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Induced representations and harmonic analysis on finite groups

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Abstract The aim of the present paper is to develop a theory of spherical functions for noncommutative Hecke algebras on finite groups. Let G be a finite group, K a subgroup and (θ, V) an irreducible, unitary K-representation. After a careful analysis of Frobenius reciprocity, we are able to introduce an orthogonal basis in the commutant of $\mathrm{Ind}_K^G V$, and an associated Fourier transform. Then we translate our results in the corresponding Hecke algebra, an isomorphic algebra in the group algebra of G. Again a complete Fourier analysis is developed. As particular cases, we obtain some classical results of Curtis and Fossum on the irreducible characters. Finally, we develop a theory of Gelfand–Tsetlin bases for Hecke algebras.

Keywords Induced representation · Frobenius reciprocity · Fourier transform · Hecke algebra · Spherical function · Gelfand–Tsetlin basis

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1 Introduction

One of the key facts in the representation theory of a finite group G is the isomorphism

$$L(G) \cong \bigoplus_{\sigma \in \widehat{G}} M_{d_\sigma, d_\sigma}(\mathbb{C}), \tag{1.1}$$

where $\widehat{G}$ is a complete set of irreducible, pairwise inequivalent unitary representations of G, $L(G)$ is its group algebra, d_σ is the dimension of $\sigma \in \widehat{G}$ and $M_{d,d}(\mathbb{C})$ is the algebra of all $d \times d$ complex matrices. Such an isomorphism is given explicitly by the Fourier transform; see [4], Section 9.5. More generally, if $K \leq G$, $X = G/K$ and λ_X is the permutation representation on $L(X)$, the of all complex valued functions defined on X, then

$$\mathrm{Hom}_G(L(X), L(X)) \cong \bigoplus_{\sigma \in J} M_{m_\sigma, m_\sigma}(\mathbb{C}), \tag{1.2}$$

where J is the set of all $\sigma \in \widehat{G}$ contained in λ_X and m_σ the multiplicity of σ in λ_X; see again [4], Section 9.4. Clearly, (1.1) is a particular case of (1.2), because $\mathrm{Hom}_G(L(G), L(G)) \cong L(G)$. The spherical Fourier transform in the setting of (1.2) has been extensively studied when (G, K) is a Gelfand pair, that is when the algebra $\mathrm{Hom}_G(L(X), L(X))$ is commutative, which is equivalent to say that $L(X)$ is multiplicity free. There are several accounts on this subject and on its many applications; see [1,4,14]. In [21–23] we constructed a spherical Fourier transform on homogeneous spaces with multiplicity and gave several applications, mainly to probability and statistics (an earlier example may be found in [20]). We showed that multiplicity freeness is not an essential tool in order to develop a satisfactory theory and to perform explicit calculations.

In the present paper we face a more general problem. Suppose that (θ, V) is an irreducible, unitary K-representation. Then we have again

$$\mathrm{Hom}_G(\mathrm{Ind}_K^G V, \mathrm{Ind}_K^G V) = \bigoplus_{\sigma \in J} \mathrm{Hom}_G(m_\sigma W_\sigma, m_\sigma W_\sigma) \cong \bigoplus_{\sigma \in J} M_{m_\sigma, m_\sigma}(\mathbb{C}), \tag{1.3}$$

where J is the set of all $\sigma \in \widehat{G}$ (with representation space W_σ) contained in $\mathrm{Ind}_K^G \theta$ and m_σ is the multiplicity of σ in $\mathrm{Ind}_K^G \theta$. The aim of the present paper is to introduce a Fourier transform that gives an explicit form of (1.3). That is, we develop a theory of spherical functions dropping two usual assumptions: (1) that the representation is induced from the trivial one; (2) that it decomposes without multiplicity (actually, 2) was already dropped in [21].

The plan of the paper is the following. Section 2 is devoted to fix notation and to introduce one of the key ideas of the paper: the use of normalized Hilbert–Schmidt scalar products in spaces of intertwining operators. This leads to several natural orthogonality relations: in Sect. 3 these are obtained by a detailed analysis of Frobenius reciprocity. Most of the single results in this section are not new, but the entire picture that we develop has never been given. In particular, for $\sigma \in J$, the *explicit* isomorphism between $\mathrm{Hom}_K(V, \mathrm{Res}^G_K W_\sigma)$ and $\mathrm{Hom}_G(W_\sigma, \mathrm{Ind}^G_K V)$ and a particular choice of an *orthonormal* basis in $\mathrm{Hom}_K(V, \mathrm{Res}^G_K W_\sigma)$ lead to an *explicit* orthogonal decomposition of the σ-isotypic component in $\mathrm{Ind}^G_K V$. This is a new result that, in the case of a Gelfand pair, corresponds to the choice of a K-invariant vector in each spherical representation and to the use of the spherical functions to decompose the permutation representation; see Section 4.6 in [4]. In Sect. 4 the results on Frobenius reciprocity are used to construct a natural orthogonal basis in $\mathrm{Hom}_G(\mathrm{Ind}^G_K V, \mathrm{Ind}^G_K V)$. The associated Fourier transform is our first explicit form of (1.3) and another important new result. In [9,10,24] the Hecke algebra was introduced as a subalgebra of $L(G)$, then, using the theory of idempotents in group algebras, $\mathrm{Ind}^G_K V$ was identified with a subspace of $L(G)$. In Sect. 5 of the present paper we use a new and different approach: the theory developed in Sect. 3 naturally yields an isometric immersion of $\mathrm{Ind}^G_K V$ in $L(G)$ and this isometry may be used as a tool to translate the harmonic analysis in Sect. 4 into a harmonic analysis in the Hecke algebra. Adapted bases for the irreducible representations of G involved in the decomposition of $\mathrm{Ind}^G_K \theta$ yield a complete set of matrix coefficients, that is of spherical functions, for noncommutative Hecke algebras. This is the most important result of the paper. The irreducible characters of the algebra $\mathrm{Hom}_G(\mathrm{Ind}^G_K V, \mathrm{Ind}^G_K V)$ were computed by C.W. Curtis and T.V. Fossum in [9]. But their results can be used only for the Fourier analysis of functions in the *center* of the algebra. In the setting of our more general theory, the results of Curtis and Fossum may be easily derived and in a more transparent form. In Sect. 6 all the theory is illustrated by an explicit example, namely when G (resp. K) is the affine group over the field $\mathbb{F}_{q^m}$ (resp. $\mathbb{F}_q$). In Sect. 7 we develop a theory of Gelfand–Tsetlin bases: when it is applicable, it leads to a natural orthonormal basis for $\mathrm{Hom}_K(V, \mathrm{Res}^G_K W_\sigma)$ and to a corresponding basis for the σ-isotypic component of $\mathrm{Ind}^G_K V$. The first one is obtained by means of iterated restrictions, while the second one is obtained by iterated inductions.

Motivations may be found in our preceding papers [20–23], where only permutation representations were studied and applied. Concrete examples of spherical functions associated with induced representations are in [16,25], but the authors of these papers consider only multiplicity free induced representations of one-dimensional representations. Hecke algebras play a fundamental role in the representation theory of finite reductive groups: see [1,15].

It should be interesting to examine the case in which K is a normal subgroup using Clifford theory (see [5]). Another direction of research might be the extension of our results for permutation representations of wreath products (see [3,6,21]) to induced representations. A parallel theory was developed by D'Angeli and Donno in [11–13] by generalizing some constructions that arise in the setting of *association schemes*.

2 Preliminaries

In this section, in order to fix notation, we recall some basic facts on finite dimensional linear operators and on the representation theory of finite groups. The scalar product on a finite dimensional Hermitian vector space V is denoted by $\langle \cdot, \cdot \rangle_V$ and the associated norm by $\|\cdot\|_V$; we usually omit the subscript if the vector space is clear from the context. All the vector spaces will be Hermitian, and therefore we will omit this adjective. Given two finite dimensional vector spaces W, U we denote by $\mathrm{Hom}(W, U)$ the vector space of all linear maps $T : W \to U$ and by T^* the adjoint of T. We define a (normalized Hilbert–Schmidt) scalar product on $\mathrm{Hom}(W, U)$ by setting

$$\langle T_1, T_2\rangle_{\mathrm{Hom}(W,U)} = \frac{1}{\dim W}\mathrm{tr}(T_2^* T_1)$$

for all $T_1, T_2 \in \mathrm{Hom}(W, U)$. Since $\mathrm{tr}(T_2^* T_1) = \mathrm{tr}(T_1 T_2^*)$ we have

$$\langle T_1, T_2\rangle = \frac{\dim U}{\dim W}\langle T_2^*, T_1^*\rangle \tag{2.1}$$

In particular, the map $\mathrm{Hom}(W, U) \ni T \mapsto \sqrt{\frac{\dim U}{\dim W}}T^* \in \mathrm{Hom}(U, W)$ is a bijective isometry. Finally, note that if $I_W : W \to W$ is the identity operator then $\|I_W\|_{\mathrm{Hom}(W,W)} = 1$.

We consider only *unitary representations* of finite groups and the adjective unitary will be usually omitted. If σ is a representation of a finite group G its dimension will be denoted by d_σ. If (σ, W) and (ρ, U) are two representations of G we denote by $\mathrm{Hom}_G(W, U) = \{T \in \mathrm{Hom}(W, U) : T\sigma(g) = \rho(g)T, \forall g \in G\}$, the space of all *intertwining operators*. Observe that if T belongs to $\mathrm{Hom}_G(W, U)$ then also T^* belongs to $\mathrm{Hom}_G(U, W)$.

If (σ, W) is irreducible and $m = \dim \mathrm{Hom}_G(W, U)$ then U contains m copies of W. In this case we say that $T_1, T_2, \ldots, T_m \in \mathrm{Hom}_G(W, U)$ give rise to an *isometric orthogonal decomposition* of the *W-isotypic component* mW of U if for every $w_1, w_2 \in W$ and $i, j \in \{1, 2, \ldots, m\}$ we have $\langle T_i w_1, T_j w_2\rangle_U = \langle w_1, w_2\rangle_W \delta_{i,j}$. This implies that the subrepresentation of U isomorphic to mW is equal to the orthogonal direct sum

$$T_1 W \oplus T_2 W \oplus \cdots \oplus T_m W$$

and each operator T_j is a isometry from W to $RanT_j \equiv T_j W$.

Lemma 2.1 *Suppose that (σ, W) is irreducible. Then the operators $T_1, T_2, \ldots, T_m$ give rise to an isometric orthogonal decomposition of the W component of U if and only if $T_1, T_2, \ldots, T_m$ form an orthonormal basis for $Hom_G(W, U)$. Moreover, if this is the case, then we have:*

$$T_j^* T_i = \delta_{i,j} I_W. \tag{2.2}$$

Proof Suppose that $T_1, T_2, \ldots, T_m$ form an orthonormal basis for $\mathrm{Hom}_G(W, U)$. Then $T_j^* \in \mathrm{Hom}_G(U, W)$. Therefore $T_j^* T_i \in \mathrm{Hom}_G(W, W)$ and, by Schur's lemma, there exist $\lambda_{i,j} \in \mathbb{C}$ such that $T_j^* T_i = \lambda_{i,j} I_W$. By taking the traces of both sides, we get

$$\delta_{i,j} d_\sigma = \mathrm{tr}(T_j^* T_i) = \lambda_{i,j} d_\sigma \Rightarrow \lambda_{i,j} = \delta_{i,j},$$

that is (2.2). Therefore, if $w_1, w_2 \in W$ then

$$\langle T_i w_1, T_j w_2 \rangle_U = \langle T_j^* T_i w_1, w_2 \rangle_W = \delta_{i,j} \langle w_1, w_2 \rangle_W .$$

The converse implication is trivial. □

Let (σ, W) be a representation of G and let $\{w_1, w_2, \dots, w_{d_\sigma}\}$ an orthonormal basis of W. The corresponding matrix coefficients are defined by setting $u^\sigma_{j,i}(g) = \langle \sigma(g) w_i, w_j \rangle$, for $i, j = 1, 2, \dots, d_\sigma$ and $g \in G$. For $\sigma, \rho \in \widehat{G}$ we set $\delta_{\sigma,\rho} = 1$, if σ and ρ are equivalent, otherwise we set $\delta_{\sigma,\rho} = 0$. Let $L(G) = \{f : G \to \mathbb{C}\}$ be the vector space of all complex valued functions defined on G, endowed with the scalar product $\langle f_1, f_2 \rangle = \sum_{g \in G} f_1(g) \overline{f_2(g)}$ for $f_1, f_2 \in L(G)$. It also has a natural structure of an algebra by defining the convolution product of $f_1, f_2 \in L(G)$ as the function $(f_1 * f_2)(g) = \sum_{g_0 \in G} f_1(g g_0^{-1}) f_2(g_0)$, for all $g \in G$. We recall (see [4]) that if $\sigma, \rho \in \widehat{G}$ then

$$\langle u^\sigma_{i,j}, u^\rho_{h,k} \rangle = \frac{|G|}{d_\sigma} \delta_{\sigma,\rho} \delta_{i,h} \delta_{j,k} \text{ (orthogonality relations)} \tag{2.3}$$

$$u^\sigma_{i,j} * u^\rho_{h,k} = \frac{|G|}{d_\sigma} \delta_{\sigma,\rho} \delta_{j,h} u^\sigma_{i,k} \text{ (convolution property).} \tag{2.4}$$

Let (σ, W) be a G-representation and denote by χ^σ its character. The following elementary formula is a generalization of (2) in Exercise 9.5.8 of [4].

