|^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}e^{it\mathcal{H}}f\|_{L^pL^q} \lesssim \|f\|_{L^2}$$

for the perturbed Dirac flow, for every $n \geq 3$.
The final chapter 5 (see [**15**]) is instead devoted to some harmonic analysis. The analysis of the dispersive properties of Schrödinger equations on non-flat waveguides (i.e. perturbations of domains of the form $\mathbb{R}^n \times \Omega$ with Ω bounded set) leads to the necessity of a weighted L^2 estimate of the form

$$\|H^\theta f\|_{L^p(w)} \leq C\|(-\Delta)^\theta f\|_{L^p(w)}, \qquad 0 \leq \theta \leq 1$$

where this time $H = -\Delta + V$. This result turns to be a corollary (see Corollary 5.4) of much deeper results of independent interest (Theorems 5.2-5.3), in which we give conditions on $L^p(w)$-boundedness for bounded functions of

Schrödinger operators or, more generally, of selfadjoint operators on $L^2(\mathbb{R}^n)$ satisfying a certain gaussian heat kernel estimate.

CHAPTER 2

The radial Dirac equation: global small solutions

1. Introduction

As it is well known, the classical strategy to study well-posedness for nonlinear dispersive equations is by the use of a fixed point argument in a suitable space, via the appropriate space-time *Strichartz estimates*. A huge literature is available on these estimates for several dispersive operators (wave, Schrödinger, Klein-Gordon, Dirac and others). For homogeneous nonlinear terms one typically finds a threshold regularity s_c in the scale of Sobolev spaces H^s, such that for *subcritical data* with $s > s_c$ solvability holds, while for *supercritical data* with $s < s_c$ one has various degrees of ill-posedness. (see [**73**] as a general reference). Data of critical regularity give rise to difficult questions which depend on the precise structure of the equation.

In this chapter we wish to investigate well-posedness for the 3D cubic non linear massless Dirac equation perturbed with a potential, that is

$$\begin{cases} iu_t - \mathcal{D}u(t,x) + V(x)u = P_3(u) \\ u(0,x) = f(x) \in \dot{H}^1 \end{cases} \tag{2.1}$$

for the spinor field $u : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{C}^4$, where we recall that $\mathcal{D}$ is the operator defined by

$$\mathcal{D} = i^{-1} \sum_{k=1}^{3} \alpha_k \partial_k = -i(\alpha \cdot \nabla)$$

while the 4×4 Dirac matrices are defined as

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3 \tag{2.2}$$

in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

We shall assume that the nonlinear term $P_3(u)$ is cubic of a very specific form, namely

$$\text{either} \quad P_3(u) = \langle \beta u, u \rangle u \quad \text{or} \quad P_3(u) = \langle u, u \rangle u, \tag{2.4}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard hermitian product in $\mathbb{C}^4$ and β is the 4×4 Dirac matrix

$$\beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}.$$

Notice that nonlinearities of the form (2.4) are the most interesting from a physical point of view (see e.g. [**61**]).

To fix the basic ideas, we begin by briefly analyzing the unperturbed case, i.e.

$$\begin{cases} iu_t - \mathcal{D}u(t,x) = P_3(u) \\ u(0,x) = f(x) \end{cases}. \tag{2.5}$$

The typical approach to the Dirac equation is based on the identity

$$(i\partial_t - \mathcal{D})(i\partial_t + \mathcal{D}) \equiv (-\partial_{tt} + \Delta)\mathbb{I}_4 \tag{2.6}$$

which follows from the anticommuting relations

$$\alpha_i\alpha_k + \alpha_k\alpha_i = 2\delta_{ik}\mathbb{I}_4, \quad i,k = 1,2,3, \tag{2.7}$$

$$\alpha_i\beta + \beta\alpha_i = 0, \quad i = 1,2,3, \qquad \beta^2 = \mathbb{I}_4, \tag{2.8}$$

so that the linear propagator $e^{it\mathcal{D}}$ associated to the Dirac operator, which can be defined via Fourier transform as the operator of symbol $e^{it\xi\cdot\alpha}$, can be written as

$$e^{it\mathcal{D}}f = \cos(t|D|)f + i\frac{\sin(t|D|)}{|D|}\mathcal{D}f, \qquad |D| = (-\Delta)^{1/2}, \tag{2.9}$$

showing that the estimates for the Dirac flow $e^{it\mathcal{D}}$ are immediate consequences of the corresponding estimates for the wave flow $e^{it|D|}$, with the same indices, restricted to the special case of dimension $n = 3$. Thus in many cases identity (2.6) allows to reduce problems for the massless Dirac equation to analogous ones for the wave equation, for which many effective tools are available.

For the wave equation on $\mathbb{R}^{1+n}$, $n \geq 3$, Strichartz estimates can be combined with Sobolev embedding and take the general form

$$\||D|^{\frac{n}{q}+\frac{1}{p}-\frac{n}{2}}e^{it|D|}f\|_{L^p_tL^q_x} \lesssim \|f\|_{L^2} \tag{2.10}$$

for all p, q such that

$$p \in [2,\infty], \qquad 0 < \frac{1}{q} \leq \frac{1}{2} - \frac{2}{(n-1)p}.$$

Notice that the limiting case $r = \infty$

$$\|e^{it|D|}f\|_{L^2L^\infty} \lesssim \||D|^{\frac{n-1}{2}}f\|_{L^2} \tag{2.11}$$

is always excluded and is indeed false for general data. See [**38**] and [**47**] for the general Strichartz estimates; concerning the limiting case $r = \infty$, see [**49**], [**35**]. The corresponding estimates for the Dirac equation are given in [**20**]; in particular, the endpoint estimate

$$\|e^{it\mathcal{D}}f\|_{L^2L^\infty} \lesssim \||D|f\|_{L^2} \tag{2.12}$$

fails.

The failure of (2.11) for generic data was first noticed for the 3D wave equation in [**49**], and the corresponding statement for the Dirac equation follows easily from relation (2.20).

A clever use of Strichartz estimates is sufficient for the study of (2.5) in the subcritical case. Local existence, and global existence for small data, was proved in [**33**] for nonlinearities of the form $\langle\beta u, u\rangle^{(p-1)/2}u$ with $p > 3$ and data in H^s with $s > 3/2-1/(p-1)$. The case of cubic nonlinearities and small data in H^s, $s > 1$, was solved in [**52**], where the result was generalized to all space dimensions. We also mention that systems involving the Dirac equation, like Dirac-Klein-Gordon and Maxwell-Dirac, have beed the object of intense attention and the subcritical theory was completed only recently (see [**25**], [**24**], [**26**]).

The failure of the endpoint estimate (2.12) means that above methods break down in the critical case of a cubic nonlinearity with H^1 data, and indeed the problem of global existence is still open in this case. To see the connection between the endpoint estimate (2.12) and the critical equation, we rewrite equation (2.5) as a fixed point problem for the map

$$v \mapsto \Phi(v) = e^{it\mathcal{D}} f + i \int_0^t e^{i(t-t')\mathcal{D}} P_3(v(t'))dt'.$$

If (2.12) were true one could write

$$\left\| \int_0^t e^{i(t-t')\mathcal{D}} v(t')^3 dt \right\|_{L^2 L^\infty} \lesssim \int_{-\infty}^{+\infty} \|e^{it\mathcal{D}} e^{-it'\mathcal{D}} P_3(v(t'))\|_{L^2 L^\infty} dt' \lesssim \|v^3\|_{L^1 H^1}$$

and in conjuction with the conservation of H^1 energy this would imply

$$\|\Phi(v)\|_{L^\infty_t H^1_x} + \|\Phi(v)\|_{L^2_t L^\infty_x} \lesssim \|f\|_{H^1} + \|v\|_{L^\infty H^1} \|v\|^2_{L^2 L^\infty}.$$

In other words, a contraction argument in the norm $\|\cdot\|_{L^2 L^\infty} + \|\cdot\|_{L^\infty H^1}$ would be enough to prove global existence of small H^1 solutions to (2.5).

The main goal of this chapter is to study a special class of solutions to equation (2.5) and more generally to a potential perturbation of the form (2.1), which can be considered as a suitable generalization of the radial solutions to the wave equation in the context of the Dirac equation. We recall that, for small potentials V with suitable decay at infinity, the full range of Strichartz estimates holds also for the perturbed flow $e^{it(\mathcal{D}+V)}$, as proved in [**21**]. Hence subcritical problems can be treated exactly as in the unperturbed case and one can extend the results of [**33**], [**52**] to (2.1) in a straightforward way. Here we focus on the more difficult case of critical $\dot{H}^1$ data with an additional symmetry assumption like

$$f = f_1 + \mathcal{D} f_2 \quad \text{with} \quad f_1 \in \dot{H}^1, \; f_2 \in \dot{H}^2, \quad f_1, f_2 \text{ radial}.$$

In addition, in order to preserve the symmetry of solutions, we need to assume that the potential $V(x)$ is *spherically symmetric* in the sense of [**74**].

Our first result is an endpoint estimate for the linear flow:

THEOREM 2.1. *Let $V(x)$ be a 4×4 matrix of the form*

$$V(x) = V_1(|x|)\mathbb{I}_4 + i\beta(\alpha \cdot \hat{x}) V_2(|x|), \qquad V_1, V_2 : \mathbb{R}^+ \to \mathbb{R}, \quad x \in \mathbb{R}^3 \quad (2.13)$$

(where $\widehat{x} = x/|x|$). Assume that for some $\sigma > 1$ and some sufficiently small $\delta > 0$

$$|V(x)| \leq \frac{\delta}{|x|^{1/2} |\log |x||^{\sigma/2} + |x|^\sigma}. \quad (2.14)$$

Then the following endpoint Strichartz estimate

$$\|e^{it(\mathcal{D}+V)} f\|_{L^2_t L^\infty_x} \lesssim \|f\|_{\dot{\mathcal{H}}^1} \quad (2.15)$$

holds for all initial data $f \in \dot{\mathcal{H}}^1$ where

$$\dot{\mathcal{H}}^1 = \{f_1 + \mathcal{D} f_2, \; f_1 \in \dot{H}^1(\mathbb{R}^3), \; f_2 \in \dot{H}^2(\mathbb{R}^3), \; f_1, f_2 \text{ radial}\}. \quad (2.16)$$

We recall that condition (2.14) is sufficient to ensure that the perturbed Dirac operator $\mathcal{D} + V$ is self-adjoint on the domain $H^1(\mathbb{R}^3)^4$. For this and many other properties of the Dirac equation, a comprehensive reference is [**74**].

The natural application of estimate (2.15) is to prove global well posedness for the critical equation (2.1). However the nonlinear term $P_3(u)$ does not operate on the space $\dot{\mathcal{H}}^1$ and additional restrictions on the algebraic structure of the data are necessary. More precisely, it is possible to decompose the space $L^2(\mathbb{R}^3)^4$ as a direct sum

$$L^2(\mathbb{R}^3)^4 \simeq \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{\substack{k_j=\\ \pm(j+1/2)}} L^2(0,+\infty;dr) \otimes \mathcal{H}_{m_j,k_j}.$$

where the spaces $\mathcal{H}_{m_j,k_j}$ are two dimensional and are generated by spherical harmonics on the sphere $\mathbb{S}^2$ (this is called a dcomposition in *partial wave subspaces*). When $j = 1/2$, we have four spaces

$$L^2(0,+\infty;dr) \otimes \mathcal{H}_{m_{1/2},k_{1/2}}$$

corresponding to the four possible choices of indices

$$(m_{1/2}, k_{1/2}) = (-1/2,-1), \quad (-1/2,1), \quad (1/2,-1), \quad (1/2,1). \tag{2.17}$$

Then we notice the mildly surprising fact that each of these four spaces is invariant not only for the Dirac operator but also for the action of cubic nonlinearities of the forms (2.4). In a sense, these spaces can be considered as a suitable generalization of radial functions adapted to the structure of the nonlinear problem (2.1). A detailed analysis of the partial wave decomposition is given in chapter 5, with explicit forms for the functions in these spaces (see Lemma 2.14 and in particular (2.67)–(2.70)).

Thanks to this invariance, we can prove the following global existence result:

THEOREM 2.2. *Consider the equation on* $\mathbb{R} \times \mathbb{R}^3$

$$iu_t - \mathcal{D}u + V(x)u = P_3(u), \qquad u(0,x) = f(x) \tag{2.18}$$

where the potential has the form

$$V = V_1(|x|)\mathbb{I}_4 + i\beta(\alpha \cdot \hat{x})V_2(|x|), \qquad \widehat{x} = x/|x|$$

and satisfies assumption (2.14)*, while*

$$P_3(u) = \langle \beta u, u \rangle u \qquad \text{or} \qquad P_3(u) = \langle u, u \rangle u.$$

Assume the initial data f belong to a space $\dot{H}^1((0,\infty),dr) \otimes \mathcal{H}_{m_{1/2},k_{1/2}}$ for one of the choices (2.17)*. Then if the $\dot{H}^1$ norm of the data is sufficiently small, problem* (2.18) *has a unique global solution in the class $C_t(\mathbb{R}, \dot{H}^1) \cap L_t^2(\mathbb{R}, L^\infty)$.*

This chapter is organized as follows. section 2.2 contains an extension of the endpoint estimate for the free Dirac operator which is then adapted in section 2.3 to a mixed endpoint-smoothing estimate with weights for the nonhomogeneous linear Dirac equation. section 2.4 is devoted to the proof of Theorem (2.1). In section 2.5 we recall the structure of the Dirac operator, the partial wave decomposition, and we investigate the interaction of the algebraic structure with the nonlinear term. chapter 6 contains the proof of Theorem (2.2).

2. The homogeneous endpoint estimate

Since as we have observed the Dirac flow does not preserve radiality, we cannot hope to adapt the simple argument used in [**49**] for the wave equation to recover the endpoint Strichartz estimate for radial initial data in the Dirac case. However we can prove an endpoint estimate for suitable classes of function; to this end we need a deeper insight in the structure of the Dirac operator expressed in radial coordinates.

Let $u = e^{it\mathcal{D}}f$ and notice that, thanks to identity (2.6), u is (formally) a solution of

$$\begin{cases} \Box u = 0 \\ u(0) = f(x) \\ u_t(0) = i\mathcal{D}f. \end{cases} \tag{2.19}$$

This gives the formula

$$u = e^{it\mathcal{D}}f = \cos(t|D|)f + i\frac{\sin(t|D|)}{|D|}\mathcal{D}f \tag{2.20}$$

and we easily see that this representation is valid for generic distribution data.

We start by proving the following result:

Proposition 2.3. *Let f belong to the space $\dot{\mathcal{H}}^1$ defined in* (2.16). *Then the following endpoint Strichartz estimate holds:*

$$\|e^{it\mathcal{D}}f\|_{L^2_tL^\infty_x} \lesssim \|f\|_{\dot{H}^1}. \tag{2.21}$$

By formula (2.20), we see that the proof is an immediate consequence of the following Lemma. Notice that the proof of the first estimate (2.22) is inspired by an argument of [**36**] which holds for all $n \geq 3$ without modification, while we fix $n = 3$ in the second estimate.

Lemma 2.4. *Let f be a radial function and let $e^{it|D|}$ be the linear propagator associated to the wave operator. Then the following estimates hold:*

$$\|e^{it|D|}f\|_{L^2_tL^\infty_x} \lesssim \|f\|_{\dot{H}^{\frac{n-1}{2}}} \qquad (n \geq 3) \tag{2.22}$$

$$\left\|\frac{e^{it|D|}}{|D|}\mathcal{D}f\right\|_{L^2_tL^\infty_x} \lesssim \|f\|_{\dot{H}^1} \qquad (n = 3). \tag{2.23}$$

Proof. Using Fourier transform in spherical coordinates and the radiality of f and setting $\rho = |x|$, $|\xi| = \lambda$, $x \cdot \xi = \rho\lambda\cos\theta$, we have

$$e^{it|D|}f = \int e^{i(x\cdot\xi+t|\xi|)}\hat{f}(\xi)d\xi = \int_0^\infty \int_0^\pi e^{i\lambda(t+\rho\cos\theta)}\hat{f}(\lambda)\lambda^{n-1}(\sin\theta)^{n-2}d\theta d\lambda. \tag{2.24}$$

With the change of variable $y = \cos\theta$ in (2.24) we obtain

$$= \int_{\mathbb{R}} d\lambda\, e^{it\lambda}g(\lambda)\int_{-1}^1 e^{i\lambda\rho y}(1-y^2)^{\frac{n-3}{2}}dy$$

with $g(\lambda) = \hat{f}(\lambda)\lambda^{n-1}H(\lambda)$ (H represents the classical Heaviside function). Now changing the order of the integrals we obtain

$$= \int_{-1}^{1} dy\,(1-y^2)^{\frac{n-3}{2}} \int_{-\infty}^{+\infty} e^{i\rho(t+\lambda y)} g(\rho) d\rho = \int_{-1}^{1} dy\,(1-y^2)^{\frac{n-3}{2}} \hat{g}(t+\lambda y)$$

Since for $n \geq 3$ one has $(1-y^2)^{\frac{n-3}{2}} \leq 1$, the change of variable $y \to y/r$ yields again

$$\leq \frac{1}{r}\int_{-r}^{r} \hat{g}(t+y)dy = M(\hat{g})(t) \tag{2.25}$$

where M denotes the standard maximal operator. Then we have for all t

$$\|e^{it|D|}f\|_{L^\infty_x} \lesssim M(\hat{g})(t), \tag{2.26}$$

and thus by the L^p-boundedness of maximal operator and Plancherel's theorem

$$\|e^{it|D|}f\|_{L^2_t L^\infty_x} \lesssim \|g\|_{L^2_t} = \left(\int_0^\infty (\lambda^{n-1}|\hat{f}|)^2 d\lambda\right)^{\frac{1}{2}} = \left(\int_0^\infty \left|\lambda^{\frac{n-1}{2}}|\hat{f}|\right|^2 \lambda^{n-1} d\lambda\right)^{\frac{1}{2}} = \tag{2.27}$$

$$= \|\lambda^{\frac{n-1}{2}}\hat{f}\|_{L^2} = \|f\|_{\dot{H}^{\frac{n-1}{2}}}$$

which gives (2.22).

We now turn to estimate (2.23), for which the calculations are similar. Indeed we can write (here we are fixing $n = 3$)

$$\frac{e^{it|D|}}{|D|}\mathcal{D}f = \int e^{i(x\cdot\xi + t|\xi|)}(\alpha\cdot\hat{\xi})\hat{f}(\xi)d\xi = \tag{2.28}$$

where

$$\alpha\cdot\hat{\xi} = \sum_{k=1}^{3}\frac{(\alpha_k\cdot\xi_k)}{|\xi|}.$$

Using spherical coordinates as before we have

$$\int_0^\infty d\lambda \int_0^{2\pi} d\phi \int_0^\pi d\theta e^{i\lambda(t+\rho\cos\theta)} A(\theta,\phi)\hat{f}(\lambda)\lambda^2\sin\theta \tag{2.29}$$

with the operator $A(\theta,\phi) = \alpha_1\cos\theta + \alpha_2\sin\theta\cos\phi + \alpha_3\sin\theta\sin\phi$. Observing that

$$\int_0^{2\pi}\alpha_2\sin\theta\cos\phi\,d\phi = \int_0^{2\pi}\alpha_3\sin\theta\sin\phi\,d\phi = 0,$$

we see that (2.29) is equal to

$$\cong \int_0^\infty d\lambda \int_0^\pi d\theta e^{i\lambda(t+\rho\cos\theta)}\alpha_1\cos\theta\hat{f}(\lambda)\lambda^2\sin\theta.$$

Setting as before $g(\lambda) = \hat{f}(\lambda)\lambda^2 H(\lambda)$, changing variable $\cos\theta \to y$ and then $y \to y/r$ yield

$$= \int_{-1}^{1} dy\,(\alpha_1\cdot y)\int_{-\infty}^{+\infty} d\rho e^{i\lambda(t+\rho y)}g(\rho) = \frac{1}{r}\int_{-r}^{r}\left(\alpha_1\cdot\frac{y}{r}\right)\hat{g}(t+y)dy \cong cM(\hat{g})(t) \tag{2.30}$$

since the term $\left(\alpha_1 \cdot \frac{y}{r}\right)$ is bounded, and so we have the bound

$$\left\| \frac{e^{it|D|}}{|D|}\mathcal{D}f \right\|_{L^\infty_x} \lesssim M(\hat{g})(t). \tag{2.31}$$

The L^p-boundedness of maximal operator and Placherel Theorem yield as above

$$\left\| \frac{e^{it|D|}}{|D|}\mathcal{D}f \right\|_{L^2_t L^\infty_x} \lesssim \|f\|_{\dot{H}^1} \tag{2.32}$$

which gives the desired estimate (2.23). □

Combining estimates (2.22) and (2.23) and using representation (2.20) for the solution of the free Dirac system we obtain estimate (2.21).

3. The mixed endpoint-smoothing estimate

We consider now the non homogeneous equation

$$iu_t - \mathcal{D}u = F(t,x), \qquad u(0,x) = 0. \tag{2.33}$$

By Duhamel's formula and the representation (2.20) we can write the solution u as

$$u(t,x) = \int_0^t e^{i(t-s)\mathcal{D}} F(s,x)ds = \tag{2.34}$$

$$= \int_0^t \left(\cos((t-s)|D|)F(s,x) + i\frac{\sin((t-s)|D|)}{|D|}\mathcal{D}F(s,x) \right) ds.$$

Thus in order to estimate the solution u to (2.33) we can deal separately with the two integrals

$$\int_0^t e^{i(t-s)|D|}F(s,x)ds \quad \text{and} \quad \int_0^t \frac{e^{i(t-s)|D|}}{|D|}\mathcal{D}F(s,x)ds.$$

We prove the following:

Proposition 2.5. *Let $n = 3$ and assume $F(t,x)$ has the structure*

$$F(t,x) = F_1(|x|)\mathbb{I}_4 + i\beta(\alpha \cdot \hat{x})F_2(|x|). \tag{2.35}$$

Then the following estimate holds

$$\left\| \int_0^t e^{i(t-s)\mathcal{D}} F(s)\, ds \right\|_{L^2_t L^\infty_x} \lesssim \|\langle x\rangle^{\frac{1}{2}+}|D|F\|_{L^2_t L^2_x}. \tag{2.36}$$

The key step in the proof of (2.36) is the following non homogeneous estimate for the wave propagator with a radial term.

Lemma 2.6. *Let $n \geq 3$, $F(t,\cdot)$ be a radial function. Then the following estimate holds*

$$\left\| \int_0^t e^{i(t-s)|D|} F(s)\, ds \right\|_{L^2_t L^\infty_x} \lesssim \|\langle x\rangle^{\frac{1}{2}+}|D|^{\frac{n-1}{2}}F\|_{L^2 t L^2_x}. \tag{2.37}$$

PROOF. We start with (2.37). Expanding u as in the homogeneous case (see formulas (2.24 and (2.25)), from the radiality of F we can estimate the L^∞ norm of the solution at fixed t as (here $\widehat{G}(s,\lambda) = \lambda^{n-1}\widehat{F}(s,\lambda)H(\lambda)$ and H is the Heaviside function)

$$\|u\|_{L^\infty_x} \lesssim \sup_r \frac{1}{r}\int_{-r}^{r}\left(\int_0^t \left|\widehat{G}(s,y+t-s)\right| ds\right) dy =$$

$$= \sup_r \frac{1}{r}\int_{-r}^{r}\left(\int_0^t \left|\widehat{G}(s,y+t-s)\right| \langle y+t-s\rangle^{\frac12+}\langle y+t-s\rangle^{-\frac12-}ds\right) dy \lesssim$$

$$\lesssim \sup_r \frac{1}{r}\int_{-r}^{r} dy \left[\left(\int_{\mathbb{R}}\left|\widehat{G}_1(s,y+t-s)\right|^2 ds\right)^{\frac12}\cdot\left(\int_0^t \langle y+t-s\rangle^{-1-}ds\right)^{\frac12}\right]$$

where in the last inequality we have used Cauchy-Schwarz inequality and G_1 is the function defined by

$$\widehat{G}_1(s,y) = \widehat{G}(s,y)\langle y\rangle^{\frac12+}.$$

Setting now $h(z) := \left(\int_{\mathbb{R}}\left|\widehat{G}_1(s,z-s)\right|^2 ds\right)^{\frac12}$ we have

$$\sup_r \frac{1}{r}\int_{-r}^{r} dy\left(\int_{\mathbb{R}}\left|\widehat{G}_1(s,y+t-s)\right|^2 ds\right)^{\frac12} = M(h)(t)$$

The L^p boundedness of the maximal operator yields

$$\|u\|_{L^2_tL^\infty_x} \lesssim \|h(t)\|_{L^2_t} = \left(\int\int\left|\widehat{G}_1(s,t-s)^2\right|^2 dsdt\right)^{\frac12} = \|\widehat{G}_1\|_{L^2_sL^2_y}$$

The last quantity is precisely

$$\|\langle y\rangle^{\frac12+}\mathcal{F}_{\lambda\to y}\left(\lambda^{n-1}\widehat{F}(s,\lambda)H(\lambda)\right)\|_{L^2_sL^2_y}$$

and to conclude the proof we need to estimate it by

$$\lesssim \|\langle x\rangle^{\frac12+}|D|^{\frac{n-1}{2}}F\|_{L^2_sL^2_x}.$$

Since $\mathcal{F}^{-1}(|D|^{\frac{n-1}{2}}f) = |\xi|^{\frac{n-1}{2}}\check{f}$ we see that it is enough to prove the general inequality (we can neglect the dependence on time)

$$\|\langle\rho\rangle^k\mathcal{F}_{\lambda\to\rho}\left(\lambda^{\frac{n-1}{2}}\widehat{f}(\lambda)H(\lambda)\right)\|_{L^2_\rho} \lesssim \|\langle x\rangle^k f\|_{L^2} \tag{2.38}$$

for $k = 1/2+$.

We prove (2.38) by interpolation. The case $k=0$ is trivial, since from Placherel's Theorem we obviously have

$$\|\mathcal{F}_{\lambda\to\rho}\left(\lambda^{\frac{n-1}{2}}\widehat{f}(\lambda)H(\lambda)\right)\|_{L^2} \cong \|\lambda^{\frac{n-1}{2}}\widehat{f}(\lambda)\|_{L^2} = \|f\|_{L^2}. \tag{2.39}$$

The case $k=1$ is just a little more complicated. Since of course $\langle\rho\rangle \le 1+|\rho|$, we need only prove that

$$\|\rho\,\mathcal{F}_{\lambda\to\rho}\left(\lambda^{\frac{n-1}{2}}\widehat{f}(\lambda)H(\lambda)\right)\|_{L^2} \lesssim \|\langle x\rangle f\|_{L^2}$$

or equivalently

$$\|\partial_\lambda\left(\lambda^{\frac{n-1}{2}}\widehat{f}(\lambda)H(\lambda)\right)\|_{L^2} \lesssim \|\langle x\rangle f\|_{L^2}. \tag{2.40}$$

We write

$$\|\partial_\lambda \left(\lambda^{\frac{n-1}{2}} \widehat{f}(\lambda)\chi_{\mathbb{R}^+}(\lambda)\right)\|_{L^2} \lesssim \|\lambda^{\frac{n-1}{2}}\partial_\lambda \hat{f}(\lambda)\|_{L^2_+} + \|\lambda^{\frac{n-3}{2}}\hat{f}(\lambda)\|_{L^2_+} = I_1 + I_2$$

with the shorthand notation $L^2_+ = L^2(0,\infty)$. For I_1 we trivially have

$$I_1 = \|\lambda^{\frac{n-1}{2}}\widehat{(\rho f)}\|_{L^2} \cong \||x|f\|_{L^2}.$$

Let's now turn to I_2. We split the norm

$$\|\lambda^{\frac{n-3}{2}}\hat{f}(\lambda)\|_{L^2_+} = \|\lambda^{\frac{n-3}{2}}\hat{f}(\lambda)\|_{L^2(\{\lambda\geq 1\})} + \|\lambda^{\frac{n-3}{2}}\hat{f}(\lambda)\|_{L^2(\{\lambda<1\})}.$$

Plancherel's Theorem yields again for the first term

$$\|\lambda^{\frac{n-3}{2}}\hat{f}(\lambda)\|_{L^2(\{\lambda\geq 1\})} \leq \|\lambda^{\frac{n-1}{2}}\hat{f}\|_{L^2(\mathbb{R})} \cong \|f\|_{L^2}.$$

To handle the second term we use Hardy's inequality:

$$\|\lambda^{\frac{n-3}{2}}\hat{f}(\lambda)\|_{L^2(\{\lambda<1\})} = \||\xi|^{-1}\hat{f}\|_{L^2(\{|\xi|<1\})} \lesssim \|\nabla_\xi \hat{f}\|_{L^2} \simeq \||x|f\|_{L^2}$$

and this concludes the case $k=1$. By interpolation with (2.39) we obtain the desired estimate (2.37). $\square$

Lemma 2.7. *Let $n = 3$ and $F(t,\cdot)$ be of the form (2.35). Then the following estimate holds*

$$\left\|\int_0^t \frac{e^{i(t-s)|D|}}{|D|^k}\mathcal{D}^k F(s)\,ds\right\|_{L^2_t L^\infty_x} \lesssim \|\langle x\rangle^{\frac{1}{2}+}|D|F\|_{L^2_t L^2_x}. \tag{2.41}$$

for $k = 0, 1$.

Proof. Since the operator $i\mathcal{D}\beta(\alpha\cdot\hat{x})\phi$ applied to a radial function ϕ produces the radial function $i\beta\phi''$, Lemma (2.6) holds, and we only need to control terms of the form (where F_{rad} denotes a radial function)

$$\left\|\int_0^t \frac{e^{i(t-s)|D|}}{|D|^{1-j}}A_j F_{rad}(s)\,ds\right\|_{L^2_t L^\infty_x}$$

with $A_j = i\beta^j(\alpha\cdot\hat{x})$, $j = 0,1$. Recalling (2.28)-(2.30), since the quantities A_j are obviously bounded, we can estimate in both cases with

$$\left|\int_0^t \frac{e^{i(t-s)|D|}}{|D|^{1-j}}A_j F_{rad}(s)ds\right| \lesssim \int_0^t \frac{1}{r}\int_{-r}^r |\widehat{G}(s,y+t-s)|\,dsdy$$

where as before $\widehat{G}(s,\lambda) = \lambda^2\widehat{F}_{rad}(s,\lambda)H(\lambda)$. Proceeding exactly as in the proof of Lemma (2.6) we obtain estimate (2.41). $\square$

Estimate (2.36) is an immediate consequence of (2.37), (2.41) and representation (2.34)

4. Proof of Theorem 2.1

We now turn to the proof of Theorem (2.1). This is based on a simple application of a smoothing estimate for a Dirac equation with potential proved in [**21**] (see also [**20**], [**19**] for related results):

THEOREM 2.8 ([**21**]). *Let $V(x) = V(x)^*$ be a 4×4 complex valued matrix and assume that for some $\sigma > 1$ and some sufficiently small $\delta > 0$ one has*

$$|V(x)| \leq \frac{\delta}{w_\sigma(x)} \qquad \text{where} \quad w_\sigma(x) = |x|(1 + |\log|x||)^\sigma \tag{2.42}$$

Then the following smoothing estimate holds:

$$\|w_\sigma^{-1/2} e^{it(\mathcal{D}+V)} f\|_{L^2_t L^2_x} \lesssim \|f\|_{L^2}. \tag{2.43}$$

It is not difficult to deduce the endpoint estimate (2.15) for the perturbed flow from the previous result and our mixed endpoint-smoothing estimate (2.36). First of all, the solution of the equation

$$iu_t = \mathcal{D}u + Vu, \qquad u(0,x) = f$$

can be written, regarding Vu as a right-hand member of the equation

$$u = e^{it(\mathcal{D}+V)} f = e^{it\mathcal{D}} f + i \int_0^t e^{i(t-s)\mathcal{D}}(Vu)ds.$$

Then we can write

$$\||D|^{-1}u\|_{L^2_t L^\infty_x} = \||D|^{-1} e^{it(\mathcal{D}+V)} f\|_{L^2_t L^\infty_x} \leq \tag{2.44}$$

$$\leq \left\||D|^{-1} e^{it\mathcal{D}} f\right\|_{L^2_t L^\infty_x} + \left\| \int_0^t \frac{e^{i(t-s)\mathcal{D}}}{|D|} (V(s) e^{is(\mathcal{D}+V)} f) ds \right\|_{L^2_t L^\infty_x}.$$

The first term can be estimated by (2.21) (notice that $|D|$ commutes with $\mathcal{D}$ and hence with the flow). In order to apply estimate (2.36) to the second term we need the following

LEMMA 2.9. *If $f \in \dot{\mathcal{H}}^1$, then $e^{it\mathcal{D}} f \in \dot{\mathcal{H}}^1$, and if V is of the form (2.13), then $V e^{it\mathcal{D}} f$ is of the form* (2.35).

