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PhD Thesis

Some analytical results for hyperbolic chemotaxis model on networks

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To my special Family.

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Contents

Introduction

The aim of this thesis is to investigate from the analytical point of view some hyperbolic-parabolic models arising in biology. Nowadays, mathematical analysis of biological phenomena has become an important tool to explore complex processes and to detect mechanisms that might be not evident to the experimenters.

In the present work, we deal with description of the movement of some cell populations on a biological fibres of extracellular matrix (namely a scaffold) under the influences of external stimulus, generally called taxis. This problem has been studied by a biological point of view in [6]. From a mathematical point of view, a scaffold is described by an oriented network $\mathcal{N}$ composed by a finite number $M > 0$ of oriented compact intervals I_i, $i = 1, \dots, M$, which are called the arcs of the network, and which intersect themselves in some vertices called nodes of the network. Thus we will consider a set of systems of hyperbolic-parabolic equations, each of them defined on an arc of the network. The first difficulty of the model we consider here is the fact that some compatibility boundary conditions must be given on the vertices and the nodes of the network. Another feature of this model is also the fact that cells can move in two different oriented directions on the same arc, with corresponding velocities $\pm\lambda_i$ on each arc I_i. This implies that on each arc we have to consider equations for both verses of movement. Our aim is to describe the evolution in time of density of cells along arcs of the network under presence of an external signal changing in time and space.

We denote by the functions $u_i^{\pm}(x, t)$ respectively the density of cells which moves on arc I_i in the oriented direction corresponding to velocity $\pm\lambda_i$. Moreover, on each arc of the network we consider the presence of an external chemoattractant $\phi^i(x, t)$ that influences the directions of cells movement.

The main novelty of this work is to consider a $1D$ model on a network. More specifically, we consider the following system of hyperbolic-parabolic equations, defined on each arc of the network

$$\begin{cases} (u_i^+)_t + \lambda(u_i^+)_x = g^+(u_i^+, u_i^-), \\ (u_i^-)_t - \lambda(u_i^-)_x = g^-(u_i^+, u_i^-), \\ \phi_t^i = D_i\phi_{xx}^i + au^i - b\phi^i, \end{cases}$$

where functions $g^{\pm}(u_i^+, u_i^-)$ are smooth functions. We deal often with the equivalent system

$$
\begin{cases} u_t^i + v_x^i = 0, \\ v_t^i + \lambda_i^2 u_x^i = G(\phi^i, \phi_x^i, u^i) - v^i, \\ \phi_t^i = D_i \phi_{xx}^i + a u^i - b \phi^i, \end{cases} \tag{0.0.1}
$$

defined on $I_i \times [0,T]$, $T > 0$, where $i = 1, \ldots, M$. Here λ_i, a, b and D_i are positive constants. In particular the coefficients λ_i represent the speed of cells, D_i the diffusion coefficients of the chemoattractants ϕ^i, a the rate of release of chemoattractant, while b the characteristic degradation in time of chemoattractant. The function G is a smooth function satisfying some suitable assumptions. Here the functions $u^i, v^i : I_i \times [0,T] \to \mathbb{R}$ are defined as

$$u^i(x,t) = u_i^+(x,t) + u_i^-(x,t)$$

and

$$v^i(x,t) = \lambda_i(u_i^+(x,t) + u_i^-(x,t)),$$

thus u^i is the total density of cells on arc I_i and v^i is the flux on arc I_i. Our aim is to study the existence and uniqueness of the solutions to the previous model, after we have coupled it with initial data and suitable boundary conditions for u^i, v^i and ϕ^i. In our studies, we always suppose that, given an initial total mass of cells, it is preserved along time. Moreover, we have to pay attemption especially to conditions we have to give on the node.

We have to specify something more about the transmission conditions on the vertices which are common to two or more intervals, i.e., on each node of the network. This is the most important point in order to study the solutions to our model, since the behaviour of the solution will be very different according to the conditions we choose. Moreover, let us recall that the coupling between the densities on the arcs are obtained through these conditions. We consider a types of transmission conditions which impose the continuity of the total fluxes of cells and chemoattractant on nodes of network rather than the continuity of the densities. Let us observe that in the study of this problem, we suppose no birth or death of cells which move on the network, thus some conditions will be given on the transmission coefficients, in order to ensure the conservation of the total mass of the system. The definition of the problem is complete after we have given also initial data and boundary conditions on external vertices of the network (the boundary points of the intervals which are common only to a single arc). More specifically we impose the following transmission condition on a node N of the network. Let $\mathcal{N}$ be a network composed of M arcs and ν

nodes. Denoting by E the set of oriented arc I_i entering in a node N, and by U the set of oriented arc I_i outgoing from the same node, we set

$$u_i^-(N,t) = \sum_{i \in E} \xi_{i,j} u_j^+(N,t) + \sum_{j \in U} \xi_{i,j} u_j^-(N,t), \tag{0.0.2}$$

if $i \in E$ for some transmission coefficients $\xi_{i,j} \in [0,1]$, $j = 1, \dots, M$, while

$$u_i^+(N,t) = \sum_{i \in E} \xi_{i,j} u_j^+(N,t) + \sum_{j \in U} \xi_{i,j} u_j^-(N,t), \tag{0.0.3}$$

if $i \in U$ for other coefficients $\xi_{i,j} \in [0,1]$, $j = 1, \dots, M$. Moreover, as we previously said, we impose the continuity of the total flux $v^i(x,t)$, $i = 1, \dots, M$ on each node N, and so we ask that

$$\sum_{i \in E} v^i(N,t) = \sum_{i \in U} v^i(N,t).$$

The transmission coefficients $\xi_{i,j}$ represent the probability to choose the j-th arc coming from the i-th. Thus we have that $0 \leq \xi_{i,j} \leq 1$, for each $i, j = 1, \dots, M$. Moreover, the condition of the continuity of the total flux implies that the transmission coefficients have to verify on each node that

$$\sum_{i \in E \cup U} \lambda_i \xi_{i,j} = \lambda_j,$$

for each j.

For the chemoattractants ϕ^i we choose the Kedem-Katchalsky transmission conditions

$$\phi_x^i(N,t) = \alpha \sum_{i \neq j} \phi^j(N,t) - \phi^i(N,t) \tag{0.0.4}$$

if $i \in E$, while

$$\phi_x^i(N,t) = \alpha \sum_{i \neq j} \phi^i(N,t) - \phi^j(N,t) \tag{0.0.5}$$

if $i \in U$. These conditions give the continuity of the spatial derivatives of the total density of the chemical signal on each node, i.e.,

$$\sum_{i \in E \cup U} \phi_x^i(N,t) = 0.$$

Analysis and main difficulties

In this thesis we investigate the existence and uniqueness of local and global solutions of (0.0.1) defined on a network in some various cases.

We begin assuming the gradient of the chemoattractants $\phi^i(x,t)$ to be constant on each arc and equal on all the arcs to a positive constant $\alpha \in \mathbb{R}^+$. Thus the last equation of (0.0.1) disappears and we consider the linear hyperbolic problem

$$
\begin{cases} (u_i^+)_t + \lambda(u_i^+)_x = \frac{1}{2\lambda_i}((\lambda_i+\alpha)u_i^- - (\lambda_i-\alpha)u_i^+), \\ (u_i^-)_t - \lambda(u_i^-)_x = -\frac{1}{2\lambda_i}((\lambda_i+\alpha)u_i^- - (\lambda_i-\alpha)u_i^+), \end{cases} \tag{0.0.6}
$$

$i = 1, \ldots, M$, coupled with initial data and boundary and transmission conditions. The unknowns of our problem are the densities of fibroblasts which migrate on the scaffold; they can move on i-th arc of network in two verses, left and right, with constant velocities respectively $\mp\lambda$.

We assume that the subcharacteristic condition $\alpha < |\lambda_i|$, on each arc I_i holds. The subcharacteristic condition is important in some problems because often guarantees that the operator associated to problem generates a semigroup of contraction in some spaces. Thanks to the transmission conditions (0.0.2), (0.0.3), (0.0.4), and (0.0.5) we have imposed on transmission coefficients on each node, we obtain that, also in this complex situation, the operator associated to this problem generates a semigroup of contraction in L^1. This problem has been studied from a numerical point of view by Natalini, Bretti and Ribot ([**?**]).

We show the global existence and uniqueness of solution (u_i^+, u_i^-), $i = 1, \ldots, M$, of the linear problem (0.0.6) belonging to $(C([0,T]; BV(I_i)))^2$ for each i. In order to do this we find a priori L^1-estimates for the solutions. Of course, the main difficulty in finding estimates is the presence of nodes. Transmission boundary conditions on each node link functions on each arc to the others, thus the only way in searching estimates is not to work separately on each arc, but with all the functions of the network together. In this way of proceeding, the conditions on transmission coefficients and in particular the assumption of continuity of total flux on each node, play a fundamental role. More precisely, we show L^1-estimate for all functions of the network, together with their spatial and temporal derivatives. In these two last estimates we must be pay attemption to consider the right transmission conditions for the derivatives of functions. By these estimates we are able to show the uniqueness of solution and a comparison result for sub/sopra-solutions. In order to prove the global existence of solutions, we build a sequence of functions, each of these solution of problem (0.0.6) with constant boundary conditions on each node for a time interval of size Δt. We show that this sequence converge in $C([0,T];L^1)$ to a vector function (u_i^+, u_i^-), $i = 1, \dots, M$ belonging to $(C([0,T];BV(I_i)))^2$. Finally we show the consistency of our approximating sequence with the (unique) solution of the problem.

To complete the analysis of the linear problem (0.0.6) we study the stationary solution of the problem and the asymptotic behaviour of solution characterized in having as initial data a small perturbation of a stationary solution.

First we show that imposing only the transmission conditions on the nodes, we may have a one-parameter family of stationary solutions, while imposing also the total mass conservation, we have a unique positive stationary solution with the fixed total mass. We recall that in our problem we have the total mass of density u^i, $i = 1, \dots, M$, preserved for all time. Given an initial total mass μ, and considered the corresponding solution $u_i^\pm(x,t)$, $i = 1, \dots, M$, we show that there always exists a stationary solution $\widetilde{u}_i^\pm(x)$, $i = 1, \dots, M$, such that we have $\sum_{i=1}^M u_i^\pm(x,t) \leq \sum_{i=1}^M \widetilde{u}_i^\pm(x)$, for each time $t \geq 0$.

Then we consider problem (0.0.6) coupled with initial data which are a small perturbation of a stationary solution, with fixed total mass μ, and we show that the solution to this problem tends asymptotically to the stationary solution with total mass μ, i.e., the stationary solution is asymptotically stable under small perturbations. To prove this fact we follow we have to pay attemption first to the fact that we work on compact closed intervals, and also to boundary conditions on node: in fact, the asymptotic behaviour of solution strictly depend on conditions on transmission coefficients.

Next we consider the complete hyperbolic-parabolic system

$$\begin{cases} u_t^i + v_x^i = 0, \\ v_t^i + \lambda_i^2 u_x^i = G(\phi^i, \phi_x^i, u^i) - v^i, \\ \phi_t^i = D_i \phi_{xx}^i + a u^i - b\phi^i, \end{cases}$$

defined on a network. In this case we consider a network composed of a single node for simplicity of calculations, but results can be extended to a general network. Of course, in this case we have to couple this system with suitable transmission conditions also for the chemoattractants ϕ^i, $i = 1, \dots, M$. As previously, we do not impose the continuity of the density of chemoattractants, but only the continuity of the flux at node. Therefore, we use the Kedem-Katchalsky permeability conditions [21], which yields the conservation of the fluxes at node.

We show the local existence and uniqueness of solution (u^i, v^i, ϕ^i), $i = 1, \dots, M$ to this problem in the functional space $(C([0,T];H^1(I_i)))^3$ under the assumption of that the source term G is locally Lipschitz continuous. In order to do this, we use some results of the semigroup theory. The most important fact is to prove that the operator A associated to this problem generates a semigroup of contraction in H^1, namely the operator has to be monotone and maximal. The main difficulty consists in proving the monotonicity of the operator because of the presence of the node. In doing this, we found that some additional conditions on transmission coefficients on the node must be required. In particular, in the case of a network composed of two arcs we find necessary and sufficient conditions for monotonicity, while in case of a general network we only found sufficient conditions. Therefore we show the maximality of the operator in H^1. At this point, the local existence of solution follows by showing first the local existence of solution to the homogeneous problem, and then by using a point fixed method to give a local solution to the non-homogeneous problem.

The last part of the Thesis is devoted to prove the global existence of solution to problem (0.0.1) choosing a particular locally Lipschitz continuous source term $G(u^i, \phi^i, \phi^i_x)$, $i = 1, \dots, M$. More specifically we choose a special form of the function G arising in chemotaxis, i.e.,

$$G(u^i, \phi^i, \phi^i_x) = u^i \chi(\phi^i)\phi^i_x,$$

where the function $\chi(\phi^i)$ belongs to $W^{1,\infty}$. To prove the global existence Theorem, we have to look for a solution (u^i, v^i, ϕ^i), $i = 1, \dots, M$ belonging to the functional space $(C([0,T]; H^2(I_i)))^3$. Thus we are forced to impose additional transmission conditions on the node for the spatial derivatives of the functions on all arcs. Also in this case, we use some results of the semigroup theory to prove the local existence of the solution in H^2. We need the monotonicity and the maximality of the operator associated to this problem in H^2, thus we have to couple the transmission coefficients with new conditions in order of having not only the monotonicity of the operator in H^1, but also the monotonicity in H^2. Then we show the maximality of the operator in this case, and the local existence and uniqueness of the solution by a point fixed method.

Then we find suitable energy estimates for the local solution. As usual,the main difficult in finding them is the presence of the node.

These estimates allow us to use the Continuation Principle estimating a particular functional $\sum_{i=1}^{M} F(u^i, v^i, \phi^i)$ in order to extend the local solution to a global ones, with small initial data. The presence of the node implies that we can have a global solution only under some relations of direct proportionality between the length of the arcs I_i and the velocities of cells $\pm\lambda_i$.

Plan of Thesis

Chapter 1 and 2 are devoted to some analytical backgrounds that we will use in the follow and to the analysis of some important models of chemotaxis. We also gives the definition of a network and we focus on the importance of the compatibility conditions that we must give on the nodes of the network in order to have a well defined model.

In Chapter 3 we consider model (0.0.1) assuming the gradients of chemoattractants ϕ_x^i to be a positive constant α. So we deal with the linear system of hyperbolic equations

$$\begin{cases} u_t^i + v_x^i = 0, \\ v_t^i + \lambda_i^2 u_x^i = \alpha u^i - v^i, \end{cases}$$

defined on the arcs I_i of a network. We complete this system with initial and boundary conditions, focusing on the reasons for the choice of transmission conditions on the nodes of the network. We work with the equivalent system of equations

$$\begin{cases} (u_i^+)_t + \lambda (u_i^+)_x = g^+(u_i^+, u_i^-), \\ (u_i^-)_t - \lambda (u_i^-)_x = g^-(u_i^+, u_i^-), \end{cases}$$

where the functions $u_i^{\pm}$ represent respectively the densities of fibroblasts which moves on the right verse or on the left ones, on the i-th arc.

First we found some a priori estimates for the solutions of this problem, and then we show a global existence and uniqueness Theorem for the solutions. Then we studied the asymptotic behaviour of small perturbations of stationary solutions: we begin giving the explicit expression of a stationary solution of (0.0.6) considering a network composed of a single node, and then we show the asymptotic stability of a small perturbation of it.

In Chapter 4 we consider the complete model (0.0.1) defined on a network composed of a single node. We formulate appropriate transmission conditions for the chemoattractants ϕ^i on the node and on external boundary points, and we complete the problem with transmission conditions introduced in Chapter 3 for u^i and v^i. Using the semigroup theory, we show that the operator associated to our problem generates a semigroup of contraction on a Hilbert space under some assumptions on transmission conditions. Then we prove the local existence of solution of (0.0.1) under the hypothesis of local lipschitzianity of the source term G.

In Chapter 5 we consider problem (0.0.1) defined on a network composed of a single node with a particular choice for the source term G. Under this choice, we extend the local solution to a global ones. To do this, we found some a priori energy estimates for the local solution, and we were be able to prove global existence of solution for small initial data.

Chapter 1

Notations and preliminaries

In this chapter we will present some notations and analytical backgrounds which will be of use in the following (for more details, see [3], [41], [13], [15]).

1.1 Notations

Let us introduce some notations.

We denote by

- $\mathbb{R}^n$, $n \geq 1$, the n-dimensional vectorial space;

- $C^k(A)$ the space of functions with k derivatives;

- $C_c^\infty(A)$ the space of infinitely differential functions with compact support on a set $A \subseteq R^n$;

- $|A|$ the cardinality of a set A;

1.2 Functional spaces

1.2.1 Some properties of L^p spaces

Let $A \subseteq \mathbb{R}^n$, $1 \leq p \leq +\infty$, and let $L^p(A)$ be the Lebesgue spaces of functions with the usual norm

$$||u||_{L^p(A)} = (\int_A |u|^p)^{\frac{1}{p}}.$$

We recall the following property for L^p spaces.

Proposition 1.2.1. *Let $A \subset \mathbb{R}^n$ a bounded set. Then*

$$L^p(A) \subset L^q(A)$$

if $p > q$, $1 \leq p,q \leq +\infty$.

In the following, we will have to work with sequences of L^p functions and we will have to establish if they are strongly compact in L^p. So we recall the following important Theorems.

Theorem 1.2.1. *Let X a Banach space with norm $||.||_X$, and let $H \subset C([0,T];X)$. If there exists a compact subset $Y \subseteq X$ such that $\forall f \in H$, $\forall t \in [0,T]$, then $f(t) \in Y$ and if H is equicontinuous, $\forall f \in H$, then H has a compact closure in $C([0,T];X)$.*

Theorem 1.2.2. *(Riesz-Fréchet-Kolmogorov) Let F be a bounded set in $L^p(\mathbb{R}^n)$, $1 \leq p \leq +\infty$. Let us suppose that $\forall f \in F$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $||f(x+h) - f(x)||_{L^p} < \varepsilon$, $\forall h \in \mathbb{R}^n$, with $|h| < \delta$. Then for each bounded and measurable set $\Omega \subset \mathbb{R}^n$, the restriction to Ω of the functions of F has compact closure in the space $L^p(\Omega)$.*

Theorem 1.2.3. *(Lebesgue average) Let us consider $B(x,r)$ the ball of radius r and centered in the point $x \in \mathbb{R}^n$. Let $U \subset \mathbb{R}^n$, and a function $u \in L^1_{loc}(U)$.*

Then

$$\lim_{r \longrightarrow 0} \frac{1}{|B|} \int_B |u(y) - u(x)|^p dy = 0, \tag{1.2.1}$$

a.e. $x \in U$.

From the previous Theorem it follows this

Corollary 1.2.1. *Let $I = (a,b)$. For each $N \in N$, $N > 0$, let $\Delta t = \frac{b-a}{N}$, $t_n = n\Delta t$, $n = 0, \ldots, N$. Let $u \in L^1(I)$, and let us define the sequence of functions*

$$u_{\Delta t}(t) = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} u(s) ds,$$

for each $t \in [t_n, t_{n+1})$. Then

$$\lim_{\Delta t \longrightarrow 0} | \int_I |u_{\Delta t}(t) - u(t)| dt = 0. \tag{1.2.2}$$

Proof. Let us firstly observe that $||u_{\Delta t}||_{L^1} \leq ||u||_{L^1}$, for each Δt.

Now, let $t \in I$. For each fixed Δt, then there exists $m_{\Delta t}$ such that $t \in [t_{m_{\Delta t}}, t_{m_{\Delta t}} + 1)$. In fact we can consider $t_{m_{\Delta t}} = m\Delta t$, $m \leq \frac{t}{\Delta t} \leq m+1$. By Lebesgue average Theorem we have that

$$\lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \int_{t_{m_{\Delta t}}}^{t_{m_{\Delta t}+1}} |u(s) - u(t)| ds \longrightarrow 0,$$

thus

$$\lim_{\Delta t \longrightarrow 0} u_{\Delta t}(t) = u(t), \tag{1.2.3}$$

a.e. $t \in I$, i.e., $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|u_{\Delta t}(t) - u(t)| < \varepsilon$. a.e. $t \in I$. Thus by Lebesgue dominated convergence Theorem we get the proof. □

1.2.2 Sobolev spaces

Let $I \subseteq \mathbb{R}$ and $u : I \to \mathbb{R}$ be a function. We will denote by u' the first derivative of u, and in general by u^h the h-derivative of u.

We will consider the following functional space (see [3]).

Definition 1.2.1. *Let $I \subseteq \mathbb{R}$ an open interval of $\mathbb{R}$, and let $p \in \mathbb{R}$, with $1 \leq p \leq +\infty$. For every integer k, the Sobolev space $W^{k,p}(I)$ is defined by*

$$W^{k,p}(I) = \{u \in L^p(I) : \exists g_1 \in L^p(I), \dots, g_k \in L^p(I) : \int_I u\phi^h = \int_I g_h \phi,$$

$$\forall \phi \in C_c^\infty(I), \textit{for each } h = 1, \dots, k\}.$$

We set

$$H^k := W^{k,2}(I),$$

and we say that $g_h = u^h$ is the h-th weak derivative of u.

In the Sobolev space $W^{k,p}(I)$ we consider the following norm:

$$||u||_{W^{k,p}(I)} = ||u||_{L^p(I)} + \sum_{h=1}^{k} ||u^h||_{L^p(I)},$$

while in the space $H^k(I)$ the scalar product

$$(u, v)_{H^k(I)} = (u, v)_{L^2(I)} + \sum_{h=1}^{k} (u^h, v^h)_{L^2(I)}$$

is well defined, and the associated norm

$$||u||_{H^k(I)} = (||u||^2_{L^2(I)} + \sum_{h=1}^{k} ||u^h||^2_{L^2(I)})^{\frac{1}{2}}$$

is equivalent to the norm of $W^{k,2}(I)$.

In the following, we will write $W^{k,p}$, H^k and L^p instead of $W^{k,p}(I)$, $H^k(I)$ and $L^p(I)$. We recall that the following Theorem holds.

Theorem 1.2.4. *(Sobolev embeddings) There exists a positive constant C (depending only on $|I| \leq +\infty$) such that*

$$||u||_{L^\infty} \leq C||u||_{W^{1,p}},$$

$\forall u \in W^{1,p}$ and $\forall 1 \leq p \leq +\infty$.

In other words, we have that $W^{1,p} \subset L^\infty$ and the injection is continue.

Moreover, if $I \subset \mathbb{R}$ is a bounded set, and denoting with $\overline{I}$ the closure of I in $\mathbb{R}$, we have

- *the injection $W^{1,p} \subset C(\overline{I})$ is compact if $1 < p \leq +\infty$*

- *the injection $W^{1,1} \subset L^q(I)$ is compact if $1 \leq p < +\infty$.*

Lax Milgram Theorem

Definition 1.2.2. *Let H be a Hilbert space, and let $a : H \times H \to H$ be a bilinear form. We say that*

- *a is continues if there exists a positive constant C such that $|a(u,v)| \leq C|u||v|$, for each $u, v \in H$;*

- *a is coercitive if there exists a positive constant D such that $|a(u,u)| \geq D|u|^2$, for each $u \in H$.*

Let us denote by H' the dual space of H, and let us denote by $< u, v >$ the scalar product on H.

Theorem 1.2.5. *(Lax Milgram) Let $a : H \times H \to H$ be a continuous and coercitive bilinear form. Then for each $w \in H'$ there exists a unique $u \in H$ such that*

$$a(u,v) =< w, v >, \text{for each } v \in H. \tag{1.2.4}$$

1.2.3 Functions of bounded variation

Let I be a bounded set of $\mathbb{R}^n$.

Definition 1.2.3. *A function $u \in L^1(I)$ has bounded variation if there exists a constant C such that*

$$|\int_I u\phi'| \leq ||\phi||_{L^\infty(I)},$$

$\forall \phi \in C_c^\infty(I)$.

Moreover, the following Proposition holds ([3]).

Proposition 1.2.2. *A function $u \in L^1(I)$, has bounded variation if and only if there exists a constant C such that for each open set $\omega \subset\subset I$ and for each $h \in \mathbb{R}$ with $|h|$ less than the distance between ω and the complementary of I,*

$$||u(x+h) - u(x)||_{L^1(\omega)} \leq C|h|.$$

These functions can be characterized in different ways. In fact they are difference of two increasing and bounded functions (eventually discontinues) on I; or analogously, they are the functions u such that there exists a constant C such that

$$\sum_{i=0}^{k-1} |u(x_{i+1}) - u(x_i)| \leq C,$$

for each sequence $x_0 < x_1 < \ldots < x_k$ of I.

Otherwise, they can be represented as the functions $u \in L^1(I)$ which have the distributional derivative a bounded measure.

We will denote the Banach space of bounded variation functions on a set $\Omega \subset \mathbb{R}^n$ as $BV(\Omega)$, $n \geq 1$.

Now let us recall the following Theorem (for more details see [3] and [15]), which is just a consequence of Riesz-Frechet-Kolmogorov Theorem.

Theorem 1.2.6. *(Helly) The injection $BV(\Omega) \hookrightarrow L^1(\Omega)$ is compact.*

1.2.4 A Trace Operator

Let $\Omega \subset \mathbb{R}^n$ be a bounded set, and $u \in C^1(\Omega)$. Then there exists a trace operator $\Gamma : u \to u_{|\Gamma}$, where $\Gamma = \partial\Omega$, well defined from the space of functions $C^1_c(\mathbb{R}^n)$ to the space $C(\Gamma)$. It has been proved that this operator is extendable by density to a linear and continuous operator from $BV(\Omega)$ to $L^1(\Gamma)$. This operator is by definition the trace of the function u on Γ, and we indicate it as $u_{|\Gamma}$ or $u(a, x)$, where $a \in I \subset \mathbb{R}$ and $x \in J \subset \mathbb{R}^{n-1}$, with $\Omega = I \times J$ (see [15], [3]).

Moreover, in general it is possible to define the trace of $u \in W^{1,p}(\Omega)$ when Ω is an open bounded set with regular and bounded boundary $\partial\Omega$. In this case the trace of u, $u_{|\Gamma} \in L^p(\Gamma)$.

If $u \in BV(\Omega)$ the following properties hold.

Proposition 1.2.3. *Let $\{u_n\}_{n \in N} \subset BV(\Omega)$ be a sequence of BV-functions, and let us suppose that it converges in the functional space $L^1(\Omega)$ to a function $u \in BV(\Omega)$.*

Then the traces $u_{|\Gamma,n} \in L^1(\Gamma)$ of the sequence u_n converge to the trace of u, $u_{|\Gamma|}$ in the norm of the space $L^1(\Gamma)$, i.e.

$$\lim_{n\to+\infty}\int_\Gamma |u_{|\Gamma,n} - u_{|\Gamma}| \to 0.$$

Proposition 1.2.4. *Let $I \subset \mathbb{R}$ be an open interval. Let $u \in C([0,T];BV(I))$ such that $(\partial_t u + a\partial_x u) \in C([0,T];BV(I))$, with $a \in \mathbb{R}$. Then $u \in BV([0,T]\times I)$.*

Proof. Let $\{u\}_{\varepsilon\in N}$ be a sequence of functions such that the total variation $TV(u) \leq TV(u)$, $u^\varepsilon \to u$ in $C([0,T];L^1(I))$, and such that $(\partial_t u^\varepsilon + a\partial_x u^\varepsilon) \in L^1(I\times[0,T])$, for each $t \in [0,T]$. Thus we have that

$$\int_0^T\int_I (|u^\varepsilon| + |au_x^\varepsilon|)dxds \leq T\sup_t \int_I (|u^\varepsilon| + |au_x^\varepsilon|)dx \leq C_1 T$$

because in particular the functions u^ε are bounded in BV for every $t\in[0,T]$ and moreover $(\partial_t u^\varepsilon + a\partial_x u^\varepsilon) \in C([0,T];BV(I))$ by definition of the sequence. Then we have that

$$\int_0^T\int_I |u_t^\varepsilon| dxds \leq \int_0^T\int_I (|u_t^\varepsilon + au_x^\varepsilon| + |au_x^\varepsilon|)dxds \leq C_2 T.$$

Thus

$$\int_0^T\int_I (|u_t^\varepsilon| + |au_x^\varepsilon| + |u^\varepsilon|)dxds \leq C_3 T.$$

By passing to limit when $\varepsilon \to 0$, we get the proof. □

Stampacchia's Lemma

Let us recall that functions in $W^{1,1}$ are also called absolutely continuous.

It is useful to recall the following Lemma (for more details we refer to [15], [3], [12]).

Lemma 1.2.1. *(Stampacchia) Let $\Omega \subseteq \mathbb{R}^n$ and let $u \in W^{1,1}_{loc}(\Omega)$. Then, for each Lipschitz function $f:\mathbb{R}\to\mathbb{R}$*

$$\partial_{x_i} f(u) = f'(u)\partial_{x_i} u(x),$$

for a.e. $x \in \Omega$

Lemma 1.2.2. *Let $I \subset \mathbb{R}$ be an open bounded interval and let $u \in L^\infty((0,T);I)$, with $(\partial_t u + a\partial_x u) \in L^\infty((0,T);I)$, $a \in \mathbb{R}$. Then*

$$sgn(u)(\partial_t u + a\partial_x u) = \partial_t|u| + a\partial_x|u|.$$

Proof. Let $\xi = x - at$ and $\tau = t$. Then $\partial_\tau u = \partial_t u + a\partial_x u$. From hypothesis $\partial_\tau u \in L^\infty$, and in particular, $u(\xi,\tau) \in W^{1,\infty}$. So, using Stampacchia's Lemma, choosing $f(u) = |u|$, we get the proof. □

1.3 Semigroup theory: some useful results

For more details, we refer to ([15]) and ([12]).

Let X be a real Banach space and consider the evolutionary ordinary problem

$$\begin{cases} \frac{du}{dt} = Au(t) \\ u(0) = u_0, \end{cases} \tag{1.3.1}$$

where $u_0 \in X$ is given and A is a linear operator. More precisely, we suppose that $D(A)$, the domain of A, is a linear subspace of X, and A is a linear unbounded operator $A : D(A) \to X$. We investigate the existence and uniqueness of the solution $u : [0, +\infty] \to X$ to problem (1.3.1). The problem consists in finding conditions on A so that (1.3.1) has a unique solution for each initial data and so that many interesting PDE can be cast into this abstract form.

Usually, the solution of (1.3.1) is written as

$$u(t) = S(t)u_0,$$

$t \geq 0$, to display explicitly the dependance of $u(t)$ on the initial value $u_0 \in X$. For each time $t \geq 0$, the map $S(t)$ is a map from X to X.

Definition 1.3.1. *A family $\{S(t)\}_{t\geq 0}$ of bounded linear operators mapping X into X is called a semigroup if the following conditions are satisfied:*

- *$S(0)u = u$, for each $u \in X$;*
- *$S(t + s)u = S(t)S(s)u = S(s)S(t)u$, for each $t, s \geq 0$, and $u \in X$;*
- *the mapping $t \to S(t)u$ is continuous from $[0, +\infty)$ to X.*

We say $\{S(t)\}_{t\geq 0}$ is a contraction semigroup if in addiction

$$||S(t)|| \leq 1,$$

$t \geq 0$*; here we denote with $||.||$ the operator norm. Thus*

$$||S(t)u|| \leq ||u||,$$

for each $t \geq 0$, and $u \in X$.

It is possible to characterize the operators A which generate contraction semigroups. Let us assume that $\{S(t)\}_{t\geq 0}$ is a contraction semigroup. We recall the following definition.

Definition 1.3.2. *Write*

$$D(A) := \{u \in X : \exists \lim_{t \to 0^+} \frac{S(t)u - u}{t}\}$$

and

$$Au := \lim_{t \to 0^+} \frac{S(t)u - u}{t},$$

$u \in D(A)$.

Let us say that the operator $A : D(A) \to X$ *is the generator of the semigroup* $\{S(t)\}_{t \geq 0}$*;* $D(A)$ *is the domain of* A.

Let us also recall the following Theorems (for the proof and details see [15]).

Theorem 1.3.1. *(Differential properties of semigroups) Let* $A : D(A) \to X$ *be the generator of a semigroup* $S(t)$*. Assume* $u \in D(A)$*. Then*

- $S(t)u \in D(A)$ *for each* $t \geq 0$*;*
- $AS(t)u = S(t)Au$ *for each* $t > 0$*;*
- *the map* $t \to S(t)u$ *is differentiable for each* $t > 0$*, and*
- $\frac{d}{dt}S(t)u = AS(t)u$*, for each* $t > 0$.

Theorem 1.3.2. *(Properties of generators) We have that*

- *the domain* $D(A)$ *is dense in* X*;*
- A *is a closed operator, i.e. for each sequence* $\{u_k\}_{k \in N} \subset D(A)$ *such that* $u_k \to u$ *and* $Au_k \to v$ *as* $k \to +\infty$*, then* $u \in D(A)$ *and* $v = u$.

In particular, we will now consider as Banach space X, a Hilbert space H. Now, we briefly introduce the concept of maximal operators (see [3]).

Definition 1.3.3. *Let* $A : D(A) \subset H \to H$ *a linear not-bounded operator. We say that* A *is a monotone operator if the scalar product*

$$(Au, u)_H \geq 0$$

for each $u \in D(A)$.

We say also that A *is maximal and monotone if we have moreover* $\forall f \in H$, $\exists u \in D(A)$ *such that*

$$u + Au = f.$$

We observe that for maximal and monotone operators the following properties holds .

Proposition 1.3.1. *Let A a maximal and monotone operator. Then*

- *$D(A)$ is closed in H;*

- *$D(A)$ is dense in H;*

- *for each $\lambda > 0$ the map $(Id + \lambda A) : D(A) \to H$ is bijective, the inverse map $(Id+\lambda A)^{-1}$ is a bounded operator (here the map Id is the identity map), and the norm of the operator in the space of the linear operators, $||(Id + \lambda A)^{-1}|| \leq 1$.*

Remark 1.3.1. *Let us observe that if A is maximal and monotone, then λA is maximal and monotone for each $\lambda > 0$.*

Now we recall the main result which connect evolution partial differential equations and maximal monotone operators.

Theorem 1.3.3. *(Hille-Yosida Theorem 1) Let A be a maximal monotone operator on a Hilbert space H. Then A for each function $u_0 \in D(A)$ there exists a unique function $u \in C^1([0, +\infty); H) \cap ([0, +\infty); D(A))$ such that*

$$\frac{du}{dt} + Au = 0,$$

with $t \geq 0$, and

$$u(0) = u_0.$$

We observe that the norm of a function in the space $D(A)$ is the Hilbertian norm

$$||u|| = (||u||_H^2 + ||Au||_H^2)^{\frac{1}{2}}.$$

It can be possible to establish a bijective correspondence between contraction semigroups and maximal and monotone operators. In fact, the following Hille-Yosida Theorem holds (see [15], [3], [12]).

Theorem 1.3.4. *(Hille-Yosida 2) Let A be a monotone operator on a Hilbert space H. Then it generates a contraction semigroup $\{S(t)\}_{t \geq 0}$ if and only if A is also a maximal operator on H.*

Chapter 2

Some mathematical models of chemotaxis

In Nature there are a lot of species (insects, animals, bacteria and so on..) which have an acute sense of smell for conveying information between members of the species. Chemicals which are involved in this process are called pheromones. For example, the acute sense of smell of many deep sea fish is particularly important for communication and predation. Generally, species move in the direction of released chemical signal, and more precisely, in the verse of increasing concentration of it. This chemically directed movement is called chemotaxis and it directs the motion up a concentration gradient. Chemotaxis is an important mean of cellular communication and determines how cells arrange and organize themselves. We can have a positive or negative chemotaxis when the chemical signal attracts or not an organism. In general, the chemical signal is called chemoattractant or chemorepellent. Chemotaxis is crucial in biological processes: for example when a bacterial infection invades the body it may be attacked by movement of cells towards the source as a result of chemotaxis, or, moreover, leukocyte cells in the blood move towards a region of bacterial inflammation, to counter it, by moving up a chemical gradient caused by the infection (see [22], [28]). It is being found to be important in an increasing range of situations. Modeling chemotaxis processes through mathematical models is very interesting because they involve fluid mechanics and filtration theory on quite different scales at the same time (Murray 1977). One of the first studies of chemotactic phenomena involve the amoeba Dictyostelium discoideum where single-cell amoebae move towards regions of relatively high concentrations of a chemical which is produced by the amoebae themselves. Interesting wavelike movement and spatial patterning are observed experimentally. Most mathematical models for spatial patterning in Dictyostelium discoideum are based on continuum models for the chemoattractants and the cells, while there are other models in which the cells are considered as discrete entities with the chemoattractant concentrations continuous (see [29], [23]). In multicellular organisms, chemotaxis is extremely important: in fact it organizes cells migration and distribution in various tissues, or it determines the cells position during the embryonic development. Focusing on our point of view, chemotaxis is also crucial in the movement of fibroblast in a wounded region to start the healing process. To derive an equation to describe chemotaxis it is supposed that the presence of a gradient in an attractant, $a(x,t)$, gives rise to a movement, of the cells say, up the gradient. The flux of cells will increase with the number of cells, $c(x,t)$, present. Thus it is reasonably to take as the chemotactic flux

$$J = c\chi(a)\nabla a,$$

where $\chi(a)$ is a function of the attractant concentration. The general conservation equation for $c(x,t)$ is

$$\frac{\partial c}{\partial t} + \nabla J = f(c),$$

where $f(c)$ represents the growth term for the cells, and J is the flux.

In general, in a lot of these processes, together with chemotaxis there is a phenomena called reaction (or diffusion).

