INVERTING A MATRIX FUNCTION AROUND A SINGULARITY VIA LOCAL RANK FACTORIZATION[*]

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Abstract. This paper proposes a recursive procedure, called the extended local rank factorization (ELRF), that characterizes the order of the pole and the coefficients of the Laurent series representation of the inverse of a regular analytic matrix function around a given point. The ELRF consists in performing a finite sequence of rank factorizations of matrices of nonincreasing dimension, at most equal to the dimension of the original matrix function. Each step of the sequence is associated with a reduced rank condition, while the termination of the ELRF corresponds to a full rank condition; this last step reveals the order of the pole. The Laurent coefficients B_n are calculated recursively as $B_n = C_n + \sum_{k=1}^{n} D_k B_{n-k}$, where C_n, D_k have simple closed form expressions in terms of the quantities generated by the ELRF. It is also shown that the ELRF characterizes the structure of Jordan pairs, Jordan chains, and the local Smith form. The procedure is easily cast in an algorithmic form, and a MATLAB implementation script is provided. It is further found that the ELRF coincides with the complete reduction process (CRP) in Avrachenkov, Haviv, and Howlett [*SIAM J. Matrix Anal. Appl.*, 22 (2001), pp. 1175–1189]. Using this connection, the results on the ELRF provide both an explicit recursive formula for B_n implied by the CRP, and the link between the CRP and the structure of the local Smith form.

Key words. matrix valued functions, matrix inversion, analytic perturbation, Laurent series expansion, local Smith form, Jordan pairs

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1. Introduction. Consider a regular analytic matrix function $A(z)$ defined on an open set $U \subseteq \mathbb{C}$, and let

$$(1.1) \qquad A(z) = \sum_{n=0}^{\infty} A_n (z - z_0)^n, \quad A_n \in \mathbb{C}^{p \times p}, \quad A_0 \neq 0, \quad z \in U,$$

be its representation around the point $z_0 \in U$. Assume that $A(z_0) = A_0$ is singular, and let the Laurent representation of the inverse of $A(z)$ be

$$(1.2) \qquad A(z)^{-1} = \sum_{n=0}^{\infty} B_n (z - z_0)^{n-m}, \qquad B_0 \neq 0.$$

This paper discusses a recursive procedure to determine the order of the pole of $A(z)^{-1}$ at z_0, m, and the Laurent coefficients, $\{B_n\}_{n=0}^{\infty}$, given the coefficients of the original matrix function, $\{A_n\}_{n=0}^{\infty}$.

A classical approach to characterizing the relation between (1.1) and (1.2) is via the local spectral theory, based on the concepts of root functions, Jordan chains, and the local Smith form; see [5, 12, 13, 17]. The case of matrix polynomials is an

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important special case; see [14, 15, 16, 28, 29, 31, 32] and [11, 30, 34, 36] for matrix polynomials of degree one. The tools derived from the local spectral theory are used in the study of similarity of matrices [11, 30, 34], for the solutions of systems of differential equations [15, 16], and in linear control theory [4, 22, 27], as well as in time series econometrics [7, 8, 9, 19, 26, 35]. The same tools are also employed in numerical algorithms, such as those in [37, 38], for calculating the global Smith form of matrix polynomials and the Laurent representation of the inverse.

Another approach to the calculation of the Laurent coefficients B_n was proposed in [3] (see also [23]) and begins by writing the identity $A(z)A(z)^{-1} = I$ as the following linear system in the A_n, B_n matrices:

$$A_0 B_0 = 0,$$
$$A_0 B_1 + A_1 B_0 = 0,$$
$$\vdots$$
(1.3)
$$A_0 B_{m-1} + \cdots + A_{m-1} B_0 = 0,$$
$$A_0 B_m + A_1 B_{m-1} + \cdots + A_m B_0 = I,$$
$$A_0 B_{m+1} + A_1 B_m + \cdots + A_{m+1} B_0 = 0,$$
$$\vdots$$

In the following, equations in system (1.3) are indexed according to the highest value of the subscript of B_n; for instance, $A_0 B_0 = 0$ is referred to as equation 0. Note that the identity appears in equation m, which is the order of the pole.

In [3], the A_n matrices (or reduced versions of them) are stacked into appropriate augmented matrices, and the resulting system is solved for the Laurent coefficients B_n. Because the system (1.3) is singular, its direct solution involves the computation of a (Moore–Penrose) generalized inverse of dimension mp. Building on the results of [22] and on the reduction technique developed in [20, 21], the authors of [3] further propose a reduction process of system (1.3) based on its singularity. This process can be applied once, giving rise to a one-step reduction process or, recursively, yielding the so-called complete reduction process (CRP).

The CRP is expected to have numerical advantages with respect to the one-step reduction process because it exploits further singularities, although an explicit formula for the resulting Laurent coefficients is not available for the CRP. In addition, the CRP also delivers the order of the pole. If one applies the direct solution or the one-step reduction, the order of the pole must be predetermined, for example using the rank test in [33]. The book [2] contains a thorough treatment of this approach and its extensions; see, e.g., [1] for the case of operator pencils on Banach spaces.

The present paper exploits rank restrictions in (1.3), which are used to rank-decompose some relevant matrices of small dimension (less than or equal to p), obtaining two new (block-)orthogonal bases. This allows one to define a relevant set of projections of the B_n coefficients, for which explicit (recursive) formulae are found; the latter solutions are then rearranged into the general expression

$$B_n = C_n + \sum_{k=1}^{n} D_k B_{n-k}, \qquad n \geq 0,$$

where C_n, D_k have simple closed form expressions. For a pole of order m, there are $m + 1$ rank conditions: the first m are reduced rank restrictions, and the last one is

a full rank condition, and this gives a way to calculate the order of the pole. This procedure is called the extended local rank factorization (ELRF), being an extension of the local rank factorization (LRF) proposed in [10].

The spirit of the ELRF is thus similar to that of Theorem 5 in [22] and also to the rank test in [33], where the order of the pole is established recursively by checking the rank of a sequence of matrices until a full rank condition is satisfied. The main difference is that instead of working on stacked system matrices whose dimension increases with m, the present approach works on matrices whose dimension is at most equal to the dimension of the original matrix function, p, and that decreases with m.

The sequence of rank factorizations is further shown to deliver the partial multiplicities and the number of partial multiplicities of a given value, i.e., the local Smith form, and to provide a construction of an extended canonical system of root functions of $A(z)$ at z_0. In this way, the structure of a Jordan pair is fully characterized and full information on Jordan chains is available. These theoretical results are translated into an algorithmic form, and a MATLAB script that implements the ELRF is provided in the supplementary material (99983_01.pdf [local/web 162KB]), which is linked from the main article webpage.

The connections with the CRP in [3] are discussed, and it is shown that the CRP coincides with the ELRF. Via this equivalence, the characteristics of the CRP are linked to the structure of the local Smith form and it is shown that the number of reductions in the CRP is equal to the number of distinct nonzero partial multiplicities and each reduction step decreases the dimension of the coefficients by the number of partial multiplicities that are equal to a given value.

The rest of the paper is organized as follows. Section 2 motivates the techniques used in the paper via an illustration on poles of orders one and two; section 3 contains the general formulation and results; section 4 discusses the connections with the local Smith form, the extended canonical system of root functions, Jordan chains, and Jordan pairs; section 5 presents the algorithmic implementation of the ELRF and its relation to the CRP in [3]; section 6 discusses the computational complexity of the algorithm; section 7 illustrates the results via a numerical example; and section 8 concludes. The supplementary material contains the MATLAB script that implements the ELRF.

2. Motivation. This section motivates and illustrates the technique presented in the paper by discussing the cases of poles of orders one and two; the general formulation is presented in section 3. In particular, the cases considered in this section make it clear how the sequence of rank factorizations can be used to define a new block-orthogonal basis, which is then employed to decompose B_n into relevant projections.

The techniques presented in the paper make repeated use of rank factorizations and projections, whose notation is introduced here: given a $p \times p$ matrix φ of rank $0 < r < p$, its rank factorization is written as $\varphi = -\alpha\beta'$, where α and β are $p \times r$ full column rank matrices that span the column space $(\operatorname{col}\varphi)$ and the row space $(\operatorname{col}\varphi')$ of φ, respectively; the negative sign is chosen for convenience in the calculations. The matrix $\varphi_\perp$ denotes a $p \times (p - r)$ full column rank matrix that spans $\operatorname{col}^\perp \varphi$, the orthogonal complement of $\operatorname{col}\varphi = \operatorname{col}\alpha$. The orthogonal projection matrix onto $\operatorname{col}\varphi$ is denoted by $P_\alpha := \bar{\alpha}\alpha' = \alpha\bar{\alpha}'$, where $\bar{\alpha} := \alpha(\alpha'\alpha)^{-1}$, and it has rank r; $P_{\alpha_\perp} := I - P_\alpha = \bar{\alpha}_\perp\alpha'_\perp = \alpha_\perp\bar{\alpha}'_\perp$ of rank $p - r$ is the orthogonal projection matrix onto $\operatorname{col}^\perp \varphi$. Similarly, for $\operatorname{col}\varphi'$, one can define P_β and $P_{\beta_\perp}$. When $r = 0$, i.e., $\varphi = 0$, one lets $\alpha = \beta = \bar{\alpha} = \bar{\beta} = 0$ and $\alpha_\perp = \beta_\perp = \bar{\alpha}_\perp = \bar{\beta}_\perp = I$. When $r = p$, i.e., φ is of

full rank, one can take either α or β equal to I and let $\alpha_\perp = \beta_\perp = \bar{\alpha}_\perp = \bar{\beta}_\perp = 0$.

Before considering the special cases of poles of orders $m = 1, 2$, some general observations are presented. Recall that the equations in (1.3) are referred to according to the appropriate order index $0, 1, \ldots$. Consider equation 0 in (1.3) and observe that, because $A_0 \neq 0$ is singular, it can be rank-decomposed as $A_0 = -\alpha_0 \beta_0'$, where α_0 and β_0 have rank $0 < r_0 := \operatorname{rank} A_0 < p$. This implies that the associated projection matrices $P_{\alpha_0}, P_{\alpha_{0\perp}}$ and $P_{\beta_0}, P_{\beta_{0\perp}}$ are all different from 0. Note that equation 0 implies $\beta_0' B_0 = 0$ so that

$$(2.1) \qquad B_0 = P_{\beta_0} B_0 + P_{\beta_{0\perp}} B_0 = P_{\beta_{0\perp}} B_0$$

can be substituted in (1.3).

Next, replacing $A_0 = -\alpha_0 \beta_0'$ in a generic equation n in (1.3) and using Kronecker's delta $\delta_{n,m}$ (defined as $\delta_{m,m} := 1$ and $\delta_{n,m} := 0$ otherwise), one can rewrite equation n as

$$(2.2) \qquad \alpha_0 \beta_0' B_n = \sum_{k=1}^{n} A_k B_{n-k} - \delta_{n,m} I.$$

Note that (2.2) implies $\beta_0' B_n = \bar{\alpha}_0' \sum_{k=1}^{n} A_k B_{n-k} - \delta_{n,m} \bar{\alpha}_0'$ and hence

$$P_{\beta_0} B_n = \bar{\beta}_0 \bar{\alpha}_0' \sum_{k=1}^{n} A_k B_{n-k} - \delta_{n,m} \bar{\beta}_0 \bar{\alpha}_0'.$$

This shows that equation n gives information on $P_{\beta_0} B_n$ but not on the complementary part $P_{\beta_{0\perp}} B_n$. Because B_n does not appear in equations 0 to $n - 1$, this information must necessarily come from the subsequent equations. The next two subsections show what happens in the cases $m = 1$ or 2.