PROOF. We write $f = f_1 + \mathcal{D} f_2$ with f_1, f_2 radial functions. Then we have, from (2.20),

$$e^{it\mathcal{D}} f = \left(\cos(t|D|) + \frac{\sin(t|D|)}{|D|} \mathcal{D} \right) (f_1 + \mathcal{D} f_2) =$$

$$= \cos(t|D|) f_1 + \sin(t|D|)|D| f_2 + \mathcal{D} \left(\cos(t|D|) f_2 + \frac{\sin(t|D|)}{|D|} f_1 \right) =$$

$$= \tilde{f}_1 + \mathcal{D} \tilde{f}_2,$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are radial functions with the appropirate regularity, and this concludes the proof of the first statement. The proof of the second statement is trivial. $\square$

We thus can estimate (2.44) with (2.21) and (2.36) obtaining

$$\lesssim \|f\|_{L^2} + \|\langle x\rangle^{\frac{1}{2}+} Vu\|_{L^2_t L^2_x}$$

Now multiplying and dividing by $w_\sigma(x)^{1/2}$ in the second norm on the right hand side yields

$$\leq \|f\|_{L^2} + \|\langle x\rangle^{1/2+} w_\sigma^{1/2} V\|_{L^\infty} \cdot \|w_\sigma^{-1/2} u\|_{L^2_t L^2_x}.$$

Notice that the weighted norm of V at the right hand side is bounded as it follows from assumption (2.14). Moreover (2.14) implies also that the assumption of Theorem 2.8 is satisfied. Then using (2.43) we conclude

$$\||D|^{-1}u\|_{L^2_t L^\infty_x} \leq \left(1 + \|\langle x\rangle^{1/2+} w_\sigma^{1/2} V\|_{L^\infty}\right) \|f\|_{L^2} \tag{2.45}$$

that gives, under hypothesis (2.14) on the potential, the desired Strichartz endpoint estimate.

5. Partial wave subspaces and radial Dirac operator

The purpose of this chapter is to construct, following [**74**], invariant subspaces for the Dirac operator with a potential having a special symmetry. To this end we use the classical decomposition of the space $L^2(\mathbb{R}^3)^4$ in the direct sum of 2-dimensional Hilbert spaces, the *partial wave subspaces*, which are invariant for the Dirac operator. We shall aslo check that the lowest order partial wave subspaces are invariant even for the cubic nonlinearities that we consider here.

We begin by recalling the basic facts on the decomposition, referring to [**74**] for more details. We shall use the standard notation for polar coordinates in $\mathbb{R}^3$

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

with the unit vectors in the directions of the polar coordinate lines given by

$$\begin{cases} e_r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = \dfrac{x}{|x|} = \hat{x} \\ e_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) = \dfrac{\partial e_r}{\partial\theta} \\ e_\phi = (-\sin\phi, \cos\phi, 0) = \dfrac{1}{\sin\theta}\dfrac{\partial e_r}{\partial\phi}. \end{cases}$$

Then we write for a function $\psi \in L^2(\mathbb{R}^3)$

$$\psi(r,\theta,\phi) = r\tilde{\psi}(x(r,\theta,\phi), y(r,\theta,\phi), z(r,\theta,\phi)). \tag{2.46}$$

Since the function $\psi(r,\cdot,\cdot)$ of the angular variables is square integrable on the unit sphere $L^2(S^2)$, the mapping $\tilde{\psi} \to \psi$ defines a unitary isomorphism

$$L^2(\mathbb{R}^3) \cong L^2((0,\infty), dr; L^2(S^2)) = L^2((0,\infty), dr) \otimes L^2(S^2).$$

Applying the transformation (2.46) on each component of the (vector valued) wavefunction, we obtain the analogous decomposition

$$L^2(\mathbb{R}^3)^4 \cong L^2((0,\infty), dr) \otimes L^2(S^2)^4.$$

The decomposition of the Hilbert space into a "radial" and an "angular" part is very useful since the angular momentum operators

$$\mathbf{L} = x \wedge (-i\nabla) \qquad \text{orbital angular momentum}$$

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \qquad \text{total angular momentum}$$

act only on the angular part $L^2(S^2)^4$ in a nontrivial way; here

$$\mathbf{S} = -1/4(\alpha \wedge \alpha)$$

is the spin angular momentum operator. Recalling the expression of ∇ in polar coordinates

$$\nabla = e_r \frac{\partial}{\partial r} + \frac{1}{r} \left(e_\theta \frac{\partial}{\partial \theta} + e_\phi \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \right) \tag{2.47}$$

we obtain that, since $x = r \cdot e_r$,

$$\mathbf{L} = i\, e_\theta \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} - i\, e_\phi \frac{\partial}{\partial \theta} \tag{2.48}$$

where the differentiation applies to each component of the wavefunction.

The Dirac operator can be written in polar coordinates as follows. Combining (2.47) and (2.48) yields

$$-i\nabla = -i\, e_r \frac{\partial}{\partial r} - \frac{1}{r}(e_r \wedge \mathbf{L})$$

and thus

$$-i\alpha \cdot \nabla = -i(\alpha \cdot e_r) \frac{\partial}{\partial r} - \frac{1}{r} \alpha \cdot (e_r \wedge \mathbf{L}). \tag{2.49}$$

By the basic property of the Dirac matrices:

$$(\alpha \cdot A)(\alpha \cdot B) = A \cdot B + 2i\mathbf{S} \cdot (A \wedge B);$$

which holds for any matrix-valued vector fields $A = (A^1, A^2, A^3)$, $B = (B^1, B^2, B^3)$ with $F^i, G^i \in \mathcal{M}_{4\times 4}(\mathbb{C})$, and

$$\gamma_5 \alpha = 2\mathbf{S},$$

where $\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we obtain

$$(\alpha \cdot A)(2\mathbf{S} \cdot B) = i\gamma_5 A \cdot B - i\alpha \cdot (A \wedge B),$$

thus equation (2.49) is equal to

$$= -i(\alpha \cdot e_r) \frac{\partial}{\partial r} + \frac{i}{r} (\alpha \cdot e_r)(2\mathbf{S} \cdot \mathbf{L}).$$

Finally, introducing the *spin orbit operator*

$$K = \beta(2\mathbf{S} \cdot \mathbf{L} + 1) \equiv \beta(J^2 - L^2 + 1/4) \tag{2.50}$$

where we used the identity $J^2 = (L + S)^2 = L^2 + 2S \cdot L + 3/4$, we arrive at the following representation:

PROPOSITION 2.10. *The 3-dimensional Dirac operator can be written as*

$$\mathcal{D} = -i(\alpha \cdot \hat{x}) \left(\frac{\partial}{\partial r} + \frac{1}{r} - \frac{1}{r} \beta K \right) \tag{2.51}$$

where K is the spin orbit operator defined in (2.50).

The key step to construct the invariant spaces is the following:

PROPOSITION 2.11. *For each choice* (j, m_j, k_j) *with* $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, .., m_j = -j, -j+1, ..., +j$, $k_j = -(j+1/2), +(j+1/2)$, *there exist precisely two eigenfunctions* $\Phi^{\pm}_{m_j,k_j} \in C^\infty(S^2)^4$ *satisfying the following relations:*

$$J^2\Phi_{m_j,k_j} = j(j+1)\Phi_{m_j,k_j},$$

$$J_3\Phi_{m_j,k_j} = m_j\Phi_{m_j,k_j},$$

$$K\Phi_{m_j,k_j} = -k_j\Phi_{m_j,k_j}.$$

The family $\Phi^{\pm}_{m_j,k_j}$ *forms an orthonomral basis of* $L^2(\mathbb{S}^2)^4$.

The functions Φ_{m_j,k_j} can be written explicitly using spherical harmonics. We first recall the following representation of 3-dimensional spherical harmonics

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(\cos\theta) \qquad \forall -l \le m \le l. \tag{2.52}$$

where P_l^m are the Legendre polynomials

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{m+l}}{dx^{m+l}}(x^2-1)^l. \tag{2.53}$$

As it is well known, the spherical harmonics form a complete orthonormal set in $L^2(S^2)$, i.e. every function $f \in L^2(S^2)$ can be written as

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m Y_l^m(\theta,\phi)$$

for some constants f_l^m; moreover, they are eigenfunctions of both the operators L^2 and L_3, i.e.

$$L^2Y_l^m = l(l+1)Y_l^m \tag{2.54}$$

$$L_3Y_l^m = mY_l^m. \tag{2.55}$$

We now define for $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, .., m_j = -j, -j+1, ..., +j$ the functions $\Psi^{m_j}_{j\mp 1/2} \in L^2(S^2)^2$:

$$\Psi^{m_j}_{j-1/2} = \frac{1}{\sqrt{2j}}\begin{pmatrix} \sqrt{j+m_j}\,Y^{m_j-1/2}_{j-1/2} \\ \sqrt{j-m_j}\,Y^{m_j+1/2}_{j-1/2} \end{pmatrix} \tag{2.56}$$

$$\Psi^{m_j}_{j+1/2} = \frac{1}{\sqrt{2j+2}}\begin{pmatrix} \sqrt{j+1-m_j}\,Y^{m_j-1/2}_{j+1/2} \\ -\sqrt{j+1+m_j}\,Y^{m_j+1/2}_{j+1/2} \end{pmatrix}. \tag{2.57}$$

These functions are, as it is easily seen, eigenfunctions of both the operators L^2 and $J^2 = L^2 + \sigma\cdot L + 3/4$ with eigenvalues $l(l+1)$ and $j(j+1)$ respectively. So we conclude, in view of (2.50), that the functions in Proposition 2.11 are given by

$$\Phi^{+}_{m_j,\mp(j+1/2)} = \begin{pmatrix} i\Psi^{m_j}_{j\mp 1/2} \\ 0 \end{pmatrix} \qquad \Phi^{-}_{m_j,\mp(j+1/2)} = \begin{pmatrix} 0 \\ \Psi^{m_j}_{j\pm 1/2}. \end{pmatrix}. \tag{2.58}$$

Thus the Hilbert space $L^2(S^2)^4$ is the orthogonal direct sum of 2-dimensional Hilbert spaces $\mathcal{H}_{m_j,k_j}$, which are spanned by simultaneous eigenfunctions $\Phi^{\pm}_{m_j,k_j}$ of J^2 and K:

$$L^2(S^2)^4 = \bigoplus_{j=\frac{1}{2},\frac{3}{2},\ldots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm(j+\frac{1}{2})} \mathcal{H}_{m_j,k_j} \tag{2.59}$$

Easy calculations show that the functions $\Psi^{m_j}_{j\pm 1/2}$ satisfy

$$(\sigma\cdot\hat{x})\Psi^{m_j}_{j\pm 1/2} = \Psi^{m_j}_{j\mp 1/2},$$

and hence

$$i(\alpha\cdot\hat{x})\Phi^{\pm}_{m_j,k_j} = \mp\Phi^{\mp}_{m_j,k_j}. \tag{2.60}$$

This proves the following:

Lemma 2.12. *The subspaces $\mathcal{H}_{m_j,k_j}$ are left invariant by the operators β and*
$\alpha\cdot\hat{x}$. With respect to the basis $\{\Phi^{+}_{m_j,k_j}\Phi^{-}_{m_j,k_j}\}$ defined above, these operators are represented by the 2×2 matrices

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad -i\alpha\cdot\hat{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.61}$$

The decomposition just shown obviously implies a similar one of $L^2(\mathbb{R}^3)^4$, in which each *partial wave subspace* $L^2((0,\infty),dr)\otimes\mathcal{H}_{m_j,k_j}$ is isomorphic to $L^2((0,\infty),dr)^2$ if we choose the basis $\{\Phi^{+}_{m_j,k_j},\Phi^{-}_{m_j,k_j}\}$. There is in fact a unitary isomorphism between the Hilbert spaces:

$$L^2(\mathbb{R}^3)^4 \cong \bigoplus L^2((0,\infty),dr)\otimes\mathcal{H}_{m_j,k_j}. \tag{2.62}$$

This decomposition and (2.51) allow us to easily calculate the action of the Dirac operator (at least on differentiable states) even in the presence of a suitable potential.

Proposition 2.13. *The Dirac operator (2.51) with the potential*

$$V(x) = V_1(|x|)\mathbb{I}_4 + i\beta(\alpha\cdot\hat{x})V_2(|x|) \tag{2.63}$$

leaves the partial wave subspaces $C_0^\infty(0,\infty)\otimes\mathcal{H}_{m_j,k_j}$ invariant. With respect to the basis $\Phi^{+}_{m_j,k_j},\Phi^{-}_{m_j,k_j}$ the Dirac operator on each subspace can be represented by the operator

$$d_{m_j,k_j} = \begin{pmatrix} V_1(|x|) & -\frac{d}{dr}+\frac{k_j}{r}+V_2(|x|) \\ \frac{d}{dr}+\frac{k_j}{r}+V_2(|x|) & V_1(|x|) \end{pmatrix} \tag{2.64}$$

which is well defined over $C_0^\infty(0,\infty)^2 \subset L^2((0,\infty),dr)^2$. Moreover, the Dirac operator $\mathcal{D}$ on $C_0^\infty(\mathbb{R}^3)^4$is unitary equivalent to the direct sum of the partial wave Dirac operators d_{m_j,k_j},

$$\mathcal{D} \cong \bigoplus_{j=\frac{1}{2},\frac{3}{2},\ldots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm(j+\frac{1}{2})} d_{m_j,k_j} \tag{2.65}$$

Remark 2.1. Proposition 2.13 holds for slightly more general potentials (see [**74**]), but we shall not need this fact here.

REMARK 2.2. The operator in (2.64) is also known as *radial Dirac operator*. It can be proved that d_{m_j,k_j} is essentially self-adjoint (for every j) on $C_0^\infty(0,\infty)$ if and only if $\mathcal{D}+V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3\backslash\{0\})$.

Thus using spherical coordinates it is possible to construct invariant spaces for the perturbed Dirac operator. What may come as a surprise is that for $j=1/2$ the partial wave subspaces are also invariant for the cubic nonlinearity, and this fact is obviously crucial for the nonlinear application we shall prove in the next chapter.

LEMMA 2.14. *Let $j=1/2$ and let $(m_{1/2},k_{1/2})$ be one of the couples (-1/2,-1), (-1/2,1), (1/2,-1), (1/2,1). Then the partial wave subspaces $C_0^\infty((0,\infty),dr)\otimes\mathcal{H}_{m_{1/2},k_{1/2}}$ are invariant for the cubic nonlinearities $P_3(u)=\langle u,u\rangle u$ and $\langle\beta u,u\rangle u$, i.e.*

$$u\in C_0^\infty((0,\infty),dr)\otimes\mathcal{H}_{m_{1/2},k_{1/2}}\Rightarrow P_3(u)\in C_0^\infty((0,\infty),dr)\otimes\mathcal{H}_{m_{1/2},k_{1/2}}. \tag{2.66}$$

PROOF. We explicitly write down the functions Φ^+, Φ^- in the four cases: a straightforward calculation using formulas (2.52), (2.53), (2.56), (2.57) and (2.58) yields

$$\Phi^+_{-1/2,-1}=\begin{pmatrix}0\\ \frac{i}{2\sqrt{\pi}}\\ 0\\ 0\end{pmatrix}\qquad \Phi^-_{-1/2,-1}=\begin{pmatrix}0\\ 0\\ \frac{1}{2\sqrt{\pi}}e^{i\phi}\sin\theta\\ -\frac{1}{2\sqrt{\pi}}\cos\theta\end{pmatrix}. \tag{2.67}$$

$$\Phi^+_{-1/2,1}=\begin{pmatrix}\frac{i}{2\sqrt{\pi}}e^{i\phi}\sin\theta\\ -\frac{i}{2\sqrt{\pi}}\cos\theta\\ 0\\ 0\end{pmatrix}\qquad \Phi^-_{-1/2,1}=\begin{pmatrix}0\\ 0\\ 0\\ \frac{1}{2\sqrt{\pi}}\end{pmatrix}. \tag{2.68}$$

$$\Phi^+_{1/2,-1}=\begin{pmatrix}\frac{i}{2\sqrt{\pi}}\\ 0\\ 0\\ 0\end{pmatrix}\qquad \Phi^-_{1/2,-1}=\begin{pmatrix}0\\ 0\\ \frac{1}{2\sqrt{\pi}}\cos\theta\\ \frac{1}{2\sqrt{\pi}}e^{i\phi}\sin\theta\end{pmatrix}. \tag{2.69}$$

$$\Phi^+_{1/2,1}=\begin{pmatrix}\frac{i}{2\sqrt{\pi}}\cos\theta\\ \frac{i}{2\sqrt{\pi}}e^{i\phi}\sin\theta\\ 0\\ 0\end{pmatrix}\qquad \Phi^-_{1/2,1}=\begin{pmatrix}0\\ 0\\ \frac{1}{2\sqrt{\pi}}\\ 0\end{pmatrix}. \tag{2.70}$$

We prove Lemma (2.14) for the couple $(1/2,1)$, i.e. for functions of the form (2.70), being the proof for the other cases completely analogous.

The generic function $u \in L^2((0,\infty),dr) \otimes \mathcal{H}_{1/2,1}$ can be written as

$$u(r,\theta,\phi) = u^+(r)\Phi^+_{1/2,1}(\theta,\phi) + u^-(r)\Phi^-_{1/2,1}(\theta,\phi)$$

for some radial functions u^+, u^-. So f takes the vectorial form

$$u = \begin{pmatrix} u^+(r)\dfrac{i}{2\sqrt{\pi}}\cos\theta \\ u^+(r)\dfrac{i}{2\sqrt{\pi}}e^{i\phi}\sin\theta \\ \dfrac{u^-(r)}{2\sqrt{\pi}} \\ 0 \end{pmatrix}. \tag{2.71}$$

Thus the Hermitian product $\langle u,u\rangle$ yields

$$\langle u,u\rangle = -\frac{1}{4\pi}\cos^2\theta\, u^+(r)^2 - \frac{1}{4\pi}\sin^2\theta\, u^+(r)^2 + \frac{1}{4\pi}u^-(r)^2 =$$
$$= -\frac{1}{4\pi}\left(u^+(r)^2 - u^-(r)^2\right)$$

that has no angular components. This proves that if $u \in C_0^\infty((0,\infty),dr) \otimes \mathcal{H}_{m_{1/2},k_{1/2}}$ then $\langle\beta u,u\rangle u \in C_0^\infty((0,\infty),dr) \otimes \mathcal{H}_{m_{1/2},k_{1/2}}$.

Minor modifications yield the same result also for the nonlinear term $\langle u,u\rangle u$. In fact we know from Lemma (2.66) that the operator β acts on the partial wave subspaces in a very simple way with respect to the basis $\{\Phi^+,\Phi^-\}$: if in fact we associate to the function u its coordinates $(u^+(r),u^-(r))$ with respect to such a basis we have $\beta u = (u^+(r),-u^-(r))$, so that

$$\langle\beta u,u\rangle = -\frac{1}{4\pi}\cos^2\theta\, u^+(r)^2 - \frac{1}{4\pi}\sin^2\theta\, u^+(r)^2 - \frac{1}{4\pi}u^-(r)^2 =$$
$$= -\frac{1}{4\pi}\left(u^+(r)^2 + u^-(r)^2\right)$$

that again has no angular components, and this shows that if $u \in C_0^\infty((0,\infty),dr)\otimes \mathcal{H}_{m_{1/2},k_{1/2}}$ then $\langle u,u\rangle u \in C_0^\infty((0,\infty),dr) \otimes \mathcal{H}_{m_{1/2},k_{1/2}}$. □

6. Global existence for the nonlinear equation

As an application of the results we have presented in the previous chapter we can now prove global existence for problem (2.18) with small initial data in one of the four partial wave subspaces $\dot{H}^1((0,\infty),dr) \otimes \mathcal{H}_{m_{1/2},k_{1/2}}$. Our goal is to prove

Theorem 2.15. *Consider the Cauchy problem for the 3-dimensional nonlinear Dirac equation*

$$iu_t - \mathcal{D}u + V(x)u = P_3(u), \qquad u(0,x) = f(x) \tag{2.72}$$

where the potential V is of the form

$$V = V_1(|x|)\mathbb{I}_4 + i\beta(\alpha\cdot\hat{x})V_2(|x|)$$

and satisfies assumption (2.14), while the nonlinear term $P_3(u)$ is either of the form $\langle\beta u,u\rangle u$ or $\langle u,u\rangle u$.

Then for every initial data $f \in \dot H^1((0,\infty),dr)\otimes \mathcal{H}_{m_{1/2},k_{1/2}}$*, with sufficiently small* $\dot H^1$ *norm, there exists a unique global solution* $u(t,x)$ *to problem (2.18) in the class* $C_t(\mathbb{R},\dot H^1)\cap L^2_t(\mathbb{R},L^\infty)$*.*

Proof. The proof is identical for both choices of the form of the nonlinear term. We rewrite (2.18) in integral form

$$u = e^{it\mathcal{D}}f + i\int_0^t e^{i(t-s)\mathcal{D}}\left(V(s)u(s)+P_3(u(s))\right)ds = \tag{2.73}$$

$$= e^{it\mathcal{D}}f + i\int_0^t e^{i(t-s)\mathcal{D}}\left(V(s)u(s)\right)ds + i\int_0^t e^{i(t-s)\mathcal{D}}\left(P_3(u(s))\right)ds =$$

$$= e^{it(\mathcal{D}+V)}f + i\int_0^t e^{i(t-s)\mathcal{D}}\left(P_3(u(s))\right)ds \equiv \Phi(u) \equiv I_1 + I_2$$

we denote by $\Phi(u)$ the RHS of (2.73) and we check that the map Φ is a contraction on the function space

$$X = L^2_tL^\infty_x \cap L^\infty_t\dot H^1_x.$$

In order to estimate the first term I_1 we use our endpoint Strichartz estimate (2.15), observing that if $f \in \dot H^1((0,\infty),dr)\otimes\mathcal{H}_{m_{1/2},k_{1/2}}$ then in particular f is of the form $f = f_1 + \mathcal{D}f_2$ with f_1, f_2 radial functions, so estimate (2.15) holds and gives

$$\|I_1\|_X \lesssim \|f\|_{\dot H^1} \tag{2.74}$$

Now we need to handle the nonlinear term I_2. By Minkowski inequality

$$\left\|\int_0^t e^{i(t-s)\mathcal{D}}P_3(u(s))ds\right\|_X \leq \int_0^\infty \|e^{it\mathcal{D}}e^{-is\mathcal{D}}P_3(u(s))\|_X ds.$$

By energy conservation we have

$$\|e^{it\mathcal{D}}e^{-is\mathcal{D}}P_3(u(s))\|_{L^\infty_t\dot H^1} = \|P_3(u(s))\|_{L^\infty_s\dot H^1}$$

On the other hand, in view of Lemma (2.14), we can use estimate (2.21) and we have

$$\|e^{it\mathcal{D}}e^{-is\mathcal{D}}P_3(u(s))\|_{L^2_tL^\infty_x} = \|P_3(u(s))\|_{\dot H^1} \leq \|P_3(u(s))\|_{L^\infty_s\dot H^1}.$$

Thus

$$\|I_2\|_X \lesssim \|P_3(u(s))\|_{L^\infty_s\dot H^1}$$

Then by Hölder inequality in t,x we obtain

$$\|P_3(u)\|_{L^1_t\dot H^1_x} \leq \|u\|^2_{L^2_tL^\infty_x}\|u\|_{L^\infty_t\dot H^1_x}$$

which implies

$$\|\Phi(u)\|_X \lesssim \|f\|_{\dot H^1} + \|u\|^3_X.$$

An analogous computation gives

$$\|\Phi(u)-\Phi(v)\|_X \lesssim (\|u\|^2_X + \|v\|^2_X)\|u-v\|_X. \tag{2.75}$$

Therefore if the data belong to a sufficiently small ball in $\dot H^1$, Φ is a contraction on that ball, and its unique fixed point is the unique global solutions to problem (2.18) in the space X. □

CHAPTER 3

Endpoint estimates with angular regularity

1. Introduction

The main topic of this chapter is again the cubic massless Dirac equation on $\mathbb{R}^{1+3}$ perturbed with a potential

$$\begin{cases} iu_t - \mathcal{D}u(t,x) + V(x)u = P_3(u) \\ u(0,x) = f(x) \end{cases} \tag{3.1}$$

(the notation here is the same as the previous chapter). As we have already widely discussed (see section 1), the cubic nonlinearity

$$P_3(u) = \langle \beta u, u \rangle u \quad \text{or} \quad P_3(u) = \langle u, u \rangle u$$

is $\dot{H}^1$-critical with respect to the scale-invariance of the Dirac operator, and since the endpoint Strichartz estimate

$$\|e^{it\mathcal{D}} f\|_{L^2 L^\infty} \lesssim \|f\|_{\dot{H}^1} \tag{3.2}$$

fails even in the free case, we cannot directly apply the standard fixed-point strategy in order to prove well-posedness for problem (3.1).
In an attempt to overcome this limitation, in recent years some refined estimates involving angular regularity have been investigated leading to interesting improvements. We introduce the natural notations

$$\|f\|_{L^a_r L^b_\omega} = \left(\int_0^\infty \|f(r\,\cdot\,)\|^a_{L^b(\mathbb{S}^{n-1})} r^{n-1} dr \right)^{\frac{1}{a}}$$

and

$$\|f\|_{L^\infty_r L^b_\omega} = \sup_{r \geq 0} \|f(r\,\cdot\,)\|_{L^b(\mathbb{S}^{n-1})}.$$

Then the following estimate for the free wave propagator is proved in [**51**]:

$$n = 3, \qquad \|e^{it|D|} f\|_{L^2_t L^\infty_r L^p_\omega} \lesssim \sqrt{p} \cdot \||D| f\|_{L^2}, \qquad \forall p < \infty \tag{3.3}$$

(we have already pointed out in chapter 2 the strict connection that exists between the Dirac and the wave equation).
Notice that the norm at the left hand side distinguishes between the integrability in the radial and tangential directions. Using estimate (3.3), Machihara et al. were able to prove global well posedness for problem (3.1) with $V = 0$ for small $\dot{H}^1$-norm data with slight additional angular regularity, and in particular for all *radial* $\dot{H}^1$ data. This is especially interesting since radial data do not correspond to radial solution for the Dirac equation (due to the fact that the operator $\mathcal{D}$ does not commute with rotations of $\mathbb{R}^3$).

Estimate (3.3) gives a bound for the standard L^2L^∞ norm via Sobolev embedding on the unit sphere $\mathbb{S}^2$

$$\|e^{it|D|}f\|_{L^2L^\infty} \lesssim \|\Lambda_\omega^\epsilon e^{it|D|}f\|_{L^2L_r^\infty L_\omega^p} \lesssim \||D|\Lambda_\omega^\epsilon f\|_{L^2}, \qquad p > \frac{2}{\epsilon} \tag{3.4}$$

where the angular derivative operator Λ_ω^s is defined in terms of the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$ as

$$\Lambda_\omega^s = (1 - \Delta_{\mathbb{S}^{n-1}})^{s/2}.$$

Using (3.4) one can prove global existence for (3.1) in the case $V = 0$, provided the norm $\||D|\Lambda_\omega^s f\|_{L^2}$ of the data is small enough for some $s > 0$. In particular, this includes all radial data with a small H^1 norm.

Our main goal here is to extend this group of results to the equation (3.1) perturbed with a small potential $V(x)$. We consider first the linear equation

$$iu_t = \mathcal{D}u + V(x)u + F(t,x). \tag{3.5}$$

The perturbative term Vu can not be handled using the inhomogeneous version of (3.3) because of the loss of derivatives. Instead, we prove new mixed Strichartz-smoothing estimates (Theorem 3.6)

$$n \geq 3, \qquad \left\| \int_0^t e^{i(t-s)|D|}F(s,x)ds \right\|_{L_t^2 L_{|x|}^\infty L_\omega^2} \lesssim \|\langle x\rangle^{\frac{1}{2}+}|D|^{\frac{n-1}{2}}\Lambda_\omega^\sigma F\|_{L_t^2 L_x^2} \tag{3.6}$$

where

$$\begin{aligned} &\textbf{for } n = 3, \qquad \sigma = 0 \\ &\textbf{for } n \geq 4, \qquad \sigma = 1 - \frac{n}{2}. \end{aligned} \tag{3.7}$$

Remark 3.1. As a byproduct of our proof, we obtain the following endpoint estimates for the wave flow with gain of angular regularity (Theorem 3.4):

$$n \geq 3, \qquad \|e^{it|D|}f\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \|\Lambda_\omega^\sigma f\|_{\dot H^{\frac{n-1}{2}}} \tag{3.8}$$

where σ is as in (3.7). Although this was not the main purpose of the paper, it is interesting to compare (3.8) with known results. In dimension $n = 3$, estimate (3.8) is just a special case of Theorem 1.1-III in [**51**] where (3.8) is proved with $\sigma = -\frac{3}{4}$; it is not known if this value is sharp, however in the same paper it is proved that the estimate is false for $\sigma < -\frac{5}{6}$. On the other hand, to our knowledge, estimate (3.8) for $n \geq 4$ and (3.6) for $n \geq 3$ are new. The literature on these kind of estimates is extensive and we refer to [**36**], [**44**] and the references therein for further information.