Unlike chemotaxis, in this kind of process particles, for example, cells, bacteria, animals and so on, usually moves around in a random way. The particles spread out as a result of this irregular individual particle's motion. When this microscopic irregular movement results in some macroscopic or gross regular motion of the group we can think of it as a diffusion process. Of course there may be interaction between particles, for example, or the environment may give some bias in which case the gross movement is not simple diffusion. Continuum models of this process has been derived in terms of a particle density or concentration. The classical approach to diffusion is the Fickian diffusion. This says that the flux, J , of material, which can be cells, amount of chemical, number of animals and so on, is proportional to the gradient of the concentration of the material. That is, in one dimension

$$J = -D\frac{\partial c}{\partial x},$$

where $c(x,t)$ is the concentration of the species and D is its diffusivity. The minus sign simply indicates that diffusion transports matter from a high to a low concentration. Equations of reaction and diffusion are characterized by the presence of the gradient of the term flux J,

$$\frac{\partial c}{\partial t} = f + \nabla J,$$

where f is a function of c, x, and t. If we want to describe a biological process in which there are both chemotaxis and diffusion we will take a flux $J = J_{diffusion} + J_{chemotaxis}$ and the equation will become

$$\frac{\partial c}{\partial t} = f(c) - \nabla(c\chi(a)\nabla a) + \nabla(D\nabla c).$$

We have said that the chemotaxis is crucial during the process of wound healing by fibroblasts. We have not said that the movement of fibroblasts is not random during this process but it follows some directions given by a the extracellular matrix (ECM). We can mathematically represent this ECM as a network. So we will have to work with a chemotaxis phenomena on a network. Wound healing is a typical examples of cellular invasion, which crucial steps are cellular migration and proliferation. The tissue engineering works to make faster this process through production of artificial "scaffolds" which play the role of the ECM on a damaged tissue. It has been shown that the production of chemoattractant by fibroblasts is strictly linked to direction of fibres of ECM and it helps the right alignment of the fibres between the old and new tissue. The ECM serves many functions such as providing support and anchorage for cells. It regulates cell's dynamic behavior, apoptosis and proliferation. It is composed of a mesh of fibrous proteins as collagen and provides directional information directly through the fibres along which cells tend to align (this process is called *contact guidance*).

In the following sections, we present the derivation of some important parabolic and hyperbolic chemotaxis models.

2.1 The Patlak-Keller-Segel Model

The spatial pattern potential of chemotaxis has been exploited in a variety of different biological contexts. Mathematical models involving chemotaxis, with reaction diffusion models, are simply part of the general area of integro-differential equation models for the development of spatial patterns. The basic KellerŰSegel continuum model was proposed by Keller and Segel (1970). Their model is one of the most important chemotaxis model and it describes chemotaxis at a macroscopic level considering the population of living species as a whole. Their model arise from their studies about the formation in the slime mould Dictyostelium discoideum. Then a discrete, more biologically based, model for the aggregation with appropriate cell signalling was developed giving oscillatory cyclic signalling in the development of this slime mould. In their original model, Keller and Segel tried to deduce the cooperative behavior of populations from their individual properties (see [28], [22], [37], [38]). The model consists of reaction-diffusion equations in which there are the interaction of four different quantities, which are mathematically represented by four functions ϕ,u,η,$c : R^n \times R^+ \to R$, with $n = 1, 2, 3$, which respectively represent:

- $\phi(x, t)$ the concentration of chemical signals at time t in the point x;

- $u(x,t)$ the density of the populations (cells);

- $\eta(x,t)$ the concentration of an enzyme (acrasinae) that degrades the chemoattractant;

- $c(x,t)$ the concentration of a substance formed from reaction of chemoattractant and the enzyme.

To derive their model, Keller and Segel made the following hypothesis:

- the total number of population remains fixed at any time t;

- the acrasinae is produced at a rate represented by a function $r(\phi, \eta)$;

- the chemical signal ϕ is produced at a rate represented by $f(\phi)$;

- η and ϕ react to form a complex c at a rate k_1 (the inverse reaction is indicated with a rate k_{-1}); c dissociates to form η and other products at a rate k_2;

- ϕ, η and c move following the Fick's law;

- population moves in the direction of increasing gradient of ϕ.

From the fourth assumption, we have that ϕ, η and c follow the Fick's law, so their associated fluxes J are given by

$$J = -D_m \nabla m,$$

with $m = \phi, \eta, c$, and where the constant D_m is the coefficient of diffusion.

From the hypothesis that cells move in the direction of increasing chemical signal, the flux J_u for the species u is different from the others, and in particular it has the form

$$J_u = -D\nabla u + C\nabla\phi;$$

the first term is given by the Fick's law, while the second term illustrates the chemotactic phenomenon in response to the chemical signal ϕ. The diffusion constant C may be positive or negative to indicate a chemoattractant or a repellent. Imposing the balance of total mass in an arbitrary domain $\Omega \subset R^n$, they found that each variable must be satisfy the differential equation

$$\partial w(x,t) = -\nabla J_w + G_w, \tag{2.1.1}$$

for $w = u, \phi, \eta, c$. Here the function G_w represent the mass of population in a unite of time per unite volume, i.e. the birth or death of population in the region. In fact, we must have that the change of mass in the total region Ω is equal to the flux out of the boundary plus the mass of population in a unite time, i.e.

$$\frac{d}{dt}\int_\Omega w(x,t)dx = -\int_{\partial\Omega} J_w n ds + \int_\Omega G_w dx; \tag{2.1.2}$$

using the divergence theorem, we obtain (2.1.1).

The total system of Patlak-Keller-Segel model is

$$\begin{cases} \partial_t u = \nabla(+D\nabla u - C\nabla\phi) \\ \partial_t \phi = \nabla(D_\phi \nabla\phi) + uf(\phi) - k_1\phi\eta + k_{-1}c \\ \partial_t \eta = \nabla(D_\eta \nabla\eta) + ug(\phi,\eta) - k_1\phi\eta + k_{-1}c + k_2 c \\ \partial_t c = \nabla(D_c \nabla c) + k_1\phi\eta - k_{-1}c - k_2 c. \end{cases} \tag{2.1.3}$$

In some articles the authors study in particular the populations' aggregation process and some of them consider aggregation as a manifestation of instability in a uniform distribution. Usually the population of cells is assumed to be homogenous but biology and studies of the models and simulations have shown that at some point of life cycle of cells the characteristic of single cell change making the distribution unstable.

Keller and Segel did not consider this case and and moreover they assumed:

- the complex c is in a steady state after a reaction;

- the total concentration of the enzyme η is a constant η_0.

With these two conditions the system (3.0.4) becomes

$$\begin{cases} \partial_t u = \nabla(D(u)\nabla u - F(u)G(\phi)H(\nabla\phi)) + f(u) \\ \partial_t \phi = \triangle\phi + ug(u) - \phi, \end{cases} \tag{2.1.4}$$

which is the general Patlak-Keller-Segel model. The model is composed of parabolic equations and its behavior is known: it has global solution in time in one dimension, while in higher dimensions, if the initial data are small, then the solution exists globally for all times. Otherwise, it explodes in finite time. The analytical and numerical studies of this model have a great interest for mathematicians and biologists because this model reflects and can be applied to a lot of biological phenomena. In particular, chemotaxis is one of the most common phenomenon in nature, because the movement of species (cells, bacteria, etc..) is usually influenced by external signals: there can be chemical signals produced by other cells which orient the direction of population, or external sources (for example food) which influence movement of cells. In the case we will study, chemotaxis is crucial because cells call each other to start healing process and the presence of fibres of ECM gives direction of movement to fibroblast. Models based on the Patlak-Keller-Segel equations have been developed to study whether chemotaxis may influence pattern formation process, as the formation of pigmentation patterns in fishes and cell colonization (see [38]). Chemotaxis is also important during formation of plaques in Alzheimer's disease, and it models some distinct stages of tumor growth, such as the invasion of cancer cells, or macrophage invasion into tumor (see [34]). There exists a lot of variations of Patlak-Keller-Segel model, especially to understand the mathematical properties of it and in particular which are the conditions under which specialization or variations of the equations of the model form finite-time blow-up or have global solutions. Some of these different models have been described basing on additional biological realism: for example, there are models in which has been introduced an additional parameter which regularize the problem such that the solution exists globally. Different values of this parameter give a bifurcations diagram in which for some of these values the problem admits a global solution, while for others pattern formation properties or non uniform solutions. Usually, different models show a global existence in time for the solution in one dimension, while a blow up in higher dimensions. Choosing small initial data and particular and smooth source terms, it can be possible to have global existence in time in a general dimension.

We recall one of the most famous model and firstly studied:

$$\begin{cases} \partial_t u + \nabla(D\nabla u - \chi u \nabla \phi) = 0, \\ \partial_t \phi = \triangle \phi + u - \phi. \end{cases}$$

This model has globally existing solutions in one space dimension and a threshold phenomenon with blow up solutions in higher dimensions.

2.2 Some biological backgrounds

In the present work, we deal with description of the movement of some cell populations on a biological fibres of a scaffold under the influences by external stimulus, generally called taxis. From a mathematical point of view, a scaffold is described by an oriented network composed of a finite number of compact intervals, the arcs of the network, which intersect in some vertices called "nodes". Thus we will consider a system of hyperbolic-parabolic equations each of these is defined on an arc of the network. Of course some compatibility boundary conditions must be given on the vertices of the network. A particularity of this model is also the fact that cells can move in two different verses on the same arc, with different velocities. Moreover, on the network we consider the presence of an external taxis that influences the directions of cells movement.

Several types of taxis are well known, but in this thesis we focus on chemotaxis, the influence of chemical substances present in the environment of the movement of cells, i.e., in our case, on the scaffold. In tissue engineering, a scaffold reproduces the fibres of Extracellular Matrix (ECM) which is a kind of environment typically produced by cells themselves on which they can move.

Scaffolds for tissue engineering play a critical role in regenerating functional tissues and organs. Ideally, a scaffold will provide a substitute extracellular matrix upon which cells can attach, proliferate, and organize as in natural tissue. It is generally recognized that both biochemical composition and microstructure of the scaffold affect cellular activity and organization ([2]). As the native ECM is comprised largely of proteins from the collagen family, emphasis has been placed on fabricating scaffolds from collagen and collagen-based composites. Gelatin, a biopolymer derived from native collagens, is potentially useful as a scaffolding material due to its low immunogenicity, biodegradability, biocompatibility, and low cost. Gelatin is widely used as a dressing for wound healing and as a scaffold for dermal tissue engineering. Gelatin can be formed via a technique called electrospinning into fibrous scaffolds at a scale similar to native ECM, which makes these scaffolds conducive for tissue engineering. Scaffolds become an environment for cells in which they can move and also proliferate. It has been experimentally observed that in addition to material composition, pore size, pore orientation, fiber structure and fiber diameter of scaffolds affect proliferation, cellular organization, and subsequent tissue morphogenesis ([11]).

One of the most important application of these biodegradable scaffolds in biology is the acceleration of the dermal wound healing process. The most important cells interested in dermal wound healing are called fibroblasts, which start the reepithelialization of the damaged tissue moving on fibres of extracellular matrix. Both wound contraction and reepithelialization from the margins of the wound play an important role in wound closure. Reepithelialization is achieved by fibroblasts proliferation and migration over the extracellular matrix. In dermal wound healing several interacting events are initiated, as inflammation, tissue formation, angiogenesis and tissue remodelling. In each of these events it is crucial the interaction between cells (fibroblasts) and the extracellular matrix. There is a basement membrane that has been shown to be an active regulator in epithelialŨmesenchymal interactions during epithelial cell development (see [17]). Wound closure has been studied both in vivo and in vitro. It has been observed that after the blood clot has formed, during the inflammatory response white blood cells invade the wound region moving through the ECM, and subsequently fibroblasts migrate into the region moving on the ECM to replace the blood clot with collagen. The new tissue is usually characterized by a new architecture that differs from the original and is less functional than it. New tissue typically has fewer blood vessels supplying the denser connective tissue which is less elastic, and it has different orientation of the fibrous matrix. The type of the ECM-cells interactions changes during the process of wound healing, and, although several of the ECM-cells interactions have been experimentally studied, this area remains poorly understood. This is partly because all interactions have not been found, but mainly because the processes involved interact in a complex manner with non linear feedback. It is important to say that one of the main questions still to be answered is how the fibroblasts are stimulated to migrate into a specific direction and how the reorganization of the cell is induced.

In the study of cells movement on a scaffold, biologists influence it by a phenomenon of chemotaxis. It is known that chemotaxis can lead to strictly oriented or partially oriented and partially tumbling movements. The movement towards a higher concentration of the chemical substance is called positive chemotaxis, while the movement towards a lower concentration is called negative chemotaxis. Substances that lead to positive chemotaxis are chemoattractants, and those leading to negative chemotaxis are repellent. In particular, chemotaxis that occurs on a scaffold is a positive chemotaxis because it has to be force and accelerate the movement of cells on the scaffold.

The idea of the acceleration of wound healing process is clearly important, but it plays a fundamental role for diabetic patients. In fact, wound healing in diabetes is a more complex process, because it is characterized by a chronic inflammation phase. The exact mechanism by which this occurs is not fully understood, and whilst several treatments for healing diabetic wounds exist, very little research has been conducted towards the causes of the extended inflammation phase. It has been known for many years that wounds in diabetic patients can take longer to heal than similar wounds in non-diabetics ([27]). Typically healing takes several months, and many wounds do not heal for $12-18$ months or more. The normal wound healing mechanism is obviously disrupted in some way, although despite intensive research a comprehensive understanding of this disruption, or its extent, has not yet been realized. There are, however, pieces of this complex process which have been identified and by combining these pieces together it becomes possible to present an initial model of the wound healing process as affected by diabetes mellitus. As we have previously said, the first stage of the wound healing process is the inflammation stage ([36]), and macrophages are among the first cells to arrive at the wound site in response to chemical signals (growth factors) released by platelets. Growth factors stimulate the chemotaxis and mitosis of both endothelial cells and fibroblasts, and are thus vital for the second stage of wound repair, the proliferative stage. In diabetic patients, macrophages are known to persist past the inflammatory stage in chronic non-healing wounds, and experimental data shows that significant numbers of these cells have been measured on day 28 of healing, long after macrophages are no longer seen in similar wounds in the control (non-diabetic) subjects. Macrophages themselves are differentiated monocytes, and result from monocytes responding to certain chemical stimuli. There are known to be three types of macrophage important to the wound environment. Each type of macrophage produces different growth factors.

It is the balance between the inflammatory and repair macrophage populations that appears to be crucial for successful wound healing. Since monocytes become repair macrophages in the presence of hyaluronan, and hyaluronan is produced by fibroblasts, it follows that if the balance between the macrophage populations is disturbed, as suspected in diabetic wounds, then the hypothesis is that there could be an insufficient amount of hyaluronan being produced by fibroblasts, resulting in the repair macrophage population being too low for healing to be completed.

The effects of diabetes on the wound healing process are the impairment of cellular proliferation for most cell types, increased apoptosis of endothelial cells, increased average blood glucose level, impairment of blood vessel regrowth, inadequate flow through blood vessels and decreased collagen deposition at the wound site. Furthermore, it is likely that growth factor expression is altered, and nitric oxide secretion and macrophage removal to the lymph nodes may also be impaired. Mathematical models of wound repair and healing have thus far been directed towards the proliferation and repair stages of the wound healing process, but it is evident that for diabetic wounds, the inflammatory phase should be modelled in the first instance, as this is when macrophages are most involved.

Mathematical modelling is a powerful tool designed to address such complex feedback mechanisms, and, in particular, the movement of bacteria under the effect of a chemical substance and the description of cell invasion during a biological process have been a widely studied topic in Mathematics in the last decades, and numerous models have been proposed. Dermal wound healing, angiogenesis and tumour invasion are typical example of cell invasion, and cell migration and proliferation are the two key cell functions responsible for cell invasion. As we previously said, in tissue engineering healthy tissue is cultured to repair damaged tissue through invasion of cells on a biodegradable scaffold. In recent years some mathematical models are developed in order to describe the interactions between cells during a wound healing process. For example, in [11], they assume that fibroblast movement is directed by the orientation of the matrix, a phenomenon known as Ścontact guidanceŠ, that the ECM affects the speed of the fibroblasts, then the composition of the ECM alters the production of different proteins by the fibroblasts, and finally that the ECM in the wound region contains a plethora of growth factors which alter fibroblast behaviour. They also assume that fibroblasts produce fibres of ECM aligned with their direction of movement and that production and degradation are balanced, so that the ECM density remains the same. Their model is the following

$$\begin{cases} f(x,t) = \sum_{i=0}^{N} w^i(x,t) \frac{f_t^i(t-\tau)}{||f_t^i(t-\tau)||}, \\ f_t^i = s\frac{v^i(t)}{||v^i(t)||}, \\ v^i(t) = (1-\rho) + c(f^i(t),t) + \rho\frac{f_t^i(t-\tau)}{||f_t^i(t-\tau)||}. \end{cases}$$

In this model, the function $f^i(x,t)$ represent the fibres of the extracellular matrix, so the first equation determines the total ECM produced by N fibroblasts. The functions $w^i(x,t)$ are weight functions in order to describe the different rates of productions of $f^i(x,t)$ for each cell i. The second and the third equations of the model govern the cell motion, where ρ and s are positive constants, with s represents cells speed, and t a time lag. The function $c(f^i(t),t)$ represents the presence of a chemoattractant gradient depending on fibres of ECM. Chemoattractant generally is produced by leucocytes and it is one of the parameters that determine trajectories of fibroblasts when they migrate towards the wound region. Using numerical approximations they show that chemoattractant gradients lead to increased fibres alignment at the interface between the wound and the healthy tissue. Results show that there is a trade-off between wound integrity and the degree of scarring. The former is found to be optimized under conditions of a large chemoattractant diffusion coefficient, while the latter can be minimized when repair takes place in the presence of a competitive inhibitor to chemoattractants.

Wound healing is usually described by continuum and discrete models are developed in few years. Some of these models are developed to capture the population-scale and cell-scale behaviour in a wound healing cell migration assay created from a scrape wound in a confluent cell monolayer ([4]). Usually, continuum models use a cell diffusivity function that decreases with cell density and a logistic proliferative growth term: thus, these models include the two dominant mechanisms and characteristics of cell migration and proliferation. Discrete models arise naturally from the continuum models. Cells are simulated as random walkers with nearest-neighbour transitions, together with a birth-and-death process ([11], [36]). The numerical simulations generated by these models capture the contact inhibition of migration effects and describe the formation of new tissue (usually called "scar tissue") in dermal wound healing. It is important to observe that often it is used a multiscale approach ([7]): the extracellular materials are modelled as continua, while fibroblasts are considered as discrete units.

In order to describe the movement of cells under the effect of a chemical substance numerous models have been proposed. Moreover it is possible to describe this biological phenomenon at different scales. For example, by considering the population density as a whole, it is possible to obtain macroscopic models of partial differential equations. One of the most celebrated model of this class is the one proposed by Patlak in 1953 [35] and subsequently by Keller and Segel in 1970 [22]. In this model, the basic unknowns are the density of individuals of the population and the concentration of chemoattractant, and the basic assumption is that dynamical of individuals is described by an equation coupled with an additional equation for the chemoattractant, chosen to be elliptic or parabolic, depending on the different regimes to be described. Model is given by a coupled reaction-advection-diffusion system for the space and time evolution of the density $u = u(x,t)$ of cells, and the chemical concentration $\phi = \phi(x,t)$ at time t and position $x \in \mathbb{R}^n$,

$$\begin{cases} \partial_t u = \nabla(-D_1 \nabla \phi + D_2 \nabla u), \\ \partial_t \phi = \nabla(D_\phi \phi) + u f(\phi) - k_\phi \phi, \end{cases}$$

where ∇ denotes the divergence respect to the spatial variable. The behavior of this system is now quite well-known: in the one-dimensional case, the solution is always global in time. In several space dimensions, if initial data are small enough in some norms, the solution will be global in time and rapidly decaying in time; while on the opposite, it will explode in finite time at least for some large initial data. The simplicity, the analytical tractability, and the capacity to replicate some of the key behaviors of chemotactic populations are the main reasons of the success of this model of chemotaxis. In particular, the ability to display auto-aggregation, has led to its prominence as a mechanism for self-organization of biological systems.

In [8] they proposed a cell migration model in presence of a chemical signal and some extracellular matrices,

$$\begin{cases} \frac{\partial u}{\partial t}(x,t,v) + v \cdot \nabla u(x,t,v) = J_m^B(x,t,v) + J_c^B(x,t,v), \\ \frac{\partial \phi}{\partial t}(x,t) = K \triangle \phi + f(u,\phi). \end{cases},$$

where $u = u(x,t,v)$ is the density of cell population, v is the velocity, $\phi = \phi(x,t)$ is the chemical signal, $J_m^B = J_m^B(x,t,v)$ is the function which represents the interaction between ECM and chemical signal, and $J_c^B = J_c^B(x,t,v)$ is the function which represents the interaction between cells and chemical signal. They have shown how macroscopic continuum models can be derived from the mesoscopic transport equation and focused on the so-called diffusive approximation; moreover, numerical simulations have been shown the ECM and environmental factors effects on cell migration.

In the following we will consider a system of equations of a particular one-dimensional case of the previous model, which reads

$$\begin{cases} u_t^i + v_x^i = 0, \\ v_t^i + \lambda_i^2 u_x^i = G(\phi^i, \phi_x^i, u^i) - v^i, \\ \phi_t^i = D_i \phi_{xx}^i + a u^i - b\phi^i, \end{cases} \tag{2.2.1}$$

where the index i varies on a finite set and where λ_i, a, b and D are positive constants. In particular the coefficients λ_i represent the speed of cells, while D_i the diffusion coefficients of the chemoattractants ϕ^i. The function G is a smooth function satisfying some suitable assumptions. In Chapter 4 we will prove a local existence Theorem for the previous system defined on a network, while in Chapter 5 we will extend local solution to a global one taking into account a particular choice of the function G.

As far the studies of models on networks, some papers concern with fluidodynamic models for traffic flow. For example, in [10] they consider a single conservation law defined on a network, and they study the corresponding evolution problem. They have some fixed rules for the distribution of traffic and an optimization criteria for the flux. They prove the existence of solutions to the Cauchy problem and moreover they show that the Lipschitz continuous dependence by initial data does not hold in general, but under special assumptions. Moreover, in [39], Valein and Zuazua study the stabilization of the wave equation on a general $1D$ network. They use an interpolation inequality to obtain the explicit decay estimate of the energy for smooth initial data and then they show that the obtained decay rate depends on the geometric and topological properties of the network.

2.3 Hyperbolic models

We have observed that the Patlak-Keller-Segel model, and all its variations, is a system of parabolic equations. There are some biological phenomena which cannot be described by these models. More precisely, Patlak-Keller-Segel model is not sufficiently precise to describe some movement of cells taking into account the fine structure of the cell density for short times. To describe these kind of phenomena, the parabolic equation for the population density u in the Patlak-Keller-Segel model has to be a hyperbolic equation. It follows that we have a hyperbolic-parabolic model, which has been widely used in recent years because it gives more realistic descriptions of biological phenomena. Hyperbolic equations can be obtained as fluid limit of transport kinetic equation with a scale parameters, usually given by $t \to \varepsilon t$ and $x \to \varepsilon x$. Another important method to obtain a hyperbolic equation is the moment closure method introduced by Hillen which reduces equations into models depending on Cattaneo's law of heat conduction (see [14], and [5]). The models obtained have finite speed propagation.

The Cattaneo-Hillen Model

From biology it is known that some cells move in a certain direction at an almost constant speed (run), suddenly they stop and choose a new direction (tumble) to continue movement. This kind of movement is called run and tumble and it can be modeled by a stochastic process which is called velocity jump process. Denoting with $\rho(x,t,v)$ the population density at spatial position $x \in \mathbb{R}^n$ at time $t \geq 0$ and with velocity v, we have that although the most meaningful space dimensions are $n = 1, 2, 3$, the theory which describes this kind of movement works for all $n \in N$. It is assumed that individuals choose any direction with bounded velocity. Then the model which describes this process is the following linear transport model

$$\partial_t \rho(x,t,v) + v\nabla\rho = -\mu\rho(x,t,v) + \mu \int T(v,v')\rho(x,t,v')dv',$$

where μ is the turning rate, and $T(v,v')$ is the probability to choose new velocity v'. It is assumed to have particle conservation. If we want to describe a diffusion process with birth and death of population included, then we have a reaction diffusion model. Depending on concrete experiments the reaction may depend on particle velocity, hence we have a nonlinear reaction-transport equation. It is interesting to observe that in biological applications, in case of no birth or death reactions, the only conserved quantity is the total particle number, but if we consider a transport equations appear in physics, for example Boltzmann equations, we have that in this physical application some quantities are conserved, for example energy, momentum and mass. A technique to understand the dynamic properties of reaction-transport equations and Boltzmann equations is due to Cattaneo and Hillen and it is called the moment method. Multiplying the previous linear transport equation by powers of v and then integrating, it is possible to derive an infinite sequence of equations for the moments of ρ. Using this technique, an important and well known problem appears because in the equation for the n-th moment, the $(n+1)$-st moment appears. So we need an approximation of the $(n+1)$-st moment. One of the most important theories to close the moment equations is due to Hillen and it is based on a minimization principle.

Cattaneo also introduced the Cattaneo Law which is a modification of Fourier's law of heat conduction. It is used to describe heat propagation with finite speed. From this law, Hillen also derives models for chemosensitive movement, for example chemotaxis of cells or bacteria. It has been observed that in general cells and bacteria change their turning rate in response to external stimuli but they do not change their turn angle distribution. So the turning rate should depend on the velocity, on the concentration of the external signal ϕ and on its gradient $\nabla\phi$, while it is assumed that the total number of cells is preserved (see [5]) In one dimensional case and assuming two constant velocities $\pm\lambda$, the Cattaneo-Hillen model is the following

$$\begin{cases} \partial_t u^+ + \lambda\partial_x u^+ = -\mu^+(\phi, \nabla\phi)u^+ + \mu^-(\phi, \nabla\phi)u^-, \\ \partial_t u^- - \lambda\partial_x u^- = \mu^+(\phi, \nabla\phi)u^+ - \mu^-(\phi, \nabla\phi)u^-, \\ \partial_t\phi - D\partial_{xx}\phi = -\beta + \alpha(u^+ + u^-). \end{cases}$$

Functions $u^\pm$Ądenote the densities of the right/leftmoving part of the total population and ϕ is the external chemotactic stimulus which influence the movement of the population itself. Parameters Ąλ and D, which are assumed to be strictly positive constants, represent speed of propagation of $u^\pm$, and the diffusion of coefficient for the chemoattractant respectively. The terms $\mu^\pm$ are called turning rates and they control the probability of transition from u+ to u. and vice versa, i.e. the change of direction in the movement of a single individual. It has been shown a first result of local and global existence for weak solutions under the assumption of turning rateĄfs boundness is proved. Recently Guarguaglini et al. have proved more general results for this model under weaker hypotheses, by showing a general result of global stability of some constant states for both the Cauchy problem on the whole real line and the Neumann problem on a bounded interval for small initial data. These results have been obtained by using the linearized operators and the accurate analysis of their nonlinear perturbations (see [18]).

The Gamba-Preziosi Model

We recall that vasculogenesis is the process of blood vessel formation by cells, endothelial cells and angioblasts. An analogous phenomenon is the angiogenesis, the physiological process involving the growth of new blood vessels from preexisting vessels. It is a normal and vital process in growth and development, as well as in wound healing and in granulation tissue. However, it is also a fundamental step in the transition of tumors from a dormant state to a malignant one. Some biologists hypothesized that, if it were possible to inhibit neovascularization, it might stop the growth of the tumour or at least contain its growth to a dormant mass of around 2 to 3 mm in diameter. In particular, Folkman suggested that such antiangiogenesis could be the basis for a new form of cancer therapy. A particularly important aspect, from a cancer therapy point of view, is that anti angiogenic therapy does not induce acquired drug resistance in experimental cancer unlike chemotherapy. The field of anti-angiogenesis is now fast growing with an increasing number of areas where modeling could be of some considerable value. In vasculogenesis, the ability to form networking capillary tubes is a cell autonomous property of the endothelial cells, which need permissive but not instructive signals from the extracellular environment. In recent years many experimental investigations have been performed on the mechanism of blood vessel formation. The in vitro studies try to replicate the type of patterns observed in vivo, so models provide information on the mechanism which operates in vivo. In general, cells are cultured on a gel matrix and their migration and aggregation are observed through videomicroscopy. The process of formation of a vascular network starting from randomly seeded cells can be accurately tracked: individual trajectories in some experiments shows marked persistence in the direction, with a small random component superimposed. The motion is directed towards a zone of higher concentration of cells, suggesting that chemotactic factors play a role. Cells migrate over distances which are an order of magnitude larger than their radius and aggregate when they get in touch with one of their neighbors. In a time of the order of 10 h they form a continuous multicellular network which can be described as a collection of nodes connected by chords. The Gamba-Preziosi model for vasculogenesis focuses on the development of vascular network formation. Their mathematical model is based on the following assumptions:

- the chemical factors released by cells diffuse and degrade in time;
- endothelial cells neither duplicate nor die during the process;
- endothelial cells show persistence in their motion;
- endothelial cells communicate via the release and absorption of a soluble growth factor;
- closely packed cells mechanically respond to avoid overcrowding;
- cells are slowed down by friction due to the interaction with the fixed substratum.

The equations are derived from the conservation laws of mass and momentum and they are

$$\begin{cases} \partial_t \rho + \nabla(\rho u) = 0 \\ \partial_t(\rho u) + \nabla(\rho u \bigotimes u) = f \\ \partial_t \phi = D \triangle \phi + a\rho - \frac{1}{\tau}\phi. \end{cases}$$

The first equation leads the mass conservation because of the assumption that cells do not undergo mitosis or apoptosis during the experimental phenomenon. The second equation assumes that cell motion can be obtained on the basis of a suitable force balance, while the last equation is a diffusion equation for the chemical factor. The source term f represents reasons which may cause a change in cell persistence and it can be of various types (see [8]).

2.4 The Preziosi-Chauvière Model

Now we want to describe an important model crucial for our studies. Firstly we focus on the biological background that is strictly connected to the development of the equation of model. Migration of cells plays a fundamental role in both normal and pathological phenomena. For example it is crucial in immune response and tissue homeostasis in mature multicellular organisms, or in tumor invasion. Migration depends on intrinsic properties of cells, or it results from their adaptation to the environment. Moreover, cell movement is influenced by external factors such as chemical signals (chemoattractant or repellent) or physical interactions between cells and ECM. In their model, Preziosi and Chauviére study the amoeboid cell migration. They consider a cell population which moves on a domain $D \subseteq R^n$, and in which each cell has a velocity $v \in V \subseteq R^n$ (V is the space of velocities). Moreover, they suppose that the cells moves in presence of fibres of extracellular matrix. The variables which interact in their model will be:

- $p(x, t, v)$ is the number of cells at time t in the point x, with velocity v;

- $v \in |V| \times S^{n-1}$ are the velocities of cells assumed to be radial (here S^{n-1} is the unite sphere in R^n);

- $m(x, n)$ is the function which represent the fibres of ECM; $n \in S^{n-1}$ is a unit vector that represent the fibre orientation.

The discrete model of cell motion is the following:

$$\frac{\partial p}{\partial t}(x, t, v) + v\nabla p(x, t, v) = M(x, t, v), \tag{2.4.1}$$

where M is an integral operator which describes cell motion. From the previous model they derive a continuum description in the following way. In fact, they introduce the new variables:

- cell density $u(x,t) := \int_V p(x,t,v)dv$, where $v \in V$;

- the mean cell velocity U, defined as $U(x,t) = \frac{\int_V p(x,t,v)dv}{u}$;

- the flux $J = uU$;

- the fiber density $F(x) = \frac{1}{2}\int_{S^{n-1}} m(x,n)dn$.

Usually, they supposed that the orientation of fibres is isotropic, so the function F of distribution of fibres does not depend on the unit vector n, i.e,

$$m(x,n) = \frac{1}{\pi}F(x).$$

There are different types of evolution equations for these macroscopic variables that characterizes cell migration. In fact, it can be possible to describe only the random migration of cells without considering the presence of chemoattractant and ECM; or they describe a model in which there are only the interaction between cells and ECM, or the interaction between cells and chemoattractant, or a complete models with all these factors.

2.4.1 Random migration

This is the simplest model about cell migration. In this model the cells move by smooth running interrupted at discrete times by an instantaneous re-orientation, which is usually influenced by environmental factors. In this kind of model, the source term of the equation (2.4.1) is

$$M(x,t,v) = -\mu p(x,t,v) + \mu \int_V T(v,v')p(x,t,v')dv', \tag{2.4.2}$$

where the term $-\mu p(x,t,v)$ describes turning cells away from velocity v with frequency μ. The function $T(v,v')$ is the probability distribution for a cell with previous velocity v' to choose the new velocity v; so it must be satisfies

$$\int_V T(v',v)dv = 1. \tag{2.4.3}$$

Furthermore, it is always required the cell number conservation which yields

$$\int_V M(x,t,v)dv = 0.$$

In [8] there are some examples of choice for T. For example, a possible choice for T is to decide that the probability to chose one velocity is the same of the probability to persist with the previous velocity. In other words, with this kind of choice, it is supposed that re-orientation has no memory of the past.

2.4.2 Contact guidance

Contact guidance is the interaction between cells and fibres of ECM, so the model has to describe the movement of a cell on some given fibres. The matrix gives a selection of preferred directions along with the cells can move. The general expression of the migration operator $M(x, t, v)$ is

$$M(x, t, v) = -L(x, t, v) + G(x, t, v),$$

where the function L is the rate at which cells turn away from velocity v, and the function G gives the rate at which cells re-orient into velocity v. The mass conservation is required also in this case. To describe contact guidance process they assume that the realignment along the fibres does not appear at a turning frequency, but it is caused by interactions between cells and ECM at a constant rate η. So, a possible choice for the functions L and G is

$$L(x, t, v) = p(x, t, v) \int_{S^{n-1}} \eta m(x, n') dn',$$

and

$$G(x, t, v) = \int_{V \times S^{n-1}} \eta \psi_m(v', v, n) p(x, t, v') m(x, n') dv' dn',$$

where the function $\psi_m(v', v, n)$ is the probability to choose velocity v for a cell with velocity v'. For simplicity they assumed that η is constant, and that the alignment process along a fibre is independent of the prior velocity v'. These assumptions imply that the migration operator M becomes

$$M(x, t, v) = \eta F(x)[\rho(x, t)\phi(v) \frac{m(x, v)}{2F(x)} - p(x, t, v)].$$

These choices permit to consider a smooth distribution ϕ which is useful for numerical simulations (see [8]) because in other case a Dirac function must be used.

2.4.3 Cell-cell interaction

It is very difficult to describe interactions between cells because they may be of different types: for example, adhesion, contact inhibition, or repulsion. In some of their works, Preziosi et al. focused on dynamical aspects and considered only the orientation effect from the interaction between two moving cells. In this case, they took as migration function M the following one:

$$M(x, t, v) = G(x, t, v) - L(x, t, v)$$

where the functions are G e L

$$L(x,t,v) = p(x,t,v) \int_V \eta p(x,t,v')dv'$$

and

$$G(x,t,v) = \int_{V\times V} \eta\phi(v',v'',v)p(x,t,v'')dv'dv'';$$

in this case the function $\phi(v',v'',v)$ defines a transition probability distribution for a moving cell with given velocity v' to choose the new velocity v when interacting with a cell with velocity v''. Basing on this model and choosing complex expressions for ϕ they are developed more sophisticated aggregation models that give numerically and analytically the formation of cell clusters.

2.4.4 Influence of environmental factors

Environmental factors can be of various nature. They can be diffusible chemicals secreted into the environment that will cause a cell movement in response. This response results from the external detection of a signal that is transducted to internal pathways. Often there are also chemical signals released by cells and felt by surrounding cells. Preziosi, Chauviére et al. have proposed a mathematical description of the cell response to environmental signals (the phenomenon called *taxis*), and focused in particular on chemotactic cue. Chemotaxis is described using transport equations, and it is introduced as a bias of the main movement which is often assumed to be random motion. A first simple description of interaction between cells and environment is the description of cell migration in ECM as a combination of random motion, contact guidance and cell-cell interaction. In this case bias is the ECM. Therefore, the equation which describes this case is (2.4.1) in which M is the sum of the two source terms which describe cell-cell contact and contact guidance described in the previous sections. To study the movement of cells in presence of an external and general bias, there is a common choice for M derived by Chauviére. The source terms studied in case of contact guidance and in case of cell-cell interaction are extended respectively to

$$M^B(x,t,v) = \eta F(x) \tag{2.4.4}$$

and

$$M^B(x,t,v) = \eta\rho(x,t) \tag{2.4.5}$$

The function $B(t,x)$ accounts for an external stimulus that modifies the rate at which a cell reorients into v". Chauviére derived the simplest expression for B which is the following gradient-based bias (see [8]):

$$B(x,t,v) = \pm\frac{\Gamma S_x(t,x)\cdot\frac{v''}{\lambda}}{\beta_s + S(t,x)}. \tag{2.4.6}$$

In the previous expression

- Γ is a parameter which reflects cell sensitivity;
- S is density of molecules's signal;
- the sign $\pm$ corresponds to a repellent or attractive effect in the direction of the gradient ∇S;
- $\beta_s > 0$ is introduced to avoid singular behavior when $S = 0$.

One of the study about these various types of equations is the derivation of macroscopic equations from the transport equation (2.4.1). This problem was firstly introduced in fluid dynamics. In one of their their work, Preziosi and Chaviére use an asymptotic method that require the existence of a small parameter ε for scaling process (Hilbert method, see [8]). Hillen and Othmer detail the diffusion limit of transport equation derived from velocity-jump processes (see [14]).

2.5 Hyperbolic models on networks

Let us introduce some important definitions.

We denote with $I_i \subset \mathbb{R}$ some closed and bounded intervals of $\mathbb{R}$ depending on indexes i which varies in a finite set. Let $N \in N$, $N > 0$, be positive integer. Now we introduce the definition of network (we refer to [39], [10], and [16] for more details).