Proposition 2.2 *If (σ, W) is irreducible, $w \in W$ is a vector of norm 1 and $\phi(g) = \langle \sigma(g) w, w \rangle$ is the diagonal matrix coefficient associated with w, then*

$$\chi^\sigma(g) = \frac{d_\sigma}{|G|} \sum_{h \in G} \phi(h^{-1} g h) \quad \forall g \in G. \tag{2.5}$$

Let K be a subgroup of G, (θ, V) a representation of K and denote by $\lambda = \mathrm{Ind}_K^G \theta$ the *induced representation* (see for instance, [1,2,7]). The representation space is given by

$$\mathrm{Ind}_K^G V = \{f : G \to V : f(gk) = \theta(k^{-1}) f(g), \ \forall g \in G, k \in K\} \tag{2.6}$$

and λ is defined by setting $[\lambda(g_0) f](g) = f(g_0^{-1} g)$, for all $f \in \mathrm{Ind}_K^G V$, $g, g_0 \in G$. Let $G = \coprod_{t \in \mathcal{T}} tK$ be a decomposition of G into left K-cosets ($\coprod$ denotes a disjoint union). For $v \in V$ we define $f_v \in \mathrm{Ind}_K^G V$ by setting

$$f_v(g) = \begin{cases} \theta(g^{-1}) v & \text{if } g \in K \\ 0 & \text{if } g \notin K. \end{cases} \tag{2.7}$$

Then for every $f \in \mathrm{Ind}_K^G V$ we have:

$$f = \sum_{t \in \mathcal{T}} \lambda(t) f_{v_t} \tag{2.8}$$

with $v_t = f(t)$. The representation $\mathrm{Ind}_K^G \theta$ is unitary with respect to the following scalar product: $\langle f_1, f_2 \rangle_{\mathrm{Ind}_K^G V} = \frac{1}{|K|} \sum_{g \in G} \langle f_1(g), f_2(g) \rangle_V$. Moreover, if $\{v_j : j = 1, 2, \ldots, d_\theta\}$ is an orthonormal basis in V then the set

$$\{\lambda(t) f_{v_j} : t \in \mathcal{T}, j = 1, 2, \ldots, d_\theta\} \tag{2.9}$$

is an orthonormal basis in $\mathrm{Ind}_K^G V$ (see [2]).

Finally, we recall the transitive property of the induction (cf. [2,7]). Let H be a subgroup of G containing K (i.e. $K \leq H \leq G$). Then

$$\mathrm{Ind}_H^G [\mathrm{Ind}_K^H V] \cong \mathrm{Ind}_K^G V. \tag{2.10}$$

Indeed, the left hand side may be seen as the set of all $F : G \times H \to V$ such that $F(gh, h_0 k) = \theta(k^{-1}) F(g, hh_0)$, for all $g \in G$, $h, h_0 \in H$ and $k \in K$. With the right hand side as in (2.6), we have that the isomorphism in (2.10) is given by the map

$$F \mapsto f \tag{2.11}$$

where $f(g) = F(g, 1_G)$ for all $g \in G$ (note that F is uniquely determined by f, because $F(g, h) = f(gh)$ for all $g \in G$ and $h \in H$).

3 Orthogonality relations for Frobenius reciprocity

Let G be again a finite group, $K \leq G$ a subgroup, (σ, W) a representation of G and (θ, V) a representations of K. Frobenius reciprocity is usually stated an explicit isomorphism between $\mathrm{Hom}_G(W, \mathrm{Ind}_K^G V)$ and $\mathrm{Hom}_K(\mathrm{Res}_K^G W, V)$.

Definition 3.1 (a) For each $T \in \mathrm{Hom}_G(W, \mathrm{Ind}_K^G V)$ we set

$$\hat{T} w = \sqrt{|G/K|}[Tw](1_G), \qquad \text{for all } w \in W.$$

(b) For each $L \in \mathrm{Hom}_K(\mathrm{Res}_K^G W, V)$ we set

$$[\overset{\vee}{L} w](g) = \frac{1}{\sqrt{|G/K|}} L\sigma(g^{-1}) w, \qquad \text{for all } w \in W, g \in G.$$

(c) For each $T \in \mathrm{Hom}_G(\mathrm{Ind}_K^G V, W)$ we set

$$\overset{\circ}{T} v = \sqrt{|G/K|} T f_v, \qquad \text{for all } v \in V.$$

(d) For each $L \in \mathrm{Hom}_K(V, \mathrm{Res}_K^G W)$ we set

$$\overset{\diamond}{L} f = \frac{1}{\sqrt{|G/K|}} \sum_{t \in \mathcal{T}} \sigma(t) L f(t), \qquad \text{for all } f \in \mathrm{Ind}_K^G V.$$

Note that $L \in \mathrm{Hom}_K(V, \mathrm{Res}^G_K W)$ implies that $\sigma(k)Lf(t) = L\theta(k)f(tk) = Lf(t)$, so that

$$\overset{\diamond}{L}f = \frac{1}{\sqrt{|G| \cdot |K|}} \sum_{t \in \mathcal{T}} \sum_{k \in K} \sigma(tk)Lf(tk) = \frac{1}{\sqrt{|G| \cdot |K|}} \sum_{g \in G} \sigma(g)Lf(g).$$

In particular, $\overset{\diamond}{L}$ does not depend on the particular choice of $\mathcal{T}$.

Theorem 3.2 (Frobenius reciprocity revisited)

(a) *For each $T \in Hom_G(W, Ind^G_K V)$ we have $\hat{T} \in Hom_K(Res^G_K W, V)$ and the map*

$$\begin{aligned} Hom_G(W, Ind^G_K V) &\longrightarrow Hom_K(Res^G_K W, V) \\ T &\longmapsto \hat{T} \end{aligned}$$

is a linear isometric isomorphism. Moreover, its inverse is given by

$$\begin{aligned} Hom_K(Res^G_K W, V) &\longrightarrow Hom_G(W, Ind^G_K V) \\ L &\longmapsto \overset{\vee}{L}. \end{aligned}$$

(b) *For each $T \in Hom_G(W, Ind^G_K V)$ we have $(T^*)^\circ = (\hat{T})^*$. In particular, the diagram*

$$\begin{array}{ccc} Hom_G(W, Ind^G_K V) & \xrightarrow{\wedge} & Hom_K(Res^G_K W, V) \\ * \downarrow & & * \downarrow \\ Hom_G(Ind^G_K V, W) & \xrightarrow{\square} & Hom_K(V, Res^G_K W) \end{array}$$

is commutative.

Proof (a) Let $T \in \mathrm{Hom}_G(W, \mathrm{Ind}^G_K V)$ and recall that $\lambda = \mathrm{Ind}^G_k \theta$. Then

$$\begin{aligned} \hat{T}\sigma(k)w &= \sqrt{|G/K|}[T\sigma(k)w](1_G) = \sqrt{|G/K|}[\lambda(k)Tw](1_G) \\ &= \sqrt{|G/K|}[Tw](k^{-1}) = \sqrt{|G/K|}\theta(k)[Tw](1_G) = \theta(k)\hat{T}w, \end{aligned}$$

for all $k \in K$ and $w \in W$ and this proves that $\hat{T} \in \mathrm{Hom}_K(\mathrm{Res}^G_K W, V)$. The identity

$$[Tw](g) = [\lambda(g^{-1})Tw](1_G) = [T\sigma(g^{-1})w](1_G) = \frac{1}{\sqrt{|G/K|}} \hat{T}\sigma(g^{-1})w \tag{3.1}$$

shows that the map $T \mapsto \hat{T}$ is injective, because T is determined by $\hat{T}$. Now we use (3.1) to show that the map is also an isometry. If $T_1, T_2 \in \mathrm{Hom}_G(W, \mathrm{Ind}_K^G V)$ and $\{w_1, w_2, \ldots, w_{d_\sigma}\}$ is an orthonormal basis of W then

$$\begin{aligned}\mathrm{tr}(T_2^* T_1) &= \sum_{i=1}^{d_\sigma} \langle T_1 w_i, T_2 w_i\rangle_{\mathrm{Ind}_K^G V} = \sum_{i=1}^{d_\sigma} \frac{1}{|K|} \sum_{g\in G} \langle [T_1 w_i](g), [T_2 w_i](g)\rangle_V \\ \text{(by(3.15))} &= \sum_{i=1}^{d_\sigma} \frac{1}{|G|} \sum_{g\in G} \langle \hat{T}_1 \sigma(g^{-1}) w_i, \hat{T}_2 \sigma(g^{-1}) w_i\rangle_V \\ &= \frac{1}{|G|} \sum_{g\in G} \mathrm{tr}[\sigma(g)(\hat{T}_2)^* \hat{T}_1 \sigma(g^{-1})] = \mathrm{tr}[(\hat{T}_2)^* \hat{T}_1],\end{aligned}$$

that is $\langle T_1, T_2\rangle = \frac{1}{d_\sigma}\mathrm{tr}(T_2^* T_1) = \frac{1}{d_\sigma}\mathrm{tr}[(\hat{T}_2)^* \hat{T}_1] = \langle \hat{T}_1, \hat{T}_2\rangle$. It is easy to see that if $L \in \mathrm{Hom}_K(\mathrm{Res}_K^G W, V)$ then $[\overset{\vee}{L} w](gk) = \theta(k^{-1})[\overset{\vee}{L} w](g)$ and $\lambda(g)\overset{\vee}{L} w = \overset{\vee}{L}\sigma(g) w$ for all $g \in G$, $k \in K$, $w \in W$, that is, $\overset{\vee}{L} w \in \mathrm{Ind}_K^G V$ and $\overset{\vee}{L} \in \mathrm{Hom}_G(W, \mathrm{Ind}_K^G V)$. Finally, by definition of $\wedge$ and $\vee$ we have

$$\left(\overset{\vee}{L}\right)^{\wedge} w = \sqrt{|G/K|}[\overset{\vee}{L} w](1_G) = Lw$$

for all $w \in W$, that is the map $T \mapsto \hat{T}$ is surjective and $L \mapsto \overset{\vee}{L}$ is its inverse.

(b) For any $T \in \mathrm{Hom}_G(W, \mathrm{Ind}_K^G V)$, $w \in W$ and $v \in V$ we have (by definition of $\circ$):

$$\begin{aligned}\frac{1}{\sqrt{|G/K|}}\langle (T^*)^\circ v, w\rangle_W &= \langle T^* f_v, w\rangle_W = \langle f_v, Tw\rangle_{\mathrm{Ind}_K^G V} = \frac{1}{|K|}\sum_{g\in G}\langle f_v(g), [Tw](g)\rangle_V \\ \text{(by (2.10))} \qquad &= \frac{1}{|K|}\sum_{k\in K}\langle \theta(k^{-1})v, [Tw](k)\rangle_V = \frac{1}{|K|}\sum_{k\in K}\langle v, \theta(k)[Tw](k)\rangle_V \\ &= \frac{1}{\sqrt{|G/K|}}\langle v, \hat{T} w\rangle_V = \frac{1}{\sqrt{|G/K|}}\left\langle (\hat{T})^* v, w\right\rangle_W.\end{aligned}$$

□

For the following corollary, see also Corollary 34.1 in [1] and Section 2.3 in [21].

Corollary 3.3 (The other side of Frobenius reciprocity)

(a) *Let $T \in Hom_G(Ind_K^G V, W)$. Then $\overset{\circ}{T} \in Hom_K(V, Res_K^G W)$ and the map*

$$\begin{aligned} Hom_G(Ind_K^G V, W) &\longrightarrow Hom_K(V, Res_K^G W) \\ T &\longmapsto \overset{\circ}{T} \end{aligned}$$

is a linear isomorphism with $\langle \overset{\circ}{T}_1, \overset{\circ}{T}_2\rangle = |G/K|\langle T_1, T_2\rangle$, for all $T_1, T_2 \in Hom_G(Ind_K^G V, W)$. The inverse is given by

$$Hom_K(V, Res^G_K W) \longrightarrow Hom_G(Ind^G_K V, W).$$
$$L \longmapsto \overset{\diamond}{L}$$

(b) *We have* $(L^*)^\vee = \left(\overset{\diamond}{L}\right)^*$ *for all* $L \in Hom_K(V, Res^G_K W)$.