2.1. Pole of order 1. In case $m = 1$, system (1.3) reads as

$$(2.3) \qquad \begin{aligned} -\alpha_0 \beta_0' B_0 &= 0, \\ -\alpha_0 \beta_0' B_1 + A_1 P_{\beta_{0\perp}} B_0 &= I, \\ -\alpha_0 \beta_0' B_2 + A_1 B_1 + A_2 P_{\beta_{0\perp}} B_0 &= 0, \end{aligned}$$

$$\vdots$$

Here it is shown that all the information on $P_{\beta_{0\perp}} B_0$ is found in equation 1. In fact, premultiply equation 1 by $P_{\alpha_{0\perp}}$ to find that

$$(2.4) \qquad (P_{\alpha_{0\perp}} A_1 P_{\beta_{0\perp}}) B_0 = P_{\alpha_{0\perp}}.$$

Because $\operatorname{rank} P_{\alpha_{0\perp}} = \operatorname{rank} P_{\beta_{0\perp}} = p - r_0$, one has $\operatorname{rank}(P_{\alpha_{0\perp}} A_1 P_{\beta_{0\perp}}) \leq p - r_0$, and hence (2.4) is consistent if and only if $P_{\alpha_{0\perp}} A_1 P_{\beta_{0\perp}}$ has maximal rank $p - r_0$. Because $P_{\alpha_{0\perp}} A_1 P_{\beta_{0\perp}} = \bar{\alpha}_{0\perp} \alpha_{0\perp}' A_1 \beta_{0\perp} \bar{\beta}_{0\perp}'$, the maximal rank condition $\operatorname{rank}(P_{\alpha_{0\perp}} A_1 P_{\beta_{0\perp}}) = p - r_0$ is equivalent to the full rank condition

$$(2.5) \qquad r_1 = r_1^{\max}, \quad \text{where} \quad r_1 := \operatorname{rank} \alpha_{0\perp}' A_1 \beta_{0\perp} \quad \text{and} \quad r_1^{\max} := p - r_0,$$

and $\alpha_{0\perp}' A_1 \beta_{0\perp}$ is a square matrix of dimension $r_1^{\max}$. This corresponds to the condition in Theorem 3 of [22] and to Theorem 4.1 in [24].

Note that in this case the full rank condition (2.5) implies that the rank factorization $\alpha'_{0\perp} A_1 \beta_{0\perp} = -\xi_1 \eta'_1$ is such that ξ_1 and η_1 are square nonsingular matrices of dimension $r_1^{\max}$; note also that, by premultiplying $\alpha'_{0\perp} A_1 \beta_{0\perp} = -\xi_1 \eta'_1$ by $\bar{\alpha}_{0\perp}$ and postmultiplying it by $\bar{\beta}'_{0\perp}$, the rank factorization can be written as

$$(2.6) \qquad P_{\alpha_{0\perp}} A_1 P_{\beta_{0\perp}} = -\alpha_1 \beta'_1, \quad \text{where} \quad \alpha_1 := \bar{\alpha}_{0\perp} \xi_1 \quad \text{and} \quad \beta_1 := \bar{\beta}_{0\perp} \eta_1.$$

Thanks to the full rank condition (2.5), one then has

$$(2.7) \qquad \operatorname{col}(\alpha_0, \alpha_1) = \operatorname{col}(\beta_0, \beta_1) = \mathbb{C}^p$$

because (i) α_0 and α_1 are orthogonal by construction, (ii) each of them has full column rank, and (iii) the sum of their number of columns $r_0 + r_1$ equals p. A similar argument holds for β_0, β_1. This implies

$$P_{\alpha_1} = P_{\alpha_{0\perp}}, \qquad P_{\beta_1} = P_{\beta_{0\perp}}, \quad \text{and hence} \quad I = P_{\alpha_0} + P_{\alpha_1} = P_{\beta_0} + P_{\beta_1}.$$

Using (2.6), one can rewrite (2.4) as $-\alpha_1 \beta'_1 B_0 = P_{\alpha_1}$, which implies $\beta'_1 B_0 = -\bar{\alpha}'_1$ and $P_{\beta_1} B_0 = -\bar{\beta}_1 \bar{\alpha}'_1$; hence

$$B_0 = P_{\beta_1} B_0 = -\bar{\beta}_1 \bar{\alpha}'_1.$$

Summing up, when $m = 1$, equation 1 contains all the information on $P_{\beta_{0\perp}} B_0$, and thus one can solve for B_0 using equations 0 and 1 only. This is in line with Theorem 1 in [3], which states that $m + 1$ equations are needed to calculate B_n given knowledge of $B_0, \dots, B_{n-1}$.

Similarly, it can be shown that the information on the second Laurent coefficient B_1 comes from equations 1 and 2. In fact, consider B_1 and decompose it into its projections on $\operatorname{col} \beta_0$ and $\operatorname{col} \beta_1$, i.e., $B_1 = P_{\beta_0} B_1 + P_{\beta_1} B_1$. From equation 1 one finds that $\beta'_0 B_1 = \bar{\alpha}'_0 (A_1 B_0 - I)$, and hence one has

$$(2.8) \qquad P_{\beta_0} B_1 = \bar{\beta}_0 \bar{\alpha}'_0 (A_1 B_0 - I).$$

Information on $P_{\beta_1} B_1$ is obtained by premultiplying equation 2 by P_{α_1} to find that $-P_{\alpha_1} A_1 B_1 = P_{\alpha_1} A_2 B_0$. Inserting $I = P_{\beta_0} + P_{\beta_1}$ between A_1 and B_1 and using (2.6), one has $\alpha_1 \beta'_1 B_1 = P_{\alpha_1} (A_2 B_0 + A_1 P_{\beta_0} B_1)$. Substituting $P_{\beta_0} B_1 = \bar{\beta}_0 \bar{\alpha}'_0 (A_1 B_0 - I)$ from (2.8) in the last expression and defining $A_{2,1} := A_2 + A_1 \bar{\beta}_0 \bar{\alpha}'_0 A_1$, one then finds that

$$(2.9) \qquad \alpha_1 \beta'_1 B_1 = P_{\alpha_1} A_{2,1} B_0 - P_{\alpha_1} A_1 \bar{\beta}_0 \bar{\alpha}'_0.$$

This implies $\beta'_1 B_1 = \bar{\alpha}'_1 (A_{2,1} B_0 - A_1 \bar{\beta}_0 \bar{\alpha}'_0)$ and hence

$$(2.10) \qquad P_{\beta_1} B_1 = \bar{\beta}_1 \bar{\alpha}'_1 (A_{2,1} B_0 - A_1 \bar{\beta}_0 \bar{\alpha}'_0).$$

Substituting (2.8) and (2.10) in $B_1 = P_{\beta_0} B_1 + P_{\beta_1} B_1$, one finds that

$$B_1 = C_1 + D_1 B_0,$$

where $C_1 := -\bar{\beta}_0 \bar{\alpha}'_0 - \bar{\beta}_1 \bar{\alpha}'_1 A_1 \bar{\beta}_0 \bar{\alpha}'_0$ and $D_1 := \bar{\beta}_0 \bar{\alpha}'_0 A_1 + \bar{\beta}_1 \bar{\alpha}'_1 A_{2,1}$. This shows that when $m = 1$ one can solve for B_1 using equations 1 and 2 and the previous expression for B_0. Similarly, for each n one can write $B_n = P_{\beta_0} B_n + P_{\beta_1} B_n$, and solve equation n for $P_{\beta_0} B_n$ and equation $n+1$ for $P_{\beta_1} B_n$, using knowledge of $B_0, \dots, B_{n-1}$.

The above illustrates the role of the "full rank condition", equivalently stated as the maximal rank of $P_{\alpha_{0\perp}} A_1 P_{\beta_{0\perp}}$ or the full rank of $\alpha_{0\perp}' A_1 \beta_{0\perp}$ (i.e., $r_1 = r_1^{\max}$), or as $\mathrm{col}(\alpha_0, \alpha_1) = \mathrm{col}(\beta_0, \beta_1) = \mathbb{C}^p$, where the latter and block-orthogonality of the bases imply the projection identities $I = P_{\alpha_0} + P_{\alpha_1} = P_{\beta_0} + P_{\beta_1}$; see (2.4), (2.5), and (2.7). Note that this full rank condition applies to equation 1 due to the presence of I on the right-hand side (RHS), corresponding to the order of the pole $m = 1$. More generally, for a pole of order m, the full rank condition applies to equation m, and this gives a way of finding the order of the pole by checking the full rank condition, as shown in section 3.

2.2. Pole of order 2. In case $m = 2$, system (1.3) reads as

$$-\alpha_0 \beta_0' B_0 = 0,$$
(2.11)
$$-\alpha_0 \beta_0' B_1 + A_1 P_{\beta_{0\perp}} B_0 = 0,$$
$$-\alpha_0 \beta_0' B_2 + A_1 B_1 + A_2 P_{\beta_{0\perp}} B_0 = I,$$
$$-\alpha_0 \beta_0' B_3 + A_1 B_2 + A_2 B_1 + A_3 P_{\beta_{0\perp}} B_0 = 0,$$

$$\vdots$$

Proceeding as above in equation 1, one finds that

$$(P_{\alpha_{0\perp}} A_1 P_{\beta_{0\perp}}) B_0 = 0,$$
(2.12)

i.e., using the rank factorization (2.6), $-\alpha_1 \beta_1' B_0 = 0$, which implies $\beta_1' B_0 = 0$ and $P_{\beta_1} B_0 = 0$. If $r_1 = r_1^{\max}$, one would have (2.7) and hence $B_0 = 0$, which in the present case $m = 2$ would give a contradiction. This means that when $m > 1$ the following reduced rank condition must hold:

$$(2.13) \qquad r_1 < r_1^{\max}, \quad \text{where} \quad r_1 := \mathrm{rank}\, \alpha_{0\perp}' A_1 \beta_{0\perp} \quad \text{and} \quad r_1^{\max} := p - r_0,$$

and in this case the rank factorization $\alpha_{0\perp}' A_1 \beta_{0\perp} = -\xi_1 \eta_1'$, or equivalently (2.6), involves a reduction of rank. Note that here ξ_1, η_1 are $r_1^{\max} \times r_1$ matrices and $\mathrm{col}\,\alpha_1$, $\mathrm{col}\,\beta_1$ are r_1-dimensional subspaces of $\mathrm{col}\,\alpha_{0\perp}$, $\mathrm{col}\,\beta_{0\perp}$, respectively.

Letting $a_2 := (\alpha_0, \alpha_1)$ and $b_2 := (\beta_0, \beta_1)$, one has $\mathrm{col}\,\alpha_{0\perp} = \mathrm{col}(\alpha_1, a_{2\perp})$ and $\mathrm{col}\,\beta_{0\perp} = \mathrm{col}(\beta_1, b_{2\perp})$, where $a_{2\perp}$ (respectively, $b_{2\perp}$) spans the $(p - r_0 - r_1)$-dimensional complementary subspace of $\mathrm{col}\,\alpha_{0\perp}$ (respectively, $\mathrm{col}\,\beta_{0\perp}$) not generated by α_1 (respectively, β_1). Hence in this case one needs to decompose the Laurent coefficients in more than two projections; in particular, note that using $I = P_{\alpha_0} + P_{\alpha_1} + P_{a_{2\perp}} = P_{\beta_0} + P_{\beta_1} + P_{b_{2\perp}}$, one has $B_n = P_{\beta_0} B_n + P_{\beta_1} B_n + P_{b_{2\perp}} B_n$.

