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The Risk Parity approach to Portfolio Construction

BY

Denis Veliu

Program Coordinator
Prof. Maria B. Chiarolla

Thesis Advisor
Prof. Fabio Tardella

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Introduction

In recent years the financial markets have been afflicted by high volatility, both equity and bonds markets. After Markowitz[1] with his first milestone work in modern portfolio theory, a number of other portfolio optimization models have been proposed in the literature. Sharpe [2] tried to linearize the portfolio optimization model. Konno and Yamazaki [3] introduced the Mean-Absolute Deviation (MAD), a different risk measure giving linear programming model instead of a quadratic programing model. The MiniMax approach, introduced by Young [4], minimizes the worst-case scenario, which is used as risk measure. Other authors introduced methods to quantify market risks, such as $VaR_\alpha(X)$ which is defined as the maximum potential change in value of a portfolio with a given probability over a certain horizon. Risk Management has used this instrument for many years, in order to evaluate the performance and regulatory requirements, and to develop methodologies to provide accurate estimates. This model doesn't allow diversification. There are many works on the alternative risk measure $CVaR_\alpha(X)$[19][20] that show why this is more preferred to $VaR_\alpha(X)$. The most important properties are that $CVaR_\alpha(X)$ is a coherent measure and convex measure[18] which allows diversification.

All these models have one problem in common: they need as an input the estimation of expected return for the assets. Models that need to estimate expected returns produce extreme weights and have significative fluctuation over time. Merton [38] shows that the Mean Variance model is too sensitive to the input parameters, specially to the expected returns. A significant variation of the input parameters can significantly change the composition of the portfolio, like in the Mean Variance portfolio. Models that rely on expected returns tend to be very concentrated on few assets and perform poorly out of sample. The Black&Litterman [41] model can be obtained using a Bayesian approach to change the estimated returns. With the passing of time, more sophisticated and advanced models were developed for the market forecasting. Thus, investors continue to use such simple allocation rules for allocating their capital across assets.

The object of study of my thesis are the models of portfolio selection under the Risk Parity criteria. More attention was focused on these models after the financial crisis of 2008 for the way they distribute the risk among the assets that compose the financial portfolio. The idea was introduced in Qian [24](2005) and it led to the creation of Risk Parity Portfolios, where we allocate an equal amount of risk to stocks and bonds in order to capture long-term risk premium embedded within various assets. Risk Parity portfolios are more efficient than the traditional 60/40 portfolios and they are truly balanced in terms of risk allocation. The first authors to formulate and discuss this argument were Sébastian Maillard, Thiery Roncalli and Jerome Teiletche [36] (2008). They showed that the volatility of Risk Parity is located between that of the minimum variance and of the equally weighted portfolio. Also they prove the uniqueness and the existence of the Risk Parity portfolio.

Risk Parity approaches are frequently used to allocate the risk of a portfolio by decomposing the total portfolio risk into the risk contribution of each component in the same quantity. In the thesis I've discussed the theoretical properties of the model, comparing them with the properties of other models. One of the biggest advantages of the Risk Parity approach is that it does not require the estimation of the expected returns. A crucial point of the thesis is the risk decomposition. Using the properties of the coherent and convex measures defined by Artzner, we can use the Euler decomposition for first order homogeneous functions.

In the Risk Parity models used in the literature, the measure of risk is the standard deviation of the financial portfolio. In the Thesis we show that is possible to apply the Risk Parity approach to a different risk measure, the Conditional Value at Risk ($CVaR_\alpha(X)$), which is a coherent and convex risk measure, that allows to apply the Euler decomposition for first order homogeneous functions. The decomposition requires the calculation of the derivatives of risk measure. In the literature this model is used under the hypothesis that the returns are distributed like a multivariate normal for the calculation of the optimal weights with historical simulation. This hypothesis is less credible due to the lack of reality.

A contribution of the thesis is the Risk Parity model with $CVaR_\alpha(X)$ as a risk measure for any (real) return distribution. This is possible with approximation methods in the calculation of the partial derivatives of the Conditional Value at Risk. We compare the Risk Parity strategies with different risk measures (standard deviation and Conditional Value at Risk). The results are very similar but the time of computation of Risk Parity with Conditional Value at Risk is significantly shorter. Starting from the studies of Colucci (2013)[34], we create a Risk Parity with Conditional Value at Risk which has no true diversification, in order to compare it with Risk Parity with $CVaR_\alpha(X)$. The models have been applied to weekly frequencies in order to have a good approximation of Risk Parity with $CVaR_\alpha(X)$.

In the thesis we developed optimization algorithms in Matlab, which is very effective in the calculation of portfolios with a large number of assets. For the Risk Parity with $CVaR_\alpha(X)$ we use an interior point algorithm with a defined number of iterations.

Since the Risk Parity approach takes into consideration all the assets with the same risk contribution, it is impossible to apply cardinality constraints. Thus, we can select a subset of assets using a criterion like minimum risk. For this we make a two steps selection of the subset of assets: to the group selected by Mean Variance the first step, we apply Risk Parity with standard deviation. We do the same procedure with Conditional Value at Risk and then, with Risk Parity-Conditional Value at Risk. In this way, we create portfolios with less assets but better diversified. In same cases this method of selection has better performance in terms of performance rations and compound return. We compare the Risk Parity methods among them and with the traditional Mean Variance and Conditional Value at Risk methods in terms of diversification using Herfindal and Bera Park indexes.

Structure of the thesis:

1. **Chapter 1:** In the first chapter we discuss the evolution of Modern Portfolio Theory. We show the main models of portfolio choice such as Mean Variance, Value at Risk $VaR_\alpha(X)$, Conditional Value at Risk $CVaR_\alpha(X)$, Mean Absolute Deviation MAD and Minimum Maximum MinMaX portfolio loss. Most of these models can be used with no particular distribution assumption. Some of them, under certain conditions, produce a selection similar to Markowitz. We put them in a chronological order to see the evolution of the Modern Portfolio Theory. We also introduce the theorems necessary for the application of the Risk Parity strategies like the Euler decomposition and the coherent measure defined by Artzner [21]. We discuss the difference between Linear Programing models and Quadratic Programing models. We describe briefly these models showing the strength and the weakness of each one.
   We suggest to pass to Risk Parity models in order to avoid the problems in estimating the expected returns.

2. **Chapter 2:** We introduce the Risk Budgeting Approach starting from the work of Maillard, Teiletche and Roncalli [36][37]. We discuss the properties of a special case of Risk Budgeting Approach, the Risk Parity approach, when the risk budgets are equal for each asset. We also discuss the existence and the uniqueness of the Risk Parity portfolio using a different formulation that leads to the same results. We mention some algorithms that already exist in literature for Risk Parity with standard deviation. We introduce Risk Parity approach to another risk measure, the $CVaR_\alpha(X)$. For this we calculate the derivatives starting from the work of Acerbi and Tasche[23][39], and we make sure that the conditional density of the returns is almost surely differentiable. This is an important step for applying the Euler decomposition for positive homogenous measures. For a comparative method, we introduce Risk Parity with $CVaR_\alpha(X)$ without true diversification[34], and we call it Risk Parity $CVaR_\alpha(X)$ Naive. Another contribution is the about the existence of the Risk Parity $CVaR_\alpha(X)$ portfolio and a special case in which there is no existence. Starting from the continuous case, we also give the conditions for applying the Law of Large Numbers in the numerical approximation for the discrete case. In the last part we describe some of the performance measures that we use in the Empirical Research.

3. **Chapter 3:** In this part of the Thesis we compare the optimization and the performance of the proposed models using groups of stocks that compose the Indexes CAC40, DAX30, Eurostoxx50, FTSE100 and NIKKEI225. We do not include all assets for missing data or interrupted series. The groups are selected with different number of assets in order to study how Risk Parity strategies perform out of sample. We compute Risk Parity with standard deviation, Risk Parity with $CVaR_\alpha(X)$, Risk Parity with CVaR Naive (no true diversification), and the classical Mean Variance and $CVaR_\alpha(X)$ portfolios.

In general we use weekly data, applying a rolling window with in - sample period of 4 past years (L=4 years or 208 weeks) and out of sample period of one month (4 weeks). In the fist part of the chapter we introduce the methodology of the analysis specifying the parameters for each performance measure. In all cases we apply models with no short selling and no leverage. In order to select a smaller subsets of assets and since we can not apply cardinality constraints, we use a different criterion to choose a small subset of the asset with the benefits of Risk Parity strategies.

An important point is the comparison of the diversification and the concentration of the portfolios, with Herfindal Index, Bera Park index, and the number of assets selected by each model.

We also compare the optimization of Bond Portfolio, a special portfolio with commodities and a mixed portfolio with 70% of stock, 24 % Bonds and 6% commodities.

Part I

From Portfolio Optimization To The Risk Parity Approach

Chapter 1

Risk Measures and Portfolio Construction

This Chapter presents a short literature review of the portfolio selection problem. We start with the "The theory of the market portfolio" published in the work of Markowitz (1952) [1]. This is the first milestone work in portfolio optimization, the step that takes in consideration every single asset, not apart but in relation with the other assets. After making the starting assumptions, we give the mathematical expression for the portfolio optimization of the Mean-Variance model.

After Markowitz, a number of other portfolio optimization models have been proposed in the literature which trying to get closer to real-life features like transaction costs, cardinality constraints, and minimum transaction lots. Sharpe [2] tried to linearize the portfolio optimization model. Konno and Yamazaki [3] introduced the Mean-Absolute Deviation (MAD) model with a different risk measure giving linear programming model instead of a quadratic programing model. The MiniMax approach, introduced by Young [4], minimizes the worst-case scenario, which is used as a risk measure. We describe shortly these models showing the strength and the weakness of each one.

The introduction of Value at Risk was a great step in quantifying risk and covering the possible portfolio losses. However, this risk measure is not effective in diversification and has some problems in computation. The Conditional Value at Risk is a better risk measure being coherent and convex and very good in diversification without the need for estimating of the covariance matrix.

The Markowitz and the CVaR models are too concentrated in a small subset of the assets. In other cases, their weights are very high so the portfolios are not stable. One of the most difficult problems to deal with is the estimation of expected returns. Models that need to estimate expected returns produce extreme weights and perform poorly out of sample.

In order to correct these problems, we introduce a method that does not rely on expected returns, so we have to deal with less issues, like the maximization estimation error and instable solution.

There are many other models in literature like, Black-Litterman [41] for example, that will not be described in the Thesis.

1.1 The Mean-Variance model

The theory of the market portfolio has been published for the first time in the work of Markowitz (1952). The Markowitz's paper [1] defines what is portfolio selection:

> "the investor does (or should) consider expected return a desiderate thing and of a variance return an undesirable thing."

An efficient portfolio minimizes the variance for a given level of return. There is only a portfolio that fulfills this condition and it is considered the optimal one. The allocation problem is formulated by quadratic optimization, that is a very useful point. One important fact of this theory is that any added asset to the portfolio must be considered in correlation with the other assets. The investors should decide on a trade-off between risk and expected return of the portfolio.

The Markowitz model assume these hypothesis:

1.The returns are considered as multivariate normal distribution of probabilities in a certain time.

2. Every single operator tries to maximize his expected utility (which is a quadratic).

3. To quantify the risks, we use variance.

4. The investments are only based on the expected return and the expected risk.

The major problem of portfolio selection with the Markowitz model is that it is too sensitive to the input parameters, and in particular to the expected returns. The investors generally use the historical data to calculate the expected return and the standard deviation of the portfolio is used to measure portfolio risk. In reality, returns are not distributed like a multivariate normal, investors do not show a quadratic utility and they do not think for just one period.

The expected return of an asset is defined as the expected price change over the certain time horizon, divided by the beginning price of the asset (the variation should consider the dividend). Markowitz [5] suggested that risk should be measured by the variance of the returns.

To calculate the optimal portfolio, one must define the vector of expected returns of the assets and the covariance matrix of asset returns. Supposing that an investor has to choose a portfolio among of n risky assets. Let $x = (x_1, x_2, x_3.....x_n)^T$ be the vector of the weights, where each weight x_i represents the percentage of the i-th asset held in the portfolio, and let be $R = (r_1, r_2, r_3.....r_n)^T$ the vector returns of a the n assets. So we have:

$$\textstyle\sum_{i=1}^{n} x_i = 1$$

or in another form:

$$x^T e = 1$$

where $e = (1, 1, \cdots, 1)^T$ with dimension $1 \times n$

Under the normal assumption we have the following expressions for the expected return R_P and for the variance σ_P^2 of the portfolio:

$$R_p = \sum_{i=1}^{n} r_i x_i$$
$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{i,j}$$

Or in the matrix form:

$$R_p = x^T R$$
$$\sigma_p^2 = x^T \Omega x$$

Where Ω is the matrix of covariances :

$$\Omega = \begin{pmatrix} \sigma_{1,1} & \cdots & \sigma_{1,n} \\ \vdots & \ddots & \vdots \\ \sigma_{n,1} & \cdots & \sigma_{n,n} \end{pmatrix}$$

Markowitz argued that for any given level of expected return, a rational investor would choose the portfolio with minimum variance from the set of all possible portfolios. The set of all possible portfolios to choose (constructed) from, is called the feasible set, minimum-variance portfolios are called mean-variance efficient portfolios, and the set of all mean-variance efficient portfolios, for different desired levels of expected return, is called the efficient frontier.

So the optimization problem is formulated as follows:

$$\min_x x^T \Omega x \tag{1}$$

$$x^T R = R_p$$
$$x^T e = 1$$

The formulation is known as the classical mean-variance optimization problem with the risk minimization formulation. This problem is a quadratic optimization problem with equality constraints.

This was the first model of Modern Portfolio Theory with his elegant solution. However, there are some weaknesses in this model. First of all, the estimation of the expected returns. Merton [38] shows that the Mean Variance model is too sensitive to the input parameters, specially to the expected returns. Models that rely on expected returns tend to be very concentrated on few assets and perform poorly out of sample. Since it is very concentrated, the portfolio has high turnover for each time of rebalancing.

1.1.1 The efficient frontier of a portfolio with n-risky assets

We now describe the analytical solution for the efficient frontier of a portfolio with n risky assets. We start with the work of Merton[6]. We solve a model where short sales are allowed. Using the same notation of the previous section :

$x = (x_1, x_2, x_3....x_n)^T$ is the vector of weights;

$R = (r_1, r_2, r_3.....r_n)^T$ is the vector of returns;

$$\Omega = \begin{pmatrix} \sigma_{1,1} & \cdots & \sigma_{1,n} \\ \vdots & \ddots & \vdots \\ \sigma_{n,1} & \cdots & \sigma_{n,n} \end{pmatrix} \text{ the covariance matrix;}$$

The feasible set is made of the combination (R_p, σ_p^2), where for each R_p, there is a minimum σ_p^2. So we want to solve the following problem (1):

The Lagrangian for problem (1) can be written as:

$$L(x, \lambda) = x^T \Omega x - 2\lambda_1(x^T R - R_P) - 2\lambda_2(x^T e - 1)$$

We use $-2\lambda_{1,2}$ instead $\lambda_{1,2}$ for easier matrix notation.

The first order conditions for $L(x, \lambda)$ are:

$$\begin{gathered} \frac{\partial L}{\partial x_k} = 0 \ \forall k \longrightarrow \Omega x - \lambda_1 R - \lambda_2 e = 0 \\ \frac{\partial L}{\partial \lambda_1} = 0 \longrightarrow x^T R = R_p \\ \frac{\partial L}{\partial \lambda_2} = 0 \longrightarrow x^T e = 1 \end{gathered}$$

From the first one we obtain :

$$\begin{gathered} \Omega x = \lambda_1 R + \lambda_2 e \\ x^* = \Omega^{-1} (\lambda_1 R + \lambda_2 e) \end{gathered}$$

Thus we obtain the minimum variance of the portfolio:

$$\begin{aligned} (\sigma_P^*)^2 = (x^*)^T \Omega x^* &= \left(\lambda_1 R^T + \lambda_2 e^T\right) \Omega^{-1} \Omega \Omega^{-1} (\lambda_1 R + \lambda_2 e) = \\ &= \lambda_1^2 R^T \Omega^{-1} R + \lambda_2^2 e^T \Omega^{-1} e + \lambda_1 \lambda_2 \left(e^T \Omega^{-1} R + R^T \Omega^{-1} e\right) \end{aligned}$$

Using the constraints form the problem (1), we obtain the following conditions:

$$\begin{gathered} R^T x^* = \lambda_1 R^T \Omega^{-1} R + \lambda_2 R^T \Omega^{-1} e = R_P \\ e^T x^* = \lambda_1 e^T \Omega^{-1} R + \lambda_2 e^T \Omega^{-1} e = 1 \end{gathered}$$

for an elegant solution, substitute:

$$\alpha = R^T \Omega^{-1} R \quad \beta = e^T \Omega^{-1} R \quad \gamma = e^T \Omega^{-1} e$$

And write the previous expression in matrix form:

$$\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} R_P \\ 1 \end{pmatrix}$$

Is easy to show that the matrix $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ is invertible since Ω is positive definite,so $\alpha > 0$ and $\gamma > 0$, and we get $det(A) \neq 0$:

$$A^{-1} = \frac{1}{\alpha\gamma - \beta^2} \begin{pmatrix} \gamma & -\beta \\ -\beta & \alpha \end{pmatrix}$$

Putting together all pieces we have the optimal weights:

$$x^* = \frac{1}{\alpha\gamma-\beta^2}\Omega^{-1}\left[(\gamma R_p - \beta)\,R + (-\beta R_p + \alpha)\,e\right]$$

And then we calculate the minimum variance of the portfolio. After all substitution we have:

$$(\sigma_P^*)^2 = (x^*)^T\Omega x^* = \frac{\gamma R_P^2 - 2\beta R_P + \alpha}{\alpha\gamma-\beta^2}$$

that's a convex function of $R_p$with the minimum variance point:
$Rp = \frac{\beta}{\gamma}$ and $\sigma_P^2 = \frac{1}{\gamma}$
with the following weights:

$$w = \frac{1}{\gamma}\Omega^{-1}e$$

This is an analytical solution to find the weights with Mean Variance portfolio with the constraints of the expected returns.

This is important in case we must know how the weights are connected to the covariance matrix.

1.2 Alternative models for portfolio optimization

In this part of the thesis we introduce other models that are applied to portfolio selections. Most of these models are linear, opposite to Markowitz's [5] which is a quadratic model.

These models come as a critic to the Mean Variance model. Since these models are linear, they are much easier to implement and to manage. We put the models in a chronological order of creation. Of course there are many other models in the literature, but we select the ones that we are going analyze applied in the Thesis. Each model has its own risk measure. Under certain conditions, some of these models have very similar results with to Mean Variance model.

Another critic to the Markowitz's model is the assumption on the return distribution. As we will see in Chapter 3, most of the markets have negative skewness and high kurtosis. Most of the alternative models can by applied without any assumption of the distribution of the returns.

1.2.1 The MAD model

At the end of the 80's Konno [7] introduced a new model of portfolio optimization which uses a different risk measure.

The risk measure can be symmetric or non-symmetric: the first one is quantified in a probability distribution weighted around a specific value (for instance the mean); the second one quantifies the risk in relation with other values that can be chosen from the risk-taker.

Konno used a symmetric measure: The mean absolute deviation from the mean of the portfolio.

The mean absolute deviation at the time t is:

$$m_t = |\sum_{i=1}^{n} x_i r_{it} - \sum_{i=1}^{n} x_i \mu_i|$$

So in this model we try to minimize the risk:

$$\begin{gathered}
\min \frac{1}{T} \sum_{t=1}^{T} m_t \\
\sum_{i=1}^{n} x_i \left(r_{it} - \mu_i\right) \leq m_t \\
\sum_{i=1}^{n} x_i \left(r_{it} - \mu_i\right) \geq -m_t \\
\sum_{i=1}^{n} x_i \mu_i = R_p \\
\sum_{i=1}^{n} x_i = 1 \\
x_i \geq 0 \qquad i = 1, \ldots, n \\
m_t \geq 0 \qquad t = 1, \ldots, T
\end{gathered}$$

where r_{it} is the return of asset i at time t and $\mu_i = \frac{1}{T} \sum_{t=1}^{T} r_{it}$ is the average of the returns of asset i.

In 1991 Konno and Yamazaki[3] showed that this model has the same results of the Markowitz model if the returns are distributed as a multivariate normal variable. The better aspects of this model w.r.t the Mean Variance model are the following:

1. It does not require the covariance matrix of the returns. This fact is helpful if there is a large number of assets.

2. Passing from a Quadratic Optimization Problem to an Linear Optimization Problem is easier to be implemented.

3. It is easier to manage the portfolio if there are few assets.

Simaan [8] discussed the advantages and disadvantages of the MAD model. Ignoring the covariance matrix is more a risk than a benefit for the performance of the portfolio. The risk measure with the Mean Variance model is more effective in small portfolios and for investors who tolerate a low risk level.

1.2.2 The MinMax model

The model created by Young [4] uses as a risk measure the minimum of the returns (loss) of the portfolio instead of the variance of the portfolio. The weights are selected in a way to minimize the maximum loss of the portfolio using the gathered historic information from the time series. In this way the problem of the quadratic utility function can be avoided in the analyses of mean variance in portfolio selection.

The results obtained from the MinMax model, for the normal multivariate distribution data, are similar to a well-diversified portfolio optimized using the Mean Variance model.

This model needs Linear Programming instead of quadratic programing, and is easier to implement. For the construction of the portfolio with n assets and T periods of observed data we use the following:

$$\begin{gathered}
\max_{M_P,x} M_P \\
\sum_{i=1}^{n} x_i r_{it} - M_P \geq 0 \\
\sum_{i=1}^{n} x_i \mu_i \geq G \\
\sum_{i=1}^{n} x_i \mu_i = R_p \\
\sum_{i=1}^{n} x_i = 1 \\
x_i \geq 0 \\
t = 1, \ldots, T \\
i = 1, \ldots, n
\end{gathered}$$

Where :
r_{it} is the return of asset i at time t
$\mu_i = \frac{1}{T} \sum_{i=1}^{T} r_{it}$ is the average return of asset i
$r_{pt} = \sum_{i=1}^{N} x_i r_{it}$ is the return of portfolio at time t
$M_P = \min r_{pt}$ is the minimum return of the portfolio

Note that the short selling are not allowed. We try to maximize the least return M_P in each period with the restriction that the total return of the portfolio must be at least equal to a certain level G.

Alternatively, we can formulate a model for maximizing the expected return of the portfolio with the restriction that the return passes never goes below to a threshold H at any time of observation:

$$\begin{gathered}
\max \sum_{i=1}^{n} x_i \mu_i \\
\sum_{i=1}^{n} x_i r_{it} \geq H \quad t = 1, \ldots T \\
\sum_{i=1}^{n} x_i = 1 \\
x_i \geq 0
\end{gathered}$$

There is no assumption on the distribution of the returns for this model. However, it produces results very similar to the CVaR model for low levels confidence α. This model has low performance for distribution with negative skewness and high kurtosis.

1.2.3 The Value at Risk (VaR) model

Value at Risk $VaR_\alpha(X)$ is a standard measure to quantify market risk for the financial analyst. Value at Risk $VaR_\alpha(X)$ measures the worst expected loss under normal market conditions over a specific time interval at a given confidence level α. Risk Management has used this instrument for many years, in order to evaluate the performance and regulatory requirements, and to develop methodologies to provide accurate estimates. The Basel Committee on Banking Supervision[9] forces to financial institutions such as banks and investment firms to meet capital requirements based on $VaR_\alpha(X)$ estimates. From the statistical point of view, Value at Risk measures the quantile of the distribution of the returns. There are different methods[10] of implementation of the VaR model and most of them differ on the estimation of distribution of the returns:
a) Parametric(RiskMetrics and GARCH)
b) Non parametric (Historical Simulation and the Hybrid model)
c) Semiparametric (Extreme Value Theory and quasi-maximum likelihood GARCH)

The criterion to decide which methodology should be used is based on the underlying assumptions and the financial data. From the work of Fama [10] we can summarize the following observations:

1. Equity returns are typically negatively skewed.

2. Financial return distributions have heavier tails and a higher peak than a normal distribution.

3. Squared returns have significant autocorrelation, i.e volatilities of market factors tend to cluster.

The Parametric method, by definition, requires the estimation of specific parameters for the behavior of the returns[12][13]. These approaches tend to underestimate $VaR_\alpha(X)$. Under the assumption that the residuals are normally distributed, the problem changes into an estimation of the parameters. There are many studies about the distribution of the residuals in different way, after they are defined or fitted, it becomes possible to write down a likelihood function and estimate the unknown parameters. After the variance of a time series is measured, the quantile for $VaR_\alpha(X)$ is obtained usually at 1%, 2% or 5% (for instance the quantile of a standard normal for 1% is 2.33 for 5% is 1.645).

The RiskMetrics[11] approach uses an Exponentially Weighted Moving Average to measure the variance. This calculation is like an Integrated $GARCH$ model and uses the assumption for the standardized residual to be normality distributed. Both these approaches seem not consistent with the behavior of financial returns because of the normally assumption of the standardized residuals. The specification of the variance equation and the distribution chosen for the likelihood or log likelihood may not be appropriate and the standardized residuals may not be i.i.d. but the main purpose of $VaR_\alpha(X)$ is for empirical problems.

The non parametric methods simply use Historical Simulation to compute the VaR and do not make any assumption about the distribution of the returns. The mathematical definition of Value at Risk can be expressed as follow:

Definition 1 *Let X be a random variable. We define the lower $\alpha-$quantile of X by :* $q_\alpha(X) = \inf\{x \in R : P[X \leq x] \geq \alpha\}$

The VaR is defined as the negative of the lower $\alpha-$quantile of distribution: $VaR_\alpha(X) = -q_\alpha(X)$

If the distribution is continuous and strictly increasing, then:

$$VaR_\alpha(X) = -F_X^{-1}(\alpha)$$

where $F_X^{-1}(\alpha)$ gives the inverse function of the distribution of X.

In banking management the Value at Risk $VaR_\alpha(X)$ gives the amount of capital needed as reserve to prevent insolvency that happens with probability α.

Let us now consider a portfolio of n assets whose random returns are described by random vector $\mathbf{R} = (r_1, ..., r_n)'$ have a joint density with finite mean $\mu = E[R]$.

Let $x = (x_{1,...,}x_n)$ be the portfolio weights, so that the total random return of the portfolio is $X = \mathbf{R}'x$. If we assume that the joint distribution of $\mathbf{R}$ is continuous and distributed like a multivariate normal (See Bertsimas [21]), we can give the following definition of $VaR_\alpha(X)$:

$$VaR_\alpha(x) = \mu' x - q_\alpha(X) \tag{1.1}$$

This is a common practice in risk management to center $VaR_\alpha(X)$ at the expected value, so that for the normal distribution is equal to the standard deviation times depending only on α. Also,this can be used in case of parametric estimations of $VaR_\alpha(X)$ with simulated data.

Historical Simulation is based on the concept of rolling windows: first we choose a window of observed data then within this window we sort the returns in ascending order and take the quantile that leaves α% on the left side and $(1-\alpha)$% on the right side. To compute the Value at Risk the next time, the whole window is moved forward by one observation and the entire procedure is repeated. Making no starting assumption will bring several problems. The returns, in this way, do not have the same distribution. Sometimes the Value at Risk based on historical simulation presents predictable jumps caused from the extreme returns.

As we will see in the next section, the Value at Risk is not a coherent risk measure and for this reason diversification is not effective. The homogeneity, monotonicity and translation invariance are clearly satisfied for this risk measure but the subadditivity property is not satisfied [18]. Since that the subadditivity is not satisfied, we can show that $VaR_\alpha(X)$ is a non-convex function and this property causes difficulty in the models.

Another big issue about this way of measuring the portfolio risk is that $VaR_\alpha(X)$ provides a lower bound for losses ignoring potential large losses beyond this limit.

During the last years, a new method[14][15] called the Extreme Value Theory (EVT) has been proposed to estimate $VaR_\alpha(X)$. It can be considered as a complement to the Central Limit Theory. There are two ways to implement EVT: the first one is very similar to the Hill estimator[16] and the second one is based on the concept of exceptions of high thresholds[17].

1.2.4 The Conditional Value at Risk (CVaR) model

The Conditional Value at Risk ($CVaR_\alpha(X)$) has many properties and is a very powerful instrument to quantify risk. The most important properties are that $CVaR_\alpha(X)$ is a coherent measure and a convex function[18]: it is easier to compute w.r.t $VaR_\alpha(X)$. There are many works on CVaR[19][20] that show why it is preferred to $VaR_\alpha(X)$. The Conditional Value at Risk coincides with tail-VaR, expected shortfall or tail loss under suitable assumptions.

Definition 2 *Let X be a random variable. The Conditional Value at Risk can be defined as follow:*
$CVaR_\alpha(X) = -E[(X|X \leq -VaR_\alpha(X)]$ *for* $\alpha \in (0, 1)$

Another way is to describe the Conditional Value at Risk as the mean of the lower $\alpha-$tail distribution of X by the following distribution function:

$$F_X^{\alpha}(x) = \left\{ \begin{array}{c} 0, \quad x < VaR_{\alpha}(X) \\ \frac{\alpha - F(-x)}{\alpha}, \quad x \geq VaR_{\alpha}(X) \end{array} \right\}$$

or in an equivalent way:

$$CVaR_{\alpha}(X) = \frac{1}{\alpha}\int_0^{\alpha} VaR_v(X)dv$$

The Conditional Value at Risk also can be seen as Expected shortfall at level α (Artzner[18]):

$$\begin{aligned} ES_{\alpha}(X) &= -\tfrac{1}{\alpha}\left\{E\left[X1_{\{X\leq -VaR_{\alpha}(X)\}}\right] - VaR_{\alpha}(X)(\alpha - P[X \leq -VaR_{\alpha}(X)]\right\} = \\ &= -\tfrac{1}{\alpha}\left\{E\left[X1_{\{X\leq q(X)\}}\right] + q_{\alpha}(X)(\alpha - P[X \leq q_{\alpha}(X)]\right\} \end{aligned}$$

For $VaR_{\alpha}(X) = -q_{\alpha}(X)$ and $P\left[X \leq q_{\alpha}(X)\right] = \alpha$
Thus, we have:

$$ES_{\alpha}(X) = -\frac{1}{P[X \leq VaR_{\alpha}(X)]}E\left[X1_{\{X\leq -VaR_{\alpha}(X)\}}\right] = -E[(X|X \leq -VaR_{\alpha}(X)]$$
$$= CVaR_{\alpha}(X)$$

For the Parametric model of $CVaR_{\alpha}(x)$ we have to make some assumption for distribution of the returns as in the same case of parametric $VaR_{\alpha}(x)$ (1.1):

$$CVaR_{\alpha}(X) = \mu' x - E[X|X \leq q_{\alpha}(X)] \quad \text{for } \alpha \in (0,1)$$

where $\mu = E[R]$ is the average of the returns.
So for the returns that are normally distributed, we can do the following passages [21]:

$$\begin{aligned} CVaR_{\alpha}(x) = \mu' x - \frac{1}{\alpha\sigma\sqrt{2\pi}}\int_{-\infty}^{q_{\alpha}(X)} x\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx = \\ -\frac{1}{\alpha\sigma\sqrt{2\pi}}\int_{-\infty}^{q_{\alpha}(X)} (x-\mu)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx \\ = -\frac{\sigma}{\alpha\sigma\sqrt{2\pi}}\int_{-\infty}^{z_{\alpha}} y\exp\left(-\frac{y^2}{2}\right)dy = \frac{\phi(z_{\alpha})}{\alpha}\sigma \end{aligned}$$

where $\phi(z_{\alpha})$ is the density of the standard normal and z_{α} is its upper $\alpha-$percentile , that is $P\left\{Z > z_{\alpha}\right\} = \alpha$ and Z is a standard normal; if the returns R are distributed as a multivariate normal distribution with mean μ and covariance matrix Ω then:

$$CVaR_{\alpha}(x) = \frac{\phi(z_{\alpha})}{\alpha}(x'\Omega x)^{1/2}$$

This is very useful because, once you estimate the covariance matrix, you can get the Historical simulated data to measure $CVaR_{\alpha}(x)$. In this thesis we will not use the case of the normally distributed data, for lack of its practical use and for the problems that came for this subjective assumptions.

If the distribution of the returns has a continuous positive density, then the gradient is given by(for more see Bertsimas[21]):

$$\nabla_x CVaR_\alpha(x) = \mu - E[R|X \leq q_\alpha(X)]$$

where $X = R'x$.
And for each every single weight:

$$\frac{\partial CVaR_\alpha(x)}{\partial x_k} = \mu_k - \frac{1}{\alpha}\frac{\partial}{\partial x_k}E[X \quad 1\{X \leq q_\alpha(X)\}]$$

the Hessian of $s_\alpha(x)$ is as follow:

$$\nabla_x^2 CVaR_\alpha(x) = \frac{f_X(q_\alpha(X))}{\alpha}Cov[R|X = q_\alpha(X)]$$

f_X is the probability density of X and $cov[R|X = q_\alpha(X)]$ is the conditional covariance matrix of R. From the Hessian we can convexity of $s_\alpha(x)$.
In optimization the CVaR problem can be described as follow:

$$\begin{gathered}\min CVaR_\alpha(x) \\ \sum_{i=1}^n x_i\mu_i = R_p \\ \sum_{i=1}^n x = 1\end{gathered}$$

In the thesis we are going to use models with no short selling and no leverage. We will discuss in the next chapter the partial derivatives for the $CVaR_\alpha(x)$.

1.3 Risk Measures

1.3.1 Coherence and convexity

We now formalize mathematically the concept of portfolio risk. Let $x = (x_1, x_2, x_3.....x_n)^T$ be the vector of the weights, where each weight x_i represents the percentage of the i-th asset held in the portfolio, and let be $R = (r_1, r_2, r_3.....r_n)^T$ the vector returns of a the n assets.
We denote X the vector of the returns:

$$X = R'x = \sum_{i=1}^n x_i r_i$$

Definition 3 *A risk measure on H is a mapping $\mathcal{R} : H \to \mathbb{R}$. We also define $\mathcal{R}$ also on the set of portfolios $H \subset \mathbb{R}^n$ by setting $\mathcal{R}(x) := \mathcal{R}(R'x)$.*

We call $\mathcal{R}(x)$ the portfolio risk and we interpret this quantity as the amount of capital that should be added to the portfolio x as a reserve in a risk-free asset in order to prevent solvency. The above definition introduces the concept of risk measure as a general real valued function on H. We need to choose a function that satisfies special properties correponding to our financial interpretation of risk.
A typical choice for H is:
$H = \left\{x = (x_1, x_2, x_3.....x_n) \in \mathbb{R}_+^n : x'1 = 1\right\}$
where short selling and leverage are not allowed.
According to Artzner [18] a risk measure $\mathcal{R}$ is coherent if it satisfies a group of properties.

Definition 4 *A risk measure $\mathcal{R} : H \to \mathbb{R}$ is called coherent on H if it satisfies the following properties:*

1. Homogeneity

For all $X \in H$ and for $\lambda > 0$ with $\lambda X \in H$: $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$

(Leveraging (deleveraging) the portfolio increases (decreases) the risk measure in the same proportion.)

2.Monotonicity

For all $X, Y \in H$, if $X \leq Y$ then $\mathcal{R}(X) \geq \mathcal{R}(Y)$

(A bigger return should have a greater risk.)

3.Translation invariance

For all $X \in H$, $m \in \mathbb{R}$: $\mathcal{R}(X + m) = \mathcal{R}(X) - m$

(Adding liquidity m to the portfolio will decrease the risk by the same amount.)

4.Subadditivity:

For all $X, Y \in H$ with $X + Y \in H$

$\mathcal{R}(X + Y) \leq \mathcal{R}(X) + \mathcal{R}(Y)$

(It means that the risk of aggregate of two portfolios should be less then adding the risk of the two separate portfolios.)

With the homogeneity and subadditivity conditions we obtain the convexity property (Follmer and Schied (2002)):

$$\mathcal{R}(\lambda X + (1 - \lambda)Y) \leq \lambda \mathcal{R}(X) + (1 - \lambda)\mathcal{R}(Y)$$

With this condition, diversification must not increase the risk.

There are different risk measures but not all of them are coherent and convex. The most common risk measure is the volatility of the portfolio:

$$\mathcal{R}(x) = \sigma(x)$$

The volatility is not a coherent risk measure because it does not fulfill the third property, which is not well suited for portfolio management and not surely the second.

The loss of a portfolio is defined as $L(x) = -r(x)$ where $r(x)$ is the return of the portfolio.

The Value at Risk :

$$\mathcal{R}(x) = VaR_\alpha(x) = \inf\{l : \Pr\{L(x) \leq l\} \geq \alpha\}$$

The VaR is the $\alpha-$quantile of the distribution $\mathbf{F}$ so:

$$VaR_\alpha(x) = F^{-1}(\alpha)$$

The expected shortfall is the average of the VaR at a certain level (Uryasev [20]):

$$\mathcal{R}(x) = CVaR_\alpha(x) = \tfrac{1}{\alpha}\int_0^\alpha VaR_z(x)dz$$

$VaR_\alpha(x)$ does not have the subadditivity property in general, and for that the diversification is not effective. Thus, it is not a convex measure and for that it is hard to minimize due existence of non global minimum. $CVaR_\alpha(x)$, as we said before, is coherent and convex risk measure. This simplifies the application and optimization of the model and its use in real markets for the diversification of portfolios.

1.3.2 Euler decomposition

An important methodology is based on the decomposition of the total risk of a portfolio into risk contributions of the individual assets in the portfolio. This methodology can be applied to a large class of risk measures if the risk measure fulfills some conditions.

These risk measures $\mathcal{R}$ can be viewed as a function on a subset $H \subset \mathbb{R}^n$ of feasible portfolios.

Definition 5 *A risk measure $\mathcal{R} : H \to \mathbb{R}$ is said to be positive homogeneous of degree α if for all $x \in H, \lambda > 0$ and $\lambda x \in H$ we have $\mathcal{R}(\lambda x) = \lambda^{\alpha}\mathcal{R}(x)$. If $\mathcal{R}$ satisfies the condition with $\alpha = 1$ we say that $\mathcal{R}$ is positive homogeneous.*

The standard deviation, $VaR_{\alpha}(x)$, $CVaR_{\alpha}(x)$ are all positive homogenous with α $= 1$.