Proof (a) Besides the statement that the map $L \mapsto \overset{\diamond}{L}$ is the inverse of $T \mapsto \overset{\circ}{T}$, everything follows from Theorem 3.2, (2.1) and the isomorphisms $T \mapsto T^*$. For all $T \in \mathrm{Hom}_G(\mathrm{Ind}^G_K V, W)$, $\phi \in \mathrm{Ind}^G_K V$, the definitions of $\overset{\diamond}{L}$ and $\overset{\circ}{T}$, with $v_t = \phi(t)$, yield:

$$(\overset{\circ}{T})^\diamond \phi = \frac{1}{\sqrt{|G/K|}} \sum_{t\in\mathcal{T}} \sigma(t)\overset{\circ}{T}\phi(t) = \sum_{t\in\mathcal{T}} \sigma(t) T f_{v_t} = \sum_{t\in\mathcal{T}} T\lambda(t) f_{v_t}$$
$$\text{(by (2.11))} = T\phi.$$

For completeness, we derive also $\langle \overset{\circ}{T}_1, \overset{\circ}{T}_2\rangle = |G/K|\langle T_1, T_2\rangle$: taking into account Theorem 3.2 and (2.1), for $T_1, T_2 \in \mathrm{Hom}_G(\mathrm{Ind}^G_K V, W)$ we have:

$$\begin{aligned}\langle \overset{\circ}{T}_1, \overset{\circ}{T}_2\rangle &= \langle [(T_1^*)^\wedge]^*, [(T_2^*)^\wedge]^*\rangle = \frac{d_\sigma}{d_\theta}\langle (T_2^*)^\wedge, (T_1^*)^\wedge\rangle \\ &= \frac{d_\sigma}{d_\theta}\langle T_2^*, T_1^*\rangle |G/K|\langle T_1, T_2\rangle.\end{aligned}$$

(b) From (b) in Theorem 3.2 it follows that $\left\{\left[(L^*)^\vee\right]^*\right\}^\circ = \left\{\left[(L^*)^\vee\right]^\wedge\right\}^* = L = \left\{\left[\left(\overset{\diamond}{L}\right)^*\right]^*\right\}^\circ$.

□

Corollary 3.4 (Orthogonality relations for Frobenius reciprocity I) *Let* m *be the dimension of* $Hom_G(W, Ind^G_K V)$ *and suppose that* $L_1, L_2, \ldots, L_m \in Hom_K(V, Res^G_K W)$. *Then the following facts are equivalent:*

(a) $\{L_1, L_2, \ldots, L_m\}$ *is an orthonormal basis of* $Hom_K(V, Res^G_K W)$;

(b) $\left\{\sqrt{|G/K|}\overset{\diamond}{L}_1, \sqrt{|G/K|}\overset{\diamond}{L}_2, \ldots, \sqrt{|G/K|}\overset{\diamond}{L}_m\right\}$ *is an orthonormal basis of* $Hom_G(Ind^G_K V, W)$;

(c) $\left\{\sqrt{\frac{d_\sigma}{d_\theta}}L_1^*, \sqrt{\frac{d_\sigma}{d_\theta}}L_2^*, \ldots, \sqrt{\frac{d_\sigma}{d_\theta}}L_m^*\right\}$ *is an orthonormal basis of* $Hom_K(Res^G_K W, V)$;

(d) $\left\{\sqrt{\frac{d_\sigma}{d_\theta}}(L_1^*)^\vee, \sqrt{\frac{d_\sigma}{d_\theta}}(L_2^*)^\vee, \ldots, \sqrt{\frac{d_\sigma}{d_\theta}}(L_m^*)^\vee\right\}$ *is an orthonormal basis of* $Hom_G(W, Ind^G_K V)$.

Corollary 3.5 (Orthogonality relations for Frobenius reciprocity II) *Suppose that* (σ, W) *and* (θ, V) *are irreducible. Then* $\sqrt{\frac{d_\sigma}{d_\theta}}(L_1^*)^\vee, \sqrt{\frac{d_\sigma}{d_\theta}}(L_2^*)^\vee, \ldots, \sqrt{\frac{d_\sigma}{d_\theta}}(L_m^*)^\vee$ *give* *rise to an isometric orthogonal decomposition of the W-component of* $\mathrm{Ind}_K^G V$ *if and only if* $L_1, L_2, \ldots, L_m$ *give rise to an isometric orthogonal decomposition of the V-component of* $\mathrm{Res}_K^G W$.

Proof It follows from Theorem 3.2 and Lemma 2.1. □

In what follows, we will often use the identity in (b), Corollary 3.3, that is $(L_j^*)^\vee = (\overset{\diamond}{L}_j)^*$. The following commutative diagram is helpful to memorize the previous results.

$$\begin{array}{ccc} \mathrm{Hom}_G(W, \mathrm{Ind}_K^G V) & \underset{\vee}{\overset{\wedge}{\rightleftarrows}} & \mathrm{Hom}_K(\mathrm{Res}_K^G W, V) \\ * \updownarrow & & \updownarrow * \\ \mathrm{Hom}_G(\mathrm{Ind}_K^G V, W) & \underset{\diamond}{\overset{\square}{\rightleftarrows}} & \mathrm{Hom}_K(V, \mathrm{Res}_K^G W) \end{array}$$

Remark 3.6 In [17], a different version of orthogonality relations for Frobenius reciprocity is developed. Actually, the author works in a more general setting: she considers representations over fields of characteristic zero and her spaces are endowed with arbitrary non-degenerate symmetric bilinear forms. However, we limit ourselves to illustrate and derive her main result in our setting. Theorem 2.1 of [17] may be expressed in the following way: under the assumption that W is G-irreducible, if $L \in \mathrm{Hom}_K(V, \mathrm{Res}_K^G W)$ is an isometry then also $\sqrt{\frac{d_\sigma}{d_\theta}}(L^*)^\vee \equiv \sqrt{\frac{d_\sigma}{d_\theta}}(\overset{\diamond}{L})^* \in \mathrm{Hom}_G(W, \mathrm{Ind}_K^G V)$ is an isometry. This is our derivation: if L is an isometry, then $\|L\| = 1$ and therefore also $\|\sqrt{\frac{d_\sigma}{d_\theta}}(L^*)^\vee\| = 1$. Arguing as in Lemma 2.1, it is easy to show that this fact implies that $\sqrt{\frac{d_\sigma}{d_\theta}}(L^*)^\vee$ is an isometry. Finally, we note that Theorem 2.4 in [17] is a version of our Corollary 3.5.

4 Harmonic analysis in $\mathrm{Hom}_G(\mathrm{Ind}_K^G V, \mathrm{Ind}_K^G V)$

Now we construct an orthonormal basis of the commutant of $\mathrm{Ind}_K^G V$ from the orthonormal bases analyzed in the previous section. This way we can introduce a Fourier transform that gives an explicit isomorphism between $\mathrm{Hom}_G(\mathrm{Ind}_K^G V, \mathrm{Ind}_K^G V)$ and $\bigoplus_{\sigma \in J} M_{m_\sigma, m_\sigma}(\mathbb{C})$.

Let (θ, V) be an irreducible representation of $K \leq G$ and (σ, W) an irreducible representation of G. Consider $L_1, L_2 \in \mathrm{Hom}_K(V, \mathrm{Res}_K^G W)$. Then we have $L_1^* \in \mathrm{Hom}_K(\mathrm{Res}_K^G W, V)$ and $(L_1^*)^\vee \in \mathrm{Hom}_G(W, \mathrm{Ind}_K^G V)$, so that $(\overset{\diamond}{L}_1)^* \overset{\diamond}{L}_2 = (L_1^*)^\vee \overset{\diamond}{L}_2 \in \mathrm{Hom}_G(\mathrm{Ind}_K^G V, \mathrm{Ind}_K^G V)$.

Lemma 4.1 *Let (σ_1, W_1) and (σ_2, W_2) be two irreducible inequivalent representations of G. Consider $L_1, L_2 \in Hom_K(V, Res_K^G W_2)$ and $L_3, L_4 \in Hom_K(V, Res_K^G W_1)$. Then*

$$\overset{\diamond}{L}_3(\overset{\diamond}{L}_1)^* = 0 \quad \textit{and} \quad \left\langle (\overset{\diamond}{L}_1)^* \overset{\diamond}{L}_2, (\overset{\diamond}{L}_3)^* \overset{\diamond}{L}_4 \right\rangle = 0. \tag{4.1}$$

Proof By Schur's lemma, $\overset{\diamond}{L}_3(\overset{\diamond}{L}_1)^* \in \mathrm{Hom}_G(W_2, W_1) = \{0\}$. Moreover, by definition of scalar product in $\mathrm{Hom}_G(\mathrm{Ind}_K^G V, \mathrm{Ind}_K^G V)$ we have

$$\left\langle (\overset{\diamond}{L}_1)^* \overset{\diamond}{L}_2, (\overset{\diamond}{L}_3)^* \overset{\diamond}{L}_4 \right\rangle = \frac{1}{\dim \mathrm{Ind}_K^G V} \mathrm{tr}\left[(\overset{\diamond}{L}_4)^* \overset{\diamond}{L}_3 (\overset{\diamond}{L}_1)^* \overset{\diamond}{L}_2 \right] = 0.$$

□

Lemma 4.2 *Let* (σ, W) *be an irreducible representation of* G *and* $\{L_1, L_2, \dots, L_m\}$ *an orthonormal basis of* $Hom_K(V, Res_K^G W)$. *Then*

$$\overset{\diamond}{L}_h(\overset{\diamond}{L}_i)^* = \frac{d_\theta}{d_\sigma} I_W \delta_{i,h}. \tag{4.2}$$

and the operators $(\overset{\diamond}{L}_i)^* \overset{\diamond}{L}_j \in Hom_G(Ind_K^G V, Ind_K^G V)$ *satisfy the orthogonality relations:*

$$\left\langle (\overset{\diamond}{L}_i)^* \overset{\diamond}{L}_j, (\overset{\diamond}{L}_h)^* \overset{\diamond}{L}_\ell \right\rangle = \delta_{i,h}\delta_{j,\ell} \frac{d_\theta}{d_\sigma |G/K|}.$$

Proof The identity (4.2) follows from (2.2) and Corollary 3.5. Moreover,

$$\begin{aligned} \left\langle (\overset{\diamond}{L}_i)^* \overset{\diamond}{L}_j, (\overset{\diamond}{L}_h)^* \overset{\diamond}{L}_\ell \right\rangle &= \frac{1}{\dim \mathrm{Ind}_K^G V} \mathrm{tr}\left[(\overset{\diamond}{L}_\ell)^* \overset{\diamond}{L}_h (\overset{\diamond}{L}_i)^* \overset{\diamond}{L}_j \right] \\ (\text{by } (4.17)) &= \frac{\delta_{i,h} d_\theta}{d_\sigma \dim \mathrm{Ind}_K^G V} \mathrm{tr}\left[(\overset{\diamond}{L}_\ell)^* \overset{\diamond}{L}_j \right] \\ (\text{by } (b) \text{ in Corollary } (3.4)) &= \delta_{i,h}\delta_{j,\ell} \frac{d_\theta}{d_\sigma |G/K|}. \end{aligned}$$

□

Now we use the notation in (1.3). For every $\sigma \in J$ select an orthonormal basis

$$\{L_{\sigma,1}, L_{\sigma,2}, \dots, L_{\sigma,m_\sigma}\} \tag{4.3}$$

of $\mathrm{Hom}_K(V, \mathrm{Res}_K^G W_\sigma)$ and set

$$U_{i,j}^\sigma = \frac{d_\sigma}{d_\theta} (\overset{\diamond}{L}_{\sigma,i})^* \overset{\diamond}{L}_{\sigma,j}, \qquad i, j = 1, 2, \dots, m_\sigma. \tag{4.4}$$

For every $T \in \mathrm{Hom}_G(\mathrm{Ind}_K^G V, \mathrm{Ind}_K^G V)$ and $\sigma \in J$, the *Fourier transform* of T at σ associated to the choice of (4.3) is the following matrix in $M_{m_\sigma, m_\sigma}(\mathbb{C})$:

$$[\mathcal{F}T(\sigma)]_{i,j} = \frac{d_\theta |G/K|}{d_\sigma} \langle T, U^\sigma_{i,j} \rangle, \qquad i, j = 1, 2, \ldots, m_\sigma.$$

We will show that the Fourier transform is an explicit form of the isomorphism (1.3). We need further notation. Every element in the algebra $\bigoplus_{\sigma \in J} M_{m_\sigma, m_\sigma}(\mathbb{C})$ may be represented in the form $\bigoplus_{\sigma \in J} A_\sigma$, where $A_\sigma \in M_{m_\sigma, m_\sigma}(\mathbb{C})$. In particular, given T in $\mathrm{Hom}_G(\mathrm{Ind}_K^G V, \mathrm{Ind}_K^G V)$, we set

$$\mathcal{F}T = \bigoplus_{\sigma \in J} \mathcal{F}T(\sigma). \tag{4.5}$$

We recall [7] that the irreducible representations of this algebra are given by the natural action of each $M_{m_\sigma, m_\sigma}(\mathbb{C})$ on $\mathbb{C}^{m_\sigma}$ and that [24] the corresponding irreducible characters are the functions $\{\varphi^\sigma : \sigma \in J\}$ given by: $\varphi^\rho \left(\bigoplus_{\sigma \in J} A_\sigma\right) = \mathrm{tr}(A_\rho)$. Under the isomorphism (1.3), the irreducible representation of $\mathrm{Hom}_G(\mathrm{Ind}_K^G V, \mathrm{Ind}_K^G V)$ corresponding to $\sigma \in J$ is given by its action on the space $\mathrm{Hom}_G(W_\sigma, \mathrm{Ind}_K^G V)$, that is by the map $S \mapsto TS$, where $T \in \mathrm{Hom}_G(\mathrm{Ind}_K^G V, \mathrm{Ind}_K^G V)$ and $S \in \mathrm{Hom}_G(W_\sigma, \mathrm{Ind}_K^G V)$. In what follows, we will also indicate by φ^σ the character of the isomorphic algebra $\mathrm{Hom}_G(\mathrm{Ind}_K^G V, \mathrm{Ind}_K^G V)$.