For $n = 0$, one finds that $B_0 = P_{b_{2\perp}} B_0$. In fact, $P_{\beta_0} B_0 = 0$ holds from equation 0 and $P_{\beta_1} B_0 = 0$ holds from equation 1 because $\alpha_1 \beta_1' B_0 = 0$ (see (2.12)) implies $P_{\beta_1} B_0 = 0$. Information on $P_{b_{2\perp}} B_0$ is found by premultiplying equation 2 by $P_{\alpha_{0\perp}}$ to find that $P_{\alpha_{0\perp}} A_1 B_1 + P_{\alpha_{0\perp}} A_2 P_{b_{2\perp}} B_0 = P_{\alpha_{0\perp}}$; inserting $I = P_{\beta_0} + P_{\beta_{0\perp}}$ between A_1 and B_1 and using (2.6), one then finds that

$$(2.14) \qquad -\alpha_1 \beta_1' B_1 + P_{\alpha_{0\perp}} A_1 P_{\beta_0} B_1 + P_{\alpha_{0\perp}} A_2 P_{b_{2\perp}} B_0 = P_{\alpha_{0\perp}}.$$

Observe that equation 1 implies $\beta_0' B_1 = \bar{\alpha}_0' A_1 P_{b_{2\perp}} B_0$, i.e., $P_{\beta_0} B_1 = \bar{\beta}_0 \bar{\alpha}_0' A_1 P_{b_{2\perp}} B_0$; hence premultiplying (2.14) by $P_{a_{2\perp}}$ and rearranging, one finds that

$$(2.15) \qquad (P_{a_{2\perp}} A_{2,1} P_{b_{2\perp}}) B_0 = P_{a_{2\perp}}, \qquad A_{2,1} := A_2 + A_1 \bar{\beta}_0 \bar{\alpha}_0' A_1.$$

This equation has the same format of (2.4), and the same reasoning applies: because $\operatorname{rank} P_{a_{2\perp}} = \operatorname{rank} P_{b_{2\perp}} = p - r_0 - r_1$, one has $\operatorname{rank}(P_{a_{2\perp}} A_{2,1} P_{b_{2\perp}}) \leq p - r_0 - r_1$, and hence (2.15) is consistent if and only if $P_{a_{2\perp}} A_{2,1} P_{b_{2\perp}}$ has maximal rank $p - r_0 - r_1$. Because $P_{a_{2\perp}} A_{2,1} P_{b_{2\perp}} = \bar{a}_{2\perp}(a'_{2\perp} A_{2,1} b_{2\perp})\bar{b}'_{2\perp}$, the maximal rank condition $\operatorname{rank}(P_{a_{2\perp}} A_{2,1} P_{b_{2\perp}}) = p - r_0 - r_1$ is equivalent to the full rank condition

$$(2.16) \quad r_2 = r_2^{\max}, \quad \text{where} \quad r_2 := \operatorname{rank} a'_{2\perp} A_{2,1} b_{2\perp} \quad \text{and} \quad r_2^{\max} := p - r_0 - r_1,$$

and $a'_{2\perp} A_{2,1} b_{2\perp}$ is a square matrix of dimension $r_2^{\max}$. This corresponds to Theorem 3 in [25].

Note that in this case the full rank condition (2.16) implies that the rank factorization $a'_{2\perp} A_{2,1} b_{2\perp} = -\xi_2 \eta'_2$ is such that ξ_2 and η_2 are square nonsingular matrices of dimension $r_2^{\max}$; observe also that, by premultiplying $a'_{2\perp} A_{2,1} b_{2\perp} = -\xi_2 \eta'_2$ by $\bar{a}_{2\perp}$ and postmultiplying it by $\bar{b}'_{2\perp}$, the rank factorization can be written as

$$P_{a_{2\perp}} A_{2,1} P_{b_{2\perp}} = -\alpha_2 \beta'_2, \quad \text{where} \quad \alpha_2 := \bar{a}_{2\perp} \xi_2 \quad \text{and} \quad \beta_2 := \bar{b}_{2\perp} \eta_2.$$

Thanks to the full rank condition (2.16), one then has

$$(2.17) \qquad\qquad \operatorname{col}(\alpha_0, \alpha_1, \alpha_2) = \operatorname{col}(\beta_0, \beta_1, \beta_2) = \mathbb{C}^p$$

because (i) $\alpha_0, \alpha_1, \alpha_2$ are orthogonal by construction, (ii) each of them has full column rank, and (iii) the sum of their number of columns $r_0 + r_1 + r_2$ equals p. A similar argument holds for $\beta_0, \beta_1, \beta_2$.[1] This implies

$$P_{\alpha_2} = P_{a_{2\perp}}, \qquad P_{\beta_2} = P_{b_{2\perp}}, \quad \text{and hence} \quad I = P_{\alpha_0} + P_{\alpha_1} + P_{\alpha_2} = P_{\beta_0} + P_{\beta_1} + P_{\beta_2}.$$

Using $P_{\alpha_2} A_{2,1} P_{\beta_2} = -\alpha_2 \beta'_2$, one can then rewrite (2.15) as $-\alpha_2 \beta'_2 B_0 = P_{\alpha_2}$, which implies $\beta'_2 B_0 = -\bar{\alpha}'_2$ and $P_{\beta_2} B_0 = -\bar{\beta}_2 \bar{\alpha}'_2$. Hence

$$B_0 = P_{\beta_2} B_0 = -\bar{\beta}_2 \bar{\alpha}'_2.$$

Summing up, when $m = 2$ there are three rank conditions: (i) $A_0 = -\alpha_0 \beta'_0$, the first reduced rank condition; (ii) (2.13), the second reduced rank condition; (iii) the full rank condition (2.16). The meaning of this sequence of rank conditions can be described as follows: (i) establishes that $m > 0$; (ii) establishes that $m > 1$; (iii) establishes that $m = 2$.

As shown in section 3 below, this is true in general: for a pole of order m there are $m+1$ rank conditions; the first $j = 1, \ldots, m$ are reduced rank conditions that establish that the order of the pole is greater than $j-1$, and the last one is the full rank condition that proves that the order of the pole is exactly m. These rank factorizations deliver two new bases of $\mathbb{C}^p$, namely $(\alpha_0, \ldots, \alpha_m)$ and $(\beta_0, \ldots, \beta_m)$, with mutually orthogonal components; this gives rise to the projection identities $I = \sum_{j=0}^{m} P_{\alpha_j} = \sum_{j=0}^{m} P_{\beta_j}$. Each Laurent coefficient B_n can then be decomposed into the sum of $m+1$ components $B_n = P_{\beta_0} B_n + \cdots + P_{\beta_m} B_n$. Equation $j + n$ of (1.3) is solved for component $P_{\beta_j} B_n$ in terms of $B_{n-1}, \ldots, B_0$. The results are then rearranged and expressed recursively as $B_n = C_n + \sum_{k=1}^{n} D_k B_{n-k}$, where C_n, D_k have simple closed form expressions in terms of the matrices defined by the rank factorizations. This is the structure of the results at the heart of the ELRF presented in the following sections.

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[1]Indeed, α_1 and β_1 can be 0; see Remark 3.2 below.

3. Laurent coefficients. This section contains the main results. Lemma 3.1 presents a rewriting of a generic equation in system (1.3) that is at the basis of the ELRF; Theorem 3.3 provides the expression for a projection of a Laurent coefficient in terms of the previous Laurent coefficients; Theorem 3.4 relates the rank conditions in system (1.3) with the order of the pole of the inverse; finally, Theorem 3.5 provides the recursive formula for the Laurent coefficients.

Results are based on the following lemma, which is proved by manipulating the identity $A(z)A(z)^{-1} = I$ written in system form as in (1.3). Recall that P_φ indicates the orthogonal projection matrix onto $\text{col}\,\varphi$ and $P_{\varphi_\perp}$ denotes the orthogonal projection matrix onto $\text{col}^\perp \varphi$; further, let $\bar\varphi := \varphi(\varphi'\varphi)^{-1}$ and set $\bar\varphi = 0$ when $\varphi = 0$.

Recall that in this paper it is assumed that $A(z_0) = A_0 \neq 0$ is singular, say of rank $0 < r_0 < p$. The following lemma defines a sequence of rank factorizations that is later shown to be relevant for the inversion of $A(z)$ at $z = z_0$.

LEMMA 3.1 (Extended local rank factorization of system (1.3)). *Consider the rank factorization $A_0 = -\alpha_0\beta_0'$ and let α_j, β_j, $j = 1, 2, \ldots$, be defined by the rank factorization*

$$(3.1) \quad P_{a_{j\perp}} A_{j,1} P_{b_{j\perp}} = -\alpha_j\beta_j', \qquad a_j := (\alpha_0, \ldots, \alpha_{j-1}), \qquad b_j := (\beta_0, \ldots, \beta_{j-1}),$$

where $A_{s,k}$ is defined by the recursions

$$(3.2) \qquad A_{s,k} := \begin{cases} A_k & \text{for } s = 1, \\ A_{s-1,k+1} + A_{s-1,1}\sum_{i=0}^{s-2} \bar\beta_i \bar\alpha_i' A_{i+1,k} & \text{otherwise.} \end{cases}$$

Then equation $n \geq j = 0, 1, \ldots$ in system (1.3) can be written as

$$(3.3) \qquad \alpha_j\beta_j' B_{n-j} = P_{a_{j\perp}} \sum_{k=1}^{n-j} A_{j+1,k} B_{n-j-k} + P_{a_{j\perp}} C_{j+1,n-j},$$

where $C_{s,k}$ is defined by the recursions

$$(3.4) \qquad C_{s,k} := \begin{cases} -\delta_{k,m} I & \text{for } s = 1, \\ C_{s-1,k+1} + A_{s-1,1}\sum_{i=0}^{s-2} \bar\beta_i \bar\alpha_i' C_{i+1,k} & \text{otherwise.} \end{cases}$$

Proof. The proof is reached by induction. For $j = 0$, (3.3) reads as $\alpha_0\beta_0' B_n = P_{a_{0\perp}} \sum_{k=1}^n A_{1,k} B_{n-k} + P_{a_{0\perp}} C_{1,n}$; by definition, $A_{1,k} = A_k$, $C_{1,n} = -\delta_{n,m} I$, and $a_0 = b_0 = 0$, which implies $P_{a_{0\perp}} = P_{b_{0\perp}} = I$. Hence (3.3) for $j = 0$ coincides with (2.2). Next assume that (3.3) holds for $j = 0, \ldots, \ell - 1$ for some $\ell > 1$; one wishes to show that it also holds for $j = \ell$. Write (3.3) for $j = \ell - 1$, $\alpha_{\ell-1}\beta_{\ell-1}' B_{n-\ell+1} = P_{a_{\ell-1\perp}} \sum_{k=1}^{n-\ell+1} A_{\ell,k} B_{n-\ell+1-k} + P_{a_{\ell-1\perp}} C_{\ell,n-\ell+1}$, premultiply by $P_{a_{\ell\perp}}$ and rearrange terms to find that

$$(3.5) \qquad 0 = P_{a_{\ell\perp}} A_{\ell,1} B_{n-\ell} + P_{a_{\ell\perp}} \sum_{k=1}^{n-\ell} A_{\ell,k+1} B_{n-\ell-k} + P_{a_{\ell\perp}} C_{\ell,n-\ell+1} := U + V,$$

where $a_\ell := (\alpha_0, \ldots, \alpha_{\ell-1})$, $U := P_{a_{\ell\perp}} A_{\ell,1} B_{n-\ell}$, and V is defined consequently. Next, let $b_\ell := (\beta_0, \ldots, \beta_{\ell-1})$ and use projections, inserting $I = P_{b_\ell} + P_{b_{\ell\perp}}$ between $A_{\ell,1}$ and $B_{n-\ell}$ in U; one finds that

$$U = P_{a_{\ell\perp}} A_{\ell,1} P_{b_{\ell\perp}} B_{n-\ell} + P_{a_{\ell\perp}} A_{\ell,1} P_{b_\ell} B_{n-\ell} =: U_1 + U_2.$$