Combining (3.6) with the techniques of [**21**] we obtain the following endpoint result for a 3D linear wave equation with singular potential. Analogous estimates can be proved for higher dimensions; here we chose to focus on the 3D case since the assumptions on V take a particular simple form:

Theorem 3.1. *Let $n = 3$ and consider the Cauchy problem for the wave equation*

$$u_{tt} - \Delta u + V(x)u = F, \qquad u(0,x) = f(x), \qquad u_t(0,x) = g(x)$$

under the assumptions:

(i) *$V(x)$ is real valued and the positive and negative parts $V_\pm$ satisfy*

$$V_+ \leq \frac{C}{|x|^{\frac{1}{2}-\epsilon}+|x|^2}, \qquad V_- \leq \frac{\delta}{|x|^{\frac{1}{2}-\epsilon}+|x|^2} \tag{3.9}$$

for some δ, ϵ sufficiently small and some $C \geq 0$;

(ii) *$-\Delta + V$ is selfadjoint;*

(iii) *0 is not a resonance for $-\Delta + V_-$ (in the following sense: if f is such that $(-\Delta + V_-)f = 0$ and $\langle x\rangle^{-1} f \in L^2$, then $f \equiv 0$).*

Then the solution $u(t,x)$ satisfies the endpoint Strichartz estimate

$$\|u\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|f\|_{\dot H^1} + \|g\|_{L^2} + \|\langle x\rangle^{\frac{1}{2}+} F\|_{L^2_t L^2_x}. \tag{3.10}$$

The next step is to prove suitable smoothing estimates for the Dirac equation with potential

$$iu_t = \mathcal{D}u + V(x)u + F(t,x)$$

(see Proposition 3.11 and Corollary 3.12). Then by a perturbative argument we obtain the following endpoint estimates for the linear flows:

Theorem 3.2. *Assume that the hermitian matrix $V(x)$ satisfies, for δ sufficiently small, C arbitrary and $\sigma > 1$, with $v(x) = |x|^{\frac{1}{2}}|\log|x||^{\frac{1}{2}+} + |x|^{1+}$,*

$$|V(x)| \leq \frac{\delta}{v(x)}, \qquad |\nabla V(x)| \leq \frac{C}{v(x)}. \tag{3.11}$$

Then the perturbed Dirac flow satisfies the endpoint Strichartz estimate

$$\|e^{it(\mathcal{D}+V)} f\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|f\|_{H^1}. \tag{3.12}$$

If the potential satisfies the stronger assumptions: for some $s > 1$,

$$\|\Lambda^s_\omega V(|x|\,\cdot\,)\|_{L^2(\mathbb{S}^2)} \leq \frac{\delta}{v(x)}, \qquad \|\Lambda^s_\omega \nabla V(|x|\,\cdot\,)\|_{L^2(\mathbb{S}^2)} \leq \frac{C}{v(x)}, \tag{3.13}$$

then we have the endpoint estimate with angular regularity

$$\|\Lambda^s_\omega e^{it(\mathcal{D}+V)} f\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|\Lambda^s_\omega f\|_{H^1} \tag{3.14}$$

and the energy estimate with angular regularity

$$\|\Lambda^s_\omega e^{it(\mathcal{D}+V)} f\|_{L^\infty_t H^1} \lesssim \|\Lambda^s_\omega f\|_{H^1} \tag{3.15}$$

We can finally apply Theorem 3.2 to the nonlinear equation (3.1) and we obtain:

Theorem 3.3. *Consider the perturbed Dirac system* (3.1)*, where the 4×4 matrix valued potential $V(x)$ is hermitian and satisfies assumptions* (3.13)*. Let $P_3(u, \overline{u})$ be a $\mathbb{C}^4$-valued homogeneous cubic polynomial. Then for any $s > 1$ there exists ϵ_0 such that for all initial data satisfying*

$$\|\Lambda^s_\omega f\|_{H^1} < \epsilon_0 \tag{3.16}$$

the Cauchy problem (3.92) *admits a unique global solution $u \in CH^1 \cap L^2 L^\infty$ with $\Lambda^s_\omega u \in L^\infty H^1$.*

In particular, problem (3.1) has a global unique solution for all radial data with sufficiently small H^1 norm.

Remark 3.2. It is clear that our methods can also be applied to nonlinear wave equations perturbed with potentials, and allow to prove global well posedness for some types of critical nonlinearities. This problem will be the object of a further note.

Remark 3.3. We did not strive for the sharpest condition on the potential V, which can be improved at the price of additional technicalities which we prefer to skip here. Moreover, differently from the previous chapter in which the structure of the non linear term was essential, the result can be extended to more general cubic nonlinearities $|P_3(u)| \sim |u|^3$.

Notice also that we need an angular regularity $s > 1$ on the data, higher than the $s > 0$ assumed in the result of [**51**]. It is possible to relax our assumptions to $s > 0$; the only additional tool we would need to prove is a Moser-type product estimate

$$\|\Lambda_\omega^s(uv)\|_{L^2_\omega} \lesssim \|u\|_{L^\infty_\omega}\|\Lambda_\omega^s v\|_{L^2_\omega} + \|\Lambda_\omega^s u\|_{L^2_\omega}\|v\|_{L^\infty_\omega}, \qquad s > 0$$

and an analogous one for $\Lambda_\omega^s|D|(uv)$. This would require a fair amount of calculus on the sphere $\mathbb{S}^2$, and here we preferred to use the conceptually much simpler algebra property of $H^s(\mathbb{S}^{n-1})$ for $s > \frac{n-1}{2}$.

On the other hand, the extension of our results to the massive case

$$iu_t = \mathcal{D}u + V(x)u + m\beta u + P_3(u), \qquad m \neq 0$$

requires a different approach and will be the object of further work.

Acknowledgments. We are indebted with Ilia Krasikov and Jim Wright for invaluable conversations which helped shape up the proofs in section 2.

2. Endpoint estimates for the free flows

To fix our notations, we recall some basic facts on spherical harmonics (see [**71**]) on $\mathbb{R}^n$, $n \geq 2$. For $k \geq 0$, we denote by $\mathcal{H}_k$ the space of harmonic polynomials homogeneous of degree k, restricted to the unit sphere $\mathbb{S}^{n-1}$. The dimension of $\mathcal{H}_k$ for $k \geq 2$ is

$$d_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2} \simeq \langle k \rangle^{n-2}$$

while $d_0 = 1$ and $d_1 = n$. $\mathcal{H}_k$ is called the space of *spherical harmonics of degree* k, and we denote by Y_k^l, $1 \leq l \leq d_k$ an orthonormal basis. Since

$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}^k$$

every function $f(x) = f(r\omega)$, $r = |x|$, can be expanded as

$$f(r) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k} f_k^l(r) Y_k^l(\omega) \tag{3.17}$$

and we have

$$\|f(r\omega)\|_{L^2_\omega} = \sum_{\substack{k\geq 0\\ 1\leq l\leq d_k}} |f_k^l|^2,$$

where we use the notation $L^2_\omega = L^2(\mathbb{S}^{n-1})$. More generally, if $\Delta_{\mathbb{S}}$ is the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$ and

$$\Lambda_\omega = (1-\Delta_{\mathbb{S}})^{1/2},$$

we have the equivalence

$$\|\Lambda^\sigma_\omega f(r\omega)\|_{L^2_\omega} \simeq \sum_{\substack{k\geq 0\\ 1\leq l\leq d_k}} \langle k\rangle^{2\sigma}|f^l_k|^2, \qquad \sigma\in\mathbb{R}.$$

As a consequence we have the equivalence

$$\|\Lambda^\sigma_\omega f\|^2_{L^2(\mathbb{R}^n)} \simeq \sum_{\substack{k\geq 0\\ 1\leq l\leq d_k}} \langle k\rangle^{2\sigma}\|f^l_k(r)r^{\frac{n-1}{2}}\|^2_{L^2_r(0,\infty)}. \tag{3.18}$$

In a similar way

$$\begin{aligned}\|\nabla f\|^2_{L^2(\mathbb{R}^n)} &= (-\Delta f, f)_{L^2} \simeq \\ &\simeq \sum_{\substack{k\geq 0\\ 1\leq l\leq d_k}} \left(\|r^{\frac{n-1}{2}}\partial_r f^l_k(r)\|^2_{L^2_r(0,\infty)} + k^2\|r^{\frac{n-3}{2}} f^l_k(r)\|^2_{L^2_r(0,\infty)}\right)\end{aligned} \tag{3.19}$$

where we used the following representation of the action of Δ

$$-\Delta f(x) = \sum Y^l_k\left(\frac{x}{|x|}\right)\left[-r^{1-n}\partial_r(r^{n-1}\partial_r f^l_k) + \frac{k(k+n-2)}{r^2}f^l_k\right], \qquad r=|x|$$

More generally we have for integer m

$$-\Delta(1-\Delta_S)^m f(x) = \sum(1+k(k+n-2))^m Y^l_k\left[-r^{1-n}\partial_r(r^{n-1}\partial_r f^l_k) + \frac{k(k+n-2)}{r^2}f^l_k\right]$$

which implies

$$\begin{aligned}\|\nabla\Lambda^m_\omega f\|^2_{L^2(\mathbb{R}^n)} &= (-\Delta(1-\Delta_S)^m f, f)_{L^2} \simeq \\ &\simeq \sum_{\substack{k\geq 0\\ 1\leq l\leq d_k}} \langle k\rangle^{2m}\left(\|r^{\frac{n-1}{2}}\partial_r f^l_k(r)\|^2_{L^2_r(0,\infty)} + k^2\|r^{\frac{n-3}{2}} f^l_k(r)\|^2_{L^2_r(0,\infty)}\right)\end{aligned} \tag{3.20}$$

and by interpolation and duality we see that (3.20) holds for all $m\in\mathbb{R}$.

We shall estimate the solution using the following norm:

$$\|f\|_{L^\infty_r L^2_\omega} = \sup_{r>0}\|f(r\omega)\|_{L^2_\omega(\mathbb{S}^{n-1})}.$$

THEOREM 3.4. *For all $n\geq 4$ the following estimate holds:*

$$\|e^{it|D|}f\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|\Lambda_\omega^{1-\frac{n}{2}} f\|_{\dot H^{\frac{n-1}{2}}}, \tag{3.21}$$

while for $n=3$ we have

$$\|e^{it|D|}f\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|f\|_{\dot H^1} \tag{3.22}$$

REMARK 3.4. In dimension $n=3$ the previous result is a special case of the stronger estimate proved in [**51**]:

$$\|e^{it|D|}f\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|\Lambda_\omega^{-3/4} f\|_{\dot H^1}. \tag{3.23}$$

Notice that it is not known if estimate (3.23) is sharp. For higher dimension, estimate (3.21) seems to be new; it is reasonable to guess that this result is not sharp and might be improved at least to

$$\|e^{it|D|}f\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \|\Lambda_\omega^{\epsilon-\frac{n-1}{2}} f\|_{\dot{H}^{\frac{n-1}{2}}}, \qquad \epsilon > 0. \tag{3.24}$$

PROOF. It is well known that the $\mathcal{H}_k$ spaces are invariant for the Fourier transform $\mathcal{F}$, and more precisely

$$\mathcal{F}\left(c(r)Y_k^l(\omega)\right)(\xi) = g(|\xi|)Y_k^l\left(\frac{\xi}{|\xi|}\right) \tag{3.25}$$

where g is given by the Hankel transform

$$g(r) = (2\pi)^{\frac{n}{2}} i^{-k} r^{-\frac{n-2}{2}} \int_0^\infty c(\rho) J_{k+\frac{n-2}{2}}(r\rho)\rho^{\frac{n}{2}} d\rho. \tag{3.26}$$

Here J_ν is the *Bessel function* of order ν which we shall represent using the Lommel integral form

$$J_\nu(y) = \frac{(y/2)^\nu}{\pi^{\frac{1}{2}}\Gamma(\nu+1/2)} \int_{-1}^1 e^{iy\lambda}(1-\lambda^2)^{\nu-\frac{1}{2}} d\lambda. \tag{3.27}$$

Now, given a function $f(x)$, we denote by $\check{f}$ its inverse Fourier transform and with $\check{f}_k^l(r)$ the coefficients of the expansion in spherical harmonics of $\check{f}$:

$$\check{f} = \sum_{k=0}^\infty \sum_{l=1}^{d_k} \check{f}_k^l(r) Y_k^l(\omega). \tag{3.28}$$

Recalling (3.25) we obtain the representation

$$f(x) = \sum (2\pi)^{\frac{n}{2}} i^{-k} |x|^{1-\frac{n}{2}} Y_k^l\left(\frac{x}{|x|}\right) \int_0^\infty \check{f}_k^l(\rho) J_{k+\frac{n-2}{2}}(|x|\rho)\rho^{\frac{n}{2}} d\rho \tag{3.29}$$

which implies

$$e^{it|D|}f = \sum (2\pi)^{\frac{n}{2}} i^{-k} |x|^{1-\frac{n}{2}} Y_k^l\left(\frac{x}{|x|}\right) \int_0^\infty e^{it\rho} \check{f}_k^l(\rho) J_{k+\frac{n-2}{2}}(|x|\rho)\rho^{\frac{n}{2}} d\rho. \tag{3.30}$$

Consider now Lommel's formula (3.27) for J_ν; since $e^{i\lambda y} = (iy)^{-k}\partial_\lambda^k(e^{i\lambda y})$, after k integration by parts we obtain

$$J_{k+\frac{n-2}{2}}(y) = c_k y^{\frac{n}{2}-1} \int_{-1}^1 e^{i\lambda y} \partial_\lambda^k \left((1-\lambda^2)^{k+\frac{n-3}{2}}\right) d\lambda \tag{3.31}$$

with

$$c_k = \frac{i^k 2^{-\frac{n}{2}-k+1}}{\pi^{\frac{1}{2}}\Gamma(\frac{n-1}{2}+k)}. \tag{3.32}$$

Thus we can write

$$\begin{aligned} |x|^{1-\frac{n}{2}} &\int_0^\infty e^{it\rho} \check{f}_k^l(\rho) J_{k+\frac{n-2}{2}}(|x|\rho)\rho^{\frac{n}{2}} d\rho = \\ &= c_k \int_{-1}^1 \partial_\lambda^k \left((1-\lambda^2)^{k+\frac{n-3}{2}}\right) \left[\int_{-\infty}^{+\infty} \mathbf{1}_+(\rho) \check{f}_k^l(\rho)\rho^{n-1} e^{i\rho(t+\lambda|x|)} d\rho\right] d\lambda \end{aligned} \tag{3.33}$$

where $\mathbf{1}_+(\rho)$ is the characteristic function of $(0,+\infty)$; regarding the inner integral as a Fourier transform we arrive at

$$= c_k \int_{-1}^{1} \partial_\lambda^k \left((1-\lambda^2)^{k+\frac{n-3}{2}} \right) \widehat{g}_k^l(t+\lambda|x|) d\lambda$$

where

$$g_k^l(\rho) = \mathbf{1}_+(\rho) \check{f}_k^l(\rho) \rho^{n-1} \tag{3.34}$$

In conclusion, we have the following representation

$$e^{it|D|}f = \sum (2\pi)^{\frac{n}{2}} i^{-k} Y_k^l \left(\frac{x}{|x|} \right) c_k \int_{-1}^{1} \partial_\lambda^k \left((1-\lambda^2)^{k+\frac{n-3}{2}} \right) \widehat{g}_k^l(t+\lambda|x|) d\lambda \tag{3.35}$$

where the constants c_k are given by (3.32) and g_k^l by (3.34). Notice that similar representations play a fundamental role also in [**36**], [**44**] In particular this gives for the L^2_ω norm of the solution at $t, |x|$ fixed the formula

$$\|e^{it|D|}f(|x|\cdot)\|^2_{L^2_\omega} \simeq \sum |c_k|^2 \left| \int_{-1}^{1} \partial_\lambda^k \left((1-\lambda^2)^{k+\frac{n-3}{2}} \right) \widehat{g}_k^l(t+\lambda|x|) d\lambda \right|^2 . \tag{3.36}$$

We now need the following estimate:

Lemma 3.5. *Let $Q_k(x)$ be the function*

$$Q_k(x) = \frac{\partial_x^k \left((1-x^2)^{k+\frac{n-3}{2}} \right)}{2^k \Gamma(k+\frac{n-1}{2})}.$$

Then we have on $x \in [-1,1]$

$$|Q_k(x)| \lesssim \langle k \rangle^{1-\frac{n}{2}} \ \text{if } n \geq 4, \qquad |Q_k(x)| \leq 1 \ \text{if } n = 3. \tag{3.37}$$

Proof. We recall that the Jacobi polynomials are defined by

$$\mathbf{P}_k^{(\alpha,\beta)}(x) = \frac{(-1)^k}{2^k k!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^k}{dx^k} \left[(1-x)^{\alpha+k} (1+x)^{\beta+k} \right]. \tag{3.38}$$

We shall use some standard properties of these polynomial which can be found in [**1**]. The function Q_k can be expressed in terms of $\mathbf{P}_k^{(\alpha,\alpha)}(x)$ with $\alpha = (n-3)/2$ as

$$|Q_k(x)| = \frac{k!(1-x^2)^{\frac{n-3}{2}}}{\Gamma(k+\frac{n-1}{2})} \left| \mathbf{P}_k^{(\frac{n-3}{2},\frac{n-3}{2})}(x) \right|. \tag{3.39}$$

Thus in order to estimate Q_k we need a bound for the function

$$T_a(x) = (1-x^2)^a \mathbf{P}_k^{(a,a)}(x), \qquad a = \frac{n-3}{2}. \tag{3.40}$$

The following approach was suggested by Ilia Krasikov, see [**50**]. Consider the second orderd differential equation

$$f''(x) + p(x)f'(x) + q(x)f(x) = 0$$

on the interval $(-1,1)$, and define the *Sonine function* as

$$S(f,x) = f(x)^2 + \frac{f'(x)^2}{q(x)}$$

under the assumption $q > 0$. It is easy to check that function S satisfies the relation

$$S' = -\left(2\frac{p}{q} + \frac{q'}{q^2}\right) f'^2.$$

The function $T_a(x)$ defined in (3.40) satisfies the differential equation

$$T_a''(x) + \frac{2(2a-1)}{1-x^2} x T_a'(x) + \frac{(k+1)(2a+k)}{1-x^2} T_a(x) = 0$$

so that the associated Sonine function

$$S_a(x) = T_a^2 + \frac{1-x^2}{(k+1)(2a+k)} {T_a'}^2$$

satisfies

$$S_a' = -\frac{2(2a-1)}{(k+1)(2a+k)} x\, T_a'^2. \tag{3.41}$$

From this identity it is clear that S_a has a maximum at $x = 0$ provided $a \geq 1/2$ i.e. $n \geq 4$. In this case we have

$$S_a(x) \leq S_a(0) = T_a(0)^2 + \frac{T_a'(0)^2}{(k+1)(2a+k)} = \mathbf{P}_k^{(a,a)}(0)^2 + \frac{{\mathbf{P}_k^{(a,a)}}'(0)^2}{(k+1)(2a+k)}$$

Now we recall that, for *even* $k \geq 2$,

$$\mathbf{P}_k^{(a,a)}(0) = \frac{\Gamma(k+a+1)}{(-2)^k \Gamma\left(\frac{k}{2}+1\right)\Gamma\left(\frac{k}{2}+a+1\right)} \simeq (-1)^k k^{-\frac{1}{2}} \qquad {\mathbf{P}_k^{(a,a)}}'(0) = 0$$

where we used the Stirling asymptotics

$$k! \simeq k^{k-1/2} e^{-k}, \qquad \Gamma(k+a+1) \simeq k^{k+a-1/2} e^{-k}.$$

In a similar way, for *odd* k,

$$\mathbf{P}_k^{(a,a)}(0) = 0, \qquad {\mathbf{P}_k^{(a,a)}}'(0) = \frac{\Gamma(k+a+1)}{(-2)^{k-1}\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{k}{2}+a+\frac{1}{2}\right)} \simeq (-1)^{k-1} k^{\frac{1}{2}}.$$

Thus for all values of $k \geq 1$ we have

$$|T_a(x)| \leq \sqrt{S_a(x)} \lesssim \frac{1}{\sqrt{k}}$$

and by (3.39) we conclude that, for $k \geq 1$ and $|x| < 1$,

$$|Q_k(x)| \lesssim k^{1-\frac{n}{2}}$$

which is precisely (3.37) for $n \geq 4$.

In the remaining case $n = 3$ we have $a = 0$ and the best we can do is to use the sharp inequality $|\mathbf{P}_k^{(0,0)}| \leq 1$ to obtain

$$|Q_k(x)| = \frac{k!}{k!} ||\mathbf{P}_k^{(0,0)}| \leq 1.$$

□

Using the Lemma, we can continue estimate (3.36) as follows

$$\|e^{it|D|} f(|x|\cdot)\|_{L^2_\omega}^2 \lesssim \sum \omega_k^2 \left(\int_{-1}^{1} |\widehat{g}_k^l(t+\lambda|x|)| d\lambda\right)^2$$

where

$$\omega_k = 1 \quad \text{if } n = 3, \qquad \omega_k = \langle k \rangle^{1-\frac{n}{2}} \quad \text{if } n \geq 4. \tag{3.42}$$

Since

$$\int_{-1}^{1} |\widehat{g}_k^l(t+\lambda|x|)|d\lambda = \frac{1}{|x|}\int_{-|x|}^{|x|} |\widehat{g}_k^l(t+\lambda)|d\lambda \leq M(\widehat{g}_k^l)(t)$$

where $M(g)$ is the centered maximal function, we obtain

$$\|e^{it|D|}f(|x|\cdot)\|^2_{L^2_\omega} \lesssim \sum \omega_k^2 M(\widehat{g}_k^l)(t)^2.$$

Now we can take the sup in $|x|$ which gives

$$\|e^{it|D|}f\|^2_{L^\infty_r L^2_\omega} \lesssim \sum \omega_k^2 M(\widehat{g}_k^l)(t)^2,$$

and integrating in time, by the L^2 boundedness of the maximal funcion, we obtain

$$\|e^{it|D|}f\|^2_{L^2_t L^\infty_r L^2_\omega} \lesssim \sum \omega_k^2 \|\widehat{g}_k^l\|^2_{L^2} \simeq \sum \omega_k^2 \|g_k^l\|^2_{L^2} \simeq \sum \omega_k^2 \|\check{f}_k^l(\rho)\rho^{n-1}\|^2_{L^2_\rho(0,\infty)}.$$

It is immediate to check that the last sum is equivalent to

$$\sum \omega_k^2 \|\check{f}_k^l(\rho)\rho^{n-1}\|^2_{L^2_\rho(0,\infty)} \simeq \||D|^{\frac{n-1}{2}}\Lambda_\omega^\sigma f\|^2_{L^2(\mathbb{R}^n)}$$

where $\sigma = 1 - n/2$ for $n \geq 4$, which proves (3.21), and $\sigma = 0$ for $n = 3$, which proves (3.22). □

Although the method of proof of Theorem 3.4 is probably not sharp for the homogeneous operator, it has the advantage that it can be adapted to handle also the nonhomogeneous term and gives the following mixed Strichartz-smoothing estimate:

THEOREM 3.6. *For any $n \geq 3$, the following estimate holds:*

$$\left\| \int_0^t e^{i(t-s)|D|}F(s,x)ds \right\|_{L^2_t L^\infty_{|x|} L^2_\omega} \lesssim \|\langle x\rangle^{\frac{1}{2}+}|D|^{\frac{n-1}{2}}\Lambda_\omega^\sigma F\|_{L^2_t L^2_x} \tag{3.43}$$

where

$$\sigma = 1 - \frac{n}{2} \quad \textit{if } n \geq 4, \qquad \sigma = 0 \quad \textit{if } n = 3.$$

PROOF. As in the proof of the previous theorem, we expand F in spherical harmonics and we obtain the representation

$$\int_0^t e^{i(t-s)|D|}F(s,x)ds =$$
$$= \sum (2\pi)^{\frac{n}{2}} i^{-k} c_k Y_k^l\left(\frac{x}{|x|}\right)\int_{-1}^{1} \partial_\lambda^k \left((1-\lambda^2)^{k+\frac{n-3}{2}}\right)\widehat{G}_k^l(s, t-s+\lambda|x|)d\lambda \tag{3.44}$$

with the constants c_k as in (3.32), where the functions G_k^l are defined as follows: denoting by $F_k^l(t,r)$ the coefficients of the expansion into spherical harmonics of the inverse Fourier transform $\check{F} = \mathcal{F}^{-1}(F)$

$$\check{F}(s,x) = \sum \check{F}_k^l(s,|x|)Y_k^l\left(\frac{x}{|x|}\right)$$

and by G_k^l the functions

$$G_k^l(s,\rho) = \mathbf{1}_+(\rho)\rho^{n-1}\check{F}_k^l(s,\rho), \tag{3.45}$$

the $\widehat{G}^l_k(s,r)$ are the Fourier transforms of G^l_k in the second variable:

$$\widehat{G}^l_k(s,r) = \int_{-\infty}^{+\infty} e^{ir\rho} G^l_k(s,\rho)d\rho.$$

Thus applying Lemma 3.5 we obtain

$$\left| \int_0^t e^{i(t-s)|D|} F(s,x)ds \right| \lesssim \sum |Y^l_k| \frac{\omega_k}{|x|} \int_{-|x|}^{|x|} d\lambda \int_0^t ds |\widehat{G}^l_k(s,t-s+\lambda)| \quad (3.46)$$

where ω_k is the same as in (3.42). We estimate the integral in s as follows

$$\begin{aligned}\int_0^t |\widehat{G}^l_k| ds &\le \int_{-\infty}^{+\infty} \langle\lambda+t-s\rangle^{\frac{1}{2}+} \langle\lambda+t-s\rangle^{-\frac{1}{2}-} |\widehat{G}^l_k(s,\lambda+t-s)|ds \\ &\lesssim \left(\int \langle\lambda+t-s\rangle^{1+} |\widehat{G}^l_k(s,\lambda+t-s)|^2 ds \right)^{\frac{1}{2}} = Q^l_k(\lambda+t),\end{aligned}$$

where

$$Q^l_k(\mu) = \left(\int_{-\infty}^{\infty} |\widehat{G}^l_k(s,\mu-s)|^2 \langle\mu-s\rangle^{1+} ds \right)^{\frac{1}{2}}.$$

Thus we see that

$$\frac{1}{|x|} \int_{-|x|}^{|x|} d\lambda \int_0^t ds |\widehat{G}^l_k(s,t-s+\lambda)| \lesssim \frac{1}{|x|} \int_{-|x|}^{|x|} Q^l_k(\lambda+t)d\lambda \le M(Q^l_k)(t).$$

Coming back to (3.46) we obtain

$$\left| \int_0^t e^{i(t-s)|D|} F(s,x)ds \right| \lesssim \sum \omega_k |Y^l_k| M(Q^l_k)(t)$$

and taking first the L^2_ω norm, then the sup in $|x|$, then the L^2_t norm, by the L^2 boundedness of the maxiaml function we have

$$\left\| \int_0^t e^{i(t-s)|D|} F(s,x)ds \right\|^2_{L^2_t L^\infty_r L^2_\omega} \lesssim \sum \omega_k^2 \|Q^l_k(t)\|^2_{L^2_t}.$$

The definition of Q^l_k implies

$$\int |Q^l_k(t)|^2 dt = \iint |\widehat{G}^l_k(s,\mu-s)|^2 \langle\mu-s\rangle^{1+} ds d\mu = \|\widehat{G}^l_k(t,r)\langle r\rangle^{\frac{1}{2}+}\|^2_{L^2_t L^2_r}$$

and hence

$$\left\| \int_0^t e^{i(t-s)|D|} F(s,x)ds \right\|^2_{L^2_t L^\infty_r L^2_\omega} \lesssim \sum \omega_k^2 \|\widehat{G}^l_k(t,r)\langle r\rangle^{\frac{1}{2}+}\|^2_{L^2_t L^2_r}. \quad (3.47)$$

Recalling the definition (3.45) of G^l_k, we see that to obtain (3.43) it is sufficient to prove the following general inequality for $s = 1/2+$ and arbitrary σ:

$$\sum \langle k\rangle^{2\sigma} \left\| \langle y\rangle^s \mathcal{F}_{\lambda\to y} \left(\mathbf{1}_+(\lambda)\lambda^{n-1} \check{f}^l_k(\lambda) \right) \right\|^2_{L^2_y} \lesssim \|\langle x\rangle^s \Lambda^\sigma_\omega |D|^{\frac{n-1}{2}} f\|^2_{L^2(\mathbb{R}^b)}. \quad (3.48)$$

Here as usual $\check{f}^l_k$ denotee the coefficients in the expansion in spherical harmonics of the inverse Fourier transform $\check{f} = \mathcal{F}^{-1} f$.

First of all, since $\mathcal{F}^{-1}(|D|^{\frac{n-1}{2}} f) = |\xi|^{\frac{n-1}{2}} \check{f}$, we see that it is enough to prove, for $0 \leq s \leq 1$ and arbitrary σ, the slightly simpler

$$\sum \langle k \rangle^{2\sigma} \left\| \langle y \rangle^s \mathcal{F}_{\lambda \to y} \left(\mathbf{1}_+(\lambda) \lambda^{\frac{n-1}{2}} \check{f}^l_k(\lambda) \right) \right\|^2_{L^2_y} \lesssim \| \langle x \rangle^s \Lambda^\sigma_\omega f \|^2_{L^2(\mathbb{R}^b)}. \tag{3.49}$$

The inequality will follow by interpolation between the cases $s = 0$ and $s = 1$; indeed, we can regard it as the statement that the operator T defined as

$$T : f \mapsto \left\{ \langle y \rangle^s \mathcal{F}_{\lambda \to y} \left(\mathbf{1}_+(\lambda) \lambda^{\frac{n-1}{2}} \check{g}^l_k \right) \right\}_{l,k}, \qquad g = \Lambda^{-\sigma}_\omega f$$

which associates to the function f the sequence of coefficients in the expansion of $\mathcal{F}^{-1}(\Lambda^{-\sigma}_\omega f)$, multiplied by $\lambda^{(n-1)/2} \mathbf{1}_+$, transformed again and multiplied by $\langle y \rangle^s$, is bounded between the weighted spaces

$$T : L^2(\langle x \rangle^{2s} dx) \to \ell^2_{\langle k \rangle^{2\sigma}}(L^2(\langle \lambda \rangle^{2s} d\lambda)).$$

When $s = 0$ we have by Plancherel's Theorem and by (3.18)

$$\sum \langle k \rangle^{2\sigma} \left\| \mathcal{F}_{\lambda \to y} \left(\mathbf{1}_+(\lambda) \lambda^{\frac{n-1}{2}} \check{f}^l_k(\lambda) \right) \right\|^2_{L^2_y} \simeq$$

$$\simeq \sum \langle k \rangle^{2\sigma} \left\| \lambda^{\frac{n-1}{2}} \check{f}^l_k(\lambda) \right\|^2_{L^2_\lambda(0,\lambda)} \simeq \| \Lambda^\sigma_\omega \check{f} \|^2_{L^2(\mathbb{R}^n)}.$$

Since Λ_ω commutes with the Fourier transform, indeed

$$\mathcal{F}(-\Delta_S f) = \mathcal{F} \sum (x_j \partial_k - x_j \partial_j)^2 f = \sum (\partial_j \xi_k - \partial_k \xi_j)^2 \mathcal{F} f,$$

again by Plancherel we obtain (3.49) for $s = 0$.