Theorem 6 (Euler's Theorem) *Let $\mathcal{R}$ be a positively homogeneous risk measure on H of degree α, and assume that H is an open such that for set for all $x \in H$, $\lambda > 0$ we have $\lambda x \in H$. If $\mathcal{R}$ is continuously differentiable respect to the x_i on H, then we have:*

$\mathcal{R}(x) = \frac{1}{\alpha}\sum_{i=1}^{n} x_i \frac{\partial \mathcal{R}}{x_i}(x)$ *for all* $x \in H$

Proof. *Consider t>0 then $\mathcal{R}(tx)$=$\mathcal{R}(tx_1, tx_2 \ldots, tx_n) \in \mathbb{R}$*

Applying the chain rule for differentiable functions in n variables we get

$\frac{d\mathcal{R}(\lambda x)}{d\lambda} = \sum_{i=1}^{n} \frac{\partial \mathcal{R}(\lambda x)}{\partial x_i} x_i$

Because of the homogeneity the left hand side becomes:

$\frac{d\mathcal{R}(\lambda x)}{d\lambda} = \frac{d}{d\lambda}\left(\lambda^{\alpha}\mathcal{R}(x)\right) = \alpha\lambda^{\alpha-1}\mathcal{R}(x)$

With $\lambda = 1$ we get the required formula. ■

This decomposition is fundamental when considering risk contributions and can be applied to all positive homogenous risk measures of order one. However, we should introduce more assumptions on the distribution of the assets returns in order to ensure differentiability. Each component $x_i \frac{\partial \mathcal{R}}{x_i}(x)$ gives the total risk contribution of each asset and each $\frac{\partial \mathcal{R}}{x_i}(x)$ the marginal contribution. The total risk contribution is the amount of risk contributed to the total risk by investing x_i in asset i. In case of positive homogenous risk measure, the sum of all these contribution gives the total amount of risk $\mathcal{R}(x)$. The marginal risk contribution represents the impact on the overall risk from a small variation in the position invested in i. This crucial part is described in the next Chapter.

Chapter 2

Risk budgeting

2.1 Introduction

The subprime crisis of 2008 has changed the point of view on investment in domestic and national financial firms. The estimation of the expected returns is more difficult during a period of crisis and the investors need to protect the value of the invested portfolios. The risk allocation has become an important aim for the portfolio managers in the investment process, thus focusing on the risk concentration and contribution to total risk of each asset. Different asset classes may have different volatilities and one can achieve diversification by taking equal amount of risk in each asset, rather than equal amount of capital.

Although powerful and elegant, the Markowitz model [1][2] could suffer from some drawbacks. Furthermore optimal portfolios could be excessively concentrated in a limited number of assets. Indeed, it is very sensitive to the input parameters, and in particular to the expected returns[38]. Models that rely on expected returns tends to produce extreme weights and perform poorly out of sample. The Mean Variance and CVaR models have high turnover of the assets. Changing the elements of the portfolio brings more fixed and variable costs of transactions.

All these aspects have given rise to a new research stream that aims at an approach that equalizing the risk contribution of each asset and without relying on expected average returns. We have to distinguish risk minimization and risk diversification: the first tends to get the lowest grade of the risk of the portfolio (the lowest volatility or CVaR) and the second tends to maximize the risk diversification.

We now introduce the risk budgeting approach, and the specifically the Risk Parity model. The idea was introduced in Qian [24] and it led to the construction of Risk Parity portfolios, where they allocate an equal amount of risk to stocks and bonds in order to capture long-term risk premium embedded within various assets. Risk Parity portfolios show better performance in terms of Sharpe ratio than the traditional 60/40 portfolios and they are better balanced in terms of risk allocation.

We start to discuss the work of Maillard, Roncalli and Teiletche [36][37] that apply Risk Parity concept to the standard deviation as risk measure of a portfolio. We recall the theoretical properties of the Risk Parity portfolio, and we show that its volatility is between those of the minimum variance and that of equally weighted portfolios. We also analyze the Risk Parity model from the view point of the optimization, discussing the conditions for the existence, the uniqueness of a solution. In all cases we assume no possibility of leverage and no short selling. For the optimization we introduce efficient algorithms for computing Risk Parity portfolio weights with standard deviation.

In addition we introduce the Risk Parity approach to another risk measure, the Conditional Value-at-Risk $CVaR_\alpha(X)$. We give some recalls regarding the partial derivatives of CVaR, starting from the work of Acerbi and Tasche[23][39]. This is a fundamental step for applying the Euler decomposition to risk measures that are positive homogeneous functions. We also investigate a new naive approach to diversify risk measured by $CVaR_\alpha(X)$. Starting from the continuous case, we also give the conditions for applying the Law of the Large Numbers in the numerical approximation for the discrete case. In the last part we describe some of the performance measures that we will compute in the Empirical Research.

2.2 The risk budgeting approach

We are going to derive the theoretical properties of the risk budgeting portfolios. An important part of this point is showing the existence and uniqueness of the optimal point in the optimization of the portfolio. Starting from the work of Maillard and Roncalli [36][37], we formulate the general case of the Risk Budgeting approach a risk measure.

We create a portfolio with n assets, each weight x_i and $\mathcal{R}(x)$ as a risk measure for the portfolio $x = (x_1, x_{2,...,}x_n)$. Using the Euler decomposition, for positive homogenous risk measures, we know that:

$$\mathcal{R}(x) = \sum_{i=1}^{n} x_i \frac{\partial \mathcal{R}(x)}{\partial x_i}$$

The Risk Budgeting approach uses the following marginal and total risk contribution of each asset:

$$MRC_i(x) = \frac{\partial \mathcal{R}(x)}{\partial x_i}$$
$$TRC_i(x) = x_i \frac{\partial \mathcal{R}(x)}{\partial x_i}$$

We consider the vector of risk budgets of all asset, $b = (b_1, b_{2...,}b_n)$, where b_i is the amount of risk in percentage of the total risk. We set $b_i \geq 0$ and $\sum_{i=1}^{n} b_i = 1$. If $b_i = 0$ it means that the asset has no risk. We do not include risk free assets in our portfolio construction, so each asset will contribute to the total risk.

For a given risk budget b, the mathematical problem for the case with no short selling and no leverage can be summarized as follows:

$$x^* \in \{x \in [0,1] : \sum_{i=1}^{n} x_i = 1, TRC_i(x) = b_i \mathcal{R}(x) \ \forall i\}$$

The difference between a risk budgeting portfolio and an optimized portfolio is that the first one does not try to maximize the utility function and the expected performance of the portfolio but it just considers the risk dimension.

The Risk Parity method is a particular case of risk budgeting when each total risk contribution is equal: in other words when $b_i = b_j = 1/n$

$$TRC_i(x) = TRC_j(x) \quad \forall i, j$$
$$x_i \frac{\partial \mathcal{R}(x)}{\partial x_i} = x_j \frac{\partial \mathcal{R}(x)}{\partial x_j} \quad \forall i, j$$

then :

$$\mathcal{R}(x) = \sum_{i=1}^{n} x_i \frac{\partial \mathcal{R}(x)}{\partial x_i} = \sum_{i=1}^{n} TRC_i(x) = nTRC_i(x)$$

In other words:

$$TRC_i(x) = \frac{\mathcal{R}(x)}{n}$$

In this way the risk is divided in the same proportion for each asset that composes the portfolio. A problem for this model consists in calculating the partial derivative of the risk $\mathcal{R}(x)$ respect to the weights x_i.

The mathematical problem for the Risk parity case can be summarized as follow:

$$x^* \in \{x \in [0,1]^n : \sum_{i=1}^{n} x_i = 1, TRC_i(x) = TRC_j(x), \forall i, j\}$$

In this thesis we will apply Risk Parity to the standard deviation and to Conditional Value at Risk. In both cases we solve the models in equal conditions, starting points in the algorithms and with no short selling or no possibility to leverage.

2.2.1 Risk Parity applied to standard deviation

In the literature, the most common use of Risk Parity is the case with the standard deviation as risk measure.

For a portfolio of n assets and weights $x = (x_1, x_{2,\ldots,}x_n)$ the standard deviation is:

$$\mathcal{R}(x) = \sigma_p(x) = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{i,j}} = \sqrt{x'\Omega x}$$

where Ω is the covariance matrix.

The marginal risk contribution of the i asset :

$$MRC_i(x) = \frac{\partial \sigma_p(x)}{\partial x_i} = \frac{x_i \sigma_i^2 + \sum_{j=1}^{n} x_i \sigma_{i,j}}{\sigma_p(x)} = \frac{(\Omega x)_i}{\sqrt{x'\Omega x}}$$

and the total risk contribution:

$$TRC_i(x) = x_i \frac{\partial \mathcal{R}(x)}{\partial x_i} = x_i \frac{x_i \sigma_i^2 + \sum_{j=1}^{n} x_i \sigma_{i,j}}{\sigma_p(x)} = x_i \frac{(\Omega x)_i}{\sqrt{x'\Omega x}}$$

It is easy to show that:

$$\sum_{i=1}^{n} TRC_i(x) = \sum_{i=1}^{n} x_i \frac{(\Omega x)_i}{\sqrt{x'\Omega x}} = \sqrt{x'\Omega x} = \sigma_p(x)$$

Recall that the solutions Mean Variance model enjoys the following problem:

$$\frac{\partial \sigma_p(x)}{\partial x_i} = \frac{\partial \sigma_p(x)}{\partial x_j}$$

In other words it to equalizes the marginal risk contributions, instead of the total risk contributions as in case of the Risk Parity:

$$TRC_i(x) = TRC_j(x) \quad \forall i, j$$

The Risk Parity model can be formulated as the following optimization problem:

$$x^* = \arg\min_x \sum_{i=1}^{n}\sum_{j=1}^{n}(TRC_i(x) - TRC_j(x))^2 \tag{2}$$

$$\begin{gathered} \textstyle\sum_{i=1}^{n} x_i = 1 \\ x \geq 0 \end{gathered}$$

Since $TRC_i(x) = x_i \frac{\partial \sigma_p(x)}{\partial x_i}$ and $TRC_i(x) = \frac{\sigma_p(x)}{n}$ we can rewrite problem (2) as follow:

$$\begin{gathered} x^* = \arg\min_x \textstyle\sum_{i=1}^{n}(x_i \frac{\partial \sigma_p(x)}{\partial x_i} - \frac{\sigma_p(x)}{n})^2 \\ \textstyle\sum_{i=1}^{n} x_i = 1 \\ x \geq 0 \end{gathered}$$

An important point is proving the existence and, after the uniqueness of the Risk Parity portfolio.

2.2.2 Existence and uniqueness of the Risk Parity Portfolio

An alternative formulation of problem (2) for finding the optimal weights is the following optimization problem:

$$y^* = \arg\min \sqrt{y^T \Omega y} \tag{3}$$

$$\begin{gathered} \textstyle\sum_{i=1}^{n} \ln y_i \geqslant c \\ y_i > 0, \quad i = 1, ..., n \end{gathered}$$

where c is an arbitrary constant. The solution x^* of problem (2) can be obtained with the following scaling of y^*:

$$x_i^* = \frac{y_i^*}{\sum_{i=1}^{n} y_i^*} \tag{4}$$

This equivalent formulation proves that the Risk Parity portfolio exists and is unique when the covariance matrix Ω is positive-definite: Indeed problem (3) requires minimization of a convex quadratic function with convex constraints. For showing the equivalent of the problem (2) and (3) we can use the first order Khun-Tucker conditions for the Lagrangian:

$$L(y, \lambda_c) = \sqrt{y\Omega y} - \lambda_c(\textstyle\sum_{i=1}^{n} \ln(y_i) - c)$$

Thus, the (unique) solution y^* to problem (3) must satisfy following conditions together with an appropriate multiplier λ_c^*:

$$\begin{gathered}\nabla_y L(y^*, \lambda_c^*) = \frac{\Omega y}{\sqrt{y\Omega y}} - \lambda_c^* \left(\frac{1}{y_1^*}, \dots, \frac{1}{y_n^*}\right) = 0 \\ \sum_{i=1}^n \ln(y_i) - c \geqslant 0 \\ \lambda_c^* (\sum_{i=1}^n \ln(y_i) - c) = 0 \ \forall i = 1, \dots, n \\ \lambda_c^* \geqslant 0\end{gathered}$$

Note that if $\lambda_c^* = 0$, it follows that $\Omega y^* = 0$, and from this, $y^* = 0$ which is clearly infeasible. Thus $\lambda_c^* > 0$ and therefore:

$$\frac{\Omega y^*}{\sqrt{y^* \Omega y^*}} = \lambda_c^* \left(\frac{1}{y_1^*}, \dots, \frac{1}{y_n^*}\right) \text{ so that } y_i^* \left(\Omega y_i^*\right) = y_j^* \left(\Omega y_j^*\right) \text{ for all } i, j$$

Thus the nomalized vector x^* obtained from y^* in (4) is the (unique) solution to the Risk Parity problem (2).

If $c = -\infty$, the optimization problem is exactly the MV problem, where the marginal contribution is the same:

$$\frac{\partial \sigma_p(x)}{\partial x_i} = \frac{\partial \sigma_p(x)}{\partial x_j}$$

Using Jensen inequality for the constraint $\sum_{i=1}^n y_i = 1$, we have that $\sum_{i=1}^n \ln y_i \leq -n \ln n$. From this we can see that the only solution is the uniform portfolio $y_i = \frac{1}{n}$. From this we can see that:

$$\sigma_{mv} \leq \sigma_{ERC} \leq \sigma_{\frac{1}{n}}$$

2.2.3 Efficient Algorithms for Computing the Risk Parity Portfolio

In this part we introduce two simple iterative algorithms to calculate the portfolio weights for a risk parity strategy [35].The two iterative algorithms presented here require only simple computations and quickly converge to the optimal solution.

Let $x = (x_1, x_{2,\dots,}x_n)$ be a vector of weights and $R_p = \sum_{i=1}^n x_i r_i = R'x$ be the return of the portfolio, where $R = (r_1, r_2, \dots, r_n)$ is the vector of returns. Then we describe the total risk contribution this way:

$$TRC_i(x) = x_i \frac{\partial \sigma_p(x)}{\partial x_i} = \sum_{j=1}^n x_i x_j \sigma_{i,j} = x_i cov(r_i, R_p)$$

We remember that the Risk Parity portfolio is obtained by equating all total risk contributions:

$$x_i \frac{\partial \sigma_p(x)}{\partial x_i} = x_j \frac{\partial \sigma_p(x)}{\partial x_j} = \lambda \quad \text{(Risk Parity) } \forall i, j$$

We also require no short selling and no possibility to leverage.

$$x_i \geq 0$$
$$x \times e = \sum_{i=1}^{n} x_i = 1$$

So the Risk Parity problem can be described using the $\beta_{i,p}$ of each asset:

$$\begin{array}{ll} x_i cov(r_i, R_p) = x_j cov(r_j, R_p) & \text{dividing both sides by } \sigma_p^2(x) \\ x_i \beta_{ip} = x_j \beta_{jp} & \text{where } \beta_{ip} = \frac{cov(r_i, R_p)}{\sigma_p^2(x)} \end{array}$$

$$\frac{x_i}{\beta_{jp}} = \frac{x_j}{\beta_{ip}}$$

In this way the weights are proportional to the inverse of the corresponding betas:

$$x_i \sim \frac{1}{\beta_{ip}}$$

The best purpose of this setting is making easy the computation as $x_i \beta_{ip} = \frac{1}{n}$, due the fact that $\sum_{i=1}^{n} x_i = 1$ and $\sum_{i=1}^{n} x_i \beta_{ip} = 1$.

To formalize the iterative process just described, we obtain the procedure for the first algorithm with the following steps:

1. Start with an initial portfolio weights $x^{(0)}$ ($x_i = \frac{1}{n}$ for instance) and a stopping criterion ε.
2. Calculate betas for all individual assets, $\beta_{ip}^{(t)}$, with respect to the current portfolio $x^{(t)}$.
3. If the condition :
   $\sqrt{\frac{1}{n-1} \sum_{i=1}^{n} \left(x^{(t)} \beta_{ip}^{(t)} - \frac{1}{n} \right)^2} < \varepsilon$
   is satisfied, stop. If not, calculate the new weights as
   $x_i = \frac{1/\beta_{ip}^{(t)}}{\sum_{i=1}^{n} 1/\beta_{ip}^{(t)}}$
   and go back to step 2.

This method does not have a mathematical proof of convergence to a solution but in many numerical applications one finds that the weights are the right one to guarantee the equal risk contribution.

The algorithms based on covariances are less efficient in terms of computation time, do not guarantee convergence to a solution but are easier to implement using non linear optimization.

The second algorithm is an application of Newton's method for solving a system of nonlinear equations $F(y) = 0$. We can write a linear approximation to this system around any point c using a Taylor expansion:

$$F(y) \approx F(c) + J(c)(y - c),$$

where $J(c)$ represents the Jacobian matrix of $F(y)$ evaluated at point c. For finding a root of the system, we set $F(y) = 0$ and solve for y :

$$y = c - J(c)^{-1} \times F(c)$$

This solution is only an approximation, but iterating the solution of the above equation will get us closer and closer to the exact solution y^* :

$$y^{(t+1)} = y^{(t)} - J(y^{(t)})^{-1} \times F(y^{(t)})$$

And the method converges $y^{(t)} \to y^*$.
We just need to adapt the Newton's method to the Risk Parity problem:

$$F(y) = F(x, \lambda) = \begin{bmatrix} \Omega x - \lambda \frac{1}{x} \\ \sum_{i=1}^{n} x_i - 1 \end{bmatrix} = 0$$
$$J(y) = J(x, \lambda) = \begin{bmatrix} \Omega + \lambda diag(\frac{1}{x^2}) & -\frac{1}{x} \\ e & 0 \end{bmatrix} = 0$$

The following steps illustrate the iterative process just described:

1. Start with an initial portfolio weights $x^{(0)}$ ($x_i = \frac{1}{n}$ for instance) , $\lambda^{(0)}(0 \leq \lambda \leq 1)$ and a stopping criterion ε. Define $y^{(0)} = [x^{(0)}, \lambda^{(0)}]$
2. Calculate $F(y^{(t)})$, $J(y^{(t)})$ and $y^{(t+1)}$.
3. If the condition :

$$\| \, y^{(t+1)} - y^{(t)} \| < \varepsilon$$

is satisfied, stop. If not, go back to step 2.

This method converges faster than the first one and we just have to deal with operations such as inverse matrix. It tends to be more robust, reaching the optimal solution even when the first algorithm fails in particular situations[35]. Both algorithms compute the same "optimal" risk parity solution as the original Maillard, Roncalli and Teiletche [36](2010) equal risk contribution solution using non-linear optimization.

2.3 Risk Parity applied to Conditional Value at Risk

2.3.1 Derivatives of the Conditional Value at Risk

To guarantee the existence of the partial derivatives of CVaR we need to impose some assumptions on the distribution of the random vector $R = (r_1, r_2.....r_n)$. Furthermore, we first deal with the problem of differentiating the quantile function $q_\alpha(X)$, and then we tackle the problem of differentiation CVaR [39].

We present sufficient conditions for quantiles of the portfolio return $X = R'x = \sum_{i=1}^{n} r_i x_i$ to be differentiable respect to the weights x_i . These conditions rely on the existence of a conditional probability density function (pdf) of the i-th asset return r_i given the others.

Definition 7 *For the random vector* $R = (r_1, r_2.....r_n)$, r_1 *has a conditional density given* $(r_2.....r_n)$ *if it exists a measurable function* $\theta : \mathbb{R}^n \longrightarrow [0, \infty)$ *such that for all* $A \in \mathcal{B}(\mathbb{R})$ *we have* $P[r_1 \in A | r_2.....r_n] = \int_A \theta(u, r_2.....r_n) du$

The existence of a joint pdf of R implies the existence of the conditional pdf but not necessarily the vice versa is true.

Lemma 8 *Assume that* r_1 *has a conditional density* θ *given* $(r_2.....r_n)$, *where* $(r_1, ..., r_n)$ *is an* $\mathbb{R}^n-$*valued random vector. For any weight vector* $x = (x_1, ..., x_n) \in \mathbb{R} \backslash \{0\} \times \mathbb{R}^{n-1}$ *we have:*

1. The random variable $X = \sum_{i=1}^{n} r_i x_i$ *has a pdf given by the following absolutely continuos functions*

$$f_X(u) = \frac{1}{x_1} E\left[\theta\left(\frac{u - \sum_{j=2}^{n} r_j x_j}{x_1}, r_2, ..., r_n\right)\right] \tag{2.1}$$

2. If $f_X(u) > 0$ *we have almost surely for* $i = 2, ..., n,$ *and for* $u \in \mathbb{R}$

$$E[r_i | \sum_{j=1}^{n} r_j x_j = u] = \frac{E\left[r_i \theta\left(\frac{1}{x_1}(u - \sum_{j=2}^{n} r_j x_j), r_2, ..., r_n)\right)\right]}{E\left[\theta\left(\frac{1}{x_1}(u - \sum_{j=2}^{n} r_j x_j), r_2, ..., r_n)\right)\right]} \tag{2.1a}$$

3. If $f_X(u) > 0$ *we have almost surely for* $u \in \mathbb{R}$

$$E[r_1 | \sum_{j=1}^{n} r_j x_j = u] = \frac{E\left[\frac{u - \sum_{i=2}^{n} r_i x_i}{x_1}\left(\frac{1}{x_1}(u - \sum_{j=2}^{n} r_j x_j), r_2, ..., r_n)\right)\right]}{E\left[\theta\left(\frac{1}{x_1}(u - \sum_{j=2}^{n} r_j x_j), r_2, ..., r_n)\right)\right]} \tag{2.1b}$$

The point 1 of the Lemma says that if there is a conditional density of r_1 given the other component, then subject of the condition $x_1 \neq 0$ the distribution $X = \sum_{i=1}^{n} r_i x_i$ is absolutely continuous with density of point 1.

Proof. 1. Consider $x_1 > 0$, then we can write:
$P[X \leq u] = E[1_{\{X \leq u\}}] = E\left[E[1_{\{X \leq u\}}] | r_2, ..., r_n\right]$
$= E\left[\int_{-\infty}^{\frac{u - \sum_{j=2}^{n} r_j x_j}{x_1}} \theta\left(v, r_2, ..., r_n\right) dv\right] = E\left[\int_{-\infty}^{u} \frac{1}{x_1} \theta\left(\frac{v - \sum_{j=2}^{n} r_j x_j}{x_1}, r_2, ..., r_n\right) dv\right]$
$= \int_{-\infty}^{u} E\left[\frac{1}{x_1} \theta\left(\frac{v - \sum_{j=2}^{n} r_j x_j}{x_1}, r_2, ..., r_n\right)\right] dv = \frac{1}{x_1} E\left[\theta\left(\frac{u - \sum_{j=2}^{n} r_j x_j}{x_1}, r_2, ..., r_n\right)\right]$
In the last step we apply the Fubini Theorem to change the order of integration. For $x_1 < 0$ we proceed in the same way.

2.

$$E[r_i|X=u] = \frac{E[r_i 1_{\{X\le u\}}]}{P[X=u]} = \lim_{\delta\to 0}\frac{\delta^{-1}E[r_i 1_{\{u<X<u+\delta\}}]}{\delta^{-1}P(u<X<u+\delta)} = \frac{\frac{\partial}{\partial u}E[r_i 1_{X\le u}]}{f_X(u)}\ ,\ \text{where } f_X(u)>0 \tag{2.2}$$

Furthermore, we have:

$$\begin{aligned}\frac{\partial}{\partial u}E[r_i 1_{X\le u}] &= \frac{\partial}{\partial u}E[E[r_i 1_{\{X\le u\}}|, r_2,...,r_n]] = \frac{\partial}{\partial u}E[r_i E[1_{\{X\le u\}}, r_2,...,r_n]] \\ &= \frac{1}{x_1}E\left[r_i\theta(\frac{u-\sum_{j=2}^n r_j x_j}{x_1}), r_2,...,r_n)\right] \qquad (2.3)\end{aligned}$$

Substituting (2.1) and (2.3) in (2.2) we obtain (2.1a)
3. We can write the expression (2.1a) and obtain the (2.1b)

$$E[r_i|X=u] = E[\frac{u-\sum_{j=2}^n r_j x_j}{x_1}|X=u]$$

■

These are possible only for these assumptions of the conditional density θ:

Assumptions 1 :
1.For fixed $r_2,...,r_n$ the mapping $t \mapsto (t, r_2,...,r_n)$ is continuous in t.
2. The map $(t,x) \mapsto E\left[\theta(\frac{u-\sum_{j=2}^n r_j x_j}{x_1}, r_2,...,r_n)\right]$ is finite valued and continuous.
3. For $i=2,...,n$ the mapping $(t,x) \mapsto E\left[r_i\theta(\frac{u-\sum_{j=2}^n r_j x_j}{x_1}, r_2,...,r_n)\right]$ is finite valued and continuous.

Theorem 9 *Assume that the distribution of the returns is such that there exists a conditional density of r_1 given $r_2.....r_n$, satisfying the above Assumptions in some open set $H\subset\mathbb{R}\backslash\{0\}\times\mathbb{R}^{n-1}$ and that $f_X(q_\alpha(x))>0$. Then q_α is partially differentiable with respect to x_i as follows:*

$$\tfrac{\partial q_\alpha}{\partial x_i}(x) = E\left[r_i|R'x = q_\alpha(x)\right]$$

Proof. Applying Lemma 8 the random variable $X=\sum_{i=1}^n r_i x_i$ has a continuos pdf conditional density of r_1 given $(r_2,...,r_n)$ as follow
$f_X(u) = \frac{1}{x_1}E\left[\theta\left(\frac{u-\sum_{j=2}^n r_j x_j}{x_1}, r_2,...,r_n\right)\right] \quad \forall x \text{ with } x_1\ge 0$

$$\alpha = P\left[X\le q_\alpha(x)\right] = E\left[\int_{-\infty}^{\frac{q_\alpha(x)-\sum_{j=2}^n r_j x_j}{x_1}}\theta\left(v, r_2,...,r_n\right)dv\right] \tag{2.4}$$

Differentiating expression (2.4) with respect to x_i for $i = 2, ..., n$, we have:

$$0 = \frac{1}{x_1} E\left[\theta(\frac{q_\alpha(x) - \sum_{j=2}^{n} r_j x_j}{x_1}, r_2, ..., r_n)\right] = f_X(u) \tag{2.5}$$

Solving (2.5) for $\frac{\partial q_\alpha(x)}{\partial x_i}$ and applying the Lemma 8 we find (2.5)

$$\frac{\partial q_\alpha}{\partial x_i}(x) = E\left[r_i | R'x = q_\alpha(x)\right]$$

■

Note that $VaR_\alpha(x) = -q_\alpha(x)$ then we can write:

$$\tfrac{\partial VaR_\alpha}{\partial x_i}(x) = -E\left[r_i | R'x = -VaR_\alpha(x)\right]$$

Applying to $VaR_\alpha(x)$ the Euler decomposition we have:

$$VaR_\alpha(x) = -\textstyle\sum_{i=1}^{n} x_i E\left[r_i | R'x = -VaR_\alpha(x)\right]$$

The calculation of the partial derivatives for the Value at Risk are crucial for the definition of the partial derivatives of the Conditional Value at Risk. Indeed, by definition of $CVaR_\alpha(x)$ we have

$$CVaR_\alpha(x) = \frac{1}{\alpha} \int_0^\alpha VaR_v(x) dv \tag{2.6}$$

Thus, using Assumptions 1 and differentiating (2.6) we obtain that:

$$\begin{aligned} \tfrac{\partial CVaR_\alpha(x)}{\partial x_i} &= \tfrac{1}{\alpha}\textstyle\int_0^\alpha \tfrac{\partial VaR_v(x)}{\partial x_i} dv = -\tfrac{1}{\alpha}\textstyle\int_0^\alpha E\left[r_i | -X = VaR_v(x)\right] dv \\ &= -\tfrac{1}{\alpha}\textstyle\int_0^\alpha E\left[r_i | X = q_v(x)\right] dv = -E\left[r_i | X \leq -VaR_\alpha(x)\right] \end{aligned} \tag{2.7}$$

For the risk measure $ES_\alpha(x)$ the partial derivatives are given by:

$$\frac{\partial ES_\alpha(x)}{\partial x_i} = -\frac{1}{\alpha}\left\{E\left[r_i 1_{\{X \leq q_v(x)\}}\right] + E\left[r_i | R'x = q_\alpha(x)\right](\alpha - P[X \leq q_\alpha(x)])\right\} \tag{2.100}$$

To show this we just apply $ES_\alpha(x) = \frac{1}{\alpha}\int_0^\alpha VaR_v(x)dv$ with the condition for finite values $E[X^-] < \infty$.

$$\begin{aligned} \frac{\partial ES_\alpha(X)}{\partial x_i} &= \frac{1}{\alpha}\int_0^\alpha \frac{\partial VaR_v(X)}{\partial x_i} dv = -\frac{1}{\alpha}\int_0^\alpha E\left[r_i | -X = VaR_v(X)\right] dv \\ &= -\frac{1}{\alpha}\int_0^\alpha E\left[r_i | X = q_v(X)\right] dv \end{aligned}$$

Furthermore,

$$\int_0^{\alpha} f\left(q_v(X)\right) dv = E\left[f\left(X\right)|1_{\{X\le q_v(X)\}}\right] - \int_0^{\alpha} f\left(q_\alpha(X)\right)\left(\alpha - P[X \le -q_\alpha(X)\right) \quad (2.8)$$

Applying $f(x) := E\left[r_i|X = x\right]$ to (2.8) we have:

$$\int_0^{\alpha} E\left[r_i|X = q_v(X)\right] dv = E[E\left[r_i|X\right]]1_{\{X\le q(X)\}}\big]+E\left[r_i|R'x = q_\alpha(x)\right]\left(\alpha - P[X \le -q_\alpha(X)]\right)$$

Using the properties of conditional expectation:

$$E[E\left[r_i|X\right]]1_{\{X\le q(X)\}}\big] = E[E\left[r_i 1_{\{X\le q(X)\}}|X\right]] = E\left[r_i 1_{\{X\le q(X)\}}\right],$$

we obtain the following Expression (2.100) (see also [39,40]

The Total Risk contribution for each asset i of a portfolio is given from the following expression:

$$\begin{aligned} TRC_i^{CVaR}(x) &= x_i \frac{\partial CVaR_\alpha(X)}{\partial x_i} \\ &= -x_i \frac{1}{\alpha}\left\{E\left[r_i 1_{\{X\le q(X)\}}\right] + E\left[r_i|R'x = q_\alpha(X)\right]\left(\alpha - P[X \le -q_\alpha(X)]\right\}\right. \end{aligned}$$

The expression in case of continuous returns distribution is the following:

$$TRC_i^{CVaR}(x) = -x_i E\left[r_i|X \le -VaR_\alpha(X)\right]$$

$$CVaR_\alpha(X) = \sum_{i=1}^{n} TRC_i^{CVaR}(x) = -\sum_{i=1}^{n} x_i E[(R_i|X \le -VaR_\alpha(X)] \quad (2.101)$$

2.3.2 Numerical approximation for estimating VaR and CVaR Risk Parity using Historical Data

In this section compute the VaR and $CVaR$ using historical scenarios of assets returns.

Suppose that the i-th asset return r_i consists of T outcomes r_{ji} with $i = 1, ..., n$ and $j = 1, ..., T$. For each portfolio $x \in \mathbb{R}^n$ where n is the number of the assets in the market, the vector of the observed portfolio returns is $R_p = (r_{p1}, ..., r_{pT})$ where:

$$r_{pj} = x' r^j \text{ with } j = 1, .., T$$

where $r^j = (r_{j1}, ..., r_{jn})$

If the number of observation T is large enough, we can apply the Law of Large Numbers for the numerical approximation of the empirical distribution of the historical portfolio return:

$$P(R_P \leq y) \approx \frac{\#(j=1,...,T|r_{pj} \leq y)}{T}$$

Therefore we compute the VaR and $CVaR$of portfolio returns as follows:

$$VaR_\alpha(x) \approx -r^{sorted}_{p\lfloor \alpha T \rfloor}$$
$$CVaR_\alpha(x) \approx -\frac{1}{\alpha T}\sum_{j=1}^{\lfloor \alpha T \rfloor} r^{sorted}_{pj}$$

where α is a specified significance level and r^{sorted}_{pj} are the sorted portfolio returns that satisfy

$$r^{sorted}_{p1} \leq r^{sorted}_{p2} \leq ...r^{sorted}_{pj} \leq .... \leq r^{sorted}_{p\lfloor \alpha T \rfloor} .$$

Using historical data, from (2.4) the approximation of the partial derivatives $CVaR_\alpha(x)$ for each asset i becomes:

$$\frac{\partial CVaR_\alpha(x)}{\partial x_i} \approx -\frac{1}{\lfloor \alpha T \rfloor}\sum_{k=1}^{\lfloor \alpha T \rfloor} r^{sorted}_{ki} \quad \forall i = 1, ..., n$$

and then the total risk contribution of asset i is

$$TRC_i^{CVaR}(x) = x_i \frac{\partial CVaR_\alpha(x)}{\partial x_i} \approx -\frac{1}{\lfloor \alpha T \rfloor} x_i \sum_{k=1}^{\lfloor \alpha T \rfloor} r^{sorted}_{ki}$$

where r^{sorted}_{ki} are the returns of asset i in the sorted portfolio returns.

2.3.3 The Risk Parity portfolio for the CVaR worse case scenario

In this section we provide a naive method to compute the Risk Parity portfolio weights when CVaR is the risk measure. This method does not require any optimization approach and it uses the CVaR convexity property.

Let us consider the vector of portfolio weights $x = (x_1, x_2, ...x_n)$ and $R = (r_1, r_2, ...r_n)$ the vector of asset returns. Combining the property of sub-additivity and positive homogeneity we obtain the CVaR is a convex function:

$$CVaR(R'x) \leq x_1 CVaR(r_1) + x_2 CVaR(r_2) + ... + x_n CVaR(r_n), \tag{2.200}$$

where the right hand side of the (2.200) represents the CVAR worst case scenario when $x_i \geq 0$ $\sum_{i=1}^n x_i = 1$. Thus we denote the absolute contribution of asset i to the maximum total risk as follows:

$$AC_i = x_i CVaR(r_i) \tag{2.201}$$

Then, the Risk Parity portfolio can be found by the following steps:

1.Start with a uniform portfolio $\frac{1}{n}$;

2.Find the portfolio upper bound risk $CVaR^U = \sum_{i=1}^n x_i CVaR(r_i)$ that corresponds to the worse case scenario;

3. We find the absolute contribution equal for every asset that belongs to the portfolio. For a fixed $CVaR^U$ compute the value of the absolute contribution of each asset in case of equality among the assets:

$$AC^u = \frac{CVaR^u}{n}$$

4. From (2.201) the Risk Parity portfolio weights are obtained by setting:

$$x_i^{**} = \frac{AC^u}{CVaR(r_i)} \quad \forall i = 1, ..., n$$

and normalizing the weights x_i^{**} to get:

$$x_i^* = \frac{x_i^{**}}{\sum_{k=1}^n x_k^{**}}$$

We call this method **Naive Risk Parity CVaR**, as it is not the true diversification.

It is possible to show that the weights of the **Naive Risk Parity CVaR** portfolio are proportional to the inverse of the $CVaR(r_i)$:

$$x_i^* = \frac{AC^u}{CVaR(r_i)} = \frac{CVaR^u}{nCVaR(r_i)}$$

thus normalizing the portfolio weights we obtain:

$$x_i^* = \frac{x_i^{**}}{\sum_{k=1}^n x_k^{**}} = \frac{\frac{CVaR^u}{nCVaR(r_i)}}{\sum_{k=1}^n \frac{CVaR^u}{nCVaR(r_k)}} = \frac{CVaR^{-1}(r_i)}{\sum_{k=1}^n CVaR^{-1}(r_k)}$$

The total risk contribution of asset i for $\sum_{i=1}^n x_i = 1$ and $x \in [0,1]$ is:

$$AC^u = TRC_i^{CVaR}(x) = x_i CVaR(r_i) = \frac{1}{\sum_{k=1}^n CVaR^{-1}(r_k)}$$

The worse case scenario of $CVaR$:

$$CVaR^u = \frac{n}{\sum_{k=1}^n CVaR^{-1}(r_k)}$$

2.3.4 On the existence of the RP-CVaR portfolio

Although in many practical cases the RP-CVaR can be found, its existence is not always guaranteed. We show this result by a counter example.

Let us assume $0 < a_1 \leq a_2 \leq .... \leq a_{n-1} \leq a_n$ and consider a portfolio of two assets with returns $r_1 = (a_1, -a_2, a_3, ..., a_{n-1}, -a_n)$ and $r_2 = (-a_1, a_2, -a_3, ..., -a_{n-1}, a_n)$:

Then $R_P = x_1 \times r_1' + x_2 \times r_2'$ where $x_1 + x_2 = 1$ and $x_1, x_2 \geq 0$

This implies that:

$R_P = [-(1-2x)\, a_1, (1-2x)\, a_2, -(1-2x)\, a_3, ..., -(1-2x)\, a_{n-1}, (1-2x)\, a_n]$

$= [-\beta a_1, \beta a_2, -\beta a_3, ..., -\beta a_{n-1}, \beta a_n]$

where $x_1 = x$ and $x_2 = 1 - x$, and $\beta = 1 - 2x$.
From (2.101) we have:

$$CVaR(R_P) = -x_1 E[(r_1|R_P \leq -VaR_\alpha(R_P)] - x_2 E[(r_2|R_P \leq -VaR_\alpha(R_P)],$$

then the RP-CVaR portfolio can be obtain by imposing that:

$$xE[(r_1|R_P \leq -VaR_\alpha(R_P)] = (1-x)E[(r_2|R_P \leq -VaR_\alpha(R_P)] \text{ with } x \in [0,1]$$

Thus, we have:

$$x_1 = x = \frac{E[(r_2|R_P \leq -VaR_\alpha(R_P)]}{E[(r_1|R_P \leq -VaR_\alpha(R_P)] + E[(r_2|R_P \leq -VaR_\alpha(R_P)]}$$

$$x_2 = 1 - x = \frac{E[(r_1|Rp \leq -VaR_\alpha(R_P)]}{E[(r_1|R_P \leq -VaR_\alpha(R_P)] + E[(r_2|R_P \leq -VaR_\alpha(R_P)]}$$

This mean that to have a solution with $x \in (0,1)$

$$E[(r_1|R_P \leq -VaR_\alpha(R_P)]E[(r_2|R_P \leq -VaR_\alpha(R_P)] > 0.$$

For $0 < x < 0.5 \implies \beta > 0$ the sorted portfolio returns are

$$R_P^{Sort} = [-\beta a_{n-1}, -\beta a_{n-3}, ..., -\beta a_1, \beta a_2, \beta a_4, ..., \beta a_{n-2}, \beta a_n]$$

then

$$E[(r_1|R_P \leq -VaR_\alpha(R_P)] = \tfrac{1}{\lceil \alpha T \rceil}\left(a_{n-1} + a_{n-3} + a_{n-5}, ...\right) > 0$$
$$E[(r_2|R_P \leq -VaR_\alpha(R_P)] = -\tfrac{1}{\lceil \alpha T \rceil}\left(a_{n-1} + a_{n-3} + a_{n-5}, ...\right) < 0$$

For $0.5 < x < 1$ the procedure is the same.
For $x = 0.5$ we have $\beta = 0$ then the sorted portfolio returns are $R_P^{Sort} = [0, 0, ...0]$, then:
$CVaR(R_P) = 0 = -0.5E[(r_1|R_P \leq -VaR_\alpha(R_P)] - 0.5E[(r_2|R_P \leq -VaR_\alpha(R_P)]$
$\implies E[(r_1|R_P \leq -VaR_\alpha(R_P)] = -E[(r_2|R_P \leq -VaR_\alpha(R_P)]$.