Substituting $P_{b_\ell} = P_{\beta_0} + \cdots + P_{\beta_{\ell-1}}$, one has $U_2 = P_{a_{\ell\perp}} A_{\ell,1} \sum_{i=0}^{\ell-1} P_{\beta_i} B_{n-\ell}$; by the induction assumption,

$$P_{\beta_i} B_{n-\ell} = \bar{\beta}_i \bar{\alpha}'_i \sum_{k=1}^{n-\ell} A_{i+1,k} B_{n-\ell-k} + \bar{\beta}_i \bar{\alpha}'_i C_{i+1,n-\ell},$$

which is derived from (3.3) replacing n with $n-\ell+j$ and j with i. Substituting in U_2, one finds that

$$U_2 = P_{a_{\ell\perp}} \sum_{k=1}^{n-\ell} \left(A_{\ell,1} \sum_{i=0}^{\ell-1} \bar{\beta}_i \bar{\alpha}'_i A_{i+1,k} \right) B_{n-\ell-k} + P_{a_{\ell\perp}} A_{\ell,1} \sum_{i=0}^{\ell-1} \bar{\beta}_i \bar{\alpha}'_i C_{i+1,n-\ell};$$

hence using (3.2) and (3.4), one has $U_2 + V = P_{a_{\ell\perp}} \sum_{k=1}^{n-\ell} A_{\ell+1,k} B_{n-\ell-k} + P_{a_{\ell\perp}} C_{\ell+1,n-\ell}$ so that substituting the rank factorization $P_{a_{\ell\perp}} A_{\ell,1} P_{b_{\ell\perp}} = -\alpha_\ell \beta'_\ell$ in U_1 and rearranging terms, (3.5) is rewritten as

$$\alpha_\ell \beta'_\ell B_{n-\ell} = P_{a_{\ell\perp}} \sum_{k=1}^{n-\ell} A_{\ell+1,k} B_{n-\ell-k} + P_{a_{\ell\perp}} C_{\ell+1,n-\ell}.$$

This shows that (3.3) holds for $j = \ell$. $\qquad\square$

Remark 3.2 (α_j and β_j). Because rank $P_{a_{j\perp}} A_{j,1} P_{b_{j\perp}} = \text{rank } a'_{j\perp} A_{j,1} b_{j\perp}$, one can replace (3.1) with the rank factorization

(3.6)
$$a'_{j\perp} A_{j,1} b_{j\perp} = -\xi_j \eta'_j, \quad a_j := (\alpha_0, \ldots, \alpha_{j-1}),$$
$$\alpha_j := \bar{a}_{j\perp} \xi_j, \quad b_j := (\beta_0, \ldots, \beta_{j-1}), \quad \beta_j := \bar{b}_{j\perp} \eta_j.$$

This shows that the rank factorizations can be performed on matrices of nonincreasing dimensions, all less than or equal to p. In the following, let $r_j := \text{rank } a'_{j\perp} A_{j,1} b_{j\perp}$ indicate the rank of the jth rank factorization and $r_j^{\max} := p - \sum_{i=0}^{j-1} r_i$ its maximal value.

It is possible that α_j, β_j, $j = 1, 2, \ldots$, in (3.1) have rank r_j equal to 0. In this case, both sides of (3.1) are equal to 0. Note also that, as j increases, $\text{col } a_j$ and $\text{col } b_j$ are nondecreasing and eventually coincide with $\mathbb{C}^p$ for some $j = s$; for subsequent values of j, $j > s$, the left-hand side (LHS) of (3.1) is equal to 0, because the relevant orthogonal complements have dimension 0, and hence all subsequent α_j, β_j are equal to 0.

Finally, observe that β_s is by construction orthogonal to β_j for $j = 0, \ldots, s-1$, i.e., $\beta'_s \beta_j = 0$; as a result, all β_j's are mutually orthogonal. A similar block-orthogonality holds for α_js. This fact is indicated in the following saying that $\{\alpha_j\}$ and $\{\beta_j\}$ are block-orthogonal sets of matrices.

Replacing $n - j$ with n in (3.3), one can equivalently write equation $n + j$ in system (1.3) as

(3.7)
$$\alpha_j \beta'_j B_n = P_{a_{j\perp}} \sum_{k=1}^{n} A_{j+1,k} B_{n-k} + P_{a_{j\perp}} C_{j+1,n};$$

this has the same format as (2.2) and implies that $\beta'_j B_n = \bar{\alpha}'_j \sum_{k=1}^{n} A_{j+1,k} B_{n-k} + \bar{\alpha}'_j C_{j+1,n}$, and hence one has proved the following result.

THEOREM 3.3 (Projection of Laurent coefficients). *One has for any j*

$$(3.8) \qquad P_{\beta_j} B_n = \bar{\beta}_j \bar{\alpha}'_j \sum_{k=1}^{n} A_{j+1,k} B_{n-k} + \bar{\beta}_j \bar{\alpha}'_j C_{j+1,n}.$$

When α_j, β_j are 0, (3.8) gives simply $0 = 0$; hence the interesting cases are for α_j, β_j different from 0. Observe also that in (3.8) the projection $P_{\beta_j} B_n$ is expressed in terms of $B_{n-1}, \ldots, B_0$ and the quantities defined in (3.2), (3.4), and (3.6).

Next, one can show that the rank factorizations in (3.6) deliver complete bases of $\mathbb{C}^p$, namely $(\alpha_0, \ldots, \alpha_s)$ and $(\beta_0, \ldots, \beta_s)$, when $s = m$; this ensures that $\sum_{j=0}^{m} P_{\alpha_j} = \sum_{j=0}^{m} P_{\beta_j} = I$ hold. Each Laurent coefficient B_n can thus be expressed as the sum of $m + 1$ orthogonal projections, $B_n = P_{\beta_0} B_n + \cdots + P_{\beta_m} B_n$, and subsequently reconstructed using (3.8). In what follows, $\mathbb{N} = \{1, 2, \ldots\}$ is the set of positive integers.

THEOREM 3.4 (Rank conditions and order of the pole). *The following statements are equivalent:*

(i) $A(z)^{-1}$ *has a pole of order m, with $m \in \mathbb{N}$, at z_0;*

$$(3.9) \qquad \text{(ii)} \qquad (P_{a_{j\perp}} A_{j,1} P_{b_{j\perp}}) B_0 = \begin{cases} 0 & \text{for } j = 0, 1, \ldots, m-1, \\ P_{a_{m\perp}} & \text{for } j = m; \end{cases}$$

$$(3.10) \qquad \text{(iii)} \qquad \begin{cases} r_j < r_j^{\max} & \text{(reduced rank condition) for } j = 0, 1, \ldots, m-1, \\ r_m = r_m^{\max} & \text{(full rank condition) for } j = m; \end{cases}$$

$$(3.11) \qquad \text{(iv)} \qquad \operatorname{col}(\alpha_0, \ldots, \alpha_m) = \operatorname{col}(\beta_0, \ldots, \beta_m) = \mathbb{C}^p.$$

Condition (iv), *together with the block-orthogonality of $\{\alpha_j\}$ and $\{\beta_j\}$, implies the projection identities*

$$I = \sum_{j=0}^{m} P_{\alpha_j} = \sum_{j=0}^{m} P_{\beta_j}.$$

Proof. (i) $\Rightarrow$ (ii). Using (3.7), write equation j in system (1.3) as $\alpha_j \beta'_j B_0 = P_{a_{j\perp}} C_{j+1,0}$; applying definition (3.4), it is straightforward to verify that

$$(3.12) \qquad C_{s,k} = \begin{cases} 0 & \text{for } s + k < m + 1, \\ -I & \text{for } s + k = m + 1, \end{cases}$$

which implies (3.9) via (3.1).

(ii) $\Rightarrow$ (i). (3.9) implies that the identity is on the RHS of equation m in (1.3), which shows that the order of the pole of the inverse is equal to m.

(ii) $\Rightarrow$ (iii). Substituting (3.1) in (3.9) one finds

$$-\alpha_j \beta'_j B_0 = \begin{cases} 0 & \text{for } j = 0, 1, \ldots, m-1, \\ P_{a_{m\perp}} & \text{for } j = m, \end{cases}$$

which implies $P_{\beta_j} B_0 = 0$ for $j = 0, 1, \ldots, m-1$. If the full rank condition in (3.10) held for some $j = s < m$, one would have $I = P_{\beta_0} + \cdots + P_{\beta_s}$ and hence $B_0 = 0$, which would give a contradiction. This shows that for $j = 1, \ldots, m-1$, r_j must be less than $r_j^{\max}$; i.e., the sequence of reduced rank conditions in (3.9) holds. For $j = m$, because $\operatorname{rank} P_{a_{m\perp}} = \operatorname{rank} P_{b_{m\perp}} = r_m^{\max}$, one has $\operatorname{rank}(P_{a_{m\perp}} A_{m,1} P_{b_{m\perp}}) \le r_m^{\max}$ and from (3.9) one finds that $P_{a_{m\perp}} A_{m,1} P_{b_{m\perp}}$ has maximal rank $r_m^{\max}$; this proves (3.10).

(iii) $\Rightarrow$ (ii). Rewrite the LHS of equation j in (1.3) as $\alpha_j \beta_j' B_0$ as above. The RHS of this rewriting of equation j is either of rank $r_j^{\max}$ or of rank 0. When $r_j < r_j^{\max}$, the RHS of equation j needs to be 0, and this happens for $j < m$. When $r_j = r_j^{\max}$, one must have hit the equation in (1.3) with the identity on the RHS; this happens when $j = m$. This implies (3.9).

(iii) $\Rightarrow$ (iv). By the definition of $r_m^{\max}$, one has $\sum_{j=0}^{m} r_j = p$; moreover, because $\{\beta_j\}$ are linearly independent due to block-orthogonality, this implies (3.11). A similar proof applies replacing $\{\beta_j\}$ with $\{\alpha_j\}$.

(iv) $\Rightarrow$ (iii). If $\mathrm{col}(\beta_0, \dots, \beta_m) = \mathbb{C}^p$, then $\sum_{j=0}^{m} r_j = p$, which implies (3.10). □

Theorems 3.3 and 3.4 allow one to build the recursive formula for the Laurent coefficients, as shown by the following result.

THEOREM 3.5 (Laurent coefficients via extended local rank factorization). *The Laurent coefficients satisfy the recursion*

$$(3.13) \quad B_n = C_n + \sum_{k=1}^{n} D_k B_{n-k}, \quad C_n := \sum_{j=0}^{m} \bar{\beta}_j \bar{\alpha}_j' C_{j+1,n}, \quad D_k := \sum_{j=0}^{m} \bar{\beta}_j \bar{\alpha}_j' A_{j+1,k},$$

where $A_{s,k}$, $C_{s,k}$ are defined in (3.2), (3.4) and α_j, β_j in (3.6); note that $C_0 = -\bar{\beta}_m \bar{\alpha}_m'$.

Proof. Using (3.11), one has $B_n = \sum_{j=0}^{m} P_{\beta_j} B_n$; substituting (3.8) in the last expression and rearranging, one finds (3.13). □

COROLLARY 3.6 (Simplifications of the Laurent coefficients). *One can rewrite (3.13) as*

$$(3.14) \quad B_n = \begin{cases} -\bar{\beta}_m \bar{\alpha}_m' & \text{if } n = 0, \\ C_n + \sum_{k=1}^{g} D_k B_{n-k} & \text{if } 1 \leq n \leq m, \\ \sum_{k=1}^{g} D_k B_{n-k} & \text{otherwise,} \end{cases} \quad g =: \min(n, \deg A(z)).$$

When $A(z)$ is a matrix function, i.e., $\deg A(z) = \infty$, the sums in (3.14) involve n terms; when $A(z)$ is a matrix polynomial, i.e., $\deg A(z) < \infty$, those sums involve at most $\deg A(z)$ terms.