To handle the case $s = 1$ we consider the quantity

$$\left\| y \mathcal{F}_{\lambda \to y} \left(\mathbf{1}_+(\lambda) \lambda^{\frac{n-1}{2}} \check{f}^l_k(\lambda) \right) \right\|^2_{L^2_y} = \left\| \partial_\lambda \left(\mathbf{1}_+(\lambda) \lambda^{\frac{n-1}{2}} \check{f}^l_k(\lambda) \right) \right\|^2_{L^2_\lambda} \lesssim$$

$$\lesssim \left\| \lambda^{\frac{n-1}{2}} \partial_\lambda \check{f}^l_k(\lambda) \right\|^2_{L^2_\lambda(0,\infty)} + \left\| \lambda^{\frac{n-3}{2}} \check{f}^l_k(\lambda) \right\|^2_{L^2_\lambda(0,\infty)}.$$

Multiplying by $\langle k \rangle^{2\sigma}$, summing over l, k and recalling (3.20), we obtain

$$\sum \langle k \rangle^{2\sigma} \left\| y \mathcal{F}_{\lambda \to y} \left(\mathbf{1}_+(\lambda) \lambda^{\frac{n-1}{2}} \check{f}^l_k(\lambda) \right) \right\|^2_{L^2_y} \lesssim \| \nabla \Lambda^\sigma_\omega \check{f} \|^2_{L^2(\mathbb{R}^n)} + \left\| \lambda^{\frac{n-3}{2}} \check{f}^0_0(\lambda) \right\|^2_{L^2_\lambda(0,\infty)}$$

where the last term can not estimated by (3.20) because of the factor k^2 which vanishes when $k = 0$. However we have

$$\check{f}^0_0(\lambda) = \int_{|\omega|=1} \check{f}(\lambda\omega) d\lambda = \int_{|\omega|=1} \Lambda^\sigma_\omega \check{f}(\lambda\omega) d\lambda$$

which implies, using Hardy's inequlity

$$\left\| \lambda^{\frac{n-3}{2}} \check{f}^0_0(\lambda) \right\|^2_{L^2_\lambda(0,\infty)} \lesssim \left\| \frac{\Lambda^\sigma_\omega \check{f}}{|\xi|} \right\|_{L^2(\mathbb{R}^n)} \lesssim \| \nabla \Lambda^\sigma_\omega \check{f} \|_{L^2}.$$

Thus we have proved

$$\sum \langle k \rangle^{2\sigma} \left\| y \mathcal{F}_{\lambda \to y} \left(\mathbf{1}_+(\lambda) \lambda^{\frac{n-1}{2}} \check{f}^l_k(\lambda) \right) \right\|^2_{L^2_y} \lesssim \| \nabla \Lambda^\sigma_\omega \check{f} \|^2_{L^2(\mathbb{R}^n)} \simeq \| |x| \Lambda^\sigma_\omega f \|^2_{L^2}$$

again by the commutation of Λ_ω with the Fourier transform. This gives (3.49) for $s = 1$ and concludes the proof of the Theorem. □

REMARK 3.5. Since the operator Λ_ω commutes with $|D|$, estimates (3.21), (3.22) and (3.43) obviously generalize to the following; for any real $s \geq 0$,

$$\|\Lambda_\omega^s e^{it|D|} f\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \|\Lambda_\omega^{s+\sigma} f\|_{\dot{H}^{\frac{n-1}{2}}} \tag{3.50}$$

and

$$\left\| \Lambda_\omega^s \int_0^t e^{i(t-s)|D|} F(s,x) ds \right\|_{L_t^2 L_{|x|}^\infty L_\omega^2} \lesssim \|\langle x \rangle^{\frac{1}{2}+} |D|^{\frac{n-1}{2}} \Lambda_\omega^{s+\sigma} F\|_{L_t^2 L_x^2} \tag{3.51}$$

where

$$\sigma = 1 - \frac{n}{2} \quad \text{if } n \geq 4, \qquad \sigma = 0 \quad \text{if } n = 3.$$

From the previous estimate for the free wave equation it is not difficult to obtain analogous endpoint Strichartz and Strichartz-smoothing estimates for the 3D Dirac system:

COROLLARY 3.7. *Let $n = 3$. Then the flow $e^{it\mathcal{D}}$ satisfies, for all $s \geq 0$, the estimates*

$$\|\Lambda_\omega^s e^{it\mathcal{D}} f\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \|\Lambda_\omega^s f\|_{\dot{H}^1}, \tag{3.52}$$

and

$$\left\| \Lambda_\omega^s \int_0^t e^{i(t-t')\mathcal{D}} F(t',x) dt' \right\|_{L_t^2 L_{|x|}^\infty L_\omega^2} \lesssim \|\langle x \rangle^{\frac{1}{2}+} |D| \Lambda_\omega^s F\|_{L_t^2 L_x^2}. \tag{3.53}$$

PROOF. If u solves the problem

$$iu_t + \mathcal{D}u = 0, \qquad u(0) = f(x), \tag{3.54}$$

by applying the operator $(i\partial_t - \mathcal{D})$, we see that u solves also

$$\Box u = 0, \qquad u(0) = f(x), \qquad u_t(0) = i\mathcal{D}f. \tag{3.55}$$

This gives the representation

$$e^{it\mathcal{D}} f = \cos(t|D|) f + i \frac{\sin(t|D|)}{|D|} \mathcal{D} f. \tag{3.56}$$

Moreover, we recall that the Riesz operators $|D|^{-1}\partial_j$ are bounded on weighted L^2 spaces with weight $\langle x \rangle^a$ for $a < n/2$. Thus in the case $s = 0$ estimates (3.52), (3.53) are immediate consequences of the corresponding estimates for the wave equation proved above.

In order to complete the proof in the case $s > 0$, we need analyze the structure of the Dirac operator $\mathcal{D}$ in greater detail. Following [**74**], we know that the space $L^2(\mathbb{R}^3)^4$ is isomorphic to an orthogonal direct sum

$$L^2(\mathbb{R}^3)^4 \simeq \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{\substack{k_j= \\ \pm(j+1/2)}} L^2(0,+\infty; dr) \otimes H_{m_j,k_j}.$$

Each space H_{m_j,k_j} has dimension two and is generated by the orthonormal basis $\{\Phi^+_{m_j,k_j}, \Phi^-_{m_j,k_j}\}$, which can be explicitly written in terms of spherical harmonics: when $k_j = j + 1/2$ we have

$$\Phi^+_{m_j,k_j} = \frac{i}{\sqrt{2j+2}} \begin{pmatrix} \sqrt{j+1-m_j} \;\; Y^{m_j-1/2}_{k_j} \\ -\sqrt{j+1+m_j} \;\; Y^{m_j+1/2}_{k_j} \\ 0 \\ 0 \end{pmatrix}$$

$$\Phi^-_{m_j,k_j} = \frac{1}{\sqrt{2j}} \begin{pmatrix} 0 \\ 0 \\ \sqrt{j+m_j} \;\; Y^{m_j-1/2}_{k_j-1} \\ \sqrt{j-m_j} \;\; Y^{m_j+1/2}_{k_j-1} \end{pmatrix}$$

while when $k_j = -(j+1/2)$ we have

$$\Phi^+_{m_j,k_j} = \frac{i}{\sqrt{2j}} \begin{pmatrix} \sqrt{j+m_j} \;\; Y^{m_j-1/2}_{1-k_j} \\ \sqrt{j-m_j} \;\; Y^{m_j+1/2}_{1-k_j} \\ 0 \\ 0 \end{pmatrix}$$

$$\Phi^-_{m_j,k_j} = \frac{1}{\sqrt{2j+2}} \begin{pmatrix} 0 \\ 0 \\ \sqrt{j+1-m_j} \;\; Y^{m_j-1/2}_{-k_j} \\ -\sqrt{j+1+m_j} \;\; Y^{m_j+1/2}_{-k_j} \end{pmatrix}.$$

The isomorphism is expressed by the explicit expansion

$$\Psi(x) = \sum \frac{1}{r}\psi^+_{m_j,k_j}(r)\Phi^+_{m_j,k_j} + \frac{1}{r}\psi^-_{m_j,k_j}(r)\Phi^-_{m_j,k_j} \tag{3.57}$$

with

$$\|\Psi\|^2_{L^2} = \sum \int_0^\infty [|\psi^+_{m_j,k_j}|^2 + |\psi^-_{m_j,k_j}|^2]dr. \tag{3.58}$$

Notice also that

$$\|\Psi\|^2_{L^2_\omega} = \sum \frac{1}{r^2}|\psi^+_{m_j,k_j}|^2 + \frac{1}{r^2}|\psi^-_{m_j,k_j}|^2. \tag{3.59}$$

Each $L^2(0,+\infty;dr) \otimes H_{m_j,k_j}$ is an eigenspace of the Dirac operator $\mathcal{D} = i^{-1}\sum \alpha_j \partial_j$ and the action of $\mathcal{D}$ can be written, in terms of the expansion (3.57), as

$$\mathcal{D}\Psi = \sum \left(-\frac{d}{dr}\psi^-_{m_j,k_j} + \frac{k_j}{r}\psi^-_{m_j,k_j}\right)\frac{\Phi^+_{m_j,k_j}}{r} + \left(\frac{d}{dr}\psi^+_{m_j,k_j} + \frac{k_j}{r}\psi^+_{m_j,k_j}\right)\frac{\Phi^-_{m_j,k_j}}{r}.$$

From decomposition (3.57) it is clear that the operator Λ^σ_ω, which acts on spherical harmonics as

$$\Lambda^\sigma_\omega Y^m_\ell = (1+\ell(\ell+1))^{\frac{\sigma}{2}} \cdot Y^m_\ell, \tag{3.60}$$

does not commute with $\mathcal{D}$. Indeed, each space H_{m_j,k_j} involves two spherical harmonics Y^m_ℓ with two values of ℓ which differ by 1, and $\mathcal{D}$ swaps them. However, the modified operator $\widetilde{\Lambda}^\sigma_\omega$ defined by

$$\widetilde{\Lambda}^\sigma_\omega \Phi^\pm_{m_j,k_j} = |k_j|^\sigma \Phi^\pm_{m_j,k_j} \tag{3.61}$$

obviously commutes with $\mathcal{D}$, thus estimates (3.52), (3.53) are trivially true if we replace Λ with $\widetilde{\Lambda}$. It remains to show that we obtain equivalent norms. The equivalence

$$\|\widetilde{\Lambda}^{\sigma}_{\omega} f\|_{L^2_\omega} \simeq \|\Lambda^{\sigma}_{\omega} f\|_{L^2_\omega}$$

follows directly from (3.60), (3.61) and (3.59). Moreover, $\widetilde{\Lambda}$ and Λ commute with Δ, hence with $|D|$, and this implies

$$\||D|\Lambda^s_\omega f\|_{L^2} \simeq \||D|\widetilde{\Lambda}^s_\omega f\|_{L^2}$$

or, equivalently,

$$\|\Lambda^s_\omega f\|_{\dot{H}^1} \simeq \|\widetilde{\Lambda}^s_\omega f\|_{\dot{H}^1}.$$

This is sufficient to prove (3.52). Since $\widetilde{\Lambda}$ and Λ also commute with radial weights we have

$$\|\langle x\rangle^{\frac{1}{2}+}|D|\Lambda^s_\omega f\|_{L^2} \simeq \|\langle x\rangle^{\frac{1}{2}+}|D|\widetilde{\Lambda}^s_\omega f\|_{L^2}$$

which gives (3.53). □

3. The wave equations with potential

Our next goal is to extend the results of previous chapter to the case of perturbed flows. This will be obtained by a perturbative argument, relying on the smoothing estimates of [**21**] and the mixed Strichartz-smoothing estimates of the previous chapter. In [**21**] smoothing estimates were proved for several classes of dispersive equations perturbed with electromagnetic potentials (while the 1D case was analyzed in [**19**]). For the wave equation in dimension $n \geq 3$ the estimates are the following:

Proposition 3.8. *Let $n \geq 3$. Assume the operator*

$$-\Delta + W(x, D) = -\Delta + a(x)\cdot\nabla + b_1(x) + b_2(x)$$

is selfadjoint and its coefficients satisfy

$$|a(x)| \leq \frac{\delta}{|x|^{1-\epsilon} + |x|^2|\log|x||^{\sigma}} \tag{3.62}$$

$$|b_1(x)| \leq \frac{\delta}{|x|^{1-\epsilon} + |x|^2}, \qquad 0 \leq b_2(x) \leq \frac{C}{|x|^{1-\epsilon} + |x|^2} \tag{3.63}$$

for some $\delta, \epsilon > 0$ sufficiently small and some $\sigma > 1/2$, $C > 0$. Moreover assume that 0 is not a resonance for $-\Delta + b_2$. Then the following smoothing estimate holds:

$$\|(|x|^{\frac{1}{2}-\epsilon} + |x|)^{-1} e^{it\sqrt{-\Delta+W}} f\|_{L^2L^2} \lesssim \|f\|_{L^2}. \tag{3.64}$$

The assumption that 0 is not a resonance for $-\Delta + b_2(x)$ here means: if $(-\Delta + b_2)f = 0$ and $\langle x\rangle^{-1} f \in L^2$ then $f \equiv 0$.

Combining Proposition 3.8 with (3.43) we obtain the following Strichartz endpoint estimate for the 3D wave equation perturbed with an electric potential:

Theorem 3.9. *Let $n = 3$ and consider the Cauchy problem for the wave equation*

$$u_{tt} - \Delta u + V(x)u = F, \qquad u(0, x) = f(x), \qquad u_t(0, x) = g(x)$$

under the assumptions:

(i) *$V(x)$ is real valued and the positive and negative parts $V_\pm$ satisfy*

$$V_+ \leq \frac{C}{|x|^{\frac{1}{2}-\epsilon}+|x|^2}, \qquad V_- \leq \frac{\delta}{|x|^{\frac{1}{2}-\epsilon}+|x|^2} \tag{3.65}$$

for some δ, ϵ sufficiently small and some $C \geq 0$;

(ii) *$-\Delta + V$ is selfadjoint;*

(iii) *0 is not a resonance for $-\Delta + V_-$.*

Then the solution $u(t,x)$ satisfies the endpoint Strichartz estimate

$$\|u\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|f\|_{\dot{H}^1} + \|g\|_{L^2} + \|\langle x\rangle^{\frac{1}{2}+} F\|_{L^2_t L^2_x}. \tag{3.66}$$

Proof. We represent $u(t,x)$ in the form

$$u(t,x) = I + II - III$$

where

$$I = \cos(t|D|)f + \frac{\sin(t|D|)}{|D|}g,$$

$$II = \int_0^t |D|^{-1}\sin((t-s)|D|)F\,ds,$$

and

$$III = \int_0^t |D|^{-1}\sin((t-s)|D|)Vu\,ds.$$

We can use (3.21) to estimate I and (3.43) to estimate II in the norm $L^2_t L^\infty_r L^2_\omega$ directly. On the other hand, applying (3.43) to III we get

$$\|III\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|\langle x\rangle^{\frac{1}{2}+} Vu\|_{L^2L^2} \leq \|\langle x\rangle^{\frac{1}{2}+}\tau_\epsilon V\|_{L^\infty_x} \|\tau_\epsilon^{-1}u\|_{L^2L^2}.$$

By assumption $\langle x\rangle^{\frac{1}{2}+}\tau_\epsilon V$ is bounded on $\mathbb{R}^n$, moreover we are allowed to use (3.64) since V satisfies the assumptions of Proposition 3.8. Notice that (3.64) implies

$$\|\tau_\epsilon^{-1}u\|_{L^2L^2} \lesssim \|f\|_{L^2} + \||D|^{-1}g\|_{L^2}$$

and in conclusion we have proved

$$\|III\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|f\|_{L^2} + \||D|^{-1}g\|_{L^2}$$

which completes the proof of (3.66). □

Analogous estimates can be proved for the Klein-Gordon equation, or in higher dimension $n \geq 3$, for first order perturbations, and for angular derivatives of the solutions. We omit the details since we prefer to focus on the Dirac equation here.

4. The Dirac equation with potential

We consider now the perturbed Dirac operator $\mathcal{D} + V(x)$ where $V(x)$ is a small 4×4 hermitian matrix valued potential. We prove here more general versions of the estimates given in [**21**], [**20**] in order to include angular regularity. We begin with the free Dirac equation:

PROPOSITION 3.10. *The free Dirac flow satisfies, for all $\sigma > 1$ and $s \geq 0$, the smoothing estimates (with $w_\sigma(x) = |x|(1 + |\log|x||)^\sigma$)*

$$\|w_\sigma^{-1/2}\Lambda_\omega^s e^{it\mathcal{D}} f\|_{L_t^2 L_x^2} \lesssim \|\Lambda_\omega^s f\|_{L^2} \tag{3.67}$$

and

$$\left\| w_\sigma^{-1/2}\Lambda_\omega^s \int_0^t e^{i(t-t')\mathcal{D}} F(t')dt' \right\|_{L_t^2 L_x^2} \lesssim \|w_\sigma^{1/2}\Lambda_\omega^s F\|_{L_t^2 L_x^2} \tag{3.68}$$

PROOF. When $s = 0$, both estimates follow from the resolvent estimate

$$\|w_\sigma^{-1/2} R_{\mathcal{D}}(z) f\|_{L^2(\mathbb{R}^3)} \leq C \|w_\sigma^{1/2} f\|_{L^2(\mathbb{R}^3)}, \qquad z \notin \mathbb{R},$$

with a constant uniform in z, proved in [**20**] using a standard application of Kato's theory (see also [**21**]). The case $s > 0$ is proved exactly as in Corollary 3.7, first by replacing Λ_ω with $\widetilde{\Lambda}_\omega$ which commutes with the flow, and then by using the equivalence of norms. □

We consider now the case of a perturbed Dirac system

$$iu_t = \mathcal{D}u + Vu$$

where $V(x)$ is a 4×4 matrix potential. If V is hermitian and its weak $L^{3,\infty}$ norm is small enough, the operator $\mathcal{D} + V$ is selfadjoint as proved in [**20**]. In all of the following results the assumptions on the potential are somewhat stronger than this, so in all cases the unitary flow $e^{it(\mathcal{D}+V)}$ will be well defined and continuous on $L^2(\mathbb{R}^3)^4$ by spectral theory.

PROPOSITION 3.11. *Let $V(x)$ be a hermitian 4×4 matrix on $\mathbb{R}^3$ such that*

$$|V(x)| \leq \frac{\delta}{w_\sigma(|x|)}, \qquad w_\sigma(r) = r \cdot (1 + |\log r|)^\sigma \tag{3.69}$$

for some $\delta > 0$ sufficiently small and some $\sigma > 1$. Then the perturbed Dirac flow $e^{it(\mathcal{D}+V)}$ satisfies the smoothing estimates

$$\|w_\sigma^{-1/2} e^{it(\mathcal{D}+V)} f\|_{L_t^2 L_x^2} \lesssim \|f\|_{L^2}, \tag{3.70}$$

$$\left\| w_\sigma^{-1/2} \int_0^t e^{i(t-t')(\mathcal{D}+V)} F(t')dt' \right\|_{L_t^2 L_x^2} \lesssim \|w_\sigma^{1/2} F\|_{L_t^2 L_x^2}. \tag{3.71}$$

If in addition V satisfies for some $s > 1$ the condition

$$\|\Lambda_\omega^s V(r\,\cdot\,)\|_{L^2(\mathbb{S}^2)} \leq \frac{\delta}{w_\sigma(r)}, \tag{3.72}$$

then we have, for all $0 \leq s \leq 2$, the estimates with angular regularity

$$\|w_\sigma^{-1/2}\Lambda_\omega^s e^{it(\mathcal{D}+V)} f\|_{L_t^2 L_x^2} \lesssim \|\Lambda_\omega^s f\|_{L^2}, \tag{3.73}$$

$$\left\| w_\sigma^{-1/2}\Lambda_\omega^s \int_0^t e^{i(t-t')(\mathcal{D}+V)} F(t')dt' \right\|_{L_t^2 L_x^2} \lesssim \|w_\sigma^{1/2}\Lambda_\omega^s F\|_{L_t^2 L_x^2}. \tag{3.74}$$

PROOF. If u solves

$$iu_t = \mathcal{D}u + Vu + F, \qquad u(0) = f$$

we can write

$$u = e^{it\mathcal{D}} f + i \int_0^t e^{i(t-t')\mathcal{D}} [Vu(t') + F(t')]dt'.$$

Using (3.67), (3.68) with $s = 0$ and assumption (3.69) we get

$$\|w_\sigma^{-1/2}u\|_{L_t^2L_x^2} \lesssim \|f\|_{L^2} + \|w_\sigma^{1/2}[Vu + F]\|_{L^2L^2} \leq$$
$$\leq \|f\|_{L^2} + \delta\|w_\sigma^{-1/2}u\|_{L^2L^2} + \|w_\sigma^{1/2}F\|_{L^2L^2}.$$

If δ is sufficiently small this implies both (3.70) and (3.71).

To prove (3.73), (3.74) we proceed in a similar way using again (3.67) and (3.68):

$$\|w_\sigma^{-1/2}\Lambda_\omega^s u\|_{L_t^2L_x^2} \lesssim \|\Lambda_\omega^s f\|_{L^2} + \|w_\sigma^{1/2}\Lambda_\omega^s(Vu + F)\|_{L^2L^2}.$$

We shall need the following fairly elementary product estimate involving the angular derivative operator Λ_ω

$$\|\Lambda_\omega^s(gh)\|_{L^2_\omega(\mathbb{S}^2)} \lesssim \|\Lambda_\omega^s g\|_{L^2_\omega(\mathbb{S}^2)}\|\Lambda_\omega^s h\|_{L^2_\omega(\mathbb{S}^2)} \tag{3.75}$$

which holds provided $s > 1$. This estimate can be proved e.g. by localizing the norm on the sphere via a finite partition of unity, and then applying in each coordinate patch a standard product estimate in the Sobolev space $H^s(\mathbb{R}^2)$, $s > 1$.

Applying (3.75), and using assumption (3.72), we have

$$\|w_\sigma^{1/2}\Lambda_\omega^s(Vu + F)\|_{L^2L^2} \leq \delta\|w_\sigma^{-1/2}\Lambda_\omega^s u\|_{L^2L^2} + \|w_\sigma^{1/2}\Lambda_\omega^s F\|_{L^2L^2}$$

and the proof is concluded as above. □

We note the following consequence of (3.70):

Corollary 3.12. *Assume that the hermitian matrix $V(x)$ satisfies, for δ sufficiently small, C arbitrary and $\sigma > 1$ (with $w_\sigma(r) = r(1 + |\log r|)^\sigma$)*

$$|V(x)| \leq \frac{\delta}{w_\sigma(|x|)}, \qquad |\nabla V(x)| \leq \frac{C}{w_\sigma(|x|)}. \tag{3.76}$$

Then besides (3.70) *we have the estimate for the derivatives of the flow*

$$\|w_\sigma^{-1/2}\nabla e^{it(\mathcal{D}+V)}f\|_{L_t^2L_x^2} \lesssim \|f\|_{H^1}. \tag{3.77}$$

If in addition we assume that, for some $s > 1$,

$$\|\Lambda_\omega^s V(r\cdot)\|_{L^2(\mathbb{S}^2)} \leq \frac{\delta}{w_\sigma(r)}, \qquad \|\Lambda_\omega^s \nabla V(r\cdot)\|_{L^2(\mathbb{S}^2)} \leq \frac{C}{w_\sigma(r)}, \tag{3.78}$$

then we have the following estimate with angular regularity

$$\|w_\sigma^{-1/2}\nabla\Lambda_\omega^s e^{it(\mathcal{D}+V)}f\|_{L_t^2L_x^2} \lesssim \|\Lambda_\omega^s f\|_{H^1}. \tag{3.79}$$

Proof. Assume at first $s = 0$ and let $u = e^{it(\mathcal{D}+V)}f$. Each derivative $u_j = \partial_j u$ satisfies an equation like

$$i\partial_t u_j = \mathcal{D}u_j + Vu_j + V_j u, \qquad V_j = \partial_j V, \qquad u_j(0, x) = f_j = \partial_j f$$

so we can represent it in the form

$$u_j = e^{it(\mathcal{D}+V)}f_j + i\int_0^t e^{i(t-s)(\mathcal{D}+V)}V_j u ds.$$

To the first term at the r.h.s. we can apply estimate (3.70) obtaining

$$\|w_\sigma^{-1/2}e^{it(\mathcal{D}+V)}f_j\|_{L_t^2L_x^2} \lesssim \|f\|_{\dot{H}^1}.$$

To handle the second term we use (3.71):

$$\left\| w_\sigma^{-1/2} \int e^{i(t-s)(\mathcal{D}+V)} V_j u \right\|_{L^2} \lesssim \|w_\sigma^{1/2} V_j u\|_{L^2_t L^2_x} \le \|w_\sigma V_j\|_{L^\infty_x} \|w_\sigma^{-1/2} u\|_{L^2_t L^2_x}$$

and again by (3.70) and by the assumption on ∇V we conclude the proof of (3.77).

For the proof of (3.79) we apply to the equation for u the operator $|D|$ which commutes with $\mathcal{D}$:

$$i\partial_t(|D|u) = \mathcal{D}(|D|u) + |D|(Vu)$$

and we use estimates (3.73), (3.74), obtaining

$$\|w_\sigma^{-1/2}|D|\Lambda^s_w u\|_{L^2L^2} \lesssim \|\Lambda^s_w f\|_{\dot{H}^1} + \|w_\sigma^{1/2}|D|\Lambda^s_w(Vu)\|_{L^2L^2}.$$

Now in the last term we commute $|D|$ with Λ and we notice that we can replace $|D|$ by ∇ obtaining an equivalent norm. This gives

$$\|w_\sigma^{1/2}|D|\Lambda^s_w(Vu)\|_{L^2L^2} \le \|w_\sigma^{1/2}\Lambda^s_w(\nabla V)u)\|_{L^2L^2} + \|w_\sigma^{1/2}\Lambda^s_w V(\nabla u)\|_{L^2L^2}.$$

We can now apply the product estimate (3.75) and assumptions (3.78); proceeding as in the first part of the proof we finally obtain (3.79). □

By a similar perturbative argument, we obtain the endpoint Strichartz estimates for the Dirac equation with potential. Notice that in the version of this Theorem given in the Introduction (Theorem 3.2) we used an equivalent formulation in terms of the potential $v_\sigma(x) = |x|^{\frac{1}{2}}|\log|x||^\sigma + \langle x\rangle^{1+\sigma}$.

THEOREM 3.13. *Assume that the hermitian matrix $V(x)$ satisfies, for δ sufficiently small, C arbitrary and $\sigma > 1$ (with $w_\sigma(r) = r(1+|\log r|)^\sigma$)*

$$|V(x)| \le \frac{\delta}{\langle x\rangle^{\frac{1}{2}+} w_\sigma(|x|)^{\frac{1}{2}}}, \qquad |\nabla V(x)| \le \frac{C}{\langle x\rangle^{\frac{1}{2}+} w_\sigma(|x|)^{\frac{1}{2}}}. \tag{3.80}$$

Then the perturbed Dirac flow satisfies the endpoint Strichartz estimate

$$\|e^{it(\mathcal{D}+V)} f\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|f\|_{H^1}. \tag{3.81}$$

If instead we make the following assumption (which implies (3.80)*): for some $s > 1$,*

$$\|\Lambda^s_\omega V(r\cdot)\|_{L^2(\mathbb{S}^2)} \le \frac{\delta}{\langle r\rangle^{\frac{1}{2}+} w_\sigma(r)^{\frac{1}{2}}}, \qquad \|\Lambda^s_\omega \nabla V(r\cdot)\|_{L^2(\mathbb{S}^2)} \le \frac{C}{\langle r\rangle^{\frac{1}{2}+} w_\sigma(r)^{\frac{1}{2}}}, \tag{3.82}$$

then we have the endpoint estimate with angular regularity

$$\|\Lambda^s_\omega e^{it(\mathcal{D}+V)} f\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|\Lambda^s_\omega f\|_{H^1} \tag{3.83}$$

and the energy estimate with angular regularity

$$\|\Lambda^s_\omega e^{it(\mathcal{D}+V)} f\|_{L^\infty_t H^1} \lesssim \|\Lambda^s_\omega f\|_{H^1} \tag{3.84}$$

PROOF. Consider first (3.81). Notice that V satisfies in particular the assumptions of Corollary 3.12. We can write

$$e^{it(\mathcal{D}+V)} f = I + II \tag{3.85}$$

with

$$I = e^{it\mathcal{D}} f, \qquad II = i\int_0^t e^{i(t-s)\mathcal{D}} Vu\, ds.$$

The term I is estimated directly using (3.52) with $s = 0$. On the other hand, applying (3.53) to the term II we get

$$\|II\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|\langle x\rangle^{\frac{1}{2}+}|D|(Vu)\|_{L^2L^2}.$$

Now we recall that the Riesz operators $|D|^{-1}\nabla$ are bounded on weighted L^2 spaces with A_2 weights, and $\langle x\rangle^s$ belongs to this class provided $s < n/2$ (see [**69**]). Thus we can continue the chain of inequalities as follows:

$$= \|\langle x\rangle^{\frac{1}{2}+}|D|^{-1}\nabla|D|(Vu)\|_{L^2L^2} \lesssim \|\langle x\rangle^{\frac{1}{2}+}\nabla(Vu)\|_{L^2L^2} \lesssim A + B$$

where

$$A = \|\langle x\rangle^{\frac{1}{2}+}(\nabla V)u\|_{L^2L^2}, \qquad B = \|\langle x\rangle^{\frac{1}{2}+}V\nabla u\|_{L^2L^2}$$

Then we have

$$A \le \|\langle x\rangle^{\frac{1}{2}+}w_\sigma^{\frac{1}{2}}\nabla V\|_{L^\infty}\|w_\sigma^{-\frac{1}{2}}u\|_{L^2L^2} \lesssim \|f\|_{L^2}$$

by the assumptions on ∇V and (3.70), while

$$B \le \|\langle x\rangle^{\frac{1}{2}+}w_\sigma^{\frac{1}{2}}V\|_{L^\infty}\|w_\sigma^{-\frac{1}{2}}\nabla u\|_{L^2L^2} \lesssim \|f\|_{H^1}$$

by (3.77). Summing up, we arrive at (3.81).

The proof of (3.83) is similar. We estimate I using (3.52). Applying (3.53) to the term II we get

$$\|\Lambda^s_\omega II\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|\langle x\rangle^{\frac{1}{2}+}|D|\Lambda^s_\omega(Vu)\|_{L^2L^2}.$$

Then we commute $|D|$ with Λ_ω, and we can replace the operator $|D|$ with ∇ since the norm is equivalent; we arrive at

$$\|\Lambda^s_\omega II\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|\langle x\rangle^{\frac{1}{2}+}\Lambda^s_\omega(\nabla V)u\|_{L^2L^2} + \|\langle x\rangle^{\frac{1}{2}+}\Lambda^s_\omega V(\nabla u)\|_{L^2L^2}.$$

Now we use the product estimate (3.75) and assumptions (3.82) to obtain

$$\lesssim C\|w_\sigma^{-1/2}\Lambda^s_\omega u\|_{L^2L^2} + \delta\|w_\sigma^{-1/2}\Lambda^s_\omega\nabla u\|_{L^2L^2}$$

and recalling the smoothing estimates (3.73), (3.79) we conclude the proof of (3.83).

It remains to prove (3.84). Consider first the free case $V \equiv 0$. We have the conservation laws

$$\|e^{it\mathcal{D}}f\|_{L^\infty L^2} \equiv \|f\|_{L^2}, \qquad \|\mathcal{D}e^{it\mathcal{D}}f\|_{L^\infty L^2} \equiv \|\mathcal{D}f\|_{L^2} \tag{3.86}$$

which imply

$$\|e^{it\mathcal{D}}f\|_{L^\infty H^1} \simeq \|f\|_{H^1}$$

since $\|\mathcal{D}f\|_{L^2} \simeq \|f\|_{\dot H^1}$. Moreover, the operator $\widetilde{\Lambda}_\omega$ introduced in (3.61) commutes with $\mathcal{D}$, so that we have for all $s \ge 0$

$$\|\widetilde{\Lambda}^s_\omega e^{it\mathcal{D}}f\|_{L^\infty H^1} \equiv \|\widetilde{\Lambda}_\omega f\|_{H^1}$$

and switching back to the equivalent operator Λ_ω as in the proof of Corollary 3.7 we obtain

$$\|\Lambda^s_\omega e^{it\mathcal{D}}f\|_{L^\infty H^1} \lesssim \|\Lambda_\omega f\|_{H^1}. \tag{3.87}$$

Consider now the case $V \not\equiv 0$. We start from

$$\|\langle x\rangle^{-\frac{1}{2}-}e^{it\mathcal{D}}f\|_{L^2L^2} \le \|f\|_{L^2} \tag{3.88}$$

which is a consequence of (3.67) (we relaxed the weight). Taking the dual of (3.88) we get

$$\left\| \int e^{-it'\mathcal{D}} F(t')dt' \right\|_{L^2} \lesssim \|\langle x \rangle^{\frac{1}{2}+} F\|_{L^2 L^2}$$

which together with (3.86) gives

$$\left\| \int e^{i(t-t')\mathcal{D}} F(t')dt' \right\|_{L^\infty L^2} \lesssim \|\langle x \rangle^{\frac{1}{2}+} F\|_{L^2 L^2}.$$

Now a standard application of Christ-Kiselev' Lemma in the spirit of [**47**] (see also [**21**] for the case of Dirac equations) allows to replace the time integral with a truncated integral and we obtain

$$\left\| \int_0^t e^{i(t-t')\mathcal{D}} F(t')dt' \right\|_{L^\infty L^2} \lesssim \|\langle x \rangle^{\frac{1}{2}+} F\|_{L^2 L^2}. \tag{3.89}$$

Recalling that the operator $\widetilde{\Lambda}_\omega$ introduced in (3.61) commutes with $\mathcal{D}$, and proceeding as in the proof of Corollary 3.7 we obtain for all $s \geq 0$

$$\left\| \Lambda_\omega^s \int_0^t e^{i(t-t')\mathcal{D}} F(t')dt' \right\|_{L^\infty L^2} \lesssim \|\langle x \rangle^{\frac{1}{2}+} \Lambda_\omega^s F\|_{L^2 L^2} \tag{3.90}$$

and finally, applying $|D|$ which commutes both with $\mathcal{D}$ and Λ_ω, we have also

$$\left\| \Lambda_\omega^s \int_0^t e^{i(t-t')\mathcal{D}} F(t')dt' \right\|_{L^\infty \dot{H}^1} \lesssim \|\langle x \rangle^{\frac{1}{2}+} \Lambda_\omega^s |D| F\|_{L^2 L^2} \tag{3.91}$$

Now we use again the representation (3.85); by (3.87) and (3.91) we can write

$$\|\Lambda_\omega^s e^{it(\mathcal{D}+V)} f\|_{L^\infty \dot{H}^1} \lesssim \|\Lambda_\omega^s f\|_{H^1} + \|\langle x \rangle^{\frac{1}{2}+} \Lambda_\omega^s |D|(Vu)\|_{L^2 L^2}$$

and proceeding exactly as in the first part of the proof we arrive at (3.84). □

5. The nonlinear Dirac equation

Theorem 3.13 contains all the necessary tools to prove global well posedness for the cubic nonlinear Dirac equation

$$iu_t = \mathcal{D}u + Vu + P_3(u, \overline{u}), \qquad u(0, x) = f(x). \tag{3.92}$$

Our result is the following:

THEOREM 3.14. *Consider the perturbed Dirac system* (3.92)*, where the* 4×4 *matrix valued potential* $V = V(|x|)$ *is hermitian and satisfies assumptions* (3.82)*. Let* $P_3(u, \overline{u})$ *be a* $\mathbb{C}^4$*-valued homogeneous cubic polynomial. Then for any* $s > 1$ *there exists* ϵ_0 *such that for all initial data satisfying*

$$\|\Lambda_\omega^s f\|_{H^1} < \epsilon_0 \tag{3.93}$$

the Cauchy problem (3.92) *admits a unique global solution* $u \in CH^1 \cap L^2 L^\infty$ *with* $\Lambda_\omega^s u \in L^\infty H^1$.