Definition 2.5.1. *A one dimensional network $\mathcal{N}$ is a connected set of $\mathbb{R}^3$, defined as*

$$\mathcal{N} = \bigcup_{i=1}^{M} I_i,$$

where $I_i = [a_i^-, a_i^+]$, $i = 1, \ldots M$ are real closed oriented interval ($a_i^- < a_i^+$), such that for $j \neq i$, the intersection $I_i \cap I_j$ is either empty or a common boundary point. We denote by $\mathcal{E} = \{I_j, 1 \leq j \leq M\}$ the set of arcs of $\mathcal{N}$. We call vertex any boundary point. In particular, a vertex belonging to two or more arcs will be called node or internal vertex, and we will denote it by N.

If $N = a_i^+$, then the corresponding interval I_i will be called entering in the node N, while if $N = a_j^-$, then the corresponding interval I_j will be called outgoing from N. The other boundary points will be called external vertices, and we will denote them simply by a_i, $i \in \{1, \dots, M\}$. Let $\mathcal{V}$ the set of the vertices of $\mathcal{N}$. We denote by$\mathcal{V}_{int} = \{N_h, h = 1, \dots, \nu, \nu < M\}$ the set of nodes of the network. For a fixed node $N_h \in \mathcal{V}_{int}$, let $\mathcal{E}_{N_h} = \{I_i : N_h \in I_i\}$ be the set of arcs having N_h as vertex.

We denote by

- E_{N_h} the set of incoming roads of a network in the node N_h;

- U_{N_h} the set of outcoming roads of a network from the node N_h;

- $\mathcal{M}_{N_h} = E_{N_h} \cup U_{N_h}$;

- $\mathcal{M}{=}\bigcup_h \mathcal{M}_{N_h}$;
  $|\mathrm{E}_{N_h}| < +\infty$ and $|U_{N_h}| < +\infty$ the cardinality of the sets E_{N_h} and U_{N_h} previously introduced.

The study of partial differential equations defined on a network is developed in recent years. This kind of problems arise from physical or biological phenomena: in particular, we will focus our attemption to the movement of some kind of population which is forced to move in certain directions. These kind of problems describe for example the traffic flow on some roads in urban traffic, or the movement of some kind of cells in presence on the fibres of the extracellular matrix. In these kind of models, junctions between the arcs of the network play a crucial role in the behavior of the solutions of the equations. Indeed the interactions occur at junctions and there the problem is undetermined. Usually, it is considered a planar network of elastic strings that undergoes small perpendicular vibrations. The control, observation and stabilization problems of these network have been the object of intensive research (see [10]). One of the problem of interest studied in this years is the stabilization of the network by means a damping term located on one single exterior node. In one of their papers, Zuazua and Valein have developed a systematic method to address this issue and to give a general result allowing to transform an observability result for the corresponding conservative system into a stabilization one for the damped one. More precisely, they studied the stabilization of the wave equation on a general one dimensional network. They transfer known observability results in the context of control of conservative systems into a weighted observability estimate for the dissipative one. Then they used an interpolation inequality to obtain the explicitly decay estimate of the energy of smooth initial data. The decay rate depends on the geometric and topological properties of the network. If $u_j(x,t)$ is the function describing the transversal displacement in time t of the arc I_j of the network, it must to satisfy the following system

$$
\text{(2.5.1)} \qquad \begin{cases} \frac{\partial^2 u_j}{\partial t^2}(x,t) - \frac{\partial^2 u_j}{\partial x^2}(x,t) = 0 \\ u_j(v,t) = u_l(v,t) \\ \sum_{j \in E_v} \frac{\partial u_j}{\partial n_j}(v,t) = 0 \\ u_{j_v}(v,t) = 0 \\ \frac{\partial u_l}{\partial x}(0,t) = \frac{\partial u_l}{\partial t}(0,t) \\ u(t=0) = u_0 \\ \frac{\partial u}{\partial t}(t=0) = u_1, \end{cases}
$$

where v is a vertex of the network and where $\frac{\partial u_j}{\partial n_j}(v,t)$ stands for the outward normal derivative of u_j at the vertex v. The above system has been considered by several authors in some particular situations (see for example [39], [10]), and explicit decay rates are obtained for networks with some special structures. From an another point of view, there are studies of mathematical models for fluid-dynamics on networks. They are based on conservation laws. In a lot of works, approximation of scalar conservation laws along arcs is carried out by using conservative methods, such as the classical Godunov scheme, or the discrete velocities kinetic schemes, with the use of suitable boundary conditions at junctions. The study of traffic flow aims to understand traffic behavior in urban context in order to answer to some questions. For example how long the cycle traffic lights should be, or where to construct entrances, exists and overpasses, where to install stop signals. Firstly, network models of transport systems are assumed to be static, but with this assumption they did not reproduce very well the urban road networks, so recently traffic engineers have started to consider some alternative models, and new mathematical models have been arisen (traffic simulation models). However, the main problem of these models is the fact that they do not properly reproduce the backward propagation of shocks. But there some macroscopic models which deal with fluid-dynamic that with some traffic regulation strategies allow to observe the evolution of movement of population on a network through waves formation (see [16]). In a single arc, the non linear model is described by the following scalar and hyperbolic conservation law:

$$
\partial_t u + \partial_x f(u) = 0,
$$

where $u(x,t)$ is the density of population (cars) and the function f is the flux of population on the road. A common choice for the flux $f(u)$ is $f(u) = uv$, where v is assumed to be a decreasing function, only depending on the density, and the flux is a concave function. This kind of model seems to be the most appropriate to explain macroscopic phenomena as shocks formation. However, they can develop discontinuities in a finite time even with smooth initial data (for more details, see [16]). We want to notice that in all classical works, only a single road is taken into account. More recently, some models have been proposed for traffic flow on networks. We have previously said that a network is composed of a finite number of arcs represented by intervals $[a_i, b_i]$ that meet at some junctions. These junctions are crucial in the study of the model, and they are undetermined even after prescribing conditions on them, usually the conservation of cars, that can be written imposing the Rankine-Hougoniot conditions. In other words, they express the equality of ingoing and outgoing fluxes. Giving boundary conditions on the boundaries that are not junctions, and initial data, a boundary problem has obtained. Some numerical methods have been developed to study the behavior of the solution of the Riemann problem of the traffic flow on a network composed of 4 arcs. In particular, Piccoli and Natalini ([32]) studied the numerical approximation of the possibly discontinuous solutions produced by this model. They introduced some suitable boundary conditions at the junctions for classical numerical schemes and they used the Godunov scheme and kinetic methods.

2.6 A hyperbolic chemotaxis model on networks

In the previous sections, we have introduced the Preziosi-Chauviére model, and we have presented different biological cases that modify the equations of the model because of the variables they have to consider to reproduce these biological phenomena. In these thesis, we consider an epidermal wound and we study the process of healing repair by fibroblasts. In normal epidermic wound healing, fibroblasts move on fibres of the extracellular matrix that they produce on the damaged tissue. To start wound healing process, fibroblasts move on the boundary of the damaged tissue and producing the extracellular matrix begin to move in the heal. Moreover, they also produce signals to call other cells if it necessary to wound healing process. At the end of the process a new tissue, called scar tissue, is formed. We want to recall that cells move along ECM-fibres in both verses. In recent years, some biologists have produced in vitro fibres of the extracellular matrix and then they have created a scaffold, a particular tissue composed of these fibres and chemoattractants for fibroblasts. Putting it on the damaged tissue, they observed an important facts: fibroblasts move faster on the damaged tissue, because of the presence of chemoattractant and because of they have not to produce fibres of ECM. So, wound healing process is faster and moreover the new tissue is is better than the scar tissue.

We want to describe the movement of fibroblast during epidermal wound healing in presence of a scaffold. So, referring to the Preziosi and Chauviére models, we have to consider the case of the cell-fibre interaction, and in presence of a chemoattractant. Fibres represent the fibres of a scaffold or the ECM-fibres. From a mathematical point of view, the movement of cells along the scaffolds is represented by a model on network, but we want to observe that in this case, we can have two possible verses of movement along each arc of the network. This case has ever been considered yet. The equations that reproduce cells movement will be transport equations, coupled with a parabolic equation that models the presence of a chemical signal. So, let us consider an oriented network composed of M arcs I_i which intersect in ν nodes $N_h \in V_{int}$. Let $u_i^{\pm} : I_i \times [0,T]$ be the functions representing the densities of cells on the arc I_i, which move respectively from left to right and viceversa. Let us suppose that on each arc cells move can with two constant velocities $\pm\lambda_i$. We suppose that positive velocity $+\lambda_i$ corresponds to positive orientation, while the negative once to negative orientation. With refer to Preziosi-Chauviére equations, let $T^i_{\alpha,\beta}$ be the function that represents the distribution of probability in the i-th road to pass from velocity α to velocity β. For each $i \in E_{N_h} \cup U_{N_h}$, denoting with $+$ or $-$ the velocity corresponding to $u_i^{\pm}$, T^i has to be satisfy the condition

$$(2.6.1) \qquad \begin{cases} T^i_{+,+} + T^i_{+,-} = 1 \\ T^i_{-,-} + T^i_{-,-} = 1. \end{cases}$$

We assume that the probability to pass from velocity $+$ to velocity $-$ is the same that the probability to stand in $-$, and that the probability to pass from $-$ to $+$ is the same that the probability to stand in $+$, and it does not depend on the point $x \in I_i$ in which we calculate the probability. So we want T^i a constant, and

$$T^i_{+,-} = T^i_{-,-} =: \frac{C_-}{2}$$

and

$$T^i_{-,+} = T^i_{+,+} =: \frac{C_+}{2};$$

from conditions (2.6.1) we must have

$$(2.6.2) \qquad C_+ + C_- = 2.$$

For each function of cell density $u_i^{\pm}$, we recall that the general expression for the source term (namely it $M^i_{\pm}(t,x)$, for each $i = 1,\ldots,M$) in the equations of the model for the density $u^i_{\pm}$ is

$$M^i_{\pm}(t,x) = -L^i_{\pm}(t,x,v) + G^i_{\pm}(t,x,v),$$

for each $i \in E \cup U$; we supposed that for each arc cells move with constant velocities $\pm\lambda_i$, so we have in particular that

$$L^i_{\pm}(t,x,v) = L^i_{\pm}(t,x)$$

and

$$G^i_{\pm}(t,x,v) = G^i_{\pm}(t,x),$$

where $L^i_{\pm}$ is the rate of change of verse and velocity caused by the interaction with ECM and the presence of chemoattractant, while $G^i_{\pm}$ is the rate of random re-orienting. On each arc i, the probability distribution T^i represents in this case the probability to choose velocity λ or $-\lambda$. Let us now consider a function $\phi^i(t,x)$ which represents the chemical signal (chemoattractant) on an arc I_i; thus the source terms $M^i_{\pm}(x,t)$ have the following expressions:

$$M^i_{\pm}(x,t) = \eta m(x)[\frac{C_{\pm}}{2}(u^+_i(t,x) + u^-_i(x,t))(1 + B^i(x,t)) - u^{\pm}_i(x,t)], \tag{2.6.3}$$

where the function $B^i(x,t)$ represents the external stimulus, and so it is a function of the chemoattractant $\phi^i(x,t)$, the cellular sensitivity Γ, and the cell velocity $\pm\lambda_i$, $B = B(\phi, \Gamma, \pm\lambda)$. We also assume the total mass conservation, and so we impose that for each index $i \in E \cup U$

$$M^i_+(x,t) + M^i_-(x,t) = -L^i_+(x,t) + G^i_+(x,t)) - L^i_-(x,t) + G^i_-(x,t) = 0.$$

Moreover, we suppose that the chemoattractant $\phi^i(x,t)$ satisfies the parabolic diffusion equation

$$\phi^i_t = D_i \triangle \phi^i + f(u^i_+, u^i_-, \phi^i), \tag{2.6.4}$$

where D is a positive constant and f a function which influences the diffusion by its dependence on cell density and chemoattractant. So, for each arc I_i, $i \in E_{N_h} \cup U_{N_h}$ and on each node $N_h \in V_{int}$ of the network, we have the following system of hyperbolic-parabolic equations.

$$\begin{cases} u^i_{+,t}(x,t) + \lambda_i u^i_{+,x}(x,t) = M^i_+(x,t) \\ u^i_{-,t}(x,t) - \lambda_i u^i_{-,x}(x,t) = M^i_-(x,t) \\ \phi^i_t = D_i \triangle \phi^i + f(u^i_+, u^i_-, \phi^i), \end{cases} \tag{2.6.5}$$

where the source terms $M^i_{\pm}$ have the general form (2.6.3) previously introduced. We consider a particular form for $M^i_{\pm}$, i.e. we choose respectively

$$M^i_\pm := \pm\frac{1}{2\lambda_i}((\phi^i+\lambda_i)u^i_- - (\lambda_i - \phi^i)u^i_+); \tag{2.6.6}$$

in other words, we choose $\eta m(x) = 1$, the coefficients $\frac{C_\pm}{2} = \frac{1}{2}$, and the function $B^i_\pm \equiv \phi^i$. Let us observe that if we introduce the quantities

$$u^i := (u^i_+ + u^i_-)$$

and the flux v^i

$$v^i := \lambda(u^i_+ - u^i_-),$$

the previous equations can be written in the equivalent way

$$\begin{cases} u^i_t + v^i_x = 0, \\ v^i_t + \lambda_i^2 u^i_x = -v^i + G(u^i, \phi^i, \phi^i_x), \\ \phi^i_t = D_i \triangle \phi^i + f(u^i, \phi^i), \end{cases} \tag{2.6.7}$$

for each $i \in E_{N_h} \cup U_{N_h}$, where G is a smooth function, adding and then subtracting the first two equations. This system models the movement of cells on a network, in presence of a chemoattractant and with the hypothesis that in each road we can have two possible verses of movement.

Our aim is the study of the existence and uniqueness of the solutions to this system. Thus we have to impose initial and boundary conditions, with a particular attention to boundary conditions on nodes N_h. In the following Chapters, we will first study the linear problem in which we suppose that the gradient of chemoattractant is constant on each arc I_i, and we will prove a global existence Theorem in case of a general oriented network. Moreover, we will study the asymptotic behaviour of perturbation of stationary solutions.

Then we will study the non linear system

$$\begin{cases} u^i_t + v^i_x = 0, \\ v^i_t + \lambda_i^2 u^i_x = -v^i + G(u^i, \phi^i, \phi^i_x), \\ \phi^i_t = D_i \phi^i_{xx} + a u^i - b\phi^i. \end{cases} \tag{2.6.8}$$

We will prove a local existence Theorem in case of local lipschitzianity of the source term $G(u^i, \phi^i, \phi^i_x)$, and then we will extend the local solution to a global solution in case of a particular choice of G.

Chapter 3

A linear hyperbolic relaxation model on a network

The aim of this chapter is the study of a linear relaxation model on a network. Let us consider an oriented network $\mathcal{N}$ composed by M arcs I_i, $i = 1, \ldots, M$, and ν nodes $N_h \in \mathcal{V}_{int}$, $h = 1, \ldots, \nu$, and external boundary points a_i, according to the definition (2.5.1). We recall that we denote by $\mathcal{E}_{N_h}$ the set of arcs having a node N_h as a vertex. For each node N_h, and arc $I_i \in \mathcal{E}_{N_h}$, let E_{N_h} be the set of arcs entering in the node N_h, and U_{N_h} be the outgoing ones, and let $\mathcal{M}_{N_h} = E_{N_h} \cup U_{N_h}$. So we have that the set of total number of arcs is denoted by $\mathcal{M} = \bigcup_{\mathcal{V}_{int}} \mathcal{M}_{N_h}$.

Given $T > 0$ and $I_i \in \mathcal{E}_{N_h}$, let $u_i^{\pm}$ be functions defined on $u_i^{\pm} : I_i \times [0, T] \to \mathbb{R}$. The aim of this chapter is the study of the following linear system of hyperbolic equations defined on $\mathcal{N}$,

$$\begin{cases} u_{i,t}^{+} + \lambda_i u_{i,x}^{+} = \frac{1}{2\lambda_i}((\alpha + \lambda_i)u_i^{-} - (\lambda_i - \alpha)u_i^{+}), \\ u_{i,t}^{-} - \lambda_i u_{i,x}^{-} = -\frac{1}{2\lambda_i}((\alpha + \lambda_i)u_i^{-} - (\lambda_i - \alpha)u_i^{+}). \end{cases} \tag{3.0.1}$$

Here $x \in I_i$, $t > 0$, and $u_i^{\pm} : I_i \times [0, T] \to \mathbb{R}$ are functions representing the population cell densities respectively on entering arcs in a node N_h and on outgoing arcs; the parameter α is the concentration of chemical signal assumed to be constant in this case. Introducing the Riemann invariants $u^i(x,t) = u_i^{+}(x,t) + u_i^{-}(x,t)$ and $v^i(x,t) = \lambda_i(u_i^{+}(x,t) - u_i^{-}(x,t))$, for each $i = 1, \ldots, M$, we can write the above linear model in the equivalent way

$$\begin{cases} u_t^{i+} + v_x^{i+} = 0, \\ v_t^i + \lambda_i^2 v_x^i = \alpha u^i - v^i, \end{cases} \tag{3.0.2}$$

$i = 1, \ldots, M$.

Our aim is to prove the global existence and uniqueness of the solution to problem (3.0.1). Then we study the asymptotic behavior of small perturbations of stationary solutions on a network composed of a single node.

The model (3.0.1) is characterized by a finite speed of propagation. In particular we denote by $\pm\lambda_i$ the velocities of cell densities on each arc I_i, respectively corresponding to densities $u_i^{\pm}(x,t)$. Thus on each arc we can have both verses of movement. Moreover we assume that

$$\alpha < |\lambda_i|, \tag{3.0.3}$$

for each index $i \in \mathcal{M}$.

This is the well known subcharactesitic condition, and it was firstly introduced by Whitham (see [40]) in order to study relaxation of hyperbolic system

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \lambda^2 \partial_x u = \frac{1}{\varepsilon}(f(u) - v), \end{cases} \tag{3.0.4}$$

where $f \in C^1(\mathbb{R})$ and λ^2 is a fixed constant. Let us firstly observe that the problem (3.0.1) is the linearized problem of (3.0.4). Relaxation systems often arise in many physical situations, for example, gases not in thermodynamic equilibrium, kinetic theory, chromatography, river flows, traffic flows, and more general waves. The study of a special class of hyperbolic systems with relaxation was developed in view of the numerical approximation of discontinuous solutions of conservation laws. The 2×2 relaxation hyperbolic systems of conservation laws were first analyzed by Liu (see [20]), who justified some nonlinear stability criteria for diffusion waves, expansion waves and traveling waves. The main stability criterion founded by Liu is the so-called subcharacteristic condition, which for the previous system reads $|f'(u)| < \lambda$, for $|u| \leq M$, $M > 0$. In the framework of general quasilinear hyperbolic relaxation problems the subcharacteristic condition gives the correct limit in the general relaxation problems. The convergence of solutions to the Cauchy problem for these systems to the unique entropy solution of the Cauchy problem for limit equation and for arbitrary large initial data was first given in [30] and [32] by using the monotonicity methods. Let us consider the functions

$$M^{\pm}(u) = \frac{1}{2}(u \mp \frac{f(u)}{\lambda}).$$

The most important role played by subcharacteristic condition is the non negative monotonicity of functions $M^{\pm}(u)$. In fact, in [31] it has been proved that the monotonicity condition is guaranteed by subcharacteristic condition; this is important because in particular the monotonicity condition implies special stability properties of system (3.0.4); thanks to the monotonicity condition, it has been proved that the evolution operator associated to this problem is contractive in the L^1-norm and the system is quasimonotone. So, under monotonicity condition, problem has a unique uniformly bounded solution for each $\varepsilon > 0$. Moreover, in [31] it has also been proved that the sequence of solutions of (3.0.4), for each $\varepsilon > 0$ is a compact sequence in $C([0,T]; L^1_{loc}(\mathbb{R}^d))$, $d \geq 1$, $T > 0$.

This Chapter is organized as follows: in the first section, we will give a presentation of the problem. The second section is devoted to some a priori estimates for the solution of the problem, while in the third section we will prove the existence and uniqueness of the solution on a network. Finally, in the last section we will study the asymptotic behavior of the solutions when the initial data are small perturbations of a stationary solution.

3.1 Formulation of the problem

Let $\mathcal{N}$ be an oriented network, composed of M arcs I_i and ν nodes $N_h \in \mathcal{V}_{int}$, according to definition (2.5.1). Here we study the movement of densities $u_i^{\pm}$ of problem (3.0.1) on a network in presence of a chemoattractant with constant gradient α and which move with constant velocities $\pm\lambda_i$ on each arc I_i. In particular, the velocities $\pm\lambda_i$ refer to functions $u_i^{\pm}$ respectively. For each $I_i \in \mathcal{E}_{N_h}$, according to definition (2.5.1), let E_{N_h} be the entering arcs in the node N_h, U_{N_h} be the outgoing ones, and for each fixed node N_h, $\mathcal{M}_{N_h} = E_{N_h} \cup U_{N_h}$. We recall that we denote by a_i the external vertices, or boundary points. Let $g^i_{\pm}(u_i^+, u_i^-)$ be the following functions

$$g^i_{\pm}(u_i^+, u_i^-) := \mp\left(\frac{1}{2\lambda_i}((\alpha+\lambda_i)u_i^- - (\lambda_i - \alpha)u_i^+)\right). \tag{3.1.1}$$

We consider the linear hyperbolic problem defined on the sets $I_i \times [0,T]$, with $T > 0$ and $i \in \mathcal{M}$,

$$\begin{cases} u^+_{i,t} + \lambda_i u^+_{i,x} = \frac{1}{2\lambda_i}((\alpha+\lambda_i)u_i^- - (\lambda_i-\alpha)u_i^+), \\ \\ u^-_{i,t} - \lambda_i u^-_{i,x} = -\frac{1}{2\lambda_i}((\alpha+\lambda_i)u_i^- - (\lambda_i-\alpha)u_i^+), \end{cases} \tag{3.1.2}$$

in which we assume that the subcharacteristic condition $\alpha < |\lambda_i|$ holds for each $i \in \mathcal{M}$. We couple the system with initial conditions

$$u_i^{\pm}(x,0) = u^{\pm}_{i,0} \in BV(I_i), \tag{3.1.3}$$

and boundary conditions as follows: on the external boundary points a_i we impose

$$u_i^+ = \beta_{a_i} u_i^- + b_{a_i}(t), \tag{3.1.4}$$

if $i \in E_N$,

$$u_i^- = \beta_{a_i} u_i^+ + b_{a_i}(t), \tag{3.1.5}$$

if $i \in U_N$, where $b_{a_i} \in L^1([0,T])$ and the coefficients $\beta_{a_i} \in \mathbb{R}$.

Now we have to formulate the transmission condition at the nodes. This is the crucial point because the behaviour of solutions to the problem strictly depends by these conditions. So we introduce some transmission coefficients $\xi_{i,j} \in [0,1]$ for every index i,j belonging to E_{N_h} and U_{N_h}, and we impose the following boundary transmission conditions on each node N_h: if $i \in E_{N_h}$, for almost every $t \geq 0$,

$$u_i^-(N_h,t) = \sum_{j \in E_{N_h}} \xi_{i,j} u_j^+(N_h,t) + \sum_{j \in U_{N_h}} \xi_{i,j} u_j^-(N_h,t) \tag{3.1.6}$$

while if $i \in U_{N_h}$, for almost every $t \geq 0$,

$$u_i^+(N_h,t) = \sum_{j \in E_{N_h}} \xi_{i,j} u_j^+(N_h,t) + \sum_{j \in U_{N_h}} \xi_{i,j} u_j^-(N_h,t). \tag{3.1.7}$$

The coefficients $\xi_{i,j}$ represent the probability to choose j-th root going from the i-th.

Here we are not interested in the continuity of density functions $u_i^\pm$, but we are interested in the continuity of the total flux on each node N_h of the network, which yields

$$\sum_{i \in E_{N_h}} \lambda_i (u_i^+(N_h,t) - u_i^-(N_h,t)) = \sum_{i \in U_{N_h}} \lambda_i (u_i^+(N_h,t) - u_i^-(N_h,t)). \tag{3.1.8}$$

By conditions (3.1.6) and (3.1.7) we have that transmission coefficients have to verify

$$\sum_{i \in \mathcal{M}_{N_h}} \lambda_i \xi_{i,j} = \lambda_j, \tag{3.1.9}$$

for each $j \in \mathcal{M}_{N_h}$, and on each node N_h.

Let us now introduce the following definition.

Definition 3.1.1. *Let $\mathcal{N}$ be a network as in definition (*2.5.1*). Let $u_0^{\pm} \in BV(I_i)$, and let $u_i^{\pm} : I_i \times [0,T] \longrightarrow \mathbb{R}$ ($T > 0$), $i \in \mathcal{M}$. For each $i \in \mathcal{M}$, we say that a couple of functions $(u_i^+, u_i^-) \in C([0,T]; BV(I_i)) \times C([0,T]; BV(I_i))$ is a BV-solution of the problem* 3.1.2 *defined on a network $\mathcal{N}$, on $I_i \times [0,T]$ ($T > 0$) if we have*

$$(3.1.10) \qquad \int_0^T \int_{I_i} (u_i^{\pm} \frac{\partial \phi_i}{\partial t} \pm \lambda \frac{\partial \phi_i}{\partial x}) dx dt + \int_{I_i} u_{i,0}^{\pm}(x) \phi_i(x,0) dx$$
$$+ \int_0^T (u_i^{\pm})_{|\Gamma} \phi_{i|\Gamma} dt = \int_0^T \int_{I_i} g_i^{\pm}(u_i^+, u_i^-) \phi_i(x,t) dx dt,$$

*where $\Gamma = (\partial I_i \times [0,T]) = (\{a_i, N_h\} \times [0,T])$, for each $N_h \in \mathcal{V}_{int}$, for each $\phi_i \in C_0^\infty(I_i, \mathbb{R}^+)$, and moreover the functions (u_i^+, u_i^-) verify the conditions (*3.1.4*), (*3.1.5*), (*3.1.6*), (*3.1.7*).*

According to definition (1.2.3) given in Chapter 1 we can define the traces $u_i^{\pm}(a_i, t)$ and $u_i^{\pm}(N_h, t)$ of functions $u_i^{\pm}(x,t)$ on $I_i \times [0,T]$. Given a solution $u_i^{\pm} : I_i \times [0,T] \longrightarrow \mathbb{R}$, then a solution of the problem (3.1.2) on the network is the vector function $U := (u_1^{\pm}, \ldots, u_M^{\pm})$.

The aim of this Chapter is to prove the following result.

Theorem 3.1.1. *(Global existence and uniqueness) Let $\mathcal{N}$ be an oriented network of arcs I_i, nodes $N \in \mathcal{V}_{int}$, and boundary points $a_i \in \mathcal{V}_{ext}$, according to definition (*2.5.1*). Let us consider the problem* (3.1.2)*, together with initial data (*3.1.3*), boundary conditions (*3.1.4*), (*3.1.5*), (*3.1.6*), (*3.1.6*), and transmission coefficients verifying $0 \leq \xi_{i,j} \leq 1$ for each $i, j \in \mathcal{M}$. If*

$$(3.1.11) \qquad |\beta_{a_i}| \leq 1,$$

then for each $T > 0$ there exists a unique global solution of (3.1.2) *on $\mathcal{N}$, $u_i^{\pm} \in C([0,T]; BV(I_i))$, $i \in \mathcal{M}$.*

3.2 A priori estimates and uniqueness

We need some a priori estimates we shall use later. Let $\mathcal{N}$ be an oriented network composed of M arcs I_i, ν nodes $N_h \in \mathcal{V}_{int}$ and boundary points a_i. From now on, let $u_0^{\pm} \in BV(I_i)$, $b_{a_i} \in BV([0,T))$ ($T > 0$).

Proposition 3.2.1. *Let $u_0^{\pm} \in BV(I_i)$, and $b_{a_i} \in BV([0,T))$, $T > 0$. Given the problem (*3.1.2*), with initial conditions (*3.1.3*), and with boundary conditions (*3.1.4*), (*3.1.5*), (*3.1.6*), (*3.1.6*) and assuming the coefficients $|\beta_{a_i}| \leq 1$, then a BV-solution of this problem verify*

$$(3.2.1)\qquad \sum_{h=1}^{\nu}\Big(\sum_{i\in\mathcal{M}_{N_h}}\int_{I_i}|u_i^+|+|u_i^-|dx\Big)\le\sum_{h}^{\nu}\Big(\sum_{i\in\mathcal{M}_{N_h}}\int_{I_i}|u_{i0}^+|+|u_{i0}^-|dx$$

$$+\sum_{i\in\mathcal{M}_N}\lambda_i\int_0^T|b_{a_i}(t)|dt\Big),$$

where $1\le\nu\le M$ is the total number of nodes.

- *Proof.* We begin with a network composed of a single node N. So E is the set of entering roads in N while U is the set of outgoing ones.

  Thanks to Lemma (1.2.2) applied to BV-functions, by considering the absolute values of the functions $u_i^{\pm}$, solutions of the linear problem (3.1.2), we have that

$$(3.2.2)\qquad \partial_t(|u_i^+|+|u_i^-|)+\lambda\partial_x(|u_i^+|+|u_i^-|)\le 0,$$

Let $i\in E$. By integrating the previous equation on the set $[a_i,N]\times[0,T]$ and using the divergence theorem we have

$$\int_{I_i}|u_i^+|+|u_i^-|dx\le\int_{I_i}|u_{i0}^+|+|u_{i0}^-|dx$$

$$+\lambda_i\int_0^T(|u_i^+(a_i)|-|u_i^-(a_i)|)dt-\lambda_i\int_0^T(|u_i^+(N)|-|u_i^-(N)|)dt$$

$$\le\int_{I_i}|u_{i0}^+|+|u_{i0}^-|dx+\lambda_i\int_0^T(|\beta_{a_i}u_i^-+b_{a_i}(t)|-|u_i^-(a_i)|)dt$$

$$-\lambda_i\int_0^T(|u_i^+(N)|-|\sum_{j\in E}\xi_{i,j}u_j^+(N,t)+\sum_{j\in U}\xi_{i,j}u_j^-(N,t)|)dt$$

$$\le\int_{I_i}|u_{i0}^+|+|u_{i0}^-|dx+\lambda_i\int_0^T(|\beta_{a_i}|-1)|u_i^-(a_i)|dt+\int_0^T\lambda_i|b_{a_i}(t)|dt$$

$$-\lambda_i\int_0^T(|u_i^+(N)|-|\sum_{j\in E}\xi_{i,j}u_j^+(N,t)+\sum_{j\in U}\xi_{i,j}u_j^-(N,t)|)dt.$$

So, using the assumption that $|\beta_{a_i}|\le 1$ we have that

$$\int_{I_i}|u_i^+|+|u_i^-|dx\le\int_{a_i}^N|u_{i0}^+|+|u_{i0}^-|dx+\int_0^T\lambda_i|b_{a_i}(t)|dt$$

$$-\lambda_i\int_0^T(|u_i^+(N)|-|\sum_{j\in E}\xi_{i,j}u_j^+(N,t)+\sum_{j\in U}\xi_{i,j}u_j^-(N,t)|)dt.$$

Analogously, we have that, if $i \in U$,

$$\int_{I_i} |u_i^+|+|u_i^-|dx \leq \int_{I_i} |u_{i0}^+|+|u_{i0}^-|dx+\lambda_i \int_0^T (|\beta_{a_i}|-1)|u_+^i(a_i)|dt+\int_0^T \lambda_i |b_{a_i}(t)|dt$$

$$+\lambda_i \int_0^T (|\sum_{j\in E} \beta_{i,j} u_+^j(N,t) + \sum_{j \in U} \gamma_{i,j} u_-^j(N,t)| - |u_-^i(N,t)|)dt$$

$$\leq \int_{I_i} |u_{i0}^+| + |u_{i0}^-|dx + \int_0^T \lambda_i |b_{a_i}(t)|dt$$

$$+\lambda_i \int_0^T (|\sum_{j\in E} \xi_{i,j} u_j^+(N,t) + \sum_{j \in U} \xi_{i,j} u_j^-(N,t)| - |u_i^-(N,t)|)dt.$$

Now, summing up on $i \in \mathcal{M}$, we obtain that

$$\sum_{i\in\mathcal{M}} \int_{I_i} |u_i^+| + |u_i^-|dx$$

$$\leq \sum_{i\in\mathcal{M}} \int_{I_i} |u_{i0}^+| + |u_{i0}^-|dx + \sum_{i\in\mathcal{M}} \int_0^T \lambda_i |b_{a_i}(t)|dt$$

$$-\sum_{i\in E} \lambda_i \int_0^T |u_i^+(N,t)|dt - \sum_{i\in U} \lambda_i \int_0^T |u_i^-(N,t)|dt$$

$$+\sum_{i\in\mathcal{M}} \int_0^T \lambda_i (|\sum_{j\in E} \xi_{i,j} u_j^+(N,t) + \sum_{j\in U} \xi_{i,j} u_j^-(N,t)|)dt$$

$$\leq \sum_{i\in\mathcal{M}} \int_{I_i} |u_{i0}^+| + |u_{i0}^-|dx + \sum_{i\in\mathcal{M}} \int_0^T \lambda_i |b_{a_i}(t)|dt$$

$$-\sum_{i\in E} \lambda_i \int_0^T |u_i^+(N,t)|dt - \sum_{i\in U} \lambda_i \int_0^T |u_i^-(N,t)|dt$$

$$+\int_0^T \sum_{j\in E} \sum_{i\in\mathcal{M}} \lambda_i \xi_{i,j} |u_j^+(N,t)| + \sum_{j\in U} \sum_{i\in\mathcal{M}} \lambda_i \xi_{i,j} |u_-^j(N,t)|)dt.$$

By the flux conservation we get

$$\sum_{i\in\mathcal{M}} \int_{I_i} |u_i^+| + |u_i^-|dx \leq \sum_{i\in\mathcal{M}} \int_{I_i} |u_{i0}^+| + |u_{i0}^-|dx$$

$$+\sum_{i\in\mathcal{M}} \int_0^T \lambda_i |b_{a_i}(t)|dt - \sum_{i\in E} \lambda_i \int_0^T |u_+^i(N,t)|dt$$

$$-\sum_{i\in U}\lambda_i\int_0^T |u_-^i(N,t)|dt+\int_0^T \sum_{j\in E}\lambda_j|u_+^j(N,t)|+\sum_{j\in U}\lambda_j|u_-^j(N,t)|)dt,$$

and we get the proof for a network with a single node.

Now, let $\mathcal{N}$ be a general network with arcs I_i, ν nodes $N_h \in \mathcal{V}_{int}$, and boundary points a_i. Firstly, let us observe that in a general network we have intervals as $[a_i, N_h]$, $[N_h, a_i]$, or intervals that connect two different nodes, $[N_h, N_k]$. Fix a node $N_h \in \mathcal{V}_{int}$. Following the passages of the previous proof with a single node, and then summing up on all nodes $N_h \in \mathcal{V}_{int}$, we have that

$$\sum_{h=1}^{\nu}(\sum_{i\in\mathcal{M}_{N_h}}\int_{I_i}|u_i^+|+|u_i^-|dx)\le\sum_{h=1}^{\nu}(\sum_{i\in\mathcal{M}_{N_h}}\int_{I_i}|u_{i0}^+|+|u_{i0}^-|dx$$

$$+\sum_{i\in\mathcal{M}_{N_h}}\int_0^T\lambda_i|b_{a_i}(t)|dt$$

$$+\lambda_i\int_0^T(|\sum_{j\in E_{N_h}}\xi_{i,j}u_j^+(N_h,t)+\sum_{j\in U_{N_h}}\xi_{i,j}u_j^-(N_h,t)|-|u_i^-(N_h,t)|)dt$$

$$-\lambda_i\int_0^T(|u_i^+(N_h)|-|\sum_{j\in E_{N_h}}\xi_{i,j}u_j^+(N_h,t)+\sum_{j\in U_{N_h}}\xi_{i,j}u_j^-(N_h,t)|))dt.$$

We observe that in the above inequality the arcs whose boundary points are both nodes N_h, N_k, i.e., $I_i = [N_h, N_k]$, are considered twice. But by conditions of flux conservation on each node terms on nodes disappear and we get the proof. □

Proposition 3.2.2. *Let $\mathcal{N}$ be a network as in definition* 2.5.1. *Let $T > 0$, and let $u_0^\pm \in BV(I_i)$, $v_0^\pm \in BV(I_i)$, $b_{a_i} \in BV((0,T))$, and $\overline{b}_{a_i} \in BV((0,T))$, for $i \in \mathcal{M}$. Let $\beta_{a_i} \in \mathbb{R}$, and $\overline{\beta}_{a_i} \in R$, $\beta, |\overline{\beta}_{a_i}| \in [0,1]$. Let us consider problem (*3.1.2*) in $(I_i\times[0,T])$, and let us assume that $u_i^\pm$, $v_i^\pm$ are BV solutions of this problem, with initial condition $u_i^\pm(x,0) = u_{i,0}^\pm(x)$, $v_i^\pm(x,0) = v_{i,0}^\pm(x)$ respectively, and respective boundary conditions $u_i^+(a_i) = \beta_{a_i}u_i^- + b_{a_i}(t)$, and $v_i^+(a_i) = \overline{\beta}_{a_i}v_i^- + \overline{b}_{a_i}(t)$, if $i \in E_{N_h}$, and $u_i^-(a_i) = \beta_{a_i}u_i^+ + b_{a_i}(t)$, $v_i^-(a_i) = \overline{\beta}_{a_i}v_i^+ + \overline{b}_{a_i}(t)$, if $i \in U_{N_h}$, and conditions on the node (*3.1.6*) and (*3.1.7*) for both. Then for each $i \in \mathcal{M}$, we have*

(3.2.3)
$$\sum_{h=1}^{\nu}\sum_{i\in\mathcal{M}_{N_h}}\int_{I_i}|u_i^+-v_i^+|+|u_i^--v_i^-|dx\le\sum_{h=1}^{\nu}(\sum_{i\in\mathcal{M}_{N_h}}\int_{I_i}|u_{i0}^+-v_{i0}^+|+|u_{i0}^--v_{i0}^-|dx$$

$$+\sum_{i\in\mathcal{M}_{N_h}}\lambda_i\int_0^T|b_{a_i}(t)-\overline{b}_{a_i}(t)|dt).$$

Proof. Let us define $w_i^{\pm}:=u_i^{\pm}-v_i^{\pm}$. They verify problem (3.1.2) with initial conditions $w_{\pm}^i(x,0)=u_{0\pm}^i(x)-v_{0\pm}^i(x)$, and boundary conditions, if $i\in E_N$

$$w_+^i=\beta_{a_i}u_-^i-\overline{\beta}_{a_i}v_-^i+b_{a_i}(t)-\overline{b}_{a_i}(t),$$

and if $i\in U_N$

$$w_-^i=\beta_{a_i}u_+^i-\overline{\beta}_{a_i}v_+^i+b_{a_i}(t)-\overline{b}_{a_i}(t),$$

and the transmission conditions (3.1.6) and (3.1.7). Proof follows from the previous Proposition (3.2.1). □

From the previous result, it easy to show the following corollary.