2.4 Performance Measures and Diversification Indices

2.4.1 Introduction

In this section we introduce the performance indices necessary for the comparison of the model. Since the introduction of the Sharpe ratio in 1966 [26], a large variety of new measures has appeared in scientific publications. We first present the class of relative measures, then the absolute measures. The last are the general measures based on specific features of the return distribution. We do not consider a few measures that take into account the investor's utility functions.

The ex post comparison of the investment portfolios helps to evaluate the real added value of the managers. Performance measures can have an impact on the inflows of funds and may be used an objective target in some asset allocation problems. The most complete and recent studies are on performance measures those of Aftalion and Poncet [25], and Bacon[22]. This is still an active area of research, and numerous approaches are continuously being created (See Caporin[27]).

2.4.2 Relative Performance Measures

In general terms, the relative performance measures can be expressed in the following way:

$$PM = \frac{E(R_P - r_f)}{\mathcal{R}(R_P - r_f)},$$

where r_f can be the risk free asset or a generic threshold. In the denominator it can be different from the numerator which expresses the performance. The denominator expresses the risk measure selected in each case and can sometimes be subject to corrections. These are often called risk adjusted performance measures since they compare the expected return in excess of a threshold for a unit of risk. It is clear that the portfolio performance is an increasing function of the measured performance and a decreasing function of risk.

The first ratio of this family was developed by Sharpe [26]. The Reward to variability ratio equalize the expected return in excess of the risk free rate over the standard deviation of returns on the same portfolio:

$$S_P = \frac{E(R_P) - r_f}{\sigma_{R_P}}$$

In relation with the Mean-Variance model it shares the same drawbacks under the assumption that the returns are normally distributed. The standard deviation equally weights positive and negative excess returns, so it can be very influential in the portfolio performance.

The Sharpe Ratio induced some authors to develop other so called Sharpe-like Performance Measures. With the introducing the bootstrap methodology, Morey and Vinod [28] created the Double Sharpe Ratio:

$$DS_P = \frac{E(R_P) - r_f}{\sigma_{R_P}} * \sigma_{S_P},$$

where σ_{S_P} is the standard deviation of the Shape ratio, obtained with the bootstrap methodology from a large number of excess returns.

Dowd [29] introduces the reward to Value at Risk ratio that permits to determine the amount of performance for the managed portfolio:

$$S_{VaR_\alpha} = \frac{E(R_P)-r_f}{VaR_\alpha(R_P)}$$

Konno and Yamazaki [4] present another way of measuring performance using the Mean Absolute Deviation (Konno 1988)[3] as a risk measure, a more robust estimator of the scale compared to the standard deviation as we showed in the section 1.2.1:

$$KY_P = \frac{E(R_P)-r_f}{MAD_{R_P}}$$

Caporin and Lisi[30] present a performance measure named Expected Return over Range ratio (ERR):

$$ERR_P = \frac{E(R_P)-r_f}{RG_{R_P}}$$

The Range (RG_{R_P}) of the portfolio is estimated as:

$$RG_{R_P} = \{\max(r_{p,i}) - \min(r_{p,i})\} \quad \text{for} \quad i \in [1\ldots,t]$$

Where $r_{p,i}$ are the portfolio returns over the time period $i \in [1\ldots,t]$ and $\max(r_{p,i})$ and $\min(r_{p,i})$ are respectively the largest and the smallest investor's portfolio returns.

This ratio measures the direct impact of market shocks on the performance of the portfolio. For example, a high value of the range measure will imply a strong sensitivity of the investor's portfolio returns to the market shocks.

Young [4] develops a similar approach to the Mean-Variance portfolio selection, based on a linear programming problem. This model uses minimum return of the investor portfolio rather than variance as a measure of risk to find the optimal portfolio, known as MiniMax portfolio as we showed in the section 1.2.2:

$$YG_P = \frac{E(R_P)-r_f}{MiniMax(R_P)}$$

Remember that this is a linear programming technique, simpler to be applied than the Mean Variance model.

If we really want to use deviations from the average as a measure of "risk", we can avoid returns that are larger than r_f, just measuring the deviation of those returns that are smaller than r_f.More precisely we we take the standard deviation of the returns below r_f:

$$\sigma_d^2 = \frac{1}{n}\sum_{i=1}^{n}\left((r_{p,i}-r_f)_+\right)^2$$

where $(z)_+ = \max(z,0)$

Thus we get the Sortino Ratio, also known as Reward to Lower Partial Moment Ratio:

$$SortR = \frac{E(R_P)-r_f}{\sigma_d}$$

There's a problem with the Sortino ratio. It may happen that no returns are less than r_f in which case $\sigma_d^2 = 0$ and $SortR = \infty$. There are other relative Performance Measure derived from the Sortino Ratio, like Sortino Satchell(2001) which is a lower partial moment of order $q > 1$:

$$SortS = \frac{E(R_P)-r_f}{\left(\frac{1}{n}\sum_{i=1}^{n}\left((r_{p,i}-r_f)_+\right)^q\right)^{\frac{1}{q}}}$$

where q denotes the order of the lower partial moment and the other notation is the same as in the Sortino ratio. The Sortino and Satchell index evaluates the portfolio performance by considering their risk profile if we place a threshold instead of risk free r_f.

Other authors use different measures of risk in the denominator like the Gini Ratio (Yitzhaki [31]), the maximum drawdown (Martin McCann [32]) and other modified versions Young [4].

Since we are not going to use the Absolute Performance Measures, based on Jensen and other Jensen-type measures, we will not describe that in the thesis, but we are introducing some Performance Measure based on the Return Distribution.

2.4.3 Performance Measures based on the Return Distribution

This family of performance measures includes measures based on some general features of the return distribution. This class of performance measures has the following form:

$$PM = \frac{\mathcal{P}^+(R_P)}{\mathcal{P}^-(R_P)}$$

where $\mathcal{P}^+(R_P)$ and $\mathcal{P}^-(R_P)$ denote respectively the right and the left part of the support of the returns density. Measures that belong to this family are based on features of the return distribution, in the first 2 moments or with some quantiles.

The Rachev ratio [43] is a performance measure based on the return distribution. This performance measure is defined as the average of quantiles of the portfolio return distribution that are above a certain target:

$$RaR_{\alpha,\beta} = \frac{CVaR_\alpha(R_P-r_f)}{CVaR_\beta(r_f-R_P)}$$

In the ex post analysis, the Rachev ratio is computed by dividing the corresponding two samples of the $CVaR_\alpha(x)$ and since the performance levels in the Rachev ratio are quantiles of the active returns distribution, they are relative levels as they adjust according to the distribution.

In the same way, Ortobelli [33] introduces two other performance measures based on Drawups-Drawdowns, where Drawups are defined similarly to Drawdowns, focusing on positive returns. The Drawups Ratio is called Rachev Average Drawup-Drawdown ratio and is computed as the average Drawup of the portfolio returns over its average Drawdown. Another ratio is the Rachev Maximum Drawup-Drowdown ratio and is calculated using the maximum operator instead of the average to compute the portfolio performance.

2.4.4 Diversification Measures

In this part of the thesis we introduce some Diversification Measures to compare the models.

For a portfolio $x = (x_1, x_2, ...x_n)$ satisfying the budget constraint $\sum_{i=1}^{n} x_i = 1$ and with short sales not allowed ($x_i \geq 0$). The first naive diversification measure is the Herfindal Index:

$$D_{Her} = 1 - xx',$$

which takes the value 0 if the portfolio is concentrated on one asset and the maximum value $1 - \frac{1}{n}$ for the equally weighted (or naive) portfolio.

For long only strategies $x_i \geq 0$, we introduce the measure proposed by Bera and Park[42]. This diversification measure can be interpreted as the probability of each weight measured in terms of entropy.

$$D_{BP} = -\sum_{i=1}^{n} x_i \log(x_i) = \sum_{i=1}^{n} x_i \log(\tfrac{1}{x_i})$$

The D_{BP} takes value between 0 (fully concentrated on one asset) and $\log(n)$ for the naive portfolio.

Another index of diversification based on the weights that compose the portfolio has been proposed by Hannah and Kay:

$$D_{HK}^{\alpha} = -\left(\sum_{i=1}^{n} x_i^{\alpha}\right)^{\frac{1}{\alpha-1}} \text{ for } \quad \alpha > 0$$

Is easy to verify that $D_{HK}^{2} = D_{Her} - 1$.These three quantities represent diversification only in terms of capital invested and do not take into account that assets contribute differently to the total portfolio volatility.

Another useful index for estimating transaction costs, is the turnover of the portfolio:

$$TO = \sum_{i=1}^{n} |x_i^{t+1} - x_i^{t}|,$$

where x_i^t denotes the weight of asset i at time t.

Part II

Empirical Research

Chapter 3

Risk Parity in the Real Markets

In this part of the thesis we compare the optimization and the performance of the models using groups of stocks that compose the Indices CAC40, DAX30, Eurostoxx50, FTSE100 and NIKKEI225. We choose a period of observation from 1/1/2000 to 4/7/2014 consisting of 756 weeks or 174 months (14.5 years). We do not include all titles because of missing data or interrupted series. The groups are selected with different numbers of assets in order to study how Risk Parity strategies perform out of sample. We compute the Risk Parity with standard deviation, the Risk Parity with $CVaR_\alpha(X)$, the Risk Parity with $CVaR_\alpha(X)$ Naive (no true diversification) and the classical Mean Variance and CVaR portfolios.

In general use weekly data, apply a rolling time window with in sample period of 4 past years (L=4 years or 208 weeks) and out of sample period of one month (4 weeks). In the first part we introduce the methodology of the analysis specifying the parameters for each performance measure.

For the first group of assets of the CAC40 index we compute the numerical solutions of the weights calculated for a large observation period, for instance L=728 weeks in order to have the maximum information of the series for monthly and weekly data.

In all cases we apply models with no short selling and no leverage. Also we do not consider weights smaller than 10^{-8}. We measure the performance in terms of compound returns, and the riskiness comparing the volatility and the $CVaR_\alpha(X)$ at 10% for each model. An important point is the comparison of the diversification and the concentration of the portfolios, with Herfindal Index and Bera Park index and last the number of assets that each model selects.

Since the Risk Parity strategies take into consideration all the assets of the portfolio in a significant way, their performance tend to be between that of the Mean Variance and of the Uniform portfolio. In order to select a smaller subsets of assets and since we can not apply the cardinality constraints for the optimization model, we select a different criterion to choose a small subset of the assets. Proceeding this way we have a smaller group of assets with the benefits of the Risk Parity strategies.

We also consider a case where we have 4 commodities and 4 foreign currencies. We choose this group of assets for the different types of distribution of the returns.

After the crisis of the European sovereign debt, the market has been polluted with uncertain condition; this new conditions bring more volatility to the market of European bonds, in particular to Greece, Portugal and Ireland. We choose a group of 9 bonds with constant maturity in 7 to 10 years for the period from January 2000 to December 2013. We see the reaction of each model during the crisis, and how to allocate the assets in terms of contribution to the risk.

The last example is a mixed portfolio obtained by combining stocks, bonds and commodities. In this way we analyze the allocation of the risk parity strategies in case of different classes of risk. This is a portfolio composed by 70 % of stocks belonging to DAX30, 24% bonds and 3% each gold and silver. The data is provided from data stream THOMSON REUTERS® and they refer to the adjusted closure.

3.1 Structures of the analysis and definition of the indices for the benchmark portfolios

Suppose that the i-th asset return r_{ji} consists of T outcomes with $i = 1, ..., n$ and $j = 1, ..., T$. For each portfolio $x \in \mathbb{R}^n$ where n is the number of the assets in the market, the vector of the observed portfolio returns $R_p = (r_{p1}, ..., r_{pT})$ has components:

$$r_{pj} = x' r^j \text{ with } j = 1, .., T$$

where $r^j = (r_{j1}, ..., r_{jn})$

In the analysis we choose an in-sample period ΔL and an out-of-sample period ΔH which are shorter than the ΔL using, generally, weekly time series.

The holding (or out-of-sample) period represents the investment horizon of the selected portfolio. The mean weekly portfolio return is:

$$\mu(R_p) = \frac{1}{T} \sum_{j=1}^{T} r_{pj}$$

The annualized mean portfolio return for the weekly observation:

$$\mu(R_p)ann = (1 + \mu(R_p))^{52} - 1$$

In this way mean returns are going to be used in order to quantify relationships between portfolio risk and return.

To quantify the total gain of the strategy we compute for $k = 1, ..., T$ the compounded return:

$$\mu_k^c(R_p) = \Pi_{j=1}^{k} (1 + r_{pj}) - 1$$

so that $\mu_T^c(R_p)$ is the compounded return over the whole period (terminal compound return).

As measures of risk we compute the sample volatility, $VaR_a(X)$ and $CVaR_a(X)$, of the weekly returns over the period.

$$\sigma(R_p) = \left(\frac{1}{T} \sum_{j=1}^{T} (r_{pj} - \mu(R_p))^2 \right)^{\frac{1}{2}}$$
$$VaR_\alpha(x) \approx -r_{p\lfloor \alpha T \rfloor}^{sorted}$$
$$CVaR_\alpha(x) \approx -\frac{1}{\alpha T} \sum_{j=1}^{\lfloor \alpha T \rfloor} r_{pj}^{sorted}$$

Note that in order to have a good approximation the tail of the observations αT, we can choose a longer period of estimation, the so called in sample period L, or, in order to reflect better and more recently the fluctuation of the market, a larger α, for instance $\alpha = 10\%$.

These represent weekly risks and can be annualized by multiplying them with $\sqrt{52}$. For the annualized risks we use the notation σ_{ann}, $VaR_\alpha ann$ and $CVaR_\alpha ann$.

We will then consider the following performance ratios: $S_\sigma = \frac{\mu(R_p)ann}{\sigma_{ann}}$, $S_{CVaR_\alpha} = \frac{\mu(R_p)ann}{CVaR_\alpha ann}$, $Svar = \frac{\mu(R_p)ann}{VaR_\alpha ann}$.

At last, we apply the Sortino ratio with risk free rate equal to zero and the Rachev ratio at the confidence level equal to $\alpha = 5\%$.

$$SortR = \frac{\mu(R_p)}{\sigma_d}$$

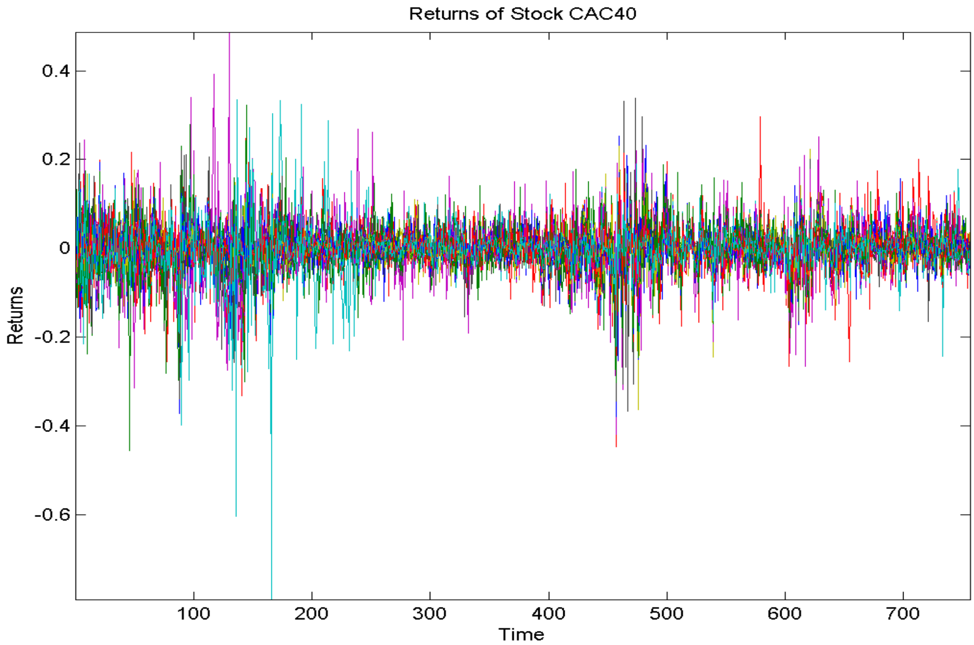
$$RaR_{\alpha,\beta} = \frac{CVaR_\alpha(R_P - r_f)}{CVaR_\beta(r_f - R_P)}$$

3.2 Portfolio optimization for the stocks of CAC40

In this part of the thesis we compare the optimization and the performance of the models using a group of stocks of the Index CAC40, the most widely-used indicator of the Paris market. This Index is composed with 40 largest equities listed in France, measured by free-float market capitalization and liquidity. We are going to choose a period of observation from 1/1/2000 to 4/7/2014 which in frequencies are 756 weeks or 174 months (14.5 years). We choose 32 stock from 40 for missing data. So we do not include the following group of stocks:

L'Air Liquide SA
Credit Agricole S.A.
Electricite de France SA
GDF SUEZ S.A.
Gemalto NV
Legrand SA
Unibail-Rodamco SE
Veolia Environnement S.A.

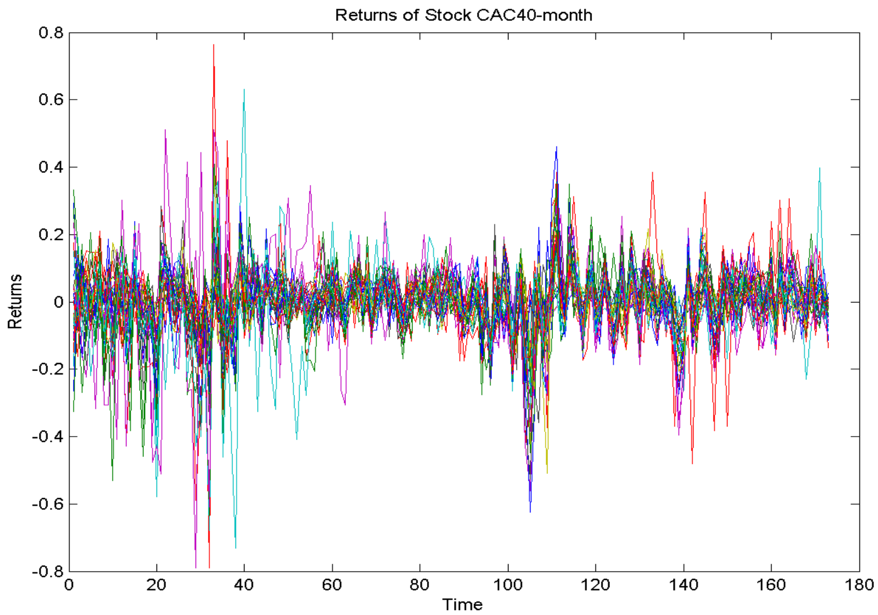
The returns of the remaining 32 stocks are as follow for the weekly frequencies:



To have a clear idea of the assets that compose our portfolio, it is better to take a glance at the characteristics of the distribution of the returns (mean, median, range, skewness and kurtosis) for the weekly case. The purpose of this analysis is to see the kind of distribution of each asset that composes the portfolio and if we can apply other models of optimization that require particular conditions for the distribution. Part of negative skewness is due to the subprime crisis of 2008 and negative skewness means that the negative returns tend to be larger in magnitude than the positive ones.

Symbol	Stock	Mean	Median	Range	Skewness	Kurtosis
AC.PA	Accor S.A.	0,00028423	0,00313312	0,53082408	-0,872246	8,9937956
AI.PA	L'Air Liquide SA	0,00123999	0,00278114	0,30096908	-0,20850102	5,5086237
ALU.PA	Alcatel-Lucent	-0,00356145	-0,0040591	0,74381843	-0,214127	5,7715358
ALO.PA	Alstom SA	-0,00321887	-0,0015124	1,12354327	-1,47931249	18,61665
MT.PA	ARCELORMITTAL RE	-0,00040513	0	0,81202587	0,020620681	7,2345109
CS.PA	AXA Group	-0,00068706	0,00228441	0,60733326	-0,35794284	7,1374411
BNP.PA	BNP Paribas SA	0,00025176	0,00238113	0,7052534	-0,39004136	10,980815
EN.PA	Bouygues SA	-0,00080092	0,00141827	0,47372841	-0,01677194	5,4696404
CAP.PA	Cap Gemini S.A.	-0,001871	-0,0024844	0,65004118	-0,26990778	6,4407835
CA.PA	Carrefour SA	-0,00127986	0,0002065	0,45060883	-0,58030912	7,0812516
SGO.PA	Compagnie de Sain	1,29E-05	0,0015706	0,5228516	-0,9001256	9,1869935
ML.PA	Compagnie Genera	0,00105039	0,00377269	0,3601384	-0,21981539	4,4785916
BN.PA	Danone	0,00090905	0,00024971	0,32286352	-0,06590759	5,5822869
EI.PA	Essilor Internation	0,00214697	0,00258565	0,37230637	-0,37676871	8,4868894
KER.PA	Kering SA	-0,00040322	0,00120105	0,55901985	-0,17071123	7,7047707
OR.PA	L'Oreal SA	0,00069669	0,00124693	0,26570513	-0,29422924	4,5457559
LG.PA	Lafarge S.A.	-0,00042473	0,00183337	0,51675529	-0,55334338	7,0364893
MC.PA	LVMH Moet Henne	0,00080786	0,00149778	0,49624151	-0,49160495	8,6642362
ORA.PA	Orange	-0,00290418	-0,0009582	0,76182403	0,635669348	13,63481
RI.PA	Pernod-Ricard SA	0,00215576	0,00288064	0,53304073	-0,2476145	11,509124
PUB.PA	Publicis Groupe SA	0,00081834	0,00163537	0,57600383	-0,05297622	8,4902929
RNO.PA	Renault Soci	0,00040931	0,00213595	0,61016668	-0,70065037	7,1097348
SAF.PA	Safran SA	0,0005679	0,00043083	0,73462859	-0,71902754	11,121952
SAN.PA	Sanofi	0,00090035	0,00277075	0,42668093	-0,74206133	8,6617005
SU.PA	Schneider Electric S	0,00077331	0,00191329	0,43784412	-0,27652893	5,2938803
GLE.PA	Societe Generale G	-0,00026654	0,00079612	0,56973093	-0,41063861	6,5801649
SOLB.PA	SOLVAY	0,00062729	0,00163353	0,33006061	-0,17826408	4,9020667
TEC.PA	Technip SA	0,00144303	0,00443783	0,67405332	-0,84879542	10,002307
FP.PA	TOTAL S.A.	0,00076893	0,00169234	0,42278428	-0,53955588	8,2360704
FR.PA	Valeo SA	0,0002658	0,00233504	0,48988647	-0,49833478	5,6427646
DG.PA	VINCI S.A.	0,00201651	0,00230264	0,4563246	-0,22256256	7,0256343
VIV.PA	Vivendi Soci	-0,0019219	0,00012176	0,93812953	-1,82794238	27,666163

We take a fast view on the monthly frequencies:



The trend is very similar, just the weekly returns have higher volatility. We are going to use them for comparison between the 2 types of data.

3.2.1 Risk Parity applied to Standard Deviation

In this part of the thesis we introduce the optimization of the portfolio with the Risk Parity strategy using the standard deviation as risk measure. To see each contribution to the risk, in this case to standard deviation, we consider the data for the period of time from 1/1/2000 to 31/12/2013 for weekly and monthly frequencies (728 weeks or 168 months). The sample period is large enough to apply the Law of Large Numbers and in this case we get the maximum information for the range.

In order to have equal risk contribution we use the following optimization model with no short sales and no leveraged positions.

$$x^* = \arg\min_x \sum_{i=1}^{n}\sum_{j=1}^{n}(TRC_i(x) - TRC_j(x))^2$$

$$\sum_{i=1}^{n} x_i = 1$$
$$x \geq 0$$

With this we obtain the following solution:

Asset i	x_i	$\sigma_{x_i}(x)$	$TRC_i(x)$ (10^{-3})	i	x_i	$\sigma_{x_i}(x)$	$TRC_i(x)(10^{-3})$
1	0.026386	0.036123	0.95317	12	0,028075	0.033876	0.95111
2	0.039850	0.023564	0.93907	13	0.054479	0.017014	0.92674
3	0.019392	0.04949	0.95974	14	0.060521	0.015158	0.91666
4	0.025611	0.03718	0.95248	15	0.025474	0.037443	0.95376
5	0.024240	0.03939	0.95497	16	0.048267	0.019272	0.93010
6	0.019379	0.04954	0.96067	17	0.024687	0.038668	0.95463
7	0.023756	0.04022	0.95561	18	0.028126	0.033815	0.95111
8	0.025886	0.03682	0.95338	19	0.038146	0.024665	0.94089
9	0.023143	0.04130	0.95604	20	0.042386	0.022097	0.93662
10	0.033958	0.02783	0.94524	21	0.028389	0.033499	0.95103
11	0.023323	0.04102	0.95680	22	0.021203	0.045211	0.95861

Asset i	x_i	$\sigma_{x_i}(x)$	$TRC_i(x)(10^{-3})$
23	0.02909	0.032656	0.9502
24	0.04526	0.020624	0.93342
25	0.02697	0.035319	0.95259
26	0.02010	0.047773	0.96026
27	0.03852	0.024419	0.94076
28	0.02685	0.035473	0.95272
29	0.03937	0.02387	0.93991
30	0.02349	0.040695	0.95623
31	0.03421	0.027612	0.94477
32	0.03141	0.030121	0.94636
SUM	1	STD	0.030336

We notice the approximate equality of the total risk contributions. The range of the weights is between 0.019392 and 0.060521, so nearly one to three times. The total contribution should be equal for each asset, so where the weights are higher the marginal contribution is lower; this means that the asset carries less risk. The sum of the total risk contributions gives the standard deviation of the portfolio.

If we apply the Mean Variance model without the return constraint(i.e, we find the Minimum Variance portfolio) we obtain the following results:

Asset i	x_i	Asset i	x_i	Asset i	x_i
1	0	12	0	23	0
2	0.070083	13	0.1760978	24	0.0408834
3	0	14	0.2816419	25	0
4	0.008541	15	0	26	0
5	0	16	0.1311123	27	0.0768854
6	0	17	0	28	0
7	0	18	0	29	0.0792715
8	0	19	0.0641720	30	0
9	0	20	0.0396101	31	0
10	0	21	0	32	0.0317012
11	0	22	0	SUM	1

We notice that the Minimum Variance is realized with a small number of assets (11 out of 32). This is one of the problems to deal with in the Markowitz's model. We approximate to 0 the weights of the assets smaller than 10^{-8}. If we show the marginal risk contribution it should be equal for each asset selected and the total risk contribution is 0 where the assets are not selected.

We also compute the Naive portfolio for comparative reasons with the weights $x = \frac{1}{N} = \frac{1}{32} = 0,03125$ and calculate the standard deviation:

$\sigma_{MV} = 0.0228725$

$\sigma_{RP} = 0.0303357$

$\sigma_{\frac{1}{N}} =\ 0.0333074$

As we see the standard deviation of the Risk Parity portfolio is larger than that of the Minimum Variance one but smaller than that of the Naive portfolio:

$$\sigma_{MV} < \sigma_{RP} < \sigma_{\frac{1}{N}}.$$

Thus, we compute some performance ratios:

Weekly CAC40	RP-Std	M-V	Uniform
$\mu(\%)$	0.0148	0.0849	-0.0060
$\mu_{ann}(\%)$	0.7747	4.5116	-0.3099
$Median$	0.2187	0.1672	0.2336
$\sigma(\%)$	3.0336	2.2873	3.3307
$VaR_{10\%}(\%)$	3.7451	2.5703	4.1391
$CVaR_{10\%}(\%)$	5.8484	4.1919	6.4533
$\sigma_{ann}(\%)$	21.875	16.494	24.018
$VaR_{10\%}ann$	27.006	18.535	29.847
$CVaR_{10\%}ann$	42.173	30.228	46.536
S_{σ}	0.0354	0.2735	-0.0129
S_{VaR}	0.0287	0.2434	-0.0104
S_{CVaR}	0.0184	0.1493	-0.0067

The expected return of the portfolio is better for Mean Variance. Also the risk measures are smaller for the Mean Variance model. The Risk Parity performance is smaller than Mean Variance but better than that of the Naive Portfolio. Thus the Risk Parity is a good trade off between Mean Variance and Naive portfolio.

To understand the relation between data frequencies and performance we repeat the procedure for the monthly frequencies at the same sample selected (168 frequencies).

In this case, due to the fact that the return in a range of time of a month may change faster than in a week, we will have more skewness. In the same way we compute the Risk Parity portfolio:

Asset i	x_i	$\sigma_{x_i}(x)$	$TRC_i(x)$ (10^{-3})	Asset i	x_i	$\sigma_{x_i}(x)$	$TRC_i(x)(10^{-3})$
1	0.023432	0.067991	0.001593	12	0.024997	0.063754	0.001594
2	0.055820	0.028531	0.001592	13	0.058254	0.027339	0.001593
3	0.012721	0.125571	0.001594	14	0.069674	0.022853	0.001592
4	0.017011	0.093697	0.001594	15	0.021478	0.07421	0.001594
5	0.019551	0.081500	0.001594	16	0.050469	0.031560	0.001593
6	0.016642	0.095772	0.001594	17	0.023441	0.067987	0.001594
7	0.024592	0.06478	0.001594	18	0.024723	0.064459	0.001594
8	0.028131	0.056782	0.001594	19	0.031112	0.051211	0.001593
9	0.017191	0.092720	0.001594	20	0.051300	0.03105	0.001593
10	0.039091	0.040753	0.001593	21	0.023835	0.066861	0.001594
11	0.021281	0.074883	0.001594	22	0.016591	0.096073	0.001594

Asset i	x_i	$\sigma_{x_i}(x)$	$TRC_i(x)(10^{-3})$
23	0.024712	0.064272	0.001593
24	0.063356	0.025134	0.001594
25	0.027823	0.057276	0.001593
26	0.018327	0.087000	0.001593
27	0.033466	0.047681	0.001593
28	0.024123	0.065930	0.001594
29	0.051964	0.030633	0.001593
30	0.018633	0.085480	0.001593
31	0.037714	0.042201	0.001594
32	0.028431	0.055910	0.001593
SUM	1	STD	0.050989

The results are very similar to the case of the weekly frequencies but we notice that the range of weights is between 0.012721 and 0.069674, one to six times. The approximation is good in calculating the Total Risk contribution.

In the same way we calculate the Mean Variance portfolio weights:

Asset i	x_i	Asset i	x_i	Asset i	x_i
1	0	12	0	23	0
2	0.162992	13	0.163208	24	0.082622
3	0	14	0.22461107	25	0
4	0	15	0	26	0
5	0	16	0.012912	27	0
6	0	17	0	28	0
7	0	18	0	29	0.239467
8	0	19	0.0463026	30	0
9	0	20	0.0459962	31	0
10	0.021887	21	0	32	0
11	0	22	0	SUM	1

This time the Mean Variance portfolio is concentrated in less assets than in the case of weekly frequencies. Some of the assets are the same as in the other case (13, 14 and 16) but the portfolio is more concentrated (nearly 80% in 4 assets).

For comparison with the other case, if we compute the same performance measures, at the same time, the results are very similar to the other case except for the fact that risk measures are higher.

Monthly CAC40	RP-Std	M-V	Uniform
$\mu(\%)$	0.1579	0.4364	-0.0056
$\mu_{ann}(\%)$	1.9113	5.3642	-0.0672
$Median$	0.7979	1.0162	0.7962
$\sigma(\%)$	5.099	3.365	6.2564
$VaR_{10\%}(\%)$	5.7381	4.1408	7.4855
$CVaR_{10\%}(\%)$	10.5769	6.4771	13.0908
$\sigma_{ann}(\%)$	17.6633	11.6568	21.6730
$VaR_{10\%}ann$	41.3778	29.86	53.9791
$CVaR_{10\%}ann$	76.2711	46.7069	94.3989
S_σ	0.1082	0.4602	-0.0031
S_{VaR}	0.2066	0.851	-0.0054
S_{CVaR}	0.1121	0.5441	-0.0031

3.2.2 Risk Parity applied to CVaR

Here we study the optimization of the portfolio with the Risk Parity strategy using the Conditional Value at Risk as risk measure. We compute the total risk contribution of each asset to Conditional Value at Risk, using the same time series of the case of the standard deviation as a risk measure (i.e, the period of time from 1/1/2000 to 31/12/2013 for weekly and monthly frequencies). We also apply the Naive Risk Parity CVaR, not the true diversification, to see the difference between these models in the contribution of the risk. For reasons that we will discuss later we choose a confidence level of 10%.

For the weekly frequencies we have the following tables with weights, marginal risk contribution and total risk contribution for Naive Risk Parity CVaR- and Risk Parity CVaR:

	Risk Parity CVaR-Naive			Risk Parity CVaR		
Asset i	x_i	$\frac{\partial CVaR_\alpha(X)}{\partial x_i}$	$TRC_i(x)$	x_i	$\frac{\partial CVaR_\alpha(X)}{\partial x_i}$	$TRC_i(x)$
1	0.03036	0.09002	0.00273	0.02416	0.07467	0.00180
2	0.04684	0.05835	0.00273	0.04133	0.04347	0.00180
3	0.01851	0.14771	0.00273	0.01764	0.10335	0.00180
4	0.01811	0.15091	0.00273	0.02259	0.07869	0.00180
5	0.02013	0.13574	0.00273	0.02548	0.07141	0.00180
6	0.02420	0.11292	0.00273	0.01867	0.09685	0.00180
7	0.02644	0.10339	0.00273	0.02260	0.07981	0.00180
8	0.02679	0.10203	0.00273	0.02471	0.0729	0.00180
9	0.02293	0.11918	0.00273	0.02297	0.07875	0.00180
10	0.03312	0.08254	0.00273	0.03380	0.05338	0.00180
11	0.02727	0.10023	0.00273	0.02347	0.07606	0.00180

Asset i	x_i	$\frac{\partial CVaR_\alpha(X)}{\partial x_i}$	$TRC_i(x)$	x_i	$\frac{\partial CVaR_\alpha(X)}{\partial x_i}$	$TRC_i(x)$
12	0.03016	0.09063	0.00273	0.02556	0.07014	0.00180
13	0.04744	0.05762	0.00273	0.05041	0.03588	0.00180
14	0.05035	0.05428	0.00273	0.06507	0.02726	0.00180
15	0.02917	0.09370	0.00273	0.02442	0.07383	0.00180
16	0.04561	0.05992	0.00273	0.04872	0.03675	0.00180
17	0.02850	0.09591	0.00273	0.02315	0.07755	0.00180
18	0.03377	0.08093	0.00273	0.02858	0.06379	0.00180
19	0.02563	0.10664	0.00273	0.04079	0.04411	0.00180
20	0.04059	0.06733	0.00273	0.04598	0.0394	0.00180
21	0.029231	0.09351	0.00273	0.03066	0.05892	0.00180
22	0.024344	0.11229	0.00273	0.01903	0.0949	0.00180

Asset i	x_i	$\frac{\partial CVaR_\alpha(X)}{\partial x_i}$	$TRC_i(x)$	x_i	$\frac{\partial CVaR_\alpha(X)}{\partial x_i}$	$TRC_i(x)$
23	0.02608	0.10480	0.00273	0.03124	0.05759	0.00180
24	0.03896	0.07016	0.00273	0.04679	0.03920	0.00180
25	0.03117	0.08769	0.00273	0.02654	0.06807	0.00180
26	0.02271	0.12039	0.00273	0.01920	0.09454	0.00180
27	0.03937	0.06943	0.00273	0.03540	0.05122	0.00180
28	0.02582	0.10587	0.00273	0.02788	0.06500	0.00180
29	0.04382	0.06237	0.00273	0.04129	0.04338	0.00180
30	0.02550	0.10722	0.00273	0.02108	0.08505	0.00180
31	0.03992	0.06846	0.00273	0.03422	0.05271	0.00180
32	0.02706	0.10103	0.00273	0.03652	0.04956	0.00180
SUM	1	CVaR	0.08748	1	CVaR	0.05773

We notice that the total risk contribution is bigger in case of Risk Parity Portfolio Naive than in case of Risk Parity CVaR. As a consequence the R.P Naive CVaR is riskier than R.P CVaR. Another interesting result is that the marginal risk contribution of Risk Parity Portfolio Naive is bigger than the corresponding assets marginal risk contribution of Risk Parity in each case.