PROOF. The proof is based on a fixed point argument in the space X defined by the norm

$$\|u\|_X := \|\Lambda_\omega^s u\|_{L_t^2 L_r^\infty L_\omega^2} + \|\Lambda_\omega^s u\|_{L_t^\infty H_x^1}. \tag{3.94}$$

Notice that in Theorem 3.13 we proved the estimate

$$\|e^{it(\mathcal{D}+V)}f\|_X \lesssim \|\Lambda^s_\omega\|_{H^1}. \tag{3.95}$$

Define $u = \Phi(v)$ for $v \in X$ as the solution of the linear problem

$$iu_t = \mathcal{D}u + Vu + P(v, \overline{v}), \qquad u(0,x) = f(x) \tag{3.96}$$

and represent u as

$$u = \Phi(v) = e^{it(\mathcal{D}+V)}f + i\int_0^t e^{i(t-t')(\mathcal{D}+V)}P(v(t'), \overline{v(t')})dt'.$$

We recall now the product estimate

$$\|\Lambda^s_\omega(gh)\|_{L^2_\omega(\mathbb{S}^2)} \lesssim \|\Lambda^s_\omega g\|_{L^2_\omega(\mathbb{S}^2)}\|\Lambda^s_\omega h\|_{L^2_\omega(\mathbb{S}^2)}$$

(see (3.75)). Then we have, by (3.95)

$$\begin{aligned}\|u\|_X &\lesssim \|\Lambda^s_\omega f\|_{H^1} + \int_0^\infty \|e^{i(t-t')\mathcal{D}}P(v(t'), \overline{v(t')})\|_X dt' \\ &\lesssim \|\Lambda^s_\omega f\|_{H^1} + \int_0^\infty \|\Lambda^s_\omega P(v(t'), \overline{v(t')})\|_{H^1}dt' \equiv \|\Lambda^s_\omega f\|_{H^1} + \|\Lambda^s_\omega P(v, \overline{v})\|_{L^1H^1}.\end{aligned}$$

By (3.75) we have

$$\|\Lambda^s_\omega(v^3)\|_{L^2_\omega(\mathbb{S}^2)} \lesssim \|\Lambda^s_\omega v\|^3_{L^2_\omega(\mathbb{S}^2)}$$

whence

$$\|\Lambda^s_\omega(v^3)\|_{L^2_x} \lesssim \|\Lambda^s_\omega v\|_{L^2_x}\|\Lambda^s_\omega v\|^2_{L^\infty_r L^2_\omega}$$

and

$$\|\Lambda^s_\omega(v^3)\|_{L^1_t L^2_x} \lesssim \|\Lambda^s_\omega v\|_{L^\infty_t L^2_x}\|\Lambda^s_\omega v\|^2_{L^2_t L^\infty_r L^2_\omega} \leq \|v\|^3_X. \tag{3.97}$$

In a similar way,

$$\|\Lambda^s_\omega \nabla(v^3)\|_{L^2_\omega(\mathbb{S}^2)} \lesssim \|\Lambda^s_\omega \nabla v\|_{L^2_\omega(\mathbb{S}^2)}\|\Lambda^s_\omega v\|^2_{L^2_\omega(\mathbb{S}^2)}$$

so that

$$\|\Lambda^s_\omega \nabla(v^3)\|_{L^2_x} \lesssim \|\Lambda^s_\omega \nabla v\|_{L^2_x}\|\Lambda^s_\omega v\|^2_{L^\infty_r L^2_\omega}$$

and

$$\|\Lambda^s_\omega \nabla(v^3)\|_{L^1_t L^2_x} \lesssim \|\Lambda^s_\omega \nabla v\|_{L^\infty_t L^2_x}\|\Lambda^s_\omega v\|^2_{L^2_t L^\infty_r L^2_\omega} \leq \|v\|^3_X. \tag{3.98}$$

In conclusion, (3.97) and (3.98) imply

$$\|\Lambda^s_\omega P(v, \overline{v})\|_{L^1H^1} \lesssim \|v\|^3_X$$

and the estimate for $u = \Phi(v)$ is

$$\|u\|_X \equiv \|\Phi(v)\|_X \lesssim \|\Lambda^s_\omega f\|_{H^1} + \|v\|^3_X.$$

An analogous computation gives the estimate

$$\|\Phi(v) - \Phi(w)\|_X \lesssim \|v - w\|_X \cdot (\|v\|_X + \|w\|_X)^2$$

and an application of the contraction mapping theorem concludes the proof. □

CHAPTER 4

Higher dimensions: virial identity and dispersive estimates

1. Introduction

The goal of this chapter is to study the dispersive properties of the Dirac equation perturbed by a magnetic field in every dimension $n \geq 1$.
First of all we thus need to clearly define the Dirac operator in the generic space dimension.
In chapter 2 we have introduced the 3D Dirac equation from a physical point of view, deriving it (following more or less the original argument) by linearization and standard quantization of the energy-momentum relation. The condition of anticommutation on the hermitian Dirac matrices $\alpha_0 = \beta, \alpha_1, ..., \alpha_3$, i.e.

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}\mathbb{I}_M, \qquad 0 \leq j,k \leq 3. \tag{4.1}$$

and also their dimension $M = 4$ were natural consequences of the physical structure.
In the general space dimension $n \geq 1$ there exist different choices of M and of matrices α_j satisfying all of the above conditions: a possible way to construct a familiy of matrices satisfying such properties is the following.
For $n = 1$ let

$$\alpha_0^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_1^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For $n \geq 2$ let

$$\alpha_j^{(n)} = \begin{pmatrix} 0 & \alpha_j^{(n-1)} \\ \alpha_j^{(n-1)} & 0 \end{pmatrix}, \quad j = 0, ..., n-1, \qquad \alpha_n^{(n)} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.$$

Notice that in this case $M = 2^n$ (for a more detailed analysis of general Dirac matrices, see [**52**], [**45**], [**58**]).
In this chapter, the Dirac equation on $\mathbb{R}^{1+n}$ will thus be a constant coefficient, hyperbolic system of the form

$$iu_t - \mathcal{D}u - m\beta u = 0 \tag{4.2}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x^n \to \mathbb{C}^M$, the *Dirac operator* is defined by

$$\mathcal{D} = i^{-1}\sum_{k=1}^{n} \alpha_k \frac{\partial}{\partial x_k} = i^{-1}(\alpha \cdot \nabla),$$

and the *Dirac matrices* $\alpha_0 \equiv \beta$, $\alpha_1, \ldots, \alpha_n$ are a set of $M \times M$ hermitian matrices satisfying the anticommutation relations

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}\mathbb{I}_M, \qquad 0 \leq j,k \leq n. \tag{4.3}$$

The quantity $m \geq 0$ is called the *mass*.
An easy consequence of the anticommutation relations is the identity

$$(i\partial_t - \mathcal{D} - m\beta)(i\partial_t + \mathcal{D} + m\beta) = (\Delta - m^2 - \partial_{tt}^2)\mathbb{I}_M. \tag{4.4}$$

which reduces the study of (4.2) to a corresponding study of the Klein-Gordon equation, or the wave equation in the massless case $m = 0$. The analysis of the important Maxwell-Dirac and Dirac-Klein-Gordon systems of quantum electrodynamics in [**9**]- [**10**] was based on this method; notice however that in the reduction step some essential details of the structure may be lost, as recently pointed out in [**25**], [**24**], [**26**].

From (4.4) one can deduce in a straightforward way the dispersive properties of the Dirac flow from the corresponding properties of the wave-Klein-Gordon flow. Based on this approach, an extensive theory of local and global well posedness for nonlinear perturbations of (4.2) was developed in [**31**], [**33**], [**52**], [**51**]; see also [**20**], [**21**] for a study of the dispersive properties of the Dirac equation perturbed by a magnetic field.

The goal of this chapter is thus to study the dispersive properties of the system (4.2) perturbed by a magnetic field, thus extending to the n-dimensional setting the smoothing and Strichartz estimates proved in [**11**] for the 3D magnetic Dirac equation. Denoting with

$$A(x) = (A^1(x), ..., A^n(x)) : \mathbb{R}^n \to \mathbb{R}^n$$

a static magnetic potential, the standard way to express its interaction with a particle is by replacing the derivatives ∂_k with their covariant counterpart $\partial_k - iA^k$, thus obtaining the *magnetic Dirac operator*

$$\mathcal{D}_A = i^{-1} \sum_{k=1}^{n} \alpha_k(\partial_k - iA^k) = i^{-1}\alpha \cdot \nabla_A, \qquad \nabla_A = \nabla - iA(x). \tag{4.5}$$

Here and in the following we denote with a dot the scalar product of two vectors of operators:

$$(P_1, \dots, P_m) \cdot (Q_1, \dots, Q_m) = \sum_{j=1}^{m} P_j Q_j.$$

We shall also use the unified notation

$$\mathcal{H} = i^{-1}\alpha \cdot \nabla_A + m\beta = \mathcal{D}_A + m\beta \tag{4.6}$$

to include both the massive and the massless case.

Thus we plan to investigate the dispersive properties of the flow $e^{it\mathcal{H}}f$ defined as the solution to the Cauchy problem

$$iu_t(t,x) + \mathcal{H}u(t,x) = 0, \qquad u(0,x) = f(x). \tag{4.7}$$

It is natural to require that the operator $\mathcal{H}$ be selfadjoint. Several sufficient conditions are known for selfadjointness (see [**74**]). For greatest generality, we prefer to make an abstract selfadjointness assumtpion; we also include a density condition which allows to approximate rough solutions with smoother ones, locally uniformly in time, and is easily verified in concrete cases. The condition is the following:

SELF-ADJOINTNESS ASSUMPTION (A). The operator $\mathcal{H}$ is essentially selfadjoint on $C_c^\infty(\mathbb{R}^n)$, and in addition for initial data $f \in C_c^\infty(\mathbb{R}^n)$ the flow $e^{it\mathcal{H}}f$ belongs at least to $C(\mathbb{R}, H^{3/2})$.

REMARK 4.1. It is easy to show, using Fourier transform, the conservation of the mass under the magnetic Dirac flow: being $e^{it\mathcal{H}}$ unitary we have indeed

$$\|e^{it\mathcal{H}}f\|_{L^2} = \|f\|_{L^2}.$$

The main tool used here is the method of Morawetz multipliers, in the version of [**23**], [**11**]. This method allows to partially overcome the smallness assumption on the potential which was necessary for the perturbative approach of [**21**]. An additional advantage is that the assumptions on the potential are expressed in terms of the *magnetic field* B rather than the vector potential A; indeed, B is a physically measurable quantity while A should be thought of as a mathematical abstraction. We recall that in dimension 3 the magnetic field B is defined as

$$B = \mathrm{curlA}.$$

In arbitrary dimension n, a natural generalization of the previous definition is the following

DEFINITION 4.1. Given a magnetic potential $A : \mathbb{R}^n \to \mathbb{R}^n$, the *magnetic field* $B : \mathbb{R}^n \to \mathcal{M}_{n\times n}(\mathbb{R})$ is the matrix valued function

$$B = DA - DA^t, \qquad B^{jk} = \frac{\partial A^j}{\partial x^k} - \frac{\partial A^k}{\partial x^j}$$

and its *tangential component* $B_\tau = \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$B_\tau = \frac{x}{|x|}B.$$

Notice indeed that $B_\tau(x)$ is orthogonal to x for all x.

REMARK 4.2. The previous definition reduces to the standard one in dimension $n = 3$; indeed the matrix B satisfies for all $v \in \mathbb{R}^3$

$$Bv = \mathrm{curlA} \wedge \mathrm{v}$$

and in this sense B can be identified with $\mathrm{curl}A$. Notice also that

$$B_\tau = \frac{x}{|x|} \wedge \mathrm{curl}A.$$

Our first result is the following (formal) virial identity for the n-dimensional magnetic Dirac equation (4.7):

THEOREM 4.2 (Virial identity). *Assume that the operator $\mathcal{H}$ defined in* (4.6) *satisfies (A), and let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a real valued function. Then any solution $u(t,x)$ of* (4.7) *satisfies the formal virial identity*

$$2\int_{\mathbb{R}^n} \nabla_A u \cdot D^2\phi \cdot \overline{\nabla_A u} - \frac{1}{2}\int_{\mathbb{R}^n} |u|^2 \Delta^2\phi + 2\int_{\mathbb{R}^n} \Im\left(u\nabla\phi \cdot B \cdot \overline{\nabla_A u}\right) +$$
$$+ \int_{\mathbb{R}^n} \overline{u} \cdot \sum_{j<k} \alpha_j\alpha_k(\nabla\phi \cdot \nabla B^{jk})u = -\frac{d}{dt}\int_{\mathbb{R}^n} \Re\left(u_t(2\nabla\phi \cdot \overline{\nabla_A u} + \overline{u}\Delta\phi)\right). \tag{4.8}$$

REMARK 4.3. If $\phi = \phi(|x|)$ is a radial function, as we shall always assume in the following, the virial identity can be considerably simplified. In particular, notice that

$$\sum_{j<k} \alpha_j\alpha_k(\nabla\phi \cdot \nabla B^{jk}) = \phi'(|x|)\sum_{j<k} \alpha_j\alpha_k\partial_r B^{jk}.$$

As a direct consequence of the previous virial identity, we can prove a smoothing estimate for the n-dimensional magnetic Dirac equation (4.7). In the following we shall denote respectively with $\nabla_A^r u$ and $\nabla_A^\tau u$ the radial and tangential components of the covariant gradient, namely

$$\nabla_A^r u := \frac{x}{|x|}\cdot\nabla_A u, \qquad \nabla_A^\tau u := \nabla_A u - \frac{x}{|x|}\cdot\nabla_A^r u$$

so that

$$|\nabla_A^r u|^2 + |\nabla_A^\tau u|^2 = |\nabla_A u|^2.$$

We shall use the notation

$$[B]_1 = \sum_{j,k=1}^n |B^{jk}|$$

to denote the ℓ^1 norm of a matrix (i.e. the sum of the absolute values of its entries), and we shall measure the size of matrix valued functions using norms like

$$\|B\|_{L^\infty} = \|[B(x)]_1\|_{L^\infty_x}$$

Then we have:

THEOREM 4.3 (Smoothing estimates). *Let $n \geq 4$. Let the operator $\mathcal{H}$ defined in (4.6) satisfies assumption (A). Let $B = DA - DA^t = B_1 + B_2$ with $B_2 \in L^\infty$, and assume that*

$$|B_\tau(x)| \leq \frac{C_1}{|x|^2}, \quad \frac{1}{2}[\partial_r B(x)]_1 \leq \frac{C_2}{|x|^3} \tag{4.9}$$

for all $x \in \mathbb{R}^n$ and for some constants C_1, C_2 such that

$$\left(\frac{9}{4}\right) C_1^2 + 3C_2 \leq (n-1)(n-3) \tag{4.10}$$

Assume moreover that

$$C_0 = \||x|^2 B_1\|_{L^\infty(\mathbb{R}^n)} < \frac{(n-2)^2}{4}.$$

Finally, in the massless case restrict the choice to $B_1 = B$, $B_2 = 0$ in the above assumptions.

Then for all $f \in L^2$ the following smoothing estimate holds

$$\sup_{R>0}\frac{1}{R}\int_{-\infty}^{+\infty}\int_{|x|\leq R} |e^{it\mathcal{H}}f|^2 dxdt \lesssim \|f\|_{L^2}^2. \tag{4.11}$$

REMARK 4.4. As in [**34**] and [**11**], a sharper estimate can be proved if inequality (4.10) is strict, but we won't deal with the details of this aspect here.

The limitation to $n \geq 3$ space dimensions is intrinsic in the multiplier method; low dimensions $n = 1, 2$ require a different approach (see e.g. [**19**] for a general result in dimension 1). In the present paper we shall only deal with the case $n \geq 4$, the 3-dimensional case being exaustively discussed in [**11**]. Notice that, as it often occurs, the three dimensional case yields different hypothesis on the potential, being slightly different the multiplicator that one needs to consider.

A natural application of the smoothing estimate (4.11) is to derive Strichartz estimates for the perturbed flow $e^{it\mathcal{H}}f$, both in the massless and massive case. Our concluding result is the following:

THEOREM 4.4 (Strichartz estimates). *Let $n \geq 4$. Assume $\mathcal{H}$, A, B are as in Theorem 4.3, and in addition assume that*

$$\sum_{j\in\mathbb{Z}} 2^j \sup_{|x|\cong 2^j} |A| < \infty. \tag{4.12}$$

Then the perturbed Dirac flow satisfies the Strichartz estimates

$$\||D|^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}} e^{it\mathcal{H}} f\|_{L^pL^q} \lesssim \|f\|_{L^2} \tag{4.13}$$

where, in the massless case $m = 0$, the couple (p, q) is any wave admissibile, non-endpoint couple i.e. such that

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}, \qquad 2 < p \leq \infty \qquad \frac{2(n-1)}{n-3} > q \geq 2, \tag{4.14}$$

while in the massive case the same bound holds for all Schrödinger adimmissible couple, non-endpoint (p, q), i.e. such that

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad 2 < p \leq \infty \qquad \frac{2n}{n-2} > q \geq 2. \tag{4.15}$$

The chapter is organized as follows: in section 2 we shall prove Theorem 4.2, deriving it from a classical virial identity for the wave equation (see Theorem 4.5) plus the algebric structure of the Dirac operator. In section 3 we shall use the multiplicator technique to prove the smoothing estimate (4.11) from Theorem 4.2. Finally in section 4 we shall derive the Strichartz estimates of Theorem 4.4 by a perturbative argument based on the smoothing estimates. section 5 is devoted to the proof of a magnetic Hardy inequality for the Dirac operator, needed at several steps in the proof of the previous theorems.

2. Proof of the virial identity

Let u be a solution to equation (4.2). Using identity

$$0 = (i\partial_t - \mathcal{H})(i\partial_t + \mathcal{H})u = (-\partial_{tt} - \mathcal{H}^2)u,$$

we see that u solves the Cauchy problem for a magnetic wave equation:

$$\begin{cases} u_{tt} + \mathcal{H}^2 u = 0 \\ u(0) = f \\ u_t(0) = i\mathcal{H}f. \end{cases} \tag{4.16}$$

In [**11**] the following general result was proved for a solution $u(t, x)$ of wave-type equations:

THEOREM 4.5 ([**11**]). *Let L be a selfadjoint operator on $L^2(\mathbb{R}^n)$, and let $u(t,x)$ be a solution of the equation*

$$u_{tt}(t,x) + Lu(t,x) = 0.$$

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ and define the quantity

$$\Theta(t) = (\phi u_t, u_t) + \mathcal{R}((2\phi L - L\phi)u, u). \tag{4.17}$$

Then $u(t,x)$ satisfies the formal virial identities

$$\dot{\Theta}(t) = \mathcal{R}([L,\phi]u, u_t) \tag{4.18}$$

$$\ddot{\Theta}(t) = -\frac{1}{2}([L,[L,\phi]]u,u). \tag{4.19}$$

In order to apply this proposition to our case we thus need to compute explicitly the commutators in (4.18), (4.41) with the choice $L = \mathcal{H}^2$. We begin by expanding the square

$$\mathcal{H}^2 = (\mathcal{H}_0 - \alpha\cdot A)^2 = \mathcal{H}_0^2 - \mathcal{H}_0(\alpha\cdot A) - (\alpha\cdot A)\mathcal{H}_0 + (\alpha\cdot A)(\alpha\cdot A),$$

and we recall that the unperturbed part of the operator

$$\mathcal{H}_0 = \mathcal{D} + m\beta = i^{-1}\alpha\cdot\nabla + m\beta$$

satisfies

$$\mathcal{H}_0^2 = (m^2 - \Delta)\mathbb{I}_M.$$

Since β anticommutes with each α_j we get

$$\mathcal{H}^2 = \mathcal{H}_0^2 - i^{-1}(\alpha\cdot\nabla)(\alpha\cdot A) - i^{-1}(\alpha\cdot A)(\alpha\cdot\nabla) + (\alpha\cdot A)(\alpha\cdot A). \tag{4.20}$$

We need a notation to distinguish the composition of the operators (multiplication by) A_k and ∂_j, which we shall denote with $\partial_j \circ A^k$, i.e.,

$$\partial_j \circ A^k u = \partial_j(A^k u)$$

and the simple derivative $\partial_j A^k$. After a few steps we obtain (we omit for simplicity the factor $\mathbb{I}_M$ in diagonal operators)

$$\mathcal{H}^2 = \mathcal{H}_0^2 + i(\nabla\cdot A) + i(A\cdot\nabla) + |A|^2 + i\sum_{j\neq k}^{n}\alpha_j\alpha_k(\partial_j\circ A^k + A^j\partial_k).$$

or equivalently

$$\mathcal{H}^2 = (m^2 - \Delta_A) + i\sum_{j\neq k}^{n}\alpha_j\alpha_k(\partial_j\circ A^k + A^j\partial_k), \tag{4.21}$$

where

$$\Delta_A = (\nabla - iA)^2 = \nabla_A^2.$$

Now we observe that

$$\begin{aligned}
&\sum_{j\neq k}\alpha_j\alpha_k(\partial_j\circ A^k+A^j\partial_k)\\
&\qquad=\sum_{j<k}\alpha_j\alpha_k[(\partial_j\circ A^k+A^j\partial_k)-(\partial_k\circ A^j+A^k\partial_j)]=\\
&\qquad=\sum_{j<k}\alpha_j\alpha_k(\partial_jA^k-\partial_kA^j)\\
&\qquad=\sum_{j<k}\alpha_j\alpha_kB^{jk}=\\
&\qquad=\frac{1}{4}\sum_{j,k=1}^{n}(\alpha_j\alpha_k-\alpha_k\alpha_j)B^{jk}
\end{aligned}$$

since B is skewsymmetric. If we introduce the matrix $S=[S_{jk}]$ whose entries are the matrices

$$S_{jk}=\frac{1}{4}(\alpha_j\alpha_k-\alpha_k\alpha_j)\equiv\frac{1}{2}\alpha_j\alpha_k$$

and we use the notation

$$[a_{jk}]\cdot[b_{jk}]=\sum_{j,k=1}^{n}a_{jk}b_{jk}$$

for the scalar product of matrices, the above identity can be compactly written in the form

$$\sum_{j\neq k}\alpha_j\alpha_k(\partial_j\circ A^k+A^j\partial_k)=S\cdot B.$$

In conclusion we have proved that

$$\mathcal{H}^2=(m^2-\Delta_A)\mathbb{I}_M+iS\cdot B \tag{4.22}$$

and hence for the massless case

$$\mathcal{D}_A^2=-\Delta_A\mathbb{I}_M+iS\cdot B. \tag{4.23}$$

Thus the commutator with ϕ reduces to

$$[\mathcal{H}^2,\phi]=[m^2,\phi]-[\Delta_A,\phi]+i[S\cdot B,\phi]=-[\Delta_A,\phi].$$

Using the Leibnitz rule

$$\nabla_A(fg)=g\nabla_Af+f\nabla g,$$

we arrive at the explicit formula

$$[\mathcal{H}^2,\phi]=-[\Delta_A,\phi]=-2\nabla\phi\cdot\nabla_A-(\Delta\phi). \tag{4.24}$$

Recalling (4.17) and (4.18) we thus obtain

$$\dot{\Theta}(t)=-\Re\int_{\mathbb{R}^n}u_t(2\nabla\phi\cdot\overline{\nabla_Au}+\overline{u}\Delta\phi). \tag{4.25}$$

We now turn to the second commutator. By formulas (4.22) and (4.24) we have

$$[\mathcal{H}^2,[\mathcal{H}^2,\phi]]=[\Delta_A,[\Delta_A,\phi]]-i[S\cdot B,[\Delta_A,\phi]]. \tag{4.26}$$

The first commutator is well known and was computed e.g. in [**34**]; taking formula (2.19) there (with $V \equiv 0$) we obtain

$$(u, [\Delta_A, [\Delta_A, \phi]]) = 4\int_{\mathbb{R}^n} \nabla_A u D^2\phi \overline{\nabla_A u} - \int_{\mathbb{R}^n} |u|^2 \Delta^2\phi + \tag{4.27}$$

$$+4\Im \int_{\mathbb{R}^n} u \nabla\phi B_\tau \cdot \overline{\nabla_A u}.$$

By (4.24) the last term in (4.26) becomes

$$\begin{aligned}
[S\cdot B, [\Delta_A, \phi]] &= 2[S\cdot B, \nabla\phi\cdot\nabla_A] = \\
&= 2(S\cdot B\nabla\phi\cdot\nabla_A - \nabla\phi\cdot\nabla_A S\cdot B) = \\
&= \sum_{j<k}\alpha_j\alpha_k B^{jk}\nabla\phi\cdot\nabla_A - \nabla\phi\cdot\nabla_A\sum_{j<k}\alpha_j\alpha_k B^{jk} = \\
&= \sum_{j<k}\alpha_j\alpha_k[B^{jk}, \nabla\phi\cdot\nabla_A] = \\
&= -\sum_{j<k}\alpha_j\alpha_k(\nabla\phi\cdot\nabla B^{jk}).
\end{aligned} \tag{4.28}$$

Identity (4.8) then follows from (4.41), (4.25), (4.26), (4.27) and (4.28).

3. Smoothing estimates

We shall use the following radial multiplier (for a detailed description see [**34**], [**11**]):

$$\tilde{\phi}_R(x) = \phi(x) + \varphi_R(x) \tag{4.29}$$

where

$$\phi(x) = |x|$$

for which we have

$$\phi'(r) = 1, \qquad \phi''(r) = 0, \qquad \Delta^2\phi(r) = -\frac{(n-1)(n-3)}{r^3}$$

with the notation $r = |x|$, and φ_R is the rescaled $\varphi_R(r) = R\varphi_0(\frac{r}{R})$, of the multiplier

$$\varphi_0(r) = \int_0^r \varphi'(s)ds \tag{4.30}$$

where

$$\varphi_0'(r) = \begin{cases} \frac{n-1}{2n}r, & r \le 1 \\ \frac{1}{2} - \frac{1}{2nr^{n-1}}, & r > 1 \end{cases} \tag{4.31}$$

and so

$$\varphi_0''(r) = \begin{cases} \frac{n-1}{2n}, & r \le 1 \\ \frac{n-1}{2nr^n}, & r > 1. \end{cases}$$

Thus we have

$$\varphi_R'(r) = \begin{cases} \frac{(n-1)r}{2nR}, & r \le R \\ \frac{1}{2} - \frac{R^{n-1}}{2nr^{n-1}}, & r > R \end{cases} \tag{4.32}$$

$$\varphi_R''(r) = \begin{cases} \frac{1}{R}\frac{n-1}{2n}, & r \le R \\ \frac{1}{R}\frac{R^n(n-1)}{2nr^n}, & r > R \end{cases}. \tag{4.33}$$

$$\Delta^2\varphi_R = -\frac{n-1}{2R^2}\delta_{|x|=R} - \frac{(n-1)(n-3)}{2r^3}\chi_{[R,+\infty)}. \tag{4.34}$$

Notice that φ_R', φ_R'', $\Delta\varphi_R \geq 0$ and moreover $\sup_{r\geq 0}\varphi'(r) \leq \frac{1}{2}$.

Thus it is easy to show the bounds for the derivatives of the perturbed multiplier

$$\sup_{r\geq 0}\tilde{\phi}_R' \leq \frac{3}{2}, \qquad \Delta\tilde{\phi}_R \leq \frac{n}{r}. \tag{4.35}$$

We separate the estimates of the LHS and the RHS of (4.8)

Estimate of the RHS of (4.8)

Consider the expression

$$\int_{\mathbb{R}^n} u_t(2\nabla\phi\cdot\overline{\nabla_A u} + u\Delta\phi) = (u_t, 2\nabla\phi\cdot\nabla_A u + \overline{u}\Delta\phi)_{L^2}$$

appearing at the right hand side of (4.8). Since u solves the equation we can replace u_t with

$$u_t = -i\mathcal{H}u = -im\beta u - i\mathcal{D}_A u.$$

By the selfadjointess of β it is easy to check that

$$\Re[-im(\beta u, 2\nabla\phi\cdot\nabla_A u) - im(\beta u, \Delta\phi u)] = 0$$

so that

$$\Re[(u_t, 2\nabla\phi\cdot\nabla_A u + u\Delta\phi) = 2\mathcal{I}(\mathcal{D}_A u, \nabla\phi\cdot\nabla_A u)] + \mathcal{I}(\mathcal{D}_A u, \Delta\phi u)$$

and by Young inequality we obtain

$$\left|\Re\left(\int_{\mathbb{R}^n} u_t(2\nabla\phi\cdot\overline{\nabla_A u} + u\Delta\phi)\right)\right| \leq \frac{3}{2}\|\mathcal{D}_A u\|_{L^2}^2 + \|\nabla\phi\cdot\nabla_A u\|_{L^2}^2 + \frac{1}{2}\|u\Delta\phi\|_{L^2}^2. \tag{4.36}$$

Now we put in (4.36) the multiplier $\tilde{\phi}$ defined in (4.29). From the boundedness of φ and the magnetic Hardy inequality (4.59) we have, with the choice $\varepsilon = (n-2)^2 - 4C_0$ which is positive in virtue of the assumption $C_0 < (n-2)^2/4$,

$$\|\nabla\tilde{\phi}\cdot\nabla_A u\|_{L^2}^2 \leq \frac{3}{2}\frac{1}{(n-2)^2 - 4C_0}\|\mathcal{D}_A u\|_{L^2}^2. \tag{4.37}$$

The third term in (4.36) can be estimated again using Hardy inequality with

$$\|u\Delta\tilde{\phi}\|_{L^2}^2 \leq \frac{4n}{(n-2)^2 - 4C_0}\|\mathcal{D}_A u\|_{L^2}^2. \tag{4.38}$$

Summing up, by (4.36), (4.37) and (4.38) we can conclude

$$\left|\Re\left(\int_{\mathbb{R}^n} u_t(2\nabla\phi\cdot\overline{\nabla_A u} + \overline{u}\Delta\phi)\right)\right| \leq c(n)\|\mathcal{D}_A u\|_{L^2}^2. \tag{4.39}$$

Estimate of the LHS of (4.8)
We shall make use of the following identity, that holds in every dimension:

$$\nabla_A u D^2\phi\overline{\nabla_A u} = \frac{\phi'(r)}{r}|\nabla_A^\tau u|^2 + \phi''(r)|\nabla_A^r u|^2. \tag{4.40}$$

For the seek of simplicity, we divide this part in two steps, first considering just the multiplier $\phi(r) = r$, for which the calculations turn out fairly straightforward, and then perturbating it to $\tilde{\phi}$.