Theorem 4.3 (a) *The set*

$$\left\{ \sqrt{\frac{d_\theta |G/K|}{d_\sigma}} U^\sigma_{i,j} : \sigma \in J, i, j = 1, 2 \ldots, m_\sigma \right\} \tag{4.6}$$

is an orthonormal basis of $Hom_G(Ind_K^G V, Ind_K^G V)$. In particular, the Fourier inversion formula is: $T = \sum_{\sigma \in J} \sum_{i,j=1}^{m_\sigma} [\mathcal{F}T(\sigma)]_{i,j} U^\sigma_{i,j}$.

(b) *Setting $T_{\sigma,i} = \sqrt{\frac{d_\sigma}{d_\theta}} (\overset{\diamond}{L}_{\sigma,i})^*$, we have the isometric orthogonal decomposition $Ind_K^G V = \bigoplus_{\sigma \in J} \bigoplus_{i=1}^{m_\sigma} T_{\sigma,i} W_\sigma$, and the corresponding explicit isomorphism*

$$\begin{aligned} Hom_G(Ind_K^G V, Ind_K^G V) &\longrightarrow \bigoplus_{\sigma \in J} M_{m_\sigma, m_\sigma}(\mathbb{C}) \\ T &\longmapsto \mathcal{F}T. \end{aligned}$$

(c) *The operator $U^\sigma_{i,j}$ intertwines the subspace $T_{\sigma,j} W_\sigma$ with $T_{\sigma,i} W_\sigma$.*

(d) *The operator $U^\sigma_{i,i}$ is the orthogonal projection of $Ind_K^G V$ onto $T_{\sigma,i} W_\sigma$.*

(e) *The irreducible characters of $Hom_G(Ind_K^G V, Ind_K^G V)$ are the functions $\{\varphi^\sigma : \sigma \in J\}$ given by:*

$$\varphi^\sigma(T) = \mathrm{tr}\,[\mathcal{F}T(\sigma)], \quad \forall\, T \in Hom_G(Ind_K^G V, Ind_K^G V).$$

Proof From Lemmas 4.1 and 4.2 we deduce that the set (4.6) is orthonormal. Moreover, $\dim \mathrm{Hom}_G(\mathrm{Ind}_K^G V, \mathrm{Ind}_K^G V) = \sum_{\sigma \in J} m_\sigma^2$ so that it is a basis. For the other assertions just note that $T_{\sigma,i}$ is an isometry and that, by (4.1) and (4.2), we have

$$U^\sigma_{i,j}U^\rho_{h,l} = \delta_{\sigma,\rho}\delta_{j,h}U^\sigma_{i,l}, \qquad U^\sigma_{i,j}T_{\rho,h} = \delta_{\sigma,\rho}\delta_{j,h}T_{\sigma,i},$$

and therefore

$$TT_{\sigma,j} = \sum_{i=1}^{m_\sigma}[\mathcal{F}T(\sigma)]_{i,j}T_{\sigma,i},$$

for all $\sigma, \rho \in J$, $i, j = 1, 2, \dots, m_\sigma$, $h, l = 1, 2, \dots, m_\rho$ and $T \in \mathrm{Hom}_G(\mathrm{Ind}_K^G V, \mathrm{Ind}_K^G V)$. □

5 Harmonic analysis in the Hecke algebra

As in the previous section, G is a finite group, $K \leq G$, (θ, V) an irreducible K-representation and (σ, W) an irreducible G-representation. The left regular representation of G is denoted by λ_G (to distinguish it from $\lambda = \mathrm{Ind}_K^G\theta$); as usual, $[\lambda_G(g)f](g_0) = f(g^{-1}g_0)$ for all $f \in L(G)$, $g, g_0 \in G$. We choose $v \in V$ with $\|v\| = 1$ and define $\psi \in L(K)$ and $T_v \in \mathrm{Hom}(\mathrm{Ind}_K^G V, L(G))$ by setting

$$\psi(k) = \frac{d_\theta}{|K|}\langle v, \theta(k)v\rangle, \qquad \text{for all } k \in K;$$

$$(T_v f)(g) = \sqrt{d_\theta/|K|}\langle f(g), v\rangle \qquad \text{for all } f \in \mathrm{Ind}_K^G V, g \in G$$

The following projection formula will be a very useful tool.

Lemma 5.1 *If $v \in V$ has norm 1, then we have*

$$\sum_{k\in K}\langle\theta(k)u, v\rangle\theta(k^{-1})v = \frac{|K|}{d_\theta}u, \quad \forall u \in V. \tag{5.1}$$

Proof Let $\{v_1, v_2, \dots, v_{d_\theta}\}$ be an orthonormal basis of V with $v = v_1$. Then, for $i, j = 1, 2, \dots, d_\theta$, we have, taking into account the orthogonality relations in $L(K)$,

$$\left\langle\sum_{k\in K}\langle\theta(k)v_i, v\rangle\theta(k^{-1})v, v_j\right\rangle = \sum_{k\in K}\langle\theta(k)v_i, v_1\rangle\overline{\langle\theta(k)v_j, v_1\rangle} = \frac{|K|}{d_\theta}\delta_{i,j}.$$

Therefore we have proved (5.1) when $u = v_i$ and the general case follows by linearity. □

Proposition 5.2 (a) *T_v belongs to $Hom_G(Ind_K^G V, L(G))$ and it is an isometry.*
(b) *Since $\psi \in L(K) \subseteq L(G)$, we can define $P : L(G) \longrightarrow L(G)$, by setting $Pf = f * \psi$ for all $f \in L(G)$. Then P is the orthogonal projection of $L(G)$ onto $T_v[Ind_K^G V]$*

Proof (a) It is obvious that $T_v\lambda(g)f = \lambda_G(g)T_v f$. To show that T_v is an isometry, we use the basis in (2.9): assuming that $v = v_1$ we have, for $t_1, t_2 \in \mathcal{T}$, $i, j = 1, 2, \ldots, d_\theta$,

$$\begin{aligned}\langle T_v\lambda(t_1)f_{v_i}, T_v\lambda(t_2)f_{v_j}\rangle_{L(G)} &= \frac{d_\theta}{|K|}\sum_{g\in G}\langle f_{v_i}(t_1^{-1}g), v\rangle\overline{\langle f_{v_j}(t_2^{-1}g), v\rangle}\\ (\text{by } (2.10)) &= \frac{d_\theta}{|K|}\delta_{t_1,t_2}\sum_{k\in K}\langle\theta(k^{-1})v_i, v_1\rangle\overline{\langle\theta(k^{-1})v_j, v_1\rangle}\\ (\text{by } (2.6)) &= \delta_{t_1,t_2}\delta_{i,j}.\end{aligned}$$

(b) First of all, note that $\psi * \psi = \psi$ and $\overline{\psi(g^{-1})} = \psi(g)$. The first identity follows from (2.4) applied to θ and ensures that P is an idempotent; from the second identity we deduce that P is selfadjoint, and therefore it is an orthogonal projection. Moreover, for all $f \in \mathrm{Ind}_K^G V$, $g \in G$ we have

$$\begin{aligned}[(T_v f) * \psi](g) &= \left(\frac{d_\theta}{|K|}\right)^{3/2}\sum_{k\in K}\langle f(gk^{-1}), v\rangle\langle v, \theta(k)v\rangle\\ (\text{by } (2.9)) &= \left(\frac{d_\theta}{|K|}\right)^{3/2}\left\langle f(g), \sum_{k\in K}\langle\theta(k)v, v\rangle\theta(k^{-1})v\right\rangle\\ (\text{by } (5.22)) &= (T_v f)(g)\end{aligned}$$

that is, $PT_v f = T_v f$ for all $f \in \mathrm{Ind}_K^G V$ (and in particular $Ran P \supseteq T_v\mathrm{Ind}_K^G V)$. Finally, for all $\phi \in L(G)$, $g \in G$, we have

$$P\phi(g) = \sum_{k\in K}\phi(gk)\psi(k^{-1}) = \frac{d_\theta}{|K|}\left\langle\sum_{k\in K}\phi(gk)\theta(k)v, v\right\rangle = T_v f(g),$$

if $f(g) = \sqrt{d_\theta/|K|}\sum_{k\in K}\phi(gk)\theta(k)v$. Since it is immediate to check that f belongs to $\mathrm{Ind}_K^G V$, we conclude that $Ran P$ is contained in (and therefore equal to) $T_v\left[\mathrm{Ind}_K^G V\right]$. □

Remark 5.3 We want to relate the operator T_v in the context of the results in Section 3. First note that the choice of v is equivalent to the choice of an isometry $L \in \mathrm{Hom}(\mathbb{C}, V)$, namely $L(\alpha) = \alpha v$ for $\alpha \in \mathbb{C}$. Then, with K, V, G, W replaced by $1_K, \mathbb{C}, K, V$ we have:

$$[(L^*)^\vee u](k) = \frac{1}{\sqrt{|K|}}\langle u, \theta(k)v\rangle \qquad \text{for all } u \in V \text{ and } k \in K$$

($L^*u = \langle u, v\rangle$ for all $u \in V$) and the map $S_v = \sqrt{d_\theta}(L^*)^\vee$ is an isometric immersion of V into $L(K)$ (this is also an easy consequence of the orthogonality relation for matrix coefficients). Since $S_v \in \mathrm{Hom}_K(V, \mathrm{Res}_K^G L(G))$ (because $L(K) \subseteq \mathrm{Res}_K^G L(G)$ in the natural way), it is easy to prove that $T_v = \sqrt{|G/K|}\overset{\diamond}{S}_v$, where in this case we apply the machinery in Sect. 3 with K, V and G as in that section, but with W replaced by $L(G)$.

In the terminology of [9,10,24], ψ is a primitive idempotent, $S_v(V) = \{f \in L(K) : f * \psi = f\}$ is the minimal left ideal in $L(K)$ generated by ψ and $T_v[\mathrm{Ind}_K^G V]$ is generated by ψ as a left ideal in $L(G)$. Then, the *Hecke algebra* $\mathcal{H}(G, K, \psi)$ is by definition

$$\mathcal{H}(G, K, \psi) = \{\psi * f * \psi : f \in L(G)\} \equiv \{f \in L(G) : f = \psi * f * \psi\}.$$

It is well known that $\mathcal{H}(G, K, \psi)$ is antiisomorphic to $\mathrm{Hom}_G(\mathrm{Ind}_K^G V, \mathrm{Ind}_K^G V)$: we now want to go further and develop a suitable harmonic analysis in $\mathcal{H}(G, K, \psi)$, by translating the results of Sect. 4. First of all, we introduce a suitable orthonormal basis in each G-irreducible representation. We divide the description of these bases in various cases.

Suppose that $\sigma \in J$ and that $\mathrm{Res}_K^G W_\sigma = m_\sigma V \oplus \left(\oplus_{\rho \in R} m_\rho U_\rho\right)$ is the decomposition of $\mathrm{Res}_K^G W_\sigma$ into irreducible K-representations, where R contains the representations different from σ. Let $\{L_{\sigma,1}, L_{\sigma,2}, \ldots, L_{\sigma,m_\sigma}\}$ be as in (4.3) and v as above. We begin by introducing an orthonormal basis in the *V-isotypic component*. The first step consists in setting

$$w_i^\sigma = L_{\sigma,i} v, \qquad i = 1, 2, \ldots, m_\sigma. \tag{5.2}$$

In the second and last step we introduce (see also Lemma 5.1) an orthonormal basis $v_1, v_2, \ldots, v_{d_\theta}$ of V with $v = v_1$ and we suppose that $\{w_h^\sigma : m_\sigma + 1 \le h \le m_\sigma d_\theta\}$ is an *arbitrary arrangement* of the vectors $\{L_{\sigma,i} v_j : 1 \le i \le m_\sigma, 2 \le j \le d_\theta\}$. The final result

$$\{w_h^\sigma : 1 \le h \le m_\sigma d_\theta\}$$

is the desired orthonormal basis in $m_\sigma V$.