Proof. As a direct consequence of (3.12), applying the definition of C_n in (3.13) one finds that $C_0 = -\bar{\beta}_m \bar{\alpha}_m'$. Moreover, note that if $A(z)$ is a matrix polynomial of degree d, i.e., $A_k = 0$ for $k > d$, then definition (3.2) implies $A_{s,k} = 0$ for $k > d$, which yields $C_{s,k} = 0$ for $k > d$ and hence $C_n = 0$ for $n \geq d + 1$. Finally, observe that definition (3.4) implies $C_{s,k} = 0$ for $k > m$, which yields $C_n = 0$ for $n \geq m + 1$. Collecting these facts together, one can rewrite (3.13) as (3.14). □

4. Local Smith form, extended canonical system of root functions, and Jordan pairs. This section links $\{\alpha_j, \beta_j, r_j, A_{j,k}\}$ with the local Smith form of $A(z)$ at z_0, its extended canonical systems of root functions, and Jordan pairs; see [13] for their definitions. In particular, it is shown that the values of j with $r_j > 0$ provide the distinct partial multiplicities of $A(z)$ at z_0 and r_j gives the number of partial multiplicities that are equal to a given j. Three main results are presented: Theorems 4.2 and 4.3 provide a characterization of the local Smith form of $A(z)$ at z_0 and a construction of an extended canonical system of root functions. As a direct consequence of them, Corollary 4.4 provides a construction of Jordan pairs of $A(z)$ at z_0.

The following notation is instrumental to state these results.

DEFINITION 4.1 (Index sets $\mathcal{J}$, $\mathcal{J}_+$, and $\mathcal{K}$). *Let $\mathcal{J} := (j : r_j > 0)$ be the ordered set that contains the $d := \#\mathcal{J}$ indexes j that correspond to nonzero ranks r_j. Indicate the elements of $\mathcal{J}$ by $(j_1, j_2, \ldots, j_d)$, and fix the reverse ordering $m = j_1 > j_2 > \cdots > j_{d-1} > j_d = 0$. Next, let $\mathcal{J}_+$ be the ordered set that contains only the positive elements of $\mathcal{J}$, i.e.,*

$$\mathcal{J}_+ := (j \in \mathcal{J} : j > 0) = (j_1, j_2, \ldots, j_{d-1}).$$

Finally, let $\mathcal{K}$ be the ordered set that contains each $j \in \mathcal{J}_+$ repeated r_j times and indicate its elements by $(k_1, k_2, \ldots, k_{p-r_0}) := \mathcal{K}$, i.e.,

$$\mathcal{K} := (\underbrace{j_1, \ldots, j_1}_{r_{j_1} \text{ times}}, \underbrace{j_2, \ldots, j_2}_{r_{j_2} \text{ times}}, \ldots, \underbrace{j_{d-1}, \ldots, j_{d-1}}_{r_{j_{d-1}} \text{ times}})$$

$$= (\underbrace{k_1, \ldots, k_{r_{j_1}}}_{\text{equal to } j_1}, \underbrace{k_{r_{j_1}+1}, \ldots, k_{r_{j_1}+r_{j_2}}}_{\text{equal to } j_2}, \ldots \underbrace{k_{\sum_{i=1}^{d-2} r_{j_i}+1}, \ldots, k_{p-r_0}}_{\text{equal to } j_{d-1}}).$$

Note that $\mathcal{J} = (\mathcal{J}_+, 0)$ and that the index set $\mathcal{J}_+$ contains at least one element (equal to m) and at most m elements, $\mathcal{J}_+ = (m, m-1, \ldots, 1)$, i.e., $1 \leq d-1 \leq m$. The index set $\mathcal{K}$ contains $\sum_{j \in \mathcal{J}_+} r_j = p - r_0$ elements. In the following, $(a_j)_{j \in \mathcal{J}}$ indicates $(a_m, \ldots, a_0)$ and $\text{diag}(a_j)_{j \in \mathcal{J}}$ indicates a block diagonal matrix with a_m, ..., a_0 on the main diagonal.

THEOREM 4.2 (Local Smith form and an extended canonical system of root functions). *Define the $p \times r_j$ matrix function*

$$(4.1) \qquad \gamma_j(z) := \left(\beta_j' - \bar{\alpha}_j' \sum_{k=1}^{m-j-1} A_{j+1,k}(z - z_0)^k \right)', \qquad j \in \mathcal{J},$$

and the $p \times p$ matrix functions

$$\Gamma(z) := (\gamma_j(z))'_{j \in \mathcal{J}} = \begin{pmatrix} \gamma_m(z)' \\ \vdots \\ \gamma_0(z)' \end{pmatrix},$$

$$(4.2) \quad \Lambda(z) := \text{diag}((z - z_0)^j I_{r_j})_{j \in \mathcal{J}} = \begin{pmatrix} (z - z_0)^m I_{r_m} & & \\ & \ddots & \\ & & (z - z_0)^0 I_{r_0} \end{pmatrix}.$$

Then

$$(4.3) \quad A(z)\Phi(z) = \widetilde{\Phi}(z)\Lambda(z), \quad \Phi(z) := \Gamma(z)^{-1}, \quad \det \Phi(z_0) \neq 0, \quad \det \widetilde{\Phi}(z_0) \neq 0;$$

i.e., $\Lambda(z)$ is the local Smith form of $A(z)$ at z_0, and $\Phi(z)$ is an extended canonical system of root functions of $A(z)$ at z_0.

Proof. The proof proceeds as follows: first one shows that

$$(4.4) \qquad \gamma_j(z)' A(z)^{-1} = (z - z_0)^{-j} \widetilde{\gamma}_j(z)', \quad j \in \mathcal{J}, \quad \widetilde{\gamma}_j(z_0) \neq 0,$$

where $\gamma_j(z)$ is as defined in (4.1). Then one can rewrite (4.4) as $\Gamma(z)A(z)^{-1} = \Lambda(z)^{-1}\widetilde{\Gamma}(z)$, where $\Gamma(z), \Lambda(z)$ are defined in (4.2) and $\widetilde{\Gamma}(z) := (\widetilde{\gamma}_j(z))'_{j \in \mathcal{J}}$, and finally one can show that $\Gamma(z_0), \widetilde{\Gamma}(z_0)$ are nonsingular. This implies that $A(z)\Gamma(z)^{-1} =$

$\widetilde{\Gamma}(z)^{-1}\Lambda(z)$, and hence (4.3) holds. Thus (see Theorems 1.2 and 1.3 in [13]), $\Lambda(z)$ is the local Smith form of $A(z)$ at z_0 and $\Phi(z)$ is an extended system of root functions of $A(z)$ at z_0.

One hence starts by proving (4.4); write (1.2) as

$$A(z)^{-1} = B_0(z-z_0)^{-m} + \sum_{n=1}^{m-j-1} B_n(z-z_0)^{n-m} + (z-z_0)^{-j}R_0(z), \qquad R_0(z_0) = B_{m-j},$$

and premultiply it by β_j' to find that

$$\beta_j' A(z)^{-1} = \beta_j' B_0(z-z_0)^{-m} + \sum_{n=1}^{m-j-1} \beta_j' B_n(z-z_0)^{n-m} + (z-z_0)^{-j}\beta_j' R_0(z).$$

First, consider $j \neq m$. From (3.14) one has $B_0 = -\bar{\beta}_m\bar{\alpha}_m'$ and hence $\beta_j' B_0 = 0$; then

$$(4.5) \quad \beta_j' A(z)^{-1} = \sum_{n=1}^{m-j-1} \beta_j' B_n(z-z_0)^{n-m} + (z-z_0)^{-j}\beta_j' R_0(z) =: U(z) + V(z),$$

where $U(z)$ is the first sum after the equal sign and

$$(4.6) \qquad\qquad \beta_j' B_n = \bar{\alpha}_j' \sum_{k=1}^{n} A_{j+1,k}B_{n-k} + \bar{\alpha}_j' C_{j+1,n}$$

follows from (3.7). Observe that for $n = 1, \ldots, m-j-1$ one has $j+1+n \leq m$, and hence $C_{j+1,n} = 0$ follows from (3.12); substituting (4.6) and rearranging terms, one can rewrite $U(z)$ as

$$U(z) = \sum_{n=1}^{m-j-1} \left(\bar{\alpha}_j' \sum_{k=1}^{n} A_{j+1,k}B_{n-k} \right)(z-z_0)^{n-m}$$

$$= \bar{\alpha}_j' \sum_{k=1}^{m-j-1} A_{j+1,k} \left(\sum_{n=k}^{m-j-1} B_{n-k}(z-z_0)^{n-m} \right).$$

Next, multiply (1.2) by $(z-z_0)^k$ and write it as

$$(z-z_0)^k A(z)^{-1} = \left(\sum_{n=k}^{m-j-1} B_{n-k}(z-z_0)^{n-m} \right) + (z-z_0)^{-j}R_k(z), \quad R_k(z_0) = B_{m-j-k};$$

then

$$U(z) = \left(\bar{\alpha}_j' \sum_{k=1}^{m-j-1} A_{j+1,k}(z-z_0)^k \right) A(z)^{-1} - (z-z_0)^{-j}\bar{\alpha}_j' \sum_{k=1}^{m-j-1} A_{j+1,k}R_k(z),$$

$$U(z) + V(z) = \left(\bar{\alpha}_j' \sum_{k=1}^{m-j-1} A_{j+1,k}(z-z_0)^k \right) A(z)^{-1}$$

$$+ (z-z_0)^{-j}\left(\beta_j' R_0(z) - \bar{\alpha}_j' \sum_{k=1}^{m-j-1} A_{j+1,k}R_k(z) \right),$$

and (4.5) is rewritten as

$$\gamma_j(z)'A(z)^{-1} = (z - z_0)^{-j}\widetilde{\gamma}_j(z)',$$

where $\gamma_j(z)$ is defined in (4.1) and

$$\widetilde{\gamma}_j(z) := \left(\beta_j' R_0(z) - \bar{\alpha}_j' \sum_{k=1}^{m-j-1} A_{j+1,k} R_k(z) \right)'.$$

Note that

$$\gamma_j(z_0) = \beta_j, \qquad \widetilde{\gamma}_j(z_0) = \left(\beta_j' B_{m-j} - \bar{\alpha}_j' \sum_{k=1}^{m-j-1} A_{j+1,k} B_{m-j-k} \right)',$$

where the last expression follows from $R_k(z_0) = B_{m-j-k}$. Now consider (3.3) for $n = m$ and write

$$\beta_j' B_{m-j} = \bar{\alpha}_j' \sum_{k=1}^{m-j} A_{j+1,k} B_{m-j-k} + \bar{\alpha}_j' C_{j+1,m-j};$$

because $C_{j+1,m-j} = -I$ (see (3.12)), one then finds that

$$\widetilde{\gamma}_j(z_0)' = \bar{\alpha}_j' A_{j+1,m-j} B_0 - \bar{\alpha}_j' = -\bar{\alpha}_j' A_{j+1,m-j}\bar{\beta}_m\bar{\alpha}_m' - \bar{\alpha}_j',$$

where the last expression follows from $B_0 = -\bar{\beta}_m\bar{\alpha}_m'$. This implies $\widetilde{\gamma}_j(z_0)'\alpha_j = -I_{r_j}$, and thus it proves that (4.4) holds for $j \neq m$.