Step 1
With the choice $\phi(r) = r$, by (4.40) we can rewrite the LHS of (4.8) as follows:

$$2\int_{\mathbb{R}^n}\frac{|\nabla_A^\tau u|^2}{|x|}dx + \frac{(n-1)(n-3)}{2}\int_{\mathbb{R}^n}\frac{|u|^2}{|x|^3}dx+ \tag{4.41}$$
$$+2\int_{\mathbb{R}^n}\Im(uB_\tau\cdot\overline{\nabla_A u})dx + \int_{\mathbb{R}^n}\overline{u}\cdot\sum_{j<k}\alpha_j\alpha_k\partial_r B^{jk}u.$$

The first thing to be done is to prove this quantity to be positive. For what concerns the perturbative term, assuming that

$$|B_\tau| \le \frac{C_1}{|x|^2}$$

we have

$$-\left|2\int_{\mathbb{R}^n}\Im(uB_\tau\cdot\overline{\nabla_A u})dx\right| \ge -2\left(\int_{\mathbb{R}^n}\frac{|u|^2}{|x|^3}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}|x|^3|B_\tau|^2|\nabla_A^\tau u|^2dx\right)^{\frac{1}{2}} \tag{4.42}$$
$$\ge -2C_1K_1K_2,$$

where

$$K_1 = \left(\int_{\mathbb{R}^n}\frac{|u|^2}{|x|^3}dx\right)^{\frac{1}{2}}$$
$$K_2 = \left(\int_{\mathbb{R}^n}\frac{|\nabla_A^\tau u|^2}{|x|}dx\right)^{\frac{1}{2}}.$$

Analogously, assuming

$$\Big\|\sum_{j<k}\alpha_j\alpha_k\partial_r B^{jk}(x)\Big\|_{M\times M} \le \frac{1}{2}[\partial_r B(x)]_1 \le \frac{C_2}{|x|^3}$$

(recall that here $\|\cdot\|_{M\times M}$ denotes the operator norm of $M\times M$ matrices and $[\cdot]_1$ denotes the sum of absolute values of the entries of a matrix) we have

$$-\left|\int_{\mathbb{R}^n}\overline{u}\cdot\sum_{j<k}\alpha_j\alpha_k\partial_r B^{jk}udx\right| \ge -\int|u|^2\Big\|\sum_{j<k}\alpha_j\alpha_k\partial_r B^{jk}\Big\|_{M\times M}dx \ge -C_2K_1^2 \tag{4.43}$$

where K_1 is as before. Thus we have reached the following estimate

$$2\int_{\mathbb{R}^n}\frac{|\nabla_A^\tau u|^2}{|x|}dx + \frac{(n-1)(n-3)}{2}\int_{\mathbb{R}^n}\frac{|u|^2}{|x|^3}dx+ \tag{4.44}$$

$$+2\int_{\mathbb{R}^n}\Im(uB_\tau\cdot\overline{\nabla_A u})dx+\int_{\mathbb{R}^n}\overline{u}\cdot\sum_{j<k}\alpha_j\alpha_k\partial_r B^{jk}u\geq$$

$$\geq 2K_2^2-2C_1K_1K_2-C_2K_1^2+\frac{(n-1)(n-3)}{2}K_1^2=:C(C_1,C_2,K_1,K_2).$$

As usual, we want to optimize the condition on the constants C_1, C_2 under which the quantity C is positive for all K_1, K_2. Fixing $K_1=1$ and requiring that

$$\left(\frac{(n-1)(n-3)}{2}-C_2\right)K_1^2-2C_1K_1+2\geq 0$$

we can easily conclude that the resulting condition on the constants is given by

$$C_1^2+2C_2\leq(n-1)(n-3). \tag{4.45}$$

Thus, if condition (4.45) is satisfied, we have that the quantity in (4.41) is positive.

Step 2

We now perturb the multiplier to complete the proof. We thus put the multiplier $\tilde{\phi}_R$ as defined in (4.29) in the LHS of (4.8), and repeat exactly the same calculations as in Step 1. Notice that multiplier φ_R with properties (4.32)-(4.34) yield the estimate, through (4.40),

$$2\int_{\mathbb{R}^n}\nabla_A uD^2\varphi_R\overline{\nabla_A u}-\frac{1}{2}\int_{\mathbb{R}^n}|u|^2\Delta^2\varphi_R\geq \tag{4.46}$$

$$\geq C(n)\left(\frac{1}{R}\int_{|x|\leq R}|\nabla_A u|^2dx+2\int\frac{|\nabla_A^\tau u|^2}{|x|}\right)+$$

$$+\frac{n-1}{4R^2}\int_{|x|=R}|u|^2d\sigma(x)+\frac{(n-1)(n-3)}{4}\int\frac{|u|^2}{|x|^3}$$

for some positive constant $C(n)$. Using now the complete multiplier $\tilde{\phi}_R$ we notice that estimates (4.42) and (4.43) still hold with the rescaled constants $\widetilde{C}_1=\frac{3}{2}C_1$, $\widetilde{C}_2=\frac{3}{2}C_2$, so that we can rewrite (4.44) as follows

$$\frac{1}{R}\int_{|x|\leq R}|\nabla_A u|^2dx+2\int_{\mathbb{R}^n}\frac{|\nabla_A^\tau u|^2}{|x|}dx+\frac{(n-1)(n-3)}{2}\int_{\mathbb{R}^n}\frac{|u|^2}{|x|^3}dx+ \tag{4.47}$$

$$+2\int_{\mathbb{R}^n}\Im(uB_\tau\cdot\overline{\nabla_A u})dx+\int_{\mathbb{R}^n}\overline{u}\cdot\sum_{j<k}\alpha_j\alpha_k\partial_r B^{jk}u\geq$$

$$\geq\frac{1}{R}\int_{|x|\leq R}|\nabla_A u|^2dx+C(\widetilde{C}_1,\widetilde{C}_2,K_1,K_2).$$

Conditions (4.9)-(4.10) on the potential ensure the positivity of $C(\widetilde{C}_1,\widetilde{C}_2,K_1,K_2)$

Thus putting all together , taking the supremum over $R>0$, integrating in time and dropping the corresponding nonnegative terms we have reached the estimate

$$2\int_{-T}^{T}dt\int_{\mathbb{R}^n}\nabla_A uD^2\phi\overline{\nabla_A u}-\frac{1}{2}\int_{-T}^{T}dt\int_{\mathbb{R}^n}|u|^2\Delta^2\phi+ \tag{4.48}$$

$$2\mathcal{I}\int_{-T}^{T}dt\int_{\mathbb{R}^n}u\phi' B_\tau\cdot\overline{\nabla_A u}+\int_{-T}^{T}dt\int_{\mathbb{R}^n}|u|^2\sum_{j<k}\alpha_j\alpha_k(\nabla\phi\cdot\nabla B^{jk})\geq$$

$$\geq\sup_{R>0}\frac{1}{R}\int_{-T}^{T}dt\int_{|x|\leq R}|\nabla_A u|^2dx\geq$$

$$\geq\sup_{R>0}\frac{1}{R}\int_{-T}^{T}dt\int_{|x|\leq R}|\mathcal{D}_A u|^2dx$$

where in the last step we have used the pointwise inequality $|\mathcal{D}_A u|\leq|\nabla_A u|$. We now integrate in time the virial idetity on $[-T,T]$, and using (4.48) and (4.39) we obtain

$$\sup_{R>0}\frac{1}{R}\int_{-T}^{T}dt\int_{|x|\leq R}|\mathcal{D}_A u|^2dx\lesssim\|\mathcal{D}_A u(T)\|_{L^2}^2+\|\mathcal{D}_A u(-T)\|_{L^2}^2. \tag{4.49}$$

Let us now consider the range of D_A: from proposition (4.7) we have that for $C_0<(n-2)^2/4$ $0\notin\ker(D_A)$, so $\mathrm{ran(D_A)}$ is either L^2 or it is dense in L^2. Fix now an arbitrary $g\in\mathrm{ran(D_A)}$, there exists $f\in D(D_A)=D(\mathcal{H})$ such that $D_Af=g$. We then consider the solution $u(t,x)$ to the problem

$$\begin{cases}iu_t=-m\beta u+\mathcal{D}_A u\\ u(0,x)=f(x)\end{cases}$$

with opposite mass, and notice that u satisifies (4.49) since no hypothesis on the sign of the mass m have been used for it. If we thus apply to this equation the operator $\mathcal{D}_A$ we obtain, by the anticommutation rules,

$$\begin{cases}i(\mathcal{D}_A u)_t=\beta m(\mathcal{D}_A u)+\mathcal{D}_A(\mathcal{D}_A u)=0\\ \mathcal{D}_A u(0,x)=\mathcal{D}_A f(x)\end{cases}$$

or, in other words, the function $v=\mathcal{D}_A u$ solves the problem

$$\begin{cases}iv_t=\mathcal{H}v\\ v(0,x)=g\end{cases}$$

so that $v=e^{it\mathcal{H}}g$. Substituting in (4.49) and letting $T\to\infty$ we conclude that, in view of remark (4.1),

$$\sup_{R>0}\frac{1}{R}\int_{-\infty}^{+\infty}\int_{|x|\leq R}|e^{it\mathcal{H}}g|^2\lesssim\|g\|_{L^2}^2$$

that is exactly (4.11) for $g\in\mathrm{ran(D_A)}$, which is as we have noticed dense in L^2. Density arguments conclude the proof.

4. Proof of the Strichartz estimates

We begin by recalling the Strichartz estimates for the free Dirac flow, both in the massless and in the massive case. They are a direct consequence of the corresponding estimates for the wave and Klein-Gordon equations:

PROPOSITION 4.6. *Let $n\geq 3$. Then the following Strichartz estimates hold:*

(i) in the massless case, for any wave admissible couple (p,q) (see (4.14))

$$\||D|^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}e^{it\mathcal{D}}f\|_{L^pL^q}\lesssim\|f\|_{L^2}; \tag{4.50}$$

(ii) in the massive case, for any Schrödinger admissible couple (p, q) *(see* (4.15)*)*

$$\||D|^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}} e^{it(\mathcal{D}+\beta)} f\|_{L^pL^q} \lesssim \|f\|_{L^2}. \tag{4.51}$$

PROOF. We restrict the proof to the case $n \geq$, refering to [**11**] for an exaustive proof of the 3-dimensional case.
Recalling identity (4.4) we immediately have that $u(t,x) = e^{it\mathcal{D}}f$ and $v(t,x) = e^{it(D+\beta)}$ satisfy the two Cauchy problems

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u(0,x) = f(x) \\ u_t(0,x) = i\mathcal{D}f, \end{cases} \tag{4.52}$$

$$\begin{cases} v_{tt} - \Delta v + mv = 0 \\ v(0,x) = f(x) \\ v_t(0,x) = i(\mathcal{D}+\beta)f, \end{cases} \tag{4.53}$$

and so each component of the M-dimensional vectors u and v satisfy the same Strichartz estimates as for the n-dimensional wave equation and Klein-Gordon equation respectively. Thus case (i) follows from the standard estimates proved in [**38**] and [**47**], while case (ii) follows from similar techniques (the details can be found e.g. in the Appendix of [**21**]). □

We turn now to the perturbed flow. In the massless case, from the Duhamel formula we can write

$$u(t,x) \equiv e^{it\mathcal{D}_A} f = e^{it\mathcal{D}} f + \int_0^t e^{i(t-s)\mathcal{D}} \alpha \cdot Au(s)ds. \tag{4.54}$$

The term $e^{it\mathcal{D}}f$ can be directly estimated with (4.50). For the perturbative term we follow the Keel-Tao method [**47**]: by a standard application of the Christ-Kiselev Lemma, since we only aim at the non-endpoint case, it is sufficient to estimate the untruncated integral

$$\int e^{i(t-s)\mathcal{D}} \alpha \cdot Au(s)ds = e^{it\mathcal{D}} \int e^{-is\mathcal{D}} \alpha \cdot Au(s)ds.$$

Using again (4.50) we have

$$\left\| |D|^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}} e^{it\mathcal{D}} \int e^{-is\mathcal{D}} \alpha \cdot Au(s)ds \right\|_{L^pL^q} \lesssim \left\| \int e^{-is\mathcal{D}} \alpha \cdot Au(s)ds \right\|_{L^2}. \tag{4.55}$$

Now we use the dual form of the smoothing estimate (4.11), i.e.

$$\left\| \int e^{-is\mathcal{D}} \alpha \cdot Au(s)ds \right\|_{L^2} \leq \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \||A| \cdot |u|\|_{L_t^2L^2(|x|\cong 2^j)}, \tag{4.56}$$

where we have used the dual of the Morrey-Campanato norm as in [**60**]. Hence by Hölder inequality, hypothesis (4.12) and estimate (4.11) we have

$$\sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \||A| \cdot |u|\|_{L_t^2L^2(|x|\cong 2^j)} \leq \sum_{j\in\mathbb{Z}} 2^j \sup_{|x|\cong 2^j} |A| \cdot \sup_{j\in\mathbb{Z}} \|u\|_{L_t^2L^2(|x|\cong 2^j)} \lesssim \|f\|_{L^2} \tag{4.57}$$

which proves (4.13). The proof in the massive case is exactly the same.

REMARK 4.5. The endpoint estimates can also be recovered, both in the massless and massive case, adapting the proof of Lemma 13 in [**43**], but we will not go into details of this aspect.

5. Magnetic Hardy inequality

This chapter is devoted to the proof of a version of Hardy's inqeuality adapted to the perturbed Dirac operator

$$\mathcal{H} = \mathcal{D}_A + m\beta, \qquad \mathcal{D}_A = i^{-1}\alpha \cdot \nabla_A \equiv i^{-1}\alpha \cdot (\nabla - iA).$$

The proof is simple but we include it for the sake of completeness.

PROPOSITION 4.7. *Let $B = DA - DA^t = B_1 + B_2$ and assume that*

$$\||x|^2 B_1\|_{L^\infty(\mathbb{R}^n)} < \infty, \quad \|B_2\|_{L^\infty(\mathbb{R}^n)} < \infty. \tag{4.58}$$

Then for every $f : \mathbb{R}^n \to \mathbb{C}^M$ such that $\mathcal{H}f \in L^2$ and any $\varepsilon < 1$ the following inequality holds when $m \neq 0$:

$$m^2 \int_{\mathbb{R}^n} |f|^2 + \left((1-\varepsilon)\frac{(n-2)^2}{4} - \frac{1}{2}\||x|^2 B_1\|_{L^\infty} \right) \int_{\mathbb{R}^n} \frac{|f|^2}{|x|^2} + \varepsilon \int_{\mathbb{R}^n} |\nabla_A f|^2 \leq \leq \left(1 + \frac{\|B_2\|_{L^\infty}}{2m^2} \right) \int_{\mathbb{R}^n} |\mathcal{H}f|^2. \tag{4.59}$$

When $m = 0$, the inequality is also true provided we choose $B_1 = B$, $B_2 = 0$ and we interpret the right hand side of (4.59) *simply as $\int |\mathcal{H}f|^2$.*

PROOF. Denote with $(\cdot, \cdot)$ the inner product in $L^2(\mathbb{R}^n, \mathbb{C}^M)$ and with $\|\cdot\|$ the associated norm. Recalling (4.22), we can write

$$\|\mathcal{H}f\|^2 = m^2\|f\|^2 + \|\nabla_A f\|^2 + i(S \cdot Bf, f)$$

where the matrix $S \cdot B = [S_{jk}] \cdot [B^{jk}]$ is skew symmetric since

$$S_{jk} = \frac{1}{2}\alpha_j\alpha_k, \qquad B^{jk} = \partial_j A^k - \partial_k A^j.$$

The selfadjoint matrices α_j have norm less than 1 (recall $\alpha_j^2 = \mathbb{I}$), so that

$$|(S \cdot Bf, f)| \leq \frac{1}{2}([B]_1 f, f)$$

where we denote by $[B]_1$ the ℓ^1 matrix norm

$$[B(x)]_1 = \sum_{j,k} |B^{jk}(x)|.$$

Now recalling assumption (4.58) we can write

$$|(S \cdot Bf, f)| \leq \frac{1}{2}\||x|^2 B_1\|_{L^\infty} \left\| \frac{f}{|x|} \right\|^2 + \frac{1}{2}\|B_2\|_{L^\infty}\|f\|^2$$

and in conclusion

$$\|\mathcal{H}f\|^2 \geq m^2\|f\|^2 + \|\nabla_A f\|^2 - \frac{1}{2}\||x|^2 B_1\|_{L^\infty} \left\| \frac{f}{|x|} \right\|^2 - \frac{1}{2}\|B_2\|_{L^\infty}\|f\|^2.$$

We now recall the magnetic Hardy inequality proved in [**34**]:

$$\frac{(n-2)^2}{4}\int_{\mathbb{R}^n}\frac{|f|^2}{|x|^2} \leq \int_{\mathbb{R}^n}|\nabla_A f|^2. \tag{4.60}$$

Observing now that

$$\|\mathcal{H}f\|^2 = (\mathcal{H}^2 f, f) = m^2\|f\|^2 + \|\mathcal{D}_A f\|^2$$

and that

$$(1-\varepsilon)\left\|\frac{f}{|x|}\right\|^2 + \varepsilon\|\nabla_A f\|^2 \leq \|\nabla_A f\|^2,$$

the proof is complete. □

CHAPTER 5

Weighted L^p-estimates for powers of self-adjoint operators

1. Introduction

The question of L^p estimates for functions of a selfadjoint operator is a delicate one. Indeed, even for a Schrödinger operator $H = -\Delta + V(x)$ with a nonnegative potential $V \in C_c^\infty$, and a bounded smooth function $f(t)$, the operator $f(H)$ defined via spectral theory does not have in general a smooth kernel and hence does not fall within the scope of the Calderòn-Zygmund theory. The first to overcome this difficulty was Hebisch [**41**] who proved the following result; we use the notation

$$S_\lambda f(t) = f(\lambda t), \qquad \lambda > 0$$

for the scaling operator, and we denote by H^s the usual L^2–Sobolev space.

THEOREM 5.1 ([**41**]). *Let H be a nonnegative selfadjoint operator on $L^2(\mathbb{R}^n)$ satisfying a gaussian estimate*

$$0 \le e^{-tH}(x,y) \le Ct^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}, \tag{5.1}$$

let $\phi \in C_c^\infty(\mathbb{R}^+)$ be a nonzero cutoff, and assume the function $F(s)$ on $\mathbb{R}^+$ satisfies

$$\sup_{t>0} \|\phi S_t F\|_{H^a} < \infty \quad \textit{for some} \quad a > \frac{n+1}{2}. \tag{5.2}$$

Then the operator $F(H)$ is bounded from L^1 to $L^{1,\infty}$ and on any L^p, $1 < p < \infty$.

Theorem 5.1 raises a few interesing questions concerning the optimality of the assumptions and the possibility of *weighted* L^p estimates for suitable classes of operators. In the case $H = -\Delta$, the classical Hörmander mutliplier theorem requires only $a > n/2$ in (5.2), and in this sense the result is not optimal. Indeed, sharper results were obtained for bounded functions of homogeneous Laplace operators acting on homogeneous groups or on groups of polynomial growth (see [**30**], [**17**], [**54**], [**2**]). In these results the conditions on the function F were sharpened to

$$\sup_{t>0} \|\phi S_t F\|_{H^a_p} < \infty \quad \text{for some} \quad a > \frac{n}{2} \tag{5.3}$$

where H^a_p is the Sobolev space with norm $\|(1-d^2/dx^2)^{\frac{a}{2}} f\|_{L^p}$, and p is equal to 2 or ∞. The criticality of the order $a = n/2$ was proved by Sikora and Wright [**66**] in the special case of imaginary powers L^{iy}, with L a positive selfadjoint operator of the form

$$L = -\sum \partial_i a_{ij} \partial_j.$$

They obtained

$$\|L^{iy}\|_{L^1\to L^{1,\infty}} \simeq (1+|y|)^{\frac{n}{2}} \tag{5.4}$$

provided L satisfies, besides the gaussian estimate, a *finite speed of propagation* property, meaning that the operator $\cos(t\sqrt{L})$ has an integral kernel $K_t(x,y)$ supported in the ball $|x-y|\le t$ for all $t\ge 0$. Notice that the norm (5.2) for $a=n/2$ and $F(s)=s^{iy}$ grows precisely like $(1+|y|)^{\frac{n}{2}}$. It was later remarked by Sikora [**65**] that the finite speed of propagation is redundant and actually equivalent to a weaker Gaussian bound, the so-called Davies-Gaffey L^2 estimate (see Remark 5.7 below).

Condition (5.3) was further improved by Duong, Ouhabaz and Sikora [**32**]. They obtained a general result for functions of a selfajoint, positive operator L on $L^2(X,\mu)$ where X is any open subset of a space of homogeneous type, μ a doubling measure, and L satisfies a generalized pointwise gaussian estimate analogous to (5.1). In particular they obtained that if F is bounded and satisfies (5.3) with $p=\infty$, then $F(L)$ is of weak type $(1,1)$ and bounded on all L^q, $1<q<\infty$. On the other hand, if (5.3) holds for some $p\in[2,\infty)$, the same result holds provided L satisfies an additional a priori condition of Plancherel type on the kernel of $F(\sqrt{L})$; see [**32**] for further results and an extensive bibliography.

Our main purpose here is to extend these results, at least in the euclidean setting, to the case of *weighted* L^p spaces. However, in order to develop our techniques, we shall first prove a precised version of Theorem 5.1, building on the ideas of [**41**], [**66**]. Concerning the operator H, as in Hebisch' result, we shall only require a gaussian bound; for further reference we state the condition as

Assumption (H). H is a nonnegative selfadjoint operator on $L^2(\mathbb{R}^n)$ satisfying a gaussiam heat kernel estimate

$$|p_t(x,y)| \le \frac{K_0}{t^{n/2}} e^{-|x-y|^2/(dt)}, \qquad d>0. \tag{5.5}$$

A rescaling $H\to\lambda H$ shows that it is not restrictive to assume $d=1$.

Remark 5.1. In chapter 4 we shall exhibit a wide class of operators satisfying (H), namely the electromagnetic Schrödinger operators

$$H = (i\nabla - A(x))^2 + V(x) \tag{5.6}$$

under very weak conditions on the potentials: more precisely, it is sufficient to assume that $A\in L^2_{loc}$ and that V is in the Kato class with a negative part V_- small enough. For related results on magnetic Schrödinger operators see also [**8**].

In order to express the smoothness conditions in an optimal way, we shall introduce two norms on functions defined on the positive real line. In the rest of the paper we fix a cutoff $\psi\in C_c^\infty(\mathbb{R})$ with support in $[-2,2]$ and equal to 1 on $[-1.1]$, and denote with ϕ the function, supported in $[1/2,2]$,

$$\phi(s) = \begin{cases} \psi(s)-\psi(2s) & \text{if } s>0,\\ 0 & \text{if } s\le 0.\end{cases} \tag{5.7}$$

As a consequence, notice the identities for $s > 0$

$$\psi(s) = \sum_{k>0} \phi(2^k s), \qquad 1 - \psi(s) = \sum_{k\le 0} \phi(2^k s). \tag{5.8}$$

Then, writing $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$, the norms μ_a, μ'_a will be defined as

$$\mu_a(g) = \sup_{\lambda>0} \|\langle\xi\rangle^a \mathcal{F}[\phi(s)S_\lambda g]\|_{L^1}, \qquad \mu'_a(g) = \sup_{\lambda>0} \|\langle\xi\rangle^a \mathcal{F}[s\phi(s)S_\lambda g]\|_{L^1}. \tag{5.9}$$

REMARK 5.2. It is easy to control μ_a with ordinary Besov or Sobolev norms:

$$\mu_a(g) \le c(n) \sup_{t>0} \|\phi S_t g\|_{B^{a+\frac{1}{2}}_{2,1}} \le c(n,\epsilon) \sup_{t>0} \|\phi S_t g\|_{H^{a+\frac{1}{2}+\epsilon}}, \qquad \epsilon > 0. \tag{5.10}$$

The last norm in (5.10) is the one used in Theorem 5.1, and using μ_a instead allows to eliminate the $1/2+$ loss of smoothness in Hebisch' result.

Our first result is the following:

THEOREM 5.2. *Let H be an operator satisfying (H) and $g(s)$ a function on $\mathbb{R}^+$ with $\mu = \mu_\sigma(g) < \infty$ for some $\sigma > n/2$. Then the following weak $(1,1)$ estimate holds:*

$$\|g(\sqrt{H})f\|_{L^{1,\infty}} \le C\|f\|_{L^1}, \qquad C = c(n,\sigma)K_0^4(1 + \mu + \|g\|^2_{L^\infty}), \tag{5.11}$$

and for all $1 < p < \infty$, with the same C,

$$\|g(\sqrt{H})f\|_{L^p} \le 6C\left(p + (p-1)^{-1}\right)\|f\|_{L^p} \tag{5.12}$$

If in addition we assume that for some $q > 1$ the following estimate holds:

$$\|\sqrt{H}g(\sqrt{H})f\|_{L^q} \le C_q\|\nabla f\|_{L^q}, \tag{5.13}$$

and $\mu' = \mu'_\sigma(g) < \infty$ for a $\sigma > 1 + n/2$, then we have also

$$\|\sqrt{H}g(\sqrt{H})f\|_{L^{1,\infty}} \le C\|\nabla f\|_{L^1}, \qquad C = c(n,\sigma,C_q)K_0^4(1 + \mu' + \|g\|^2_{L^\infty}), \tag{5.14}$$

and for all $1 < p \le q$, with the same C,

$$\|\sqrt{H}g(\sqrt{H})f\|_{L^p} \le \frac{c(q)}{p-1}C\|\nabla f\|_{L^p}. \tag{5.15}$$

REMARK 5.3. As mentioned above, in [**32**] it was proved that the weak $(1,1)$ estimate holds under the sole assumption

$$\sup_{t>0} \|\phi S_t g\|_{H^a_\infty} < \infty$$

for some $a > n/2$ (see Theorem 3.1 and Remark 1 in that paper). Since obviously

$$\sup_{t>0} \|\phi S_t g\|_{H^a_\infty} \lesssim \mu_a(g),$$

we see that estimate (5.11) can be obtained as a special case of that result, with a slightly different form of the constant which we made explicit in terms of the gaussian constant K_0. On the other hand, estimate (5.15), which uses Auscher's Calderon-Zygmund decomposition for Sobolev functions [**3**], seems to be new.

REMARK 5.4. As evidenced by the previous discussion, the constant in (5.11) is close to optimal in the following sense: if we choose $g(s) = s^{2iy}$, we have

$$\mu_a(g) \leq C(1+|y|)^a, \qquad a \geq 0;$$

(the proof is trivial for integer values of a and follows by interpolation for real values). This implies that, for all $\epsilon > 0$ and $1 < p < \infty$,

$$\|H^{iy} f\|_{L^p} \leq C(p, n, \epsilon)(1+|y|)^{\frac{n}{2}+\epsilon} \|f\|_{L^p} \tag{5.16}$$

which is close to the optimal bound (5.4). Notice also that the strict condition $\sigma > n/2$ can be further optimized to a logarithmic condition, but we prefer not to pursue this idea here.

After it was made clear by the results of Hebisch and others that kernel smoothness is not a necessary condition for L^p boundedness, alternative weaker conditions where thoroughly investigated, also in connection with the Kato problem. A fairly complete answer was given by Auscher and Martell who developed a general theory in a series of papers (see in particular [**4**], [**5**] and the references therein). By combining the techniques of Auscher and Martell with ideas from the proof of Theorem 5.2, we are able to extend the previous estimates to weighted spaces $L^p(w)$. In the following we use the notation

$$\|f\|_{L^p(w)} = \left(\int |f|^p w(x) dx \right)^{1/p}$$

and we recall that a measurable function $w(x) > 0$ belongs to the *Muckenhoupt class* A_p, $1 < p < \infty$, if the quantity

$$\|w\|_{A_p} = \sup_{Q \text{ cube}} \left(\fint_Q w \right) \left(\fint_Q w^{1-p'} \right)^{p/p'} < \infty. \tag{5.17}$$

is finite. Then the main result of this paper is

THEOREM 5.3. *Let H be an operator satisfying (H), and let g be a bounded function on $\mathbb{R}^+$ such that $\mu = \mu_\sigma(g)$ is finite for some $\sigma > n/2$. Then, given any $1 < p < \infty$ and any weight $w \in A_p$, the operator $g(\sqrt{H})$ satisfies, for all $1 < q < \infty$ with $q > p \cdot \max\{1, n/\sigma\}$*

$$\|g(\sqrt{H})f\|_{L^q(w)} \leq c(n, \sigma, p, \psi, w) K_0^{1+2p^2} (1+\mu+\|g\|^2_{L^\infty}) \cdot q \cdot \|f\|_{L^q(w)}. \tag{5.18}$$

REMARK 5.5. It is well known that if $w \in A_p$ for some $p > 1$, then we have also $w \in A_{p-\epsilon}$ for some $\epsilon > 0$ depending only on $\|w\|_{A_p}$ (for a quantitative estimate of ϵ see [**48**]). Thus in the statement of Theorem 5.3 the condition on q can be relaxed to

$$q > (p - \epsilon) \max \left\{ 1, \frac{n}{\sigma} \right\}. \tag{5.19}$$

In particular, if $\sigma \geq n$, we have that $g(\sqrt{H})$ is bounded on $L^q(w)$ for all $w \in A_p$ and all $q > p - \epsilon$, which includes the case $q \geq p$.

REMARK 5.6. The original motivation for the present work was the need for an estimate

$$\|\langle x \rangle^{-1-\epsilon} H^\theta g\|_{L^2} \leq C(V) \|\langle x \rangle^{-1-\epsilon} (-\Delta)^\theta g\|_{L^2}, \qquad \theta = \frac{1}{4}, \qquad H = -\Delta + V(x) \tag{5.20}$$

for fractional powers of a selfadjoint Schrödinger operator H, with explicit bounds on the constant $C(V)$. For the case $\theta = 1/2$, and operators in divergence form, similar estimates are included in the results of [**5**] (see also [**4**]) concerning reverse estimates for square roots of an elliptic operator. However, other values of θ, different forms of H, and the need for precise bounds on the constant, forced us to go beyond the existing theory.

It may be interesting to recall briefly the line of investigation leading to (5.20). An analysis of the dispersive properties of Schrödinger equations on non-flat waveguides (i.e. perturbations of domains of the form $\mathbb{R}^n \times \Omega$ with Ω a bounded open set, see [**29**] for details) leads to a family of perturbed Schrödinger equations

$$iu_t + \Delta_x u - V_j(x)u = 0, \qquad u(0,x) = f_j(x), \qquad j \geq 1,\ x \in \mathbb{R}^n. \quad (5.21)$$

Here $u = u_j$ is a component of the expansion in a distorted Fourier series of a function $u(t,x,y) = \sum \phi_j(y)u_j(t,x)$. Writing for short $H_j = -\Delta + V_j$ and representing the solution as

$$u_j = e^{itH_j} f_j,$$

one expects to estimate each component separately and sum over j. Notice that a precise bound on the growth in j of the constants is essential, since this will translate into y-derivatives after summing over j. To this end we can use *smoothing estimates* of the form

$$\|\langle x\rangle^{-1-\epsilon}(-\Delta)^{1/4}e^{itH_j}f_j\|_{L^2_tL^2_x} \leq C\|H_j^{1/4}f_j\|_{L^2}. \quad (5.22)$$

which can be proved by multiplier techniques and give a complete control on the growth of the constants, and then deduce, in a standard way, *Strichartz estimates*, which are the basci tool for applications to nonlinear problems. This is possible provided we can "simplify" the powers of $-\Delta$ and H_j appearing in (5.22) and obtain the L^2–level estimate

$$\|\langle x\rangle^{-1-\epsilon}e^{itH_j}f_j\|_{L^2_tL^2_x} \leq C\|f_j\|_{L^2}. \quad (5.23)$$

But of course $(-\Delta)^{1/4}$ and e^{itH_j} do not commute, hence this step is not trivial. We need a *weighted L^2 estimate* of the form

$$\|\langle x\rangle^{-1-\epsilon}H_j^{1/4}g\|_{L^2} \leq C(V_j)\|\langle x\rangle^{-1-\epsilon}(-\Delta)^{1/4}g\|_{L^2} \quad (5.24)$$

so that we can replace $(-\Delta)^{1/4}$ by $H_j^{1/4}$ in the LHS of (5.22), commute it with e^{itH_j}, and obtain (5.23). From the previous discussion, it is clear that we need also a precise control on the constant in (5.24).