Corollary 3.2.1. *(Comparison and uniqueness) In the same assumption of the previous proposition, if we have that*

$$u_{i,0}^{\pm}(x)\le v_{i,0}^{\pm}(x),$$

almost everywhere in I_i, and $\beta_{a_i}\le\overline{\beta_{a_i}}$, and $b_{a_i}(t)\le\overline{b_{a_i}(t)}$, almost everywhere in $[0,T]$, then

$$u_i^{\pm}\le v_i^{\pm}, \tag{3.2.4}$$

almost everywhere in $I_i\times[0,T]$, and for every $i\in\mathcal{M}$. In particular, from (3.2.4), we obtain the uniqueness to the solution of problem (3.1.2).

Now we estimate the time derivatives of solutions.

Proposition 3.2.3. *Let $\mathcal{N}$ be an oriented network according to definition (2.5.1). Given the linear problem (3.1.2), with initial conditions (3.1.3), and boundary conditions (3.1.4), (3.1.5), (3.1.6), and (3.1.7) and assumed that the coefficients $|\beta_{a_i}|\le 1$, then every BV-solution to the problem verifies*

$$\sum_{h=1}^{\nu}(\sum_{i\in\mathcal{M}_{N_h}}\int_{I_i}|\partial_t u_i^+|+|\partial_t u_i^-|dx) \tag{3.2.5}$$

$$\le\sum_{h=1}^{\nu}(\sum_{i\in\mathcal{M}_{N_h}}\int_{I_i}|g_i^+(x,0)-\lambda_i\partial_x u_{i0}^+|+|g_i^-(x,0)+\lambda_i\partial_x u_{i0}^-|dx$$

$$+\sum_{i\in\mathcal{M}_{N_h}}\lambda_i\int_0^T|b'_{a_i}(t)|dt)$$

Proof. Following proof of Proposition (3.2.1), we begin considering a network composed of a single node N. Let us define the functions $w_i^{\pm}(x,t) := \partial_t u_i^{\pm}(x,t)$. Then, from the linearity of our problem, we have that these functions verify

$$\begin{cases} w_i^+{}_t + \lambda_i w_i^+{}_x = g^+(w_i^+, w_i^-) \\ w_i^-{}_t - \lambda_i w_i^-{}_x = g^-(w_i^+, w_i^-), \end{cases}$$

for each $i \in \mathcal{M}$. Moreover the corresponding initial conditions are $w_i^{\pm}(x,0) = g^{\pm}(u_{i,0}^+, u_{i,0}^-) \mp \lambda \partial_x u_{i,0} \pm (x)$, while they verify on the boundary point a_i, if $i \in E$,

$$w_i^+(a_i) = \beta_{a_i} w_i^-(a_i) + b'_{a_i}(t),$$

and if $i \in U$,

$$w_i^-(a_i) = \beta_{a_i} w_i^+(a_i) + b'_{a_i}(t),$$

while on the node N, if $i \in E$

$$w_i^-(N,t) = \sum_{j \in E} \xi_{i,j} w_j^+(N,t) + \sum_{j \in U} \xi_{i,j} w_j^-(N,t)$$

and, if $i \in U$

$$w_i^+(N,t) = \sum_{j \in E} \xi_{i,j} w_j^+(N,t) + \sum_{j \in U} \xi_{i,j} w_j^-(N,t).$$

Considering the absolute values of the functions, and thanks to Lemma (1.2.2), we have that

$$\partial_t(|w_i^+| + |w_i^-|) + \lambda \partial_x(|w_i^+| + |w_i^-|) \leq 0.$$

Let us consider $i \in E$.

Integrating on the set $[a_i, N] \times [0,T]$ we have

$$\int_{a_i}^{N} |w_i^+| + |w_i^-| dx \leq \int_{a_i}^{N} |w_{i0}^+| + |w_{i0}^-| dx$$

$$+\lambda_i \int_0^T (|w_i^+(a_i)| - |w_i^-(a_i)|) dt - \lambda_i \int_0^T (|w_i^+(N)| - |w_i^-(N)|) dt$$

and, using the boundary condition in a_i,

$$\int_{a_i}^{N} |w_i^+| + |w_i^-| dx \leq \int_{a_i}^{N} |w_{i0}^+| + |w_{i0}^-| dx + \int_0^T \lambda_i |b'_{a_i}(t)| dt$$

$$-\lambda_i \int_0^T (|w_i^+(N)| - |\sum_{j \in E} \xi_{i,j} w_j^+(N,t) + \sum_{j \in U} \xi_{i,j} w_j^-(N,t)|) dt;$$

Analogously, if $i \in U$, we obtain that

$$\int_{a_i}^{N} |w_i^+| + |w_i^-| dx \leq \int_{a_i}^{N} |w_{i,0}^+| + |w_{i,0}^-| dx + \int_0^T \lambda_i |b'_{a_i}(t)| dt$$

$$+\lambda_i \int_0^T (|\sum_{j\in E} \xi_{i,j} u_j^+(N,t) + \sum_{j\in U} \xi_{i,j} u_j^-(N,t) - u_-^i(N,t)|) dt.$$

If we sum on $i \in E \cup U$, we have

$$\sum_{i\in E\cup U} \int_{a_i}^{N} |w_i^+| + |w_i^-| dx \leq \sum_{i\in E\cup U} \int_{a_i}^{N} |w_{i,0}^+| + |w_{i,0}^-| dx + \sum_{i\in E\cup U} \int_0^T \lambda_i |b'_{a_i}(t)| dt$$

$$-\sum_{i\in E} \lambda_i \int_0^T |w_i^+(N,t)| dt - \sum_{i\in U} \lambda_i \int_0^T |w_i^-(N,t)| dt$$

$$+\sum_{i\in E\cup U} \int_0^T \lambda_i (|\sum_{j\in E} \xi_{i,j} w_j^+(N,t) + \sum_{j\in U} \xi_{i,j} w_j^-(N,t)|) dt.$$

By arguing as in the proof (3.2.1) we get the estimate on a single node an moreover in a general network. □

Now we find a priori estimate for the spatial derivatives of a solution to problem (3.1.2).

Proposition 3.2.4. *Let $\mathcal{N}$ be an oriented network according to definition (2.5.1). Given the problem (3.1.2), with initial and boundary conditions (3.1.3), (3.1.4), (3.1.5), (3.1.6), (3.1.7) and assumed that coefficients $|\beta_{a_i}| \leq 1$, for each index i, then the following estimate holds:*

(3.2.6)

$$\sum_{h=1}^{\nu} (\sum_{i\in\mathcal{M}_{N_h}} \int_{I_i} |\partial_x u_i^+| + |\partial_x u_i^-| dx) \leq \sum_{h=1}^{\nu} (\sum_{i\in\mathcal{M}_{N_h}} \int_{I_i} |\partial_x u_{i,0}^+| + |\partial_x u_{i,0}^-| dx$$

$$+ \sum_{i\in\mathcal{M}_{N_h}} \int_0^T |b'_{a_i}(t)| dt)$$

Proof. Following proof of Proposition (3.2.1), let us consider firstly a network composed of a single node N. Let us define the functions $v_i^\pm(x,t) := \partial_x u_i^\pm(x,t)$. From the linearity of the problem we have that $v_i^\pm(x,t)$ verify

$$\begin{cases} v_{i,t}^+ + \lambda_i v_{i,x}^+ = g^+(v_i^+, v_i^-) \\ v_{i,t}^- - \lambda_i v_{i,x}^- = g^-(v_i^+, v_i^-), \end{cases}$$

for each $i \in \mathcal{M}$, and we have that $v_i^\pm(x,0) = \partial_x u_{i,0}^\pm(x)$. Moreover, on the boundary points, we have that, if $i \in E$,

$$v_i^+(a_i) = \frac{1}{\lambda_i}(g_i^+ - \beta_{a_i} u_{i,t}^-(a_i) - b'_{a_i}(t)),$$

if $i \in U$,

$$v_i^-(a_i) = \frac{1}{\lambda_i}(-g^- + \beta_{a_i} u_{i,t}^+(a_i) + b'_{a_i}(t)),$$

and conditions on the node accordingly modified. As in proposition (3.2.1) we have that the functions $v_i^{\pm}(x,t)$ verify

$$\int_{a_i}^N |v_i^+| + |v_i^-| dx \le \int_{a_i}^N |v_{i,0}^+| + |v_{i,0}^-| dx$$

$$+\lambda_i \int_0^T (|v_i^+(a_i)| - |v_i^-(a_i)|)dt - \lambda_i \int_0^T |v_i^+(N)| - |v_i^-(N)| dt.$$

In the following, we denote $u_{i,t}^{\pm} := \partial_t u_i^{\pm}$. Using the boundary and initial conditions we can write the previous inequality as

$$\int_{a_i}^N |v_i^+| + |v_i^-| dx \le \int_{a_i}^N |v_{i0}^+| + |v_{i0}^-| dx$$

$$+ \int_0^T |g_i^+(u_+^i(a_i), u_-^i(a_i)) - \beta_{a_i} \partial_t u_-^i| dt + \int_0^T |b'_{a_i}(t)| dt$$

$$- \int_0^T |g_i^-(u_+^i(a_i), u_-^i(a_i)) + \partial_t u_-^i| dt$$

$$-\lambda_i \int_0^T |v_i^+(N)| - |v_i^-(N)| dt \le \int_{a_i}^N |v_{i,0}^+| + |v_{i,0}^-| dx + \int_0^T |b'_{a_i}(t)| dt$$

$$+ \int_0^T |\beta_{a_i} \partial_t u_i^- + g^-(u_i^+(a_i), u_i^-(a_i))|$$

$$- \int_0^T |\partial_t u_i^- + g^-(u_i^+(a_i), u_i^-(a_i))| dt - \lambda_i \int_0^T |v_i^+(N)| - |v_i^-(N)| dt$$

$$\le \int_{a_i}^N |v_{i,0}^+| + |v_{i,0}^-| dx$$

$$+ \int_0^T |b'_{a_i}(t)| dt + \int_0^T \max\{|\beta_{a_i}|, 1\} |\partial_t u_i^- + g^-(u_i^+(a_i), u_i^-(a_i))|$$

$$-|\partial_t u_i^- + g^-(u_i^+(a_i), u_i^-(a_i))| dt - \lambda_i \int_0^T |v_i^+(N)| - |v_i^-(N)| dt$$

$$\le \int_{a_i}^N |v_{i,0}^+| + |v_{i,0}^-| dx + \int_0^T |b'_{a_i}(t)| dt - \lambda_i \int_0^T |v_i^+(N)| - |v_i^-(N)| dt.$$

If $i \in U$, analogously we have

$$\int_{I_i} |v_i^+| + |v_i^-| dx \le \int_{I_i} |v_{i0}^+| + |v_{i0}^-| dx + \int_0^T |b'_{a_i}(t)| dt$$
$$+\lambda_i \int_0^T |v_i^+(N)| - |v_i^-(N)| dt.$$

Using the equations for $\lambda_i u_x^{\pm,i}$, and adding the above inequalities on $i \in \mathcal{M}$, we obtain

$$\sum_{i\in\mathcal{M}} \int_{I_i} |v_i^+| + |v_i^-| dx \le \sum_{i\in\mathcal{M}} \int_{I_i} |v_{i0}^+| + |v_{i0}^-| dx + \sum_{i\in\mathcal{M}} \int_0^T |b'_{a_i}(t)| dt$$
$$-\sum_{i\in E} \int_0^T \lambda_i(|v_i^+(N)| - |v_i^-(N)|) dt + \sum_{i\in U} \int_0^T \lambda_i(|v_i^+(N)| - |v_i^-(N)|) dt$$
$$= \sum_{i\in\mathcal{M}} \int_{I_i} |v_{i0}^+| + |v_{i0}^-| dx + \sum_{i\in\mathcal{M}} \int_0^T |b'_{a_i}(t)| dt$$
$$-\sum_{i\in E} \frac{1}{\lambda_i} \int_0^T (|\lambda_i(-u_{i,t}^+ + g^+)| - |\lambda_i(u_{i,t}^- + g^+)|) dt + \sum_{i\in U} \int_0^T \frac{1}{\lambda_i} (|\lambda_i(-u_{i,t}^+ + g^+)| - |\lambda_i(u_{i,t}^- + g^+)|) dt.$$

Now, substituting the boundary conditions on the node N in $u_{i,t}^-$, if $i \in E$, and in $u_{i,t}^+$, if $i \in U$, we obtain that

$$\sum_{i\in\mathcal{M}} \int_{I_i} |v_i^+| + |v_i^-| dx \le \sum_{i\in\mathcal{M}} \int_{I_i} |v_{i0}^+| + |v_{i0}^-| dx + \sum_{i\in\mathcal{M}} \int_0^T |b'_{a_i}(t)| dt \tag{3.2.7}$$
$$-\sum_{i\in E} \frac{1}{\lambda_i} \int_0^T |\lambda_i(u_{i,t}^+ + g^+)| dt - \sum_{i\in U} \frac{1}{\lambda_i} \int_0^T |\lambda_i(u_{i,t}^- + g^+)| dt$$
$$+\sum_{i\in E} \frac{1}{\lambda_i} \int_0^T |\sum_{j\in E} \lambda_i \xi_{i,j} u_{j,t}^+ + \sum_{j\in U} \lambda_i \xi_{i,j} u_{j,t}^- + \lambda_i g^+| dt$$
$$+\sum_{i\in U} \frac{1}{\lambda_i} \int_0^T |\sum_{j\in E} \lambda_i \xi_{i,j} u_{j,t}^+ + \sum_{j\in U} \lambda_i \xi_{i,j} u_{j,t}^- + \lambda_i g^+| dt$$
$$\le \sum_{i\in\mathcal{M}} \int_{I_i} |v_{i0}^+| + |v_{i0}^-| dx + \sum_{i\in\mathcal{M}} \int_0^T |b'_{a_i}(t)| dt$$
$$-\sum_{i\in E} \frac{1}{\lambda_i} \int_0^T |\lambda_i(u_{i,t}^+ + g^+)| dt - \sum_{i\in U} \frac{1}{\lambda_i} \int_0^T |\lambda_i(u_{i,t}^- + g^+)| dt$$

$$+\sum_{i\in\mathcal{M}}\frac{1}{\lambda_i}\int_0^T|\sum_{j\in E}\lambda_i\xi_{i,j}u^+_{j,t}+\sum_{j\in U}\lambda_i\xi_{i,j}u^-_{j,t}+\lambda_i g^+|dt.$$

Now, using the conditions of flux conservation for the coefficients $\xi_{i,j}$, we get the claim. Then, following the proof of (3.2.1), we get the estimate on a general network. □

3.3 Existence of the solution on a network

The aim of this section is to prove Theorem (3.1.1). So, let $\mathcal{N}$ be a network composed of M arcs I_i, ν nodes $N_h \in \mathcal{V}_{int}$ and boundary points a_i, according to definition (2.5.1). We recall that $\mathcal{V}$ is the set of all vertices of $\mathcal{N}$, E_{N_h} is the set of entering arcs a node N_h, while U_{N_h} is the set of outgoing ones and $\mathcal{M}_{N_h} = E_{N_h} \cup U_{N_h}$; so the total number of arcs is given by $\mathcal{M} = \bigcup_{h=1}^{\nu} \mathcal{M}_{N_h}$. Let us consider the linear problem

$$\text{(3.3.1)} \qquad \begin{cases} u^+_{i,t} + \lambda_i u^+_{i,x} = \frac{1}{2\lambda_i}((\alpha+\lambda_i)u^-_i - (\lambda_i-\alpha)u^+_i) \\ u^-_{i,t} - \lambda_i u^-_{i,x} = -\frac{1}{2\lambda_i}((\alpha+\lambda_i)u^-_i - (\lambda_i-\alpha)u^+_i), \end{cases}$$

$i \in \mathcal{M}$, with initial conditions $u^\pm_i(x,0) = u^\pm_{i,0}(x)$ and boundary conditions as follows: on the boundary points a_i we set

$$\text{(3.3.2)} \qquad u^+_i = \beta_{a_i} u^-_i + b_{a_i}(t),$$

if $i \in E_N$, and

$$\text{(3.3.3)} \qquad u^-_i = \beta_{a_i} u^+_i + b_{a_i}(t),$$

if $i \in U_N$, while on each node N_h we set

$$\text{(3.3.4)} \qquad u^-_i(N_h,t) = \sum_{j\in E_{N_h}} \xi_{i,j} u^+_j(N_h,t) + \sum_{j\in U_{N_h}} \xi_{i,j} u^-_j(N_h,t)$$

if $i \in E_{N_h}$, and

$$\text{(3.3.5)} \qquad u^+_i(N_h,t) = \sum_{j\in E_{N_h}} \xi_{i,j} u^+_j(N_h,t) + \sum_{j\in U_{N_h}} \xi_{i,j} u^-_j(N_h,t),$$

if $i \in U_{N_h}$; we assume the continuity of the total flux on each node N_h, which leads

$$\text{(3.3.6)} \qquad \sum_{i\in\mathcal{M}_{N_h}} \lambda_i \xi_{i,j} = \lambda_j,$$

for each $j \in \mathcal{M}_{N_h}$. Moreover, we recall the we impose the continuity of the total flux on each node N_h, i.e.,

$$\sum_{i \in E_{N_h}} \lambda_i(u_i^+(N,t) - u_i^-(N,t)) = \sum_{i \in U_{N_h}} \lambda_i(u_i^+(N,t) - u_i^-(N,t)). \tag{3.3.7}$$

This condition forces the transmission coefficients $\xi_{i,j}$ to verify

$$\sum_{i \in \mathcal{M}_{N_h}} \lambda_i \xi_{i,j} = \lambda_j, \tag{3.3.8}$$

for each $j \in \mathcal{M}_{N_h}$.

3.3.1 Case of one arc

We begin by reviewing the simple case $M = 1$, i.e. we study the existence of the solution of our problem defined in the single interval $[a, N] \times [0, T]$. In [33] authors studied a similar problem defined on the set $[0, +\infty) \times [0, T]$ but this is easily extendible to a compact set $[a, N] \times [0, T]$. They consider the hyperbolic equation

$$u_t + (f(u))_x = 0,$$

for $(x, t) \in [0, +\infty) \times [0, T]$, $T > 0$, and for a smooth flux function f, with initial condition $u(x, 0) = u_0(x)$ and boundary condition $u(0, t) = a_0(t)$. He proves uniform a priori estimates and convergence of relaxation approximation as the relaxation parameter tends to zero. So they prove the global existence of solution of this problem. As a consequence of the result shown in [33] we have that the following Proposition holds.

Proposition 3.3.1. *Let $a^\pm \in L^1([0, T])$, $T > 0$. Given the problem (3.1.2) defined on $[a, N] \times [0, T)$, with initial and boundary conditions $u^\pm(., 0) = u_0^\pm \in BV([a, N])$, and boundary conditions*

$$u^+(a, t) = \beta^+(t) u^-(a, t) + b^+(t),$$

and

$$u^-(N, t) = \beta^-(t) u^+(N, t) + b^-(t)$$

then, there exists a unique solution $u^\pm \in C([0, T]; BV([a, N]))$, $T > 0$.

3.3.2 The general case

The aim of this section is to prove the existence of a bounded variation solution of the problem (3.1.2) defined on a network $\mathcal{N}$ with ν nodes $N_h \in \mathcal{V}_{int}$ and M arcs I_i.

Let $N_h \in \mathcal{V}_{int}$. Let us consider a finite partition of the interval $[0,T]$ of length Δt, with $\Delta t = \frac{T}{K} = |t_{n+1} - t_n|$, $n = 0, \ldots, K$, $K \in N$, and $t_0 = 0$, $t_K = T$.

Now let us consider the problem (3.1.2) defined on the set $I_i \times [0, t_1)$, for each $i \in \mathcal{M}$. For this problem, we impose initial conditions $u_i^{\pm,1}(x,0) = u^{\pm}(x,0)$, and the following boundary conditions: on the external boundary points a_i we impose

$$u_i^{+,1}(a_i,t) = \beta_{a_i} u_i^{-,1}(a_i,t) + b_{a_i}(t),$$

if $i \in E_{N_h}$, and

$$u_i^{-,1}(a_i,t) = \beta_{a_i} u_i^{+,1}(a_i,t) + b_{a_i}(t)$$

if $i \in U_{N_h}$.

Now, let us fix the transmission conditions on the node N_h and set $\Delta x_i = \frac{\lambda_i}{\Delta t}$. On the node N_h we impose, for each $t \in (0, \Delta t)$:

$$u_i^{-,1}(N_h,t) = \sum_{j \in E_{N_h}} \xi_{i,j} \frac{1}{|\Delta x|} \int_{N_h - \Delta x}^{N_h} u_{j,0}^{+}(x)dx + \sum_{j \in U_{N_h}} \xi_{i,j} \frac{1}{\Delta x} \int_{N_h}^{N_h + \Delta x} u_{j,0}^{-}(x)dx,$$

if $i \in E_{N_h}$ and,

$$u_i^{+,1}(N_h,t) = \sum_{j \in E_{N_h}} \xi_{i,j} \frac{1}{\Delta x} \int_{N_h - \Delta x}^{N_h} u_{j,0}^{+}(x)dx + \sum_{j \in U_{N_h}} \xi_{i,j} \frac{1}{\Delta x} \int_{N_h}^{N_h + \Delta x} u_{i,0}^{-}(x)dx$$

if $i \in U_{N_h}$. The above problem is well defined and from the previous section it admits a unique solution $u^{\pm,1} \in C([0,t_1); BV(I_i))$, $i \in \mathcal{M}_{N_h}$, $N_h \in \mathcal{V}_{int}$.

Now let us consider problem (3.1.2) defined on the sets $I_i \times [t_n, t_{n+1})$, $n \geq 1$, and for each index i, coupled with initial conditions $u_i^{\pm,n+1}(x,t_n) = u^{\pm,n}(x,t_n)$ and the following boundary conditions: on the boundary points a_i, on each interval $[t_n, t_{n+1})$, and for each $t \in [t_n, t_{n+1})$, we impose the boundary conditions

$$u_i^{+,n+1}(a_i,t) = \beta_{a_i} u_i^{-,n+1}(a_i,t) + b_{a_i}(t),$$

if $i \in E_{N_h}$, and

$$u_i^{-,n+1}(a_i,t) = \beta_{a_i} u_i^{+,n+1}(a_i,t) + b_{a_i}(t)$$

if $i \in U_{N_h}$, while on the nodes N_h we impose,

$$u_i^{-,n+1}(N_h,t) = \sum_{j\in E_{N_h}} \xi_{i,j} \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} u_j^{+,n}(N_h,s)ds + \sum_{j\in U_{N_h}} \xi_{i,j} \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} u_j^{-,n}(N_h,s)dsdt,$$

if $i \in E_{N_h}$ and,

$$u_{i,h}^{+,n+1}(N_h,t) = \sum_{j\in E_{N_h}} \xi_{i,j} \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} u_j^{+,n}(N_h,s)ds + \sum_{j\in U_{N_h}} \xi_{i,j} \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} u_i^{-,n}(N_h,s)ds$$

if $i \in U_{N_h}$. Because of the problem (3.1.2) defined on the set $I_i \times [0,t_1)$ admits a unique solution $u_i^{\pm,1} \in C([0,t_1);BV(I_i))$, $i \in \mathcal{M}_{N_h}$, by iteration we have that the (finite) sequence of problems defined on sets $I_i \times [t_n,t_{n+1})$ is well defined and each problem admits a unique solution $u_i^{\pm,n+1} \in C([t_n,t_{n+1});BV(I_i))$, $i \in \mathcal{M}_{N_h}$.

Now, for each fixed partition of length Δt we define on the set $I_i \times [0,T]$, for each index i, the functions $u_{i,\Delta t}^{\pm}(x,t)$ defined as

$$u_{i,\Delta t}^{\pm}(x,t) = u_i^{\pm,n+1}(x,t), \text{ if } t \in [t_n,t_{n+1}),\ n \geq 1, \tag{3.3.9}$$

and

$$u_{i,\Delta t}^{\pm}(x,t) = u_i^{\pm,1}(x,t), \text{ if } t \in [0,t_1).$$

By construction, the function $u_{i,\Delta t}^{\pm}(x,t)$ is a solution of (3.1.2) defined on $I_i \times [0,T)$, and it verifies initial conditions $u_{i,\Delta t}^{\pm}(x,0) = u_{i,0}^{\pm}(x)$ and boundary conditions on a_i

$$u_{i,\Delta t}^{+}(a_i,t) = \beta_{a_i} u_{i,\Delta t}^{-}(a_i,t) + b_{a_i}(t), \tag{3.3.10}$$

if $i \in E_{N_h}$, and

$$u_{i,\Delta t}^{-}(a_i,t) = \beta_{a_i} u_{i,\Delta t}^{+}(a_i,t) + b_{a_i}(t), \tag{3.3.11}$$

if $i \in U_{N_h}$, while on each node N_h it verifies, on each interval $[t_n,t_{n+1})$ and for each $t \in [t_n,t_{n+1})$ $(n \geq 1)$:

(3.3.12)

$$u_{i,\Delta t}^{-}(N_h,t) = \sum_{j\in E_{N_h}} \xi_{i,j} \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} u_{i,\Delta t}^{+}(N_h,s)ds + \sum_{j\in U_{N_h}} \xi_{i,j} \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} u_{i,\Delta t}^{-}(N_h,s)ds,$$

if $i \in E_{N_h}$ and

(3.3.13)

$$u_{i,\Delta t}^{+}(N_h,t) = \sum_{j\in E_{N_h}} \xi_{i,j} \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} u_{i,\Delta t}^{+}(N_h,s)ds + \sum_{j\in U_{N_h}} \xi_{i,j} \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} u_{i,\Delta t}^{-}(N_h,s)ds,$$

if $i \in U_{N_h}$.

Now we show the following result of compactness which will be useful in the follow. In the following Proposition we denote by $TV(u(x,t))$ the total variation with respect to the variable x of a function $u(x,t)$

Proposition 3.3.2. *(Compactness) Let $T > 0$, and let us consider coefficients $\beta_{a_i} \in \mathbb{R}$, $|\beta_{a_i}| \le 1$, and functions $b_{a_i} \in BV([0,T])$, and $u^{\pm}_{i,0} \in BV(I_i)$, for each $i \in \mathcal{M}$. For each fixed Δt, let the sequence $\{u^{\pm}_{i,\Delta t}\}$ be a solution of the problem (3.1.2), with the initial conditions $u^{\pm,n}_i(x,0) = u^{\pm}_{i,0}(x)$, and boundary conditions (3.3.10),(3.3.12), (3.3.11), (3.3.13). Moreover, let us assume that the transmission coefficients $0 \le \xi_{i,j} \le 1$ satisfy conditions (3.3.8).*

Then we have that

(3.3.14)
$$\sum_{h=1}^{\nu}\Big(\sum_{i\in\mathcal{M}_{N_h}}\int_{I_i}|u^+_{i,\Delta t}|+|u^-_{i,\Delta t}|dx\Big) \le \sum_{h=1}^{\nu}\Big(\sum_{i\in\mathcal{M}_{N_h}}\int_{I_i}|u^+_{i,0}|+|u^-_{i,0}|dx+\lambda_i\int_0^T|b_{a_i}(t)|dt\Big),$$

$$\sum_{h=1}^{\nu}\Big(\sum_{i\in\mathcal{M}_{N_h}}|TV(u^+_{i,\Delta t})(t)|+|TV(u^-_{i,\Delta t})(t)|\Big) \tag{3.3.15}$$

$$\le \exp(C\Delta t)\sum_{h=1}^{\nu}\sum_{i\in\mathcal{M}_{N_h}}|TV(u^+_{i,\Delta t})(t-\Delta t)|+|TV(u^-_{i,\Delta t})(t-\Delta t)|,$$

for a positive constant C.

Proof. Fix $N_h \in \mathcal{V}_{int}$ and let $i \in E_{N_h}$. Following the Proposition (3.2.1) we have that

$$\sum_{i\in E_{N_h}}\int_{I_i}|u^+_{i,\Delta t}|+|u^-_{i,\Delta t}|dx \le \sum_{i\in E_{N_h}}\int_{I_i}|u^+_{i,0}|+|u^-_{i,0}|dx$$

$$+\sum_{i\in E_{N_h}}\lambda_i\int_0^T(|u^+_{i,\Delta t}(a_i)|-|u^-_{i,\Delta t}(a_i)|)dt-\sum_{i\in E_{N_h}}\lambda_i\int_0^T(|u^-_{i,\Delta t}(N_h)|-|u^+_{i,\Delta t}(N_h)|)dt$$

$$\le \sum_{i\in E_{N_h}}\Big(\int_{I_i}|u^+_{i,0}|+|u^-_{i,0}|dx+\int_0^T\lambda_i|b_{a_i}(t)|dt-\lambda_i\int_0^T(|u^-_{i,\Delta t}(N_h)|-|u^+_{i,\Delta t}(N_h)|)dt\Big),$$

where in the last inequality we have used the definition of boundary conditions on a_i.

Analogously, if $i \in U_{N_h}$ we have that

$$\sum_{i\in U_{N_h}}\int_{I_i}|u^+_{i,\Delta t}|+|u^-_{i,\Delta t}|dx\leq\sum_{i\in U_{N_h}}\int_{I_i}|u^+_{i,0}|+|u^-_{i,0}|dx+\int_0^T\lambda_i|b_{a_i}(t)|dt$$

$$+\sum_{i\in U_{N_h}}\lambda_i\int_0^T|u^-_{i,\Delta t}(N_h)|-|u^+_{i,\Delta t}(N_h)|dt.$$

Now, summing up on $i\in\mathcal{M}_{N_h}$ we get

$$\sum_{i\in \mathcal{M}_{N_h}}\int_{I_i}|u^+_{i,\Delta t}|+|u^-_{i,\Delta t}|dx\leq\sum_{i\in \mathcal{M}_{N_h}}(\int_{I_i}|u^+_{i,0}|+|u^-_{i,0}|dx+\int_0^T\lambda_i|b_{a_i}(t)|dt)$$

$$-\sum_{i\in E_{N_h}}\lambda_i\int_0^T(|u^-_{i,\Delta t}(N_h)|-|u^+_{i,\Delta t}(N_h)|)dt),+\sum_{i\in U_{N_h}}\lambda_i\int_0^T|u^-_{i,\Delta t}(N_h)|-|u^+_{i,\Delta t}(N_h)|dt.$$

Using conditions of flux conservation (3.3.8) on transmission coefficients $\xi_{i,j}$, we get the proof for a single node. Then, summing up on the vertices $\mathcal{V}_{int}$ we get the proof on a general network. We recall that when we sum up on all the internal vertices $\mathcal{V}_{int}$, the arcs connecting two nodes are considered twice, but terms on each node vanish by condition of total flux conservation.

Now we get the estimate (3.3.15). Let $i\in E_{N_h}$ (calculus for $i\in U_{N_h}$ are analogous). First we have to observe that if $i\in E_{N_h}$, by definition of the sequence $\{u^\pm_{i,\Delta t}\}$, the functions $u^\pm_{i,\Delta t}(x,t)$ are not continuous through the characteristics straight lines outgoing from the points (N_h,t_n) and (a_i,t_n), $n\geq 1$.

Let us consider a partition of the interval I_i of length $\Delta x=[x_l,x_l+h]$, such that each interval $[x_l,x_l+h]$ intersects exactly one characteristic. Now, let us consider the explicit expression of the functions $u^\pm_{i,\Delta t}(x,t)$, in each interval $[t-\Delta t,t)$ of the partition of $[0,T]$, i.e.,

$$u^+_{i,\Delta t}(x,t)=u^+_{i,\Delta t}(x-\lambda_i\Delta t,t-\Delta t)\exp(K^+_i t)+K^-_i\int_{t-\Delta t}^t\exp(-K^+_i(t-s))u^-_{i,\Delta t}(x-\lambda_i(t-s),s)ds,$$

and

$$u^-_{i,\Delta t}(x,t)=u^-_{i,\Delta t}(x+\lambda_i\Delta t,t-\Delta t)\exp(K^-_i t)+K^+_i\int_{t-\Delta t}^t\exp(-K^-_i(t-s))u^+_{i,\Delta t}(x+\lambda_i(t-s),s)ds,$$

for some positive constants $K^\pm_i$, $i\in\mathcal{M}$.

Now let us consider the function $u^+_{i,\Delta t}(x,t)$. First we have to observe that if $x\in[a_i,a_i+\frac{\Delta x}{\lambda_i}]$, from boundary conditions we have that

$$u^+_{i,\Delta t}(x,t) = \beta_{a_i} u^-_{i,\Delta t}(a_i, \frac{a_i - x}{\lambda_i} + t)\exp(K^+_i(\frac{a_i - x}{\lambda_i})) + b_i(t)$$

$$+K^-_i \int_s^t \exp(-K^+_i(\frac{a_i - x}{\lambda_i} - \tau))u^-_{i,\Delta t}(a_i,\tau)d\tau,$$

for $s \in [t - \Delta t, t)$. Thus we have that

$$|u^+_{i,\Delta t}(x_l + h, t) - u^+_{i,\Delta t}(x_l, t)| \leq \exp(K^+_i t)TV(u^+_{i,\Delta t})(t - \Delta t)$$

$$+K^-_i \int_{t-\Delta t}^t \exp(-K^+_i(t - s))TV(u^-_{i,\Delta t})(s)ds$$

$$+\beta_{a_i}\exp(K^+_i(\frac{a_i - x}{\lambda_i}))TV(u^-_{i,\Delta t}(a_i,.)) + K^-_i \int_s^t \exp(-K^+_i(\frac{a_i - x}{\lambda_i} - \tau))TV(u^-_{i,\Delta t}(a_i,.))d\tau.$$

We have now to estimate the total variation in time for the function $u^-_{i,\Delta t}(a_i, s)$, $s \in [t - \Delta t, t)$. We have that

$$u^-_{i,\Delta t}(a_i, s) = u^-_i(a_i + \lambda_i[(s - t) + \Delta t], t - \Delta t)\exp(K^-_i t)$$

$$+K^+_i \int_{t-\Delta t}^t \exp(-K^-_i(t - \theta))u^+_{i,\Delta t}(a_i + \lambda_i(t - \theta), \theta)d\theta;$$

thus

$$TV(u^-_{i,\Delta t}(a_i)) = |u^-_{i,\Delta t}(a_i, s + k) - u^-_{i,\Delta t}(a_i, s)| \leq \exp(K^-_i t)TV(u^-_{i,\Delta t})(t - \Delta t)$$

$$+K^+_i \int_{t-\Delta t}^t \exp(-K^-_i(t - \theta))TV(u^+_{i,\Delta t})(\theta)d\theta.$$

Collecting together the above estimates we get

$$TV(u^+_{i,\Delta t})(t) = |u^+_{i,\Delta t}(x_l + h, t) - u^+_{i,\Delta t}(x_l, t)| \leq \exp(K^+_i t)TV(u^+_{i,\Delta t})(t - \Delta t)$$

$$+K^-_i \int_{t-\Delta t}^t \exp(-K^+_i(t - s))TV(u^-_{i,\Delta t})(s)ds$$

$$+\beta_{a_i}\exp(K^-_i t + K^+_i(\frac{a_i}{\lambda_i}))TV(u^-_{i,\Delta t})(t - \Delta t)$$

$$+(K^+_i\beta_{a_i}\exp(K^+_i(\frac{a_i}{\lambda_i})) + K^-_i) \int_{t-\Delta t}^t \exp((K^+_i + K^-_i)(t - \theta))(TV(u^+_{i,\Delta t})(\theta) + TV(u^-_{i,\Delta t})(\theta))d\theta$$

$$+K^-_i K^+_i \int_{t-\Delta t}^t \exp(K^+_i + K^-_i(t - \theta))TV(u^+_{i,\Delta t})(\theta)d\theta.$$

Now, let us consider the total variation of the function $u^-_{i,\Delta t}(x,t)$. In this case, we have that, if $x \in [N_h - \frac{\Delta x}{\lambda_i}, N_h]$, from transmission conditions on the node N_h, we have that

$$u_i^-(x,t) = \exp(K_i^-(\frac{x-N_h}{\lambda_i}))(\sum_{j\in E_{N_h}} \frac{1}{\Delta t}\int_{t-\Delta t}^t \xi_{i,j} u^+_{j,\Delta t} ds + \sum_{j\in U_{N_h}} \frac{1}{\Delta t}\int_{t-\Delta t}^t \xi_{i,j} u^-_{j,\Delta t} ds)$$

$$+K_i^+ \int_s^t \exp(-K_i^-(\frac{x-N_h}{\lambda_i} - \tau))u_i^+(N_h,\tau)d\tau,$$

for $s \in [t-\Delta t, t)$; let us observe that, by definition of the sequence, the function $u^-_{i,\Delta t}(N_h,t)$ is constant on each interval (N_h,t), $t \in [t-\Delta t, t]$, thus its total variation in time is zero. So we have that

$$TV(u^-_{i,\Delta t})(t) = |u^-_{i,\Delta t}(x_l+h,t) - u^+_{i,\Delta t}(x_l,t)| \leq \exp(K_i^- t)TV(u^-_{i,\Delta t})(t-\Delta t)$$

$$+K_i^+ \int_{t-\Delta t}^t \exp(-K_i^-(t-s))TV(u^+_{i,\Delta t})(s)ds$$

$$+K_i^+ \exp(K_i^+(\frac{N_h}{\lambda_i}))\int_{t-\Delta t}^t \exp((K_i^+ + K_i^-)(t-\theta))TV(u_i^+)(\theta)d\theta$$

$$+K_i^- K_i^+ \int_{t-\Delta t}^t \exp(K_i^+ + K_i^-(t-\theta))TV(u_i^-)(\theta)d\theta.$$

Collecting together the above estimates we obtain that, for each $i \in \mathcal{M}$,

$$(TV(u^+_{i,\Delta t})(t) + TV(u^-_{i,\Delta t})(t)) \leq \exp(C_{i,1}t)(TV(u^+_{i,\Delta t})(t-\Delta t) + TV(u^-_{i,\Delta t})(t-\Delta t))$$

$$+C_{i,2}\int_{t-\Delta t}^t \exp(C_{i,3}(t-s))(TV(u^+_{i,\Delta t})(s) + TV(u^-_{i,\Delta t})(s))ds,$$

where

$$C_{i,1} = \max\{(K_i^+ + K_i^-), \frac{K_i^+ a_i}{\lambda_i}\},$$

$$C_{i,2} = \max\{K_i^+ K_i^-, K_i^+ \exp(\frac{K_i^+ N_h}{\lambda_i}) + K_i^-\},$$

and

$$C_{i,3} = \max\{K_i^+ + K_i^-\}.$$

Thus, applying the Gronwall's Lemma, and summing up on $h = 1, \ldots, \nu$ and on $i \in \mathcal{M}$, we get the proof. □

Now, we need to recall the following important Lemma due to Kruzkov (see [31] and [32]).