We also compute the Conditional Value at Risk portfolio at the same level of confidence 10% without the constraint on the expected return of the portfolio. In this way we obtain the minimum risk portfolio with CVaR =0.0408872 and the following weights:

Asset i	x_i	Asset i	x_i	Asset i	x_i
1	0	12	0	23	0
2	0.0425367	13	0.1794414	24	0.00639
3	0	14	0.3519022	25	0
4	0	15	0	26	0
5	0	16	0.061152	27	0
6	0	17	0	28	0
7	0	18	0	29	0.093637
8	0	19	0.071346	30	0
9	0	20	0.1549116	31	0.014627
10	0	21	0	32	0.024057
11	0	22	0	SUM	1

Like in the Mean Variance model, the Conditional Value at Risk is concentrated in 10 out of 32 possible assets, with more than the two thirds in just three assets (13, 14 and 20).

The order of the risks using CVaR (or the standard deviation) as a risk measure is the following.

$$CVaR_{cvar}(X) < CVaR_{rp-cvar}(X) < CVaR_{rp-cvarnaive}(X) < CVaR_{1/n}(X)$$

Where $CVaR_{1/n}(X) = 6.4533\%$ is the Conditional Value at Risk for the Naive portfolio.

We report the performance of the various models:

Weekly	R.P-CVaR Naive	R.P-CVaR	CVaR	Uniform
$\mu(\%)$	0.0211	0.0168	0.0963	-0.0060
$\mu_{ann}(\%)$	1.1027	0.8790	5.1348	-0.3099
$Median$	0.2425	0.1990	0.2037	0.2336
$\sigma(\%)$	3.0984	3.0084	2.3230	3.3307
$VaR_{10\%}(\%)$	3.8343	3.7111	2.6223	4.1391
$CVaR_{10\%}(\%)$	5.9836	5.7726	4.1567	6.4533
$\sigma_{ann}(\%)$	22.3425	21.6937	16.7514	24.0183
$VaR_{10\%}ann(\%)$	27.6497	26.7608	18.9099	29.8472
$CVaR_{10\%}ann(\%)$	43.1486	41.6270	29.974	46.5355
S_σ	0.0494	0.0405	0.3065	-0.0129
S_{VaR}	0.0399	0.0328	0.2715	-0.0104
S_{CVaR}	0.0256	0.0211	0.1713	-0.0067

We also repeat the procedure with the same time period but using the monthly data series

	Risk Parity CVaR-Naive			Risk Parity CVaR		
Asset i	x_i	$\frac{\partial CVaR_\alpha(X)}{\partial x_i}$	$TRC_i(x)$	x_i	$\frac{\partial CVaR_\alpha(X)}{\partial x_i}$	$TRC_i(x)$
1	0.028854	0.17625	0.00509	0.022686	0.140481	0.0032
2	0.058304	0.08722	0.00509	0.056073	0.052139	0.0030
3	0.013385	0.37993	0.00509	0.011510	0.285497	0.0033
4	0.013657	0.37235	0.00509	0.015433	0.208748	0.0032
5	0.014527	0.35006	0.00509	0.016202	0.209711	0.0033
6	0.020836	0.24407	0.00509	0.015536	0.210078	0.0032
7	0.026921	0.18890	0.00509	0.02503	0.130967	0.0032
8	0.029252	0.17385	0.00509	0.029084	0.10637	0.0032
9	0.018161	0.28002	0.00509	0.01743	0.190211	0.0033
10	0.036052	0.14106	0.00509	0.03645	0.09329	0.0034
11	0.025693	0.19793	0.00509	0.01919	0.166868	0.0032
Asset i	x_i	$\frac{\partial CVaR_\alpha(X)}{\partial x_i}$	$TRC_i(x)$	x_i	$\frac{\partial CVaR_\alpha(X)}{\partial x_i}$	$TRC_i(x)$
12	0.030985	0.16413	0.00509	0.028116	0.112002	0.0032
13	0.049279	0,1032	0.00509	0.051344	0.060472	0.0031
14	0.057388	0.08861	0.00509	0.061055	0.044992	0.0028
15	0.02409	0.21111	0.005089	0.022014	0.151135	0.0033
16	0.04501	0.11299	0.00509	0.059081	0.055023	0.0033
17	0.028164	0.18057	0.00509	0.022387	0.145514	0.0033
18	0.031426	0.16182	0.00509	0.025689	0.128739	0.0033
19	0.018176	0.27979	0.00509	0.031101	0.110468	0.0034
20	0.049710	0.10230	0.00509	0.052612	0.058844	0.0031
21	0.027437	0.18535	0.00509	0.028088	0.108923	0.0031
22	0.020021	0.254009	0.00509	0.015684	0.215645	0.0034
Asset i	x_i	$\frac{\partial CVaR_\alpha(X)}{\partial x_i}$	$TRC_i(x)$	x_i	$\frac{\partial CVaR_\alpha(X)}{\partial x_i}$	$TRC_i(x)$
23	0.022859	0.2225	0.00509	0.03311	0.08817	0.0030
24	0.046717	0.10886	0.00509	0.06323	0.04789	0.0030
25	0.034406	0.14781	0.00509	0.02808	0.10821	0.0030
26	0.023243	0.2188	0.00509	0.01852	0.17829	0.0033
27	0.036966	0.13757	0.00509	0.03569	0.08784	0.0031
28	0.024729	0.20566	0.00509	0.02379	0.13356	0.0032
29	0.053904	0.09434	0.00509	0.05406	0.05705	0.0031
30	0.022206	0.22903	0.00509	0.01974	0.16455	0.0033
31	0.042411	0.11991	0.00509	0.03689	0.08565	0.0032
32	0.025223	0.20163	0.00509	0.02508	0.12771	0.0032
SUM	1	CVaR	0.16274	1	CVaR	0.102

As we see the total and the marginal risk contribution is higher in the case of Naive Risk Parity with CVaR for each asset in consideration. Like in the case of weekly frequencies, The Risk parity Naive is riskier than the second portfolio.

We also compute the Conditional Value at Risk of the portfolio with the same confidence level 10% and obtain the following of CVaR = 0.07481 and the following table of weights:

Asset i	x_i	Asset i	x_i	Asset i	x_i
1	0	12	0	23	0
2	0.3454596	13	0.0999993	24	0.1224938
3	0	14	0.1827802	25	0
4	0	15	0	26	0
5	0	16	0	27	0
6	0	17	0	28	0
7	0	18	0	29	0.1277246
8	0	19	0.0388661	30	0
9	0	20	0.08267632	31	0
10	0	21	0	32	0
11	0	22	0	SUM	1

In this case the portfolio is more concentrated in less assets, 7 out of 32 possible choices and the CVaR is higher.

On the basis of the above results we deduced that the monthly frequencies are not adequate to study these model for the small number of elements they use and because of greater variability. In the next studies we will thus, use weekly data.

3.2.3 Comparison between models

A crucial part of the thesis is the comparison of the out of sample performance of the models. In particular we compare Risk Parity with standard deviation and Risk Parity CVaR as alternative method. To have a complete frame of the performances we also compute the Mean Variance, CVaR (at confidence level 10%), the Naive (also known as Uniform) and the Naive Risk Parity CVaR as a special case.

We create a rolling time window with in sample period L=4 year (208 observation) and out of sample period H=4 weeks for the data series form 1/1/2000 to 4/7/2014.

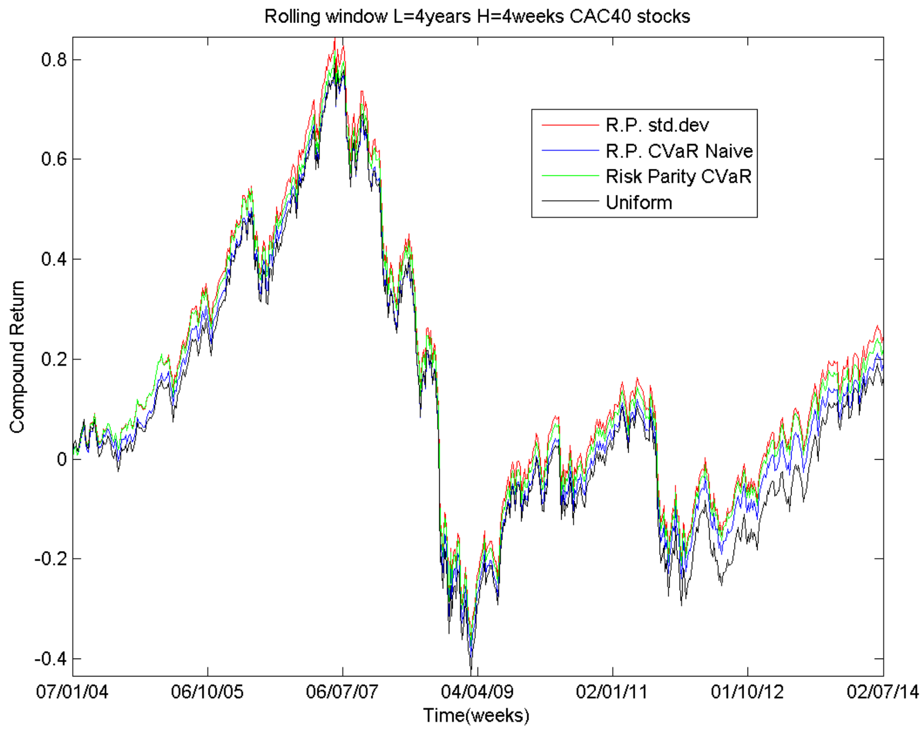
The performance of the models can be described in two parts: The first is before the subprime crisis of 2008 and second after the crisis. We notice that Mean Variance and CVaR, that are heavily concentrared, in the first part have the same trajectory and after the crisis the mean Variance dominates all the model in the performance.

Weekly CAC40	RP-Std	M-V	RP-CVaR N.	RP-CVaR	CVaR	Uniform
$\mu(\%)$	0.0869	0.1065	0.0797	0.0827	0.0693	0.0830
$\mu_{ann}(\%)$	4.6219	5.6934	4.2307	4.3943	4.3163	4.4066
$\mu^c(\%)$	24.8172	52.8936	19.3473	22.3240	32.5524	16.4396
$Median$	0.3365	0.1804	0.3015	0.3268	0.1602	0.3422
$\sigma(\%)$	3.0107	2.3738	3.0437	2.9933	2.4049	3.2829
$VaR_{10\%}(\%)$	3.5847	2.5777	3.7063	3.5274	2.5174	3.9244
$CVaR_{10\%}(\%)$	5.83	4.2962	5.9031	5.8012	4.3936	6.3618
$\sigma_{ann}(\%)$	21.7107	17.1178	21.9486	21.5851	17.3421	23.6731
$VaR_{10\%}ann$	25.8499	18.5881	26.7263	25.4364	18.1529	28.2995
$CVaR_{10\%}ann$	42.0407	30.9805	42.5677	41.8329	31.6829	45.8755
S_σ	0.2129	0.3326	0.1928	0.2036	0.2489	0.1861
S_{VaR}	0.1788	0.3063	0.1583	0.1728	0.2378	0.1557
S_{CVaR}	0.1099	0.1838	0.0994	0.1050	0.1362	0.0961
Sortino Ratio	0.0382	0.0597	0.0347	0.0364	0.0445	0.0336
Rachev Ratio	0.8255	0.8479	0.8294	0.8165	0.8038	0.8344

The compounded return shows the increase of the capital invested during the entire period of study and the models that are composed from less selected tittles usually achieve a higher one (CVaR and Mean Variance).

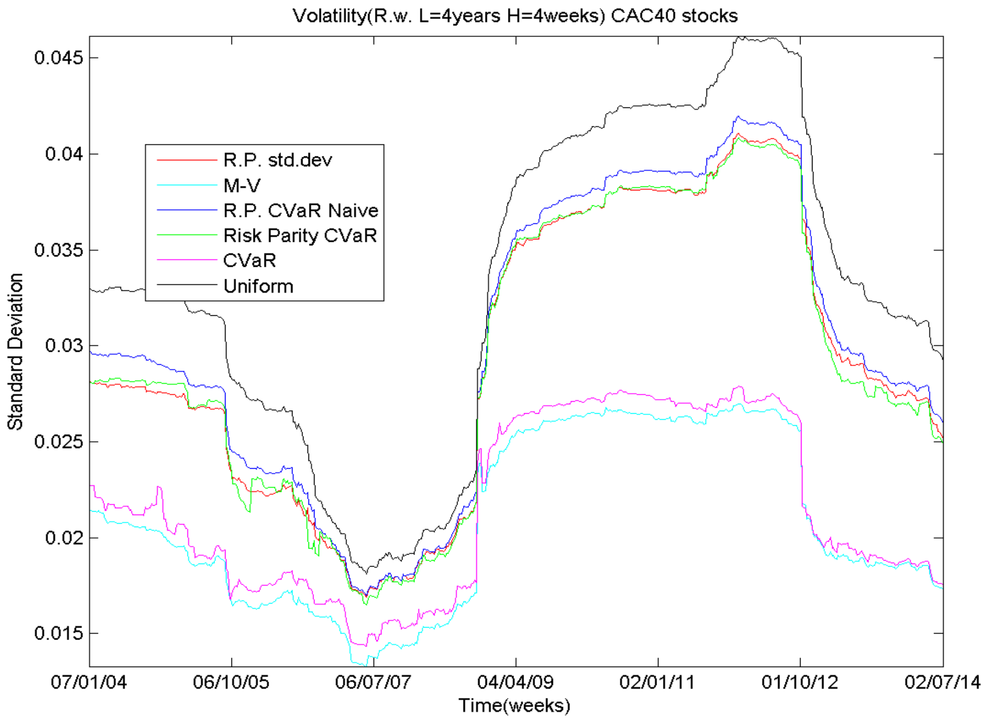
We calculate different risks measures in order to have a clear idea of the performance of each model. The Sortino Ratio in all cases is calculated with risk free equal to zero and the Rachev ratio is calculated with alpha equal to 0.05. Is clear that the order of preferences will have Mean Variance in the first place, than CVaR but if we take into consideration the diversification things changes.

For the other models (with Risk Parity strategies) there is not such a large difference and for that we should take a look closer in the following graph:



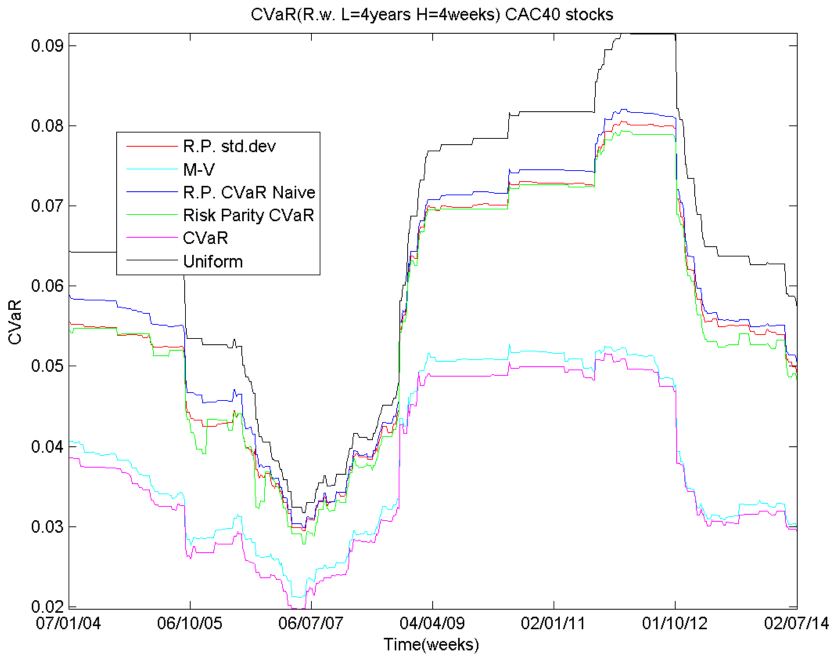
We notice that in the first period the Risk Parity with standard deviation and Risk Parity with CVaR alternate the dominance: In the second period the Risk Parity CVaR dominates Risk Parity with standard deviation but just by a small amount. Risk Parity with CVaR Naive and Naive portfolio take into consideration all the 32 assets but have the smallest performances.

If we study the risk level from the point of view of volatility, the Mean Variance model has some advantage for the simple reason that it tries to minimize volatility. In our case, the Mean Variance model gives the minimum portfolio volatility without the expected return constraint. If we compute the volatility out of sample we find the following:



We clearly see that Mean Variance and CVaR are less risky than other models. The R.P. with standard deviation and R.P. with CVaR are more or less at the same level of risk. The R.P. CVaR- Naive is more risky than the others due the fact of no true diversification. To have a complete frame of the level of volatility we also compute the Naive portfolio.

To evaluate of the order of riskiness we measure the CVaR out of sample for each model:

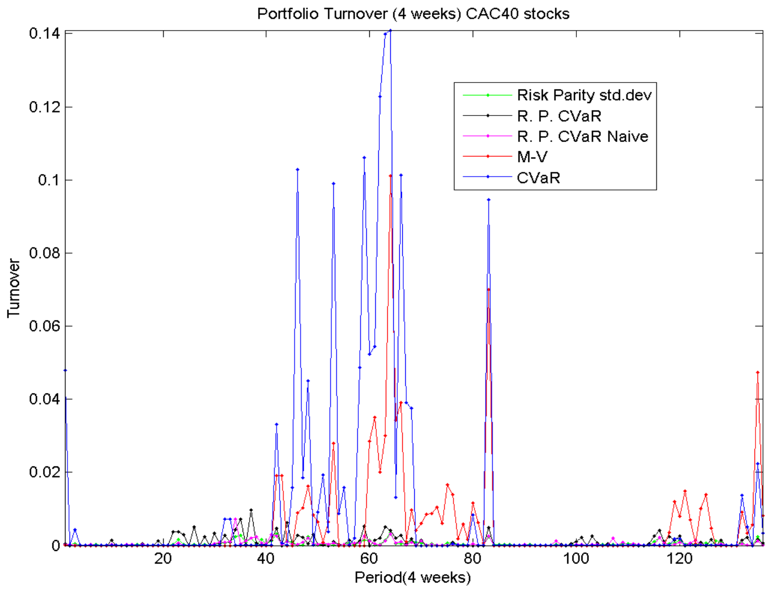


The CVaR model has a lower level of risk than the others by definition, the CVaR of the Mean Variance portfolio is a little higher. R.P. with standard deviation and R.P. with CVaR are at the same level of risk.

To get closer to the real markets when measuring performance we have to deal with the transaction cost, fixed or variable in all cases. For that we must consider the portfolio turnover for each period where we recalculate the optimal weights. As we know the CVaR model and the Mean Variance model are concentrated in small groups of assets and for that they suffer from high turnover.

The other models have lower turnover but for that we should see the amount of capital invested and the fixed costs.

In the following graph we show the portfolio's turn over for each model:



As the turn over index is measured in absolute values, some of the proportion of the amount invested should be decreased (sell asset) and some should be increased.

Here we show the average turnover for each period of rebalancing (every a week) :

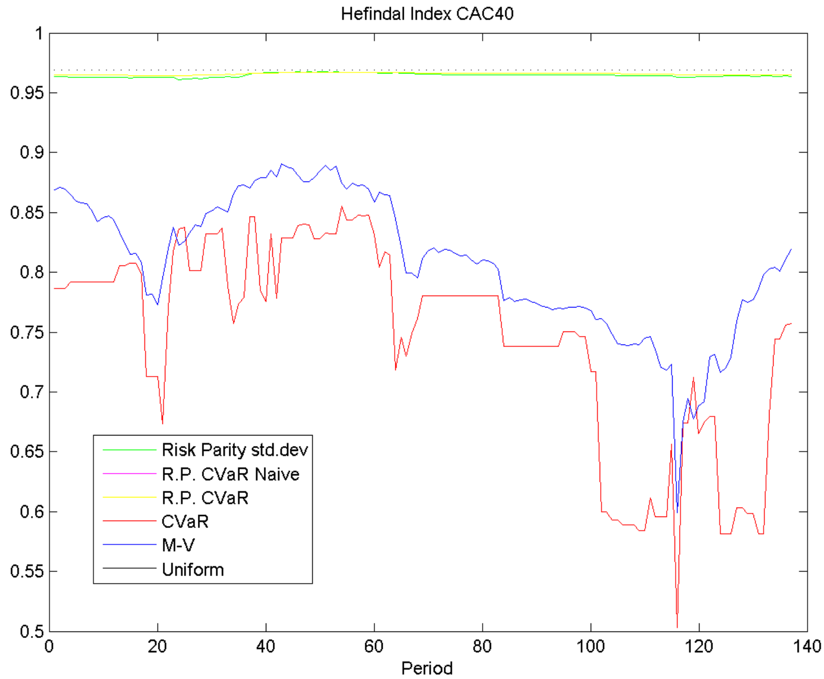
CAC40 weekly	RP-Std	M-V	RP-CVaR Naive	RP-CVaR	CVaR
Average turnover	0.05261%	0.546274%	0.05193%	0.1052%	1.0649%

The problem with the average is that in some cases the portfolios do not change the composition so we have to take a closer look.

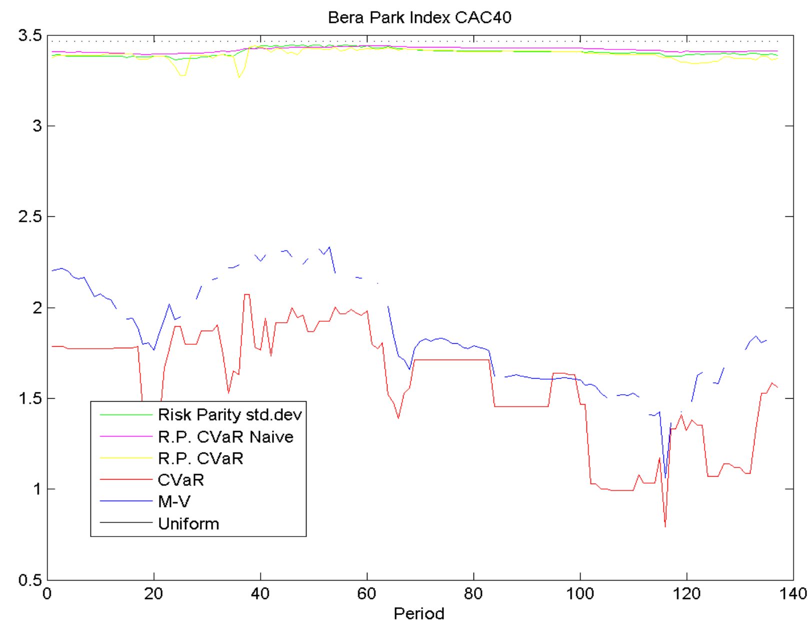
In the diversification of the portfolio we start with the Herfindal index:

$$D_{Her} = 1 - xx'$$

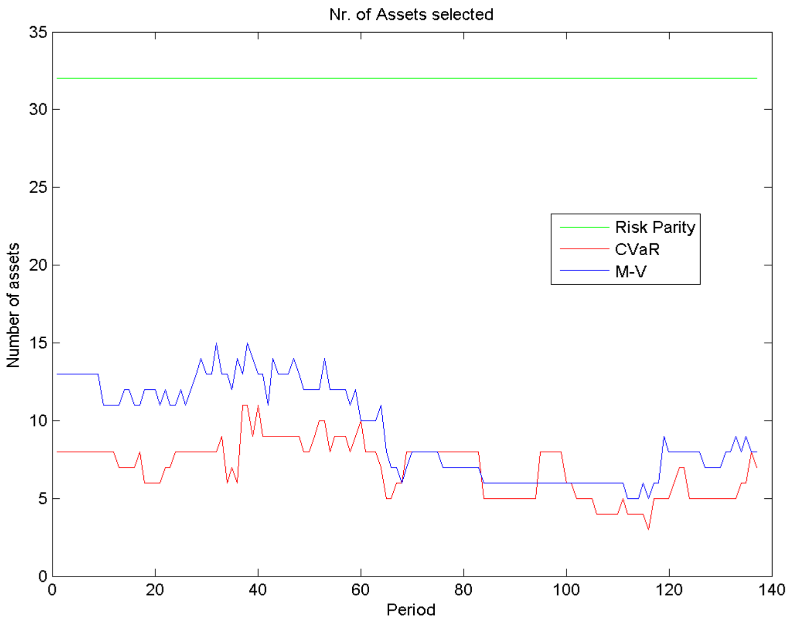
As we described in section 2.4.4, the Herfindal index takes the value 0 if the portfolio is concentrated in one asset and the maximum value $1-\frac{1}{n}$ for the naive portfolio. So for the naive(or Uniform) portfolio we have the maximum value 0,96875 for the Herfindal Index. The more the portfolio is concentrated, like CVaR and Mean Variance, the lower is the index .



Another way to study diversification is to apply the Bera Park Index, which is similar to the Herfindal index. The only problem to deal with using this measure is when the portfolio assumes the position 0 for a certain asset and there we have to adapt the quantities equal to 0 in a way to apply the index. As we see from the graph, the most concentrated are the CVaR and the Mean Variance.



As the last point we consider the number of assets that each model selected with a reasonable quantity(we do not consider the weights smaller than 10^{-6} . Since Risk Parity models and the naive portfolio consider all the assets we will show just one of them.



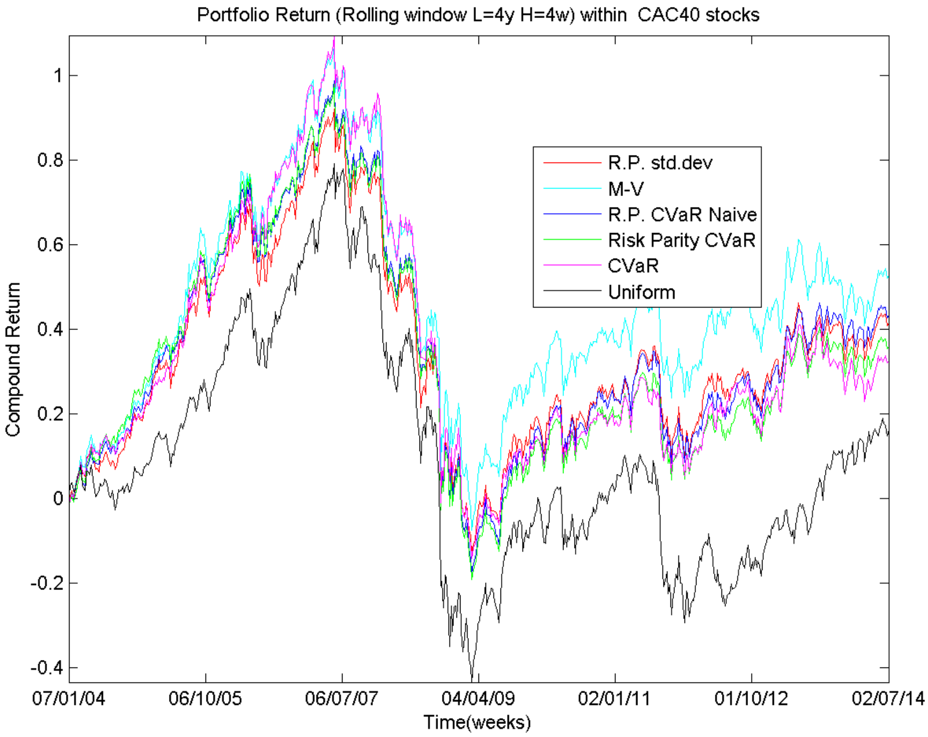
It is clear that the CVaR is very concentrated in a small number of assets and reaches the minimum in 4 assets of the 32 possible. The Mean Variance model chooses between 6 and 15 assets out of the 32 possible. With this subset selected we obtain the minimum risk for each model. In the last part of this section we apply risk parity to this groups of assets to study the performance.

3.2.4 Portfolio subset selection

The Risk Parity strategies take into consideration every asset of the market in order to contribute to the risk in the same quantity. We can not choose a smaller subset of assets applying the cardinality constraints. This pushes us to develop other methods of selection of a subset of assets.

Starting from the last point of the previous section, we choose the subset selected with Mean Variance and apply the Risk Parity with the standard deviation, and from the subset of CVaR apply Risk Parity with the CVaR and R.P. CVaR- Naive. This is just a matter of selection of a subset from all possible assets in order to have minimum risk with benefits of diversification.

Using the same rolling time window, with in sample L=208weeks (4 years) and out of sample H=4 weeks, we get the following result of the compounded returns of the portfolios:

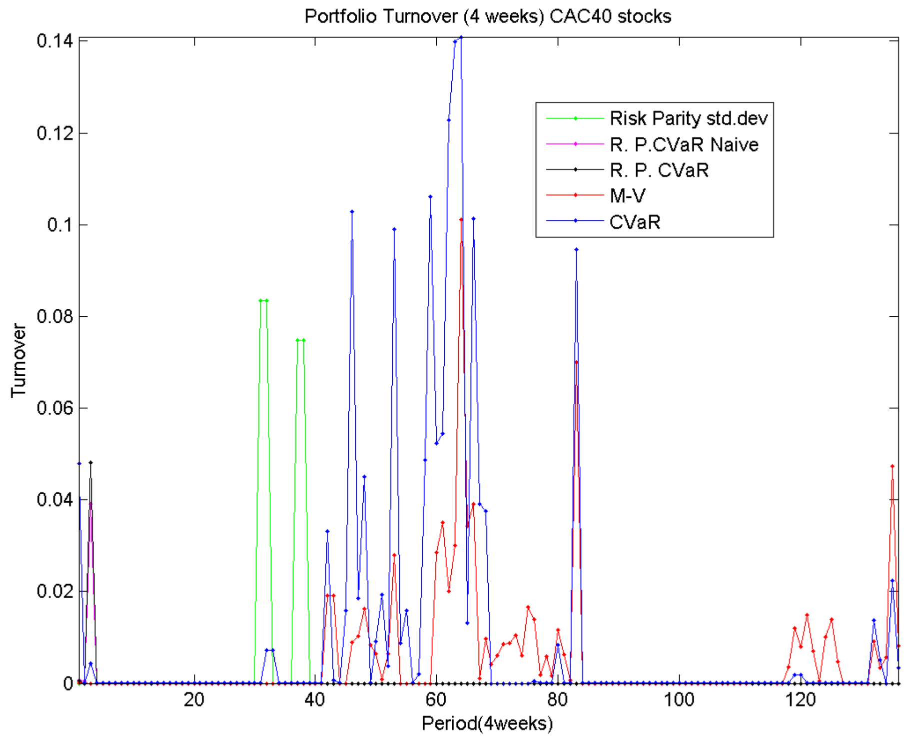


Like in the previous section we can divide the graph in two parts: Before the crisis of 2008 the models show no difference between Mean Variance and CVaR and the Risk Parity group. After the crisis the Mean Variance model recovers faster the values and the other three strategies of R.P. have the same performance of CVaR. For more details we compute again the following table:

Weekly	RP-Std	M-V	RP-CVaR N.	RP-CVaR	CVaR	Uniform
$\mu(\%)$	0.0935	0.1065	0.0952	0.0857	0.0813	0.0830
$\mu_{ann}(\%)$	4.9813	5.6934	5.0726	4.5578	4.3163	4.4066
$\mu^c(\%)$	41.6789	52.8936	43.3175	35.8540	32.5524	16.4396
$Median$	0.3101	0.1804	0.2859	0.3155	0.1602	0.3422
$\sigma(\%)$	2.4125	2.3738	2.3929	2.4056	2.4049	3.2829
$VaR_{10\%}(\%)$	2.78	2.5777	2.576	2.6422	2.5174	3.9244
$CVaR_{10\%}(\%)$	4.4512	4.2962	4.4872	4.5246	4.3936	6.3618
$\sigma_{ann}(\%)$	17.3969	17.1178	17.2558	17.3467	17.3421	23.6731
$VaR_{10\%}ann(\%)$	20.0465	18.5881	18.5761	19.0529	18.1529	28.2995
$CVaR_{10\%}ann(\%)$	32.0979	30.9805	32.3575	32.6273	31.6829	45.8755
S_σ	0.2863	0.3326	0.294	0.2627	0.2489	0.1861
S_{VaR}	0.2485	0.3063	0.2731	0.2392	0.2378	0.1557
S_{CVaR}	0.1552	0.1838	0.1568	0.1397	0.1362	0.0961
Sortino Ratio	0.0512	0.0597	0.0520	0.0465	0.0445	0.0336
Rachev Ratio	0.8077	0.8479	0.798	0.7912	0.8038	0.8344

The best model in this case remains Mean Variance, but it is interesting how the other models of Risk Parity perform better than CVaR, not only with respect to the compounded return but also to the other performance ratios.

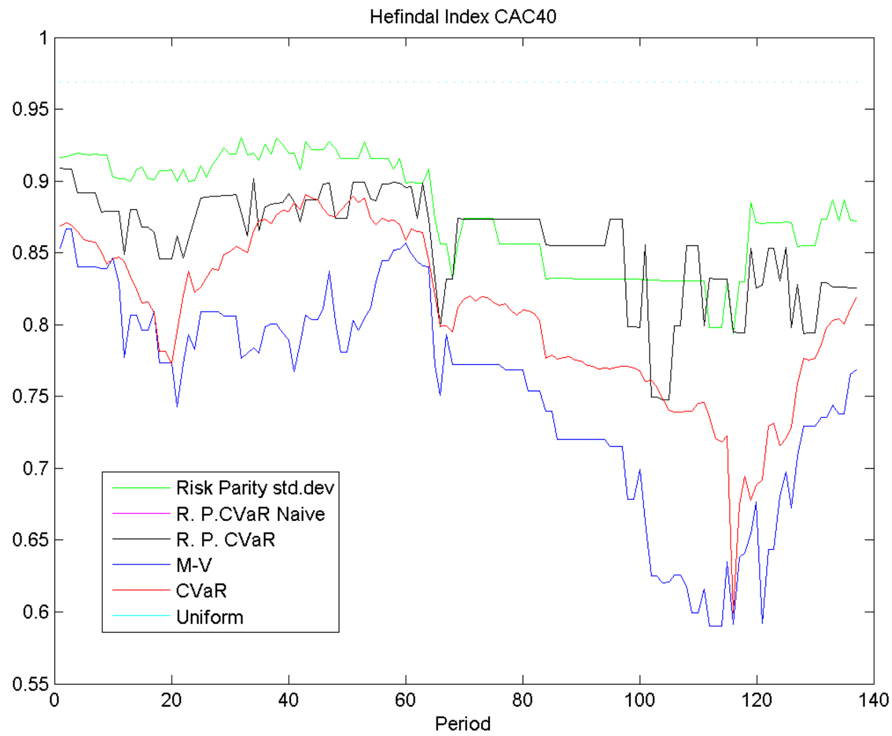
To discover the consequences on the transaction cost, we measure the turnover of the portfolios.



CaC40	RP-Std	M-V	RP-CVaR Naive	RP-CVaR	CVaR
Average Turnover (%)	0.2328	0.5463	0.0296	0.03579	1.0649

The average turnover for the rebalancing is still higher for Mean Variance and CVaR, but R.P. with the standard deviation increases from 0.052 % to 0.2328%.

Now we take a short glance at diversification. Since the Herfindal and the Bera Park indices are similar we just apply the first one.

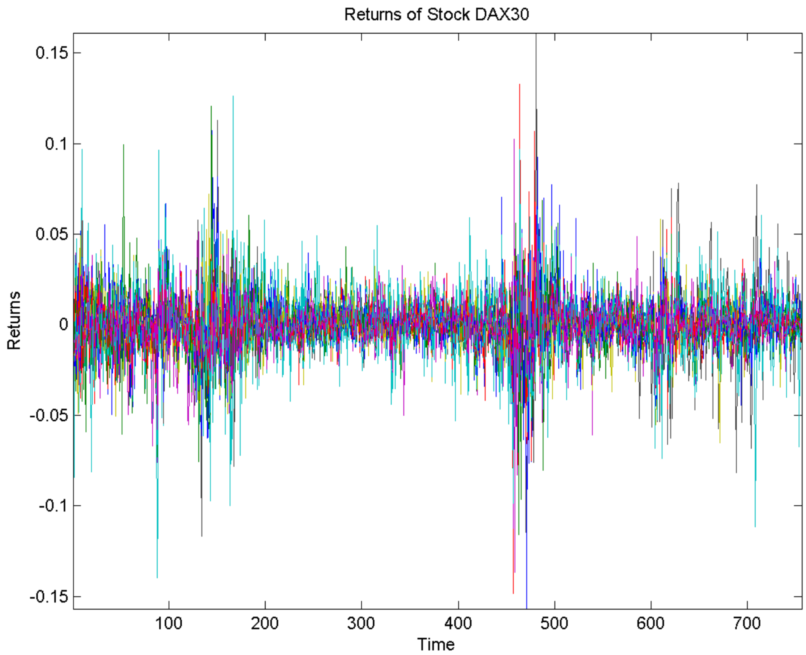


For the Risk parity group of strategies we have a higher Herfindal index for each model so the portfolio are less concentrated and better diversified.

In the further estimation we compute the volatility and CVaR out of sample and there is no significative difference from the previous case.

3.3 Portfolio optimization for the stocks of DAX30

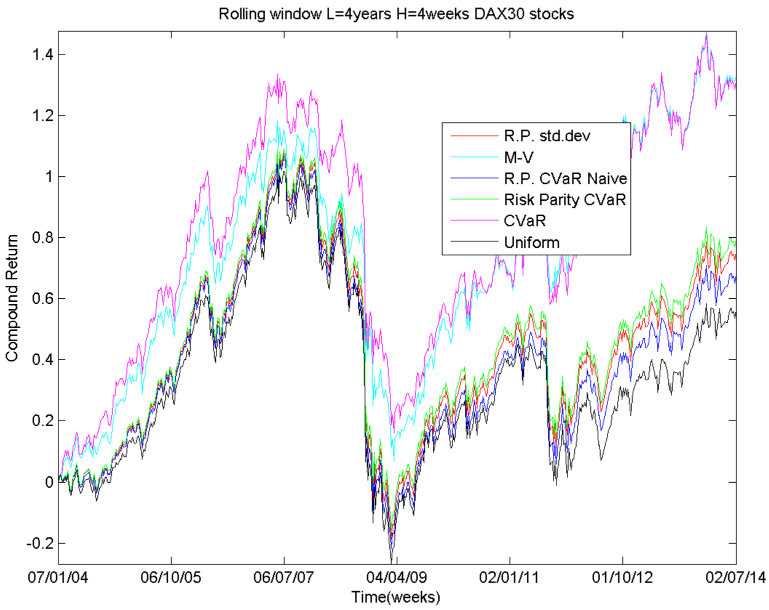
In this part of the thesis we will study the case of DAX30 that has a different number of asset with repect to the CAC40 and a different trend. To apply the Law of large numbers we use just the weekly time series in order to have a better approximation. We consider the time series for the period of time from 1/1/2000 to 04/07/2014 for weekly observation of 26 stocks from the DAX30 (we do not include 4 assets for missing data).