Our weighted estimates, via complex interpolation, allow us to give a partial answer to the original problem (5.20). Indeed, for a Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$

$$H = -\Delta + V(x), \qquad V \geq 0$$

we obtain the bounds

$$\|\langle x\rangle^{-s}H^\theta f\|_{L^p} \leq C(n,p,s)\cdot[1 + \|V\|_{L^{n/2,\infty}}]^\theta \cdot \|\langle x\rangle^{-s}(-\Delta)^\theta f\|_{L^p} \quad (5.25)$$

for all θ, p, s in the range

$$0 \leq \theta \leq 1, \qquad 1 < p < \frac{n}{2\theta}, \qquad s > -\frac{n}{p}.$$

More generally, we can prove (see the beginning of chapter 4 for the definition of Kato classes):

Corollary 5.4. *Consider the operator*

$$H = (i\nabla - A(x))^2 + V(x)$$

on $L^2(\mathbb{R}^n)$, $n \geq 3$, *under the assumptions that* $A \in L^2_{loc}(\mathbb{R}^n, \mathbb{R}^n)$, $V_+ = \max\{V, 0\}$ *is of Kato class,* $V_- = \max\{-V, 0\}$ *has a small Kato norm*

$$\|V_-\|_K < c_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} - 1\right)}, \tag{5.26}$$

and

$$|A|^2 - i\nabla \cdot A + V \in L^{n/2,\infty}, \qquad A \in L^{n,\infty}. \tag{5.27}$$

Then H *satisfies assumption (H), and for all* $0 \leq \theta \leq 1$ *the following estimate holds:*

$$\|H^\theta f\|_{L^p(w)} \leq C\|(-\Delta)^\theta f\|_{L^p(w)} \tag{5.28}$$

for all weights $w \in A_p$ *provided*

$$1 < p < \frac{n}{2\theta}.$$

The constant in (5.28) *has the form*

$$C = \frac{C(n, p, w)}{(1 - \|V_-\|_K/c_n)^{c(p)}} \left[1 + \||A|^2 - i\nabla \cdot A + V\|_{L^{n/2,\infty}} + \|A\|_{L^{n,\infty}}\right]^\theta.$$

The chapter is organized as follows. In section 2 we build the necessary kernel estimates for functions of an operator and apply them to the proof of the L^p estimates of Theorem 5.2; section 3 is devoted to the proof of the main result, Theorem 5.3, concerning weighted L^p estimates; the application to magnetic Schródinger operators is contained in sections 4 and 5. We added an appendix containing a slightly adapted version of the Auscher-Martell maximall lemma in order to make the paper self contained. In forthcoming papers we plan to apply our estimates to questions of local smoothing and dispersion for evolution equations, in the spirit of [**29**], [**28**].

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2. Kernel estimates and proof of Theorem 5.2

Throughout the proof, ϕ and ψ are the functions fixed in (5.7)–(5.8). Given an operator A with kernel $A(x, y)$, we denote its Schur norm with

$$\|A\| = \|A(x, y)\| \equiv \max\left\{\sup_x \int |A(x, y)|dy, \quad \sup_y \int |A(x, y)|dx\right\};$$

notice the product inequality

$$\|AB\| \leq \|A\| \cdot \|B\| \tag{5.29}$$

which follows from the identity

$$(AB)(x, y) = \int A(x, z)B(z, y)dy. \tag{5.30}$$

Following [**41**], for any nonnegative function $w(x)$ on $\mathbb{R}^n$ we can define a weighted version of the above norm as

$$\|A\|_w = \|A(x,y)w(x-y)\|. \tag{5.31}$$

REMARK 5.7. In the proof of the following Lemma we shall use the finite speed of propagation property of the kernel $\cos(\xi\sqrt{H})(x,y)$, namely the property

$$\cos(t\sqrt{H})(x,y) = 0 \quad \text{for } |x-y| > t \geq 0. \tag{5.32}$$

Adam Sikora in [**65**] proved the remarkable fact that (5.32) is equivalent to the following estimate: for all functions f_1, f_2 supported in the balls $B(x_1, r_1)$ and $B(x_2, r_2)$ respectively, and for any r with

$$|x_1 - x_2| - (r_1 + r_2) > r \geq 0 \tag{5.33}$$

one must have

$$\left|(e^{-tH} f_1, f_2)_{L^2}\right| \leq Ce^{-r^2/t} \|f_1\|_{L^2} \|f_2\|_{L^2}. \tag{5.34}$$

Estimates of the form (5.34) are usually called L^2 *estimates* of *Davies-Gaffey* type. Notice that the pointwise estimate in assumption (H) implies immediately (5.34) and hence (5.32).

For the sake of completeness, we recall here the elementary argument from [**65**] which allows to deduce (5.32) from (5.34). Let f_1, f_2 be two functions as in (5.33), and define

$$w(t) = \mathbf{1}_{\mathbb{R}^+}(t) \cdot 2(\pi t)^{-\frac{1}{2}} (\cos(\sqrt{tH}) f_1, f_2)_{L^2}.$$

Notice that $w(t)$ is a tempered distribution on $\mathbb{R}$ and so are the products $e^{ty} w(t)$ for any $y \leq 0$. Thus the Fourier-Laplace transform

$$v(z) = \int w(t) e^{-izt} dt$$

is well defined and analytic on the half complex plane $\Im z < 0$. Recalling the subordination formula

$$(e^{-sH} f_1, f_2)_{L^2} = \int_0^\infty (\cos(t\sqrt{H}) f_1, f_2)_{L^2} \frac{2}{\sqrt{\pi s}} e^{-\frac{t^2}{4s}} dt,$$

via the changes of variables $t \to \sqrt{t}$ and $s \to 1/(4s)$, we see that $v(z)$ can be computed explicitly as

$$v(z) = (iz)^{-\frac{1}{2}} (e^{-\frac{H}{4iz}} f_1, f_2)_{L^2}.$$

Now introduce the analytic function

$$F(z) = z^{\frac{1}{2}} e^{ir^2 z} v(z) \quad \text{on } \Im z < 0 \tag{5.35}$$

for some fixed r satisfying (5.33). By spectral calculus we have easily the bound

$$|v(z)| \leq |z|^{-\frac{1}{2}} \|f_1\| \cdot \|f_2\|$$

(all the norms in this proof are L^2 norms) which implies the growth rate

$$|F(z)| \leq \|f_1\| \cdot \|f_2\| \cdot e^{r^2|z|}. \tag{5.36}$$

If we fix a $y_0 < 0$, again by spectral calculus we obtain the bound

$$|F(x + iy_0)| \leq \|f_1\| \cdot \|f_2\| \tag{5.37}$$

along the line $z = x + iy_0$, $x \in \mathbb{R}$. Finally, along the half line $z = it$, $t < 0$, we obtain by assumption (5.34)

$$|F(it)| \leq C\|f_1\| \cdot \|f_2\|. \tag{5.38}$$

Now we can apply the Phragmén-Lindelöf theorem on the two sectors $\Im z \leq y_0$ and $\Re z \geq 0$ or $\Re z \leq 0$ (see Theorem IV.3.4 in [**70**]) and we obtain that $F(z)$ satisfies a bound like (5.38) on the whole half plane $\Im z \leq y_0$. This implies an exponential growth rate

$$|v(z)| \leq |z|^{-\frac{1}{2}} e^{r^2 \Im z} \|f_1\| \cdot \|f_2\|, \qquad \Im z \leq y_0 < 0 \tag{5.39}$$

for the transform of $w(t)$. To conclude the proof, it is sufficient to use the Paley-Wiener theorem (see Theorem 7.4.3 in [**42**]) which implies that the support of $w(t)$ must be contained in the closed convex set

$$\operatorname{supp} w \subseteq [r^2, +\infty) \tag{5.40}$$

and this gives (5.32) as claimed.

LEMMA 5.5. *Assume H satisfies (H) and let g be an even function with* $\operatorname{supp} g \subseteq [-R, R]$. *Then we have for all $a \geq 0$*

$$\|g(\sqrt{H})\|_{\langle x\rangle^a} \leq c(n,a,R) \cdot K_0 \|\langle\xi\rangle^{a+n/2} \widehat{g}\|_{L^1} \tag{5.41}$$

$$\|\sqrt{H} g(\sqrt{H})\|_{\langle x\rangle^a} \leq c(n,a,R) \cdot K_0 \|\langle\xi\rangle^{a+n/2} \widehat{g}'\|_{L^1} \tag{5.42}$$

where $c(n,a,R)$ is independent of the operator H and K_0 is defined in (5.75).

PROOF. It is sufficient to estimate the quantity

$$\sup_y \int \left| g(\sqrt{H})(x,y) \langle x-y\rangle^a \right| dx$$

since the symmetric one follows from the same computation applied to the adjoint kernel $g(\sqrt{H})^*(x,y) = \overline{g}(\sqrt{H})(y,x)$. Let $G(s) = g(s)e^{s^2}$. Since G is an an even function, apart from a $(2\pi)^{-1}$ factor we can write

$$G(t) = \int_{-\infty}^{+\infty} \widehat{G}(\xi) \cos(t\xi) d\xi$$

and we have

$$g(\sqrt{H}) = G(\sqrt{H}) e^{-H} = \int \widehat{G}(\xi) \cos(\xi\sqrt{H}) e^{-H} d\xi.$$

We decompose G using a non homogeneous Paley-Littlewood partition of unity $\chi_j(\xi)$, $j \geq 0$ (the support of $\chi_j(s)$ being $s \sim 2^j$) as

$$G = \sum_{j \geq 0} G_j, \qquad \widehat{G_j}(s) = \chi_j(s) \widehat{G}.$$

Then we have to estimate the integrals

$$I_j = \int |G_j(\sqrt{H}) e^{-H}(x,y)| \langle x-y\rangle^a dx \leq \int |\widehat{G_j}(\xi)| \int |\cos(\xi\sqrt{H}) e^{-H}| \langle x-y\rangle^a dx d\xi.$$

The innermost integral can be written in full

$$II = \int \left| \int \cos(\xi\sqrt{H})(x,z) e^{-H}(z,y) dz \right| \langle x-y\rangle^a dx$$

We introduce a a partition of $\mathbb{R}^n$ in almost disjoint unit cubes Q and denote with $\mathbf{1}_Q$ their characteristic functions. Then we can write

$$II \leq \sum_Q II_Q, \qquad II_Q = \int \left| \int \cos(\xi\sqrt{H})(x,z)e^{-H}(z,y)\mathbf{1}_Q(z)dz \right| \langle x-y\rangle^a dx.$$

If z_q is the center of the cube Q we have

$$|x - z_Q| \lesssim \langle \xi \rangle$$

by the finite speed of propagation for $\cos(\xi\sqrt{H})(x,z)$ (see Remark 5.7), and recalling that $\xi \in \operatorname{supp} \widehat{G}_j$ we have also

$$\langle x-y\rangle \leq \langle x - z_Q\rangle\langle z_Q - y\rangle \lesssim \langle\xi\rangle\langle z_Q - y\rangle \lesssim 2^j\langle z_Q - y\rangle.$$

Thus by Cauchy-Schwartz in dx we obtain

$$II_Q^2 \lesssim \langle\xi\rangle^{n+2a}\langle z_Q - y\rangle^{2a} \int \left| \int \cos(\xi\sqrt{H})(x,z)e^{-H}(z,y)\mathbf{1}_Q(z)dz \right|^2 dx.$$

Using the unitarity of $\cos(\xi\sqrt{H})$ and the gaussian estimate, this gives

$$II_Q^2 \lesssim 2^{j(n+2a)}\langle z_Q - y\rangle^{2a} \int \left|e^{-H}\mathbf{1}_Q\right|^2 dz \lesssim 2^{j(n+2a)}K_0^2 \int_Q e^{-2|z-y|^2}\langle z-y\rangle^{2a}dz$$

and hence, taking square roots and summing over Q we conclude

$$II \leq c(n,a)\cdot 2^{(a+n/2)j}K_0$$

independently of y. Inserting this into I_j we see that

$$I_j \leq c(n,a)K_0 2^{(a+n/2)j} \int |\widehat{G_j}(\xi)|d\xi \leq c_1(n,a)K_0\|\langle\xi\rangle^{a+n/2}\widehat{G_j}(\xi)\|_{L^1}$$

and summing over j

$$\|g(\sqrt{H})\|_{\langle x\rangle^a} \leq c(n,a)\|\langle\xi\rangle^{a+n/2}\widehat{G}(\xi)\|_{L^1}.$$

Finally we can write

$$G(s) = g(s)e^{s^2} = g(s)\cdot\chi(s)e^{s^2}$$

with $\chi(s)$ a cutoff function equal to 1 on $[-R,R]$. Then we have

$$\widehat{G} = \widehat{g} * \widehat{(\chi e^{s^2})} \qquad \Longrightarrow \qquad \|\langle\xi\rangle^s\widehat{G}\|_{L^1} \leq c(s,R)\|\langle\xi\rangle^s\widehat{g}(s)\|_{L^1}$$

whence (5.41) follows; indeed, the symmetric quantity obtained by switching x,y in I is estimated in an identical way.

The proof of (5.42) is similar: we must estimate now the integrals

$$I_j' = \int |\sqrt{H}G_j(\sqrt{H})e^{-H}|\cdot\langle x-y\rangle^a dy \leq \iint |\widehat{G_j}'(\xi)|\cdot|\cos(\xi\sqrt{H})e^{-H}|\langle x-y\rangle^a d\xi dy$$

where we used that $\widehat{sG(s)} = i\widehat{G}'(\xi)$. Proceeding as above we obtain

$$\|\sqrt{H}g(\sqrt{H})\|_{\langle x\rangle^a} \leq c(n,a)\|\langle\xi\rangle^{a+n/2}\widehat{G}'(\xi)\|_{L^1}$$

and to conclude it is sufficient to remark that

$$\widehat{G}' = \widehat{g}' * \widehat{(\chi e^{s^2})} \qquad \Longrightarrow \qquad \|\langle\xi\rangle^s\widehat{G}\|_{L^1} \leq c(s,R)\|\langle\xi\rangle^s\widehat{g}'\|_{L^1}.$$

□

LEMMA 5.6. *Assume H satisfies (H) and ϕ is given by (5.7). Let g be a function on $\mathbb{R}^+$, and define, for $j \in \mathbb{R}$, $g_j(s) = \phi(2^j s)g(s)$. Then for any $a \geq 0$*

$$\|g_j(\sqrt{H})\|_{\langle 2^{-j}x\rangle^a} \leq c(n,a)K_0 \cdot \|\langle\xi\rangle^{a+\frac{n}{2}}\mathcal{F}[\phi(s)S_{2^{-j}}g]\|_{L^1}, \tag{5.43}$$

$$\|\sqrt{H}g_j(\sqrt{H})\|_{\langle 2^{-j}x\rangle^a} \leq c(n,a)K_0 \cdot \|\langle\xi\rangle^{a+\frac{n}{2}}\mathcal{F}[s\phi(s)S_{2^{-j}}g]\|_{L^1} \cdot 2^{-j}. \tag{5.44}$$

PROOF. Extend $g(s)$ for $s \leq 0$ as an even function; notice that the values of g on $(-\infty, 0]$ are irrelevant in the definition of $g(\sqrt{H})$. We can write

$$g_j(\sqrt{H}) = S_{2^{-j}}G_j(\sqrt{H_j})S_{2^j} \tag{5.45}$$

where

$$G_j(s) = \phi(s)g(2^{-j}s) = \phi S_{2^{-j}}g$$

and

$$H_j = 2^{2j}S_{2^j}HS_{2^{-j}}.$$

It is easy to check by rescaling that the operator H_j satisfies the conditions in Assumption (H) with the same constants. Thus we can apply Lemma 5.5 and obtain

$$\|G_j(\sqrt{H_j})\|_{\langle x\rangle^a} \leq c(n,a,R)K_0\|\langle\xi\rangle^{a+\frac{n}{2}}\mathcal{F}[\phi S_{2^{-j}}g]\|_{L^1}.$$

As a consequence of (5.45), the kernels of $G_j(\sqrt{H_j})$ and $g_j(\sqrt{H})$ are related by

$$g_j(\sqrt{H})(x,y) = G_j(\sqrt{H_j})(2^{-j}x, 2^{-j}y) \cdot 2^{-jn}.$$

and this implies (5.43). Since we have also

$$\sqrt{H_j} = 2^j S_{2^j}\sqrt{H}S_{2^{-j}}$$

(5.44) follows immediately from (5.42). □

LEMMA 5.7. *Assume H satisfies (H), let $\alpha \in C_c^\infty(\mathbb{R})$ be an even function, and for $r > 0$ write $\alpha_r(s) = \alpha(rs)$. Then, for all $m \geq 0$,*

$$|\alpha_r(\sqrt{H})(x,y)| \leq C(n,m,\alpha)K_0^2 \cdot \left\langle \frac{x-y}{r} \right\rangle^{-m} r^{-n}, \tag{5.46}$$

$$|\sqrt{H}\alpha_r(\sqrt{H})(x,y)| \leq C(n,m,\alpha)K_0^2 \cdot \left\langle \frac{x-y}{r} \right\rangle^{-m} r^{-n-1}. \tag{5.47}$$

PROOF. By rescaling, as in the proof of the previous lemma, we can reduce to the case $r = 1$. Then define $G(s) = \alpha(s)e^{s^2}$ so that, using the inequality

$$\langle x-y\rangle \leq \langle x-z\rangle\langle z-y\rangle,$$

we can write

$$\langle x-y\rangle^m|\alpha(\sqrt{H})(x,y)| \leq \int |G(\sqrt{H})(x,z)|\langle x-z\rangle^m \cdot |e^{-H}(z,y)|\langle z-y\rangle^m dz.$$

Now we have

$$|p_1(z,y)| \cdot \langle z-y\rangle^m \leq K_0 \cdot c(n,m)$$

and this implies

$$\langle x-y\rangle^m|\psi(\sqrt{H})(x,y)| \leq c(n,m)K_0\|G(\sqrt{H})\|_{\langle x\rangle^m}.$$

Applying (5.41) with $a = m$ we obtain

$$\|G(\sqrt{H})\|_{\langle x\rangle^m} \le c(n, m, \alpha)K_0$$

and (5.46) follows. Analogously, (5.47) follows from (5.44). □

We can now conclude the proof of (5.12) in a similar way as [**41**]. Let $f \in L^1$, $\lambda > 0$ and consider the Calderòn-Zygmund decomposition of f: a sequence of disjoint cubes Q_j and functions h, f_j with $\operatorname{supp} f_j \subseteq Q_j$, $j \ge 1$, such that

$$f = h + \textstyle\sum_j f_j, \qquad |h| \le C\lambda, \qquad \int |f_j| \le C\lambda|Q_j|, \qquad \sum |Q_j| \le C\lambda^{-1}\|f\|_{L^1}.$$

Then we can write $g(\sqrt{H})f$ as

$$g(\sqrt{H})f = g(\sqrt{H})h + \textstyle\sum_j g(\sqrt{H})\psi_{r_j}(\sqrt{H})f_j + \sum_j (1 - \psi_{r_j}(\sqrt{H}))f_j \quad (5.48)$$

where

$$2^{r_j} = 4\operatorname{diam}(Q_j).$$

For the first term in (5.48) we have, by the spectral theorem,

$$|\{|g(\sqrt{H})h| > \lambda\}| \le \lambda^{-2}\|g(\sqrt{H})h\|_{L^2}^2 \le \lambda^{-2}\|g\|_{L^\infty}^2\|h\|_{L^2}^2 \le C\lambda^{-1}\|g\|_{L^\infty}^2\|h\|_{L^1}$$

and hence

$$|\{g(\sqrt{H})h > \lambda\}| \le C\|g\|_{L^\infty}^2\|f\|_{L^1} \cdot \lambda^{-1} \quad (5.49)$$

since $\|h\|_{L^1} \le C\|f\|_{L^1}$. To handle the second term, we consider the product with $\gamma(x) \in L^2$

$$|(\psi_{r_j}(\sqrt{H})f_j, \gamma)_{L^2}| \le CK_0^2 \iint \left\langle \frac{x-y}{r_j} \right\rangle^{-m} r_j^{-n}|\gamma(x)f_j(y)|dxdy$$

where we have used estimate (5.46) for the kernel. Now we nortice that for all $y \in Q_j$ we have

$$\left\langle \frac{x-y}{r_j} \right\rangle^{-m} \le c(m, n)\int_{Q_j} \left\langle \frac{x-z}{r_j} \right\rangle^{-m} dz \cdot |Q_j|$$

with a constant independent of j. Thus, using $\int |f_j(y)|dy \le C\lambda|Q_j|$,

$$|(\psi_{r_j}(\sqrt{H})f_j, \gamma)_{L^2}| \le CK_0^2\lambda \int_{Q_j} dz \int \left\langle \frac{x-z}{r_j} \right\rangle^{-m} r_j^{-n}|\gamma(x)|dx.$$

The innermost integral is bounded by $c_n M\gamma(z)$ provided we choose e.g. $m = n + 1$, so that

$$\sum_j |(\psi_{r_j}(\sqrt{H})f_j, \gamma)_{L^2}| \le CK_0^2\lambda \cdot \int_{Q_j} M\gamma(z)dz \le CK_0^2\lambda\|M\gamma\|_{L^2}\|\textstyle\sum \mathbf{1}_{Q_j}\|_{L^2}$$

and noticing that $\|\sum \mathbf{1}_{Q_j}\|_{L^2} \le C\lambda^{-1/2}\|f\|_{L^1}^{1/2}$ we find

$$\sum_j |(\psi_{r_j}(\sqrt{H})f_j, \gamma)_{L^2}| \le CK_0^2\lambda^{1/2}\|f\|_{L^1}^{1/2}\|\gamma\|_{L^2}.$$

This implies

$$\|g(\sqrt{H})\sum_j \psi_{r_j}(\sqrt{H})f_j\|_{L^2}^2 \le CK_0^4\|g\|_{L^\infty}\lambda\|f\|_{L^1}$$

and proceeding as for the first piece we obtain

$$|\{|g(\sqrt{H})\sum_j \psi_{r_j}(\sqrt{H})f_j| > \lambda\}| \leq CK_0^4 \|g\|_{L^\infty}^2 \|f\|_{L^1} \cdot \lambda^{-1} \tag{5.50}$$

Finally, consider the third piece in (5.48)

$$III = \sum_j (1 - \psi_{r_j}(\sqrt{H}))f_j.$$

Recalling that

$$1 - \psi(s) = \sum_{k \leq 0} \phi(2^k s) \quad \text{for } s > 0,$$

using the notation $\lg r = \log_2 r$,

$$1 - \psi_{r_j}(s) = 1 - \psi(r_j s) = \sum_{k \leq 0} \phi(2^k r_j s) \equiv \sum_{k \leq 0} \phi(2^{k + \lg r_j} s) \quad \text{for } s > 0$$

we can write

$$III = \sum_{k \leq 0} g_{k + \lg r_j}(\sqrt{H}), \qquad g_j(s) = g(s)\phi(2^j s).$$

Now, if $4Q_j$ is a cube with the same center as Q_j but with sides multiplied by 4, and $A = \cup 4Q_j$,

$$|\{|III| > \lambda\}| \leq |A| + \lambda^{-1} \sum_j \sum_{k \leq 0} \int_{\mathbb{R}^n \setminus A} |g_{k + \lg r_j}(x, y)| \cdot |f_j(y)| dy.$$

We shall estimate the kernel of $g_{k+\lg r_j}$ as follows: let $a = \sigma - n/2$ (recall that by assumption $\mu = \mu_\sigma(g) < \infty$ for some $\sigma > n/2$, so that $a > 0$), then we can write

$$|g_{k+\lg r_j}(x, y)| \leq \|g_{k+\lg r_j}\|_{\langle x/2^k r_j \rangle^a} \cdot \left\langle \frac{x - y}{2^k r_j} \right\rangle^{-a} \leq c(n, a) K_0 \mu \cdot 2^{a(k-j)}$$

where we have used (5.43), and the fact that for $x \notin A$ and $y \in Q_j$ we have $|x - y| \geq 2^j r_j$. Notice also that $|A| \leq c(n) \sum |Q_j|$. Thus we obtain

$$|\{|III| > \lambda\}| \leq c(n)\lambda^{-1}\|f\|_{L^1} + c(n, a)K_0\mu\lambda^{-1} \sum_j \sum_{k \leq 0} 2^{a(k-j)} \|f_j\|_{L^1}.$$

Since $a > 0$, we can sum over $k \leq 0$ and we conclude

$$|\{|III| > \lambda\}| \leq c(n, a)(1 + K_0\mu)\lambda^{-1}\|f\|_{L^1}. \tag{5.51}$$

Summing (5.49), (5.50) and (5.51) we obtain (5.11).

Estimate (5.12) for general p can be obtained in a standard way by real interpolation with the L^2 trivial estimate and duality. Notice however that the constant in the Marcinkiewicz interpolation theorem diverges at both ends: if $p = (1 - \theta)/p_0 + \theta/p_1$ and the linear operator T satisfies weak L^{p_j} estimates with constants C_j, $j = 0, 1$, then T satisfies a strong L^p estimate with a norm

$$\|T\|_{L^p \to L^p} \leq 2 \left(\frac{p}{p - p_0} + \frac{p}{p_1 - p} \right)^{1/p} C_0^{1-\theta} C_1^\theta$$

(see e.g. [**40**]). Thus a second (complex) interpolation step between two strong estimates is necessary in order to get (5.12).

The proof of (5.14) requires a variant of the Calderòn-Zygmund decomposition for Sobolev functions due to Auscher [**3**]: given f with $\|\nabla f\|_{L^1} < \infty$ and $\lambda > 0$, there exists a sequence of cubes Q_j with controlled overlapping (i.e. $\sum \mathbf{1}_{Q_j} \leq N = N(n)$), and functions h, f_j with $f_j \in W^1_0(Q_j)$ such that

$$f = h + \sum_j f_j, \qquad |\nabla h| \leq C\lambda, \qquad \int |\nabla f_j| \leq C\lambda |Q_j|, \qquad \sum |Q_j| \leq C\lambda^{-1} \|\nabla f\|_{L^1}.$$

We list the modifications necessary in the preceding proof. The decomposition is obviously

$$\sqrt{H} g(\sqrt{H}) f = \sqrt{H} g(\sqrt{H}) h + \sum_j \sqrt{H} g(\sqrt{H}) \psi_{r_j}(\sqrt{H}) f_j + \sum_j \sqrt{H}(1 - \psi_{r_j}(\sqrt{H})) f_j \tag{5.52}$$

with r_j as above. The first piece is estimated using (5.13) instead of the elementary L^2 bound, which gives

$$|\{|g(\sqrt{H})h| > \lambda\}| \leq \lambda^{-q} C_q^q \|\nabla h\|^q_{L^q} \leq CC_q^q \lambda^{-1} \|\nabla h\|_{L^1} \leq CC_q^q \lambda^{-1} \|\nabla f\|_{L^1}.$$

For the second piece we write as before, but using now the kernel estimate (5.47),

$$|(\sqrt{H} \psi_{r_j}(\sqrt{H}) f_j, \gamma)_{L^2}| \leq CK_0^2 \iint \left\langle \frac{x-y}{r_j} \right\rangle^{-m} r_j^{-n-1} |\gamma(x) f_j(y)| dx dy.$$

Notice that Poincaré's inequality implies

$$\int |f_j(y)| dy \leq C r_j \int |\nabla f_j| | dy \leq C r_j \lambda |Q_j|$$

and the factor r_j cancels the additional power in r_j^{-n-1}. Thus we arrive at

$$\sum_j |(\sqrt{H} \psi_{r_j}(\sqrt{H}) f_j, \gamma)_{L^2}| \leq CK_0^2 \lambda \cdot \int_{Q_j} M\gamma(z) dz$$

and as above this implies

$$|\{|\sqrt{H} g(\sqrt{H}) \sum_j \psi_{r_j}(\sqrt{H}) f_j| > \lambda\}| \leq CK_0^4 \|g\|^2_{L^\infty} \|\nabla f\|_{L^1} \cdot \lambda^{-1}. \tag{5.53}$$

The third piece is decomposed again as

$$III' = \sum_{k \leq 0} \sqrt{H} g_{k + \lg r_j}(\sqrt{H}), \qquad g_j(s) = g(s) \phi(2^j s).$$

Using the kernel estimate (5.44) we get now, with $a = \sigma - n/2$ (so that $a > 1$ now)

$$|\{|III'| > \lambda\}| \leq c(n) \lambda^{-1} \|\nabla f\|_{L^1} + c(n,a) K_0 \mu' \lambda^{-1} \sum_j \sum_{k \leq 0} 2^{a(k-j)} \|f_j\|_{L^1} \cdot 2^{-k} r_j^{-1}.$$

Since $a > 1$ the sum in k converges with sum bounded by a constant $c(a)$, and another application of Poincaré's inequality cancels the power r_j^{-1}. In conclusion

$$|\{|III'| > \lambda\}| \leq c(n,a)(1 + K_0 \mu') \lambda^{-1} \|\nabla f\|_{L^1}$$

and the proof is complete.

3. Bounded functions of the operator: Theorem 5.3

3.1. The Auscher-Martell maximal lemma. We reproduce here the maximal lemma of [**6**], in a version slightly simplified for our needs (i.e., in the original Lemma a finer decomposition in condition (5.60) is permitted). We decided to include a short but complete proof in the Appendix, since we needed to keep track precisely of the constants appearing in the final estimate (5.63); this gives the additional bonus of making the paper self-contained. We also took the liberty of introducing some minor simplifications in the final step of the proof.

In the statement of Lemma 5.8 below, the quantity a^q/K^q in (5.62) must be interpreted as 0 when $q = \infty$, MF denotes the uncentered maximal operator over balls B

$$Mf(x) = \sup_{B \ni x} \fint_B |f(x)| dx, \tag{5.54}$$

and c_q is its norm in the weak (q, q) bound

$$\sup_{\lambda > 0} \lambda^q |\{Mf > \lambda\}| \leq c_q \|f\|^q_{L^q}, \qquad 1 \leq q < \infty, \qquad c_\infty \equiv 1. \tag{5.55}$$

We also recall that a weight $w(x) > 0$ belongs the *reverse Hölder class* RH_q, $1 < q < \infty$, if there exists a constant C such that for every cube Q

$$\left(\fint_Q w^q \right)^{1/q} \leq C \fint_Q w dx. \tag{5.56}$$

while RH_∞ is defined by the condition

$$w(x) \leq C \fint_Q w dx \quad \text{for a.e. } x \in Q. \tag{5.57}$$

The best constant C in these inequalities is denoted by $\|w\|_{RH_q}$. We shall use the following consequence of the previous definition: if $w \in RH_{s'}$ for some $1 \leq s < \infty$, then there exists C such that for every cube Q and every measurable subset $E \subseteq Q$

$$\frac{w(E)}{w(Q)} \leq \|w\|_{RH_{s'}} \left(\frac{|E|}{|Q|} \right)^{\frac{1}{s}} \tag{5.58}$$

Indeed, for $s' < \infty$ one can write

$$\frac{w(E)}{w(Q)} \leq \frac{|Q|}{w(Q)} \left(\fint_Q w^{s'} \right)^{\frac{1}{s'}} \left(\frac{|E|}{|Q|} \right)^{\frac{1}{s}} \leq \|w\|_{RH_{s'}} \left(\frac{|E|}{|Q|} \right)^{\frac{1}{s}}$$

while for $s' = \infty$ the proof is even more elementary.