Then we repeat the construction for each $U_\rho, \rho \in R$, without an initial choice of a vector in U_ρ (we avoid the first step): we select an orthonormal basis $\{M_{\rho,1}, M_{\rho,2}, \ldots, M_{\rho,m_\rho}\}$ for $\mathrm{Hom}_K(U_\rho, \mathrm{Res}_K^G W_\sigma)$, an orthonormal basis $\{u_1^\rho, u_2^\rho, \ldots, u_{d_\rho}^\rho\}$ in U_ρ and we suppose that

$$\{w_h^\sigma : m_\sigma d_\theta + 1 \le h \le d_\sigma\}$$

is an *arbitrary arrangement* of the vectors $\{M_{\rho,i} u_j^\rho : \rho \in R, 1 \le i \le m_\rho, 1 \le j \le d_\rho\}$. The final result is an orthonormal basis $\{w_h^\sigma : 1 \le h \le d_\sigma\}$ for W_σ: we say that it is *adapted to the choice of* v *and of* $\{L_{\sigma,1}, L_{\sigma,2}, \ldots, L_{\sigma,m_\sigma}\}$.

If $\sigma \notin J$ then $\{w_h^\sigma : 1 \le h \le d_\sigma\}$ is an *arbitrary* orthonormal basis of W_σ. The importance of such bases is in the following properties.

Lemma 5.4 (a) *If* $\sigma \in J$, $1 \leq j \leq m_\sigma$ *and* $1 \leq h \leq d_\sigma$ *then*

$$L^*_{\sigma,j} w^\sigma_h = \begin{cases} v_\ell & if\ w^\sigma_h = L_{\sigma,j} v_\ell\ for some\ \ell \in \{1, 2, \ldots, d_\theta\} \\ 0 & otherwise. \end{cases} \tag{5.3}$$

(b)

$$\sum_{k \in K} \psi(k)\sigma(k) w^\sigma_i = \begin{cases} \frac{|K|}{d_\theta} w^\sigma_i & if\ \sigma \in J,\ 1 \leq i \leq m_\sigma \\ 0 & otherwise. \end{cases} \tag{5.4}$$

Proof (a) This is a consequence of (2.2) and the definition of the vectors w^σ_h.
(b) First of all, note that

$$\sum_{k \in K} \psi(k)\sigma(k) w^\sigma_i = \frac{d_\theta}{|K|} \sum_{k \in K} \langle \theta(k)v, v\rangle \sigma(k^{-1}) w^\sigma_i.$$

If $\sigma \in J$ and $1 \leq i \leq m_\sigma$, then we may apply (5.1) since $\sigma(k^{-1}) w^\sigma_i = \sigma(k^{-1}) L_{\sigma,i} v = L_{\sigma,i} \theta(k^{-1}) v$. Otherwise, we can argue as in the proof of (5.1): the bases are adapted to the choice of v and to the decomposition of $\mathrm{Res}^G_K W_\sigma$ and therefore we may use the orthogonality relations for the matrix coefficients in $L(K)$. For instance, if $\sigma \in J$ and $m_\sigma < i \leq m_\sigma d_\theta$ then $w^\sigma_i = L_{\sigma,h} v_j$ for some $1 \leq h \leq m_\sigma$, $2 \leq j \leq d_\theta$, and therefore, for all $\ell = 1, 2, \ldots, d_\sigma$

$$\left\langle \sum_{k \in K} \langle \theta(k)v, v\rangle \sigma(k^{-1}) L_{\sigma,h} v_j, w^\sigma_\ell \right\rangle = \sum_{k \in K} \langle \theta(k)v, v\rangle \langle \theta(k^{-1}) v_j, L^*_{\sigma,h} w^\sigma_\ell \rangle = 0$$

because if $L^*_{\sigma,h} w^\sigma_\ell \neq 0$ by (5.3) it is equal to one of the $v_1, v_2, \ldots, v_{d_\theta}$ and $j \geq 2$. In the other cases we are dealing with matrix coefficients corresponding to inequivalent K-representations.

□

The matrix coefficients of σ corresponding to the bases chosen above are $u^\sigma_{i,j}(g) = \langle \sigma(g) w^\sigma_j, w^\sigma_i \rangle$, $\sigma \in \widehat{G}$, $i, j = 1, 2, \ldots, d_\sigma$, $g \in G$. We define the convolution operators

$$\widetilde{U}^\sigma_{i,j} f = \frac{d_\sigma}{|G|} f * \overline{u^\sigma_{j,i}}, \quad \text{where } f \in L(G), \quad \sigma \in J, i, j = 1, 2, \ldots, m_\sigma.$$

Now we show that $\widetilde{U}^\sigma_{i,j}$ corresponds to $U^\sigma_{i,j}$ in (4.4) under the isometry T_v.

Theorem 5.5 *For all* $\sigma \in J$, $i, j = 1, 2, \ldots, m_\sigma$ *and* $f \in Ind^G_K V$ *we have:*

$$T_v U^\sigma_{i,j} f = \widetilde{U}^\sigma_{i,j} T_v f,$$

that is, the following diagram is commutative:

$$\begin{array}{ccc} Ind_K^G V & \xrightarrow{T_v} & T_v[Ind_K^G V] \\ U_{i,j}^{\sigma} \downarrow & & \downarrow \widetilde{U}_{i,j}^{\sigma} \\ Ind_K^G V & \xrightarrow{T_v} & T_v[Ind_K^G V]. \end{array}$$

Proof For all $g \in G$ we have:

$$\begin{aligned}
[T_v U_{i,j}^{\sigma} f](g) &= \frac{d_\sigma}{\sqrt{|K|d_\theta}} \langle [(L_{\sigma,i}^*)^{\vee} \overset{\diamond}{L}_{\sigma,j} f](g), v\rangle \\
&= \frac{d_\sigma}{|G|\sqrt{|K|d_\theta}} \sum_{g_1 \in G} \langle L_{\sigma,i}^* \sigma(g^{-1}g_1) L_{\sigma,j} f(g_1), v\rangle \\
(L_{\sigma,i} v = w_i^{\sigma}) &= \frac{d_\sigma}{|G|\sqrt{|K|d_\theta}} \sum_{g_1 \in G} \sum_{h=1}^{d_\sigma} \langle f(g_1), L_{\sigma,j}^* w_h^{\sigma}\rangle \overline{\langle \sigma(g_1^{-1}g) w_i^{\sigma}, w_h^{\sigma}\rangle} \\
(\text{by } (5.24)) &= \frac{d_\sigma}{|G|\sqrt{|K|d_\theta}} \sum_{g_1 \in G} \sum_{\ell=1}^{d_\theta} \langle f(g_1), v_\ell\rangle \overline{\langle \sigma(g_1^{-1}g) w_i^{\sigma}, L_{\sigma,j} v_\ell\rangle} \\
(\text{by } (5.22) \text{ with } u = v_\ell) &= \frac{d_\sigma}{|G|} \frac{\sqrt{d_\theta}}{|K|^{3/2}} \sum_{\ell=1}^{d_\theta} \sum_{g_1 \in G} \sum_{k \in K} \langle f(g_1), \theta(k^{-1}) v\rangle \cdot \\
&\quad \cdot \langle v, \theta(k) v_\ell\rangle \overline{\langle \sigma(g_1^{-1}g) w_i^{\sigma}, L_{\sigma,j} v_\ell\rangle} \\
(g_1 = g_2 k) &= \frac{d_\sigma}{|G|} \frac{\sqrt{d_\theta}}{|K|^{3/2}} \sum_{\ell=1}^{d_\theta} \sum_{g_2 \in G} \langle f(g_2), v\rangle \\
&\quad \times \overline{\left\langle \sigma(g_2^{-1}g) w_i^{\sigma}, L_{\sigma,j} \sum_{k \in K} \langle \theta(k^{-1}) v, v_\ell\rangle \theta(k) v_\ell \right\rangle} \\
&= \frac{d_\sigma \sqrt{d_\theta}}{|G|\sqrt{|K|}} \sum_{g_2 \in G} \langle f(g_2), v\rangle \overline{\langle \sigma(g_2^{-1}g) w_i^{\sigma}, w_j^{\sigma}\rangle} \\
(\text{by } (5.22) \text{ with } u, v \text{replacedby} v, v_\ell) &= \frac{d_\sigma \sqrt{d_\theta}}{|G|\sqrt{|K|}} \sum_{g_2 \in G} \langle f(g_2), v\rangle \overline{u_{j,i}^{\sigma}(g_2^{-1}g)} \\
&= [\widetilde{U}_{i,j} T_v f](g).
\end{aligned}$$

□

Lemma 5.6 *We have*

$$\psi * \overline{u_{i,j}^{\sigma}} * \psi = \begin{cases} \overline{u_{i,j}^{\sigma}} & if \sigma \in J \text{ } and \text{ } 1 \le i, j \le m_\sigma \\ 0 & otherwise. \end{cases}$$

that is, $u_{ij}^{\sigma} \in \mathcal{H}(G, K, \psi)$ *if and only* $\sigma \in J$ *and* $1 \le i, j \le m_\sigma$.

Proof For all $g \in G$,

$$\begin{aligned} \overline{u^\sigma_{i,j}} * \psi(g) &= \sum_{k\in K} u^\sigma_{j,i}(kg^{-1})\psi(k) \\ (\text{by definition of } u^\sigma_{i,j}) &= \left\langle \sigma(g^{-1})w^\sigma_i, \sum_{k\in K}\psi(k)\sigma(k)w^\sigma_j \right\rangle \\ (\text{by}(5.25)) &= \begin{cases} \overline{u^\sigma_{i,j}}(g) & \text{if } \sigma \in J \text{ and } 1 \le j \le m_\sigma \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.5}$$

The reader can complete the proof by computing in a similar way $\psi * \overline{u^\sigma_{i,j}}$. □

We set

$$\phi^\sigma_{i,j} = \overline{u^\sigma_{i,j}} \tag{5.6}$$

for $\sigma \in J$, $i, j = 1, 2, \dots, m_\sigma$, that is, $\phi^\sigma_{i,j}(g) = \langle w^\sigma_i, \sigma(g)w^\sigma_j\rangle$ for all $g \in G$, and these are the *spherical matrix coefficients* of the Hecke algebra. Compare with Definition 9.4.5. of [4] and Definition 2.10 of [21]. If $f \in \mathcal{H}(G, K, \psi)$, its *Fourier transform* at $\sigma \in J$ is the $m_\sigma \times m_\sigma$ complex matrix whose i, j-entry is

$$[\mathcal{F}f(\sigma)]_{i,j} = \langle f, \phi^\sigma_{i,j}\rangle_{L(G)}.$$

As in (4.5), we set $\mathcal{F}f = \bigoplus_{\sigma\in J}\mathcal{F}f(\sigma)$. We denote by χ^σ the character of the G-irreducible representation (σ, W_σ). Moreover, $\chi^\sigma(f) = \sum_{g\in G}\chi^\sigma(g)f(g)$ for all $f \in L(G)$.

Theorem 5.7 (a) *The set* $\{\phi^\sigma_{i,j} : \sigma \in J, i, j = 1, 2, \dots, m_\sigma\}$ *is an orthogonal basis for* $\mathcal{H}(G, K, \psi)$ *and* $\|\phi^\sigma_{i,j}\|^2_{L(G)} = \frac{|G|}{d_\sigma}$. *In particular, the Fourier inversion formula is*

$$f = \frac{1}{|G|}\sum_{\sigma\in J} d_\sigma \sum_{i,j=1}^{m_\sigma}[\mathcal{F}f(\sigma)]_{i,j}\,\phi^\sigma_{i,j}.$$

(b) *The map*

$$\begin{aligned} \mathcal{H}(G, K, \psi) &\longrightarrow \bigoplus_{\sigma\in J} M_{m_\sigma,m_\sigma}(\mathbb{C}) \\ f &\longmapsto \mathcal{F}f \end{aligned}$$

is an isomorphism of algebras.

(c) *Set* $\phi^\sigma = \sum_{i=1}^{m_\sigma}\phi^\sigma_{i,i}$ *and suppose that* φ^σ *is the irreducible character of* $\mathcal{H}(G, K, \psi)$ *corresponding to* $M_{m_\sigma,m_\sigma}(\mathbb{C})$. *Then*

$$\varphi^\sigma(f) = \chi^\sigma(f) = \sum_{g\in G} f(g)\overline{\phi^\sigma(g)}, \qquad \textit{for all } f \in \mathcal{H}(G, K, \psi). \tag{5.7}$$

Moreover, the ϕ^σ's satisfy the following orthogonality relations:

$$\langle \phi^\sigma, \phi^\rho \rangle = \delta_{\sigma,\rho} \frac{|G| m_\sigma}{d_\sigma}. \tag{5.8}$$

Proof (a) It follows from Lemma 5.6 and the usual orthogonality relations (2.3).
(b) The usual convolution properties of the matrix coefficients (2.4) yields

$$\phi^\sigma_{i,j} * \phi^\rho_{h,\ell} = \frac{|G|}{d_\sigma} \delta_{\sigma,\rho} \delta_{j,h} \phi^\sigma_{i,\ell}.$$

Then from the inversion it follows that $\mathcal{F}$ is an isomorphism.
(c) For all $f \in \mathcal{H}(G, K, \psi)$ and $\sigma \in J$, we have

$$\begin{aligned} \varphi^\sigma(f) &= \sum_{i=1}^{m_\sigma} [\mathcal{F} f(\sigma)]_{i,i} = \sum_{i=1}^{m_\sigma} \langle f, \phi^\sigma_{i,i} \rangle = \sum_{g \in G} f(g) \overline{\phi^\sigma(g)} \\ &= \sum_{i=1}^{m_\sigma} \sum_{g \in G} f(g) u^\sigma_{i,i}(g) = \chi^\sigma(f), \end{aligned}$$

where the last equality follows from $\langle f, u^\sigma_{i,i} \rangle = 0$ if $i > m_\sigma$. (5.8) is obvious.