Now consider $j = m$. From (3.14) one has $\beta_m' B_0 = -\bar{\alpha}_m' \neq 0$ and hence $\gamma_m(z)'A(z)^{-1} = (z - z_0)^{-m}\widetilde{\gamma}_m(z)'$, where $\gamma_m(z) := \beta_m$ and $\widetilde{\gamma}_m(z_0) = -\bar{\alpha}_m \neq 0$. This completes the proof of (4.4). Finally, one can show that $\Gamma(z_0), \widetilde{\Gamma}(z_0)$ are nonsingular. Because $\Gamma(z_0) = (\beta_j)'_{j\in\mathcal{J}}$, the first statement follows from (3.11). Because $\widetilde{\gamma}_j(z_0)'\alpha_j = -I_{r_j}$ and $\widetilde{\gamma}_j(z_0)'\alpha_h = 0$ for $h < j \in \mathcal{J}$, $\widetilde{\Gamma}(z_0)(\alpha_j)_{j\in\mathcal{J}}$ is a lower triangular matrix with identities on the main diagonal, and hence it is nonsingular. Because $(\alpha_j)_{j\in\mathcal{J}}$ is nonsingular (see (3.11)), the same holds for $\widetilde{\Gamma}(z_0)$. $\qquad\square$

Theorem 4.2 links the rank structure of $A(z)$ at z_0, derived in Lemma 3.1 and Theorem 3.4, to the local spectral characteristics of $A(z)$ at z_0. In particular, it delivers the structure of the local Smith form, $\Lambda(z)$, and provides an extended canonical system of root functions, $\Phi(z)$. From $\Lambda(z)$ in (4.2) one sees that the distinct partial multiplicities of $A(z)$ at z_0 are equal to the elements of $\mathcal{J}$ and further that there are exactly r_j partial multiplicities for each $j \in \mathcal{J}$. Note that $\mathcal{J}_+$ contains the nonzero partial multiplicities; information on them is also presented in the index set $\mathcal{K}$, which lists them with the corresponding repetitions. That is, full information on the orders and the number of root functions of the same nonzero order in an extended canonical system is available given $\mathcal{K}$.

The next result expresses the coefficients of $\Phi(z)$ in terms of $\{\alpha_j, \beta_j, r_j, A_{j,k}\}$.

THEOREM 4.3 (Coefficients of the extended canonical system of root functions). *Write $\Phi(z)$ in (4.2) as $\Phi(z) = \sum_{n=0}^{\infty} \Phi_n(z - z_0)^n$; then*

$$(4.7) \qquad \Phi_0 = (\bar{\beta}_j)_{j\in\mathcal{J}}, \quad \Phi_n = \sum_{k=1}^{\min(n,m-1)} D_{m-k,k}\Phi_{n-k}, \quad n \geq 1,$$

where $D_{m-k,k} := \sum_{j=0}^{m-k-1} \bar{\beta}_j\bar{\alpha}_j' A_{j+1,k}$.

Proof. From (4.1) one has that the degree of $\gamma_j(z)$ is $m - j - 1$, and thus $\Gamma(z)$ in (4.2) has degree $m - 1$. Writing $\Gamma(z) = \Gamma_0 - \sum_{k=1}^{m-1} \Gamma_k (z - z_0)^k$, one then has

$$\Gamma_0 = (\beta_j)'_{j \in \mathcal{J}} = \begin{pmatrix} \beta'_m \\ \vdots \\ \beta'_0 \end{pmatrix}, \qquad \Gamma_k = \begin{pmatrix} 0 \\ \Psi_k \end{pmatrix},$$

$$\Psi_k := (A'_{j+1,k} \bar{\alpha}_j)'_{j \in \mathcal{J}_k} = \begin{pmatrix} \bar{\alpha}'_{m-k-1} A_{m-k,k} \\ \vdots \\ \bar{\alpha}'_0 A_{1,k} \end{pmatrix},$$

where $\mathcal{J}_k := \{j \in \mathcal{J} : j \leq m - k - 1\}$ and in the last equality it is assumed that $m - k - 1$ belongs to $\mathcal{J}$.

Using the identity $\Gamma(z)\Phi(z) = I$, one finds that

$$\Phi_0 = \Gamma_0^{-1}, \quad \Phi_n = \Gamma_0^{-1} \sum_{k=1}^{\min(n,m-1)} \Gamma_k \Phi_{n-k}, \quad n \geq 1;$$

since $\Gamma_0^{-1} = (\bar{\beta}_j)_{j \in \mathcal{J}}$, one has $\Gamma_0^{-1} \Gamma_k = \sum_{j \in \mathcal{J}_k} \bar{\beta}_j \bar{\alpha}'_j A_{j+1,k}$ and thus the statement. $\square$

Observe that even though the coefficients of $\Phi(z)$ depend on the choice of bases in the rank factorizations, any other choice delivers an alternative extended canonical system of root functions.

Given $\Phi(z)$ and $\mathcal{K}$, one can construct a Jordan pair of $A(z)$ at z_0 as follows.

COROLLARY 4.4 (Jordan pair). *Let $\phi_{i,n}$ be the ith column of Φ_n, and let k_i be the ith element in the index set $\mathcal{K}$; for $i = 1, \ldots, p - r_0$, define*

$$X_i := (\phi_{i,n})_{n=0}^{k_i-1}, \qquad J_{k_i} := \begin{pmatrix} z_0 & 1 & & \\ & \ddots & \ddots & \\ & & z_0 & 1 \\ & & & z_0 \end{pmatrix},$$

respectively, of dimensions $p \times k_i$ and $k_i \times k_i$. Then the columns of X_i form a Jordan chain of maximal length k_i and J_{k_i} is the corresponding Jordan block. Collecting the Jordan chains and the Jordan blocks, respectively, in

$$X := (X_i)_{i=1}^{p-r_0}, \qquad J := \mathrm{diag}(I_{r_j} \otimes J_j)_{j \in \mathcal{J}_+},$$

one has that (X, J) is a Jordan pair of $A(z)$ at z_0.

Proof. The proof is a direct consequence of Theorems 4.2 and 4.3 and the definition of Jordan pairs in [13]. $\square$

5. Algorithmic implementation. This section contains an algorithmic implementation of the ELRF. This is formulated in Definition 5.3. A MATLAB script that implements the ELRF is provided in the supplementary material (99983_01.pdf [local/web 162KB]). The connections with the CRP in [3] are discussed in Theorem 5.10, where it is shown that the CRP coincides with the ELRF, and in Corollary 5.11, which links the characteristics of the CRP to the structure of the local Smith form of $A(z)$ at z_0.

Before describing the ELRF algorithm, it is useful to discuss the implementation of the rank factorizations, which constitute its building blocks. In the following, $\{r, \xi, \eta\}$

are said to be given by the rank factorization of φ when $\varphi = -\xi\eta'$, and ξ and η are full column matrices with r columns. It is also assumed that $\bar{\xi}, \bar{\eta}, \xi_\perp, \eta_\perp, \bar{\xi}_\perp, \bar{\eta}_\perp$ can be simultaneously computed, as illustrated in the following remarks.

Remark 5.1 (Rank factorization via SVD). Several standard matrix procedures can be used to perform the rank factorization $\{r, \xi, \eta\}$ of φ; here computations are illustrated using the singular value decomposition (SVD); see, e.g., [18]. Let $\varphi = USV'$ represent the SVD of φ, where $U'U = V'V = I$ and $S = \mathrm{diag}(s_1^2, \ldots, s_p^2)$, with $s_1^2 \geq \cdots \geq s_p^2 \geq 0$. The rank of φ is numerically computed as the largest integer r for which $s_1^2 \geq \cdots \geq s_r^2 > 0$ and $s_{r+1}^2 = \cdots = s_p^2 = 0$. Given r, one can define $\xi = -U_1$, $\eta = V_1 S_1$, $\xi_\perp = U_2$, $\eta_\perp = V_2$, where $U = (U_1, U_2)$ and $V = (V_1, V_2)$ are partitioned into blocks of the first r columns (with subscript 1) and the last $p - r$ columns (with subscript 2), and $S_1 = \mathrm{diag}(s_1^2, \ldots, s_r^2)$. With this choice, one has $\xi'\xi = I_r$, $\xi'_\perp\xi_\perp = \eta'_\perp\eta_\perp = I_{p-r}$ so that $\bar{\xi} = \xi$, $\bar{\xi}_\perp = \xi_\perp$, $\bar{\eta}_\perp = \eta_\perp$; that is, no matrix inversion is involved when computing the "bar" operation $\bar{\zeta} := \zeta(\zeta'\zeta)^{-1}$ in these cases. Moreover, one has $\bar{\eta} = V_1 \mathrm{diag}(s_1^{-2}, \ldots, s_r^{-2})$. This requires the inversion of the diagonal matrix S_1, which can be computed elementwise. Note that this is one possible choice of bases of the various spaces; this specific choice is convenient because the only matrix inversion involved can be performed elementwise.

Remark 5.2 (Rank factorization via QR). As an alternative to the SVD, one can consider using the QR decomposition; see, e.g., [18]. Let $\varphi = QR$ represent the QR decomposition of φ, where $Q'Q = QQ' = I$ and R is upper triangular. The rank r of φ is numerically computed as the number of nonzero rows of R. Given r, one can define $\xi = -Q_1$, $\xi_\perp = Q_2$, $\eta = R_1$, where $Q = (Q_1, Q_2)$ and $R' = (R_1, R_2)$ are partitioned into blocks of the first r columns (with subscript 1) and the last $p - r$ columns (with subscript 2). With this choice, one has $\xi'\xi = I_r$ and $\xi'_\perp\xi_\perp = I_{p-r}$ so that $\bar{\xi} = \xi$ and $\bar{\xi}_\perp = \xi_\perp$. However, unlike the SVD, the QR decomposition does not return directly $\eta_\perp$, which can be computed using a second QR decomposition to η, namely $\eta = Q^\circ R^\circ$. With this choice, one has $\eta_\perp = Q_2^\circ$, where $Q^\circ = (Q_1^\circ, Q_2^\circ)$ is partitioned into blocks of the first r columns and the last $p - r$ columns, which also satisfies $\eta'_\perp\eta_\perp = I_{p-r}$ so that $\bar{\eta}_\perp = \eta_\perp$. Hence, the same output obtained via the SVD can be computed from the QR decomposition, provided this is applied twice.

The next definition contains an algorithmic implementation of the ELRF.

DEFINITION 5.3 (The ELRF algorithm).
INPUT: *The inputs are the $p \times p$ matrices $\{A_n\}_{n=0}^\infty$ and the number q of Laurent coefficients $B_0, \ldots, B_{q-1}$ to be computed.*[2]
OUTPUT: *The outputs are the scalar m and the Laurent coefficients $B_0, \ldots, B_{q-1}$.*
INITIALIZATION: *Set $j = 0$, $r_0^{\max} = p$, $\mathcal{J}_0 = 0$, and $a_0 = b_0 = 0$. Compute $\{r_0, \xi_0, \eta_0\}$ as the rank factorization of A_0,*

$$(5.1) \qquad\qquad\qquad A_0 = -\xi_0\eta_0',$$

and set $a_1 = \alpha_0 = \xi_0$, $b_1 = \beta_0 = \eta_0$, $\theta_0 = \bar{\beta}_0\bar{\alpha}_0'$. Go to RECURSION.
RECURSION: *If $r_j = r_j^{\max}$, then go to* FINAL LOOPS; *else increase j by 1 and perform all the following computations: Set $r_j^{\max} = r_{j-1}^{\max} - r_{j-1}$, compute $A_{1,j} = A_j$ and $D_{1,j} = \theta_0 A_{1,j}$, and for $s = 2, \ldots, j$ compute $A_{s,j-s+1}$ and $D_{s,j-s+1}$ using*

$$(5.2) \qquad A_{s,k} = A_{s-1,k+1} + A_{s-1,1} D_{s-1,k}, \qquad D_{s,k} = D_{s-1,k} + \theta_{s-1} A_{s,k}.$$

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[2]Because A_j is used in RECURSION j, in practice only a finite number of A_n matrices is needed as input to the algorithm.