The returns of the assets show a high Kurtosis and a negative Skewness and this is not useful in case of expected returns.

3.3.1 Comparison between models

We proceed in the rolling time window approach with in sample L=208 weeks (4 years) and out of sample H=4 weeks, as in the previous case. We first study the compounded returns for understanding the trend of each model.

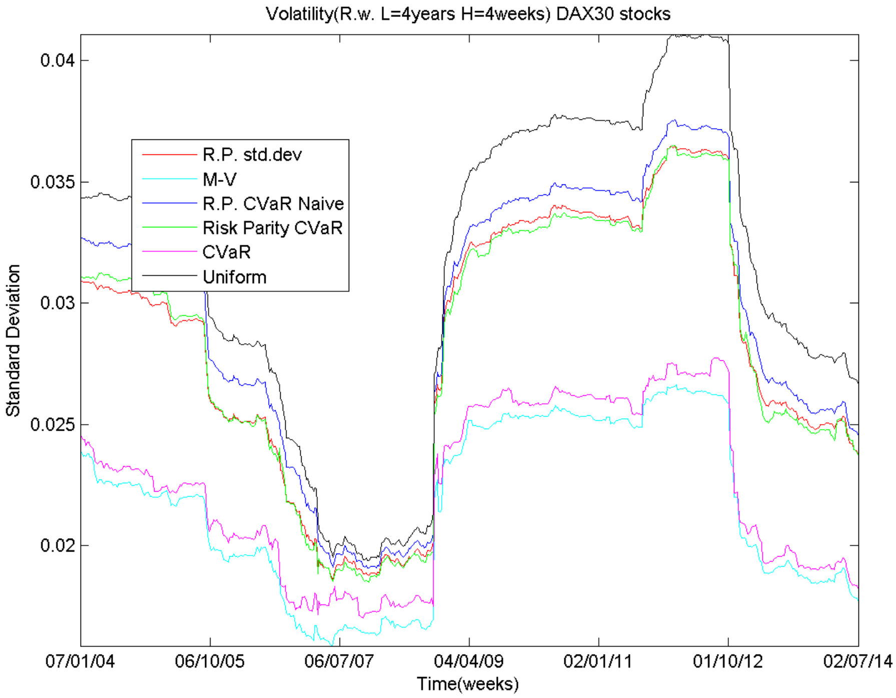


As we notice the CVaR and the Mean Variance have a better performance than the group of Risk Parity strategies. There's no significative difference between Risk Parity CVaR and Risk Parity with standard deviation. Here's the summary table:

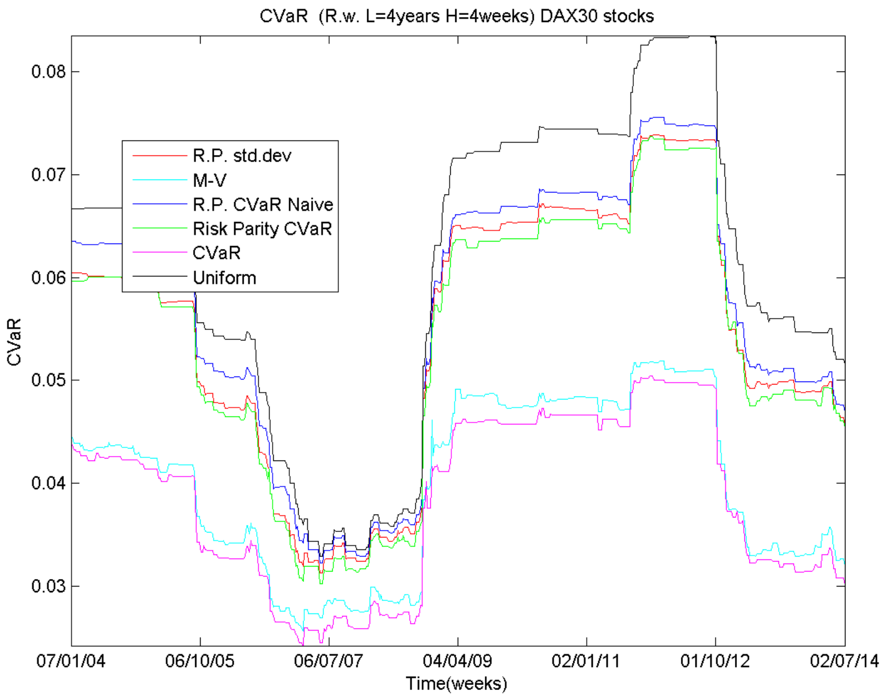
Weekly	R.P.-Std.	M-V	RP-CVaR N.	RP-CVaR	CVaR	Uniform
$\mu(\%)$	0.1424	0.1822	0.1360	0.1456	0.1811	0.1277
$\mu_{ann}(\%)$	7.6805	9.9259	7.3217	7.8587	9.8635	6.86
$\mu^c(\%)$	75.5371	132.7693	68.0539	79.7658	131.1289	56.4026
$Median$	0.4207	0.3124	0.4302	0.4181	0.2936	0.3837
$\sigma(\%)$	2.7728	2.3271	2.8256	2.7329	2.3372	2.9888
$VaR_{10\%}(\%)$	2.8649	2.3753	2.9520	2.8757	2.3806	3.1186
$CVaR_{10\%}(\%)$	5.2718	4.2556	5.3743	5.1987	4.2435	5.7270
$\sigma_{ann}(\%)$	19.9952	16.7810	20.3757	19.7069	16.8536	21.5523
$VaR_{10\%}ann(\%)$	20.6592	17.1288	21.2875	20.7369	17.1665	22.4886
$CVaR_{10\%}ann(\%)$	38.0152	30.6875	38.7543	37.4882	30.6003	41.2982
S_σ	0.3841	0.5915	0.3593	0.3988	0.5852	0.3183
S_{VaR}	0.3718	0.5795	0.3439	0.3790	0.5746	0.3050
S_{CVaR}	0.202	0.3235	0.1889	0.2096	0.3223	0.1661
Sortino Ratio	0.0673	0.1042	0.0630	0.0699	0.1043	0.0562
Rachev Ratio	0.7746	0.7759	0.7792	0.7737	0.8154	0.7863

The terminal compounded return and the performance ratios are very similar between CVaR and Mean Variance, and between the Risk Parity with CVaR and R.P standard deviation.

Comparing the risk measures, starting with volatility out of sample, we observe that Mean Variance has a lower risk as in the other cases. We note the same level of risk between the Risk Parity with standard deviation and Risk Parity with CVaR.

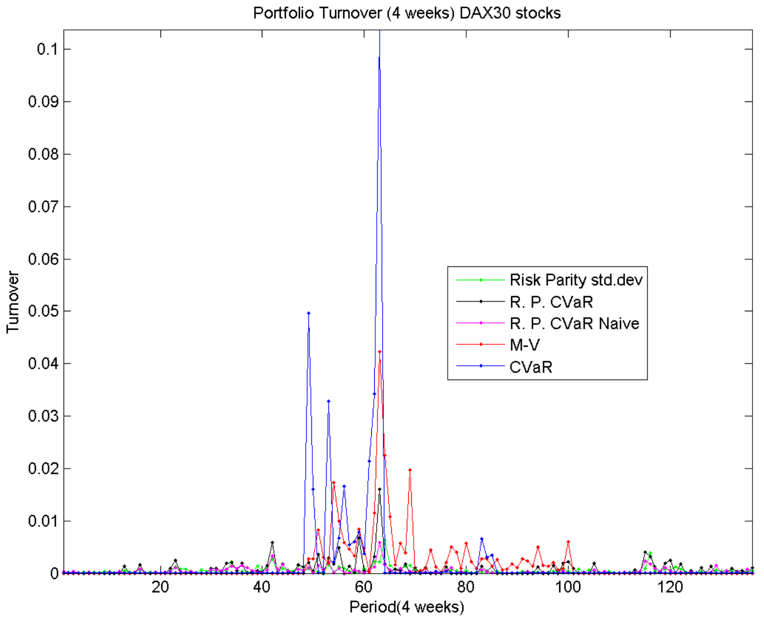


If we compute CVaR out of sample, Mean Variance and CVaR switch places, but the rest is very similar to the volatility case.



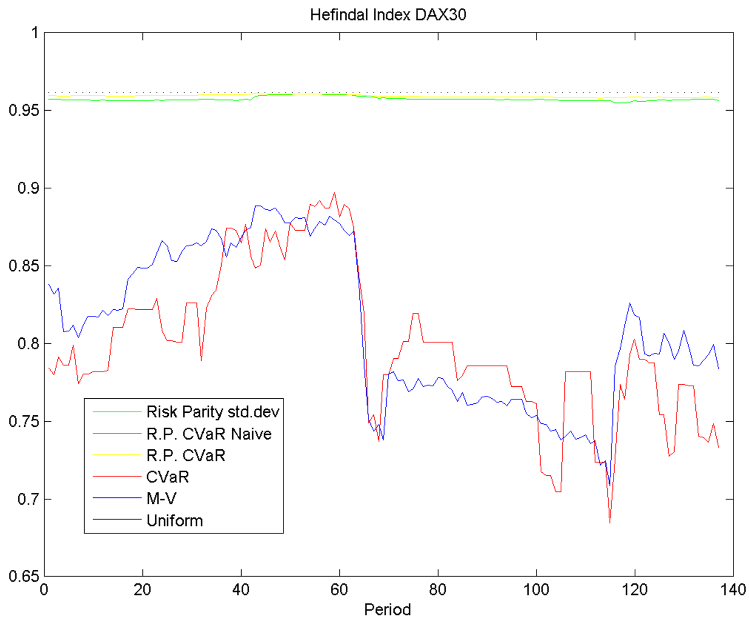
In both cases, the Risk Parity with CVaR has more or less the same level of risk as Risk Parity with standard deviation.

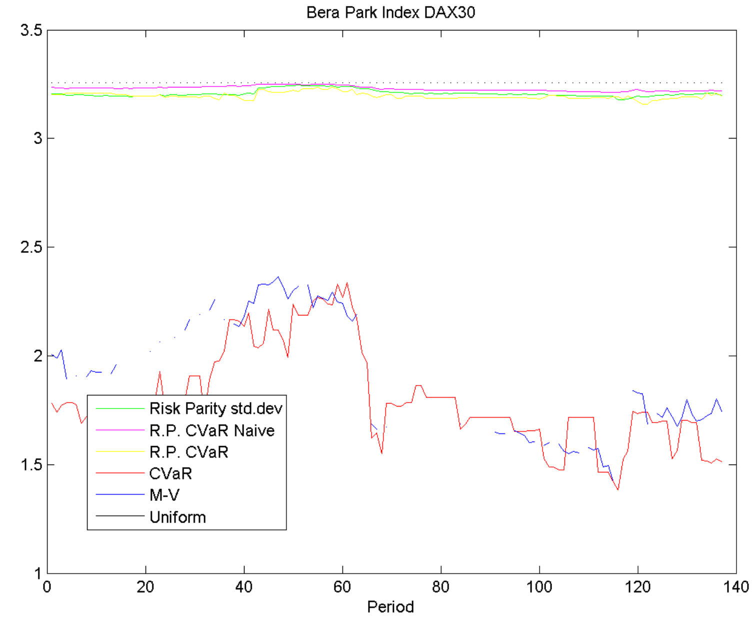
The turnover of Mean Variance and CVaR is 4 to 6 times higher than the Risk Parity group.

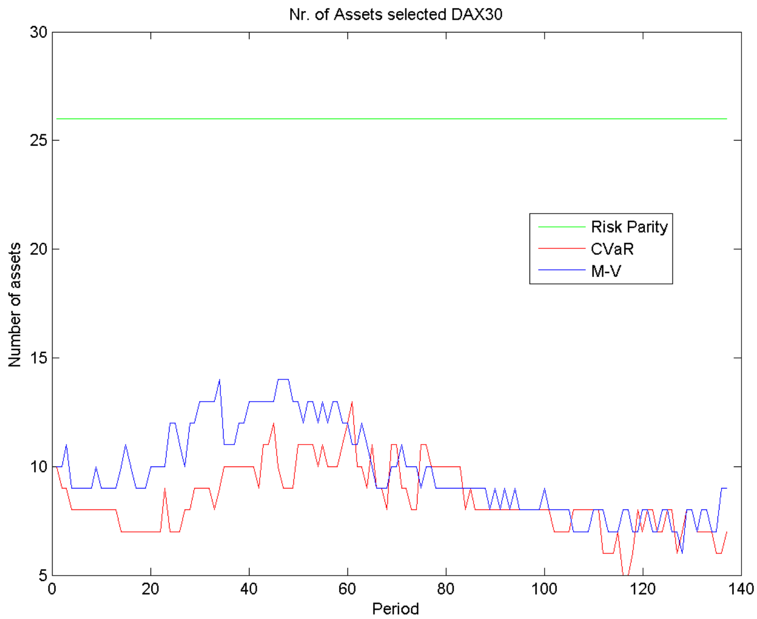


Weekly DAX30	RP-Std	M-V	RP-CVaR N.	RP-CVaR	CVaR
Average Turnover (%)	0.0529	0.1893	0.0438	0.0847	0.2356

Studying the diversification we will expect that CVaR and Mean Variance will concentrate in a smaller group of assets and that the Herfindal and Bera Park will have smaller values.







The CVaR portfolio is more concentrated than Mean Variance and it selects a group of 5-14 assets. To these selected subsets we apply the Risk Parity strategies and study the performance:

If we use an in sample of L=730 weeks we obtain the following results:

Weekly	RP-Std	M-V	RP-CVaR	CVaR	Naive
D_{Her}	0.958641	0.8697264	0.9604169	0.824862	0.961538
D_{BP}	3.222061	2.1591398	3.2436518	1.917492	3.2581
N. Assets	26	11	26	9	26

For the monthly frequencies:

Monthly	RP-Std	M-V	RP-CVaR	CVaR	Naive
D_{Her}	0.95654	0.82976	0.95877	0.70699	0.96153
D_{BP}	3.19911	1.96021	3.22339	1.4898	3.25809
N. Assets	26	9	26	6	26

We only consider assets that have weights higher than 10^{-6}.

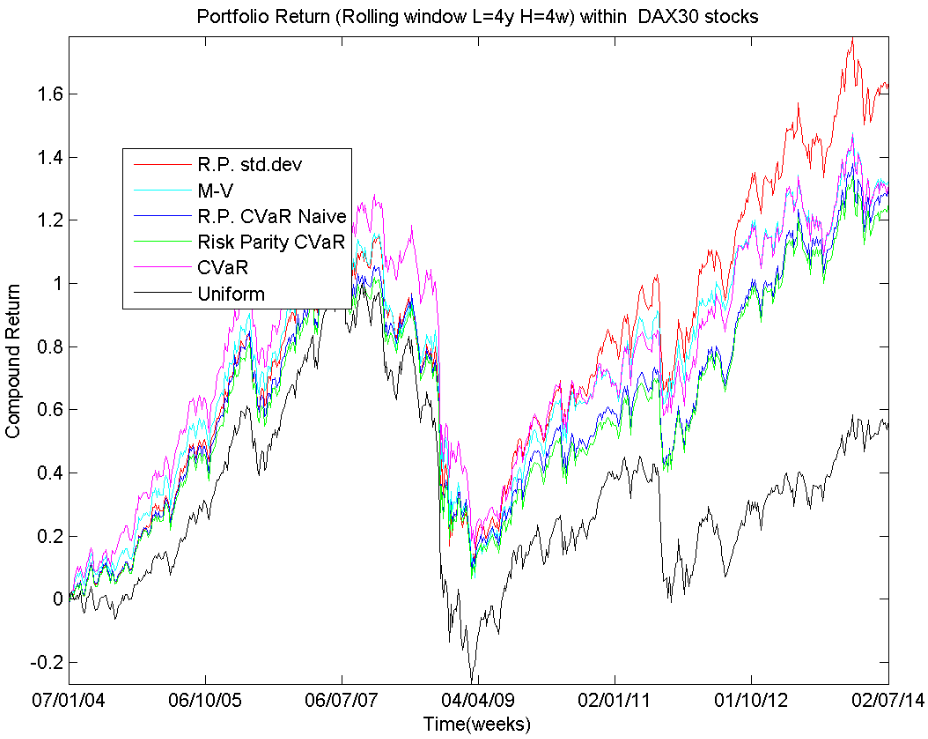
3.3.2 Portfolio subset selection

For the same period, from 1/1/2000 till 4/7/2014, we take the subset selected from Mean Variance and to this group of assets we apply the Risk Parity with standard deviation. We do the same with CVaR and study Risk Parity with CVaR and CVaR-Naive.

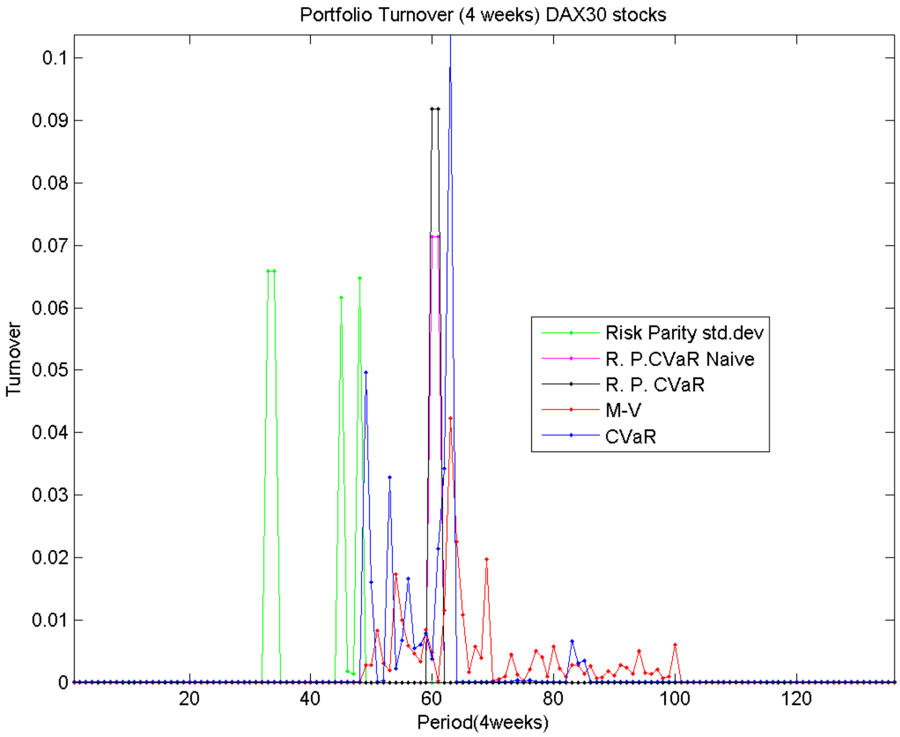
We create the rolling time window under the same conditions as in the previous case, with in sample period of 208 weeks and out of sample of 4 weeks. The results are very surprising:

Weekly	RP-Std	M-V	RP-CVaR Naive	RP-CVaR	CVaR	Uniform
$\mu(\%)$	0.2063	0.1822	0.1815	0.1769	0.1811	0.1277
$\mu_{ann}(\%)$	11.311	9.9259	9.8857	9.6280	9.8635	6.86
$\mu^c(\%)$	163.9363	132.7693	142.3350	125.2608	131.1289	56.4026
$Median$	0.4214	0.3124	0.3742	0.3838	0.2936	0.3837
$\sigma(\%)$	2.3783	2.3271	2.3758	2.3591	2.3372	2.9888
$VaR_{10\%}(\%)$	2.2759	2.3753	2.4555	2.3985	2.3806	3.1186
$CVaR_{10\%}(\%)$	4.3847	4.2556	4.4349	4.4123	4.2435	5.727
$\sigma_{ann}(\%)$	17.1499	16.781	17.1321	17.0115	16.8536	21.5523
$VaR_{10\%}ann(\%)$	16.4115	17.1288	17.7072	17.2957	17.1665	22.4886
$CVaR_{10\%}ann(\%)$	31.6186	30.6875	31.9803	31.8174	30.6	41.2982
S_σ	0.6595	0.5915	0.577	0.566	0.5852	0.3183
S_{VaR}	0.6892	0.5795	0.5583	0.5567	0.5746	0.3050
S_{CVaR}	0.3577	0.3235	0.3091	0.3026	0.3223	0.1661
Sortino Ratio	0.1159	0.1042	0.1012	0.0993	0.1043	0.0562
Rachev Ratio	0.7779	0.7759	0.7722	0.7701	0.8154	0.7863

In this case, the Risk Parity with standard deviation has a higher terminal compound return and a better performance. We notice that there is not a significant difference for the other models Risk Parity CVaR, CVaR and Mean Variance.

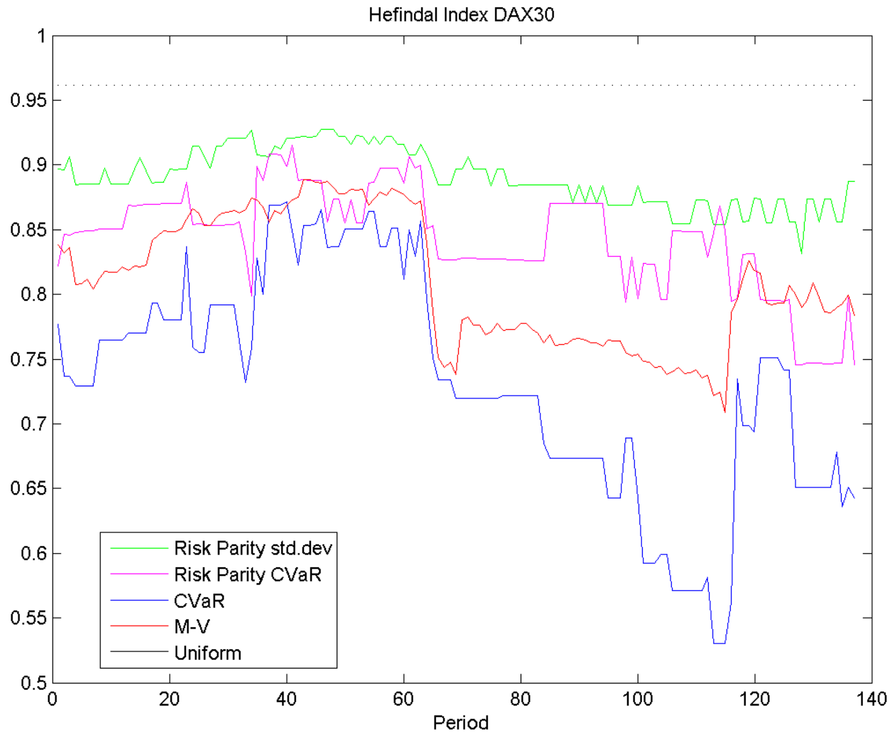


This performance will lead to a higher turnover for the Risk Parity with standard deviation and in case with more transaction costs.



	RP-Std	M-V	RP-CVaR Naive	RP-CVaR	CVaR
Average Turnover (%)	0.1921	0.1893	0.105	0.1351	0.2356

We find a low Herfindal Index for the Risk Parity models, but it is higher, because of the extreme values, for CVaR and Mean Variance.



3.4 Portfolio optimization for the stocks of Eurostoxx 50

In this section we will study the Euro big cap index of Eurostoxx 50. We actually select a group of 44 stocks, avoiding 6 for non continuous data. In order to have a homogeneous study we select the same range of time series data from 1/1/2000 to 4/7/2014 and we show just the results of the weekly time series.

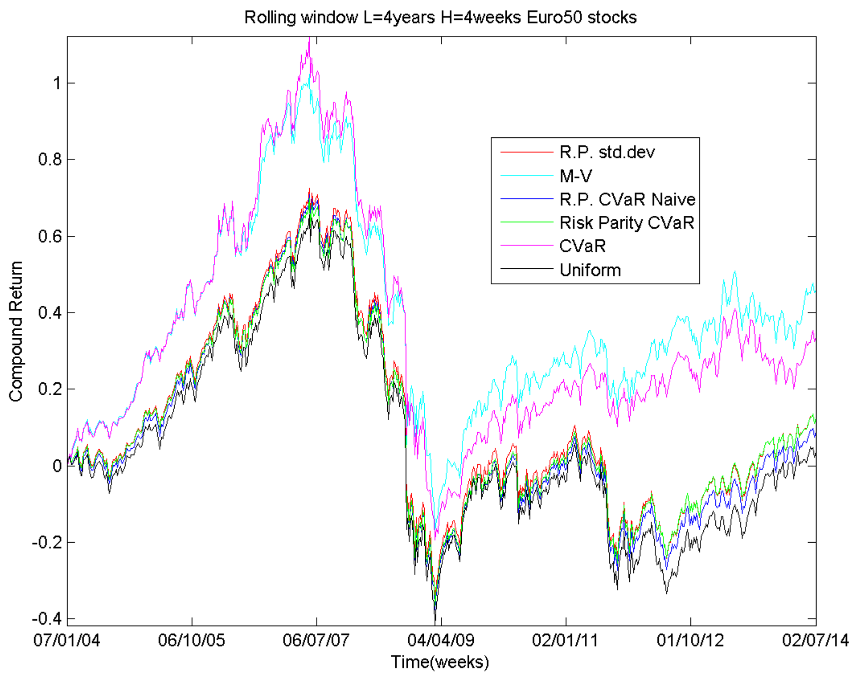
3.4.1 Comparison between models

Creating a rolling time window with in sample of L=208 weeks and out of sample of H=4 week we obtain the following results.

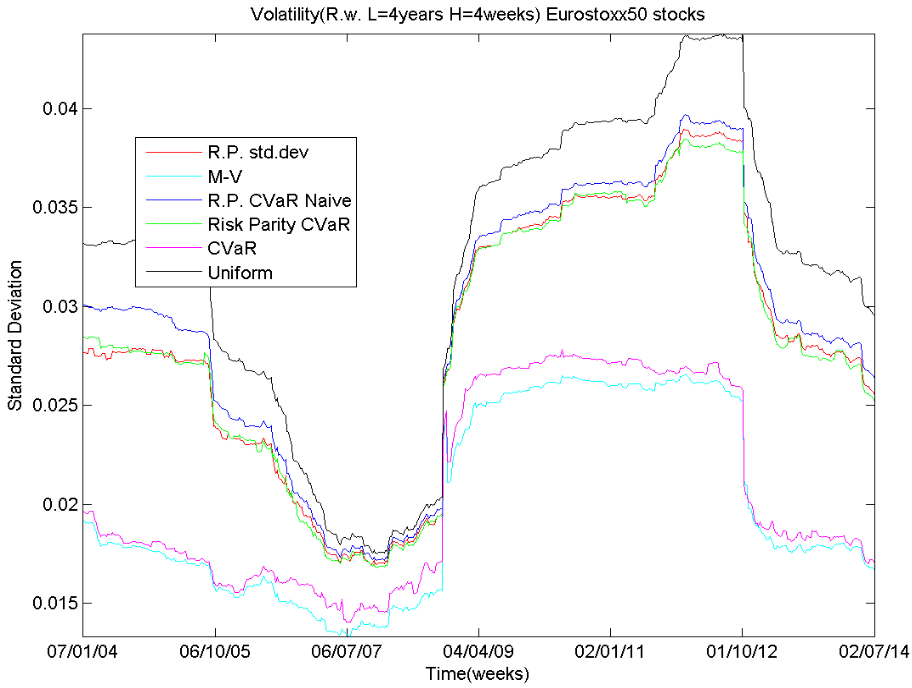
Weekly Euro50	RP-Std	M-V	RP-CVaR N.	RP-CVaR	CVaR	Uniform
$\mu(\%)$	0.0645	0.0980	0.0595	0.0635	0.0818	0.0570
$\mu_{ann}(\%)$	3.4114	5.2281	3.1439	3.3565	4.341	3.0058
$\mu^c(\%)$	12.530	46.0789	8.8183	12.4130	33.6318	3.6832
$Median$	0.4084	0.2874	0.3634	0.4102	0.2897	0.3956
$\sigma(\%)$	2.8835	2.3551	2.9222	2.8555	2.363	3.1253
$VaR_{10\%}(\%)$	3.2348	2.5195	3.3222	3.1785	2.5093	3.4154
$CVaR_{10\%}(\%)$	5.4675	4.2722	5.541	5.4211	4.3819	5.9139
$\sigma_{ann}(\%)$	20.7932	16.9825	21.0724	20.5913	17.0399	22.5367
$VaR_{10\%}ann(\%)$	23.3262	18.1683	23.9567	22.9207	18.0952	24.6288
$CVaR_{10\%}ann(\%)$	39.4267	30.8073	39.9567	39.0920	31.5983	42.6455
S_σ	0.1641	0.3079	0.1492	0.163	0.2548	0.1334
S_{VaR}	0.1462	0.2878	0.1312	0.1464	0.2399	0.1220
S_{CVaR}	0.0865	0.1697	0.0787	0.0859	0.1374	0.0705
Sortino Ratio	0.0291	0.0539	0.0265	0.0289	0.0452	0.0239
Rachev Ratio	0.7713	0.8119	0.7699	0.7709	0.7989	0.7877

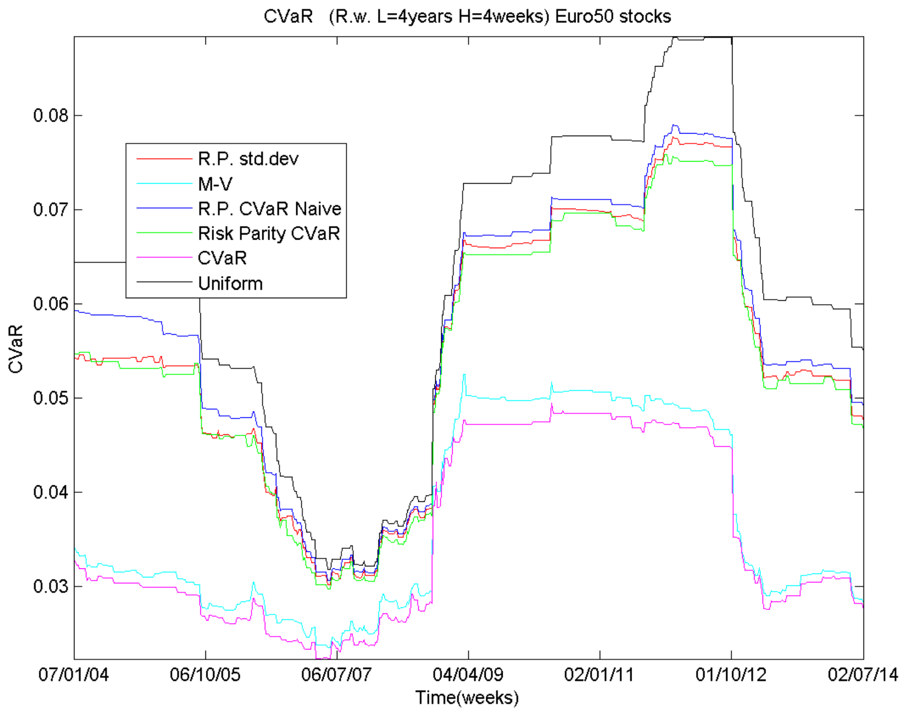
The Mean Variance and CVaR perform better than the other models. The Risk Parity with standard deviation and CVaR are almost identical in performance.

Form the compounded return graph we see that the risk parity group is almost in the same area. This is due to the fact that they take into consideration all the 44 assets, and some of these had a poor performance.

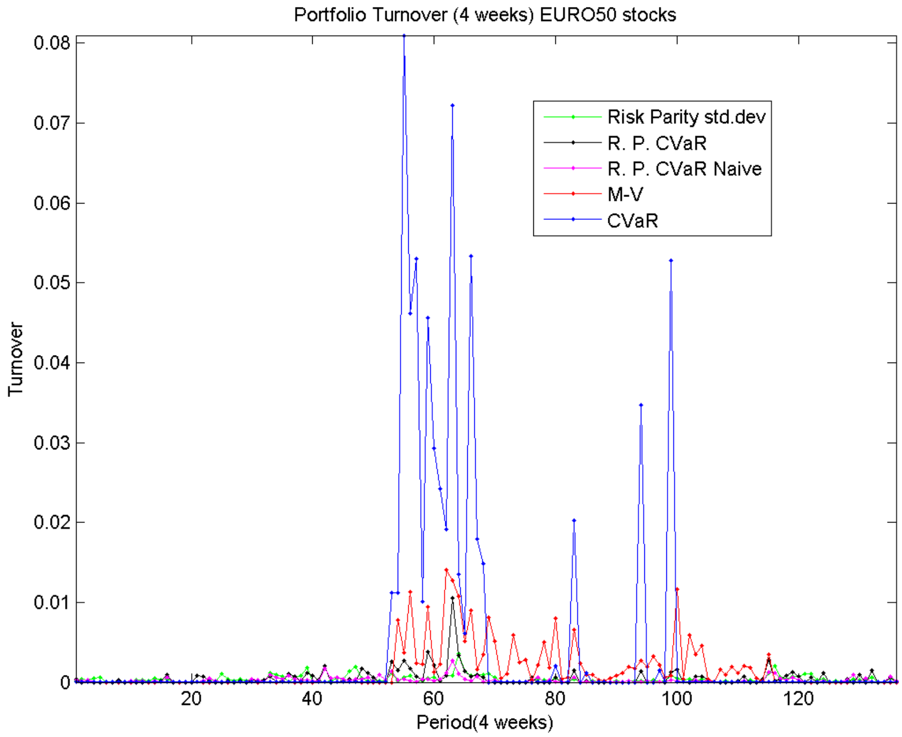


In terms of riskiness, if consider both standard deviation and CVaR (10%), we have the same situation as in the previous cases. This means that there is persistence in the order of riskiness.



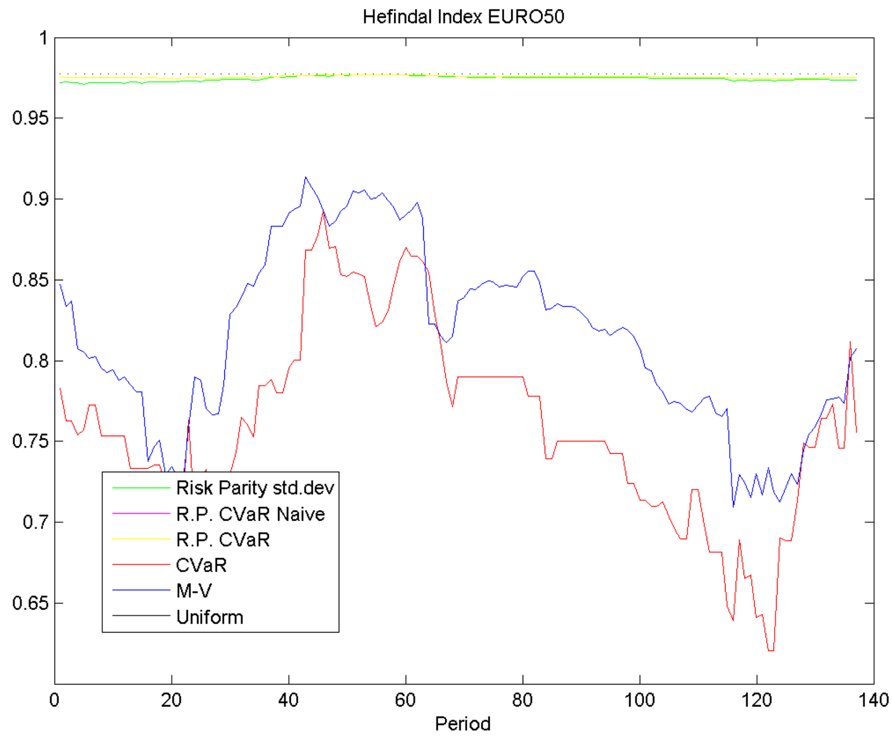


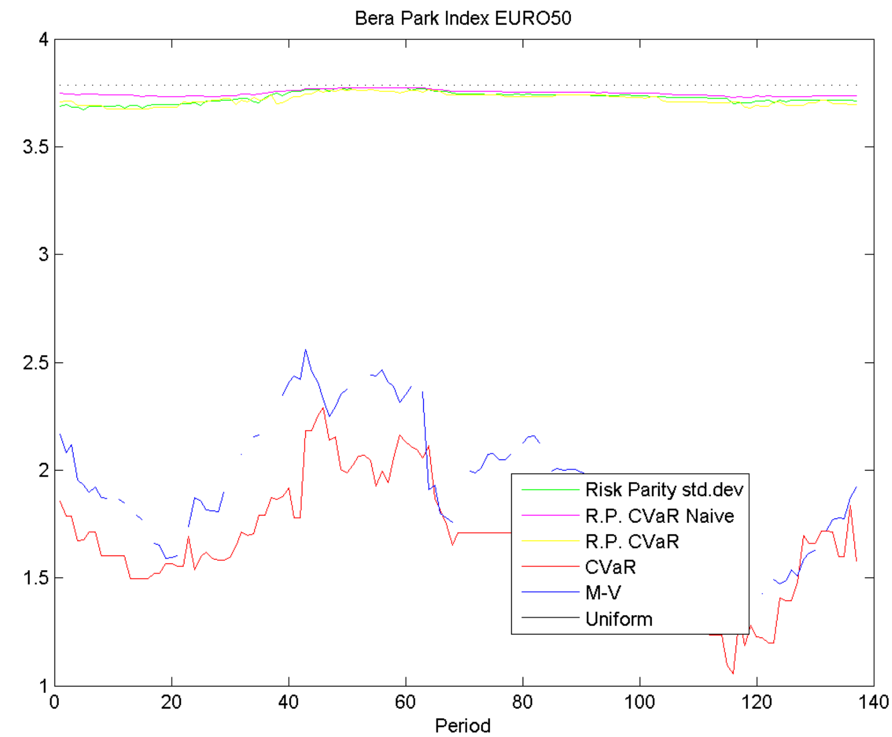
Passing to the study of the turnover we see that due to the fact that CVaR and Mean Variance are more concentrated, they have a higher turnover (5 to 9 time more concentrated in average).