□

Remark 5.8 To each $\phi \in L(G)$ we can associate the convolution operator $T_\phi f = f * \psi$, for all $f \in L(G)$. Since $T_{\phi_1 * \phi_2} = T_{\phi_2} T_{\phi_1}$, the map $\phi \mapsto T_\phi$ is an *antiisomorphism* between $L(G)$ and $\mathrm{Hom}_G(L(G), L(G))$, see Exercise 4.2.2. in [4]. It follows that the map $U^\sigma_{i,j} \mapsto \widetilde{U}^\sigma_{i,j}$ in Theorem 5.5 yields the *antiisomorphism* $U^\sigma_{i,j} \mapsto \phi^\sigma_{j,i}$ between $\mathrm{Hom}_G(\mathrm{Ind}_K^G V, \mathrm{Ind}_K^G V)$ and $\mathcal{H}(G, K, \psi)$ (see also [9,10,24]). Actually, these algebras are isomorphic, because they are both isomorphic to $\bigoplus_{\sigma \in J} M_{m_\sigma, m_\sigma}(\mathbb{C})$. Moreover, all other results in Theorem 4.3 may be translated in the present setting. For instance, if we define $\widetilde{T}_{\sigma,i} : W_\sigma \to L(G)$ by setting

$$(\widetilde{T}_{\sigma,i} w)(g) = \sqrt{d_\sigma / |G|} \langle w, \sigma(g) w^\sigma_i \rangle, \qquad \text{for all } w \in W, g \in G,$$

then it is easy to check that $T_v T_{\sigma,i} = \widetilde{T}_{\sigma,i}$ and $T_v[\mathrm{Ind}_K^G V] = \bigoplus_{\sigma \in J} \bigoplus_{i=1}^{m_\sigma} \widetilde{T}_{\sigma,i} W_\sigma$ is an isometric orthogonal decomposition. Moreover, $\widetilde{U}^\sigma_{i,j}$ intertwines $\widetilde{T}_{\sigma,j} W_\sigma$ with $\widetilde{T}_{\sigma,i} W_\sigma$ and $\widetilde{U}^\sigma_{i,i}$ is the orthogonal projection onto $\widetilde{T}_{\sigma,i} W_\sigma$.

We now prove some formulas that relate χ^σ, φ^σ and ϕ^σ [see also (5.7)]. We recall that δ_g is the Dirac function centered at g, that is $\delta_g(g_0) = \begin{cases} 1 & \text{if } g = g_0 \\ 0 & \text{otherwise.} \end{cases}$

Theorem 5.9 *We have:*

$$\chi^\sigma(g) = \frac{d_\sigma}{|G| m_\sigma} \sum_{h \in G} \overline{\phi^\sigma(h^{-1} g h)} \tag{5.9}$$

and

$$\phi^\sigma(g) = \overline{\varphi^\sigma(\psi * \delta_g * \psi)}, \qquad \textit{for all } \sigma \in J \textit{ and } g \in G. \tag{5.10}$$

Proof Clearly, (5.9) follows from (2.5), since ϕ^σ is the sum of the conjugate of m_σ diagonal matrix coefficients. Now we prove (5.10). Starting from (5.7) we get:

$$\begin{aligned}
\varphi^\sigma(\psi * \delta_g * \psi) &= \sum_{g_1 \in G} (\psi * \delta_g * \psi)(g_1)\overline{\phi^\sigma(g_1)} \\
&= \sum_{\substack{g_1 \in G: \\ g_1 k_2^{-1} g^{-1} \in K}} \sum_{k_2 \in K} \left[\psi(g_1 k_2^{-1} g^{-1})\psi(k_2)\right] \overline{\phi^\sigma(g_1)} \\
(k_1 = g_1 k_2^{-1} g^{-1}) &= \sum_{k_1, k_2 \in K} \psi(k_1)\psi(k_2)\overline{\phi^\sigma(k_1 g k_2)} \\
(\text{by Lemma 5.6}) &= \overline{\phi^\sigma(g)}.
\end{aligned}$$

□

Remark 5.10 In [9] (see also [10,24]), a formula, that expresses χ^σ in terms of φ^σ, is proved . In our notation it reads:

$$\chi^\sigma(g) = \frac{|G|}{|\mathcal{C}(g)|}\varphi^\sigma(\psi * \mathbf{1}_{\mathcal{C}(g)} * \psi)\left[\sum_{h \in G} \varphi^\sigma(\psi * \delta_{h^{-1}} * \psi) \cdot \varphi^\sigma(\psi * \delta_h * \psi)\right]^{-1} \tag{5.11}$$

where $\mathcal{C}(g)$ denotes the conjugacy class of $g \in G$ and $\mathbf{1}_{\mathcal{C}(g)}$ its characteristic function. We now want to deduce (5.11) from the results in the present paper. First note that by (5.10)

$$\begin{aligned}
\sum_{h \in G} \varphi^\sigma(\psi * \delta_h * \psi)\varphi^\sigma(\psi * \delta_{h^{-1}} * \psi) &= \sum_{h \in G} \overline{\phi^\sigma(h)}\,\overline{\phi^\sigma(h^{-1})} = \sum_{h \in G} |\phi^\sigma(h)|^2 \\
(\text{by (2.6)}) &= \frac{m_\sigma |G|}{d_\sigma}.
\end{aligned} \tag{5.12}$$

Moreover, from the equality $\mathbf{1}_{\mathcal{C}(g)} = \frac{|\mathcal{C}(g)|}{|G|} \sum_{h \in G} \delta_{h^{-1}gh}$, using (5.10) and then (5.9) we get

$$\begin{aligned}
\varphi^\sigma(\psi * \mathbf{1}_{\mathcal{C}(g)} * \psi) &= \frac{|\mathcal{C}(g)|}{|G|} \sum_{h \in G} \varphi^\sigma(\psi * \delta_{h^{-1}gh} * \psi) = \frac{|\mathcal{C}(g)|}{|G|} \sum_{h \in G} \overline{\phi^\sigma(h^{-1}gh)} \\
&= \frac{|\mathcal{C}(g)| m_\sigma}{d_\sigma} \chi^\sigma(g).
\end{aligned} \tag{5.13}$$

Then (5.11) follows from (5.12) and (5.13).

The spherical functions of a finite Gelfand pair satisfy the following functional identity $\frac{1}{|K|} \sum_{k \in K} \phi(gkh) = \phi(g)\phi(h)$, for all $g, h \in G$ (see Theorem 4.5.3 in [4]). We prove an analogous identity for the spherical matrix coefficients $\phi^\sigma_{i,j}$.

Proposition 5.11 *For $\sigma \in J$, $i, j = 1, 2, \ldots, m_\sigma$ and $g, h \in G$, we have*

$$\sum_{k \in K} \phi^\sigma_{i,j}(gkh)\overline{\psi(k)} = \sum_{\ell=1}^{m_\sigma} \phi^\sigma_{i,\ell}(g)\phi^\sigma_{\ell,j}(h).$$

Proof

$$\begin{aligned}\sum_{k \in K} \phi^\sigma_{i,j}(gkh)\overline{\psi(k)} &= \sum_{k \in K} \langle w^\sigma_i, \sigma(gkh)w^\sigma_j\rangle\overline{\psi(k)} \\ &= \sum_{\ell=1}^{d_\sigma} \langle \sigma(g^{-1})w^\sigma_i, w^\sigma_\ell\rangle \sum_{k \in K} \overline{u^\sigma_{\ell,j}(kh)\psi(k)} \\ \text{(by (5.26))} &= \sum_{\ell=1}^{m_\sigma} \phi^\sigma_{i,\ell}(g)\phi^\sigma_{\ell,j}(h).\end{aligned}$$

□

Remark 5.12 If $\chi : K \to \{z \in \mathbb{C} : |z| = 1\}$ is a one-dimensional K-representation then

$$\mathcal{H}(G, K, \chi) = \{f \in L(G) : f(k_1 g k_2) = \overline{\chi(k_1)\chi(k_2)} f(g), \ \forall k_1, k_2 \in K, \ g \in G\}.$$

See [24] for the easy details. If $G = \coprod_{s \in \mathcal{S}} KsK$ is the decomposition of G into double K-cosets, then a function $f \in \mathcal{H}(G, K, \psi)$ is determined by its values on $\mathcal{S}$. In particular, the orthogonality relations for $\phi^\sigma_{i,j}$ and ϕ^σ take the form:

$$\sum_{s \in \mathcal{S}} |KsK|\phi^\sigma_{i,j}(s)\overline{\phi^\rho_{\ell,r}(s)} = \frac{|G|}{d_\sigma}\delta_{\sigma,\rho}\delta_{i,\ell}\delta_{j,r} \qquad \text{and}$$

$$\sum_{s \in \mathcal{S}} |KsK|\phi^\sigma(s)\overline{\phi^\rho(s)} = \frac{m_\sigma|G|}{d_\sigma}\delta_{\sigma,\rho}.$$

From the last formula, it is easy to deduce the orthogonality relations in 2, Theorem 4.24 in [24], (the original sources are Theorem 2.4 in [9] or (ii), Theorem 11.32 in [10]).

6 A worked example

In this section, we describe an example in which all the operators and functions of Sects. 3, 4 and 5 are explicitly computable. In order to simplify the notation, we will use the same symbol to denote a one-dimensional representation and its representation space. Denote by $\mathbb{F}_q$ the finite field on $q = p^n$ elements (p is a prime and $n \geq 1$), by $\mathbb{F}^*_q$ the multiplicative group of non zero elements and by $\widehat{\mathbb{F}}_q$ the set of all additive characters on $\mathbb{F}_q$. We recall that the *finite affine group* over $\mathbb{F}_q$ is the semidirect product $\mathbb{F}^*_q \ltimes \mathbb{F}_q$;

if $(x,y),(a,b)\in\mathbb{F}_q^*\ltimes\mathbb{F}_q$ their product is given by: $(x,y)(a,b)=(xa,xb+y)$. This group has exactly $q-1$ one-dimensional representations and one $(q-1)$-dimensional representation that we call θ. If $\chi\in\widehat{\mathbb{F}}_q$ is nontrivial, then

$$\theta=\mathrm{Ind}_{\mathbb{F}_q}^{\mathbb{F}_q^*\ltimes\mathbb{F}_q}\chi. \tag{6.1}$$

See [8,26]. The corresponding representation space is

$$V=\{f:\mathbb{F}_q^*\ltimes\mathbb{F}_q\to\mathbb{C}\mid f(x,y)=\chi(-yx^{-1})f(x,0)\text{ for all }(x,y)\in\mathbb{F}_q^*\ltimes\mathbb{F}_q\}. \tag{6.2}$$

The action is given by:

$$[\theta(a,b)f](x,y)=f(a^{-1}x,a^{-1}y-a^{-1}b), \tag{6.3}$$

and the scalar product is: $\langle f_1,f_2\rangle_V=\sum_{x\in\mathbb{F}_q^*}f_1(x,0)\overline{f_2(x,0)}$. We fix χ as above and for $z\in\mathbb{F}^*$ we define $\chi_z(x)=\chi(xz)$, for all $x\in\mathbb{F}_q$. Then $\{\chi_z:z\in\mathbb{F}_q^*\}$ is exactly the set of all nontrivial additive characters of $\mathbb{F}_q$. Moreover, if we denote by V_z the space (6.2) with χ replaced by χ_z then it is easy to check that the map $T_z:V\to V_z$, defined by setting $T_zf(x,y)=f(xz^{-1},y)$, for all $f\in V$ and $(x,y)\in\mathbb{F}_q^*\ltimes\mathbb{F}_q$, is an isometric isomorphism of representations.