Next, calculate $\{r_j, \xi_j, \eta_j\}$ *as the rank factorization of* $a'_{j\perp} A_{j,1} b_{j\perp}$,

$$(5.3) \qquad\qquad\qquad a'_{j\perp} A_{j,1} b_{j\perp} = -\xi_j \eta'_j,$$

where $a_{j\perp} = a_{j-1\perp} \xi_{j-1\perp}$ *and* $b_{j\perp} = b_{j-1\perp} \eta_{j-1\perp}$. *If* $r_j = 0$, *define* $\mathcal{J}_j = \mathcal{J}_{j-1}$, $a_{j+1} = a_j$, $b_{j+1} = b_j$, *and* $\theta_j = 0$; *else (i.e.,* $1 \le r_j \le r_j^{\max}$) *set* $\mathcal{J}_j = (j, \mathcal{J}_{j-1})$, $\alpha_j = \bar{a}_{j\perp} \xi_j$, $\beta_j = \bar{b}_{j\perp} \eta_j$, $\theta_j = \bar{\beta}_j \bar{\alpha}'_j$, $a_{j+1} = (\alpha_j, a_j)$, *and* $b_{j+1} = (\beta_j, b_j)$.
FINAL LOOPS: *Set* $m = j$, $\mathcal{J} = \mathcal{J}_j$, $a = a_{j+1}$, $b = b_{j+1}$ *and compute* $D_{m+1,k}$ *using* (5.2) *for* $k = 1, \ldots, q-1$. *Next, let* $C_{1,k} = -\delta_{k,m} I$, $E_{1,k} = -\theta_0 \delta_{k,m}$ *and compute* $E_{m+1,k}$ *for* $k = 0, \ldots, q-1$ *using the following recursions to compute* $E_{s,m+1+k-s}$ *for* $s = 2, \ldots, m+1$:

$$(5.4) \qquad C_{s,k} = C_{s-1,k+1} + A_{s-1,1} E_{s-1,k}, \qquad E_{s,k} = E_{s-1,k} + \theta_{s-1} C_{s,k}.$$

Finally, use (3.14) *to compute* B_n, $n = 0, \ldots, q-1$, *with* $C_k = E_{m+1,k}$, $D_k = D_{m+1,k}$.

A few remarks are in order.

Remark 5.4 (Extension). The recursion of the ELRF coincide with the LRF of [10]; the extension refers to the final loops, which allow one to calculate the Laurent coefficients B_n, $n = 1, \ldots, q-1$.

Remark 5.5 (Definitions). The matrices in (5.1), (5.2), (5.3), and (5.4) coincide with those defined in Lemma 3.1 and Theorem 4.3.

Remark 5.6 (Order of the pole). The ELRF algorithm determines the order of the pole m by checking the ranks of the $r_j^{\max} \times r_j^{\max}$ matrices $a'_{j\perp} A_{j,1} b_{j\perp}$ in (5.3) until full rank is found. This stopping condition terminates the recursion and determines the order of the pole. This exploits the results in Theorem 3.4.

Remark 5.7 (Dimension of rank factorizations). Successive rank decompositions are performed on matrices of nonincreasing dimension, i.e., $r_j^{\max} \le r_{j-1}^{\max} \le p - r_0$, where p is the dimension of A_n and r_0 is the rank of A_0.

Remark 5.8 (Nonuniqueness of factors in rank factorizations). Because in a rank decomposition the factors are not unique, one of them can be chosen to be orthonormal, for instance the first one as in Remark 5.1. In this case, one has $\xi_j = \bar{\xi}_j$ ($\alpha_j = \bar{\alpha}_j$) so that only $\bar{\eta}_j$ ($\bar{\beta}_j$) needs to be computed. Similarly, because $\xi_{j\perp}$, $\eta_{j\perp}$ are any bases of the orthogonal complements of $\text{col}\,\xi_j$ and $\text{col}\,\eta_j$, one can choose them to be orthonormal. In this case, because $a_{j\perp}$ can also be chosen orthonormal, one finds that $a_{j+1\perp} = a_{j\perp} \xi_{j\perp}$. Similar remarks apply to $\eta_{j\perp}$ and $b_{j\perp}$.
Despite the nonuniqueness of factors in the rank factorizations, the outputs of the ELRF are invariant with respect to the choice of bases. Hence, in general, this nonuniqueness can be exploited in an advantageous way for subsequent calculations.

Remark 5.9 (The ELRF and Moore–Penrose inverses). Note that (5.2) and (5.4) coincide with (3.2) and (3.4). These expressions involve Moore–Penrose inverses of dimension p because $\bar{\beta}_j \bar{\alpha}'_j$ is the Moore–Penrose inverse of $\alpha_j \beta'_j$; see, e.g., Theorem 5 on p. 48 of [6]. However, in practice one can compute $\bar{\beta}_j \bar{\alpha}'_j$ using Moore–Penrose inverses of matrices of dimension $r_j^{\max} \le r_{j-1}^{\max} \le p - r_0$; in fact, because $\bar{\alpha}_j = a_{j\perp} \bar{\xi}_j$ and $\bar{\beta}_j = b_{j\perp} \bar{\eta}_j$, one has $\bar{\beta}_j \bar{\alpha}'_j = b_{j\perp} \bar{\eta}_j \bar{\xi}_j a'_{j\perp}$. Here $\bar{\eta}_j \bar{\xi}'_j$ is the Moore–Penrose inverse of $\xi_j \eta'_j$ of dimension $r_j^{\max}$. It can also be noted that if one performs the rank factorizations as illustrated in Remark 5.1, the only matrix inversion involved in $\bar{\eta}_j \bar{\xi}'_j$ is that of a $r_j^{\max} \times r_j^{\max}$ diagonal matrix, which can be performed elementwise.

Next, attention is turned to the relation between the ELRF and the CRP in [3]. Given that in Theorem 5.10 below the two procedures are shown to coincide, it follows that (3.13) provides the explicit recursive formula to compute the Laurent coefficients when the CRP is performed.

THEOREM 5.10 (CRP and ELRF). *The CRP coincides with the ELRF.*

Proof. Substitute $\alpha_j = \bar{a}_{j\perp}\xi_j$, $\beta_j = \bar{b}_{j\perp}\eta_j$ in (3.7) and premultiply by $a'_{j\perp}$ to find that

$$\xi_j \eta'_j \bar{b}'_{j\perp} B_n = a'_{j\perp} \sum_{k=1}^{n} A_{j+1,k} B_{n-k} + a'_{j\perp} C_{j+1,n};$$

inserting the projection identity $I = P_{b_{j\perp}} + P_{b_j}$ between $A_{j+1,k}$ and B_{n-k}, one then finds that

(5.5) $\quad \xi_j \eta'_j \bar{b}'_{j\perp} B_n = \sum_{k=1}^{n} a'_{j\perp} A_{j+1,k} P_{b_{j\perp}} B_{n-k} + \sum_{k=1}^{n} a'_{j\perp} A_{j+1,k} P_{b_j} B_{n-k} + a'_{j\perp} C_{j+1,n}.$

Note that (5.5) for $j = 0$ gives the original system (1.3). One can next show that, given the system (5.5), the application of one reduction step in the sense of [3] leads to the next matrix rank factorization in the ELRF. This shows that the CRP coincides with the ELRF.

First, observe that (5.5) can be written in the format of equations (8.0)–(8.t) in [3],[3]

(5.6) $$C_0 V_n + \sum_{k=1}^{n} C_k V_{n-k} = R_n,$$

by setting $C_0 = \xi_j \eta'_j$, $C_k = -a'_{j\perp} A_{j+1,k} b_{j\perp}$, $V_n = \bar{b}'_{j\perp} B_n$, and

$$R_n = \sum_{k=1}^{n} a'_{j\perp} A_{j+1,k} P_{b_j} B_{n-k} + a'_{j\perp} C_{j+1,n}.$$

One can next apply a reduction step to (5.6) premultiplying it by a basis of the left null space of C_0 to find that

$$\sum_{k=1}^{n} \xi'_{j\perp} C_k V_{n-k} = \xi'_{j\perp} R_n.$$

Observe that $\xi'_{j\perp} C_k V_{n-k} = -\xi'_{j\perp} a'_{j\perp} A_{j+1,k} P_{b_{j\perp}} B_{n-k} = -a'_{j+1\perp} A_{j+1,k} P_{b_{j+1\perp}} B_{n-k} - a'_{j+1\perp} A_{j+1,k} P_{\beta_j} B_{n-k}$, where the last equality follows by definition from $a_{j+1\perp} = a_{j\perp} \xi_{j\perp}$ and $P_{b_{j\perp}} = P_{b_{j+1\perp}} + P_{\beta_j}$. Rearranging terms and setting $s = n - 1$, one has

$$-a'_{j+1\perp} A_{j+1,1} P_{b_{j+1\perp}} B_s - \sum_{k=1}^{s} a'_{j+1\perp} A_{j+1,k+1} P_{b_{j+1\perp}} B_{s-k} = S_s,$$

where $S_s = \xi'_{j\perp} R_{s+1} + \sum_{k=1}^{s+1} a'_{j+1\perp} A_{j+1,k} P_{\beta_j} B_{s+1-k}$. This can be rewritten in the format of equations (10.0)–(10.$t - 1$) in [3],

(5.7) $$D_0 W_s + \sum_{k=1}^{s} D_k W_{s-k} = S_s,$$

---

[3]Here and in the proof of Corollary 5.11 the letters C and D are used to match the notation in [3], and they do not refer to (3.13) as in the rest of the paper.

where $D_k = -a'_{j+1\perp} A_{j+1,k+1} b_{j+1\perp}$, $W_{s-k} = \bar{b}'_{j+1\perp} B_{s-k}$ for $k = 0, \ldots s$, and $s = 0, \ldots, m-1$. Because the reduced system (5.7) is again reducible if and only if $D_0 = a'_{j+1\perp} A_{j+1,1} b_{j+1\perp}$ is of reduced rank, which is the rank condition in (5.3), this proves that the CRP coincides with the ELRF. $\square$

Finally, the characteristics of the CRP are linked to the structure of the local Smith form of $A(z)$ at z_0. In Theorem 4.2, the latter is shown to be characterized by the ELRF; via Theorem 5.10, the characteristics of the CRP in [3] are thus linked to the structure of the local Smith form.

COROLLARY 5.11 (CRP and structure of the local Smith form). *The CRP consists of $1 \leq d - 1 \leq m$ reduction steps, where d is the number of distinct partial multiplicities of $A(z)$ at z_0 (see Definition 4.1); reduction step $i = 1, \ldots, d-1$ decreases the dimension of the coefficients by $r_{j_{d+1-i}}$, the number of partial multiplicities that are equal to the given value j_{d+1-i}.*

Proof. In the proof of Theorem 5.10 it is shown that dimension of the C_k coefficients of the reducible system (5.6) is $r_j^{\max} \times r_j^{\max}$, where $r_j^{\max} = p - \sum_{i=0}^{j-1} r_i$, and the dimension of the D_k coefficients of the reduced system (5.7) is $r_{j+1}^{\max} \times r_{j+1}^{\max}$, where $r_{j+1}^{\max} = r_j^{\max} - r_j$ and $r_j = \operatorname{rank} C_0$. Hence a reduction occurs if and only if $r_j > 0$, i.e., $j \in \mathcal{J}$, and the dimension of the coefficients is decreased by r_j. Because each and every $j \in \mathcal{J}$ is a partial multiplicity of $A(z)$ at z_0 and there are r_j partial multiplicities that are equal to j (see $\Lambda(z)$ in (4.2)), the statement is proved. $\square$

6. Computational complexity. In this section the computational complexity of the ELRF algorithm is discussed in terms of floating point operations (flops). Because of Theorem 5.10, this corresponds to the computational complexity of the CRP. In particular, this confirms that the flops associated to the one-step reduction process are always greater than or equal to those of the CRP.