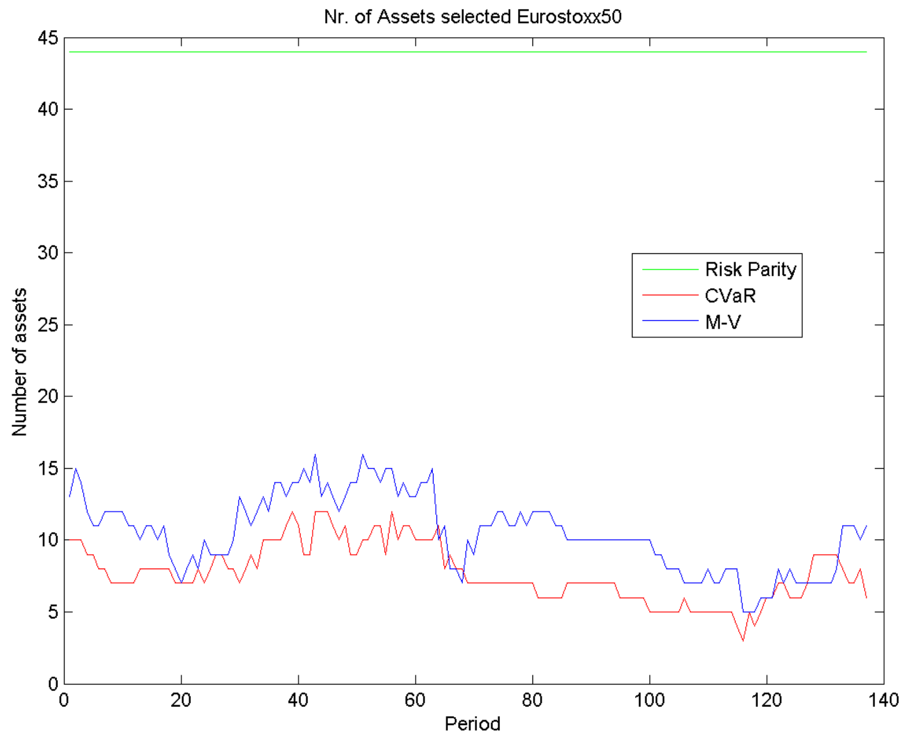


	RP-Std	M-V	RP-CVaR Naive	RP-CVaR	CVaR
Average Turnover(%)	0.0392	0.1546	0.0267173	0.0530	0.4568

In the diversification part we have the same results like in the previous cases due the fact that Mean Variance and CVaR produce extreme weights.


Hefindal Index EURO50
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0
20
40
60
80
100
120
140
Period
Risk Parity std.dev
R.P. CVaR Naive
R.P. CVaR
CVaR
M-V
Uniform


Bera Park Index EURO50
4
3.5
3
2.5
2
1.5
1
0
20
40
60
80
100
120
140
Period
Risk Parity std.dev
R.P. CVaR Naive
R.P. CVaR
CVaR
M-V
Uniform



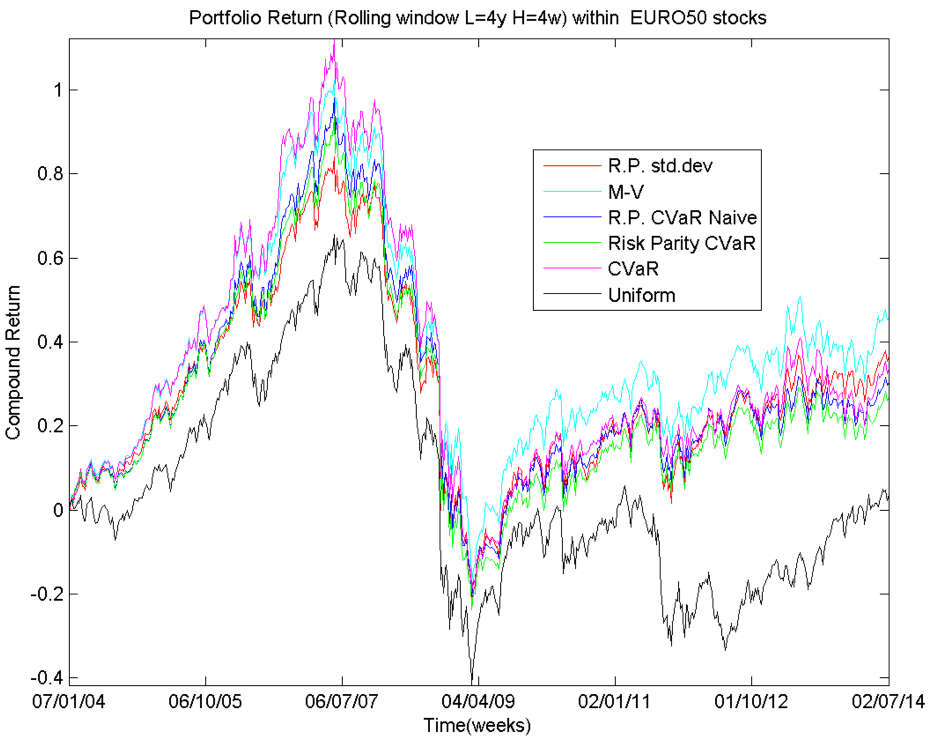
As in the other cases the CVaR is more concentrated for the minimum risk portfolio. In the next section, we will apply the Risk Parity strategies to this group of assets with respect to the corresponding risk measure.

3.4.2 Portfolio subset selection

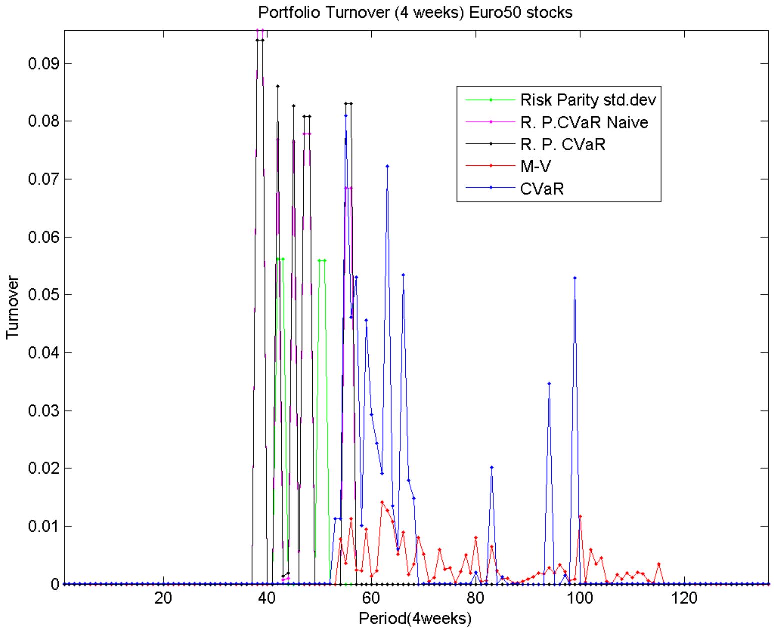
After the selection of a subset of assets, we apply the risk parity criteria with the corresponding risk measures. We use the same time series and the same Rolling window (L=208 and H=4).

Weekly	RP-Std	M-V	RP-CVaR Naive	RP-CVaR	CVaR	Uniform
$\mu(\%)$	0.0865	0.098	0.0766	0.0718	0.0818	0.057
$\mu_{ann}(\%)$	4.6002	5.2281	4.0626	3.8002	4.3410	3.0058
$\mu^c(\%)$	36.6637	46.0789	30.5511	26.9905	33.6318	3.6832
$Median$	0.3349	0.2874	0.2218	0.2445	0.2897	0.3956
$\sigma(\%)$	2.388	2.3551	2.3246	2.3313	2.363	3.1253
$VaR_{10\%}(\%)$	2.5316	2.5195	2.4905	2.5020	2.5093	3.4154
$CVaR_{10\%}(\%)$	4.4633	4.2722	4.3908	4.4128	4.3819	5.9139
$\sigma_{ann}(\%)$	17.2201	16.9825	16.7626	16.8111	17.0399	22.5367
$VaR_{5\%}ann(\%)$	18.2554	18.1683	17.9595	18.0425	18.0952	24.6288
$CVaR_{5\%}ann(\%)$	32.1850	30.8073	31.6629	31.8215	31.5983	42.6455
S_σ	0.2671	0.3079	0.2424	0.2261	0.2548	0.1334
S_{VaR}	0.252	0.2878	0.2262	0.2106	0.2399	0.122
S_{CVaR}	0.1429	0.1697	0.1283	0.1194	0.1374	0.0705
Sortino Ratio	0.0469	0.0539	0.0425	0.0396	0.0452	0.0239
Rachev Ratio	0.7785	0.8119	0.7684	0.7659	0.7989	0.7877

From the table above we notice that the Risk Parity with standard deviation, like in the Mean Variance model, has a significative improvement in order of terminal compounded returns and performance ratios. In the graph we notice that Mean Variance still performs better than the others but Risk Parity with standard deviation is getting closer.

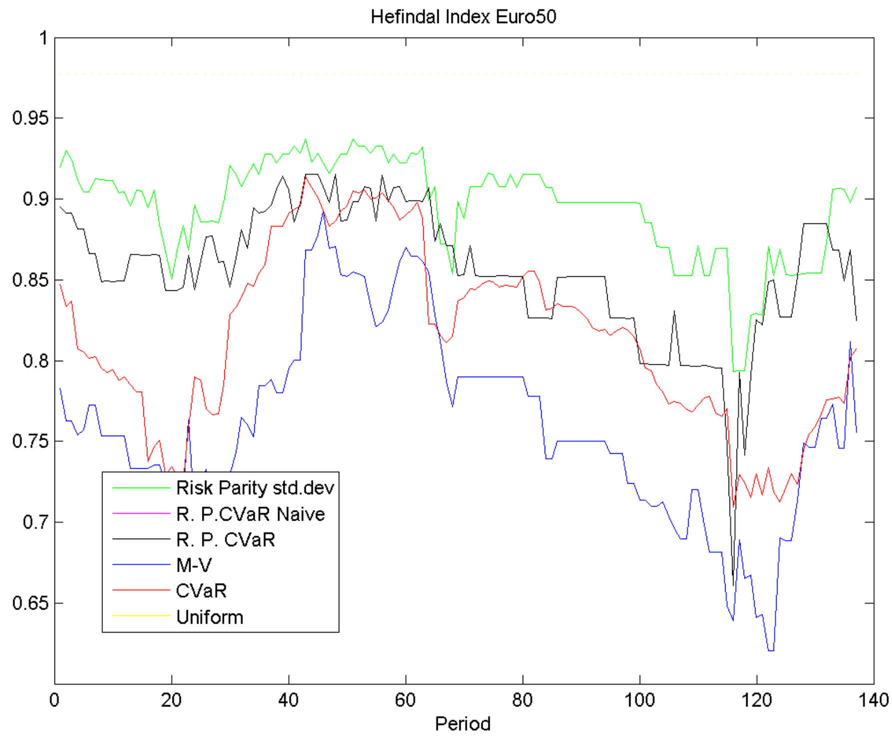


An interesting fact is that the Risk Parity with standard deviation and Risk parity with CVaR now have a higher turnover than the corresponding measure of risk.



	RP-Std	M-V	RP-CVaR Naive	RP-CVaR	CVaR
Average Turnover (%)	0.1649	0.1546	0.47	0.5056	0.4568

In terms of diversification the risk Parity strategies are better due to the concentration of the CVaR and Mean Variance model.



3.5 Portfolio optimization for stocks of FTSE100

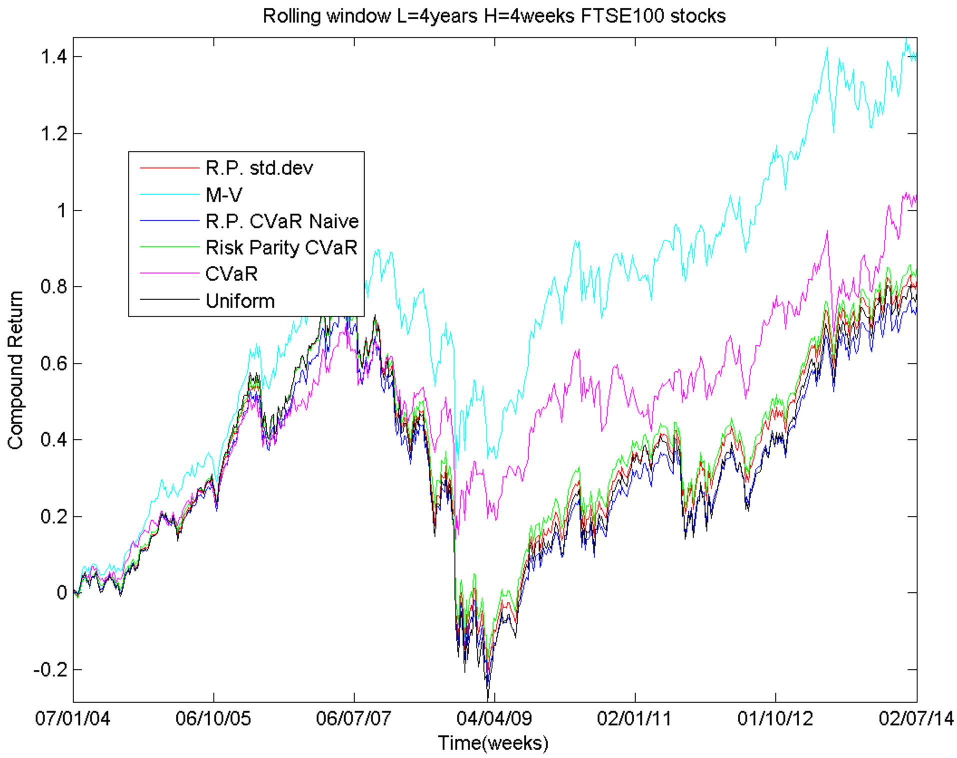
In this section we study FTSE100 of the London Stock Exchange. The purpose is to study how the Risk Parity strategies perform with different numbers of assets and in different markets. As before, we use the same length of data series 1/1/2000- 4/7/2014. We take 77 assets from the 100 possible. As the other European markets this has the same trend through time but the number of assets that consider this time is higher.

3.5.1 Comparison between models

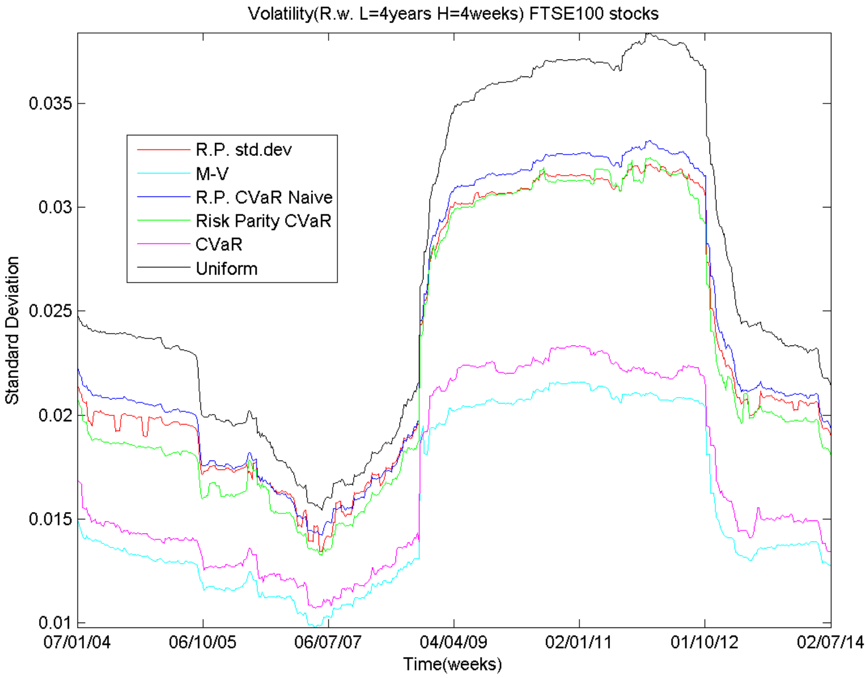
We proceed like in the previous sections creating a rolling time window with in sample period L=4 years and out of sample period H=4 weeks.

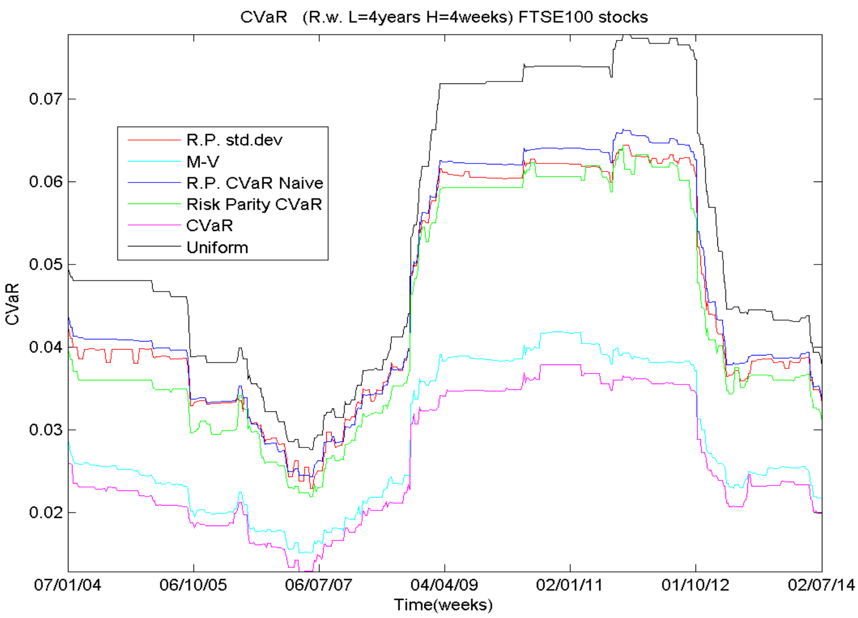
Weekly	RP-Std	M-V	RP-CVaR Naive	RP-CVaR	CVaR	Uniform
$\mu(\%)$	0.1398	0.1801	0.1334	0.1418	0.1501	0.1441
$\mu_{ann}(\%)$	7.5361	9.8103	7.1764	7.6473	8.1093	7.7729
$\mu^c(\%)$	82.0888	141.9759	75.2453	85.2133	104.829	78.8365
$Median$	0.3133	0.3379	0.3308	0.3254	0.2609	0.3518
$\sigma(\%)$	2.4360	1.9135	2.4573	2.3899	1.9386	2.7232
$VaR_{10\%}(\%)$	2.3094	1.778	2.3703	2.2484	2.1232	2.6946
$CVaR_{10\%}(\%)$	4.5973	3.3798	4.6373	4.4810	3.519	5.1982
$\sigma_{ann}(\%)$	17.5665	13.7984	17.7198	17.2342	13.9791	19.6374
$VaR_{10\%}ann(\%)$	16.6535	12.8216	17.0925	16.2133	15.3106	19.4312
$CVaR_{10\%}ann(\%)$	33.1517	24.3724	33.4402	32.3128	25.3759	37.4851
S_σ	0.4290	0.7110	0.4050	0.4437	0.5801	0.3958
S_{VaR}	0.4525	0.7651	0.4199	0.4717	0.5297	0.4
S_{CVaR}	0.2273	0.4025	0.2146	0.2367	0.3196	0.2074
Sortino Ratio	0.0765	0.1280	0.0722	0.0791	0.106	0.0708
Rachev Ratio	0.7953	0.8585	0.7903	0.8	0.8633	0.8098

From the summary table we notice that there is no significative difference between Risk Parity with the standard deviation and Risk Parity with CVaR. Yet again the Mean Variance model leads the performance. The Naive Risk Parity with CVaR is like the Uniform portfolio since it has no benefits of true diversification. This is a particular case because we have chosen a portfolio with 77 assets; since the Risk Parity strategies group and the uniform portfolios must take into consideration every single asset, it appears to have a lower performance than the portfolios that are more concentrated.

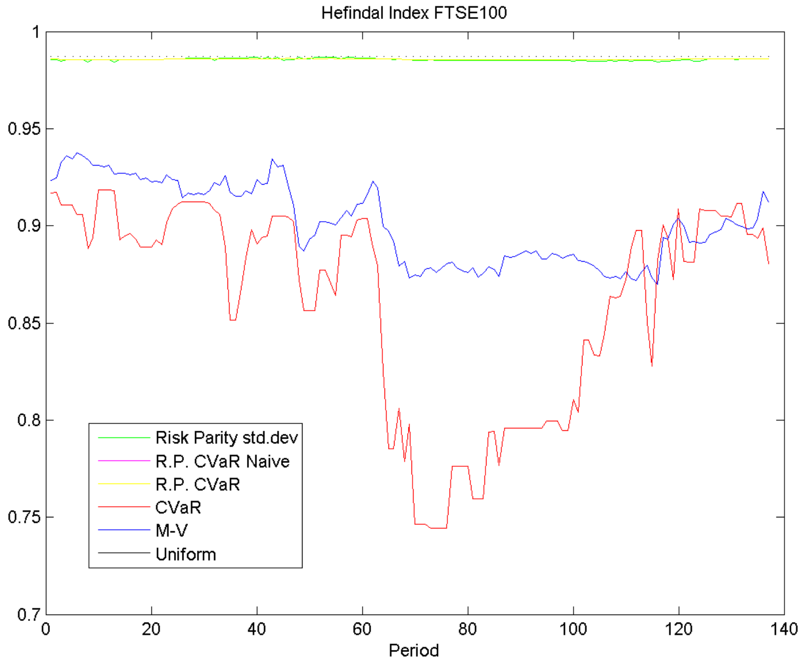


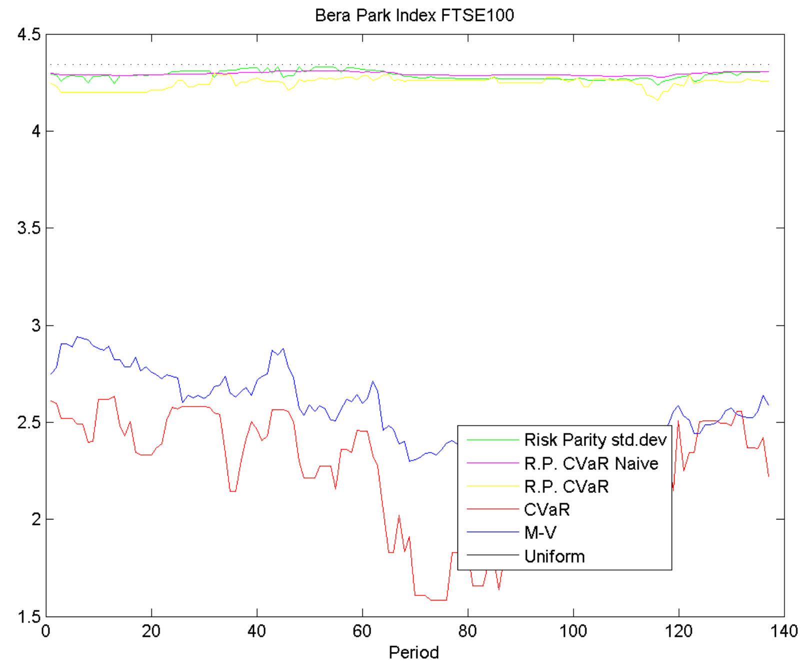
Considering risk, in terms of volatility and CVaR, in this case we still find persistence with the other cases. The Risk Parity with standard deviation has the same level of risk as Risk Parity with CVaR. Since we have a larger portfolio than in the other cases, is easier to see the distance between CVaR and Mean Variance models.

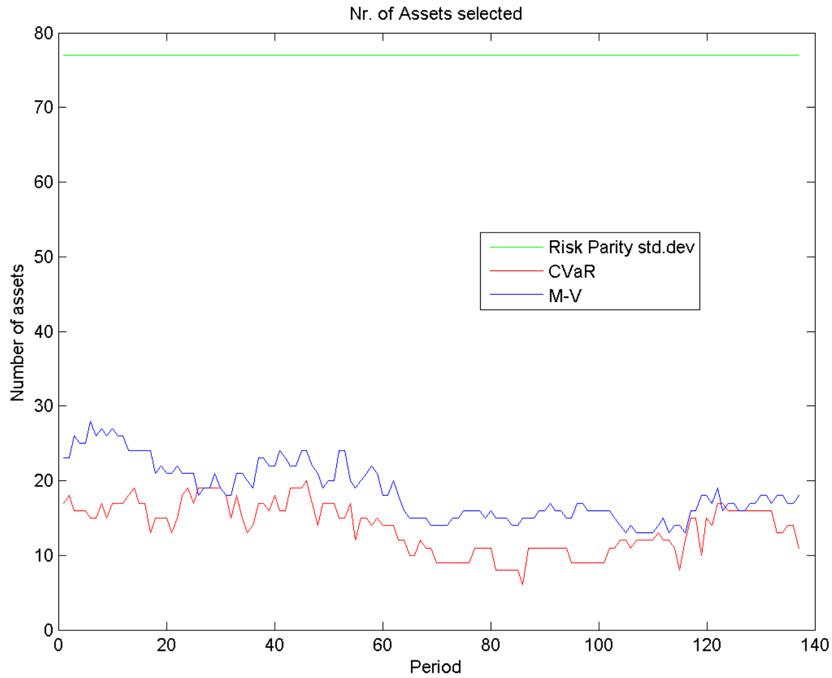




The turnover is very similar to the previous cases and for that we will not show it, but we will focus on the diversification measures.


Hefindal Index FTSE100
1
0.95
0.9
0.85
0.8
0.75
0.7
0
20
40
60
80
100
120
140
Period
Risk Parity std.dev
R.P. CVaR Naive
R.P. CVaR
CVaR
M-V
Uniform


Bera Park Index FTSE100
4.5
4
3.5
3
2.5
2
1.5
0
20
40
60
80
100
120
140
Period
Risk Parity std.dev
R.P. CVaR Naive
R.P. CVaR
CVaR
M-V
Uniform



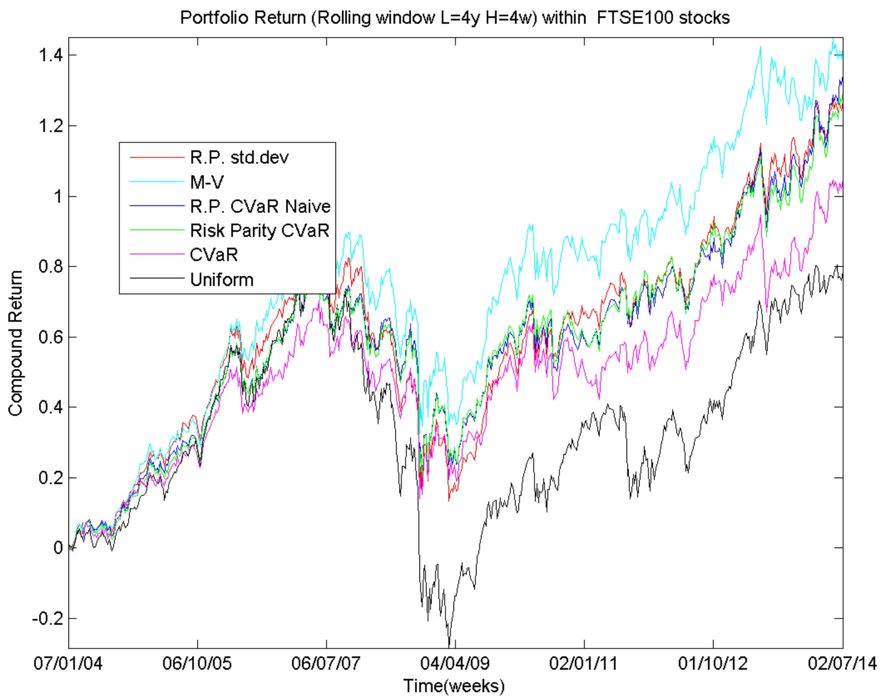
The Conditional Value at Risk tends to be more concentrated (8-20 assets), and with high values of the weights. The Mean Variance model is less concentrated (18 to 28 assets), and from the Herfindal index we can say that it does not have weights too concentrated in one asset. The Uniform and the Risk Parity group are the best in diversification.

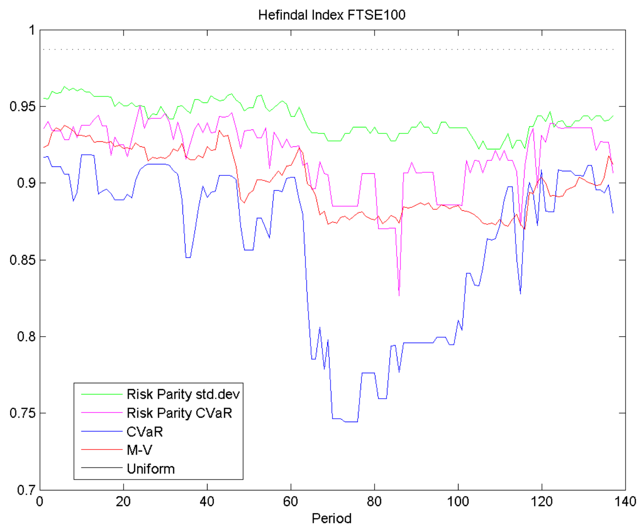
3.5.2 Portfolio subset selection

Starting from the subset of assets selected from Mean Variance and CVaR models, we apply the Risk Parity strategies using the same time series and the Rolling time window of the case above.

This time the Risk parity with CVaR is applied to a much smaller number of assets. This correction improves the performance (terminal compounded return from 85.2133 to 129.7467) and the diversification.

Weekly	RP-Std	M-V	RP-CVaR Naive	RP-CVaR	CVaR	Uniform
$\mu(\%)$	0.1689	0.1801	0.1746	0.1707	0.1501	0.1441
$\mu_{ann}(\%)$	9.1707	9.8103	9.4943	9.2715	8.1093	7.7729
$\mu^c(\%)$	126.9397	141.976	134.749	129.7467	104.829	78.84
$Median$	0.3819	0.3379	0.2784	0.2781	0.2609	0.3518
$\sigma(\%)$	1.9383	1.9135	1.9143	1.9164	1.9386	2.7232
$VaR_{10\%}(\%)$	1.891	1.778	1.8651	1.9422	2.1232	2.6946
$CVaR_{10\%}(\%)$	3.5378	3.3798	3.4347	3.4472	3.519	5.1982
$\sigma_{ann}(\%)$	13.9772	13.7984	13.8045	13.8194	13.9791	19.6374
$VaR_{10\%}ann(\%)$	13.6365	12.8216	13.4497	14.0053	15.3106	19.4312
$CVaR_{10\%}ann(\%)$	25.5112	24.3724	24.7678	24.8581	25.3759	37.4851
S_σ	0.6561	0.7110	0.6878	0.6709	0.5836	0.3958
S_{VaR}	0.6725	0.7651	0.7059	0.662	0.5297	0.4
S_{CVaR}	0.3595	0.4025	0.3833	0.373	0.3196	0.2074
Sortino Ratio	0.1169	0.128	0.124	0.1213	0.106	0.0708
Rachev Ratio	0.7952	0.8585	0.8195	0.8153	0.8633	0.8098





Proceeding this way we limit the selection of assets in a smaller group obtaining a better performance and a better diversification for the Risk Parity strategies.

3.6 Portfolio optimization for stocks of Nikkei 225

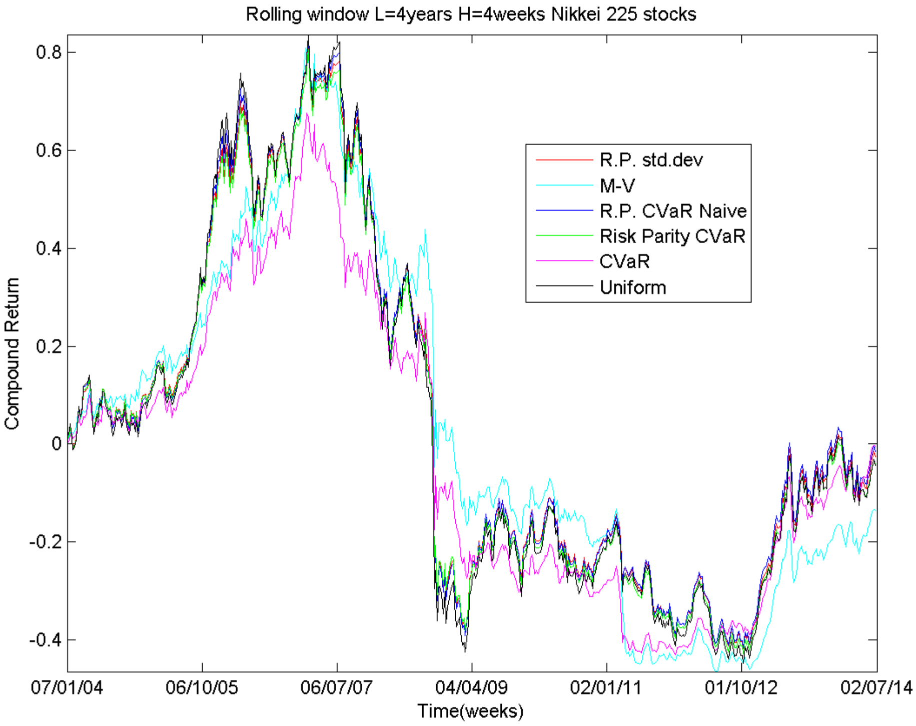
For the last case in stock markets we consider a market with a very large number of assets. We choose the stocks that compose the Japan index of Nikkei225.

We take 188 stocks and do not consider the others for missing data. Since there is a large number of assets, it will better represent the impact of the crisis of 2008 and for that the performance of the models should be divided in two parts: before the crisis and after the crisis. If we consider the whole period of time, it will take more for the portfolios to regain the value before the crisis, and so they will have negative compounded returns.

3.6.1 Comparison between models

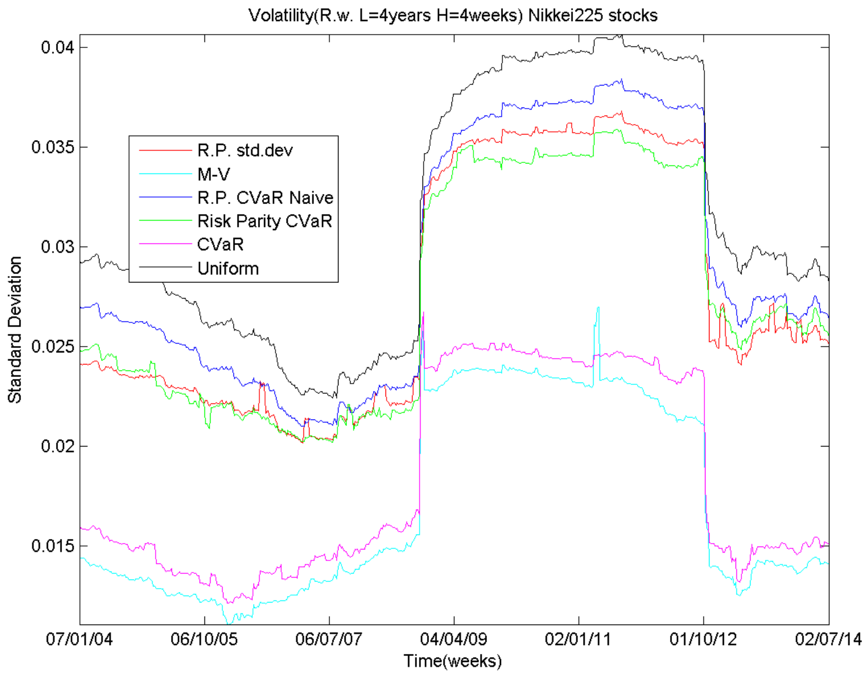
If we take the same time series reference from 1/1/2000 to 4/7/2014 and apply the rolling time window for in sample period L=208 weeks and out of sample period H=4 weeks. The table of results is shown below:

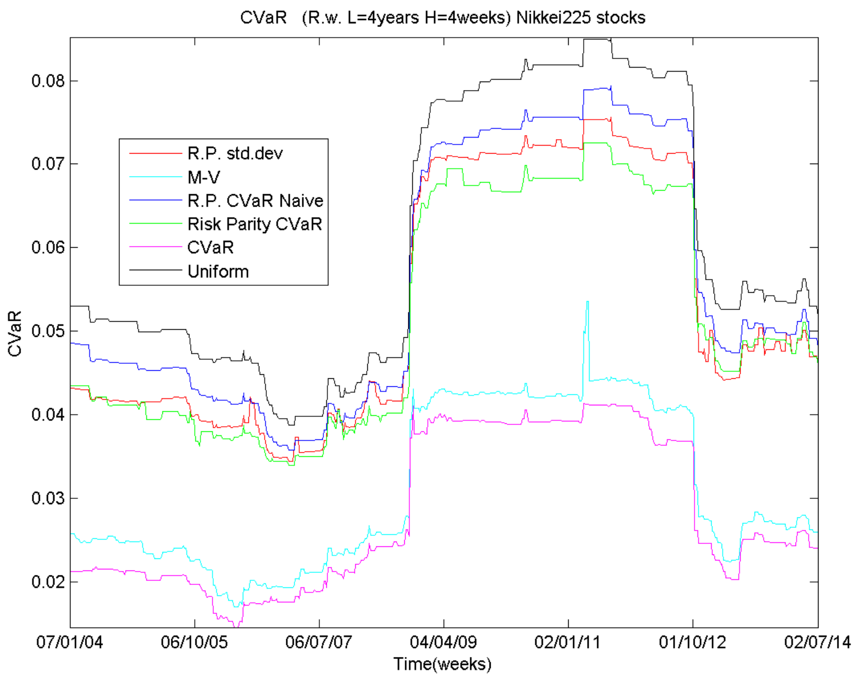
Weekly	RP-Std	M-V	RP-CVaR Naive	RP-CVaR	CVaR	Uniform
$\mu(\%)$	0.0405	0.0031	0.0446	0.0357	0.0272	0.0449
$\mu_{ann}(\%)$	2.1286	0.1628	2.3472	1.8741	1.4255	2.3621
$\mu^c(\%)$	-2.5187	-13.6596	-1.5350	-4.2388	-0.4710	-4.3594
$Median$	0.2801	0.0839	0.3060	0.24	0.162	0.3892
$\sigma(\%)$	2.9439	2.3808	3.0211	2.8927	2.3160	3.1991
$VaR_{10\%}(\%)$	3.1821	2.1751	3.3622	3.0762	2.3414	3.6494
$CVaR_{10\%}(\%)$	5.5735	4.2026	5.7176	5.4369	4.1498	6.1132
$\sigma_{ann}(\%)$	21.2289	17.1683	21.7858	20.8595	16.7009	23.069
$VaR_{10\%}ann$	22.9465	15.6851	24.2455	22.1828	16.8839	26.3166
$CVaR_{10\%}ann$	40.1914	30.3055	41.23	39.2059	29.924	44.0831
S_σ	0.1003	0.0095	0.1077	0.0898	0.0854	0.1024
S_{VaR}	0.0928	0.0104	0.0968	0.0845	0.0844	0.0898
S_{CVaR}	0.053	0.0054	0.0569	0.0478	0.0476	0.0536
Sortino Ratio	0.0177	0.0016	0.0191	0.0159	0.015	0.0182
Rachev Ratio	0.7274	0.762	0.7354	0.7355	0.7903	0.7401



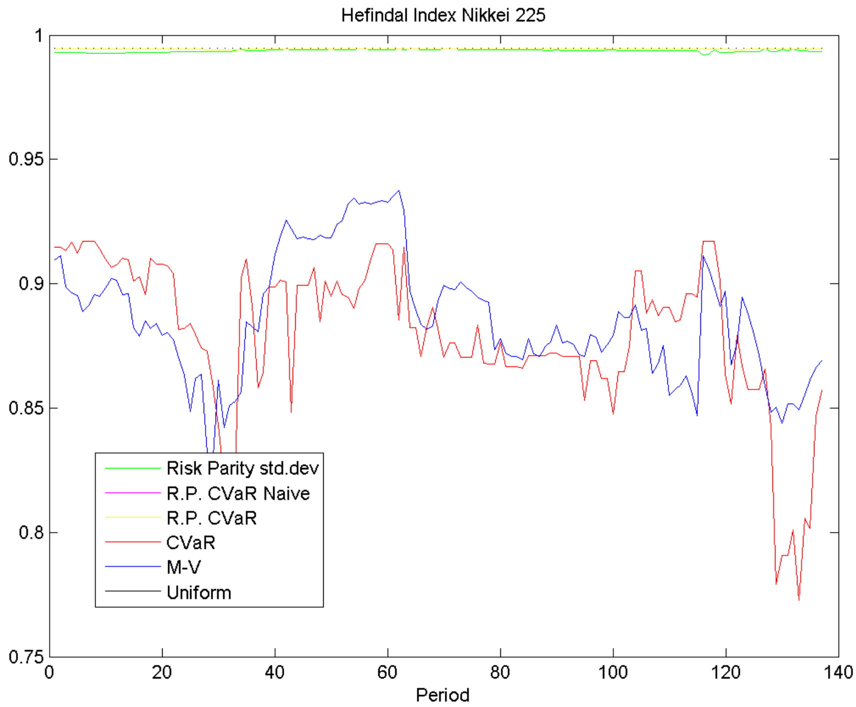
Before the crisis The Risk Parity models perform better than CVaR; due to the large number of assets, the Risk Parity has a bigger drawdown during the crisis.