LEMMA 5.8 ([**6**]). *Let F, G be positive measurable functions on $\mathbb{R}^n$, $1 < q \leq \infty$, $a \geq 1$, $1 \leq s < \infty$, $w \in RH_{s'}$. Assume that for every ball B there exist G_B, H_B positive functions such that*

$$F \leq G_B + H_B \qquad \text{a.e. on } B, \tag{5.59}$$

$$\|H_B\|_{L^q(B)} \leq a(MF(x) + G(y)) \cdot |B|^{\frac{1}{q}} \quad \text{for every } x, y \in B, \tag{5.60}$$

$$\|G_B\|_{L^1(B)} \leq G(x) \cdot |B| \quad \text{for every } x \in B. \tag{5.61}$$

Then for all $\lambda > 0$, $0 < \gamma < 1$, $K \geq 2^{n+2}a$, *we have, with* $C_0 = 2^{6(n+q)}(c_1 + c_q)$,

$$w\{MF > K\lambda,\ G \leq \gamma\lambda\} \leq C_0 \|w\|_{RH_{s'}} \cdot \left(\frac{\gamma}{K} + \frac{a^q}{K^q}\right)^{\frac{1}{s}} \cdot w\{MF > \lambda\}. \quad (5.62)$$

As a consequence, if F *is* L^1 *and* $1 \leq p < q/s$,

$$\|MF\|_{L^p(w)} \leq C_1 \|G\|_{L^p(w)}, \qquad C_1 = \left[(8C_0\|w\|_{RH_{s'}} + 2^{n+3})a^p\right]^{\frac{s}{1-ps/q}}. \quad (5.63)$$

3.2. Proof of Theorem 5.3. Assume for the moment $w \in RH_{s'}$ for some $1 \leq s < \infty$; at the end of the proof we shall optimize the choice in order to handle a generic weight in A_r. Moreover, fix a $\nu > 1$ so large that $\sigma > n/\nu$ i.e. $\nu > n/\sigma$.

Given any test function f, set $F(x) = |g(\sqrt{H})f|^\nu$, which is in L^1 by Theorem 5.2. Then, for any ball B define, with $\psi_r(s) = \psi(rs)$,

$$G_B = 2^\nu |g(\sqrt{H})(1 - \psi_r(\sqrt{H}))f|^\nu, \quad H_B = 2^\nu |g(\sqrt{H})\psi_r(\sqrt{H})f|^\nu$$

where r is the radius of the ball B. We will show now that with these choices the assumptions of the maximal lemma are satisfied. Clearly we have $F \leq G_B + H_B$ a.e. on $\mathbb{R}^n$.

We check that assumption (5.60) holds with $q = \infty$. For any $z \in B$ we have, writing for short $T = g(\sqrt{H})$,

$$|T\psi_r(\sqrt{H})f(z)| \leq \int |\psi_r(\sqrt{H})(z, y)| \cdot |Tf(y)| dy = I.$$

We can apply Lemma 5.7 with $m = n+1$; writing $B_j = 2^j B$, $j \geq 0$, $B_{-1} = \emptyset$, we have

$$I \leq C(n, \psi) K_0^2 r^{-n} \sum_{j \geq 0} \int_{B_j \setminus B_{j-1}} \left\langle \frac{z - y}{r} \right\rangle^{-n-1} |Tf(y)| dy$$

and using $\langle |z - y|/r \rangle \geq 2^{j-1}$ and $|B_j| = 2^{nj} r^n \omega_n$, we obtain

$$I \leq C(n, \psi) K_0^2 2^{n+1} \omega_n \sum_{j \geq 0} 2^{-j} \fint_{B_j} |Tf(y)| dy.$$

Now if $x \in B$ and $B' = B(x, r)$, $B'_j = 2^j B'$, we have

$$\fint_{B_j} |Tf(y)| dy \leq c(n) \left(\fint_{B'_{j+1}} |Tf(y)|^\nu dy \right)^{\frac{1}{\nu}} \leq c(n) \cdot MF(x)^{1/\nu}$$

and we obtain (5.60) with $q = \infty$:

$$|H_B(z)| = 2^\nu |T\psi_r(\sqrt{H})f(z)|^\nu \leq aMF(x), \qquad a = c(n, \psi, \nu) K_0^{2\nu}. \quad (5.64)$$

Consider now the remaining term, which we split as

$$G_B = 2^\nu |g(\sqrt{H})(1 - \psi_r(\sqrt{H}))f|^\nu \leq 4^\nu (II^\nu + III^\nu)$$

where

$$II = |g(\sqrt{H})(1 - \psi_r(\sqrt{H}))f_1|, \qquad III = |g(\sqrt{H})(1 - \psi_r(\sqrt{H}))f_2|,$$

$$f_1 = f \cdot \mathbf{1}_{4B}, \qquad f_2 = f \cdot \mathbf{1}_{\mathbb{R}^n \setminus 4B}.$$

For the piece II we use Theorem 5.2 (recall that we can take $\nu >> 1$):

$$\|II\|_{L^\nu(B)} \leq \nu \cdot c(n,\sigma) K_0^4 (1+\mu+\|g\|^2_{L^\infty}) \|(1-\psi_r(\sqrt{H}))f_1\|_{L^\nu}.$$

Notice that

$$\|(1-\psi_r(\sqrt{H}))f_1\|_{L^\nu} \leq \|\psi_r(\sqrt{H})f_1\|_{L^\nu} + \|f_1\|_{L^\nu}$$

and using (5.46) with $m=n+1$ we see that

$$\|\psi_r(\sqrt{H})f_1\|_{L^\nu} \leq c(n,\psi)K_0^2\|f_1\|_{L^\nu}$$

which implies

$$\|II\|_{L^\nu(B)} \leq cK_0^6(1+\mu+\|g\|^2_{L^\infty})\|f_1\|_{L^\nu}.$$

Estimating with the maximal function we obtain

$$\|II\|_{L^\nu(B)} \leq c(n,\sigma,\psi)K_0^6(1+\mu+\|g\|^2_{L^\infty}) \cdot r^{n/\nu} \cdot M(|f|^\nu)(x)^{1/\nu} \qquad \forall x \in B. \tag{5.65}$$

We can now focus on the piece III; we write

$$1-\psi(s) = \sum_{k\leq 0} \phi(2^k s) \quad \text{for } s>0$$

and hence, using the notation $\lg r = \log_2 r$,

$$1-\psi_r(s) = 1-\psi(rs) = \sum_{k\leq 0}\phi(2^k rs) \equiv \sum_{k\leq 0}\phi(2^{k+\lg r}s) \quad \text{for } s>0$$

which implies

$$g(\sqrt{H})(1-\psi_r(\sqrt{H})) = \sum_{k\leq 0} g_{k+\lg r}(\sqrt{H}), \qquad g_j(s) = g(s)\phi(2^j s).$$

Denote by $a_k(x,y)$ the kernel of $g_{k+\lg r}(\sqrt{H})$, then we have ($B_j = 2^j B$)

$$\|g_{k+\lg r}(\sqrt{H})f_2\|_{L^2(B)} \leq \sum_{j\geq 3}\left\| \int_{B_j\setminus B_{j-1}} |a_k(z,y)f_2(y)|dy\right\|_{L^2_z(B)}.$$

Now by Hölder's inequality

$$\left\|\int_A |a(z,y)f(y)|dy\right\|_{L^\nu_z(B)} \leq C\|f\|_{L^\nu(A)}$$

where

$$C = \max\left\{\sup_{z\in A}\left(\int_B |a(z,y)|dy\right), \sup_{z\in B}\left(\int_A |a(z,y)|dy\right)\right\}. \tag{5.66}$$

Moreover, Lemma 5.6 and assumption (5.9) ensure that

$$\|a_k\|_{\langle 2^k r^{-1}x\rangle^\sigma} \leq c(n,\sigma)K_0\mu. \tag{5.67}$$

We notice that for $z\in B$ and $y \in B_j \setminus B_{j-1}$, $j\geq 2$, $k\leq 0$, one has

$$\frac{|z-y|}{2^k r} \geq 2^{j-k-2} \geq 1 \implies \left\langle \frac{z-y}{2^k r}\right\rangle^\sigma \geq 4^{-\sigma}2^{\sigma(j-k)}$$

which together with (5.67) implies for (5.66)

$$C \leq c(n,\sigma)K_0\mu \cdot 2^{\sigma(k-j)}$$

amd hence

$$\left\| \int_{B_j \setminus B_{j-1}} |a_k(z,y) f_2(y)| dy \right\|_{L^\nu_z(B)} \le c(n,\sigma) K_0 \mu \cdot 2^{\sigma(k-j)} \|f\|_{L^\nu(B_j \setminus B_{j-1})}.$$

Now let $x \in B$ arbitrary and $B' = B(x,r)$, $B'_j = 2^j B$, then

$$\|f\|_{L^\nu(B_j \setminus B_{j-1})} \le \|f\|_{L^\nu(B'_{j+1})} \le c_n 2^{nj/\nu} r^{n/\nu} \cdot M(|f|^\nu)(x)^{1/\nu},$$

thus we have proved for all $x \in B$

$$\left\| \int_{B_j \setminus B_{j-1}} |a_k(z,y) f_2(y)| dy \right\|_{L^\nu_z(B)} \le c(n,\sigma) K_0 \mu \cdot 2^{\sigma(k-j)} 2^{nj/\nu} r^{n/\nu} M(|f|^\nu)(x)^{1/\nu}.$$

Summing over $j \ge 3$, since $\sigma > n/\nu$ we get

$$\|g_{k+\lg r}(\sqrt{H}) f_2\|_{L^2(B)} \le c(n,\sigma) K_0 \mu \cdot 2^{k\sigma} r^{n/\nu} \cdot M(|f|^\nu)(x)^{1/\nu}. \tag{5.68}$$

and summing over $k \le 0$, and recalling (5.65), we conclude

$$\begin{aligned} \|G_B\|_{L^1(B)} \le& 4^\nu \|II\|^\nu_{L^\nu(B)} + 4^\nu \|III\|^\nu_{L^\nu(B)} \\ \le& \nu^\nu c(n,\sigma)^\nu K_0^\nu (1 + \mu + \|g\|^2_{L^\infty})^\nu \cdot M(|f|^\nu)(x) \cdot |B|. \end{aligned} \tag{5.69}$$

This proves (5.61) with the choice

$$G(x) = \nu^\nu c(n,\sigma)^\nu K_0^\nu (1 + \mu + \|g\|^2_{L^\infty})^\nu \cdot M(|f|^\nu)(x) \tag{5.70}$$

We are finally in position to apply Lemma 5.8 and we obtain, for all $1 \le p < \infty$, and any weight $w \in RH_{s'}$ for some $1 \le s < \infty$,

$$\|F\|_{L^p(w)} \le \|MF\|_{L^p(w)} \le C_1 \|G\|_{L^p(w)} \tag{5.71}$$

where in our case

$$C_1 = c(n,\sigma,\psi,p,s)(\|w\|_{RH_{s'}} + 1)^s K_0^{2ps\nu},$$

that is to say

$$\|g(\sqrt{H}) f\|^\nu_{L^{p\nu}(w)} \le C_2 \|M(|f|^\nu)\|_{L^p(w)} \tag{5.72}$$

where

$$C_2 = \nu^\nu c(n,\sigma,\psi,p,s)^\nu (\|w\|_{RH_{s'}} + 1)^s K_0^{\nu+2ps\nu} (1 + \mu + \|g\|^2_{L^\infty})^\nu$$

Now, assume the weight is in some A_p; recalling that $\cup_{1 \le p < \infty} A_p = \cup_{1 < q \le \infty} RH_q$, we have also $w \in RH_{s'}$ for some $1 \le s < \infty$, and all the previous computations apply. Since the maximal operator is bounded on $L^p(w)$, we deduce from (5.72)

$$\|g(\sqrt{H}) f\|_{L^{p\nu}(w)} \le C_3 \|f\|_{L^{p\nu}(w)}$$

where

$$C_3 = \nu \cdot c(n,\sigma,\psi,p,w) K_0^{1+2p^2} (1 + \mu + \|g\|^2_{L^\infty}).$$

Let $q = \nu p$; since we can take $\nu > n/\sigma$ (provided $\nu > 1$) arbitrarily large, we see that we have proved (5.18) for all $q > \max\{p, pn/\sigma\}$, with a constant

$$\frac{q}{p} \cdot c(n,\sigma,\psi,p,w) K_0^{1+2p^2} (1+\mu+\|g\|^2_{L^\infty}) = c'(n,\sigma,\psi,p,w) K_0^{1+2p^2} (1+\mu+\|g\|^2_{L^\infty}) q$$

as claimed.

4. The electromagnetic laplacian

In this chapter we verify that an electromagnetic Laplacian

$$H = (i\nabla - A(x))^2 + V(x)$$

satisfies Assumption (H), under suitable (very weak) regularity and integrability conditions on the coefficients. We recall that a measurable function V on $\mathbb{R}^n$ is in the *Kato class* when

$$\sup_x \lim_{r\downarrow 0} \int_{|x-y|<r} \frac{|V(y)|}{|x-y|^{n-2}} dy, \qquad (n \geq 3)$$

while the *Kato norm* is defined by

$$\|V\|_K = \sup_x \int \frac{|V(y)|}{|x-y|^{n-2}} dy \qquad (n \geq 3)$$

(replace $|x-y|^{2-n}$ with $\log|x-y|$ in dimension $n = 2$).

Our conditions will be based on the following result, which is obtained by combining an heat kernel estimate from [**27**] with Simon's diamagnetic inequality:

PROPOSITION 5.9. *Consider the Schrödinger operator $H = (i\nabla - A(x))^2 + V(x)$ on $L^2(\mathbb{R}^n)$, $n \geq 3$. Assume that $A \in L^2_{loc}(\mathbb{R}^n, \mathbb{R}^n)$, moreover the positive and negative parts $V_\pm$ of V satisfy*

$$V_+ \text{ is of Kato class}, \tag{5.73}$$

$$\|V_-\|_K < c_n = \pi^{n/2}/\Gamma(n/2 - 1). \tag{5.74}$$

Then H has a unioque nonnegative selfadjoint extension, e^{-tH} is an integral operator whose kernel satisfies the pointwise estimate

$$|e^{-tH}(x,y)| \leq \frac{K_0}{t^{n/2}} e^{-|x-y|^2/(8t)}, \qquad K_0 = \frac{(2\pi)^{-n/2}}{1 - \|V_-\|_K/c_n}. \tag{5.75}$$

PROOF. Simon's diamagnetic pontwise inequality (see Theorem B.13.2 in [**67**]), which holds under weaker assumptions, states that for any test function $\phi(x)$,

$$|e^{t[(\nabla - iA(x))^2 - V]}\phi| \leq e^{t(\Delta - V)}|\phi|.$$

By choosing a delta sequence ϕ_ϵ of test functions, this implies an analogous pointwise inequality for the corresponding heat kernels. Now we can apply the second part of Proposition 5.1 in [**27**] which gives precisely estimate (5.75) for the heat kernel of $e^{-t(\Delta - V)}$ under (5.73), (5.74). □

5. Fractional powers: proof of Corollary 5.4

Theorem 5.4 will be proved via Stein-Weiss interpolation for a suitable analytic family of operators We need the following lemma:

LEMMA 5.10. *Assume $n \geq 3$, $1 < p < n/2$, and let $w(x)$ be a weight of class A_p. Then the operator $H = (i\nabla - A)^2 + V$ satisfies the estimate*

$$\|Hg\|_{L^p(w)} \leq c(n,p,w) \cdot (\||A|^2 - i\nabla \cdot A + V\|_{L^{n/2}} + \|A\|_{L^n} + 1)\|(-\Delta)g\|_{L^p(w)} \tag{5.76}$$

PROOF. Setting $w = v^p$, the right hand side of (5.76) can be written $\|vHg\|_{L^p}$. If we expand the operator H and use Hölder's inequality for Lorentz spaces we find

$$\|vHg\|_{L^p} \leq \||A|^2 - i\nabla \cdot A + V\|_{L^{n/2,\infty}} \|vg\|_{L^{p^{**},p}} + 2\|A\|_{L^{n,\infty}} \|v\nabla g\|_{L^{p^*,p}}$$

where

$$p^* = \frac{np}{n-p}, \qquad p^{**} = \frac{np}{n-2p}.$$

We can use now the weighted version of Sobolev embeddings proved by Muckenhopt and Wheeden (see [**56**] and [**7**]). Recall also the definition of the reverse Hölder class (5.56) – (5.57).

THEOREM 5.11. *For* $1 < p \leq q < \infty$ *we have*

$$\|v(-\Delta)^{-\alpha/2} g\|_{L^q} \leq C\|vg\|_{L^p}$$

provided $\dfrac{\alpha}{n} = \dfrac{1}{p} - \dfrac{1}{q}$ *and* $v \in A_{2-\frac{1}{p}} \cap RH_q$.

By real interpolation the preceding estimates extend easily to Lorentz spaces as follows

$$\|v(-\Delta)^{-\alpha/2} g\|_{L^{q,p}} \leq C\|vg\|_{L^p}, \tag{5.77}$$

under the same conditions on p, q, w. Notice that this result for $\alpha = 1, 2$, combined with the boundedness of the Riesz operator $\nabla(-\Delta)^{-1/2}$ in weighted spaces, gives precisely the estimates we need:

$$\|vg\|_{L^{p^{**},p}} \leq C\|v(-\Delta)g\|_{L^p}, \qquad \|v\nabla g\|_{L^{p^*,p}} \leq C\|v(-\Delta)g\|_{L^p}$$

as soon as the weights are in the appropriate classes. In order to apply Theorem 5.11 we must require that

$$v = w^{1/p} \in A_{2-\frac{1}{p}} \cap RH_{\frac{np}{n-p}} \cap RH_{\frac{np}{n-2p}}$$

We now use a few basic properties of weighted spaces and reverse Hölder classes (for more details see [**37**]). First of all, for $1 \leq r \leq \infty$ and $1 < q < \infty$ one has

$$v \in A_r \cap RH_q \Leftrightarrow v^q \in A_{q(r-1)+1}.$$

Setting $q = p = q(r-1) + 1$, which implies $r = 2 - 1/p$, we obtain

$$v \in A_{2-\frac{1}{p}} \cap RH_p \Leftrightarrow w = v^p \in A_p.$$

Since the classes RH_q are decreasing in q, i.e.

$$RH_\infty \subset RH_q \subset RH_p, \quad \text{for } 1 < p \leq q \leq \infty$$

and $p < p^* < p^{**}$, all conditions on v collapse to $w \in A_p$ and the proof is concluded. □

Now fix $1 < p_0 < \infty$, $1 < p_1 < n/2$, and two weights $w_0 \in A_{p_0}$, $w_1 \in A_{p_1}$, and consider the family of operators for z in the strip $0 \leq \Re z \leq 1$

$$T_z = w_z H^z (-\Delta)^{-z} w_z^{-1}, \qquad w_z^{\frac{1}{p_z}} = w_0^{\frac{1-z}{p_0}} w_1^{\frac{z}{p_1}}, \qquad \frac{1}{p_z} = \frac{1-z}{p_0} + \frac{z}{p_1}.$$

We follow here the standard theory of [**71**] (see Theorem V.4.1), and in particular the operators T_z are defined on simple functions ϕ belonging to $L^1(\mathbb{R}^n)$, with values into measurable functions. Moreover, we have

$$|T_{1+iy}\phi| = w_1^{\frac{1}{p_1}} |H^{iy} H(-\Delta)(-\Delta)^{-iy} w_1^{-\frac{1}{p_1}} (w_0^{1/p_0} w_1^{-1/p_1})^{iy} \phi|.$$

The function $g(s) = s^{2iy}$ satisfies $\mu_\sigma(g) \leq C(1+|y|)^\sigma < \infty$ for all σ (see Remark 5.4), so choosing e.g. $\sigma = n+1$, by the weighted estimate (5.18) we have that H^{iy} is bounded on $L^q(w)$ for all $w \in A_p$ and all $q \geq p$ (actually $q > p - \epsilon$ as per Remark 5.5). This applies also to the special case of the operator $(-\Delta)^{iy}$. Combining (5.18) with Lemma 5.10, we deduce

$$\|T_{1+iy}\phi\|_{L^{p_1}} \leq c(n,p_1,w_1) K_0^{1+2p_1^2} C(A,V)(1+|y|)^{n+1} \|\phi\|_{L^{p_1}},$$

where

$$C(A,V) = \||A|^2 - i\nabla \cdot A + V\|_{L^{n/2}} + \|A\|_{L^n} + 1. \tag{5.78}$$

Notice in particular the polynomial growth in y which ensures that T_z is an admissible family in the sense of [**71**]. On the other hand we have

$$|T_{iy}\phi| = w_0^{\frac{1}{p_0}} |H^{iy}(-\Delta)^{-iy} w_0^{-\frac{1}{p_0}} (w_0^{1/p_0} w_1^{-1/p_1})^{iy} \phi|$$

and by a similar argument we deduce

$$\|T_{iy}\phi\|_{L^{p_0}} \leq c(n,\epsilon,p_0,w_0) K_0^{1+2p_0^2} (1+|y|)^n \|\phi\|_{L^{p_0}}.$$

Thus we are in position to apply complex interpolation for the family T_z, and we conclude that, for $0 < \theta < 1$,

$$\|T_\theta \phi\|_{L^{p_\theta}} \leq c(n,p_j,w_j) K_0^{2(1+p_0^2+p_1^2)} C(A,V)^\theta \|\phi\|_{L^{p_\theta}}$$

which is equivalent to

$$\|H^\theta \phi\|_{L^{p_\theta}(w_\theta)} \leq c(n,\epsilon,p_j,w_j) K_0^{2(1+p_0^2+p_1^2)} C(A,V)^\theta \|(-\Delta)^\theta \phi\|_{L^{p_\theta}(w_\theta)}.$$

Notice that

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \tag{5.79}$$

and since $1 < p_0 < \infty$, $1 < p_1 < n/2$ are arbitrary, p_θ can be any index in the range $1 < p < n/(2\theta)$.

Summing up, we have proved inequality (5.28) for all choices of $0 < \theta < 1$, $1 < p < n/(2\theta)$ and all weights $w(x)$ which can be represented in the form

$$w = w_0^{p_\theta \frac{1-\theta}{p_0}} w_1^{p_\theta \frac{\theta}{p_1}}, \tag{5.80}$$

with $w_j \in A_{p_j}$. The indices p_0, p_1 must be such that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

and of course $1 < p_0 < \infty$, $1 < p_1 < n/2$. It is clear that the weights of the form (5.80) belong to A_p (using e.g. the characterization in therms of maximal estimates). Conversely, it is not difficult to see that any A_p weight can be represented in the form (5.80). Indeed, recall the following characterization of Muckenhoupt weights (see [**69**]): $w \in A_p$, $1 \leq p < \infty$, if and only if there exist two weights $a(x), b(x) \in A_1$ with $w = a \cdot b^{1-p}$. Then if we choose

$$w_0(x) = a(x)b(x)^{1-p_0}, \qquad w_1(x) = a(x)b(x)^{1-p_1}$$

we see that (5.80) is satisfied, and of course $w_j \in A_{p_j}$. This concludes the proof.

5.1. Proof of Lemma 5.8. The following proof follows [**6**] closely, with some minor modifications and simplifcations as explained at the beginning of chapter 3. We denote by $\mathbf{1}_A$ the characteristic function of a set A, and, given a ball B, by mB the ball with the same center and radius multiplied by a factor m. Consider the sets

$$U_\lambda = \{MF > K\lambda,\ G \le \gamma\lambda\} \quad \subseteq \quad E_\lambda = \{MF > \lambda\}.$$

E_λ is open and we can decompose it in a sequence of disjoint Whitney cubes $E = \bigcup_j Q_j$ with $4Q_j \cap (\mathbb{R}^n \setminus E_\lambda) \neq \emptyset$, so that

$$\exists x_j \in 4Q_j \quad \text{with} \quad MF(x_j) \le \lambda. \tag{5.81}$$

To each Q_j we associate a ball B_j with the same center as Q_j and radius equal to 16 times the side of Q_j. Clearly we have also $U_\lambda = \bigcup_j E_\lambda \cap Q_j$. In the following we shall discard the cubes such that $U_\lambda \cap Q_j = \emptyset$, and select an arbitrary $y_j \in U_\lambda \cap Q_j$, so that

$$y_j \in Q_j, \qquad MF(y_j) > K\lambda, \qquad G(y_j) \le \gamma\lambda. \tag{5.82}$$

We remark that from the above choices it follows

$$|\{MF > K\lambda\} \cap Q_j| \le |\{M(F\mathbf{1}_{B_j}) > K\lambda/2\}|. \tag{5.83}$$

Indeed, take any point $x \in \{MF > \lambda\} \cap Q_j$ and a ball B containing x with $\int_B |F| > K\lambda|B|$. If $B \subseteq B_j$ we have

$$\int_{Q\cap B_j} |F| = \int_B |F| > K\lambda|B| \implies M(F\mathbf{1}_{B_j})(x) > K\lambda;$$

if on the other hand $B \not\subseteq B_j$, it is easy to chack that $2B$ must contain x_j and this implies (recalling that $MF(x_j) \le \lambda$)

$$\int_{B\setminus B_j} |F| \le \int_{2B} |F| \le \lambda|2B|$$

so that, using $K \ge 2^{n+2}a \ge 2^{n+2}$,

$$\int_{B\cap B_j} |F| > K\lambda|B| - |2B|\lambda \ge (K - 2^n) \cdot |B \cap B_j| \cdot \lambda \ge \frac{K\lambda}{2} \cdot |B \cap B_j|.$$

In order to prove inequality (5.62), we rewrite it as

$$w(U_\lambda) \le \|w\|_{RH_{s'}} C_0 \cdot \left(\frac{\gamma}{K} + \frac{a^q}{K^q}\right)^{\frac{1}{s}} \cdot w(E_\lambda)$$

which is implied by

$$w(U_\lambda \cap Q_j) \le \|w\|_{RH_{s'}} C_0 \cdot \left(\frac{\gamma}{K} + \frac{a^q}{K^q}\right)^{\frac{1}{s}} \cdot w(Q_j) \quad \text{for every } j.$$

Thus, recalling (5.58), we see that it is sufficient to prove

$$|U_\lambda \cap Q_j| \le C_0 \cdot \left(\frac{\gamma}{K} + \frac{a^q}{K^q}\right)|Q_j| \quad \text{for every } j. \tag{5.84}$$

Now, by (5.83), we can write

$$|U_\lambda \cap Q_j| \le |\{MF > K\lambda\} \cap Q_j| \le |\{M(F\mathbf{1}_{Bj}) > K\lambda/2\}|$$

and using $F\mathbf{1}_{B_j} \le G_{B_j}\mathbf{1}_{B_j} + H_{B_j}\mathbf{1}_{B_j}$ we obtain

$$|U_\lambda \cap Q_j| \le |\{M(G_{B_j}\mathbf{1}_{B_j}) > K\lambda/4\}| + |\{M(H_{B_j}\mathbf{1}_{B_j}) > K\lambda/4\}| = I + II. \tag{5.85}$$

To the term I we apply the weak bound (5.55) for $q = 1$:

$$|\{M(G_{B_j}\mathbf{1}_{B_j}) > K\lambda/4\}| \le \frac{4c_1}{K\lambda}\int_{B_j}|G_{B_j}| \le \frac{4c_1}{K\lambda}|B_j|G(y_j) \le \frac{2^{5n+2}c_1}{K}|Q_j|\gamma \tag{5.86}$$

where we used (5.61), (5.82) and $|B_j| \le 2^{5n}|Q_j|$.

Consider then the term II in (5.85). When $q = \infty$ we can write by (5.60), (5.81), (5.82) and $K \ge 2^{n+1}a$

$$\|M(H_{B_j}\mathbf{1}_{B_j})\|_{L^\infty} \le \|H_{B_j}\mathbf{1}_{B_j}\|_{L^\infty} \le a(MF(x_j) + MG(y_j)) \le 2a\lambda \le \frac{K\lambda}{4}$$

so that $II \equiv 0$. When $q < \infty$, we use the weak (q,q) bound (5.55), (5.60) and (5.81) to obtain

$$II \le \frac{4^q c_q}{(K\lambda)^q}\|H_{B_j}\|^q_{L^q(B_j)} \le \frac{4^q c_q}{(K\lambda)^q}\cdot|B_j|\cdot a^q[MF(x_j)+G(y_j)]^q \le \frac{2^{5(n+q)}c_q a^q}{K^q}|Q_j|$$

which together with (5.86) implies (5.84) and concludes the proof of (5.62).

We now prove (5.63); we can assume that the right hand side is finite. First we choose K large enough and γ small enough that

$$C_0 \cdot \left(\frac{\gamma}{K} + \frac{a^q}{K^q}\right)^{\frac{1}{s}} \cdot \|w\|_{RH_{s'}} \le \frac{1}{2K^p};$$

to obtain this, it is sufficient to set

$$K^{q-ps} = 4^s(C_0\|w\|_{RH_{s'}} + 2^n)^s a^q, \qquad \gamma = 4^{-s}(C_0\|w\|_{RH_{s'}} + 2^n)^{-s} \cdot K^{1-ps}. \tag{5.87}$$

With this choice, (5.62) implies (after a rescaling $\lambda \to \lambda/K$)

$$w\{MF > \lambda\} \le \frac{1}{2K^p}w\{MF > \lambda/K\} + w\{MG > \gamma\lambda/K\}. \tag{5.88}$$

Now define, for $j \in \mathbb{Z}$,

$$c_j = \int_{K^j}^{K^{j+1}} p\lambda^p w\{MF > \lambda\}\frac{d\lambda}{\lambda}, \qquad d_j = \int_{\gamma K^{j-1}}^{\gamma K^j} p\lambda^p w\{MG > \lambda\}\frac{d\lambda}{\lambda}.$$

Multiplying (5.88) by $p\lambda^p$ and integrating in $d\lambda/\lambda$ we obtain that c_j, d_j are finite and satisfy

$$c_j \le \frac{1}{2}c_{j-1} + \left(\frac{K}{\gamma}\right)^p d_j. \tag{5.89}$$

Summing from $-N$ to N, $N > 0$, we have, with $C' = (K/\gamma)^p$,

$$\sum_{-N}^{N} c_j \le \frac{1}{2} \sum_{-N-1}^{N-1} c_j + C' \sum_{-N}^{N} d_j \le \frac{1}{2} \sum_{-N}^{N} c_j + \frac{1}{2} c_{-N-1} + C' \sum_{-N}^{N} d_j$$

and hence

$$\sum_{-N}^{N} c_j \le c_{-N-1} + 2C' \sum_{-N}^{N} d_j \implies \sum_{-\infty}^{+\infty} c_j \le \limsup_{j \to -\infty} c_j + 2C' \sum_{-\infty}^{+\infty} d_j.$$

If we can show that c_j is uniformly bounded for $j < 0$, this implies that the series in c_j converges and hence the limsup is actually 0, implying

$$\sum_{-\infty}^{+\infty} c_j \le 2 \left(\frac{K}{\gamma} \right)^p \sum_{-\infty}^{+\infty} d_j$$

which gives (5.63) and concludes the proof. The bound on c_j is easy if the weight w is an L^∞ function: using the weak $(1,1)$ estimate for MF we have

$$c_j \le \|w\|_{L^\infty} \|F\|_{L^1} \int_{K^{j-1}}^{K^j} p\lambda^{p-1} d\lambda$$

which is bounded uniformly for $j < 0$ since $K > 1$ and $p \ge 1$. If w is not in L^∞, we first prove the estimate for the truncated weight $w_R = \inf\{w, R\}$ for all $R > 0$, then observe that the constant in the estimate depends only on the quantity $\|w_R\|_{RH_{s'}}$, which is bounded uniformly in $R \ge 1$ since $w \in RH_{s'}$, and does not depend on the L^∞ norm of the weight. Letting $R \to \infty$ we obtain (5.63).

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