Lemma 3.3.1. *(Kruskov's Lemma) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex open set and set $\Omega_{k_0} = \{x \in \mathbb{R}^n : d(x,\Omega) < k_0\}$ for some fixed $k_0 > 0$. Let $u \in BV(\Omega_{k_0} \times [0,T))$, $T > 0$, and let $u_\Omega \in C([0,k_0))$ be a non decreasing function, with $u_\Omega(0) = 0$, such that for every $t \in (0,T)$ and $|k| \le k_0$*

$$\int_{\Omega_{k_0}} |u(x+k,t) - u(x,t)|dx \le u_\Omega(|k|). \tag{3.3.16}$$

Assume the following condition holds

$$|\int_\Omega (u(x,t+\tau) - u(x,t))\phi(x)dx| \le C_\Omega \tau ||\phi||_{C^2}, \tag{3.3.17}$$

for any $t,(t+\tau) \in (0,T)$, $\tau > 0$, for any $\phi \in C^2(\Omega)$ and some constant C_Ω. Then for any $0 \le t \le t+\tau \le T$ we have that

$$\int_\Omega |u(x,t+\tau) - u(x,t)|dx \le \tau C|k|, \tag{3.3.18}$$

for some positive constant C.

Let us observe that from the previous Compactness Lemma, for each fixed Δt the function $u^{\pm}_{i,\Delta t}$ verifies hypothesis of Kruzkov's Lemma. As a consequence of it we have that the sequence $\{u^{\pm}_{i,\Delta t}\}_{\Delta t}$ is time equicontinuos. In fact the following Proposition holds.

Proposition 3.3.3. *(Time equicontinuity) Given the sequence $\{u^{\pm}_{i,\Delta t}\}_{\Delta t}$, then it verifies*

$$\sup_t \sum_{i\in\mathcal{M}} \int_{I_i} |u^{\pm}_{i,\Delta t}(x,t+k) - u^{\pm}_{i,\Delta t}(x,t)|dx \le Ck,$$

for a positive constant C and for each $k > 0$.

Proof. Firstly let us observe that $u^{\pm}_{i,\Delta t}$ verifies (3.3.16) thanks to the estimate (3.3.14). We have to show that $u^{\pm}_{i,\Delta t}$ verifies (3.3.17). So, for each $i \in \mathcal{M}$, let $u^i_{\Delta t} = u^+_{i,\Delta t} + u^-_{i,\Delta t}$ and $v^i_{\Delta t} = \lambda_i(u^+_{i,\Delta t} - u^-_{i,\Delta t})$. Thus $(u^i_{\Delta t})_t + \lambda_i(v^i_{\Delta t})_x = 0$. We have

$$\sum_{i\in\mathcal{M}_{N_h}} |\int_{I_i} (u^i_{\Delta t}(x,t+\tau) - u^i_{\Delta t}(x,t))\phi^i(x)dx| \le \sum_{i\in\mathcal{M}_{N_h}} \tau |\int_{I_i} (u^i_{\Delta t})_t(x,t)\phi^i(x)dx|$$

$$= \sum_{i\in\mathcal{M}_{N_h}} \tau\lambda_i |\int_{I_i} (v^i_{\Delta t})_x(x,t)\phi^i(x)dx| = \tau(\sum_{i\in\mathcal{M}_{N_h}} \lambda_i |\int_{I_i} v^i_{\Delta t}\phi^i_x dx$$

$$+\sum_{i\in E_{N_h}}\lambda_i v^i_{\Delta t}(N_h)\phi^i(N_h)-\sum_{i\in U_{N_h}}\lambda_i v^i_{\Delta t}(N_h)\phi^i(N_h)-\sum_{i\in E_{N_h}}\lambda_i v^i_{\Delta t}(a_i)\phi^i(a_i)$$

$$+\sum_{i\in U_{N_h}}\lambda_i v^i_{\Delta t}(a_i)\phi^i(a_i)|);$$

now, using the assumption that $|\beta_{a_i}|\leq 1$ and by definition of the functions $u^{\pm}_{i,\Delta t}(N_h,t)$ we have that

$$\sum_{i\in\mathcal{M}_{N_h}}|\int_{I_i}(u^i_{\Delta t}(x,t+\tau)-u^i_{\Delta t}(x,t))\phi^i(x)dx|\leq\tau(\sum_{i\in\mathcal{M}_{N_h}}\lambda_i(||v^i||_{L^1}||\phi^i||_{C^2(I_i)}$$

$$+||b_{a_i}||_{L^1[0,T]}||\phi^i||_{C^2(I_i)}$$

$$+||\phi^i||_{C^2(I_i)}\lambda_i(\sum_{i\in E_{N_h}}\xi_{i,j}\sum_n\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}u_i^{+,n}dt+\sum_{i\in U_{N_h}}\xi_{i,j}\sum_n\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}u_i^{-,n}dt)$$

$$\leq\tau\sum_{i\in\mathcal{M}_{N_h}}\lambda_i^2(||u^i_{\Delta t}||_{L^1}+||b_{a_i}||+\sum_{i\in E_{N_h}}\xi_{i,j}\sum_n\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}u_i^{+,n}dt$$

$$+\sum_{i\in U_{N_h}}\xi_{i,j}\sum_n\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}u_i^{-,n}dt))||\phi^i||_{C^2(I_i)}.$$

Summing up on $h=1,\dots,\nu$ we get the estimate on all the network.

Now, from the Kruskov's Lemma we get the proof. □

Now we can prove the following Theorem.

Theorem 3.3.1. *(Existence of a solution on a network) Let $\mathcal{N}$ be a network composed of M arcs I_i, $i=1,\dots,M$, ν nodes $N_h\in\mathcal{V}_{int}$, and external boundary points a_i. Let $T>0$, $\beta_{a_i}\in\mathbb{R}$, $|\beta_{a_i}|\leq 1$, $b_{a_i}\in BV([0,T])$, and $u^{\pm}_{i,0}\in BV(I_i)$, $i\in\mathcal{M}$. Given the problem (*3.1.2*), defined on sets $I_i\times[0,T]$, $i\in\mathcal{M}$, with initial conditions (*3.1.3*) and boundary conditions (*3.1.4*), (*3.1.5*), (*3.1.6*) and (*3.1.7*), then there exists a unique solution $u_i^{\pm}\in C([0,T];BV(I_i))$, $i\in\mathcal{M}$, $T>0$.*

Proof. Fix a node $N_h\in\mathcal{V}_{int}$. Let $\{u^{\pm}_{i,\Delta t}\}_{\Delta t}$ be the sequence defined in (3.3.9). By construction, for each fixed partition of length Δt the function $u^{\pm}_{i,\Delta t}$ is a BV solution of (3.1.2) with initial conditions $u^{\pm}_{i,\Delta t}(x,0)=u^{\pm}_{i,0}(x)$ and boundary conditions (3.3.10), (3.3.11), (3.3.12), (3.3.13). In particular, it verifies

$$\int_0^T(\int_{I_i}(u^{\pm}_{i,\Delta t}\frac{\partial\phi_i}{\partial t}\pm\lambda\frac{\partial\phi_i}{\partial x})dxdt+\int_{I_i}u^{\pm}_{i,0}(x)\phi_i(x,0)dx \tag{3.3.19}$$

$$+\int_0^T (u^{\pm}_{i,\Delta t})_{|\Gamma}\phi_{i|\Gamma}dt = \int_0^T \int_{I_i} g^{\pm}(u^+_{i,\Delta t}, u^-_{i,\Delta t})\phi_i dxdt,$$

where $\Gamma = \{a_i, N_h\} \times [0,T]$, for each $i \in \mathcal{M}$ and for each function $\phi_i \in C_c^\infty(I_i \times [0,T])$. Now, given the sequence $\{u^{\pm}_{i,\Delta t}\}_{\Delta t}$, let us assume that for each index i it converges to a function $(u_i^+, u_i^-) \in (C([0,T]; BV(I_i)))^2$ when $\Delta t \to 0$. Thus we have that

$$\int_0^T \int_{I_i} |u^{\pm}_{i,\Delta t} - u_i^{\pm}| dxdt \longrightarrow 0,$$

when $\Delta t \longrightarrow 0$, and in particular, by the traces properties of BV functions, we have that the traces of the sequence converge to the traces of the limit function $u_i^{\pm}$, i.e.

$$\int_0^T |u^{\pm}_{i,\Delta t}(a_i,t) - u_i^{\pm}(a_i,t)| dt \longrightarrow 0,$$

$$\int_0^T |u^{-}_{i,\Delta t}(N_h,t) - u_i^{-}(N_h,t)| dt \longrightarrow 0.$$

We claim that if the limit function $u_i^{\pm}$ exists, then it is the unique BV solution of problem (3.1.2), and so it verifies (3.1.10), and the conditions (3.1.4), (3.1.5), (3.1.6), and (3.1.7). From the condition (3.3.19) the first assert follows immediately passing to the limit on $\Delta t \longrightarrow 0$; so $u_i^{\pm}$ verifies (3.1.10). We have to show now that the limit function verifies the boundary conditions, i.e.

$$\int_0^T |u_i^+(a_i,t) - \beta_{a_i} u_i^-(a_i,t) - b_{a_i}(t)| dt = 0,$$

if $i \in E_{N_h}$,

$$\int_0^T |u_i^-(a_i,t) - \beta_{a_i} u^+_{i,h}(a_i,t) - b_{a_i}(t)| dt = 0,$$

if $i \in U_{N_h}$, and

$$\int_0^T |u_i^-(N_h,t) - \sum_{j \in E_{N_h}} \xi_{i,j} u_i^+(N_h,t) - \sum_{j \in U_{N_h}} \xi_{i,j} u_i^-(N_h,t)| dt = 0,$$

if $i \in E_{N_h}$,

$$\int_0^T |u_i^+(N_h,t) - \sum_{j \in E_{N_h}} \xi_{i,j} u_i^+(N_h,t) - \sum_{j \in U_{N_h}} \xi_{i,j} u_i^-(N_h,t)| dt = 0,$$

if $i \in U_{N_h}$. Let $i \in E_{N_h}$. On the boundary points a_i we immediately get the claim by definition of boundary conditions of the function $u^{\pm}_{i,\Delta t}$ and the convergence of the traces of $u^\pm_{i,\Delta t}$ to the traces of $u^\pm_i$. Let us study the boundary conditions each node $N_h \in \mathcal{V}_{int}$. Let $i \in E_{N_h}$. We have

$$\int_0^T |u_i^-(N_h,t) - \sum_{j\in E_{N_h}} \xi_{i,j} u_j^+(N_h,t) - \sum_{j\in U_{N_h}} \xi_{i,j} u_j^-(N_h,t)|dt$$

$$\leq \sum_n \int_{t_n}^{t_{n+1}} |u_i^-(N_h,t) - \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} u_i^-(N_h,s)ds|dt$$

$$+\sum_n \int_{t_n}^{t_{n+1}} |\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} u_i^-(N_h,s)ds - u^-_{i,\Delta t}(N_h,t)|dt$$

$$+\sum_n \int_{t_n}^{t_{n+1}} \sum_{j\in E_{N_h}} \xi_{i,j}|\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} u^+_{j,\Delta}(N_h,s)ds - u^+_{j,\Delta}(N_h,t)|dt$$

$$+\sum_n \int_{t_n}^{t_{n+1}} \sum_{j\in U_{N_h}} \xi_{i,j}|\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} u^-_{j,\Delta}(N_h,s)ds - u^-_{j,h,\Delta}(N_h,t)|dt$$

$$+\sum_n \int_{t_n}^{t_{n+1}} \sum_{j\in E_{N_h}} \xi_{i,j}|u^+_{j,\Delta}(N_h,t) - u^+_j(N_h,t)|dt$$

$$+\sum_n \int_{t_n}^{t_{n+1}} \sum_{j\in U_{N_h}} \xi_{i,j}|u^-_{j,\Delta t}(N_h,t) - u^-_j(N_h,t)|dt$$

$$\leq \sum_n \int_{t_n}^{t_{n+1}} |u_i^-(N_h,t) - \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} u_i^-(N_h,s)ds|dt$$

$$+\sum_n \int_{t_n}^{t_{n+1}} |u_i^-(N_h,t) - u^-_{i,\Delta t}(N_h,t)|dt + \sum_n \int_{t_n}^{t_{n+1}} |u_i^-(N_h,t) - u^-_{i,\Delta t}(N_h,t)|dt$$

$$+\sum_n \int_{t_n}^{t_{n+1}} \sum_{j\in E_{N_h}} \xi_{i,j}|\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} u^{+,n}_j(N_h,s)ds - u^{+,n}_j(N_h,t)|dt$$

$$+\sum_n \int_{t_n}^{t_{n+1}} \sum_{j\in U_{N_h}} \xi_{i,j}|\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} u^{-,n}_j(N_h,s)ds - u^{-,n}_j(N_h,t)|dt$$

$$+\int_0^{t_1} \sum_{j\in E_{N_h}} \xi_{i,j}|u^+_{j,\Delta t}(N_h,t+\Delta t) - u^+_{j,\Delta t}(N_h,t)|dt$$

$$+\int_0^{t_1} \sum_{j\in U_{N_h}} \xi_{i,j}|u^-_{j,\Delta t}(N_h,t+\Delta t) - u^-_{j,\Delta t}(N_h,t)|dt$$

$$+\sum_n \int_{t_n}^{t_{n+1}} \sum_{j\in E_{N_h}} \xi_{i,j}|u^+_{j,\Delta}(N_h,t)-u^+_j(N_h,t)|dt$$

$$+\sum_n \int_{t_n}^{t_{n+1}} \sum_{j\in U_{N_h}} \xi_{i,j}|u^-_{j,\Delta t}(N_h,t)-u^-_j(N_h,t)|dt.$$

Thanks to the property of trace convergence of bounded variation functions, we have that the quantities

$$\sum_n \int_{t_n}^{t_{n+1}} |u^-_i(N_h,t)-u^-_{i,\Delta t}(N_h,t)|dt \to 0,$$

$$\sum_n \int_{t_n}^{t_{n+1}} \sum_{j\in E_{N_h}} \xi_{i,j}|u^+_{j,\Delta}(N_h,t)-u^+_j(N_h,t)|dt \to 0,$$

$$\sum_n \int_{t_n}^{t_{n+1}} \sum_{j\in U_{N_h}} \xi_{i,j}|u^-_{j,\Delta t}(N_h,t)-u^-_j(N_h,t)|dt \to 0,$$

and

$$\sum_n \int_{t_n}^{t_{n+1}} |u^-_{i,\Delta t}(N_h,t)-u^-_i(N_h,t)|dt \to 0,$$

go to zero when $\Delta t \to 0$.

Then, using the average Lebesgue Theorem (1.2.3) introduced in Chapter 1, we have that the quantities

$$\sum_n \int_{t_n}^{t_{n+1}} \sum_{j\in E_{N_h}} \xi_{i,j}|\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} u^{+,n}_j(N_h,s)ds - u^{+,n}_j(N_h,t)|dt \to 0,$$

$$\sum_n \int_{t_n}^{t_{n+1}} \sum_{j\in U_{N_h}} \xi_{i,j}|\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} u^{-,n}_j(N_h,s)ds - u^{-,n}_j(N_h,t)|dt \to 0,$$

go to zero when $\Delta t \to 0$.

Moreover, thanks to Theorem (1.2.1), we have that

$$\sum_n \int_{t_n}^{t_{n+1}} |u^-_i(N_h,t) - \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} u^-_i(N_h,s)ds|dt \to 0,$$

goes to zero when $\Delta t \to 0$.

Then let us observe that the following quantities

$$\int_0^t \sum_{j\in E_{N_h}} \xi_{i,j}|u^+_{j,\Delta t}(N_h,t+\Delta t)-u^+_{j,\Delta t}(N_h,t)|dt \to 0$$

and

$$\int_0^t \sum_{j\in U_{N_h}} \xi_{i,j}|u^-_{j,\Delta t}(N_h, t+\Delta t) - u^-_{j,\Delta t}(N_h, t)|dt \to 0$$

go to zero when $\Delta t \to 0$ by the shift property of L^1-functions.

Then, if $i \in U_{N_h}$ the way of proceeding is the same.

In this way we have shown that if the sequence $\{u^{\pm}_{i,\Delta t}\}_{\Delta t}$ converges to a limit in the space of functions $C([0,T]; L^1(I_i))$, for each $i \in \mathcal{M}$, then this limit function belongs to $C([0,T]; BV(I_i))$ and it is a solution of the problem we are considering (3.1.2). We have to show now that the sequence $\{u^{\pm}_{i,\Delta t}\}$ admits a limit $u^{\pm}_i$ in the space $C([0,T]; L^1(I_i))$ for each $i \in \mathcal{M}$. Thanks to the Compactness's Lemma previously proved, we have that the sequence $\{u^{\pm}_{i,\Delta t}\}$ is bounded in the space of functions $C([0,T); L^1(I_i))$, with its derivatives. Moreover we have shown that it is equicontinuos in time, thus by the Riesz-Frechet-Kolmogorov Theorem it has compact closure in $C([0,T]; L^1(I_i))$, for each $i \in \mathcal{M}$. Then there exists a subsequence of it which converges to a function (u^+_i, u^-_i) in $(C([0,T]; L^1(I_i)))^2$. But we have previously shown that the function (u^+_i, u^-_i) must be a BV solution of our problem. Then, by the boundness of the sequence $\{u^{\pm}_{i,\Delta t}\}_{\Delta t}$, it cannot admit diverging subsequences; moreover because of the uniqueness of solution there is not a subsequences converging to a limit function different from the solution of our problem. So all the sequence converges and the claim follows. $\square$

3.4 Asymptotic behavior of perturbation of stationary solutions

In this section we consider an oriented network composed of M arcs and only one node N. So, this section is devoted to the study of the asymptotic behavior of small perturbations of stationary solutions of the linear problem (3.1.2) defined on an oriented network $\mathcal{N}$, composed of M arcs and a single node N, according to definition (2.5.1). Let E and U be respectively the entering arcs in the node N and the outgoing ones, and let $\mathcal{M} := E \cup U$. We begin observing that introducing the new variables

$$u^i := u^+_i + u^-_i,$$

and

$$v^i := \lambda_i(u^+_i - u^-_i),$$

we can write model (3.1.2) in the equivalent way

$$\begin{cases} u^i_t + v^i_x = 0 \\ v^i_t - \lambda_i^2 u^i_x = \alpha u^i - v^i, \end{cases} \tag{3.4.1}$$

for each $i \in \mathcal{M}$. In these new variables, the initial conditions become $u^i(x,0) = u^+_{i,0} + u^-_{i,0}$, and $v^i(x,0) = \lambda_i(u^+_{i,0} - u^-_{i,0})$ and boundary conditions (3.1.4), (3.1.5), (3.1.6), and (3.1.7) must be opportunely transformed. Let us recall that the transmission coefficients $\xi_{i,j} \in [0,1]$, for each $i,j \in \mathcal{M}$, satisfy the condition of conservation of total flux in the node N, which yields

$$\sum_{i\in\mathcal{M}} \lambda_i \xi_{i,j} = \lambda_j, \tag{3.4.2}$$

for each $j = 1,\dots,M$. This condition ensures that the global mass is conserved along the time, namely

$$\sum_{i\in\mathcal{M}} \int_{I_i} u^i(x,t) = \mu_0 := \sum_{i\in\mathcal{M}} \int_{I_i} u^i(x,0), \tag{3.4.3}$$

for all $t > 0$.

We need some Lemmas which will be useful in the following.

Lemma 3.4.1. *Let $u^{\pm}_{i,1}$ and $u^{\pm}_{i,2}$, $i \in \mathcal{M}$ be solutions of problem (3.1.2) with initial conditions $u^{\pm}_{i,1}(x,0) = u^{\pm}_{i,1,0}(x) \in BV(I_i)$ and $u^{\pm}_{i,2}(x,0) = u^{\pm}_{i,2,0}(x) \in BV(I_i)$ and boundary conditions (3.1.4), (3.1.5), (3.1.6), (3.1.7) for both. Then*

(3.4.4)

$$\sum_{i\in\mathcal{M}} \int_{I_i} |u^+_{i,1} - u^+_{i,2}| + |u^-_{i,1} - u^-_{i,2}| dx \le \sum_{i\in\mathcal{M}} \int_{I_i} |u^+_{i,1,0} - u^+_{i,2,0}| + |u^-_{i,1,0} - u^-_{i,2,0}| dx$$

$$- \sum_{i\in\mathcal{M}} \int_0^t \int_{I_i} L_i(x,s)\,dx\,ds,$$

where

$$L_i(x,s) := [sgn(u^+_{i,1} - u^+_{i,2}) - sgn(u^-_{i,1} - u^-_{i,2})][\frac{\lambda_i - \alpha}{2\lambda_i}(u^+_{i,1} - u^+_{i,2}) - \frac{\lambda_i + \alpha}{2\lambda_i}(u^-_{i,1} - u^-_{i,2})].$$

Proof. Let $z^{\pm}_i = u^{\pm}_{i,1} - u^{\pm}_{i,2}$. Then the functions $z^{\pm}_i$ verifies problem (3.1.2) with initial conditions $z^{\pm}_i(x,0) = u^{\pm}_{i,1,0} - u^{\pm}_{i,2,0}$ and zero boundary conditions.

Now, let us multiply equation for z^+_i by $sgn(u^+_{i,1} - u^+_{i,2})$ and equation for z^-_i by $sgn(u^-_{i,1} - u^-_{i,2})$; adding the two equations and following the proof of estimate (3.2.1), we get the proof. □

Remark 3.4.1. *Let us observe that from the subcharacteristic condition*

$$\alpha < |\lambda_i|,$$

for each $i \in \mathcal{M}$, we have that the quantity

$$-\frac{\lambda_i + \alpha}{2\lambda_i} \leq 0, \textit{ and } \tfrac{\lambda_i - \alpha}{2\lambda_i} \geq 0.$$

Thus we have that the quantity

$$L_i(x, s) \geq 0$$

for each $i \in \mathcal{M}$.

As a consequence of this fact and of the above estimate (3.4.4), we are able to prove that also in this new case the semigroup generated by the problem (3.1.2) is contractive in $(L^1(I_i))^2$, $i \in \mathcal{M}$. Let us observe that in proving this, the transmission conditions on the node play a fundamental role.

Now let us recall some definitions and results which will be useful in what follows.

Definition 3.4.1. *Let $M_n(\mathbb{R})$ be the space of square matrix of order n, $n \geq 1$, and let $A = \{a_{i,j}\} \in M_n(\mathbb{R})$. We say that A is reducible if there exists a permutation matrix P such that*

$$PAP^T = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

where B, C and D are square matrices.
We say that A is irriducible if is not reducible.

Definition 3.4.2. *Let $A = \{a_{i,j}\} \in M_n(\mathbb{R})$, with enters $a_{i,j} \geq 0$. We say that A is primitive if there exists $m > 0$ such that $A^m > 0$.*

Theorem 3.4.1. *(Perron-Frobenius) Let $A = \{a_{i,j}\} \in M_n(\mathbb{R})$, $a_{i,j} \geq 0$, be a square irriducible and primitive matrix of order n $n \geq 1$. Then*

- *the largest eigenvalue ν is real and positive;*
- *ν has algebraic multiplicity equal to one;*
- *the corresponding eigenvector to ν has positive components;*
- *the corresponding eigenvector is the only non negative eigenvector of the matrix.*

In order to study the asymptotic behaviour of perturbations of stationary solutions, we observe that the general stationary solutions of the problem (3.4.1) are the functions

$$(\overline{u}^i, \overline{v}^i) = (C_i \exp(\frac{\alpha x}{\lambda_i^2}) - \frac{v^i(x)}{\lambda_i^2}, v^i(x)), \tag{3.4.5}$$

$i \in \mathcal{M}$, for some constants C_i, $i \in \mathcal{M}$, depending on the boundary conditions. We have that the following Proposition holds.

Proposition 3.4.1. *Let us consider the problem (*3.4.1*) coupled with conditions (*3.1.6*), (*3.1.7*), with coefficients $\xi_{i,j}$ satisfying (*3.4.2*), and no flux boundary conditions at the boundary points a_i, i.e.,*

$$u_i^+(a_i, t) = u_i^-(a_i, t), \tag{3.4.6}$$

$i \in \mathcal{M}$. Let us assume that the transmission coefficients are strictly positives, $0 < \xi_{i,j} \leq 1$, for each $i, j \in \mathcal{M}$. Then the general stationary solution

$$(\overline{u}^i, \overline{v}^i) = (C_i \exp(\frac{\alpha x}{\lambda_i^2}), 0)$$

to the problem is a one-parameter stationary solution.

Proof. The general stationary solutions of (3.4.1) are

$$(\overline{u}^i, \overline{v}^i) = (C_i \exp(\frac{\alpha x}{\lambda_i^2}) - \frac{v^i(x)}{\lambda_i^2}, v^i(x)),$$

for some constants C_i which depend on boundary conditions. From the no-flux boundary condition $v^i(a_i, t) = 0$ for each i follows that the only stationary solution for the function v^i is the null function

$$v^i(x, t) \equiv 0,$$

for each $(x, t) \in I_i \times [0, T]$, $i \in \mathcal{M}$, which implies that the stationary solution $u^i(x, t) = u^i(x)$ is the function

$$\overline{u}^i(x) = C_i e^{\frac{\alpha x}{\lambda_i^2}},$$

and that the stationary solutions $\overline{u}_i^{\pm}(x) = \frac{\overline{u}^i(x)}{2}$ for each $i \in \mathcal{M}$. In order to determine the constants C_i, we recall that the stationary solutions must be verify the boundary conditions on the node N. Thus, we insert the stationary solutions $\overline{u}_i^{\pm}(x)$ in the boundary conditions at the node N, (3.1.6), and (3.1.7). In this way we obtain a system in the unknowns C_i, i.e., we have that for each $i \in \mathcal{M}$

$$0 = C_i \exp(\frac{\alpha N}{\lambda_i^2}) - \sum_{j \in \mathcal{M}} \xi_{i,j} C_j \exp(\frac{\alpha N}{\lambda_j^2}) \tag{3.4.7}$$

$$= C_i \exp(\frac{\alpha N}{\lambda_i^2})(1 - \xi_{i,i}) - \sum_{j \neq i} \xi_{i,j} \exp(\frac{\alpha N}{\lambda_j^2}) C_j.$$

This is a linear system of $\mathcal{M}$ equations in the $\mathcal{M}$ unknown constants C_i. The matrix of the system is

$$B = \begin{pmatrix} (1-\xi_{1,1})\exp(\frac{\alpha N}{\lambda_1^2}) & -\xi_{1,2}\exp(\frac{\alpha N}{\lambda_2^2}) & ... & -\xi_{1,M}\exp(\frac{\alpha N}{\lambda_M^2}) \\ -\xi_{2,1}\exp(\frac{\alpha N}{\lambda_1^2}) & (1-\xi_{2,2})\exp(\frac{\alpha N}{\lambda_2^2}) & ... & -\xi_{2,M}\exp(\frac{\alpha N}{\lambda_M^2}) \\ \vdots & ... & \ddots & ... \\ \vdots & ... & \ddots & ... \\ -\xi_{M,1}\exp(\frac{\alpha N}{\lambda_1^2}) & ... & ... & (1-\xi_{M,M})\exp(\frac{\alpha N}{\lambda_M^2}) \end{pmatrix}$$

We claim that the rank of B is $M-1$. To do this we first prove that the rows are linearly dependent. In fact consider the following vector linear combination of the rows with coefficients λ_i, $i = 1, \ldots, M$, we have that

$$(\exp(\frac{\alpha N}{\lambda_1^2})(\lambda_1 - \sum_{i \in \mathcal{M}} \lambda_i \xi_{i,1}), \ldots, \exp(\frac{\alpha N}{\lambda_M^2})(\lambda_M - \sum_{i \in \mathcal{M}} \lambda_i \xi_{i,1})) = (0, \ldots, 0)$$

by conditions (3.4.2). Thus we have that the rank of B is at most $M-1$. Now let us consider the following matrix of a system equivalent to (3.4.7):

$$\overline{B} = \begin{pmatrix} \lambda_1(\xi_{1,1}-1) & \lambda_1\xi_{1,2} & ... & \lambda_1\xi_{1,M} \\ \lambda_2\xi_{2,1} & \lambda_2(\xi_{2,2}-1) & ... & \lambda_2\xi_{2,M} \\ \vdots & ... & \ddots & ... \\ \vdots & ... & \ddots & ... \\ \lambda_M\xi_{M,1} & ... & ... & \lambda_M(\xi_{M,M}-1) \end{pmatrix}$$

Let us denote by $\overline{B}^T$ the transpose matrix of $\overline{B}$. We have to show that $\dim Ker\overline{B} = 1$ or equivalently that $\dim Ker\overline{B}^T = 1$. So let $v \in Ker\overline{B}^T$, $v = (v_1, \ldots, v_M)$. Computing $\overline{B}^T v = 0$, we have that

$$\sum_{i=1}^{M} \lambda_i \xi_{i,j} v_i = \lambda_j v_j,$$

for each $j = 1, \dots, M$. Let us observe that we can rewritten the above expression as

$$\sum_{i=1}^{M} \frac{\lambda_i}{\lambda_j} \xi_{i,j} v_i = v_j.$$

The above equalities imply that we have that $v \in Ker\overline{B}^T$ if and only if v is an eigenvector associated to the eigenvalue $\nu = 1$ for the matrix $A = \{a_{i,j}\}$ with elements $a_{i,j} = \frac{\lambda_j}{\lambda_i}\xi_{j,i}$. Now, from conditions of flux conservation

$$\sum_{i=1}^{M} \lambda_i \xi_{i,j} = \lambda_j, \tag{3.4.8}$$

we have that the norm of A, $||A||_\infty$,

$$||A||_\infty := \max_{i=1,\dots,M} \sum_{j=1}^{M} |a_{i,j}| = \max_{i=1,\dots,M} \sum_{j=1}^{M} |\frac{\lambda_j}{\lambda_i}\xi_{j,i}| = 1.$$

So, denoting by $\nu_1, \dots, \nu_s$, $s \geq 1$, the eigenvalues of A, from $||A||_\infty = 1$ we deduce that the spectral radius $\rho(A) = \max_{1 \leq i \leq s}(|\nu_i|) \leq 1$. Let us also observe that conditions (3.4.8) imply that the vector $v_\nu = (1, \dots, 1)^T$ is an eigenvector associated to the eigenvalue $\nu = 1$. Therefore, we have the exact equality $\rho(A) = 1$.

Now, we have that the elements of the matrix A, $a_{i,j} = \frac{\lambda_j}{\lambda_i}\xi_{j,i}$ are all strictly positives, thus A is a primitive matrix. Moreover, under assumption that $0 < \xi_{i,j} \leq 1$, clearly A is irriducible. Then by the Perron-Frobenius Theorem we have that the eigenvalue $\nu = 1$ has algebraic multiplicity equal to one, i.e., is a simple eigenvalue. Because of $v \in Ker\overline{B}^T$ if and only if v is an eigenvector associated to the eigenvalue $\nu = 1$ for A, we have that the dimension of the eigenspace generated by the eigenvector v associated to eigenvalue $\nu = 1$ is equal to $\dim Ker\overline{B}^T$. Thus $\dim Ker\overline{B}^T = \dim Ker\overline{B} = 1$.

Thus the system (3.4.7) has a one-parameter solution $(C_1, \dots, C_M)$ not identically equal to zero, and we get the proof. □

Now we have that the following Proposition holds.

Proposition 3.4.2. *Let us consider the one-parameter stationary solution of (3.4.1) and let us assume that $0 < \xi_{i,j} < 1$ for each $i, j \in \mathcal{M}$. Then, given $\mu_0 > 0$, there exists a unique positive stationary solution $(\overline{u}^i(x), 0)$ of (3.4.1) such that $\sum_{i \in \mathcal{M}} \overline{u}^i(x) = \mu_0$.*

Proof. By definition of one-parameter solution of a linear and homogeneous system, we have that the coefficients $(C_1, \dots, C_M)$ of the stationary solution of (3.4.1) are of kind $(r_1 C_j, r_2 C_j, \dots, C_j, \dots, r_M C_M)$, where $r_i \in \mathbb{R}$,

$i = 1, \dots, M-1$. Let us observe that from the system (3.4.7) and using conditions $0 < \xi_{i,j} < 1$, by substitution one obtains that r_i is positive for each $i = 1, \dots, M-1$.

In order to fix the parameter C_j from which the solution is depending, we have to couple the system (3.4.7) with another condition. Imposing the condition of mass conservation (3.4.3), we have

$$\sum_{i \in \mathcal{M}} \overline{u}^i(x)dx = \sum_{i \in \mathcal{M}} C_i \exp(\frac{\alpha x}{\lambda_i^2})dx = \mu_0;$$

so

$$C_j(\sum_{i \neq j} \int_{I_i} r_i \exp(\frac{\alpha x}{\lambda_i^2})dx + \int_{I_j} \exp(\frac{\alpha x}{\lambda_j^2})dx) = \mu_0,$$

and therefore

$$C_j = \frac{\mu_0}{\sum_{i \neq j} \int_{I_i} r_i \exp(\frac{\alpha x}{\lambda_i^2})dx + \int_{I_j} \exp(\frac{\alpha x}{\lambda_j^2})dx}.$$

We have that $C_j > 0$, and so we get the proof. □

From now on let $(\overline{u}^i(x), 0)$, $i \in \mathcal{M}$, the stationary solution such that $\sum_{i \in \mathcal{M}} \overline{u}^i(x)dx = \mu_0$, for a constant μ_0. We recall that in the variables $\overline{u}_i^{\pm}(x)$ we have $\overline{u}_i^{+}(x) = \overline{u}_i^{-}(x) = \frac{\overline{u}^i(x)}{2}$.

Now we want to prove an uniform estimate for functions $u_i^{\pm}(x,t)$, for each $i \in \mathcal{M}$.

Proposition 3.4.3. *Let $\mathcal{N}$ be a network composed of M arcs and a single node N. Given a solution of (3.1.2) coupled with initial conditions $u_{i,0}^{\pm} \in L^{\infty}(I_i)$ $i \in \mathcal{M}$, and boundary conditions (3.4.6), (3.1.6), and (3.1.7), then there exist stationary solutions $\overline{u}^{i,1}(x)$ and $\overline{u}^{i,2}(x)$ such that*

$$u_i^{\pm}(x,t) \leq \overline{u}_i^{\pm,1}(x),$$

and

$$|(u_i^{\pm}(x,t))_t| \leq \overline{u}_i^{\pm,2}(x),$$

for each $i \in \mathcal{M}$, and $t \geq 0$.

Proof. Given a positive constant μ_0, we have previously shown that condition

$$\sum_{i \in \mathcal{M}} \int_{I_i} C_i \exp(\frac{\alpha x}{\lambda_i^2})dx = \mu_0, \tag{3.4.9}$$

implies that the coefficients C_i, $i = 1, \dots, M$ are all positive, so the unique stationary solution $\overline{u}^i(x)$ with total mass μ_0 is positive for each $i = 1, \dots, M$. Thus, from (3.4.9) we deduce that

(3.4.10) if the total mass $\mu_0 \to +\infty$, then $C_i \to +\infty$, for each $i \in \mathcal{M}$.

Now, let us consider the initial data of (3.1.2), $u^{\pm}_{i,0} \in L^\infty(I_i)$, $i \in \mathcal{M}$, and let us define $K := \max_{i\in\mathcal{M}}\{||u^+_{i,0}||_{L^\infty} + ||u^-_{i,0}||_{L^\infty}\}$. By property (3.4.10) we deduce that

$$\text{given } K > 0,\ \exists \mu_{0,K} > 0 \text{ such that } \min_{i\in\mathcal{M}}\{C_i\} > K,\ \forall \mu > \mu_{0,K}.$$

Thus there exists a stationary solution $\overline{u}^{\pm,1}_i(x)$, $i \in \mathcal{M}$ such that

$$\max_{x\in I_i} u^{\pm}_{i,0}(x) \le \overline{u}^{\pm,1}_i(x),$$

for each $i \in \mathcal{M}$. Then by Proposition (3.2.1) we get the first estimate.