Now suppose suppose that $m>1$; considering $\mathbb{F}_{q^m}$ as a vector space over the subfield $\mathbb{F}_q$, we can fix a decomposition $\mathbb{F}_{q^m}=\mathbb{F}_q\oplus\mathbb{F}_q^{m-1}$, so that any $x\in\mathbb{F}_{q^m}$ may be written uniquely in the form: $x=x_1+x_2$, with $x_1\in\mathbb{F}_q$ and $x_2\in\mathbb{F}_q^{m-1}$. Similarly, every $\psi\in\widehat{\mathbb{F}}_{q^m}$ may be written uniquely in the form: $\psi=\chi\xi$, where $\chi\in\widehat{\mathbb{F}}_q$ and $\xi\in\widehat{\mathbb{F}}_q^{m-1}$. Now we fix $\chi\in\widehat{\mathbb{F}}_q$ nontrivial and we apply the theory developed in the previous sections to case:

$$G=\mathbb{F}_{q^m}^*\ltimes\mathbb{F}_{q^m},\qquad K=\mathbb{F}_q^*\ltimes\mathbb{F}_q,\qquad \sigma=\mathrm{Ind}_{\mathbb{F}_{q^m}}^{\mathbb{F}_{q^m}^*\ltimes\mathbb{F}_{q^m}}(\chi\mathbf{1}),\qquad \theta=\mathrm{Ind}_{\mathbb{F}_q}^{\mathbb{F}_q^*\ltimes\mathbb{F}_q}\chi$$

where $\mathbf{1}$ is the trivial character of $\mathbb{F}_q^{m-1}$. Since the restriction of the one-dimensional representations of G cannot contain θ, by Frobenius reciprocity $\mathrm{Ind}_K^G\theta$ contains only σ: $\mathrm{Ind}_K^G\theta\cong\frac{q^m-1}{q-1}\sigma$. Therefore we can avoid the superscript σ.

The space V is as in (6.2) and W has a similar structure. We choose a subset $Z\subset\mathbb{F}_{q^m}^*$ such that: $|Z|=q^{m-1}$ and

$$\{\chi\xi:\xi\in\widehat{\mathbb{F}}_q^{m-1}\}=\{(\chi\mathbf{1})_z:z\in Z\}.$$

That is, we have a bijection between $\widehat{\mathbb{F}}_q^{m-1}$ and Z: if ξ corresponds to z then $(\chi\xi)(x)=(\chi\mathbf{1})(zx)$, for all $x\in\mathbb{F}_{q^m}$. In particular, if $x\in\mathbb{F}_q$ then $(\chi\mathbf{1})_z(x)=\chi(x)$.

Lemma 6.1 *$\{z\mathbb{F}_q^*:z\in Z\}$ is a family of pairwise disjoint subsets of $\mathbb{F}_{q^m}^*$.*

Proof First of all, note that $\mathrm{Ker}(\chi\xi)$ is equal to the set of all $x=x_1+x_2\in\mathbb{F}_{q^m}$, with $x_1\in\mathbb{F}$ and $x_2\in\mathbb{F}_q^m$ such that: $\xi(x_2)=\overline{\chi(x_1)}$ (the values of χ and ξ are p-th roots of the unit). In particular, $\mathrm{Ker}\,(\chi\mathbf{1})_z = \mathrm{Ker}\,(\chi\xi) = \{x \in \mathbb{F}_{q^m} : zx \in \mathbb{F}_q^{m-1}\}$ determines z. Now we show that if $z_1, z_2 \in Z$, $z_1 \neq z_2$ then $z_1^{-1}z_2 \notin \mathbb{F}_q^*$. By contradiction, assume that $z_2 = az_1$ with $a \in \mathbb{F}_q^*$. If $x \in \mathbb{F}_{q^m}$, $z_1x = y_1 + y_2$ and $z_2x = u_1 + u_2$, with $y_1, u_1 \in \mathbb{F}_q$ and $y_2, u_2 \in \mathbb{F}_q^{m-1}$, then $u_1 = ay_1$. Therefore $y_1 = 0$ if and only if $u_1 = 0$ so that $\mathrm{Ker}\,(\chi\mathbf{1})_{z_1} = \mathrm{Ker}\,(\chi\mathbf{1})_{z_2}$. □

Now with each $z \in Z$ we associate an operator $L_z : V \to W$ defined by setting:

$$(L_z f)(x, y) = \begin{cases} (\chi\mathbf{1})\,(-x^{-1}y)f(zx, 0) & \text{if } zx \in \mathbb{F}_q^* \\ 0 & \text{otherwise,} \end{cases}$$

for all $(x, y) \in G$ and $f \in V$.

Proposition 6.2 *The set $\{L_z : z \in Z\}$ is an orthonormal basis of $Hom_K(V, Res_K^G W)$.*

Proof First of all, we show that L_z intertwines the actions of K on V and W. Suppose that $f \in V$, $(x, y) \in G$ and $(a, b) \in K$. If $xz \in \mathbb{F}_q^*$ then

$$\begin{aligned}(\chi\mathbf{1})\,(-x^{-1}y + x^{-1}b) &= (\chi\mathbf{1})\,(-x^{-1}y)\,(\chi\mathbf{1})_z\,(x^{-1}z^{-1}b) \\ (x^{-1}z^{-1}b \in \mathbb{F}_q) &= (\chi\mathbf{1})\,(-x^{-1}y)\chi(x^{-1}z^{-1}b)\end{aligned}$$

and therefore

$$\begin{aligned}[L_z\theta(a, b)f](x, y) &= (\chi\mathbf{1})\,(-x^{-1}y)[\theta(a, b)f](zx, 0) \\ \text{(by (6.36) and (6.37))} &= (\chi\mathbf{1})\,(-x^{-1}y)\chi(x^{-1}z^{-1}b)f(a^{-1}zx, 0) \\ &= (\chi\mathbf{1})\,(-x^{-1}y + x^{-1}b)f(a^{-1}zx, 0) \\ \text{(by (6.36) and (6.37) applied to } \sigma) &= [\sigma(a, b)L_z f](x, y).\end{aligned}$$

Similarly, if $zx \notin \mathbb{F}_q^*$ then $[L_z\theta(a, b)f](x, y) = 0 = [\sigma(a, b)L_z f](x, y)$. From Lemma 6.1 it follows that the operators $\{L_z : z \in Z\}$ form an orthogonal family because the supports of their images are disjoint; it is easy to check that these operators are also isometric immersions. □

By transitivity of induction, we have the isomorphism $\mathrm{Ind}_K^G\theta = \mathrm{Ind}_K^G\mathrm{Ind}_{\mathbb{F}_q}^K\chi \cong \mathrm{Ind}_{\mathbb{F}_q}^G\chi$. More precisely, taking into account (2.11) a function $F : G\times K \to \mathbb{C}$ belongs to $\mathrm{Ind}_K^G V$ if and only if $F(x, y; a, b) = F(xa, xb + y; 1, 0)$ and $F(x, y; a, b) = \chi(-a^{-1}b)F(x, y; a, 0)$, for all $(x, y) \in G$, $(a, b) \in K$ and the isomorphism is given by the map $F \mapsto f$ where

$$f(x, y) = F(x, y; 1, 0) \quad \text{and} \quad F(x, y; a, b) = f(xa, xb + y). \tag{6.4}$$

It is easy to see that this isomorphism is also an isometry; in conclusion, we identify $\mathrm{Ind}_K^G V$ with $\mathrm{Ind}_{\mathbb{F}_q}^G\chi$, that is with the space of all $f : G \to \mathbb{C}$ such that

$$f(x, xy_1 + xy_2) = \chi(-y_1)f(x, xy_2) \tag{6.5}$$

for all $x \in \mathbb{F}_{q^m}^*$, $y_1 \in \mathbb{F}_q$ and $y_2 \in \mathbb{F}_q^{m-1}$.

Theorem 6.3 *For every $z \in Z$, define a linear operator $T_z : W \to L(G)$ by setting: $T_z f(x, y) = \frac{1}{\sqrt{|G/K|}} f(xz^{-1}, y)$, for all $f \in W$, $(x, y) \in G$. Then*

$$T_z \in Hom_G(W, Ind_{\mathbb{F}_q}^G \chi); \tag{6.6}$$

$$(L_z^*)^\vee = T_z; \tag{6.7}$$

$$\overset{\diamond}{L}_z f(x, y) = \frac{1}{q\sqrt{|G/K|}}(\chi\mathbf{1})(-yx^{-1}) \sum_{y' \in \mathbb{F}_{q^m}} (\chi\mathbf{1})(y'x^{-1}) f(zx, y') \tag{6.8}$$

for all $f \in Ind_{\mathbb{F}_q}^G \chi$, $(x, y) \in G$.

Proof First of all, we show that $T_z f \in \mathrm{Ind}_{\mathbb{F}_q}^G \chi$ by verifying (6.5):

$$\begin{aligned}(T_z f)(x, xy_1 + xy_2) &= f(xz^{-1}, xy_1 + xy_2)\\ (\text{by (6.36) applied to } \sigma) &= (\chi\mathbf{1})(-zy_1 - zy_2) f(xz^{-1}, 0)\\ &= (\chi\mathbf{1})_z(-y_1) f(xz^{-1}, xy_2)\\ (y_1 \in \mathbb{F}_q) &= \chi(-y_1) f(xz^{-1}, xy_2).\end{aligned}$$

It is immediate to check that T_z intertwines σ and $\mathrm{Ind}_{\mathbb{F}_q}^G \chi$ and this ends the proof of (6.6). By Theorem 3.2 and Corollary 3.3, (6.7) is equivalent to $L_z^* = \hat{T}_z$. In order to give an explicit form of $\hat{T}_z$, it suffices to use the second identity in (6.4) and Definition 3.1.(a): $(\hat{T}_z f_1)(a, b) = f_1(az^{-1}, b)$ for all $f_1 \in W$ and $(a, b) \in K$. If also $f_2 \in V$ then:

$$\langle \hat{T}_z f_1, f_2 \rangle_V = \sum_{a \in \mathbb{F}_q^*} f_1(az^{-1}, 0)\overline{f_2(a, 0)} = \sum_{x \in z^{-1}\mathbb{F}_q^*} f_1(x, 0)\overline{f_2(zx, 0)} = \langle f_1, L_z f_2 \rangle_W$$

and this ends the proof of (6.7). Finally, we can deduce (6.8) from the identity $\overset{\diamond}{L}_z = T_z^*$ and the fact that, for $f_1 \in W$ and $f_2 \in \mathrm{Ind}_{\mathbb{F}_q}^G \chi$, the equality $\langle T_z f_1, f_2 \rangle_{\mathrm{Ind}_{\mathbb{F}_q}^G \chi} = \langle f_1, T_z^* f_2 \rangle_W$ is equivalent to

$$\frac{1}{q} \sum_{x \in \mathbb{F}_{q^m}^*} f_1(x, 0) \sum_{y' \in \mathbb{F}_{q^m}} (\chi\mathbf{1})(-y'x^{-1})\overline{f_2(zx, y')} = \sum_{x \in \mathbb{F}_{q^m}^*} f_1(x, 0)\overline{(T_z^* f_2)(x, 0)}.$$

□

Now we are in position to describe the operators $U_{i,j}^\sigma$ defined in (4.4) in the present setting. For $z_1, z_2 \in Z$ we set $U_{z_1,z_2} = \frac{d_\sigma}{d_\theta} T_{z_1} \overset{\diamond}{L}_{z_2}$. The explicit formula is easy to determine:

$$\left[U_{z_1,z_2} f\right](x, y) = \frac{1}{q^m}(\chi\mathbf{1})(-yx^{-1}z_1) \sum_{y' \in \mathbb{F}_{q^m}} (\chi\mathbf{1})(y'x^{-1}z_1) f(z_2 z_1^{-1}x, y')$$

for all $f \in \mathrm{Ind}_{\mathbb{F}_q}^G \chi$, $(x, y) \in G$. On the other hand, if we choose the vector $v \in V$ of Sect. 5 by setting $v(1, b) = \overline{\chi(b)}$ for all $b \in \mathbb{F}_q$ and $v(a, b) = 0$ for $b \in \mathbb{F}_q$, $a \in \mathbb{F}_q$, $a \neq 1$, then it is easy to check that $T_v \mathrm{Ind}_K^G V$ coincides with $\mathrm{Ind}_{\mathbb{F}_q}^G \chi$, that is $(T_v F)(x, y) = \frac{1}{\sqrt{q}} F(x, y; 1, 0)$, for all $F \in \mathrm{Ind}_K^G V$ [(in the notation of (6.4), $T_v F = \frac{1}{\sqrt{q}} f$]. Then, according to (5.2), we set $w_z = L_z v$ for all $z \in Z$. Elementary calculations show that

$$w_z(x, y) = \begin{cases} (\chi\mathbf{1})(-yz) & \text{if } x = z^{-1} \\ 0 & \text{if } x \neq z^{-1}. \end{cases}$$

Moreover, the spherical functions (5.6) have the expression:

$$\phi_{z_1,z_2}(x, y) = \begin{cases} (\chi\mathbf{1})(-yz_1) & \text{if } x = z_2 z_1^{-1} \\ 0 & \text{if } x \neq z_2 z_1^{-1}. \end{cases}$$

It is also easy to verify that $U_{z_1,z_2} f = \frac{d_\sigma}{|G|} f * \phi_{z_2,z_1}$, according to Theorem 5.5 (in view of the identification of $\mathrm{Ind}_K^G V$ with $\mathrm{Ind}_K^G \chi \equiv T_v \mathrm{Ind}_K^G V$, now the operators $\widetilde{U}$'s coincide with the U's).