The $AB + C$ operation, where A, B, and C are $p \times p$ matrices, requires $O(p^3)$ flops; the same order of complexity holds for the rank decomposition of a $p \times p$ matrix via SVD (see, e.g., p. 18 and p. 254 in [18]) or via QR (see p. 233 in [18]).[4] In each recursion, j operations of the type $AB + C$ are performed to compute $A_{s,k}$ in (5.2) and the same number of $AB + C$ operations is required for $D_{s,k}$ in (5.2). Hence the total number of $AB + C$ operations is $2 \sum_{j=1}^{m} j = m(m + 1)$, corresponding to $O(m^2 p^3)$ flops. The total complexity of the rank decompositions is always less than $O(mp^3)$ flops because it consists of $O(p^3)$ flops for (5.1) and of $O((r_j^{\max})^3)$ flops for (5.3), where $r_j^{\max} \leq p - r_0$.

Next, consider the final loops; each iteration involves $AB + C$ operations to compute $A_{s,k}$ and $D_{s,k}$ in (5.2) and $C_{s,k}$ and $C_{s,k}$ in (5.4). Hence this requires $O(m^2 p^3)$ flops. Because there are $q - 1$ final loops, this leads to a total of $O((q-1)m^2 p^3)$ flops. Summing up, it can be seen that the ELRF computes m and $B_0, \ldots, B_{q-1}$ with $O(qm^2 p^3)$ flops. Note that this complexity is determined by the $AB + C$ operations and not by the matrix rank decompositions. Note also that this estimate does not include the simplifications due to the presence of zero matrices; see Corollary 3.6.

In [3], it is shown that (for known order of the pole) the one-step reduction process computes B_0 with $O(\max\{m^2 p^3, m^3(p - r_0)^3\})$ flops. Hence when $\max\{m^2 p^3, m^3(p - r_0)^3\} = m^2 p^3$, i.e., if $m \leq p^3/(p - r_0)^3$, the computational complexity of the one-step reduction coincides with that of the ELRF (which, however, also provides the order

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[4]For upper Hessenberg matrices, the flops order is lower; see [18, p. 228]. For generality, however, the rest of the section refers to the generic case of unstructured matrices.

of the pole), as can be seen by setting $q = 1$. When $m > p^3/(p - r_0)^3$, there is a computational gain in using the ELRF, i.e., the CRP, with respect to the one-step reduction process. This gain arises because stacking matrices in a potentially large system and performing a Moore–Penrose inverse on it dominates the computational complexity of the $AB + C$ operations as m increases.

7. Example. This section illustrates results using a numerical example; the calculations are performed using the MATLAB implementation of the ELRF provided in the supplementary material (99983_01.pdf [local/web 162KB]). Consider the matrix polynomial

$$A(z) = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A_0} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}}_{A_1} z + \underbrace{\begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}}_{A_2} z^2 + \underbrace{\begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix}}_{A_3} z^3,$$

and observe that $A(0) = A_0$ is singular. The ELRF at $z_0 = 0$ is used to determine the order of the pole of $A(z)^{-1}$ at $z_0 = 0$, to compute the coefficients of its principal part, and to construct the local Smith form and a Jordan pair of $A(z)$ at 0.

INIZIALIZATION delivers $r_0^{\max} = 3$, $\mathcal{J}_0 = 0$, and

$$A_0 = - \underbrace{\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}}_{\xi_0} \underbrace{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}}_{\eta_0'}, \quad a_1 = \alpha_0 = \xi_0, \quad b_1 = \beta_0 = \eta_0,$$

$$\theta_0 = \bar{\beta}_0 \bar{\alpha}_0' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Given that $1 = r_0 < r_0^{\max} = 3$, the counter j is increased to $j = 1$ and RECURSION 1 delivers $r_1^{\max} = 2$,

$$A_{1,1} = A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad D_{1,1} = \theta_0 A_{1,1} = 0,$$

$$a_{1\perp}' A_{1,1} b_{1\perp} = - \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\xi_1} \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\eta_1'}, \quad \mathcal{J}_1 = (1, 0),$$

$$\alpha_1 = \bar{a}_{1\perp} \xi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \beta_1 = \bar{b}_{1\perp} \eta_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \theta_1 = \bar{\beta}_1 \bar{\alpha}_1' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $a_2 = (\alpha_1, \alpha_0)$, $b_2 = (\beta_1, \beta_0)$, $a_{2\perp} = b_{2\perp} = (0, 0, 1)'$. Since $1 = r_1 < r_1^{\max} = 2$, the counter j is incremented to $j = 2$ and RECURSION 2 delivers $r_2^{\max} = 1$,

$$A_{1,2} = A_2 = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad D_{1,2} = \theta_0 A_{1,2} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and because $D_{1,1} = 0$,

$$A_{2,1} = A_{1,2} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \qquad D_{2,1} = \theta_1 A_{2,1} = 0.$$

Next, one finds that $a'_{2\perp} A_{2,1} b_{2\perp} = 0$ and hence $\mathcal{J}_2 = \mathcal{J}_1$, $a_3 = a_2$, $b_3 = b_2$, and $\theta_2 = 0$. Because $0 = r_2 < r_2^{\max} = 1$, the counter is upgraded to $j = 3$ and RECURSION 3 delivers $r_3^{\max} = 1$,

$$A_{1,3} = A_3 = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix}, \qquad D_{1,3} = \theta_0 A_{1,3} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_{2,2} = A_{1,3} + A_{1,1} D_{1,2} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_{2,2} = D_{1,2} + \theta_1 A_{2,2} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$A_{3,1} = A_{2,2} + A_{2,1} D_{2,1} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D_{3,1} = D_{2,1} + \theta_2 A_{3,1} = 0.$$

Hence one finds that $a'_{3\perp} A_{3,1} b_{3\perp} = 0$ so that $\mathcal{J}_3 = \mathcal{J}_2$, $a_4 = a_3$, $b_4 = b_3$, and $\theta_3 = 0$. Because $0 = r_3 < r_3^{\max} = 1$, the counter j is raised to $j = 4$ and RECURSION 4 delivers $r_4^{\max} = 1$, $A_{1,4} = A_4 := 0$, $D_{1,4} = \theta_0 A_{1,4} = 0$,

$$A_{2,3} = A_{1,4} + A_{1,1} D_{1,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}, \quad D_{2,3} = D_{1,3} + \theta_1 A_{2,3} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_{3,2} = A_{2,3} + A_{2,1} D_{2,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad D_{3,2} = D_{2,2} + \theta_2 A_{3,2} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_{4,1} = A_{3,2} + A_{3,1} D_{3,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \qquad D_{4,1} = D_{3,1} + \theta_3 A_{4,1} = 0.$$

Hence one has $a'_{4\perp} A_{4,1} b_{4\perp} = \frac{1}{2}$, $\xi_4 = 1$, $\eta_4 = -\frac{1}{2}$, $\mathcal{J}_4 = (4, 1, 0)$, and

$$\alpha_4 = \bar{a}_{4\perp} \xi_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \beta_4 = \bar{b}_{4\perp} \eta_4 = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \quad \theta_4 = \bar{\beta}_4 \bar{\alpha}'_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$a_5 = (\alpha_4, \alpha_1, \alpha_0)$, $b_5 = (\beta_4, \beta_1, \beta_0)$.

Given that the full rank condition $1 = r_4 = r_4^{\max} = 1$ is reached, the algorithm enters the FINAL LOOPS and defines $m = j = 4$, $\mathcal{J} = \mathcal{J}_4 = (4, 1, 0)$, $a = a_5$, $b = b_5$. Setting $q = m = 4$, one can then compute $D_k = D_{5,k}$ for $k = 1, 2, 3$ and $C_k = E_{5,k}$ for $k = 0, 1, 2, 3$ using (5.2) and (5.4); one finds that

$$D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \qquad D_2 = D_3 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad C_1 = C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence (see Theorem 3.5), $A(z)$ has a pole of order $m = 4$ at $z_0 = 0$ and the coefficients of the principal part of $A(z)^{-1}$ at $z_0 = 0$ are given by

$$B_0 = C_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad B_1 = C_1 + D_1 B_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix},$$

$$B_2 = C_2 + D_1 B_1 + D_2 B_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

$$B_3 = C_3 + D_1 B_2 + D_2 B_1 + D_3 B_0 = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{4} & 0 & -\frac{1}{4} \end{pmatrix}.$$

A direct computation shows that (1.3) is satisfied. Because $m = 4$, $p = 3$, and $p - r_0 = 2$, one has $m > \frac{p^3}{(p-r_0)^3}$, and thus a computational gain arises by performing the complete reduction process instead of the one-step reduction.

Finally, construct the local Smith form and a Jordan pair of $A(z)$ at 0 using Theorems 4.2 and 4.3 and Corollary 4.4. From $\mathcal{J} = (4, 1, 0)$ and $r_4 = r_1 = r_0 = 1$ one has $\mathcal{K} = \mathcal{J}_+ = (4, 1)$; see Definition 4.1.

From (4.2) the local Smith form of $A(z)$ at $z_0 = 0$ is

$$\Lambda(z) := \mathrm{diag}((z - z_0)^j I_{r_j})_{j \in \mathcal{J}} = \begin{pmatrix} z^4 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From Corollary 4.4 one has that a Jordan pair (X, J) of $A(z)$ at $z_0 = 0$ is

$$X := (X_i)_{i=1}^{p-r_0} = (X_1, X_2), \qquad J := \mathrm{diag}(I_{r_j} \otimes J_j)_{j \in \mathcal{J}_+} = \begin{pmatrix} J_4 & 0 \\ 0 & J_1 \end{pmatrix},$$

where

$$X_1 := (\phi_{1,n})_{n=0}^{k_1-1} = (\phi_{1,0}, \phi_{1,1}, \phi_{1,2}, \phi_{1,3}), \qquad J_4 = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix},$$

$$X_2 := (\phi_{2,n})_{n=0}^{k_2-1} = \phi_{2,0}, \qquad J_1 = 0,$$

and $\phi_{i,n}$ is the ith column of Φ_n. Using (4.7), the latter is computed as

$$\Phi_0 = (\bar{\beta}_4, \bar{\beta}_1, \bar{\beta}_0), \qquad \Phi_1 = D_{3,1}\Phi_0,$$
$$\Phi_2 = D_{3,1}\Phi_1 + D_{2,2}\Phi_0, \Phi_3 = D_{3,1}\Phi_2 + D_{2,2}\Phi_1 + D_{1,3}\Phi_0$$

and found to be

$$\Phi_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}, \qquad \Phi_1 = 0, \qquad \Phi_2 = \Phi_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

hence

$$X_1 = (\phi_{1,0}, \phi_{1,1}, \phi_{1,2}, \phi_{1,3}) = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}, \qquad X_2 = \phi_{2,0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Application of Theorem 7.1 on p. 184 in [15] confirms that (X, J) is a Jordan pair of $A(z)$ at 0.

8. Conclusion. The ELRF delivers a recursive formula to compute the order of the pole and the Laurent coefficients of the inverse of a regular analytic matrix function, without stacking coefficients in potentially large linear systems. The procedure consists in performing a finite sequence of rank factorizations of matrices of nonincreasing dimension at most equal to the dimension of the original matrix function.

The sequence of rank factorizations is shown to deliver the partial multiplicities and the number of partial multiplicities of a given value, i.e., the local Smith form of the original matrix function, and to provide a construction of an extended canonical system of root functions of $A(z)$ at z_0. In this way, the structure of Jordan pairs is fully characterized and full information on Jordan chains is available.

Moreover, the CRP in [3] is shown here to coincide with the ELRF; via this equivalence, one has that the number of reductions in the CRP is equal to the number of distinct nonzero partial multiplicities and each reduction step decreases the dimension of the coefficients by the number of partial multiplicities that are equal to a given value. This links the characteristics of the reduction process to the structure of the local Smith form. Finally, it is shown that the computational complexity of the ELRF compares favourably with the one of the one-step reduction process.

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