Now, let us consider the time derivatives of $u^{\pm}_i$, $(u^{\pm}_i)_t$, $i \in \mathcal{M}$. They verify problem (3.1.2) with boundary conditions (3.4.6), (3.1.6), and (3.1.7), and initial data given by

$$(u^{\pm}_i(x,0))_t = g^{\pm}(u^+_{i,0}, u^-_{i,0}) \mp \lambda_i (u^{\pm}_{i,0})_x.$$

Thus

$$|(u^{\pm}_i(x,0))_t| \le |g^{\pm}(u^+_{i,0}, u^-_{i,0})| + \lambda_i |(u^{\pm}_{i,0})_x|.$$

Following the previous proof there exists a stationary solution $\overline{u}^{\pm,2}_i(x)$ such that

$$\max_{x\in I_i}(|g^{\pm}(u^+_{i,0}(x), u^-_{i,0}(x))| + \lambda_i |(u^{\pm}_{i,0}(x))_x|) \le \overline{u}^{\pm,2}_i(x),$$

and we get the second uniform estimate for the time derivative of solution $u^{\pm}_i(x,t)$, $i \in \mathcal{M}$. □

The aim of this section is to prove the following Theorem.

Theorem 3.4.2. *Let us consider the linear problem (3.1.2) defined on a network $\mathcal{N}$ composed of M arcs I_i and a single node N, and coupled with initial conditions $u^{\pm}_i(x,0) = u^{\pm}_{i,0} \in BV(I_i)$, $i \in \mathcal{M}$, and boundary conditions (3.4.6), (3.1.6), and (3.1.7). Let $\psi^{\pm}_i \in BV(I_i)$ and $\mu \in \mathbb{R}^+$. Let $u^{\pm}_{i,0} := \frac{\overline{u}^i}{2} + \psi^{\pm}_i(x)$ be the initial data such that*

$$\sum_{i\in\mathcal{M}} \int_{I_i} (\psi^+_i(x) + \psi^-_i(x))dx = \mu, \tag{3.4.11}$$

and let $u^{\pm}_i(x,t)$, $i \in \mathcal{M}$, the corresponding solution. Then

$$\lim_{t \to +\infty} \sum_{i \in \mathcal{M}} \int_{I_i} |u_i^+(x,t) - \frac{\widetilde{u}^i}{2}| + |u_i^-(x,t) - \frac{\widetilde{u}^i}{2}| dx = 0, \tag{3.4.12}$$

where $\widetilde{u}^i(x)$ is the stationary solution corresponding to the mass $\mu_0 + \mu$.

The first part of the proof of the above Theorem follows the tecniques used in [24]: we consider the set of accumulation points of the solution to problem (3.1.2), and we have to show that it is composed of the stationary solution $\widetilde{u}^i(x)$ only. The novelty is in the second part of the proof: in order to show that the set of accumulation points is composed only by the stationary solution, we show that the solutions to problem (3.1.2), $u_i^{\pm}(x,t)$, are continuous because they are uniformly bounded together to their spatial derivatives. Then, using the fact that we know the explicit expression for the stationary solution, and the L^1 contractivity of the evolution operator associated to this problem, we find our goal.

Proof. For each $i \in \mathcal{M}$, let $u^i(x,t) = u_i^+(x,t) + u_i^-(x,t)$. Let us first observe that from the condition of mass conservation (3.4.3) we have that

$$\sum_{i \in \mathcal{M}} \int_{I_i} u^i(x,t) dx = \mu_0 + \mu = \sum_{i \in \mathcal{M}} \int_{I_i} \widetilde{u}^i(x) dx$$

for each $t \geq 0$. From Propositions (3.2.1), and (3.2.4) we have that the sequence $\{u_i^+(.,t), u_i^-(.,t)\}_{t>0}$ is compact in $BV(I_i)^2$, for each $i \in \mathcal{M}$, so it admits convergence subsequences. Thus, for each $t \geq s$ the set B_s^i of accumulation points of the sequence $\{u_i^+(.,t), u_i^-(.,t)\}_{t \geq s}$ is not empty for each $i \in \mathcal{M}$. Moreover, the set

$$A : \bigcup_{i \in \mathcal{M}} A^i := \bigcup_{i \in \mathcal{M}} \cap_{s \geq 0} B_s^i,$$

it is not empty too, since the sets A^i are the intersection of a decreasing family of not empty compact sets. Our aim is to show that $A \equiv \{\widetilde{u}_1^+(x), \widetilde{u}_1^-(x), i = 1, \ldots, M\}$.

So, let $a_i^{\pm}(x) \in A^i$, $i \in \mathcal{M}$; by definition of A^i there exists a subsequence $\{t_n\}_{n \in N}$ such that

$$\lim_{n \to +\infty} \int_{I_i} |u_i^{\pm}(x,t_n) - a_i^{\pm}(x)| dx = 0.$$

Now, from the Proposition (3.2.1) applied to functions $u_i^{\pm}(x,t)$ and $\widetilde{u}_i^{\pm}(x)$ we have that the application

$$t \longrightarrow \sum_{i \in \mathcal{M}} \int_{I_i} |u_i^{\pm}(x,t) - \widetilde{u}_i^{\pm}(x)| dx \tag{3.4.13}$$

is decreasing in t and thus admits limit $K \geq 0$ as $t \to +\infty$. Now, considering the above subsequence $\{u_i^{\pm}(x, t_n)\}_{n \in N}$, and taking into account (3.4.13), we have that

$$\sum_{i \in \mathcal{M}} \int_{I_i} (|a_i^+(x) - \widetilde{u}_i^+(x)| + |a_i^-(x) - \widetilde{u}_i^-(x)|)dx = K;$$

let us observe function $\widetilde{u}_i^{\pm}(x)$ verify the boundary conditions (3.1.6) and (3.1.7) in the node. Now let us observe that by construction we have

$$\sum_{i \in \mathcal{M}} \int_{I_i} a_i^+(x) + a_i^-(x)dx = \mu_0 + \mu = \sum_{i \in \mathcal{M}} \int_{I_i} \widetilde{u}^i(x)dx.$$

Our aim is to prove that $K \equiv 0$ and that all the sequence $\{u_i^{\pm}(x, t)\}_{t \geq 0}$ converge to $a_i^{\pm}(x)$ for each $i \in \mathcal{M}$.

So, let $a_i^{\pm}(x, t)$, with $i = 1, \ldots, M$, be the solution of the problem (3.1.2) with initial data $a_i^{\pm}(x)$ and boundary conditions (3.4.6), (3.1.6), and (3.1.7). Let us observe that the functions $a_i^{\pm}(x, t)$ belong to A^i, for each $i \in \mathcal{M}$ and for each $t \geq 0$, because the set A is invariant under the flow generated by the contraction semigroup of (3.1.2); so for each fixed $t \geq 0$, $a_i^{\pm}(x, t)$ also are accumulation points for $u_i^{\pm}(x, t)$, and in particular we have that

$$\sum_{i \in \mathcal{M}} \int_{I_i} |a_i^+(x, t) - \widetilde{u}_i^+(x)| + |a_i^-(x, t) - \widetilde{u}_i^-(x)|dx = K.$$

To show that $a_i^{\pm}(x, t) = \widetilde{u}_i^{\pm}(x)$, for each $t \geq 0$ and $i \in \mathcal{M}$, first of all we remark that on the boundary points this is true by construction.

Then, we show that functions $u_i^{\pm}(x, t)$ are continuous and Lipschitz in I_i; the goal is obtained if we prove that the family $\{u_i^{\pm}(x, t)\}_{t \geq 0}$ is uniformly bounded together with their spatial derivatives $(u_i^{\pm}(x, t))_x$. From Proposition (3.4.3) we have that functions $u_i^{\pm}(x, t)$ and $(u_i^{\pm}(x, t))_t$ are uniformly bounded for each $i \in \mathcal{M}$ and $t \geq 0$. Then, by equations for $u_i^{\pm}$, we have that

$$(u_i^{\pm})_x = \frac{\mp(u_{i,t}^{\pm}) + g^{\pm}(u_i^+, u_i^-)}{\lambda_i},$$

so they are uniformly bounded too.

Thus functions $u_i^{\pm}$ are continuous in I_i, $i \in \mathcal{M}$, and as a consequence we have that $a_i^{\pm}(x, t)$ are continuous too.

Now, applying Proposition (3.4.1) to solutions $a_i^{\pm}(x, t)$ and $\widetilde{u}_i^{\pm}(x)$ we have that

$$K = \sum_{i \in \mathcal{M}} \int_{I_i} |a_i^{\pm}(x, t) - \widetilde{u}_i^{\pm}(x)|dx \leq \sum_{i \in \mathcal{M}} \int_{I_i} |a_i^{\pm}(x) - \widetilde{u}_i^{\pm}(x)|dx - \int_0^T \int_{I_i} I(x, s)dxds$$

$$= K - \int_0^T \sum_{i\in\mathcal{M}} \int_{I_i} L_i(x,s)dxds$$

for each $t \geq 0$, so since $I(x,s) \geq 0$, we must have that $L_i(x,s) = 0$ for each $x \in I_i$, $t \geq 0$, $i \in \mathcal{M}$. By definition of $L_i(x,s)$ this implies that

$$sgn(a_i^+(x,t) - \widetilde{u}_i^+(x)) = sgn(a_i^-(x,t) - \widetilde{u}_i^-(x)), \tag{3.4.14}$$

or

$$a_i^\pm(x,t) = \widetilde{u}_i^\pm(x), \tag{3.4.15}$$

since functions $a_i^\pm(x,t)$ and $\widetilde{u}_i^\pm(x)$ are continuous. Let us assume (3.4.14) to be true while (3.4.15) to be false. So, denoting by $(\widetilde{u}_i^\pm)^{-1}(x)$ the inverse of function $\widetilde{u}_i^\pm(x)$, for each $i \in \mathcal{M}$, let us consider the following functions:

$$h_i^\pm(x,t) := (\widetilde{u}_i^\pm)^{-1}(a_i^\pm(x,t)) = \frac{2\log(\frac{\alpha a_i^\pm(x,t)}{\lambda_i^2})}{C_i}.$$

The above functions are well defined because constants $C_i > 0$ for each $i \in \mathcal{M}$. We claim that $h_i^+(x,t) = h_i^-(x,t)$ for each index i. In fact, assuming for example $h_i^+ < h_i^-$, for some index i and for some $(x,t) \in I_i \times \mathbb{R}^+$ and let $h_i^+(x,t) < h_{i,0}(x,t) < h_i^-(x,t)$. Then, by continuity of $\widetilde{u}_i^\pm(x)$, we have

$$sgn(a_i^\pm(x,t) - \widetilde{u}_i^\pm(h_{i,0}(x,t))) = sgn(\widetilde{u}_i^\pm(h_i^\pm(x,t)) - \widetilde{u}_i^\pm(h_{i,0}(x,t)))$$
$$= sgn(h_i^\pm(x,t) - h_{i,0}(x,t)) = \mp 1,$$

but this is impossible because, in the above expression, (3.4.14) is violated. So we have that $h_i^+(x,t) = h_i^-(x,t)$ for each $i \in \mathcal{M}$ and $(x,t) \in I_i \times \mathbb{R}^+$, which implies that

$$a_i^+(x,t) = a_i^-(x,t),$$

for each $(x,t) \in I_i \times \mathbb{R}^+$. In particular, we have that $v^i(x,t) = \lambda_i(a_i^+ - a_i^-)(x,t) \equiv 0$, thus, because of functions $a_i^\pm$ are solutions of the problem (3.1.2), from the first equation of it, we deduce that $a_i^+(x,t) = a_i^-(x,t) = a_{i,0}^\pm(x)$ for each $t \geq 0$. Then $a_i^\pm$ must a stationary solution of (3.1.2). Because of the total mass of $a_i^\pm$ is equal to the total mass of $\widetilde{u}_i^\pm(x)$, by the uniqueness of a stationary solution of a fixed positive total mass, we conclude that

$$a_i^\pm(x,t) = \widetilde{u}_i^\pm(x),$$

for each $(x,t) \in I_i \times \mathbb{R}^+$, $i \in \mathcal{M}$, and this conclude the proof. □

Chapter 4

The hyperbolic-parabolic model on a network

This chapter is devoted to the study of existence of solutions of a non linear hyperbolic-parabolic problem defined on a network $\mathcal{N}$ composed of M arcs I_i, $i = 1, \ldots, M$, and a single node N, according to the definition (2.5.1). Before starting we resume some results about hyperbolic problems with boundary conditions.

4.0.1 Boundary values problems

Let us consider the general semilinear boundary values problem of the form

$$\begin{cases} \partial_t u + \nabla v = 0, \\ \partial_t v + \lambda^2 \nabla u = F(u) - v, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), \\ v(0,t) = v_b(t), \end{cases} \tag{4.0.1}$$

where in general the functions $u, v \in I \subset \mathbb{R}^n \times [0,T]$, $(T > 0,\ n \geq 1)$, for $x \geq 0$, $t \geq 0$ and $F \in C^1(I)$. The parameter λ is the characteristic finite speed of propagation of the densities u and v. This model is characterized by a finite speed of propagation and it is based on the so-called Cattaneo system. The numerical approximation of the above initial boundary value problem for conservation law has been studied by several authors. It has been studied numerical schemes for multi-dimensional discretizations for scalar equations and their convergence has been proved in various situations using for example the finite volume method, or the finite element method. Usually, the hyperbolic problem (4.0.1) is approximated by a sequence of problems of the form

$$
(4.0.2) \qquad \begin{cases} \partial_t u^\varepsilon + \partial_x v^\varepsilon = 0, \\ \partial_t v^\varepsilon + \lambda^2 \partial_x u^\varepsilon = \frac{1}{\varepsilon}(F(u^\varepsilon) - v^\varepsilon), \end{cases}
$$

and it is coupled with boundary and initial conditions:

$$
\begin{cases} v^\varepsilon(0,t) = v_b(t), \\ u^\varepsilon(x,0) = u_0(x), \\ v^\varepsilon(x,0) = F(u_0(x)). \end{cases}
$$

It has been shown numerically the existence and uniqueness of the solution of this problem. Convergence of the method holds under strong restrictions: in fact the initial boundary data must be a small perturbation of a constant state $(\widetilde{u}, \widetilde{v})$, which is supposed such that $F'(\widetilde{u}) \neq 0$. In case of particular boundary data, Natalini and Terracina have been proved convergence of the scheme without the previous restrictions (see [33], [31], [19]). Moreover, their results have been extended to the case of any set of velocities by Milisic ([25]). The case of systems is an open problem. In a recent work ([26]), Milisic has discretized the problem (4.0.2) with a particular choice for F, in such a way that when $\varepsilon \to 0$, he obtains a convergent numerical scheme for the initial boundary value problem for the conservation law (4.0.1). Such scheme is a kinetic scheme firstly introduced in the framework of Boltzmann approach of hydrodynamics problems.

We want to recall that it is known that the boundary conditions of the general system (4.0.1) cannot be imposed in general, and one looks for a condition which is to be effective only in the inflow part of the boundary. Several attempts have been done in this direction in the scalar case as well as for systems. In some cases, it has been proved global existence and uniqueness of bounded variation solutions thanks to their property of having strong traces. More recently, Otto gave a formulation of boundary conditions which allows to prove existence and uniqueness for bounded and measurable data ([9]). Another possibility is to consider the half Riemann problem in the quarter space $x > 0$, $t > 0$ and it has been used Godunov scheme to solve this boundary value problem.

In the previous Chapter we have studied the existence and uniqueness of solution of a linear hyperbolic system of equations defined on a network $\mathcal{N}$ composed of M arcs I_i, and nodes $N_h \in \mathcal{V}_{int}$, and we have shown that transmission boundary conditions must be imposed in a way which ensures the total flux conservation on each node of the network. This model arise from biology and it describes the movement of a cell population on fibres of extracellular matrix, namely a scaffold, and with two possible verses of movement on each arc I_i characterized by two possible velocities $\pm\lambda_i$. The movement of cells is generally influenced by presence of a chemoattractant $\phi^i(x,t)$, but in the previous Chapter we have assumed to be characterized by a constant gradient all over the network.

In this Chapter we will study the non linear hyperbolic-parabolic system arising after considering a chemical signal $\phi^i(x,t)$ variable in space and time on each arc I_i of the network. Chemical signal diffusion follows heat diffusion at rate D_i, so we assume that it verifies on each arc I_i the parabolic heat equation

$$\phi^i_t = D_i\phi^i_{xx} + a(u^i_+ + u^i_-) - b\phi^i,$$

where D_i, a and b are positive constants. Such kind of models are based on an adaptation to the chemotactic case of the so-called hyperbolic heat or Cattaneo or telegraph equation, adding a source term accounting for the chemotactic motion in the equation for the flux.

So, let $\mathcal{N}$ be an oriented network composed of M arcs I_i , $i = 1, \ldots, M$, and a single node N. According to definition (2.5.1), let E be the set of incoming road in the node N, and U be the outcoming ones, and let $E \cup U = \mathcal{M}$. We model the movement of a cell population u^i, $i \in \mathcal{M}$ on $\mathcal{N}$, influenced on each arc by the presence of a chemical signal $\phi^i(x,t)$. On each arc of the network we have two possible verses of movement for the densities of cells u_i^+ and u_i^-, with velocities $\pm\lambda_i$ respectively. Let $u^i = u_i^+ + u_i^-$, and $v^i = \lambda_i(u_i^+ - u_i^-)$, $i \in \mathcal{M}$. We consider the following hyperbolic-parabolic system of equations

$$\begin{cases} u^i_t + v^i_x = 0, \\ v^i_t + \lambda_i^2 u^i_x = -v^i + G(u^i, \phi^i, \phi^i_x), \\ \phi^i_t = D_i\phi^i_{xx} + au^i - b\phi^i, \end{cases} \tag{4.0.3}$$

for $i \in \mathcal{M}$, where $x \in I_i$, $t \geq 0$. The functions $u^i : I_i \times \mathbb{R}^+ \to \mathbb{R}$, represent the total density of cell population on each arc, $v^i : I_i \times \mathbb{R}^+ \to \mathbb{R}$ the fluxes of population on each arc, and the functions $\phi^i : I_i \times \mathbb{R}^+ \to \mathbb{R}$ the concentrations of the chemical signal on each arc I_i, which influence movement of cells. The parameters λ_i represent the finite speeds of propagation of the cells on each arc I_i. The positive constants D_i are the diffusion coefficients of the chemoattractants, while the positive coefficients a and b, are respectively the production and the degradation rate. The source term $G(u^i, \phi^i, \phi^i_x)$ is a smooth function. Then, system (4.0.3) is coupled with initial and boundary conditions opportunely defined.

As previously observed, this problem is a more general case of (3.1.2) studied in Chapter 3, because previously we have supposed that the gradient of chemoattractant, ϕ^i_x, to be constant on each arc and equal to α; in this more general case we have a variable chemical signal $\phi(x,t)$ in space and time.

In this Chapter we will prove the local and global existence of solutions of this problem under some assumptions about the source terms $G(u^i, \phi^i, \phi^i_x)$, $i \in \mathcal{M}$. Chapter is organized as follows: in the first section we give an exactly formulation of the problem. Then we begin proving local existence and uniqueness of solution of the problem under the assumption of the local lipschitzianity of the function G. To do this, we will use some results of semigroup theory and a fixed point theorem; in particular we will show the monotonicity and maximality of the operator associated to problem (4.0.3). Then, in the last section we will show that it is possible to extend local solution to a global one in case of a particular choice of the source term $G(u^i, \phi^i, \phi^i_x)$.

4.1 Formulation of the problem

Let us consider an oriented network $\mathcal{N}$ with M arcs I_i, $i = 1, \ldots, M$, and a single node N and let $u^\pm_i = \frac{1}{2}(u^i \pm \frac{v^i}{\lambda_i})$ such that $u^i = u^+_i + u^-_i$ and $v^i = \lambda_i(u^+_i - u^-_i)$. We consider system (4.0.3) written in the equivalent diagonal way

$$(4.1.1)\qquad \begin{cases} u^+_{i\,t} + \lambda_i u^+_{i\,x} = \frac{1}{2\lambda_i}((g^-(\phi^i, \phi^i_x, u^-_i) - g^+(\phi^i, \phi^i_x, u^+_i)), \\ \\ u^-_{i\,t} - \lambda_i u^-_{i\,x} = -\frac{1}{2\lambda_i}(g^-(\phi^i, \phi^i_x, u^-_i) - g^+(\phi^i, \phi^i_x, u^+_i)), \\ \\ \phi^i_t = D_i \phi^i_{xx} + a(u^+_i + u^-_i) - b\phi^i, \end{cases}$$

where $u^\pm_i \in C(\mathbb{R}^+; H^1(I_i))$, $\phi^i \in C(\mathbb{R}^+; H^2(I_i))$, the coefficients λ_i,D_i, b, a are positive constants, and $g^\pm(\phi^i, \phi^i_x, u^\pm_i)$ are smooth functions; together with this system we consider initial data

$$(4.1.2)\qquad u^i(x,0) = u^+_i(x,0) + u^-_i(x,0) = (u^+_{i,0} + u^-_{i,0}) \in H^1(I_i),$$

$$(4.1.3)\qquad v^i(x,0) = \lambda_i(u^+_i(x,0) - u^-_i(x,0)) = \lambda_i(u^+_{i,0} - u^-_{i,0}) \in H^1(I_i),$$

$$(4.1.4)\qquad \phi^i_0(x) = \phi^i_0(x) \in H^2(I_i),$$

and boundary conditions as follows. On the outer boundary points a_i for each $i \in \mathcal{M}$ we impose flux null conditions $v^i(a_i,t) = 0$, i.e.,

$$u_i^+(a_i,t) = u_i^-(a_i,t). \tag{4.1.5}$$

On the outer boundaries, we also consider no-flux Neumann boundary conditions for ϕ^i,

$$\phi^i_x(a_i,t) = 0, \tag{4.1.6}$$

$i \in \mathcal{M}$. This condition could be generalized, for example in the case when we assume that there is a production of cells on the boundary. Let us describe how to define the conditions at a node; this is an important point, since the behaviour of the solution will be very different according to the conditions we choose. So, on the node N we impose the following transmission conditions for the functions $u_i^\pm$:

$$u_i^-(N,t) = \sum_{j\in E} \xi_{i,j} u_j^+(N,t) + \sum_{j\in U} \xi_{i,j} u_j^-(N,t) \tag{4.1.7}$$

if $i \in E$, and

$$u_i^+(N,t) = \sum_{j\in E} \xi_{i,j} u_j^+(N,t) + \sum_{j\in U} \xi_{i,j} u_j^-(N,t), \tag{4.1.8}$$

if $i \in U$, where the constant $\xi_{i,j} \in [0,1]$ for each index i, j are the transmission coefficients. We are not interested in having the continuity of the densities at the node but we are interested in having the continuity of the fluxes at the node, which yields

$$\sum_{i\in E} \lambda_i (u_i^+(N,t) - u_i^-(N,t)) = \sum_{i\in U} \lambda_i (u_i^+(N,t) - u_i^-(N,t)), \tag{4.1.9}$$

so transmission coefficients have to verify

$$\sum_{i\in\mathcal{M}} \xi_{i,j}\lambda_i = \lambda_j, \tag{4.1.10}$$

for each $j \in \mathcal{M}$. This condition leads the global mass conservation at any time $t > 0$,

$$\sum_{i\in\mathcal{M}} \int_{I_i} u^i(x,t)dx = \sum_{i\in\mathcal{M}} \int_{I_i} u^i_0(x). \tag{4.1.11}$$

Now let us consider the transmission conditions for ϕ^i. Also in this case, we do not impose the continuity of the density of the chemoattractants, but only the continuity of the flux at node N. Therefore, we use the Kedem-Katchalsky permeability conditions [21], which has been first proposed in the case of fluxes through a membranes: denoting with $\partial_n \phi^i$ the partial derivative along the normal direction of a surface, they set

$$D_i \partial_n \phi^i = \sum_{j \neq i} k_{i,j} (\phi^j - \phi^i),$$

$i \in \mathcal{M}$. The condition $k_{i,j} = k_{j,i}$, $i, j = 1, \ldots, M$ yields the conservation of the fluxes through the membrane, i.e.,

$$\sum_{i \in \mathcal{M}} D_i \partial_n \phi^i = 0. \tag{4.1.12}$$

Here we choose $k_{i,j} \equiv \alpha$, where α is a positive constant. So we set

$$D_i \phi^i_x(N, t) = \alpha \sum_{j \neq i} (\phi^j(N, t) - \phi^i(N, t)), \tag{4.1.13}$$

if $i \in E$, and

$$D_i \phi^i_x(N, t) = \alpha \sum_{j \neq i} (\phi^i(N, t) - \phi^j(N, t)), \tag{4.1.14}$$

if $i \in U$.

In the following we turn to variables u^i, v^i, ϕ^i, $i \in \mathcal{M}$, and so we turn to consider the equivalent problem (4.0.3)

$$\begin{cases} u^i_t + v^i_x = 0 \\ v^i_t - \lambda_i^2 u^i_x = -v^i + G(u^i, \phi^i, \phi^i_x) \\ \phi^i_t = D_i \phi^i_{xx} + a u^i - b \phi^i, \end{cases} \tag{4.1.15}$$

with initial and boundary conditions opportunely transformed.

4.2 The homogeneous case

Let $\mathcal{N}$ be an oriented network composed of M arcs I_i, $i = 1, \ldots, M$, and a single node N, and let E be the set of incoming arcs in the node, and U be the outcoming ones, and $E \cup U = \mathcal{M}$. The aim of this section is to prove the uniqueness and global existence of the solutions of the following linear problem

(4.2.1)
$$\begin{cases} u^i_t + v^i_x = 0, \\ v^i_t - \lambda_i^2 u^i_x + v^i = 0, \\ \phi^i_t = D_i \phi^i_{xx} - b\phi^i, \end{cases}$$

defined on $\mathcal{N}$, with initial and boundary conditions (4.1.2), (4.1.3), (4.1.4), (4.1.5), (4.1.6), (4.1.7), (4.1.8), (4.1.13), and (4.1.14) previously introduced. To prove this, we will use some results of the semigroups theory (see Chapter 1). We will show that the evolution operator A associated to system (4.2.1) generates a contraction semigroup $S(t)$ on an Hilbert space. Thanks to semigroup theory (see Chapter 2), we have that the solution $W_i = (u^i, v^i, \phi^i)$, $i = 1, \dots, M$ of (4.2.1) can be written, for each $t > 0$, as

$$W_i(t) = S(t)W_i(0).$$

We will prove that A is a monotone and maximal operator on a particular Hilbert space. In this way, we can use the Hille Yosida Theorem to have the existence of the solution of (4.2.1) through the generation of a contraction semigroup.

4.2.1 Monotonicity of the operator

We need some preliminary results which we will use in the following. Let us first focus on the fact that in order to show the monotonicity and maximality of the operator associated to problem (4.2.1), e need to work with the symmetrized problem of (4.2.1). Thus, from now on, we denote by

$$u^i = u_i^+ + u_i^-,$$

and

$$v^i = u_i^+ - u_i^-.$$

Making this change of variable, the problem (4.2.1) is equivalent to the symmetric problem

(4.2.2)
$$\begin{cases} u^i_t + \lambda_i v^i_x = 0 \\ v^i_t + \lambda_i u^i_x + v^i = 0 \\ \phi^i_t = D_i \phi^i_{xx} - b\phi^i, \end{cases}$$

with initial and boundary conditions opportunely transformed. For each index $i \in \mathcal{M}$ we are looking for solutions (u^i, v^i, ϕ^i) belonging to the Banach space $(C(\mathbb{R}^+; H^1(I_i)))^3$, $i \in \mathcal{M}$.

Remark 4.2.1. *For each fixed $t \geq 0$, the norm of a vector $Z^i(.,t) = (u^i(.,t), v^i(.,t), \phi^i(.,t)) \in (C(\mathbb{R}^+; H^1(I_i)))^3$ is defined as*

$$||Z^i(t)|| := ||u^i(.,t)||_{H^1(I_i)} + ||v^i(.,t)||_{H^1(I_i)} + ||\phi^i(.,t)||_{H^1(I_i)}.$$

Now, let us observe that for each $i = 1, \ldots, M$, we can rewrite (4.2.2) as the evolution problem

$$Z_t^i + A_i Z^i = 0 \tag{4.2.3}$$

where for each index i the function Z^i is the vector $Z^i := (u^i, w^i, \phi^i)$. We have denoted by A_i the differential operator given by the following matrix

$$\begin{pmatrix} 0 & \lambda_i \partial_x & 0 \\ \lambda_i \partial_x & 1 & 0 \\ 0 & 0 & -D_i \partial_{xx} + b \end{pmatrix},$$

Definition 4.2.1. *For each $i \in \mathcal{M}$, let $D(A_i)$ the domain of the operator A_i (4.2.1). We have that*

$$A_i : D(A_i) \to L^2(I_i) \times L^2(I_i) \times L^2(I_i), \tag{4.2.4}$$

for each $i \in \mathcal{M}$; let us define

$$D(A_i) := \{(u^i, v^i, \phi^i) \in H^1(I_i) \times H^1(I_i) \times H^2(I_i) \text{such that } (4.1.5), (4.1.6), (4.1.7), (4.1.8), (4.1.13), (4.1.14) \text{ hold}\}. \tag{4.2.5}$$

Moreover let us define the set

$$D(A) := \bigcup_{i \in \mathcal{M}} D(A_i), \tag{4.2.6}$$

and let A be the differential operator

$$A : D(A) \to \bigcup_{i \in \mathcal{M}} L^2(I_i) \times L^2(I_i) \times L^2(I_i),$$

such that

$$A_{|D(A_i)} = A_i. \tag{4.2.7}$$

We introduce the following notation. Let $W = (w^1, \ldots, w^k)$ and $Z = (z^1, \ldots, z^k)$. We denote by $W \cdot Z$ the inner product of W and Z in the space $L^2(I_i) \times \ldots, L^2(I_i)$, i.e.,

$$W \cdot Z := \sum_{i=1}^{k} \int_I z^i w^i dx.$$

Our aim is to find conditions on the transmission coefficients $\xi_{i,j}$, $i, j = 1, \ldots, M$, in order to have the monotonicity of the operator A (4.2.7) on its domain $D(A)$, which yields

$$\sum_{i=1}^{M} A^i Z^i \cdot Z^i \geq 0,$$

where $Z^i = (u^i, w^i, \phi^i)$, $i = 1, \ldots, M$.

So, computing $\sum_{i=1}^{M} A^i Z^i \cdot Z^i$ we obtain

$$\sum_{i=1}^{M} A_i W_i \cdot W_i = \sum_{i \in E \cup U} \int_{I_i} \lambda_i (u^i v^i)_x dx - \int_{I_i} D_i \phi^i_{xx} \phi dx + \int_{I_i} (v^i)^2 dx + \int_{I_i} b(\phi^i)^2 dx$$

$$\geq \sum_{i \in \mathcal{M}} (\int_{I_i} \lambda_i (u^i v^i)_x dx - \int_{I_i} D_i \phi^i_{xx} \phi dx) \geq \sum_{i \in \mathcal{M}} (\lambda_i u^i(N) v^i(N) - D_i \phi^i_x(N) \phi^i(N))$$

$$= \sum_{i \in E} (\lambda_i ((u_i^+(N))^2 - (u_i^-(N))^2) - \sum_{i \in U} (\lambda_i ((u_i^+(N))^2 - (u_i^-(N))^2)$$

$$- \sum_{i \in E} D_i \phi^i_x(N) \phi^i(N) + \sum_{i \in U} D_i \phi^i_x(N) \phi^i(N).$$

From now on, we denote by

$$S_1 := \sum_{i \in E} (\lambda_i ((u_i^+(N))^2 - (u_i^-(N))^2) - \sum_{i \in U} (\lambda_i ((u_i^+(N))^2 - (u_i^-(N))^2)$$

$$- \sum_{i \in E} D_i \phi^i_x(N) \phi^i(N) + \sum_{i \in U} D_i \phi^i_x(N) \phi^i(N).$$

Case of two arcs

We begin studying the simpler case $M = 2$. In other words, we are considering the simpler network composed of only two arcs represented by the closed intervals $[a_1, N]$ and $[N, a_2]$.

In this case, the boundary conditions on the node N are

$$u_1^-(N,t) = \xi_{1,1}u_1^+ + \xi_{1,2}u_2^-,$$

$$u_2^+(N,t) = \xi_{2,1}u_1^+ + \xi_{2,2}u_2^-,$$

$$D_1\phi_x^1(N,t) = \alpha(\phi^2(N,t) - \phi^1(N,t)),$$

and

$$D_2\phi_x^2(N,t) = \alpha(\phi^1(N,t) - \phi^2(N,t)).$$

The condition of flux conservation becomes in this case

$$\lambda_1 = \lambda_1\xi_{1,1} + \lambda_2\xi_{1,2}, \;\; \lambda_2 = \lambda_1\xi_{2,1} + \lambda_2\xi_{2,2}; \tag{4.2.8}$$

while the term S_1 becomes

$$S_1 = \lambda_1((u_+^1(N))^2 - (u_-^1(N))^2) - \lambda_2((u_+^2(N))^2 - (u_-^2(N))^2) \\ -D_1\phi_x^1(N)\phi^1(N) + D_2\phi_x^2(N)\phi^2(N). \tag{4.2.9}$$

In this case we can find necessary and sufficient conditions in order to have $S_1 \geq 0$. In fact we have that the following Proposition holds.

Proposition 4.2.1. *Let $M = 2$ and let us assume that condition (4.2.8) holds. Given the differential operator A (4.2.7), then it is monotone on its domain $D(A)$ if and only if the following conditions hold:*

$$\xi_{2,2} = \frac{\lambda_1(\xi_{1,1} - 1)}{\lambda_2} + 1 \tag{4.2.10}$$

$$\max\{0, \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\} \leq \xi_{1,1} \leq 1, \tag{4.2.11}$$

Proof. Let us study the non negativity of the term S_1. Let us assume that (4.2.11), and (4.2.10) hold. Substituting the transmission conditions on the node in 4.2.9 we get

$$S_1 = \lambda_1((u_+^1(N))^2 - (\xi_{1,1}u_+^1 + \xi_{1,2}u_-^2)^2) - \lambda_2((\xi_{2,1}u_+^1 + \xi_{2,2}u_-^2)^2 - (u_-^2(N))^2)$$

$$+(\alpha(\phi^2(N,t)-\phi^1(N,t)))(\phi^2(N,t)-\phi^1(N,t)) = \lambda_1((u^1_+(N))^2-(\xi_{1,1}u^1_++\xi_{1,2}u^2_-)^2)$$
$$-\lambda_2((\xi_{2,1}u^1_+ + \xi_{2,2}u^2_-)^2 - (u^2_-(N))^2) + \alpha(\phi^2(N,t) - \phi^1(N,t))^2;$$

now let us observe that thanks to the conditions on the node for the functions ϕ^1_x and ϕ^2_x and the choice $\alpha > 0$, we obtain that the quantity $\alpha(\phi^2(N,t) - \phi^1(N,t))^2$ is always non negative. So we have to study the non negativity of the quantity

$$\lambda_1((u^1_+(N))^2 - (\xi_{1,1}u^1_+ + \xi_{1,2}u^2_-)^2) - \lambda_2((\xi_{2,1}u^1_+ + \xi_{2,2}u^2_-)^2 - (u^2_-(N))^2).$$

We have that

$$\lambda_1((u^1_+(N))^2 - (\xi_{1,1}u^1_+ + \xi_{1,2}u^2_-)^2) - \lambda_2((\xi_{2,1}u^1_+ + \xi_{2,2}u^2_-)^2 - (u^2_-(N))^2).$$
$$= (\lambda_1(1-\xi^2_{1,1}) - \lambda_2\xi^2_{1,2})(u^1_+)^2 - 2(\lambda_1\xi_{1,1}\beta_{2,1} + \lambda_2\xi_{1,2}\xi_{2,2})u^1_+u^2_-$$
$$+(\lambda_2(1-\xi^2_{2,2}) - \lambda_1\xi^2_{2,1})(u^2_-)^2.$$

Thus we have obtained a bilinear form in the variables (u^1_+,u^2_-), and we want it is semidefinite positive. Denoting by $B_1 := \lambda_1(1-\xi^2_{1,1}) - \lambda_2\xi^2_{1,2}$, $B_2 := \lambda_1\xi_{1,1}\xi_{2,1} + \lambda_2\xi_{1,2}\xi_{2,2}$, and $B_3 := \lambda_2(1-\xi^2_{2,2}) - \lambda_1\xi^2_{2,1}$, then the matrix B associates to bilinear form is

$$\begin{pmatrix} B_1 & B_2 \\ B_2 & B_3 \end{pmatrix}$$

From the condition of flux conservation (4.2.8) we have

$$\xi_{1,2} = \frac{\lambda_1}{\lambda_2}(1-\xi_{1,1}),\ \xi_{2,1} = \tfrac{\lambda_2}{\lambda_1}(1-\xi_{2,2}), \tag{4.2.12}$$

so we have

$$B_1 = \lambda_1(1-\beta^2_{1,1}) - \lambda_2\beta^2_{1,2}$$
$$= \lambda_1(1-\beta_{1,1})(1+\beta_{1,1}) - \lambda_2\frac{\lambda_1^2}{\lambda_2^2}(1-\beta_{1,1})^2$$
$$= \frac{\lambda_1}{\lambda_2}(1-\beta_{1,1})(\lambda_2(1+\beta_{1,1}) - \lambda_1(1-\beta_{1,1}));$$

now, substituting condition (4.2.11) we get $B_1 \geq 0$. Analogously, substituting (4.2.12) and (4.2.10) in B_3 we get $B_3 \geq 0$. Now we study $\det(B)$:

$$\det(B) = B_1B_3 - B_2^2$$

$$= (\lambda_1(1-\xi_{1,1}^2) - \lambda_2\xi_{1,2}^2)(\lambda_2(1-\xi_{2,2}^2) - \lambda_1\xi_{2,1}^2) - (\lambda_1\xi_{1,1}\beta_{2,1} - \lambda_2\xi_{1,2}\xi_{2,2})^2$$

$$= (1-\xi_{1,1})(1-\xi_{2,2})((\lambda_2-\lambda_1)+(\lambda_2+\lambda_1)\xi_{1,1})((\lambda_1-\lambda_2)+(\lambda_1+\lambda_2)\xi_{2,2})-(\lambda_1\xi_{1,1}\xi_{2,1}-\lambda_2\xi_{1,2}\xi_{2,2})^2.$$

Defining the new variables $x := \xi_{1,1}$, $y := \xi_{2,2}$ and $\nu := \frac{\lambda_1}{\lambda_2} > 0$, from conditions (4.2.11) and (4.2.16) we have that

$$\frac{\nu-1}{\nu+1} \leq x \leq 1, \tag{4.2.13}$$

$$\frac{1-\nu}{\nu+1} \leq y \leq 1, \tag{4.2.14}$$

and $\det(B)$ as

$$\det(B) = (1-x)(1-y)[(1-\nu)+(1+\nu)x][(\nu-1)+(\nu+1)y] \geq [x(1-y)+\nu y(1-x)]^2.$$

Let us rewrite $\det(B)$ defining the function

$$F_\nu(x,y) = (1-x)(1-y)[(1-\nu)+(1+\nu)x][(\nu-1)+(\nu+1)y]-[x(1-y)+\nu y(1-x)]^2; \tag{4.2.15}$$

developing the above expression with respect to powers of ν, we find

$$-\nu^2(1-x)^2 + 2(1-x)(1-y)\nu - (1-y)^2 \geq 0,$$

i.e.,

$$-(\nu(1-x) + (1-y))^2 \geq 0$$

which is verified if and only if

$$\nu(1-x) + (1-y) = 0,$$

i.e., condition (4.2.10) holds. Now, let us assume the positivity of the quantity (4.2.9), i.e., the matrix B previously defined is semidefinite positive. Thus we have that $B_1 \geq 0$, $B_3 \geq 0$ and $\det(B) \geq 0$. It is easy to see that the positivity of B_1 imply condition (4.2.11), while we have previously shown that $\det(B) \geq 0$ if and only if (4.2.10) holds, studying the sign of the function (4.2.15). Moreover, condition $B_3 \geq 0$ implies that

$$\max\{0, \frac{\lambda_2-\lambda_1}{\lambda_1+\lambda_2}\} \leq \xi_{2,2} \leq 1, \tag{4.2.16}$$

which is satisfied if (4.2.10) holds. □

Case of M arcs

Let us consider the case of a network $\mathcal{N}$ composed of M arcs I_i, $M \geq 3$, and a single node N. In this case the monotonicity of the operator A (4.2.7) is given by the non negativity of the quantity

$$S_1 = \sum_{i\in E}(\lambda_i((u_i^+(N))^2 - (u_i^-(N))^2) - \sum_{i\in U}(\lambda_i((u_i^+(N))^2 - (u_i^-(N))^2) - \sum_{i\in E} D_i\phi_x^i(N)\phi^i(N) + \sum_{i\in U} D_i\phi_x^i(N)\phi^i(N). \tag{4.2.17}$$

We observe that in this general case we find only sufficient conditions for the non negativity of term S_1. In fact the following Proposition holds.