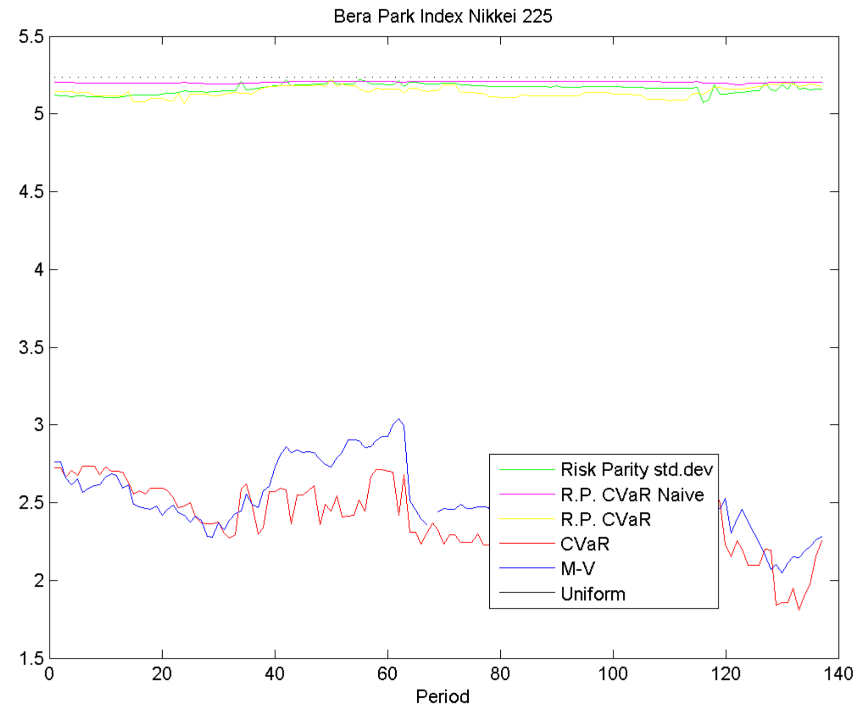
The order of riskiness will be the same, as in the previous:

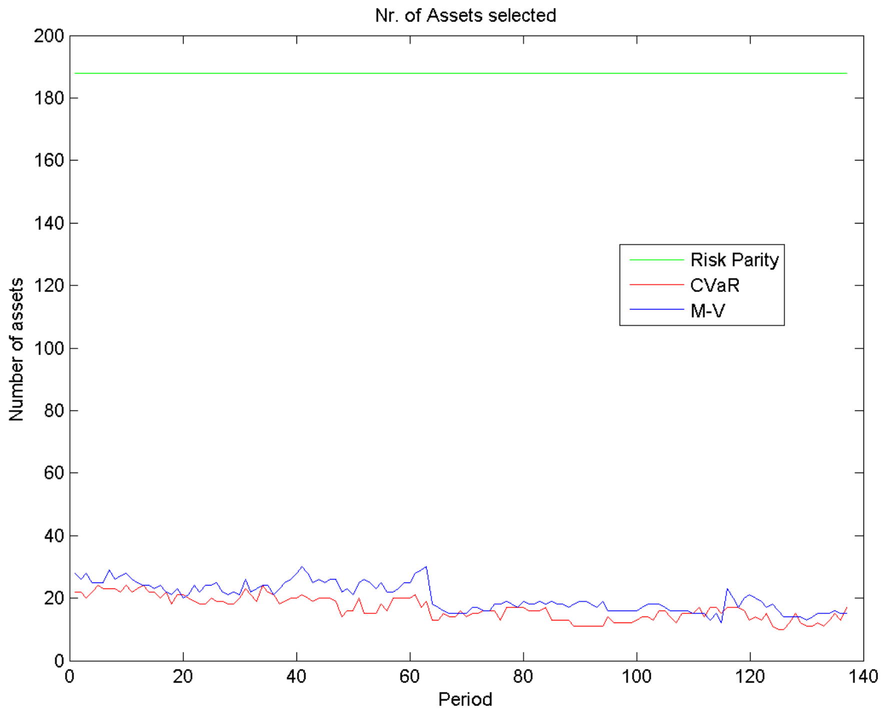




The diversification will be better for the Risk Parity strategy in the case of a large portfolio. The Bera Park index gives a clearer view of the concentration of portfolios.


Hefindal Index Nikkei 225
1
0.95
0.9
0.85
0.8
0.75
0
20
40
60
80
100
120
140
Period
Risk Parity std.dev
R.P. CVaR Naive
R.P. CVaR
CVaR
M-V
Uniform

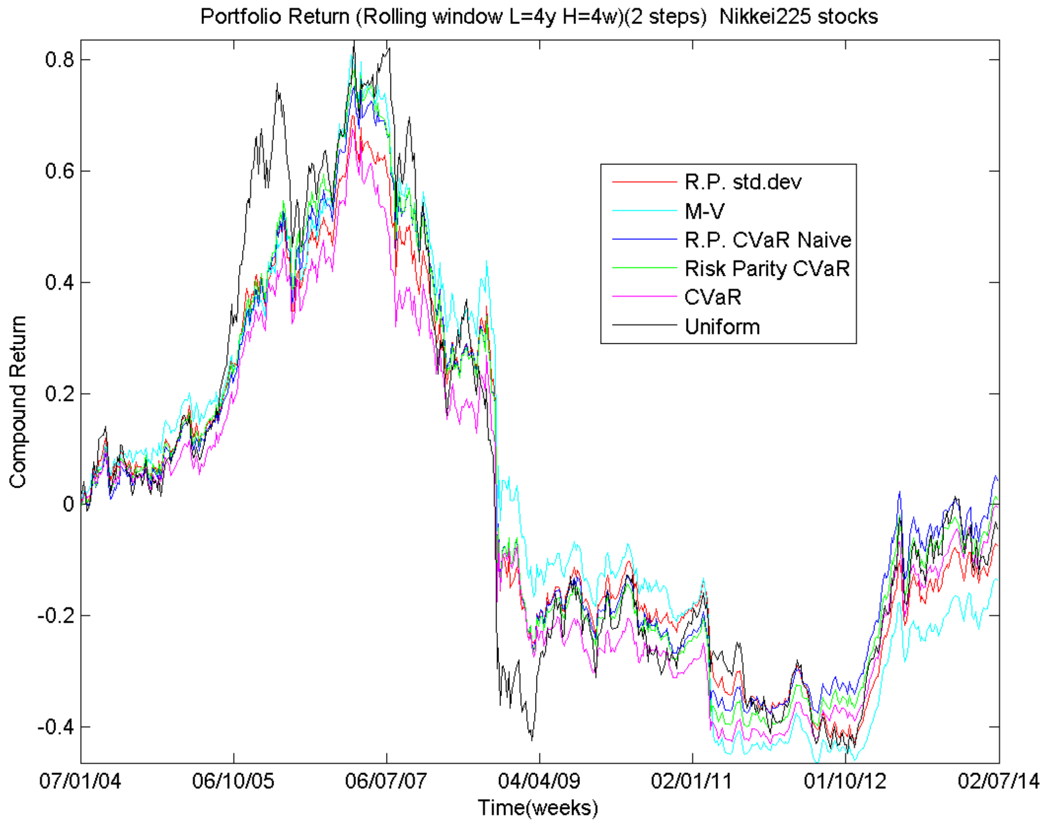

Bera Park Index Nikkei 225
5.5
5
4.5
4
3.5
3
2.5
2
1.5
0
20
40
60
80
100
120
140
Period
Risk Parity std.dev
R.P. CVaR Naive
R.P. CVaR
CVaR
M-V
Uniform



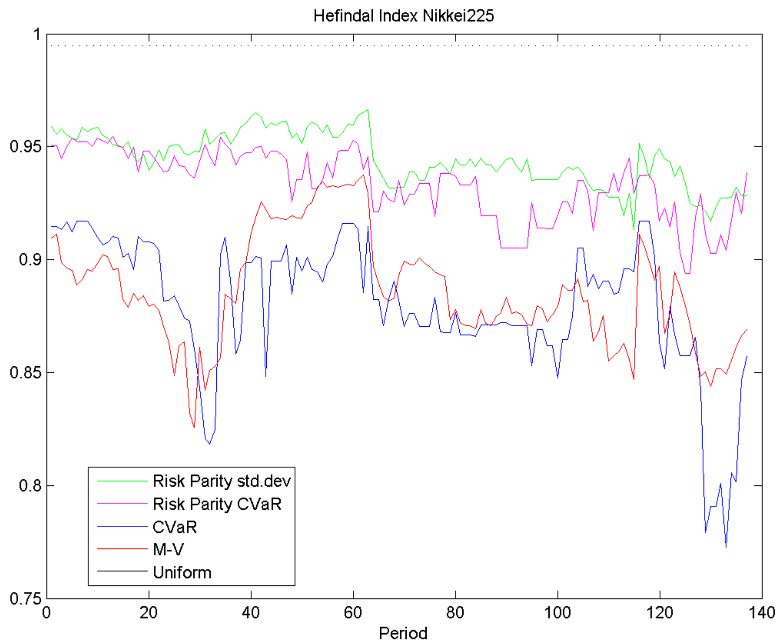
3.6.2 Portfolio subset selection

We notice that Mean Variance selects 20-30 assets from the 188 possible choices and CVaR even less. If we apply the Risk Parity strategy to this subset selected (R.P. with standard deviation to the subset selected with Mean Variance and R.P. with CVaR to the group selected with CVaR model) and use a rolling time window with the same time horizon, we obtain the following :

Weekly 2	RP-Std	M-V	RP-CVaR Naive	RP-CVaR	CVaR	Uniform
$\mu(\%)$	0.0158	0.0031	0.0377	0.0324	0.0272	0.0449
$\mu_{ann}(\%)$	0.823	0.1628	1.9812	1.6973	1.4255	2.3621
$\mu^c(\%)$	-7.3846	-13.6596	4.2832	1.0310	-0.4710	-4.3594
$Median$	0.1743	0.0839	0.1384	0.1412	0.1620	0.3892
$\sigma(\%)$	2.3748	2.3808	2.3865	2.3974	2.316	3.1991
$VaR_{10\%}(\%)$	2.5266	2.1751	2.3695	2.3342	2.3414	3.6494
$CVaR_{10\%}(\%)$	4.3395	4.2026	4.3346	4.3360	4.1498	6.1132
$\sigma_{ann}(\%)$	17.1252	17.1683	17.2091	17.2876	16.7009	23.0691
$VaR_{10\%}ann$	18.2196	15.6851	17.0866	16.8319	16.8839	26.3166
$CVaR_{10\%}ann$	31.2927	30.3055	31.2575	31.2675	29.9246	44.0831
S_σ	0.0481	0.0095	0.1151	0.0982	0.0854	0.1024
S_{VaR}	0.0452	0.0104	0.1160	0.1008	0.0844	0.0898
S_{CVaR}	0.0263	0.0054	0.0634	0.0543	0.0476	0.0536
Sortino Ratio	0.0083	0.0016	0.0198	0.0168	0.0150	0.0182
Rachev Ratio	0.7423	0.7622	0.7144	0.7020	0.7903	0.7401



In this case the Risk Parity with CVaR- Naive and Risk Parity with CVaR perform better than the other models. And the diversification will be as follows:



So we obtain portfolios with a better performance and better diversified.

3.7 Portfolio optimization with Commodities

In this section we consider portfolios of Commodities in the period from January 2000 to end of September 2014 .

The goal is the study of Risk Parity strategies for a small group of assets with a particular distribution.

We consider a portfolio of 4 Commodities and 4 foreign currencies:

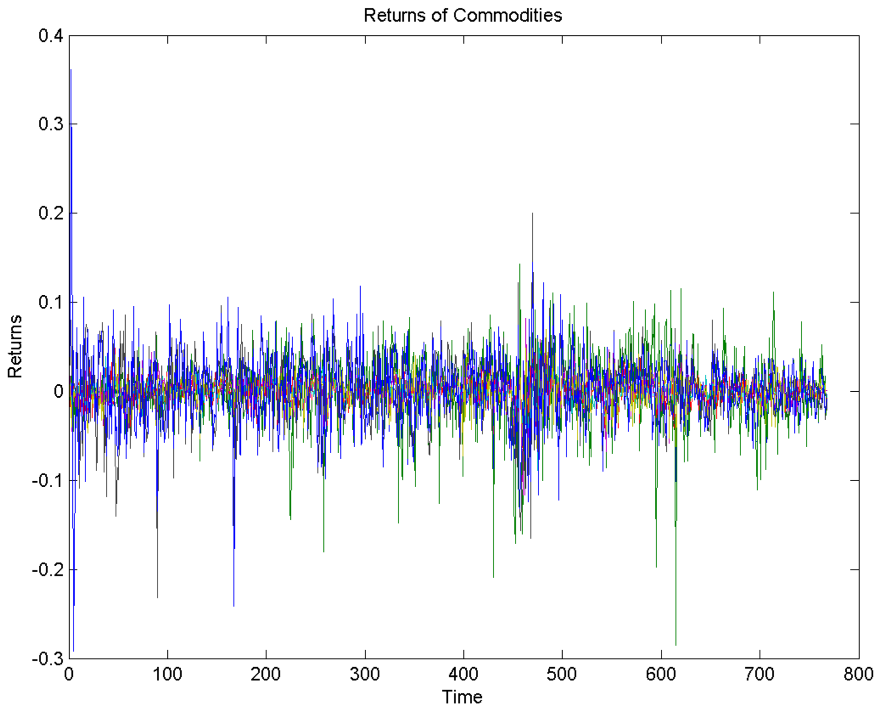
Gold
Silver
Oil
Heat Oil
Euro
Pounds
Australian Dollar
New Zealand Dollar

The price of the commodities and the exchange rate is in dollars. In the real markets the foreign currencies exchange rates have a different behavior from the commodities.

Returns of the commodities and currencies have the following distribution:

	Mean	Median	Std. dev.	Min	Max	Range	Kurtosis	Skewn
Gold	0,002	0,00234	0.0252	-0.1056	0.1170	0,2226	4,8774	-0,286
Silver	0,0018	0,00386	0.0434	0.2854	0.1425	0,4279	7,6291	-1,036
Euro	0,00035	0,00106	0.0144	-0.0632	0.0563	0,1195	4,3344	-0,3016
GBPo	0.00005	0,0004	0.0135	-0.0890	0.0551	0,1441	8,8843	-0,8867
Au Doll	0,00047	0,0014	0.0185	-0.1165	0.0816	0,1985	8,4358	-0,9257
New Zea.	0,00066	0,0026	0.0191	-0.0898	0.0696	0,1593	4,8085	-0,5965
Oil	0,0018	0,00447	0.0414	-0.2316	0.2002	0,4318	5,670	-0,615
Heat Oil	0,00175	0,0024	0.0448	-0.2914	0.3617	0,653	12,473	0,1222

Most of the asset's returns have higher Kurtosis but small negative Skewness. It is clear that gold and the Euro currency are more stable than the others.



From the graph we see that the Euro and Great Britain Pound are more fluctuating than the other assets. We will proceed with the application of the Risk parity models and analyze their performance comparing them with the base models.

3.7.1 Risk Parity applied to Standard Deviation

To have a clear idea of how the weights are distributed we take a static view, for instance the in sample for L=730 observations for 8 assets.

The weights for the Risk Parity with the standard deviation are the following:

Asset i	x_i	$\sigma_{x_i}(x)$	$TRC_i(x)$ (10^{-3})
1	0,113473	0.015907	0.0018
2	0,068762	0.028407	0.0019
3	0,179930	0.009821	0.0018
4	0,189567	0.008611	0.0017
5	0,132533	0.013914	0.0018
6	0,137012	0.013475	0.0018
7	0,091176	0.021057	0.0019
8	0,087544	0.020943	0.0018
SUM	1	Std. Dev.	0.0146

Since the silver has higher volatility, the marginal risk contribution is higher, so the weight is smaller. The Oil and Heat Oil are less preferred from the model. So the portfolio is composed mostly of gold and currencies.

The Mean Variance model has the following weights:

Asset i	x_i
1	0.0485475
2	0
3	0.3075421
4	0.49843959
5	0
6	0.05856998
7	0.03859674
8	0.04830397
SUM	1

The Mean Variance portfolio does not include silver and Australian dollar. Half of the portfolio is composed from Great Britain pound and 30% from Euro. It is interesting to observe that gold is not preferred by the model.

3.7.2 Risk Parity applied to CVaR

In this part we compute the Risk Parity with CVaR and Risk Parity CVaR Naive at the same confidence level of 10%. We remember that Risk Parity CVaR Naive has no true diversification. We take a static view computing an in sample period of L=730 weeks:

	Risk Parity CVaR-Naive				Risk Parity CVaR		
Asset i	x_i	$\frac{\partial CVaR_\alpha(X)}{\partial x_i}$	$TRC_i(x)$	Asset i	x_i	$\frac{\partial CVaR_\alpha(X)}{\partial x_i}$	$TRC_i(x)$
1	0.11237	0.047361	0.00532	1	0.111893	0.030177	0.00337
2	0.06199	0.085857	0.00532	2	0.055594	0.060804	0.00338
3	0.19439	0.027378	0.00532	3	0.204942	0.016445	0.00337
4	0.20878	0.025491	0.00532	4	0.188135	0.017920	0.00337
5	0.15001	0.035476	0.00532	5	0.1275434	0.026466	0.00338
6	0.13879	0.038344	0.00532	6	0.1263714	0.026712	0.00338
7	0.06627	0.080308	0.00532	7	0.0975126	0.034637	0.00338
8	0.06739	0.078967	0.00532	8	0.0880085	0.038461	0.00338
SUM	1	CVaR	0.04258	SUM	1	CVaR	0.02701

Like in the case of Risk Parity with standard deviation, silver is not preferred and the marginal risk contribution for this asset is higher than in the others models. In each case, the marginal risk contribution for Risk Parity CVaR naive is higher than the corresponding marginal contribution in Risk Parity with CVaR as risk measure. The selection is very similar to Risk Parity with standard deviation but we have more gold this time. The level of risk of Risk Parity CVaR naive is higher than R.P with CVaR.

We apply the Conditional Value at Risk CVaR at level 10%

Asset i	x_i
1	0.067424248796
2	0
3	0.418914070069
4	0.392852208649
5	0
6	0
7	0.068190282803
8	0.052619189682
SUM	1

The results are very similar to the Mean Variance portfolio due the fact that most of the portfolio is composed from Great Britain Pound and Euro currency.

Here is the summary of the portfolios created with these models:

Weekly	Naive RP-CVaR	RP-CVaR	CVaR	Uniform
$\mu(\%)$	0.0222	0.0308	0.0249	0.035
$\mu_{ann}(\%)$	1.1613	1.6117	1.3031	1.8368
$\sigma(\%)$	1.1765	1.2986	1.1748	1.5587
$VaR_{10\%}(\%)$	1.4329	1.6318	1.5925	1.9604
$CVaR_{10\%}(\%)$	2.2684	2.5612	2.1494	3.1709
$\sigma_{ann}(\%)$	8.4836	9.3640	8.4718	11.24
$VaR_{10\%}ann(\%)$	10.3327	11.7671	11.4839	14.1369
$CVaR_{10\%}ann(\%)$	16.3579	18.4694	15.4997	22.8658
S_σ	0.1369	0.1721	0.1538	0.1634
S_{VaR}	0.1124	0.1370	0.1135	0.1299
S_{CVaR}	0.0710	0.0873	0.0841	0.0803

The order of preference is the following: CVaR, Risk Parity with CVaR and the other two models.

Calculating the diversification index:

	RP-Std	M-V	RP-CVaR	CVaR	Naive
D_{Her}	0.861770183	0.6473655145	0.85145159	0.65821342	0.875
D_{BP}	2.02665387	1.294736234	1.98374733	1.25143768	2.07944154
N. Assets	8	6	8	5	8

Mean Variance and CVaR portfolios suffer from extreme weights for The Great Britain Pound and the Euro currencies due the fact that these assets are more stable.

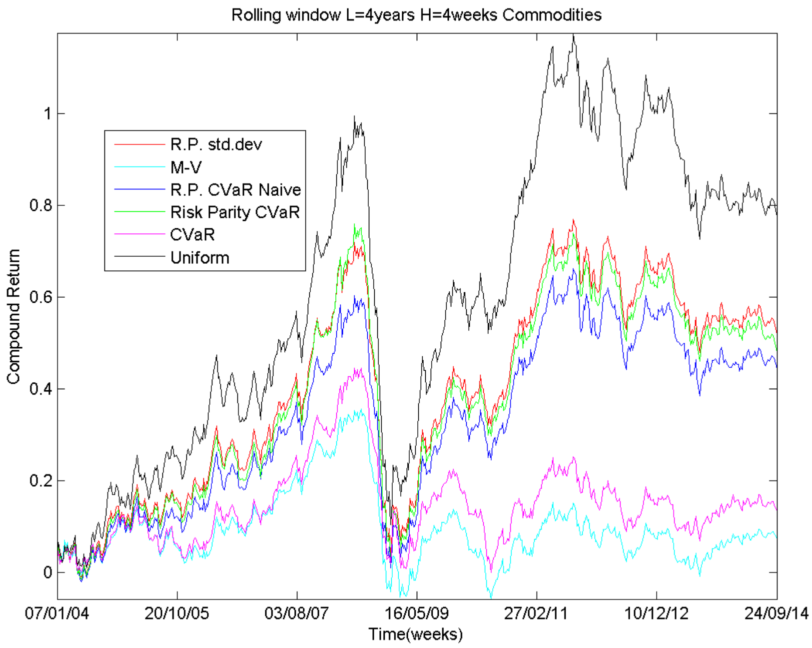
3.7.3 Comparison between models

In this part of the thesis we compare the performance of the models and their levels of risk. We remember that this is a particular case of a portfolio of 4 commodities and 4 foreign currencies.

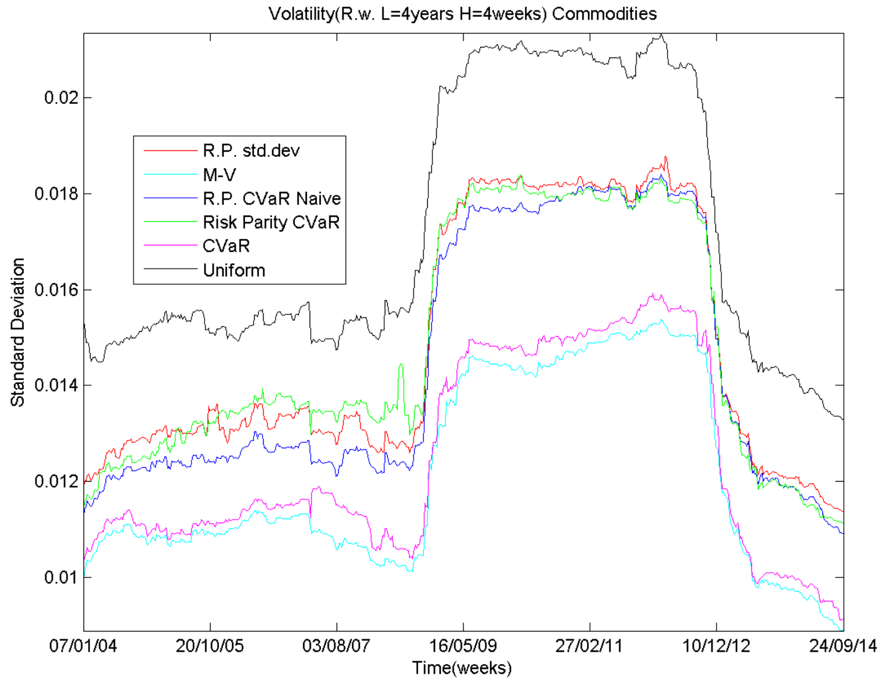
We create a Rolling window for in sample L=208 and H=4 out of sample for the period from January 2000 to end of September 2014.

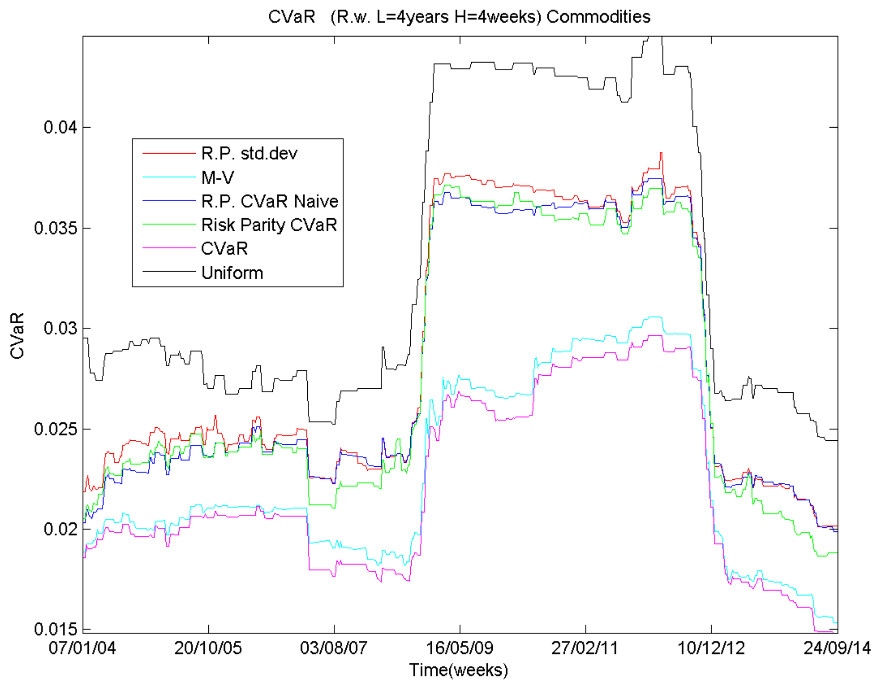
Weekly	RP-Std	M-V	RP-CVaR Naive	RP-CVaR	CVaR	Uniform
$\mu(\%)$	0.0858	0.0209	0.0763	0.0813	0.0313	0.1170
$\mu_{ann}(\%)$	4.5618	1.0920	4.0446	4.3166	1.6396	6.2695
$\mu^c(\%)$	51.8831	7.3882	44.3673	48.1149	13.4721	77.4398
$Median$	0.2192	0.0735	0.2059	0.1947	0.1075	0.2103
$\sigma(\%)$	1.4902	1.2746	1.4578	1.4884	1.3189	1.7012
$VaR_{10\%}(\%)$	1.5935	1.4680	1.6432	1.7068	1.5318	1.8715
$CVaR_{10\%}(\%)$	2.8343	2.3956	2.8173	2.8232	2.4384	3.2076
$\sigma_{ann}(\%)$	10.7457	9.1912	10.5125	10.7333	9.5110	12.2673
$VaR_{10\%}ann(\%)$	11.4908	10.5857	11.8495	12.3082	11.0461	13.4955
$CVaR_{10\%}ann(\%)$	20.4382	17.2748	20.3161	20.3587	17.5833	23.1305
S_σ	0.4245	0.1188	0.3847	0.4022	0.1724	0.5111
S_{VaR}	0.397	0.1032	0.3413	0.3507	0.1484	0.4646
S_{CVaR}	0.2232	0.0632	0.1991	0.212	0.0932	0.2711
Sortino Ratio	0.0787	0.0221	0.0709	0.0748	0.0329	0.0949
Rachev Ratio (5%)	0.8168	0.8231	0.7946	0.8391	0.8955	0.8085

In this case, the Naive portfolio has a better performance with a higher level or risk. The Risk Parity strategies have similar performance and higher values than Mean Variance and CVaR.



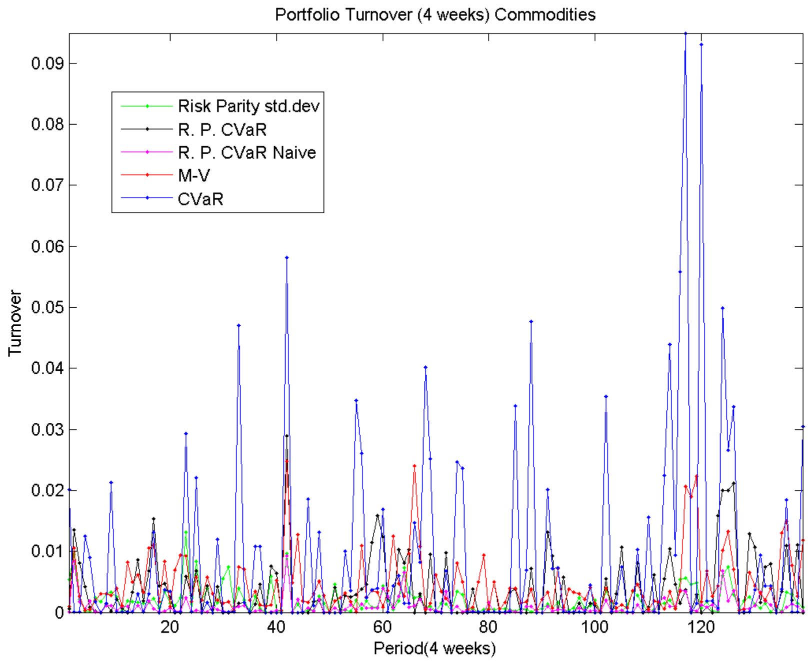
Comparing the riskiness we have the same situation in both cases with volatility and CVaR. There is no significant difference between Risk Parity with standard deviation and Risk Parity with CVaR.





In particular we notice that Risk Parity with CVaR and Risk Parity with CVaR-Naive have the same levels of risk.

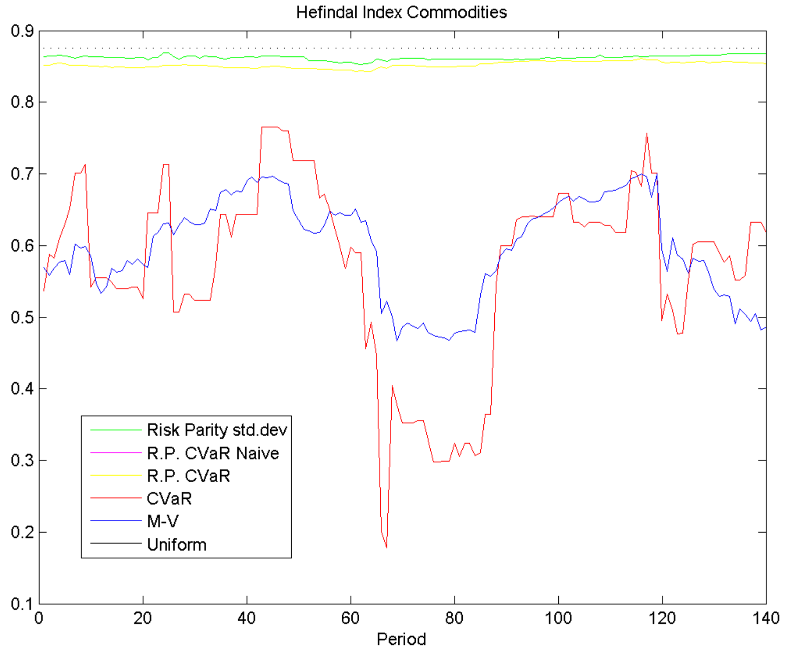
	RP-Std	M-V	RP-CVaR Naive	RP-CVaR	CVaR
Average Turnover(%)	0.192943	0.45228	0.0958	0.3875	0.941

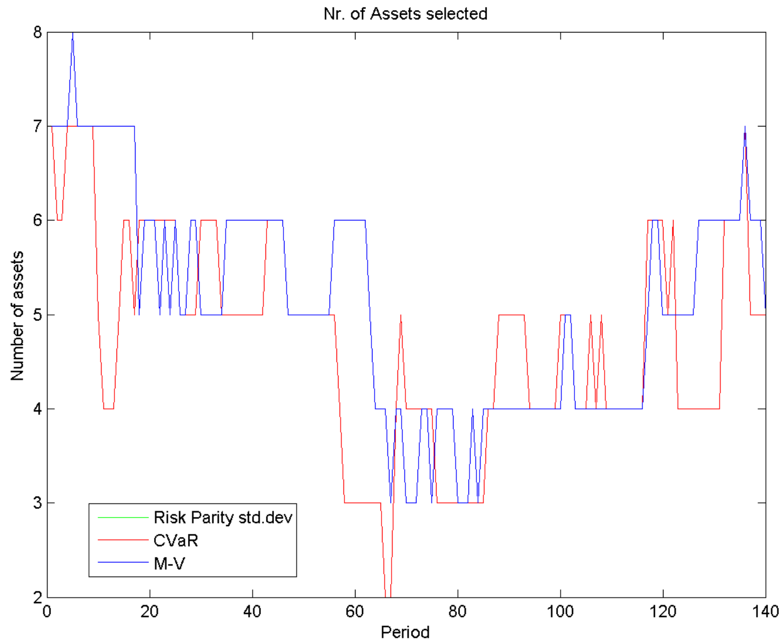


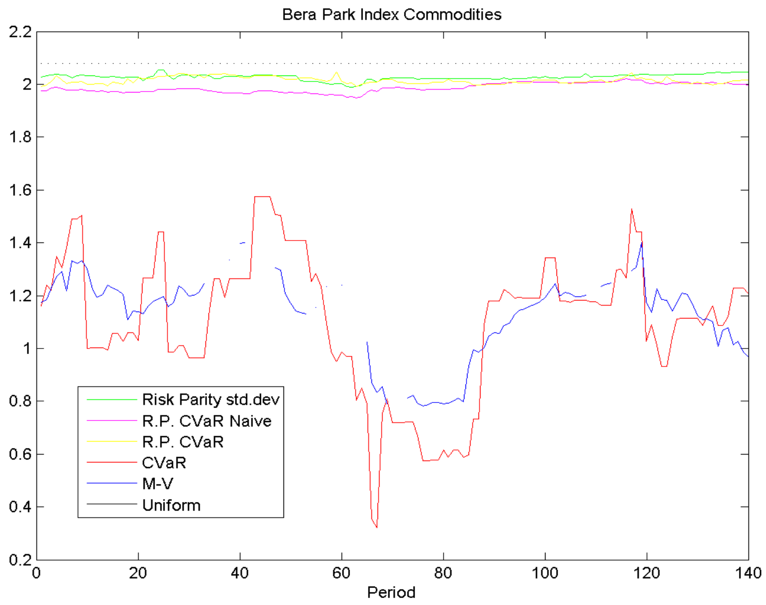
Since the Risk Parity with CVaR Naive has no true diversification, it will have a lower turnover. The turnover of Risk Parity with standard deviation is half of Mean Variance due the fact that there are just 8 assets.

In the same situation we have Risk Parity with CVaR and the CVaR portfolio.

The Risk Parity strategies are better diversified and, as we know, take into consideration all 8 assets.


Hefindal Index Commodities
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
20
40
60
80
100
120
140
Period
Risk Parity std.dev
R.P. CVaR Naive
R.P. CVaR
CVaR
M-V
Uniform


Nr. of Assets selected
8
7
6
5
4
3
2
0
20
40
60
80
100
120
140
Period
Number of assets
Risk Parity std.dev
CVaR
M-V



The Mean Variance model and CVaR are in all cases more concentrated with high weights in Euro and Great Britain Pound Currencies.