7 Gelfand–Tsetlin bases

We now extend to our setting the classical theory of Gelfand–Tsetlin bases (cf. [7, 18, 22]), that yields a natural choice for the orthonormal basis in Corollary 3.4. We continue to use the notation of the previous sections (in particular Sects. 3 and 4). First we prove a preliminary result that examines the correspondence $L \mapsto (L^*)^\vee$ in relation to the induction in stages. Let H be a subgroup of G containing K (i.e. $K \leq H \leq G$) and denote by (ρ, U) an irreducible H-representation. If $L_1 \in \mathrm{Hom}_K(V, \mathrm{Res}_K^H U)$ and $L_2 \in \mathrm{Hom}_H(U, \mathrm{Res}_H^G W)$ then $L_2 L_1 \in \mathrm{Hom}_K(V, \mathrm{Res}_K^G W)$. Since $(L_1^*)^\vee \in \mathrm{Hom}_H(U, \mathrm{Ind}_K^H V)$, we can consider $(L_1^*)^\vee U$ as a subspace of $\mathrm{Ind}_K^H V$. Therefore, $\mathrm{Ind}_H^G[(L_1^*)^\vee U]$ is a subspace of $\mathrm{Ind}_H^G[\mathrm{Ind}_K^H V]$ which can be identified with $\mathrm{Ind}_K^G V$ by the isomorphism (2.11).

Theorem 7.1 *Under the isomorphism (2.11), we have*

$$\left[(L_2 L_1)^*\right]^\vee W \leq \mathit{Ind}_H^G \left[(L_1^*)^\vee U\right].$$

Proof The space $\mathrm{Ind}_H^G[(L_1^*)^\vee U]$ is made up of all functions $F \in \mathrm{Ind}_H^G[\mathrm{Ind}_K^H V]$ such that, for every fixed $g \in G$, the function $h \mapsto F(g, h)$ belongs to $(L_1^*)^\vee U$, i.e. there exists $u_g \in U$ such that

$$F(g, h) = \left[(L_1^*)^\vee u_g\right](h). \tag{7.1}$$

For $w \in W$, $g \in G$ we have:

$$\left\{\left[(L_2L_1)^*\right]^\vee w\right\}(g) = \frac{1}{\sqrt{|G/K|}} L_1^* L_2^* \sigma(g^{-1})w$$

and therefore

$$\begin{aligned}
\left\{\left[(L_2L_1)^*\right]^\vee w\right\}(gh) &= \frac{1}{\sqrt{|G/K|}} L_1^* L_2^* \sigma(h^{-1})\sigma(g^{-1})w \\
(L_2 \in \mathrm{Hom}_H(U, \mathrm{Res}^G_H W)) &= \frac{1}{\sqrt{|G/K|}} L_1^* \rho(h^{-1})[L_2^*\sigma(g^{-1})w] \\
&= \frac{\sqrt{|H/K|}}{\sqrt{|G/K|}} \left\{(L_1^*)^\vee [L_2^*\sigma(g^{-1})w]\right\}(h) \\
&= \left\{(L_1^*)^\vee [(L_2^*)^\vee w](g)\right\}(h).
\end{aligned}$$

This means that, with respect to (2.11), the function $f = (L_2L_1)^{*\vee} w \in \mathrm{Ind}^G_K V$ corresponds to an $F(g,h)$ of the form (7.1), with $u_g = [(L_2^*)^\vee w](g)$. Therefore, $f \in \mathrm{Ind}^G_H[(L_1^*)^\vee U]$. $\square$

Suppose now that there exists a chain of subgroups of G of the form

$$K = H_1 \leq H_2 \leq \cdots \leq H_{m-1} \leq H_m = G. \tag{7.2}$$

Define recursively $J_\ell \subseteq \widehat{H_\ell}$, $1 \leq \ell \leq m$, by setting $J_1 = \{\theta\}$ and $J_{\ell+1}$ equal to the set of all $\eta \in \widehat{H_{\ell+1}}$ such that η is contained in $\mathrm{Ind}^{H_{\ell+1}}_{H_\ell}\rho$, for some $\rho \in J_\ell$, $\ell = 1, 2, \ldots, m-1$. We say that the chain (7.2) satisfies the *Gelfand–Tsetlin* condition if for all $1 \leq \ell \leq m-1$, $\rho \in J_\ell$ and $\eta \in J_{\ell+1}$ the multiplicity of $\eta \in \mathrm{Ind}^{H_{\ell+1}}_{H_\ell}\rho$ (equivalently, the multiplicity of ρ in $\mathrm{Res}^{H_{\ell+1}}_{H_\ell}\eta$) is at most 1; we write $\eta \to \rho$ when the multiplicity is equal to 1. If the Gelfand–Tsetlin condition is satisfied, the associated *Bratteli diagram* is the finite oriented graph whose vertex set is $\coprod_{\ell=1}^m J_\ell$ and the edge set is formed by the pairs (η, ρ) such that $\eta \to \rho$. A *path* in the Bratteli diagram is a sequence $C: \rho_m \to \rho_{m-1} \to \cdots \to \rho_2 \to \rho_1$, where $\rho_1 = \theta$ and $\rho_m \in J$. For every $\sigma \in J$, we denote by $\mathcal{P}(\sigma)$ the set of all paths $C: \rho_m \to \rho_{m-1} \to \cdots \to \rho_2 \to \rho_1$ such that $\rho_m = \sigma$. Fix now $\sigma \in J$ and denote by W its representing space. We define recursively a chain of subspaces

$$W_m \geq W_{m-1} \geq \cdots \geq W_2 \geq W_1$$

as follows. We set $W_m = W$ and for $\ell = m-1, m-2, \ldots, 1$, we denote by W_ℓ the *unique* subspace of $\mathrm{Res}^{H_{\ell+1}}_{H_\ell} W_{\ell+1}$ isomorphic to the representation space of ρ_ℓ. This way, $W_1 \sim V$ as a K-representation; we set $V_C = W_1$. If $\widetilde{C}: \widetilde{\rho}_m \to \widetilde{\rho}_{m-1} \to \cdots \to \widetilde{\rho}_2 \to \widetilde{\rho}_1$ is a different path in $\mathcal{P}(\sigma)$, then there exists $2 \leq \ell \leq m$ such that $\rho_i \sim \widetilde{\rho}_i$ for $i = m, m-1, \ldots, \ell$ and $\rho_{\ell-1} \nsim \widetilde{\rho}_{\ell-1}$. Therefore, if $\widetilde{W}_m \geq \widetilde{W}_{m-1} \geq \cdots \geq \widetilde{W}_2 \geq \widetilde{W}_1$ is the chain of subspaces associated with $\widetilde{C}$ then $W_i = \widetilde{W}_i$, $i = m, m-1, \ldots, \ell$, but $W_{\ell-1}$ and $\widetilde{W}_{\ell-1}$ are orthogonal, because they afford inequivalent representations. This implies that also V_C and $V_{\widetilde{C}}$ are orthogonal. Finally, by induction on m, it is easy to prove that

$$\bigoplus_{C\in\mathcal{P}(\sigma)} V_C \tag{7.3}$$

is an orthogonal decomposition of the θ-isotypic component of $\mathrm{Res}^G_K W$. Let $L_{\sigma,C} \in \mathrm{Hom}_K(V, \mathrm{Res}^G_K W)$ be an isometry with $L_{\sigma,C}V = V_C$. The operator $L_{\sigma,C} : V \to W$ is defined up to a complex constant of modulus 1 (the *phase factor*) and, by Lemma 2.1 the set

$$\{L_{\sigma,C} : C \in \mathcal{P}(\sigma)\} \tag{7.4}$$

is an orthonormal basis for $\mathrm{Hom}_K(V, \mathrm{Res}^G_K W)$.

Similarly, with each $C \in \mathcal{P}(\sigma)$, $C : \rho_m \to \rho_{m-1} \to \cdots, \rho_2 \to \rho_1$, we can associate the following sequence of spaces: $Z_1 = V$, and recursively, $Z_{\ell+1}$ is the unique subspace of $\mathrm{Ind}^{H_{\ell+1}}_{H_\ell} Z_\ell$, that affords $\rho_{\ell+1}$; finally, we set $W_C = Z_m$. Clearly, W_C is a subspace of $\mathrm{Ind}^G_K V$ and

$$\bigoplus_{C\in\mathcal{P}(\sigma)} W_C \tag{7.5}$$

is an orthogonal decomposition of the σ-isotypic component of $\mathrm{Ind}^G_K V$. Indeed, we have

$$\mathrm{Ind}^G_K V = \mathrm{Ind}^{H_m}_{H_{m-1}} \mathrm{Ind}^{H_{m-1}}_{H_{m-2}} \cdots \mathrm{Ind}^{H_2}_{H_1} V$$

and at each stage the induction is *multiplicity free*.

We now show that the decomposition (7.3) and (7.5) are closely related as in Corollary 3.5.

Theorem 7.2 *The orthonormal basis*

$$\left\{\sqrt{\frac{d_\sigma}{d_\theta}}(L^*_{\sigma,C})^\vee : C \in \mathcal{P}(\sigma)\right\}$$

of $\mathrm{Hom}_G(W, \mathrm{Ind}^G_K V)$ *gives rise precisely to the isometric orthogonal decomposition* (7.5)*, that is*

$$W_C = \sqrt{\frac{d_\sigma}{d_\theta}}(L^*_{\sigma,C})^\vee W$$

for every $C \in \mathcal{P}(\sigma)$.

Proof It follows from Corollary 3.5 and a repeated application of Theorem 7.1, by induction on m. □

7.1 A worked example: part II

Now we analyze the results in Sect. 6 from another point of view.

Example 7.3 The chain $\mathbb{F}_q \leq \mathbb{F}_{q^m} \leq \mathbb{F}^*_{q^m} \ltimes \mathbb{F}_{q^m}$ satisfies the Gelfand–Tsetlin condition with respect to the nontrivial additive character χ of $\mathbb{F}_q$. Indeed, in the notation of Sect. 6,

$$\mathrm{Ind}_{\mathbb{F}_q}^G \chi = \mathrm{Ind}_{\mathbb{F}_{q^m}}^G \mathrm{Ind}_{\mathbb{F}_q}^{\mathbb{F}_{q^m}} \chi = \bigoplus_{z \in Z} \mathrm{Ind}_{\mathbb{F}_{q^m}}^G (\chi \mathbf{1})_z.$$

It is easy to see that $T_z W$ coincides with $\mathrm{Ind}_{\mathbb{F}_{q^m}}^G (\chi \mathbf{1})_z$ as a subspace of $\mathrm{Ind}_{\mathbb{F}_q}^G \chi$ and therefore of $L(G)$. In other words, the decomposition of $\mathrm{Ind}_K^G V$ given by the basis in Theorem 6.3 coincides with the decomposition given by the above Gelfand–Tsetlin construction. The diagram

$$\begin{array}{ccc} & G & \\ \nearrow & & \nwarrow \\ \mathbb{F}_{q^m} & & K \\ \nwarrow & & \nearrow \\ & \mathbb{F}_q & \end{array} \tag{7.6}$$

summarizes the two ways in which we can induce χ from $\mathbb{F}_q$ to G: the right path has been described in Sect. 6; the left path yields immediately the same decomposition of $\mathrm{Ind}_{\mathbb{F}_q}^G \chi \cong \mathrm{Ind}_K^G V$. Actually, the Gelfand–Tsetlin decomposition leads to a slightly different approach to calculations in Sect. 6: it yields immediately the expression of T_z and then one can obtain the expression of L_z because it is equal to $\frac{1}{\sqrt{|G/K|}} \left(\hat{T}_z \right)^*$.

Open Problem 7.4 *Set $K = GL(2, \mathbb{F}_q)$, $G = GL(2, \mathbb{F}_{q^m})$ and let χ be again a nontrivial additive character of $\mathbb{F}_q$. Then $\mathbb{F}_q \leq \mathbb{F}_{q^m} \leq G$ is a Gelfand–Tsetlin chain for χ. This depends on the fact that $\mathrm{Ind}_{\mathbb{F}_{q^m}}^G (\chi\xi)$ (which is called the Gelfand-Graev representation) decomposes without multiplicity, for every $\xi \in \widehat{\mathbb{F}}_q^{m-1}$* [1,19]. *Therefore, also in this situation, the left path of the diagram* (7.6) *yields a decomposition of $\mathrm{Ind}_{\mathbb{F}_q}^G \chi$. Now the situation in the right path is more difficult than in Sect. 6, and for every $\theta \in \widehat{K}$ contained in $\mathrm{Ind}_{\mathbb{F}_q}^K \chi$ we have the problem to decompose $\mathrm{Ind}_K^G \theta$ and develop the relative machinery of intertwining operators and spherical functions.*

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