Proposition 4.2.2. *If we have that*

$$\sum_{j\in\mathcal{M}} \xi_{i,j} = 1, \tag{4.2.18}$$

then the operator A (4.2.7) is monotone on its domain $D(A)$.

Proof. Let us study the non negativity of the term S_1. Substituting the boundary conditions for the functions $u_\pm^i$ and ϕ^i on the node N and developing the calculus in S_1, we find

$$S_1 = \sum_{j\in E}\lambda_j(u_j^+)^2 + \sum_{k\in U}\lambda_k(u_k^-)^2 - \sum_{(j,k)\in E\times E}(\sum_{i\in E\cup U}\lambda_i\xi_{i,j}\xi_{i,k})u_j^+u_k^+$$
$$-\sum_{(j,k)\in U\times U}(\sum_{i\in E\cup U}\lambda_i\xi_{i,j}\xi_{i,k})u_j^-u_k^- - 2\sum_{(j,k)\in E\times U}(\sum_{i\in E\cup U}\lambda_i\xi_{i,j}\xi_{i,k})u_j^+u_k^-$$
$$+\alpha\sum_{i\neq j}(\phi^j(N,t) - \phi^i(N,t))^2.$$

Thanks to the choice $\alpha > 0$, and by the structure of transmission conditions for ϕ_x^i on the node, we get that the quantity

$$\alpha\sum_{i\neq j}(\phi^j(N,t) - \phi^i(N,t))^2 \geq 0.$$

Now, let $h_{j,k} = \sum_{i\in E\cup U}\lambda_i\xi_{i,j}\xi_{i,k}$, and let us assume that the incoming arcs are denoted by $1,\ldots,R$ while the outcoming ones by $R+1,\ldots,M$. We have that term in $u_i^\pm$ is a bilinear form in the variables $u_1^+,\ldots,u_R^+,u_{R+1}^-,\ldots,u_M^-$;

we want sufficient conditions in having it semidefinite positive. The diagonal of the matrix of this bilinear is composed by terms λ_j, $j \in E$, λ_k, $k \in U$, and $h_{j,k}$, so we note that a necessary condition to have the non negativity of the form is that for each $i \in \mathcal{M}$, we must have

$$\lambda_i - h_{i,i} \geq 0.$$

Under the previous condition, a sufficient condition is that the matrix must be diagonal dominant and so to have the positive of the matrix we put

$$\lambda_i \geq \sum_{j \in E \cup U} h_{i,j}, \tag{4.2.19}$$

for each $i \in E \cup U$. Using the condition of flux conservation we have

$$\sum_{j \in \mathcal{M}} \lambda_j \xi_{i,j} = \lambda_i \geq h_{i,i} = \sum_{i \in \mathcal{M}} \lambda_j \xi_{i,j}^2,$$

namely

$$\sum_{j \in \mathcal{M}} \lambda_j \xi_{i,j}(1 - \xi_{i,j}) \geq 0.$$

This is satisfied if $0 \leq \xi_{i,j} \leq 1$ for each $i, j \in \mathcal{M}$. We also rewrite the condition (4.2.19) as

$$\sum_{j \in \mathcal{M}} \lambda_j \xi_{i,j} = \lambda_i \geq \sum_{i \in \mathcal{M}} h_{i,j} = \sum_{i \in \mathcal{M}} \lambda_j \xi_{i,j}^2,$$

namely

$$\sum_{j \in \mathcal{M}} \lambda_j \xi_{i,j}(1 - \sum_{i \in \mathcal{M}} \xi_{i,j}) \geq 0;$$

these conditions are satisfied if

$$0 \leq \sum_{j \in \mathcal{M}} \xi_{i,j} \leq 1,$$

for each $i \in \mathcal{M}$. Now, using the flux conservation and summing up for $j = 1, \ldots, M$, we obtain

$$\sum_{j \in \mathcal{M}} \lambda_j = \sum_{j \in \mathcal{M}} \sum_{i \in \mathcal{M}} \lambda_i \xi_{i,j},$$

so we have that

$$\sum_{j \in \mathcal{M}} \lambda_j = \sum_{j \in \mathcal{M}} \sum_{i \in \mathcal{M}} \lambda_i \xi_{i,j} \quad \leq \quad \sum_{i \in \mathcal{M}} \lambda_i.$$

Since $\lambda_i > 0$, we can therefore conclude that the previous inequalities are all equalities, and so we have that sufficient conditions for monotonicity of A are

$$\sum_{j\in\mathcal{M}} \xi_{i,j} = 1,$$

for all $i \in \mathcal{M}$. Thus we get the proof. □

4.2.2 Global existence of solutions of the homogeneous problem

Let $\mathcal{N}$ be an oriented network composed of M arcs I_i and a single node N, according to definition (2.5.1), and let us consider the problem (4.2.2) with initial and boundary conditions (4.1.2), (4.1.3), (4.1.4), (4.1.7), (4.1.8), (4.1.13), (4.1.14), (4.1.5), and (4.1.6). In the previous section we have found that if the transmission coefficients $\xi_{i,j}$, $i, j \in \mathcal{M}$ satisfy

$$\sum_{j\in\mathcal{M}} \xi_{i,j} = 1, \tag{4.2.20}$$

then the linear and unbounded operator A is monotone on its domain $D(A)$. Now we will prove that the operator A is monotone and maximal on $D(A)$ (for more details see definitions of Chapter 1).

Proposition 4.2.3. *Let us consider the problem* (4.2.2) *with initial and boundary conditions* (4.1.7)*,* (4.1.8)*,* (4.1.13)*,* (4.1.14)*,* (4.1.5)*, and* (4.1.6)*, and let $\xi_{i,j}$, $i, j \in \mathcal{M}$, verifying* (4.2.20)*. Let A be the linear and unbounded operator* (4.2.7)*. Then A is monotone and maximal on its domain $D(A)$.*

Proof. We have previously shown that A is monotone on $D(A)$. Now, for each $i = 1, \ldots, M$, let $f^i = (f_1^i, f_2^i, f_3^i) \in L^2(I_i) \times L^2(I_i) \times L^2(I_i)$. We have to show that for each fixed f^i, $i \in \mathcal{M}$, the problem

$$\begin{cases} u^i + \lambda_i v_x^i = f_1^i, \\ v^i + \lambda_i u_x^i + v^i = f_2^i, \\ \phi^i - D_i \phi_{xx}^i + b\phi^i = f_3^i, \end{cases} \tag{4.2.21}$$

$i \in \mathcal{M}$, coupled with boundary conditions (4.1.7), (4.1.8), (4.1.13), (4.1.14), (4.1.5), and (4.1.6), admits a unique solution $(u^i, v^i, \phi^i) \in (H^1(I_i)^2 \times H^2(I_i))$, for each $i \in \mathcal{M}$. Let us firstly consider the last equation, which does not depend by each others,

$$\phi^i - D_i \phi_{xx}^i + b\phi^i = f_3^i, \tag{4.2.22}$$

with its boundary conditions. The linear and non-limited operator associated to this form is

$$A_\phi(\phi^1, \dots, \phi^M) := (-D_1\phi^1_{xx}, \dots, -D_M\phi^M_{xx})$$

, for each $i \in \mathcal{M}$.

Let

$$D(A_\phi) = \{H^2(I_1) \times \dots H^2(I_M)\text{such that (4.1.13),(4.1.14),(4.1.6) hold}\}.$$

Let us consider the bilinear form $a : (H^2(I_1) \times \dots H^2(I_M)^2 \to \mathbb{R}$ defined as

$$\sum_{i\in\mathcal{M}} \int_{I_i} (1+b)\phi^i\psi^i + D_i\phi^i_x\psi^i_x dx - \sum_{i\in E} D_i\phi^i_x(N)\psi^i(N)$$
$$+ \sum_{i\in E} D_i\phi^i_x(N)\psi^i(N).$$

We have that

$$\sum_{i\in\mathcal{M}} \int_{I_i} (1+b)\phi^i\psi^i + D_i\phi^i_x\psi^i_x dx - \sum_{i\in E} D_i\phi^i_x(N)\psi^i(N)$$
$$+ \sum_{i\in E} D_i\phi^i_x(N)\psi^i(N) \le \sum_{i\in\mathcal{M}} \frac{1+b+D_i}{2} ||\phi^i||^2_{H^2(I_i)}||\psi^i||^2_{H^2(I_i)},$$

thus the bilinear form a is continue on $(H^2(I_1) \times \dots H^2(I_M)^2$. Moreover

$$\sum_{i\in\mathcal{M}} \int_{I_i} (1+b)(\phi^i)^2 + D_i(\phi^i_x)^2 dx - \sum_{i\in E} D_i\phi^i_x(N)\phi^i(N)$$
$$+ \sum_{i\in E} D_i\phi^i_x(N)\phi^i(N) \ge \sum_{i\in\mathcal{M}} (1+b+D_i||\phi^i||^2_{H^2(I_i)};$$

in the above inequality we have used the fact that, thanks to the transmission conditions on the node for $D_i\phi^i_x$, we have that the following equality holds:

$$-\sum_{i\in E} D_i\phi^i_x(N)\phi^i(N) + \sum_{i\in E} D_i\phi^i_x(N)\phi^i(N) = \alpha \sum_{i\neq j} (\phi^j(N,t) - \phi^i(N,t))^2.$$

Thus the bilinear form a is coercitive too. So applying the Lax-Milgram Theorem we have that (4.2.22) has a unique solution $(\phi^1, \dots, \phi^M) \in H^2(I_i)^M$.

Now let us consider the system

$$\begin{cases} u^i + \lambda_i v^i_x = f^i_1, \\ v^i + \lambda_i u^i_x + v^i = f^i_2, \end{cases} \tag{4.2.23}$$

$i \in \mathcal{M}$, with boundary conditions previously defined. Proceeding as above, we have that the associated operator is

$$A_{u,v}((u^1, v^1), \dots, (u^M, v^M)) = \{(\lambda_i v_x^i, \lambda_i u_x^i), i \in \mathcal{M}\}.$$

Let

$$D(A_{u,v}) := \{(H^1(I_1) \times H^1(I_1)), \dots, (H^1(I_M) \times H^1(I_M)) \\ \text{such that (4.1.7),(4.1.8),(4.1.5) hold}\}.$$

Let us consider the bilinear form $a : ((H^1(I_1) \times H^1(I_1)), \dots, (H^1(I_M) \times H^1(I_M)))^2 \to \mathbb{R}$ defined as

$$a((u^i, v^i), (\overline{u}^i, \overline{v}^i)) = \sum_{i\in\mathcal{M}} \int_{I_i} u^i\overline{u}^i + v^i\overline{v}^i + \lambda_i(v_x^i\overline{u}^i + u_x^i\overline{v}^i)dx.$$

We claim that this form is continuous and coercitive on $D(A)$. In fact we have

$$|a((u^i, v^i), (\overline{u}^i, \overline{v}^i))| \leq \sum_{i\in\mathcal{M}} (1 + \lambda_i)(||u^i||_{H^1}||\overline{u}^i||_{H^1} + ||v^i||_{H^1}||\overline{v}^i||_{H^1})$$

and moreover

$$|a((u^i, v^i), (u^i, v^i))| \geq \sum_{i\in\mathcal{M}} (||u^i||^2_{H^1} + ||v^i||^2_{H^1}).$$

To obtain the above inequality we have used the fact that the quantity

$$\lambda_i(v_x^i\overline{u}^i + u_x^i\overline{v}^i)dx \geq 0$$

is non negative thanks to conditions (4.2.20), as we have shown in the Proposition (4.2.2). Thus by Lax-Milgram Theorem (see Chapter 1) there exists a unique solution of (4.2.23) belonging in $(H^1(I_1) \times H^1(I_1)), \dots, (H^1(I_M) \times H^1(I_M))$. Observing that the solution $\{(u^i, v^i, \phi^i), i = 1, \dots, M\}$ belongs to $D(A)$ we get the proof. □

We have now proved that the operator A is monotone and maximal on its domain $D(A)$. Thanks to proposition (1.3.1) in Chapter 1, we get in particular the density of the domain $D(A)$ and the closure of A in the Hilbert space $\bigcup_{i\in\mathcal{M}}(L^2(I_i))^3$. Now we are ready to prove the main Theorem of this Chapter.

Theorem 4.2.1. *Let us consider the problem (4.2.2), with initial and boundary conditions (4.1.2), (4.1.3), (4.1.4), (4.1.7), (4.1.8), (4.1.13), (4.1.14), (4.1.5), and (4.1.6), and let $\xi_{i,j}$, $i, j \in \mathcal{M}$, verifying (4.2.20), i.e, $\sum_{j\in\mathcal{M}} \xi_{i,j} = 1$. Then there exists a unique global solution $w = (w^1, \dots, w^M)$, such that, for each i the function $w^i = (u^i, v^i, \phi^i)$ verifies*

$$w^i \in (C^1([0, +\infty); H^1(I_i)))^3 \cap ([0, +\infty); D(A_i)).$$

Proof. We have previously shown that under sufficient conditions (4.2.20) the operator A is monotone, closed and that its domain $D(A)$ is dense in the functional space $\bigcup_{i\in\mathcal{M}}(H^1(I_i))^3$. So A generates a semigroup of contraction $S(t)$ on $D(A)$; thus applying the Hille-Yosida Theorem, we get the proof. Moreover we observe that we can write the solution of the problem as $w(t) = S(t)w(0)$. □

Remark 4.2.2. *Let us observe that the Hille-Yosida Theorem ensures that if we choose an initial data $w_0^i \in D(A_i)$, $i \in \mathcal{M}$, then the solution $w(t)$ to the problem (4.2.2) belongs to $D(A)$ for each time of existence $t \geq 0$.*

4.3 Local existence of solutions to the general non-homogeneous case

Let $\mathcal{N}$ be an oriented network composed of M arcs I_i, $i = 1, \ldots, M$, and a single node N. As in the previous sections, let E the incoming arcs in the node, U the outcoming ones and $E \cup U = \mathcal{M}$, as in definition (2.5.1). We turn to consider the non homogeneous problem

$$\begin{cases} u_t^i + \lambda_i v_x^i = 0, \\ v_t^i + \lambda_i u_x^i = -v^i + G(u^i, \phi^i, \phi_x^i), \\ \phi_t^i - D_i \phi_{xx}^i = au^i - b\phi^i, \end{cases} \tag{4.3.1}$$

for each $i = 1, \ldots, M$, where the functions u^i,v^i,$\phi^i : I_i \times R^+ \to R$, and the coefficients $\lambda_i > 0$ and a,$b > 0$, while $G : H^1 \times H^1 \times H^1 \to H^1$ is a locally Lipschitz map. We complete the problem with initial and boundary conditions (4.1.2), (4.1.3), (4.1.4),(4.1.7), (4.1.8), (4.1.13), (4.1.14), (4.1.5), and (4.1.6), and transmission coefficients $\xi_{i,j}$, $i, j \in \mathcal{M}$, verifying $\sum_{j\in\mathcal{M}} \xi_{i,j} = 1$.

From now on, for each $i \in \mathcal{M}$, let us denote by

$$F^i = (0, G(u^i, \phi^i, \phi_x^i), au^i),$$

the source terms of problems (4.3.1).

4.3.1 Local existence of the solution

Our aim is to prove the existence and uniqueness of the solution of problem (4.3.1) using a fixed point method.

Theorem 4.3.1. *Let us consider the problem (4.3.1), coupled with initial data and boundary conditions (4.1.2), (4.1.3), (4.1.4),(4.1.7), (4.1.8), (4.1.13), (4.1.14), (4.1.5), and (4.1.6), and transmission coefficients $\xi_{i,j}$, $i, j \in \mathcal{M}$,* *verifying* $\sum_{j\in\mathcal{M}} \xi_{i,j} = 1$. *Let us assume that the function* $G(u^i, \phi^i, \phi^i_x)$ *is a locally Lipschitz map for each* $i \in \mathcal{M}$. *Then there exists a time* $T^* > 0$ *such that the problem* (4.3.1) *has a unique solution* (u^i, v^i, ϕ^i), $i \in \mathcal{M}$.

$$w^i = (u^i, v^i, \phi^i) \in (C([0,T]; H^1(I_i))^3, \tag{4.3.2}$$

for each $i = 1, \ldots, M$.

The proof follows two steps:

1. firstly we show the global existence of the solution under the assumption that the functions G is a globally Lipschitz map;

2. then we consider the case of local lipschitzianity.

Proof. Let $w^i_0(x,0) = (u^i_0, v^i_0, \phi^i_0) \in D(A_i)$, $i \in \mathcal{M}$, be the initial data of (4.3.1), and let $S(t)$ be the semigroup of contraction generated by the operator A on $D(A)$. By the monotonicity of the operator A (4.2.7) on its domain $D(A)$ follows that, by the semigroup theory and the Duhamel principle, we can write the solution of our problem as

$$\sum_{i\in\mathcal{M}} w^i(x,t) = \sum_{i\in\mathcal{M}} S(t) w^i_0(x) + \int_0^t S(t-s) F^i ds,$$

where $w^i(x,t) = (u^i(x,t), v^i(x,t), \phi^i(x,t))$, $w^i_0(x) = (u^i_0, v^i_0, \phi^i_0)$, and $\{S(t)\}_{t\geq 0}$ is the contraction semigroup generates by the operator A. In the following we will write $||.||_{L^2} = ||.||$, when no confusion arises.

Let G be a globally Lipschitz map with Lipschitz constant $K > 0$. Thus G verifies

$$\sum_{i\in\mathcal{M}} ||G(u^i_1, \phi^i_1, \phi^i_{1,x}) - G(u^i_2, \phi^i_2, \phi^i_{2,x})||$$
$$\leq \sum_{i\in\mathcal{M}} K(||u^i_1 - u^i_2|| + ||\phi^i_1 - \phi^i_2|| + ||\phi^i_{1,x} - \phi^i_{2,x}||),$$

for each vector $w^i = (u^i_1, \phi^i_1, \phi^i_{1,x})$ and $\widetilde{w}^i = (u^i_2, \phi^i_2, \phi^i_{2,x})$. First of all we show that for each $i \in \mathcal{M}$, F^i is globally Lipschitz too. In fact we have

$$\sum_{i\in\mathcal{M}} ||F^i(w^i) - F^i(\widetilde{w^i})||$$
$$= \sum_{i\in\mathcal{M}} a||u^i - \widetilde{u^i}|| + ||G(u^i, \phi^i, \phi^i_x) - G(\widetilde{u^i}, \widetilde{\phi^i}, \widetilde{\phi^i_x})||$$
$$\leq \sum_{i\in\mathcal{M}} a||u^i - \widetilde{u^i}|| + K(||u^i - \widetilde{u^i}|| + ||\phi^i - \widetilde{\phi^i}|| + ||\phi^i_x - \widetilde{\phi^i_x}||)$$

$$\leq \sum_{i\in\mathcal{M}}(a+K)||u^i-\widetilde{u^i}||+K||\phi^i-\widetilde{\phi^i}||+K||\phi^i_x-\widetilde{\phi^i_x}||)$$

$$\leq \sum_{i\in\mathcal{M}}\max\{a+K,K\}(||u^i-\widetilde{u^i}||+||\phi^i-\widetilde{\phi^i}||).$$

Let $\overline{K}=\max\{a^2+K,K\}$.

Our aim is to use a fixed point method to prove the existence of the solution, thus we have to fix a set in which solutions are well defined. Let $w=(u^1,v^1,\phi^1,\dots,u^M,v^M,\phi^M)$ and let $L>0$. Let us define the set

$$B=\{w\in D(A)\text{such that }\sum_{i\in\mathcal{M}}(||u^i(t)-u^i_0||+||v^i(t)-v^i_0||+||\phi^i(t)-\phi^i_0||)\leq L\};$$

this is a non empty set because the initial data w^i_0 belongs to B for each index $i\in\mathcal{M}$. Now let us define the map M on B such that

$$M(w):=\widetilde{w}=\sum_{i\in\mathcal{M}}(S(t)w^i_0(x)+\int_0^t S(t-s)F^i(w^i,\phi^i_x)ds).$$

We have to prove that, for small times $t>0$, the map M is a contraction on B. Firstly we prove that B is an invariant set. We have that

$$\sum_{i\in\mathcal{M}}(||\widetilde{u}^i(t)-u^i_0||+||\widetilde{v}^i(t)-v^i_0||+||\widetilde{\phi}^i(t)-\phi^i_0||)\leq\sum_{i\in\mathcal{M}}||S(t)w^i_0-w^i_0||$$

$$+\sum_{i\in M}||\int_0^t S(t-s)F^i(w^i,\phi^i_x)ds||.$$

Because of $S(t)$ is a contraction semigroup, then there exists $\overline{t}>0$ such that, for each $t<\overline{t}$,

$$\sum_{i\in\mathcal{M}}||S(t)w^i_0-w^i_0||\leq\frac{L}{2}.$$

Thus we have

$$\sum_{i\in\mathcal{M}}(||\widetilde{u}^i(t)-u^i_0||+||\widetilde{v}^i(t)-v^i_0||+||\widetilde{\phi}^i(t)-\phi^i_0||)\leq\frac{L}{2}+\sum_{i\in\mathcal{M}}\int_0^t||S^i(t-s)F^i(w^i,\phi^i_x)||ds$$

$$\leq\frac{L}{2}+\sum_{i\in\mathcal{M}}\overline{K}\int_0^t(||u^i||+||v^i||+||\phi^i||)ds\leq\frac{L}{2}+\sum_{i\in\mathcal{M}}t\overline{K}\sup_{0\leq s\leq t}(||u^i||+||v^i|+||\phi^i||).$$

Let us observe that the vector function $(u^1,v^1,\phi^1,\dots,u^M,v^M,\phi^M)\in B$, and so we have

$$\sum_{i\in\mathcal{M}}||w^i(t)||\leq\sum_{i\in\mathcal{M}}(||u^i(t)-u^i_0||+||v^i-v^i_0||+||\phi^i-\phi^i_0||)$$

$$+\sum_{i\in\mathcal{M}}(||u_0^i||+||v_0^i||+||\phi_0^i||)\leq\sum_{i\in\mathcal{M}}3L+(||u_0^i||+||v_0^i||+||\phi_0^i||);$$

so

$$\sum_{i\in\mathcal{M}}||\widetilde{w^i}(t)-w_0^i||\leq\frac{L}{2}+\sum_{i\in\mathcal{M}}t\overline{K}(3L+||u_0^i||+||v_0^i||+||\phi_0^i||).$$

If we choose a time t^* such that $t^*\leq\frac{L}{2(\overline{K}(3L+\sum_i||u_0^i||+||v_0^i||+||\phi_0^i||))}$, then we have

$$\sum_{i\in\mathcal{M}}||\widetilde{w^i}(t)-w_0^i||\leq L;$$

thus B is an invariant set. Now we have to prove that the map M is contraction on the set B, i.e. $||M(w_1)-M(w_2)||\leq h||w_1-w_2||$, with the positive constant $h<1$. For each $w_1,w_2\in B$, we have

$$||M(w_1)-M(w_2)||\leq\sum_{i\in\mathcal{M}}\int_0^t||S(t-s)(F^i(w_1^i,\phi_{1,x}^i)-F^i(w_2^i,\phi_{2,x}^i))||ds$$

$$\leq\sum_{i\in\mathcal{M}}\overline{K}\int_0^t(||u_1^i-u_2^i||+||v_1^i-v_2^i||+||\phi_1^i-\phi_2^i||)ds$$

$$\leq\sum_{i\in\mathcal{M}}\overline{K}t\sup_{0\leq s\leq t}(||u_1^i-u_2^i||+||v_1^i-v_2^i||+||\phi_1^i-\phi_2^i||).$$

If we choose t^* such that $t^{'}<\frac{1}{\overline{K}}$, then for $t\in(0,t^{'})$,

$$||M(w_1)-M(w_2)||\leq\sum_{i\in\mathcal{M}}h\sup_{0\leq s\leq t}(||u_1^i-u_2^i||+||v_1^i-v_2^i||+||\phi_1^i-\phi_2^i||),$$

with $h<1$, therefore for small t the map M is a contraction on the set B. Thus by the fixed point Theorem, the map M has a unique fixed point $\overline{w}$ on B. Moreover, the fixed point $\overline{w}$ belongs to $D(A)$ because the initial data $w_0^i\in D(A_i)$, $i\in\mathcal{M}$, i.e. the problem (4.3.1) has a unique solution in $(0,\min\{t^*,t^{'})\}$. Thanks to the global lipschitzianity of the functions F^i, we can iterate this procedure considering problem (4.3.1) with initial data $w^i(x,\min\{t^*,t^{'})\})$; thus we obtain a global solution in time.

Now let $w_0^i\in D(A_i)$, and let us assume that the map $G(u^i,\phi^i,\phi_x^i)$ is locally Lipschitz, i.e., for each compact set C_L

$$C_L=\{(u^i,\phi^i,\phi_x^i)\text{such that}\sum_{i\in\mathcal{M}}||u^i||+||\phi^i||+||\phi_x^i||\leq L\},$$

there exists a constant K_L such that

$$\sum_{i\in\mathcal{M}}||G(u_1^i,\phi_1^i,\phi_{1,x}^i)-G(u_2^i,\phi_2^i,\phi_{2,x}^i)||\leq\sum_{i\in\mathcal{M}}K_L(||u_1^i-u_2^i||+||\phi_1^i-\phi_2^i||+||\phi_{1,x}^i-\phi_{2,x}^i||),$$

for each vector $(u_1^i, \phi_1^i, \phi_{1,x}^i)$ and $(u_2^i, \phi_2^i, \phi_{2,x}^i)$ belonging in C_L. Let us observe that local lipschitzianity of G implies local lipschitzianity of F^i for each i, with Lipschitz constant $\overline{K_L} = \max\{a + K_L, K_L\}$. Now let us fix $L_0 > 0$ and let us consider the set

$$B_{L_0} = \{(u^i, v^i, \phi^i) : \sum_{i \in \mathcal{M}} ||u^i|| + ||v^i|| + ||\phi^i|| \leq L_0\};$$

let us denote with $\overline{K_{L_0}}$ the Lipschitz constant of F^i on B_{L_0}; now let $w_0^i = (u_0^i, v_0^i, \phi_0^i) \in B_{L_0}$ be the initial data of (4.3.1). Now, let $\chi_{[0,1]} : \mathbb{R}^+ \to \mathbb{R}$ be the characteristic function of interval $[0, 1]$. For each $i \in \mathcal{M}$, let us define the function

$$\overline{F^i}(u^i, \phi^i, \phi_x^i) = \chi(\frac{||u^i|| + ||\phi^i|| + ||\phi_x^i||}{L_0})F^i(u^i, \phi^i, \phi_x^i) =$$

$$\begin{cases} F^i, & \text{on } B_{L_0} \\ 0, & \text{on } B_{L_0}^c; \end{cases}$$

Let us denote by $\overline{F}^i = (0, \overline{G}(u^i, \phi^i, \phi_x^i), au^i)$, where

$$\overline{G}(u^i, \phi^i, \phi_x^i) = \begin{cases} G, & \text{on } B_{L_0} \\ 0, & \text{on } B_{L_0}^c; \end{cases}$$

Let us consider the problem (4.3.1) with source term $\overline{F^i}$, i.e.

$$\begin{cases} \overline{u^i}_t + \lambda_i \overline{v^i}_x = 0 \\ \overline{v^i}_t + \lambda_i \overline{u^i}_x + \overline{v^i} = \overline{G}(\overline{u^i}, \overline{\phi^i}, \overline{\phi^i}_x) \\ \overline{\phi^i}_t - D_i \overline{\phi^i}_{xx} = a\overline{u^i} - b\overline{\phi^i}, \end{cases}$$

for each $i \in \mathcal{M}$, with boundary conditions (4.1.7), (4.1.8), (4.1.13), (4.1.14), (4.1.5), and (4.1.6), and initial data $(u_0^i, v_0^i, \phi_0^i) \in B_{L_0}$. The new map $\overline{F^i}$ is globally Lipschitz on B_{L_0}, so, from the above proof, this new problem has a unique global solution $(\overline{u^i}, \overline{v^i}, \overline{\phi^i})$, $i = 1, \ldots, M$. We claim that there exists a time t^* such that the solution $(\overline{u^i}, \overline{v^i}, \overline{\phi^i})$, $i = 1, \ldots, M$ belongs to B_{L_0} for $t \in [0, t^*)$. In fact, let $\overline{w^i} = (\overline{u^i}, \overline{v^i}, \overline{\phi^i})$, $i = 1, \ldots, M$. We have that

$$\begin{aligned} \sum_{i \in \mathcal{M}} ||\overline{w^i}|| \leq \sum_{i \in \mathcal{M}} ||S(t)w_0^i|| &+ \int_0^t ||S(t-s)\overline{F^i}(\overline{u^i}, \overline{v^i}, \overline{\phi^i})||ds \\ &+ \int_0^t ||S(t-s)\overline{F^i}(\overline{u^i}, \overline{v^i}, \overline{\phi^i})||ds \\ \leq \sum_{i \in \mathcal{M}} ||w_0^i|| &+ \sum_{i \in \mathcal{M}} \overline{K_{L_0}} t \sup_{0 \leq s \leq t} (||\overline{w^i}||). \end{aligned}$$

Thus we have that

$$\sum_{i\in\mathcal{M}} \sup_{0\le s\le t} (||\overline{w^i}||) \le \frac{\sum_{i\in\mathcal{M}} ||w_0^i||}{1 - t\overline{K_{L_0}}},$$

with $t < \frac{1}{K_{L_0}}$, which implies that if $t^* < \frac{1}{K_{L_0}}$, then

$$\sum_{i\in\mathcal{M}} \sup_{0\le s\le t} (||\overline{w^i}||) \le L_0.$$

By definition of $\overline{F^i}$, we have that $\overline{F^i} = F^i$, $i \in \mathcal{M}$, on B_{L_0}, then from the uniqueness of the solution, we have that $\overline{w^i} = w^i$, for each $i = 1, \dots, M$ for $t \in [0, t^*)$. Moreover, we have that w^i belongs to $D(A_i)$, $i \in \mathcal{M}$, because $w_0^i \in D(A_i)$, i.e. we get a local solution of the problem (4.3.1), and this conclude the proof. □

4.4 Global existence of solutions in the case of a quadratic source term

4.4.1 The H^2-semigroup

Let $\mathcal{N}$ be an oriented network according to definition (2.5.1) of Chapter 2, composed of M oriented arcs I_i, $i = 1, \dots, M$, and a single node N. As before, we denote by E the set of arcs entering in the node, while by U the set of outgoing ones and we denote by $\mathcal{M} = E \cup U$. Moreover, let a_i, $i = 1, \dots, M$, be the outer boundary points of $\mathcal{N}$; we denote by $[a_i, N]$ an entering arc, and by $[N, a_i]$ an outgoing arc. In the previous section we have proved the uniqueness and local existence of the solution to problem (4.3.1), with initial data and boundary conditions respectively given by (4.1.2), (4.1.3), (4.1.4), (4.1.7), (4.1.8), (4.1.13), (4.1.14), (4.1.5), and (4.1.6) under the assumption of local lipschitzianity of the source term $G(u^i, \phi^i, \phi^i_x)$, for each $i \in \mathcal{M}$. In doing so it was sufficient to work with functions which for each $t \ge 0$ take values in the functional space $H^1(I_i)$, $u^i(t), v^i(t), \phi^i(t) \in H^1(I_i)$, and so we found a local solution $(u^i, v^i, \phi^i) \in (C([0, t^*]; H^1(I_i)))^3$, $t^* > 0$, for each $i \in \mathcal{M}$.

The aim of this section is to extend local solutions to the problem (4.3.1) to a global solutions. To do this we have to work with functions which take values in the functional space $H^2(I_i)$ to have a better control of the conditions at node, as we shall see in the following. In the following, given a network $\mathcal{N}$, we denote by $(u, v, \phi) \in (C([0, T]; X(\mathcal{N})))^3$ a solution defined on $\mathcal{N}$, which take values on a Banach space X.

So we are looking for a solution $(u, v, \phi) \in (C([0, T]; H^2(\mathcal{N})))^3$. First we prove the existence and uniqueness of the local solution of (4.3.1) in $H^2(I_i)$, $i \in \mathcal{M}$, i.e., $(u^i, v^i, \phi^i) \in ([0, T]; H^2(I_i)))^3$, $T > 0$, for each $i \in \mathcal{M}$.

To achieve this result we have to introduce some supplementary boundary conditions, which manage the transmission of the derivatives. Therefore we extend our solutions to global solutions in time by considering the following map $G(u^i, \phi^i, \phi^i_x)$:

$$G(u^i, \phi^i, \phi^i_x) = u^i \chi(\phi^i)\phi^i_x, \tag{4.4.1}$$

for each $i \in \mathcal{M}$, where the function $\chi(\phi^i)$ verifies some properties. So, for each $i \in \mathcal{M}$, we consider the symmetric hyperbolic-parabolic problem

$$\begin{cases} u^i_t + \lambda_i v^i_x = 0, \\ v^i_t + \lambda_i u^i_x = -v^i + u^i\chi(\phi^i)\phi^i_x, \\ \phi^i_t = D_i \phi^i_{xx} + au^i - b\phi^i, \end{cases} \tag{4.4.2}$$

where u^i is the density on each arc I_i, v^i is the average of the flux on I_i, and ϕ^i the density of the chemical signal. Parameters λ_i, D_i, a, and b are, respectively, the finite speed of propagation, the diffusion coefficient of chemical signal, the rate of release, and the rate of degradation on each interval I_i.

Let us assume that the function $\chi : I_i \times \mathbb{R}^+ \to R$ belongs to $W^{1,\infty}(I_i \times \mathbb{R}^+)$, i.e., that there exist positive real constants χ_0, χ_1 and χ_2 such that

$$|\chi(\phi)| \le \chi_0, |\chi'(\phi)| \le \chi_1. \tag{4.4.3}$$

We have seen that the difficult of this model comes from the presence of the node which forced us to choose specific transmission conditions on transmission in order to have the existence of the solution. In this Chapter we will prove a global existence results for solutions to (4.4.2), finding suitable energy estimates of functions in H^2.