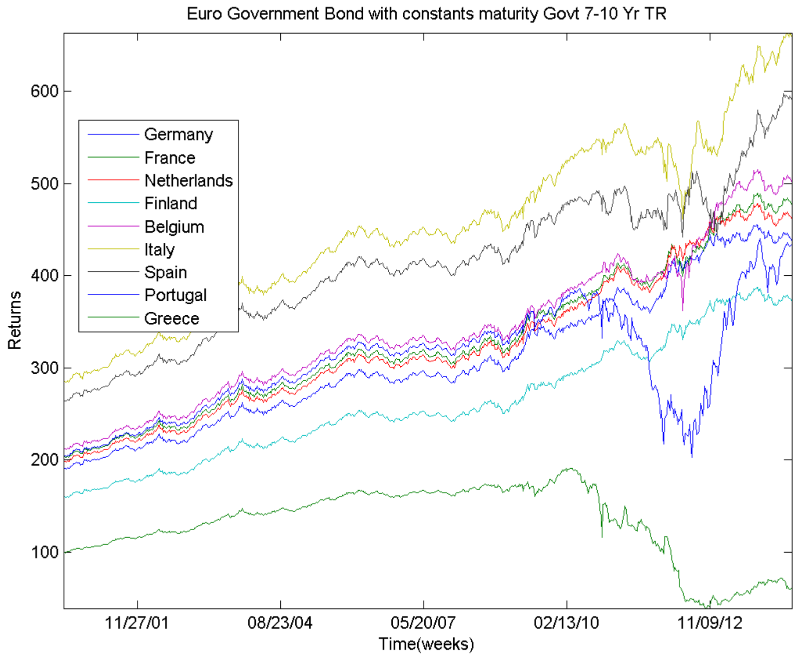
3.8 Portfolio optimization for Bond Portfolio

Some investors are interested in portfolios with low risk profile. In the past, most of the investors tended to choose Euro Bonds for low risk portfolios. After the crisis of the European sovereign debt, the market has been polluted with uncertain condition; this new conditions brings more fluctuation to the market of European bond, in particular to Greece, Portugal and Ireland.

We choose the following group of bonds with constant maturity of 7 to 10 years for the period from January 2000 to December 2013 :

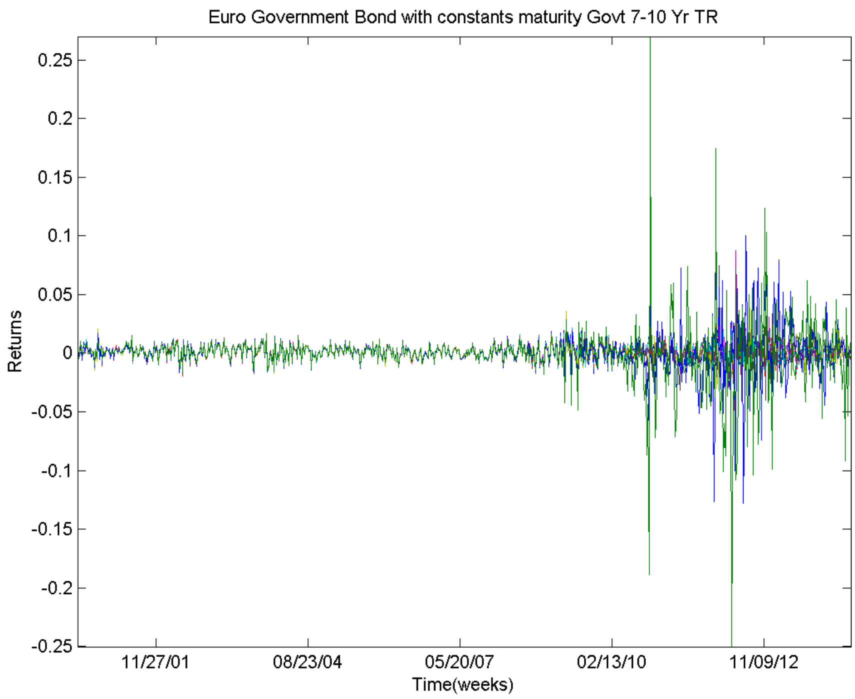
1. Germany Govt 7-10 Yr TR ;
2. France Govt 7-10 Yr TR ;
3. Netherlands Govt 7-10 Yr TR
4. Finland Govt 7-10 Yr TR
5. Belgium Govt 7-10 Yr TR
6. Italy Govt 7-10 Yr TR
7. Spain Govt 7-10 Yr TR
8. Portugal Govt 7-10 Yr TR
9. Greece Govt 7-10 Yr TR

Here's the price of the bonds:



Starting from the year 2010, the crisis of the European sovereign debt reduced the prices of Greece and Portugal bonds. After the intervention of the European Bank, the bonds recovered prices.

Lets take a look to the returns to have a clearer idea.



Is clear that before 2010 the markets have the same trends. The Greece and Portugal bonds have more troubles in terms of Skewness and Kurtosis and for that the returns have higher fluctuation:

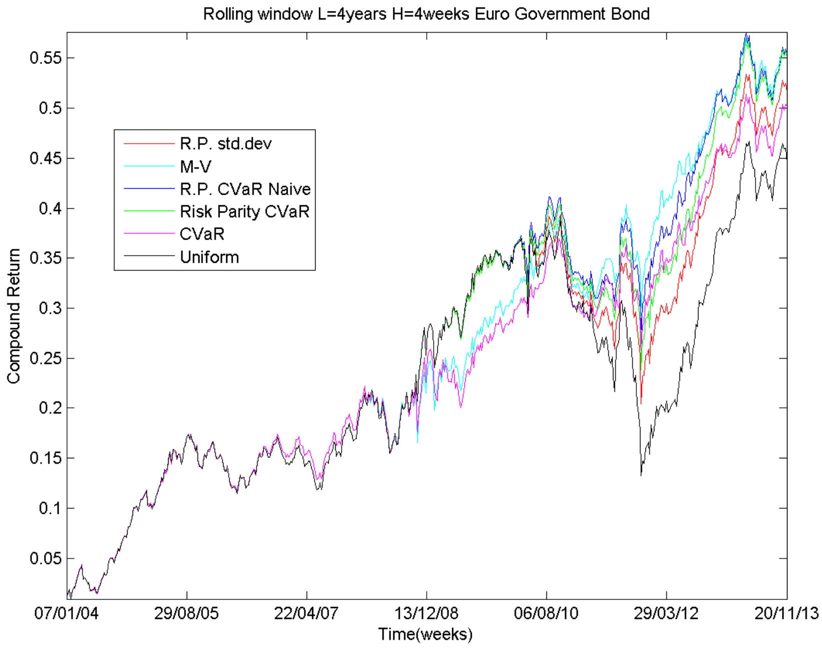
Returns	Mean(%)	Median(%)	Std.dev (%)	Range(%)	Skewness	Kurtosis
Germany Govt	0.11314	0.16452	0.71508	4.741	-0.10468	3.2934
France Govt	0.11615	0.14787	0.70294	5.8302	-0.040433	4.2597
Netherlands Govt	0.11501	0.16036	0.70289	5.3219	-0.15195	3.9649
Finland Govt	0.11545	0.15631	0.69775	5.223	-0.16742	3.9265
Belgium Govt	0.11748	0.16501	0.85997	16.29	0.22863	26.312
Italy Govt	0.1142	0.14543	0.88761	12.088	0.72532	12.767
Spain Govt	0.10953	0.12328	1.0363	12.116	1.1888	14.809
Portugal Govt	0.10256	0.11703	1.8438	24.508	0.061703	17.858
Greece Govt	-0.067916	0.12434	2.6085	51.999	-0.016151	38.329

3.8.1 Comparison between models

We create a rolling time window with an in sample period of 208 weeks and out of sample 4 weeks. So we calculate the weights 130 times to create the Rolling Window, after that we proceed with the calculation of the returns of the portfolio and measure the risk.

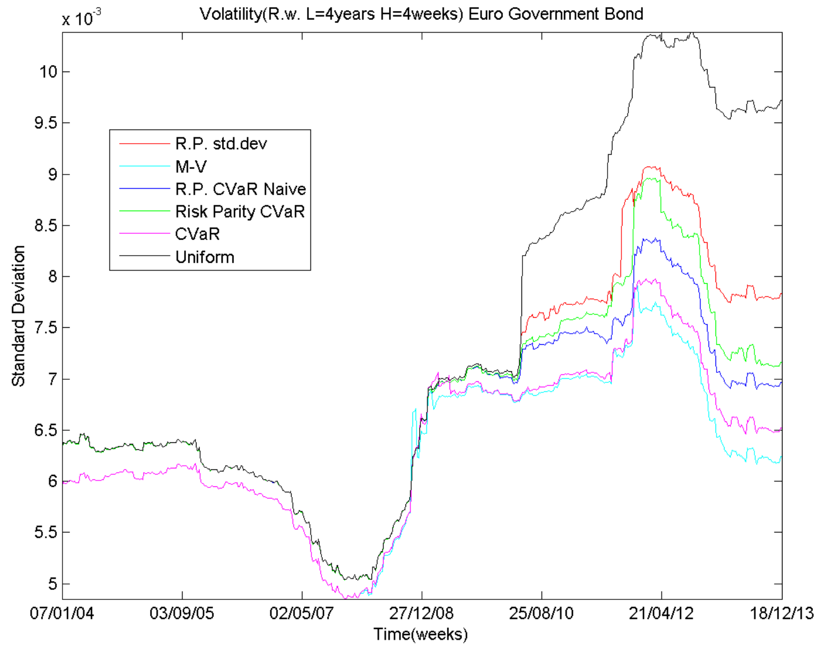
Weekly Bonds	RP-Std	M-V	RP-CVaR Naive	RP-CVaR	CVaR	Uniform
$\mu(\%)$	0.0836	0.0875	0.0880	0.0876	0.0812	0.0746
$\mu_{ann}(\%)$	4.4386	4.6552	4.6823	4.6603	4.3096	3.9512
$\mu^c(\%)$	51.6637	55.1765	55.5069	55.0750	50.2005	44.5242
$Median$	0.1221	0.1285	0.1278	0.1224	0.1223	0.1125
$\sigma(\%)$	0.7502	0.6857	0.6988	0.7182	0.6794	0.7956
$VaR_{10\%}(\%)$	0.7483	0.7226	0.7094	0.7319	0.7171	0.8682
$CVaR_{10\%}(\%)$	1.2368	1.1523	1.1537	1.1769	1.1539	1.3423
$\sigma_{ann}(\%)$	5.4099	4.9443	5.0393	5.1788	4.8992	5.7369
$VaR_{10\%}ann(\%)$	5.3963	5.2106	5.1157	5.2776	5.1714	6.2603
$CVaR_{10\%}ann(\%)$	8.9188	8.3096	8.3196	8.4871	8.3208	9.6796
S_σ	0.8205	0.9415	0.9292	0.8999	0.8797	0.6887
S_{VaR}	0.8225	0.8934	0.9153	0.8830	0.8334	0.6312
S_{CVaR}	0.4977	0.5602	0.5628	0.5491	0.5179	0.4082
Sortino Ratio	0.1706	0.1874	0.1934	0.1875	0.1773	0.1419
Rachev Ratio (5%)	1.1	1.02	1.1203	1.1199	1.0411	1.0411

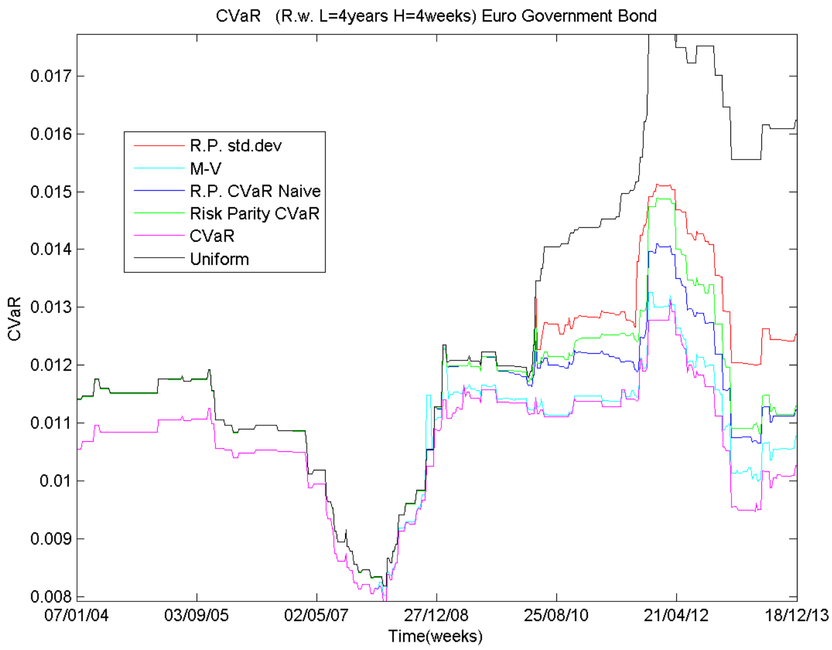
The fact that the Euro bonds perform in the same way for 10 year reflects in the performance. We have to focus on how they behave during the crisis:



It is interesting that the compounded return of the models is the same till the crisis of 2008. The models with better diversification have a better response to the crisis of 2008 until the problem of the crisis of the European sovereign debt at the beginning of 2010. The Risk Parity model with CVaR has a better performance than Risk Parity with standard deviation.

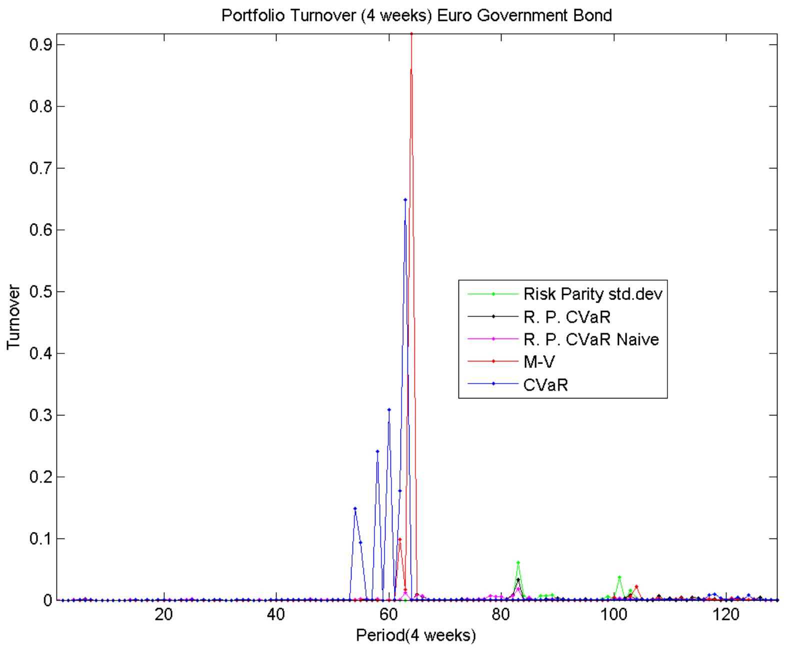
Let's take a look at the level of risk measured by volatility, and Conditional Value at Risk at the level of 10%:





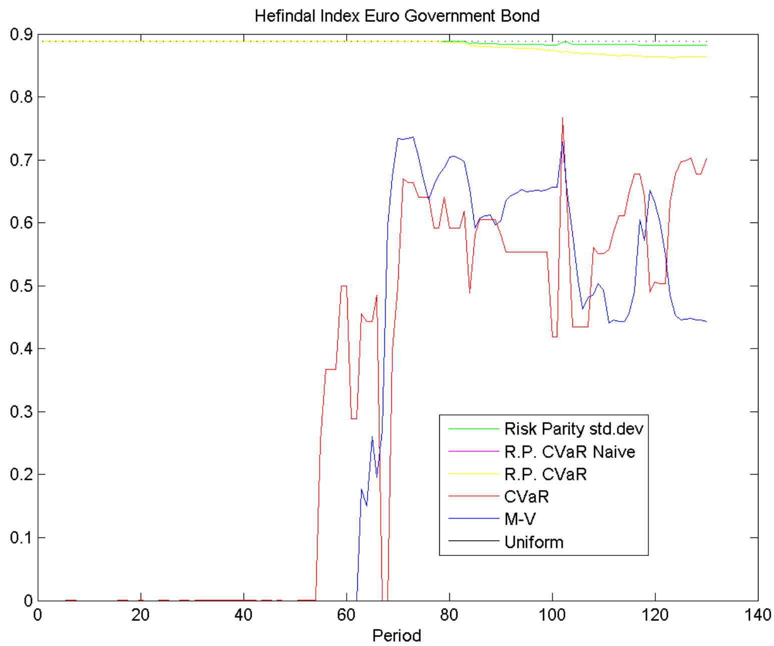
The uniform portfolio and the Risk Parity strategies have the same risk before the crisis of 2008, Mean Variance and CVaR have lower risk. With the starting of the crisis of the sovereign debts, the volatility of the Uniform portfolio increases much quicker than in the other models. It is interesting that Risk Parity with CVaR-Naive, since we have a very small number of assets, has the same level of risk as Risk Parity with CVaR.

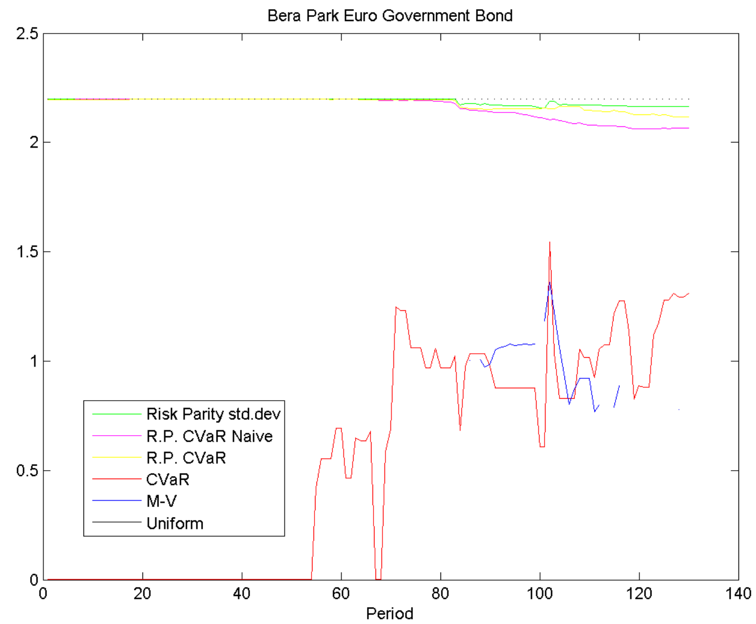
Before we study the diversification let's take a glance at the portfolio turnover:

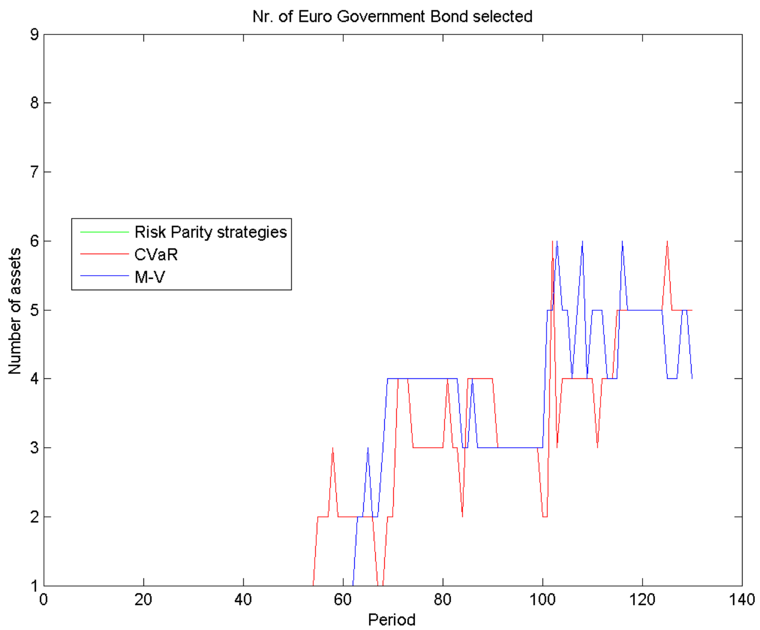


	RP-Std	M-V	RP-CVaR Naive	RP-CVaR	CVaR
Average Turnover(%)	0.1292	0.8485	0.0987	0.1202	1.2815

Risk Parity strategies have a lower turnover, precisely 6-10 times lower. The Mean Variance model without the return constraint before the crisis concentrates more in one asset, the one with less volatility.







The Bera Park and the Herfindal index for Mean Variance and CVaR are equal to 0 before the crisis. This means that both portfolios have just one asset. Looking at the turnover we understand that this asset is the same for the first 60 periods (5 years). Then they take into consideration more assets to minimize the risk.

The Risk Parity strategies have better diversification and for than they are not concentrated. We have just 9 bond, so applying the subset selection, as in the previous cases, is not useful.

3.9 The portfolio optimization for mixed portfolios

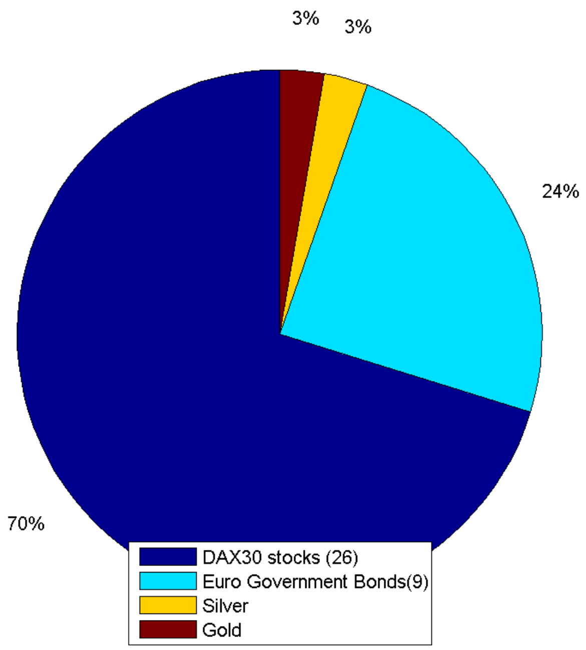
As the last environment we consider a mixed portfolio with stocks, bonds, and commodities.

The target of this study is to show the behaviour of the Risk Parity strategies for a set of assets with different classes of risk.

We consider the period from January 2000 to December 2013 for the following assets:

- 26 stocks of DAX30
- 9 Euro Government Bond
- Gold
- Silver

Their percentages are given in the following chart:

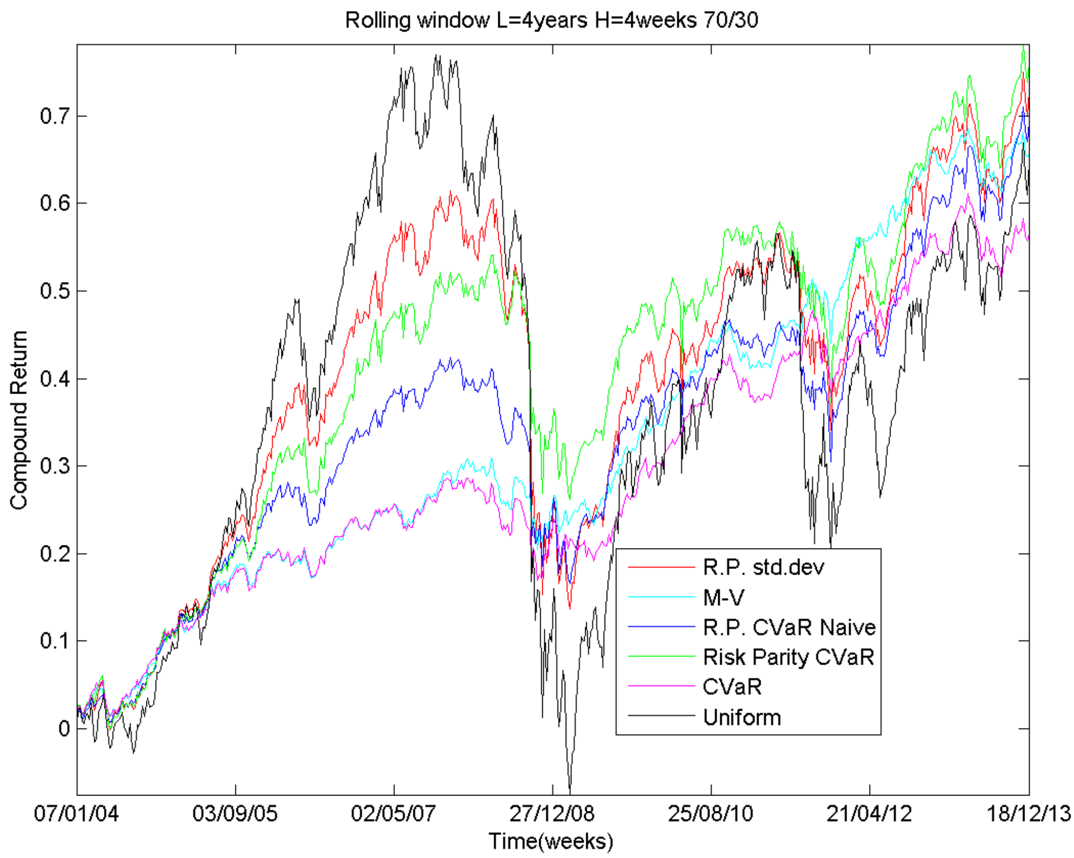


3.9.1 Comparison between models

We do the same analysis for the mixed portfolio creating a rolling time window of 4 year in sample period (208weekly observations) data and rebalancing every month (4 week out of sample).We present the portfolio statistics, returns, volatilities and total turnover diagrams of the R.P. strategy and the usual benchmarks.

Weekly Mixed	RP-Std	M-V	RP-CVaR N.	RP-CVaR	CVaR	Uniform
$\mu(\%)$	0.1142	0.0990	0.1059	0.1149	0.0881	0.1193
$\mu_{ann}(\%)$	6.1136	5.2819	5.6559	6.1508	4.6878	6.3970
$\mu^c(\%)$	72.7785	65.8115	69.4239	76.1627	56.5337	64.885
$Median$	0.2258	0.1328	0.1680	0.1552	0.1224	0.3251
$\sigma(\%)$	1.33	0.5899	0.9376	1.0863	0.6215	2.1310
$VaR_{10\%}(\%)$	1.232	0.6609	0.8287	1.0212	0.6750	2.1818
$CVaR_{10\%}(\%)$	2.4233	1.0247	1.6459	1.938	1.0849	4.0388
$\sigma_{ann}(\%)$	9.5905	4.2540	6.7611	7.8331	4.4818	15.3665
$VaR_{10\%}ann(\%)$	8.8839	4.7655	5.9761	7.3643	4.8677	15.7332
$CVaR_{10\%}ann(\%)$	17.4749	7.3894	11.8687	13.9753	7.8235	29.1242
S_σ	0.6375	1.2416	0.8365	0.7852	1.046	0.4163
S_{VaR}	0.6882	1.1084	0.9464	0.8352	0.963	0.4066
S_{CVaR}	0.3499	0.7148	0.4765	0.4401	0.5992	0.2196
Sortino Ratio	0.1151	0.2512	0.1583	0.1528	0.2122	0.0745
Rachev Ratio (5%)	0.8364	1.0102	0.8694	0.9480	0.9918	0.7986

The terminal compounded return is better for the Risk Parity with CVaR but in terms of performance ratio Mean Variance is better than the others.

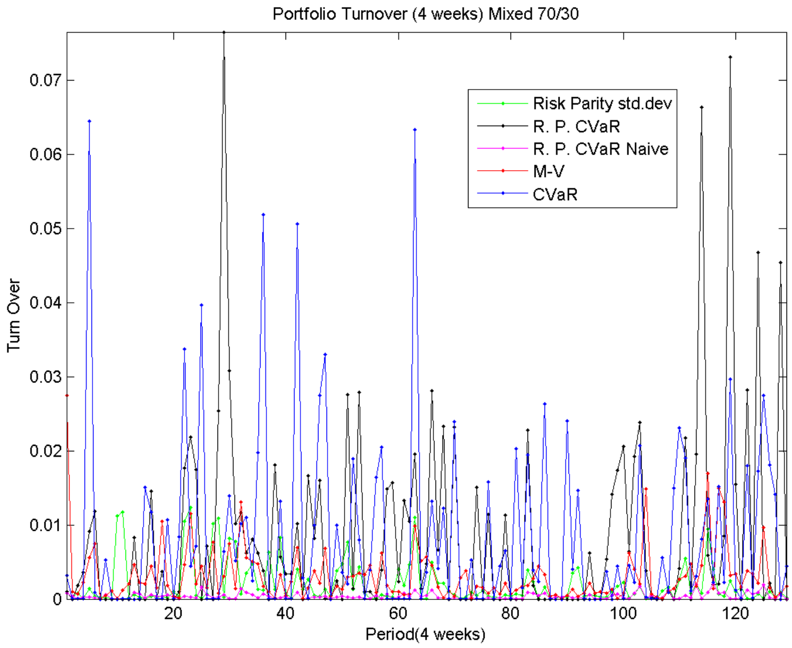


The compounded returns of the Risk Parity strategies and the naive portfolio are better before the crisis. Then there is a drawdown of the naive portfolio. After the crisis of 2008 the Risk Parity with CVaR and Risk Parity with standard deviation perform better than the other models.

In this case there is a particular fact about the turnover:

	RP-Std	M-V	RP-CVaR Naive	RP-CVaR	CVaR
Average Turnover(%)	0.1985	0.3166	0.0387	0.8762	0.8449

Since the Mean Variance will be more concentrated to risk free assets, will have a lower turnover .The RP-CVaR almost the same turnover as the CVaR.



Chapter 4

Conclusion and future research

In this Thesis we discussed the Risk Parity models with different types of risk measures. In literature, the most used case is Risk Parity with standard deviation. For comparative reasons and in order to have a complete idea of the Risk Parity strategies, we have described in detail how the Risk Parity portfolios with standard deviation are constructed. Also we introduce two simple iterative algorithms to calculate the portfolio weights for a risk parity strategy.

We have introduced the Risk Parity approach for the Conditional Value at Risk with no particular distribution assumption on the returns. Passing from standard deviation to Conditional Value at Risk in the Risk Parity strategies is important due the fact that CVaR is a coherent and convex risk measure (section 1.3).

To implement Risk Parity with CVaR, we have calculated the derivatives of the Conditional Value at Risk in section 2.3.1 making some assumptions for the numerical approximation, in order to switch from the continuous case to the discrete case. We have discussed how Risk Parity portfolios are constructed and we have performed some evaluations based on historical data. We also give an example where the Risk Parity with $CVaR_\alpha(X)$ is not possible to be calculated.

We have analyzed 5 cases of stock portfolios with different samples (26, 32, 44, 77 and 188 stocks) of different markets (DAX30, CAC40, Eurostoxx50, FTSE100, Nikkei225), a Bond portfolio with 9 bonds (Euro government bonds), a mixed portfolio (26 stock, 9 bonds and 2 commodities) and a special case of 8 commodities where we have considered 4 foreign currencies .

We have compared the Risk Parity standard deviation with Risk Parity -$CVaR_\alpha(X)$, also with a special case of Risk Parity $CVaR_\alpha(X)$ (Naive Risk Parity $CVaR_\alpha(X)$ (section 2.3.3)) where we describe as the worst case scenario from the point of view of diversification. For a better comparison, we also have computed the Mean Variance and the $CVaR_\alpha(X)$ with $\alpha = 10\%$ without the constraint of the expected returns. In this way, we have the lowest risk portfolios with Mean Variance and $CVaR_\alpha(X)$.

For each group of assets selected, we analyze the portfolio allocation creating a rolling time window with in sample period of 4 years (208 week) and out of sample period of (4 weeks) with weekly data.

In all empirical results, we notice that there is no significative difference between the Risk Parity with the standard deviation and the Risk Parity with Conditional Value at Risk from the point of view of performance ratios and of compounded returns.

From section 2.2.2 we know that the volatility of Risk Parity with standard deviation is between the Mean Variance's and the uniform $\frac{1}{n}$:

$$\sigma_{mv} \leq \sigma_{ERC} \leq \sigma_{\frac{1}{n}}$$

We have measured the riskiness of the portfolios in terms of volatility and $CVaR_{\alpha}(X)$.

From the empirical results, we have a similar inequality result for the Risk Parity with $CVaR_{\alpha}(X)$ in each case we have found:

$$CVaR_{cvar}(X) \leq CVaR_{rp-cvar}(X) \leq CVaR_{1/n}(X)$$

This is also true in the volatility case:

$$\sigma_{cvar} \leq \sigma_{rp-cvar} \leq \sigma_{\frac{1}{n}}$$

From the point of view of riskiness and performance, the Risk Parity with $CVaR_{\alpha}(X)$ is a good trade off between $CVaR_{\alpha}(X)$ and the Uniform portfolio.

The larger the portfolio is, the more time is needed calculate the approximations.

In general, we showed that the Risk Parity strategies lower have turnover than Mean Variance and $CVaR_{\alpha}(X)$. This is important in terms of transaction costs, when the investors try to rebalance the portfolios.

In the mixed portfolio combining assets with very different risk levels, we can guarantee good performance with the Risk Parities portfolios.

For the practical implementation and the empirical research, we have used the Matlab 2011 Version 7.13 ® software. Also the computation time of Risk parity with $CVaR_{\alpha}(X)$ is significantly shorter than Risk Parity with standard deviation using the same starting conditions.

For further research we need to prove the uniqueness of the Risk Parity with $CVaR_{\alpha}(X)$. We may compare the case of Risk Parity $CVaR_{\alpha}(X)$ with the Monte Carlo simulated data where we suppose that the returns are distributed like a normal multivariate. We may also prove analytically that $CVaR_{rp-cvar}(X)$ is between $CVaR_{\alpha}(X)$ and $CVaR_{1/n}(X)$. Another point is the comparison of the drawdowns of these groups of portfolios and specific cases where we have fixed and variable costs.

Appendix A

This appendix is for the explanation of the Matlab code for the creation of the Risk Parity with Conditional Value at Risk.

For the practical implementation and the empirical research, we have used the Matlab 2011 Version 7.13 ® software licensed by Univerisita degli studi La Sapienza in a Windows 7 64 bits operating system running on a HP computer with processor Intel Core i5 M430 2.27 Ghz and 6 GB RAM.

The code for the Risk Parity with standard deviation is created from Prof. Farid Moussaoui http://mfquant.net/erc_portfolio.html

There are many other authors that implement the Risk Parity with standard deviation in different ways for instance the algorithm of Spinu.

Taking in account the assumption for the Numerical approximation for estimating Risk Parity CVaR we have to solve this optimization problem:

$$x^* = \arg\min_x \sum_{i=1}^{n}\sum_{j=1}^{n}(TRC_i(x) - TRC_j(x))^2$$

$$\sum_{i=1}^{n} x_i = 1$$
$$x \geq 0$$

where $TRC_i^{CVaR}(x) = x_i \frac{\partial CVaR_\alpha(X)}{\partial x_i} \approx -\frac{1}{\lfloor \alpha T \rfloor} x_i \sum_{k=1}^{\lfloor \alpha T \rfloor} r_{ki}^{sorted}$

and r_{ki}^{sorted} are the asset returns corresponding to the sorted portfolio returns (all the details in section 2.3.2) .

We create a code that minimizes the distance between the total risk contributions:

```
clear all;
close all;
dati=load('DAX_weekly_elab.txt');
dati=dati(:,2:end);
R=price2ret(dati);
[T,n]=size(R);
Aeq = ones(1,n);
  Beq = 1;
```

```
  lb = zeros(n,1);
  ub = ones (n,1);

   qoptions = optimset('Display', 'iter', ...
                       'Algorithm','interior-point', ...
                       'MaxFunEvals', 500000, ...
                       'TolFun', 1e-20) ;
  n1 = 1.0/n;
  w0 = repmat(n1, n, 1) ;
  NLLfunction = @(x) gabimi(R, x) ;

  [weights,fval]= fmincon(NLLfunction, w0,[], [], Aeq, Beq, lb, ub, [],qoptions) ;
  weights
save 'weights_rpcvar1.txt' weights -ascii
x=weights';
conflevel=0.10;
threshold = floor(conflevel*T);
 mx=R.*repmat(x,T,1);
 mx=sum(mx,2);
 [srp,IX] = sort(mx,1);
 VaR= -srp(threshold);
 td=srp(1:threshold,:);
 cVaR= -mean(td)
 indici=IX(1:threshold);
%%%% Here we check the CVaR are the same
%index=repmat(indici,1,n);
 DDD=R(indici,:);
 meanRi=-mean(DDD);%%%
 x=x';
 y=x.*meanRi';
  CVaR=sum(y)
```

The distance from the total risk contributions is in the following function:

```
function fval=gabimi(R,x);  %%%fval=gabimi(CVaR,x);

%y=x.*CVaR; %%%%RC(i)=x(i)*CVaR(i); TOTAL RISK CONTRIBUTION
 fval = 0 ;
 %x=ones(n,1)./n;
 [T,n]=size(R);
 conflevel=0.10;
threshold = floor(conflevel*T);
 x=x';
 mx=R.*repmat(x,T,1);
 mx=sum(mx,2);
 [srp,IX] = sort(mx,1);
```

```
 indici=IX(1:threshold);
 DDD=R(indici,:);
 meanRi=-mean(DDD);%%%% derivates
 x=x';
 y=x.*meanRi';

  for i = 1:n;
   for j = i+1:n;
      xij  = y(i) - y(j) ;
      fval = fval + xij*xij ;
    end
  end
end
```

The algorithm is required to perform at maximum 1000 iterations to have a good approximation (optimset part). The smaller is the portfolio, the smaller is the time of computation. We start from the naive portfolio $1/n$. In the case of the Risk Parity with CVaR, they just use 10-30% of iterations and the time is shorter.

Here we will shortly summarize the time of computation and number iterations for the rolling window with in sample 4 years (208 weeks) and out of sample 1 month (4weeks)from January 2000 to June 2014.

So there are (755-208)/4=137 iterations.

In the case of 32 stocks CAC40 here is the time and number of iterations:

Model	Nr. Iterations	Time
R. P. std dev	137*1000	≈17 min
R. P.CVaR	137*60	≈ 2min
Mean Variance	137	≈ 1min
CVaR	137	≈ 1min

The Risk Parity with standard deviation uses all the possible iterations, so the time of computation is longer.

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