Let $u^i = u^+_- + u^-_i$, $v^i = u^+_- - u^-_i$ as in (4.2.1), and let us consider problem (4.4.2) coupled with initial conditions

(4.4.4)
$$u^i(x,0) = u^i_0 \in H^2(I_i), v^i(x,0) = v^i_0 \in H^2(I_i),\ \phi^i(x,0) = \phi^i_0(x) \in H^3(I_i)$$

and boundary conditions as follows. On the outer boundary points we put no flux conditions

$$u^i_+(a_i,t) = u^i_-(a_i,t), \tag{4.4.5}$$

and

$$\phi^i_x(a_i,t) = 0. \tag{4.4.6}$$

On the node N we impose, as previously, the transmission conditions

$$u_i^-(N,t) = \sum_{j\in E} \xi_{i,j} u_j^+(N,t) + \sum_{j\in U} \xi_{i,j} u_j^-(N,t), \tag{4.4.7}$$

if $i \in E$, and

$$u_i^+(N,t) = \sum_{j\in E} \xi_{i,j} u_j^+(N,t) + \sum_{j\in U} \xi_{i,j} u_j^-(N,t), \tag{4.4.8}$$

if $i \in U$. Let us assume that transmission coefficients $\xi_{i,j}$ verify condition

$$\sum_{i\in\mathcal{M}} \lambda_i \xi_{i,j} = \lambda_j. \tag{4.4.9}$$

for each $j \in \mathcal{M}$ which guarantees the continuity of total flux on the node N, i.e.,

$$\sum_{i\in E} v^i(N,t) = \sum_{i\in U} v^i(N,t), \tag{4.4.10}$$

and the global mass conservation at any time $t > 0$,

$$\sum_{i\in\mathcal{M}} \int_{I_i} u^i(x,t)dx = \sum_{i\in\mathcal{M}} \int_{I_i} u_0^i(x).$$

Let us also assume the transmission coefficients verify the monotonicity condition that

$$\sum_{j\in\mathcal{M}} \xi_{i,j} = 1. \tag{4.4.11}$$

On the node N we also impose the continuity of the total flux of ϕ^i. For each index $i \in \mathcal{M}$ we impose the Kedem-Kadtchasky transmission conditions

$$D_i \phi_x^i(N,t) = \alpha \sum_{j\neq i} (\phi^j(N,t) - \phi^i(N,t)), \tag{4.4.12}$$

if $i \in E$, and

$$D_i \phi_x^i(N,t) = \alpha \sum_{j\neq i} (\phi^i(N,t) - \phi^j(N,t)), \tag{4.4.13}$$

if $i \in U$, where $\alpha > 0$ is a positive constant. This condition implies that

$$\sum_{i\in U} D_i \phi_x^i(N,t) = \sum_{i\in E} D_i \phi_x^i(N,t).$$

Now let us observe that the choice of looking for solutions in $H^2(I_i)$ forced us to impose other transmission conditions at the node for the problem. We first impose that

$$D_i \phi^i_{xx}(N,t) = \beta \sum_{i \neq j} (\phi^j_x(N,t) - \phi^i_x(N,t)), \tag{4.4.14}$$

if $i \in E$, and

$$D_i \phi^i_{xx}(N,t) = \beta \sum_{i \neq j} (\phi^i_x(N,t) - \phi^j_x(N,t)), \tag{4.4.15}$$

if $i \in U$, for a positive constant $\beta > 0$.

Then, as in the case of transmission conditions for functions $u_i^{\pm}$, we want that on each arc I_i, each spatial derivative of $u_i^{\pm}$ is a linear combination of each other spatial derivatives of functions on the other arcs. So we are looking for some coefficients $\widetilde{\xi_{i,j}}$ such that

$$\partial_x(u_i^-)(N,t) = \sum_{j \in E} \widetilde{\xi_{i,j}} \partial_x(u_j^+)(N,t) + \sum_{j \in U} \widetilde{\xi_{i,j}} \partial_x(u_j^-)(N,t) \tag{4.4.16}$$

if $i \in E$, and

$$\partial_x(u_i^+)(N,t) = \sum_{j \in E} \widetilde{\xi}_{i,j} \partial_x(u_j^+)(N,t) + \sum_{j \in U} \widetilde{\xi_{i,j}} \partial_x(u_j^-)(N,t), \tag{4.4.17}$$

if $i \in U$.

Now, we recall that we are looking for global solutions, and in order to do this, we impose the following crucial transmission condition on the node N:

$$\sum_{i \in E} \lambda_i^2 (\partial_x(v^{i,2}))(N,t) = \sum_{i \in U} \lambda_i^2 (\partial_x(v^{i,2}))(N,t). \tag{4.4.18}$$

Up to now, this condition has only a technical meaning but its role will be clear in the following. Computing the above conditions on the fluxes, we find that the transmission coefficients $\widetilde{\xi}_{i,j}$, $i,j \in \mathcal{M}$, have to satisfy some compatibility conditions.

We first begin showing conditions on transmission coefficients $\widetilde{\xi}_{i,j}$ in the case of two arcs.

Case of two arcs: transmission conditions for coefficients $\widetilde{\xi}_{i,j}$

We have that in this case the conditions for the spatial derivatives of functions $u^-_{1,x}(N,t)$ and $u^+_{2,x}(N,t)$ on the node are

$$u^-_{1,x}(N,t) = \widetilde{\xi}_{1,1} u^+_{1,x}(N,t) + \widetilde{\xi}_{1,2} u^-_{2,x}(N,t), \tag{4.4.19}$$

and

$$u^+_{2,x}(N,t) = \widetilde{\xi}_{2,1} u^+_{1,x}(N,t) + \widetilde{\xi}_{2,2} u^-_{2,x}(N,t), \tag{4.4.20}$$

for some coefficients $\widetilde{\xi}_{i,j}$, $i,j = 1,2$. Then, in this case, condition (4.4.18) becomes

$$\lambda_1^2 v^1(N,t) v^1_x(N,t) = \lambda_2^2 v^2(N,t) v^2_x(N,t); \tag{4.4.21}$$

From the condition of flux conservation (4.4.10) we have that $\lambda_1 v^1(N,t) = \lambda_2 v^2(N,t)$, thus we must have that

$$\lambda_1 v^1_x(N,t) = \lambda_2 v^2_x(N,t), \tag{4.4.22}$$

i.e., we are asking the continuity of the spatial derivatives of total flux on the node N. Substituting in the above expression the transmission conditions (4.4.19) and (4.4.19) we find the following conditions on coefficients $\widetilde{\xi}_{i,j}$, $i,j = 1,2$:

$$\lambda_1 \widetilde{\xi}_{1,1} + \lambda_2 \widetilde{\xi}_{2,1} = \lambda_1, \tag{4.4.23}$$

and

$$\lambda_1 \widetilde{\xi}_{1,2} + \lambda_2 \widetilde{\xi}_{2,2} = \lambda_2. \tag{4.4.24}$$

Let us observe that the above conditions are of the same type that the condition (4.4.9) for coefficients $\xi_{i,j}$ that we found in order to have the continuity of the total flux at node.

Case of M arcs: transmission conditions for the coefficients $\widetilde{\xi}_{i,j}$

In this case, we want to show that condition (4.4.18) can be satisfied, i.e.: there exist some transmission coefficients $\widetilde{\xi}_{i,j}$, $i,j \in \mathcal{M}$ such that

$$\sum_{i \in E} \lambda_i^2 v^i(N,t) v^i_x(N,t) = \sum_{i \in U} \lambda_i^2 v^i(N,t) v^i_x(N,t). \tag{4.4.25}$$

Let us introduce the new variables

$$\beta_{i,j} := \begin{cases} \xi_{i,i} - 1, & \text{if } i = j \\ \xi_{i,j}, & \text{if } i \neq j \end{cases}$$

$$\text{and } \gamma_{i,j} := \begin{cases} \widetilde{\xi}_{i,i} - 1, & \text{if } i = j \\ \widetilde{\xi}_{i,j}, & \text{if } i \neq j \end{cases}$$

Let us observe that from conditions (4.4.9) and (4.4.11) on transmission coefficients $\xi_{i,j}$, $i,j \in \mathcal{M}$, we have that

$$\sum_{i\in\mathcal{M}} \lambda_i \beta_{i,j} = 0, \text{for each } j \in \mathcal{M}$$

and

$$\sum_{j\in\mathcal{M}} \beta_{i,j} = 0, \text{for each } i \in \mathcal{M}.$$

Now, from conditions (4.4.7), (4.4.8), (4.4.16), (4.4.17) we have that in the above variables

$$v^i(N,t) = -\sum_{i\in E} \beta_{i,j} u_j^+(N,t) - \sum_{i\in U} \beta_{i,j} u_j^-(N,t), \text{if } i \in E,$$
$$v^i(N,t) = \sum_{i\in E} \beta_{i,j} u_j^+(N,t) + \sum_{i\in U} \beta_{i,j} u_j^-(N,t), \text{if } i \in U$$

and

$$v_x^i(N,t) = -\sum_{i\in E} \gamma_{i,j} u_{j,x}^+(N,t) - \sum_{i\in U} \gamma_{i,j} u_{j,x}^-(N,t), \text{if } i \in E,$$
$$v_x^i(N,t) = \sum_{i\in E} \gamma_{i,j} u_{j,x}^+(N,t) + \sum_{i\in U} \gamma_{i,j} u_{j,x}^-(N,t) \text{if } i \in U.$$

Thus, computing (4.4.25) yields

$$\sum_{i\in E} \lambda_i^2 v^i(N,t) v_x^i(N,t) - \sum_{i\in U} \lambda_i^2 v^i(N,t) v_x^i(N,t)$$
$$= \sum_{i\in E} \lambda_i^2 \Big(\sum_{i\in E} \beta_{i,j} u_j^+(N,t) + \sum_{i\in U} \beta_{i,j} u_j^-(N,t)\Big)\Big(\sum_{i\in E} \gamma_{i,j} u_{j,x}^+(N,t) + \sum_{i\in U} \gamma_{i,j} u_{j,x}^-(N,t)\Big)$$
$$- \sum_{i\in U} \lambda_i^2 \Big(\sum_{i\in E} \beta_{i,j} u_j^+(N,t) + \sum_{i\in U} \beta_{i,j} u_j^-(N,t)\Big)\Big(\sum_{i\in E} \gamma_{i,j} u_{j,x}^+(N,t) + \sum_{i\in U} \gamma_{i,j} u_{j,x}^-(N,t)\Big) = 0.$$

The above expression is an indefinite bilinear form in the variables $(u_i^+, u_j^-, u_{i,x}^+, u_{j,x}^-)$, $i \in E$, $j \in U$. Therefore it is identically null if and only if all its coefficients are equal to zero. By developing calculation we obtain the following result.

Proposition 4.4.1. *Let us consider the coefficients $\beta_{i,j}$ and $\gamma_{i,j}$, $i,j \in \mathcal{M}$, defined as*

$$\beta_{i,j} := \begin{cases} \xi_{i,i} - 1, & \text{if } i = j \\ \xi_{i,j}, & \text{if } i \neq j \end{cases}$$

$$\text{and } \gamma_{i,j} := \begin{cases} \widetilde{\xi}_{i,i} - 1, & \text{if } i = j \\ \widetilde{\xi}_{i,j}, & \text{if } i \neq j \end{cases}$$

The condition (4.4.18) is satisfies if the following condition holds

$$\sum_{i\in E} \lambda_i^2 \beta_{i,j}\gamma_{i,h} - \sum_{i\in U} \lambda_i^2 \beta_{i,j}\gamma_{i,h} = 0, \tag{4.4.26}$$

for each $j,h \in \mathcal{M}$. Moreover, for each set of coefficients $\beta_{i,j}$, $i,j \in \mathcal{M}$, such that (4.4.26) holds, it is possible to find a non empty set of values for the coefficients $\gamma_{i,j}$, $i,j \in \mathcal{M}$.

Proof. We have shown that condition (4.4.18) is satisfies if (4.4.26). Now we have that for each fixed $h \in \mathcal{M}$, the expression (4.4.26) is a linear and homogeneous system in the variables $(\gamma_{1,h}, \dots, \gamma_{M,h})$, whose associated matrix is

$$\begin{pmatrix} \lambda_1^2\beta_{1,1} & \dots & \lambda_{|E|}^2\beta_{|E|,1} & -\lambda_{|E|+1}^2\beta_{|E|+1,1} & \dots & -\lambda_M^2\beta_{M,1} \\ \vdots & \dots & \dots & \ddots & \dots & \dots \\ \vdots & \dots & \dots & \ddots & \dots & \dots \\ \vdots & \dots & \dots & \ddots & \dots & \dots \\ \vdots & \dots & \dots & \ddots & \dots & \dots \\ \lambda_M^2\beta_{1,M} & \dots & \lambda_{|E|}^2\beta_{|E|,M} & -\lambda_{|E|+1}^2\beta_{|E|+1,M} & \dots & -\lambda_M^2\beta_{M,M} \end{pmatrix}$$

The rows of the above matrix are linearly dependent thanks to condition that $\sum_{j\in E}\beta_{i,j} = 0$, for each $i \in \mathcal{M}$. Thus, for each $h \in \mathcal{M}$, the above system admits non trivial solutions. □

The reminder of this section is organized as follows: first we will show the uniqueness and local existence of solution to (4.4.2), $(u,v,\phi) \in (C(\mathbb{R}^+; H^2(\mathcal{N})))^3$ under the new boundary conditions with a monotonicity requirement on coefficients. Then we will find suitable energy estimates for the local solutions (u,v,ϕ) in order to prove the global existence Theorem. Proof of this Theorem is based on a Continuation Principle and on the estimates a particular functional in the vector variables (u^i, v^i, ϕ^i), $i \in \mathcal{M}$.

Local existence of smooth solutions

Let us consider the restriction of the differential operators A^i associated to problem (4.4.2) and defined in (4.2.1) to the set $D_\chi(A^i)$,

$$D_\chi(A^i) := \{(u^i, v^i, \phi^i) \in H^2(I_i) \times H^2(I_i) \times H^3(I_i) : (4.4.5), (4.4.6), (4.4.27)$$

(

4.4.7), (4.4.16), (4.4.8), (4.4.17), (4.4.12), (4.4.13), (4.4.14), (4.4.15), $u_x^i(a_i, t) = 0$ hold$\}$.

and the restriction of the operator A defined in (4.2.1) to the set $D_\chi(A)$,

$$D_\chi(A) := \bigcup_{i \in \mathcal{M}} D_\chi(A^i) \subset D(A). \tag{4.4.28}$$

We have that

(4.4.29)

$$A^i(D_\chi(A^i)) \subset (H^1(I_i) \times H^1(I_i) \times H^1(I_i)) \subset (L^2(I_i) \times L^2(I_i) \times L^2(I_i)),$$

and

(4.4.30)

$$A(D_\chi(A)) \subset \bigcup_{i \in \mathcal{M}} (H^1(I_i) \times H^1(I_i) \times H^1(I_i)) \subset \bigcup_{i \in \mathcal{M}} (L^2(I_i) \times L^2(I_i) \times L^2(I_i)).$$

The aim of this section is to prove the following Theorem.

Theorem 4.4.1. *Let us consider the hyperbolic-parabolic problem* (4.4.2)*, with initial and boundary conditions* (4.4.4)*,* (4.4.5)*,* (4.4.6)*,* (4.4.7)*,* (4.4.16)*,* (4.4.8)*,* (4.4.17)*,* (4.4.12)*,* (4.4.13)*,* (4.4.14)*,* (4.4.15)*, and the function* $\chi \in W^{2,\infty}$*, which verifies* (4.4.3)*. Let us assume that the transmission coefficients verify* (4.4.11) *for each* $i \in \mathcal{M}$*, and that*

$$\sum_{j \in \mathcal{M}} |\widetilde{\xi}_{i,j}| \leq 1, \tag{4.4.31}$$

for each $i \in \mathcal{M}$*. Then there exists a time* t^* *such that there exists a unique solution* $(u^i, v^i, \phi^i) \in (C([0, t^*); H^2(I_i)))^3$*,* $i \in \mathcal{M}$*.*

The strategy of the proof is the following: we show the monotonicity and maximality of the operator A on its domain $D_\chi(A)$, thus A generates a contraction semigroup on $D(A)$; then we use a point fixed Theorem to prove local existence of solution of (4.4.2). Let us observe that to prove local existence of solutions we do not need of condition (4.4.18). This condition is only needed to prove the global existence of solutions.

Proof. First we have to show that the operator A is monotone and maximal on its domain $D_\chi(A)$. Let us observe that the maximality of A on $D_\chi(A)$ follows as in the proof of Proposition (4.2.3). Moreover, in the previous section we have shown that A is monotone on $D(A)$, thus to establish the monotonicity of A on $D_\chi(A)$ we have to prove that the inner product

$$\sum_{i\in\mathcal{M}}(A^iZ^i)_x\cdot Z^i_x\geq 0, \tag{4.4.32}$$

where $Z^i=(u^i,v^i,\phi^i)$, $i\in\mathcal{M}$. Computing the first derivatives with respect to the variable x of the problem (4.4.2) we obtain that

$$\sum_{i\in\mathcal{M}}(A^iZ^i)_x\cdot Z^i_x=\sum_{i\in E}\lambda_i u^i_x v^i_x(N)-\sum_{i\in U}\lambda_i u^i_x v^i_x(N) \tag{4.4.33}$$
$$-\sum_{i\in E}D_i\phi^i_{xx}(N)\phi^i_x(N)+\sum_{i\in U}D_i\phi^i_{xx}(N)\phi^i_x(N).$$

We recall that $u^i_x(a_i,t)=0$. First let us consider the hyperbolic part, namely

$$S_2=\sum_{i\in E}\lambda_i u^i_x(N)v^i_x(N)-\sum_{i\in U}\lambda_i u^i_x(N)v^i_x(N).$$

We recall that for each $i\in\mathcal{M}$ we have $u^i=u^+_i+u^-_i$ and $v^i=u^+_i-u^-_i$, so substituting these quantities in the above expression we get

$$S_2=\sum_{i\in E}\lambda_i((u^{+,i}_x)^2-(u^{-,i}_x)^2)-\sum_{i\in U}\lambda_i((u^{+,i}_x)^2-(u^{-,i}_x)^2); \tag{4.4.34}$$

We recall that conditions at the node are

$$\partial_x(u^-_i)(N,t)=-\sum_{j\in E}\widetilde{\xi}_{i,j}\partial_x(u^+_j)(N,t)+\sum_{j\in U}\widetilde{\xi}\partial_x(u^-_j)(N,t)$$

if $i\in E$, and

$$\partial_x(u^+_i)(N,t)=\sum_{j\in E}\widetilde{\xi}_{i,j}\partial_x(u^+_j)(N,t)-\sum_{j\in U}\widetilde{\xi}_{i,j}\partial_x(u^-_j)(N,t),$$

if $i \in U$. Substituting conditions at the node in (4.4.34), we find the following bilinear form

$$\sum_{j\in E}\lambda_j(u^+_{j,x})^2+\sum_{k\in U}\lambda_k(u^-_{k,x})^2-\sum_{(j,k)\in E\times E}(\sum_{i\in\mathcal{M}}\lambda_i\widetilde{\xi}_{i,j}\widetilde{\xi}_{i,k})u^+_{j,x}u^+_{k,x}$$

$$-\sum_{(j,k)\in U\times U}(\sum_{i\in\mathcal{M}}\lambda_i\widetilde{\xi}_{i,j}\widetilde{\xi}_{i,k})u^-_{j,x}u^-_{k,x}$$

$$-2\sum_{(j,k)\in E\times U}(\sum_{i\in\mathcal{M}}\lambda_i\widetilde{\xi}_{i,j}\widetilde{\xi}_{i,k})u^+_{j,x}u^-_{k,x}.$$

Following the proof of monotonicity of A (4.2.2), we have that a sufficient condition in order to have the non negativity of the above quantity is

$$\sum_{j\in\mathcal{M}}|\widetilde{\xi}_{i,j}|\leq 1,$$

i.e., condition (4.4.31) is satisfied. Thus we have concluded the study of the hyperbolic part.

Now we have to study the non negativity of the quantity

$$S_\phi=-\sum_{i\in E}D_i\phi^i_{xx}(N)\phi^i_x(N)+\sum_{i\in U}D_i\phi^i_{xx}(N)\phi^i_x(N).$$

Substituting the conditions (4.4.14), and (4.4.15) in the above expression, and following the proof of (4.2.2) we obtain that

$$S_\phi=\beta\sum_{i\neq j}(\phi^j_x(N,t)-\phi^i_x(N,t))^2,$$

which is always non negative thank to choice of $\beta>0$.

So we have obtained that the operator A is monotone on $D_\chi(A)$. Now, let us observe that in this case the boundary condition $u^i_x(a_i,t)=0$ follows from the boundary conditions $v^i(a_i,t)=0$ and $\phi^i_x(a_i,t)=0$. Moreover, we observe that the source terms $u^i\chi(\phi^i)\phi^i_x$, $i\in\mathcal{M}$, are locally Lipschitz maps. So, following the proof of Theorem (4.3.1), we have that there exists a time $t^*>0$ such that we get the local existence and the uniqueness of solution to (4.4.2), $(u^i,v^i,\phi^i)\in(C([0,t^*);H^2(I_i)))^3$. $i\in\mathcal{M}$. □

4.4.2 Global existence of smooth solutions

The aim of this section is to prove the global existence of solution to problem (4.4.2). To prove this fact, we observe that condition (4.4.18) for the total flux in the node N is crucial.

In order to do this, first of all we recall that a local solution to (4.4.2), $(u^i,v^i,\phi^i)\in(C([0,t^*);H^2(I_i)))^3$, $t^*>0$, exists and in particular it belongs to $D_\chi(A)$ for each time of its existence. Then we will find a priori uniform in time estimates for the local solution. We observe that we are working with the norm of the codomain of the operator A, $\bigcup_{i\in\mathcal{M}}(H^1(I_i))^3$. Therefore, thanks to these uniform estimates, we can extend the local solution to a global one. Finally, this solution belongs to $D(A)$ for each time $t \geq 0$ thanks to Theorem (1.3.1) (see Chapter 1).

Thus we want to prove the following Theorem.

Theorem 4.4.2. *Let us consider the hyperbolic-parabolic problem* (4.4.2)*, coupled with initial and boundary conditions* (4.4.4)*,* (4.4.5)*,* (4.4.6)*,* (4.4.7)*,* (4.4.16)*,* (4.4.8)*,* (4.4.17)*,* (4.4.13)*,* (4.4.14)*,* (4.4.15)*, and the function* $\chi \in W^{2,\infty}$*, which verifies* (4.4.3)*. Let us assume that condition* (4.4.18) *is satisfied, and that transmission coefficients* $\xi_{i,j}$ *and* $\widetilde{\xi}_{i,j}$ *satisfy conditions* (4.4.11) *and* (4.4.31)*.*

Then there exists a unique global solution $u^i \in (C([0,\infty]; H^2(I_i))$*,* $v^i \in (C([0,\infty]; H^2(I_i))$*,* $\phi^i \in (C([0,\infty]; H^2(I_i))$*,* $i \in \mathcal{M}$*.*

For the following, we recall the following Continuation Principle for local solutions (see [19] and [1]).

Theorem 4.4.3. *(Continuation Principle) Let* $T = T_{\max} < +\infty$ *the maximal time of existence of a local solution of the problem* (4.4.2)*, coupled with initial and boundary conditions* (4.4.4)*,* (4.4.5)*,* (4.4.6)*,* (4.4.7)*,* (4.4.8)*,* (4.4.16)*,* (4.4.17)*,* (4.4.13)*,*(4.4.14)*,* (4.4.15)*, and the function* $\chi \in W^{2,\infty}$*, which verifies* (4.4.3)*. Then*

$$\limsup_{t\to T^-} \sum_{i\in\mathcal{M}} (||u^i||_{H^2(I_i)} + ||v^i||_{H^2(I_i)} + ||\phi^i||_{H^2(I_i)}) = +\infty.$$

Proof. Let (u^i, v^i, ϕ^i), $i = 1, \ldots, M$ a local solution on the maximal interval of time $[0, T_{\max})$. Let us consider a time $T > T_{\max}$, and suppose that there exists an a priori estimate for the solution (u^i, v^i, ϕ^i) such that

$$\sum_{i\in\mathcal{M}} \sup_{[0,T_{\max})} (||u^i||_{H^2(I_i)} + ||v^i||_{H^2(I_i)} + ||\phi^i||_{H^2(I_i)}) \leq K.$$

Let (u_0^i, v_0^i, ϕ_0^i), $i = 1, \ldots, M$, such that $\sum_{i\in\mathcal{M}} ||u_0^i||_{H^2(I_i)} + ||v_0^i||_{H^2(I_i)} + ||\phi_0^i||_{H^2(I_i)}) \leq K$ be initial data for our problem, and let $\frac{T_L}{2} > 0$ be the maximal time of existence of the corresponding solution of this problem. Then there exists a time $\widetilde{t} \in (T_{\max} - \frac{T_L}{2}, T_{\max})$ such that we can consider our problem with initial data $(u^i(x, \widetilde{t}), v^i(x, \widetilde{t}), \phi^i(x, \widetilde{t}))$, with $i = 1, \ldots, M$. So, the corresponding solution of this new problem extends the solution with initial data (u_0^i, v_0^i, ϕ_0^i), $i = 1, \ldots, M$, and exists for a maximal time $\widetilde{T} = \widetilde{t} + \frac{T_L}{2} > T_{\max}$, but this is a contradiction. □

By the Continuation Principle, if we find a priori uniform estimates in time for the local solution to (4.4.2), then it exists for all times $t \geq 0$.

Now, let $w^i(x,t) := (u^i, v^i, \phi^i)$, $i = 1, \dots, M$, be the local solution of (4.4.2). For each $i = 1, \dots, M$, let us define the following functionals:

$$F_i^2(t) := \sup_{0 \leq s \leq t} ||w^i(s)||_{H^1}^2 + \int_0^t ||w^i(s)||_{H^1}^2 ds, \tag{4.4.35}$$

and

$$F^2(t) = \sum_{i \in \mathcal{M}} F_i^2(t). \tag{4.4.36}$$

The proof of the global existence Theorem (4.4.2) follows two principal steps:

1. for each fixed time $t \geq 0$, we will find suitable energy estimates in the space $H^1(I_i) \times H^1(I_i) \times H^1(I_i)$ for functions $(u^i(t), v^i(t), \phi^i(t))$, $i = 1, \dots, M$;

2. then, we will find and estimate for the functional $F^2(t)$ using the energy estimates for (u^i, v^i, ϕ^i), $i = 1, \dots, M$, in order to use the Continuation Principle to extend the local solution to (4.4.2) to a global one.

We recall the following Lemma due to Nishida (see [19]).

Lemma 4.4.1. *(Nishida) Let $F^2(t)$ be the functional defined in (4.4.36). Let $T > 0$ and let $w^i = (u^i, v^i, \phi^i)$, $i \in \mathcal{M}$ be a local solution to the problem (4.4.2) in the interval $[0, T]$. Let us assume that there exist positive constants $k > 0$ and $C > 0$ such that if $F^2(t) \leq k$, then*

$$F^2(t) \leq C(F^2(0) + F^3(T)). \tag{4.4.37}$$

Then the solution w^i, $i \in \mathcal{M}$ exists for all times $t \in [0, +\infty]$.

We start with an energy estimate for the function ϕ^i, $i \in \mathcal{M}$, which will be useful in the following.

Lemma 4.4.2. *(Energy estimate for $\int_0^t ||\phi_{xx}^i||_{L^2(I_i)}^2 ds$) Let ϕ^i, $i \in \mathcal{M}$, be a local solution to the problem (4.4.2). Then for each $i \in \mathcal{M}$ it verifies*

(4.4.38)
$$\int_0^t ||\phi_{xx}^i||_{L^2(I_i)}^2 ds \leq (\frac{a(a+b)}{D_i} + \alpha a) \int_0^t ||u^i||_{H^1}^2 + \frac{b(a+b)}{D_i} \int_0^t ||\phi^i||_{L^2}^2$$
$$+\alpha(D_i \alpha\beta + a + b) \int_0^t \sum_{i \in \mathcal{M}} ||\phi^i||_{H^1}^2 ds + ||\phi_0^i||_{H^1}^2.$$

Proof. From the equation for ϕ^i, we have that, for each $i \in \mathcal{M}$,

$$D_i \phi^i_{xx} = \phi^i_t - au^i + b\phi^i.$$

Let $i \in E$ (calculus for $i \in U$ are analogous). Multiplying by ϕ^i_{xx} and integrating on I_i and $[0,t]$, we have that, using the Cauchy-Schwartz inequality,

$$D_i \int_0^t ||\phi^i_{xx}||^2_{L^2} \leq \frac{a}{2\varepsilon} \int_0^t ||u^i||^2_{L^2} + \frac{b}{2\varepsilon} \int_0^t ||\phi^i||^2_{L^2} + \frac{\varepsilon(a+b)}{2} \int_0^t ||\phi^i_{xx}||^2_{L^2} + \int_0^t \int_{I_i} \phi^i_t \phi^i_{xx} dxds. \tag{4.4.39}$$

We have

$$\int_0^t \int_{I_i} \phi^i_t \phi^i_{xx} dxds = \int_0^t \phi^i_x(N,s)\phi^i_t(N,s)ds - \frac{1}{2}\int_0^t \int_{I_i} \partial_t((\phi^i_x)^2)dxds$$
$$= \int_0^t \phi^i_x(N,s)(D_i\phi^i_{xx}(N,s) + au^i(N,s) - b\phi^i(N,s))ds + \frac{1}{2}||\phi^i_0||^2_{H^1}.$$

We have

$$D_i \int_0^t \phi^i_x(N,s)\phi^i_{xx}(N,s)ds \leq D_i\alpha\beta \int_0^t \sum_{j\neq i}(\phi^j(N,s) - \phi^i(N,s)) \sum_{j\neq i}(\phi^j_x(N,s) - \phi^i_x(N,s))$$
$$\leq \frac{D_i\alpha^2\beta}{2} \int_0^t \sum_{i\in\mathcal{M}} ||\phi^i||^2_{H^1(I_i)} ds.$$

Moreover, we have that

$$a \int_0^t \phi^i_x(N,s)u^i(N,s)ds \leq \frac{\alpha a}{2} \int_0^t \sum_{i\in\mathcal{M}} ||\phi^i||^2_{H^1(I_i)} + ||u^i||^2_{H^1} ds,$$

and

$$b \int_0^t \phi^i_x(N,s)\phi^i(N,s)ds \leq \frac{\alpha b}{2} \int_0^t \sum_{i\in\mathcal{M}} ||\phi^i||^2_{H^1(I_i)} ds.$$

Substituting the above inequalities in (4.4.39) we get

$$(D_i - \frac{\varepsilon(a+b)}{2} \int_0^t ||\phi^i_{xx}||^2_{L^2} \leq \frac{a}{2\varepsilon} \int_0^t ||u^i||^2_{L^2} + \frac{b}{2\varepsilon} \int_0^t ||\phi^i||^2_{L^2}$$
$$+ \frac{D_i\alpha^2\beta + \alpha a + \alpha b}{2} \int_0^t \sum_{i\in\mathcal{M}} ||\phi^i||^2_{H^1(I_i)} ds$$
$$+ \frac{\alpha a}{2} \int_0^t ||u^i||^2_{H^1} ds.$$

Now, choosing $\varepsilon = \frac{a+b}{4D_i}$ we get the proof. □

Lemma 4.4.3. *(Zero order estimate for the functions (u^i, v^i, ϕ^i), $i = 1, \dots, M$) Let (u^i, v^i, ϕ^i), $i = 1, \dots, M$, be the local solution to the problem (4.4.2). Then it verifies*

$$\sum_{i\in\mathcal{M}}(||w^i||^2_{L^2} + \int_0^t ||v^i||^2_{L^2} ds) \le \sum_{i\in\mathcal{M}}(2||w^i_0||^2_{L^2} + \frac{a}{2}\int_0^t ||u^i||^2_{L^2} ds$$
$$+(\frac{a}{2} - b)\int_0^t ||\phi^i||^2_{L^2} + \frac{\chi_0}{2\lambda_i}\sup_{0\le s\le t}||u^i||_{H^1}\int_0^t ||\phi^i_x||^2_{L^2} + ||v^i||^2_{L^2} ds). \tag{4.4.40}$$

Proof. For each $i \in \mathcal{M}$, let $w_i := (u^i, v^i, \phi^i)$ be the local solution to (4.4.2). Multiplying the problem (4.4.2) for the vector w^i and integrating on I_i we have that

$$\frac{\partial t}{2}\int_{I_i}(w^i)^2 dx \le -\int_{I_i}(v^i)^2 dx + \int_{I_i} u^i\chi(\phi^i)\phi^i_x v^i dx + a\int_{I_i} u^i\phi^i dx - b\int_{I_i}\phi^{i,2} dx;$$

we observe that we have no influence from the boundary terms for (u^i, v^i, ϕ^i) thanks to the monotonicity of the operator A (4.2.7) founded in the previous sections. By the Cauchy-Schwartz inequality and the boundness of the function $\chi(\phi^i)$, and then integrating on 0 and t, we get the following estimate:

$$||w^i||^2_{L^2} + \int_0^t ||v^i||^2_{L^2} ds \le 2||w^i_0||^2_{L^2} + \frac{a}{2}\int_0^t ||u^i||^2_{L^2} ds$$
$$+(\frac{a}{2} - b)\int_0^t ||\phi^i||^2_{L^2} + \frac{\chi_0}{2\lambda_i}\sup_{0\le s\le t}||u^i||_{H^1}\int_0^t\int_0^t ||\phi^i_x||^2_{L^2} + ||v^i||^2_{L^2} ds. \tag{4.4.41}$$

Summing up on $i \in \mathcal{M}$ we obtain

$$\sum_{i\in\mathcal{M}}(||w^i||^2_{L^2} + \int_0^t ||v^i||^2_{L^2} ds) \le \sum_{i\in\mathcal{M}}(2||w^i_0||^2_{L^2} + \frac{a}{2}\int_0^t ||u^i||^2_{L^2} ds$$
$$+(\frac{a}{2} - b)\int_0^t ||\phi^i||^2_{L^2} + \frac{\chi_0}{2\lambda_i}\sup_{0\le s\le t}||u^i||_{H^1}\int_0^t\int_0^t ||\phi^i_x||^2_{L^2} + ||v^i||^2_{L^2} ds), \tag{4.4.42}$$

and we get the proof. □

Lemma 4.4.4. *(First order estimate for the functions (u^i, v^i, ϕ^i), $i = 1, \dots, M$) For each $i \in \mathcal{M}$, let $w^i := (u^i, v^i, \phi^i)$ be the local solution to the problem (4.4.2). Let us assume that transmission coefficients $\widetilde{\xi}_{i,j}$, $i, j \in \mathcal{M}$, verify (4.4.31). Then w^i, $i \in \mathcal{M}$, satisfies*

$$\sum_{i\in\mathcal{M}}(||w^i_x||^2_{L^2} + \int_0^t ||v^i_x||^2_{L^2} ds) \le \sum_{i\in\mathcal{M}}(2||(w^i_0)_x||^2_{L^2} + \frac{a}{2}\int_0^t ||u^i_x||^2_{L^2} ds \tag{4.4.43}$$

$$+(\frac{a}{2}-b)\int_0^t ||\phi_x^i||_{L^2}^2 ds + \frac{\chi_0}{2}\int_0^t ||u_x^i||_{L^2}^2 ds + \frac{\chi_0}{4}\int_0^t ||v_x^i||_{L^2}^4 ds$$
$$+\frac{\chi_0}{4}\int_0^t ||\phi_x^i||_{L^2}^4 ds + \frac{\chi_1}{2}\sup_{0\le s\le t}||u^i||_{H^1}\int_0^t (||v_x^i||_{L^2}^2 + ||\phi_x^i||_{L^2}^4) ds$$
$$+\frac{\chi_0}{2}\sup_{0\le s\le t}||u^i||_{H^1}\int_0^t (||v_x^i||_{L^2}^2 + ||\phi_{xx}^i||_{L^2}^2) ds.$$

Proof. Deriving with respect the spatial variable x the system 4.4.2 we obtain

$$\begin{cases} u_{xt}^i + \lambda_i v_{xx}^i = 0, \\ v_{xt}^i + \lambda_i u_{xx}^i = -v_x^i + u_x^i \chi(\phi^i)\phi_x^i + u^i \chi'(\phi^i)\phi_x^{i,2} + u^i \chi(\phi^i)\phi_{xx}^i, \\ \phi_{xt}^i = D_i \phi_{xxx}^i + a u_x^i - b\phi_x^i, \end{cases} \tag{4.4.44}$$

for each $i \in \mathcal{M}$. Multiplying the above equations for the vector $w_x^i = (u_x^i, v_x^i, \phi_x^i)$, and integrating with respect to I_i and $s \in [0,t]$ and summing up on $i \in \mathcal{M}$ we have that

$$\sum_{i\in\mathcal{M}}(\int_{I_i}(w_x^i)^2 dx + \int_0^t\int_{I_i}(\lambda_i (u_x^i v_x^i)_x - D_i \phi_{xxx}^i \phi_{xx}^i) dx ds$$
$$+\int_0^t\int_{I_i}(v_x^i)^2 dx) \le \sum_{i\in\mathcal{M}}(2||(w_0^i)_x||_{L^2}^2 + \frac{a}{2}\int_0^t ||u_x^i||_{L^2}^2 ds$$
$$+(\frac{a}{2}-b)\int_0^t ||\phi_x^i||_{L^2}^2 ds + \frac{\chi_0}{2}\int_0^t ||u_x^i||_{L^2}^2 ds + \frac{\chi_0}{4}\int_0^t ||v_x^i||_{L^2}^4 ds$$
$$+\frac{\chi_0}{4}\int_0^t ||\phi_x^i||_{L^2}^4 ds + \frac{\chi_1}{2}\sup_{0\le s\le t}||u^i||_{H^1}\int_0^t (||v_x^i||_{L^2}^2 + ||\phi_x^i||_{L^2}^4) ds$$
$$+\frac{\chi_0}{2}\sup_{0\le s\le t}||u^i||_{H^1}\int_0^t (||v_x^i||_{L^2}^2 + ||\phi_{xx}^i||_{L^2}^2) ds,$$

where we have used the Cauchy-Schwartz inequality.

Now let us consider the term

$$S_3 = \sum_{i\in\mathcal{M}}\int_0^t\int_{I_i}(\lambda_i (u_x^i v_x^i)_x - D_i \phi_{xxx}^i \phi_{xx}^i) dx ds.$$

We have that

$$S_3 = \int_0^t \sum_{i\in E}\lambda_i u_x^i v_x^i(N) - \sum_{i\in U}\lambda_i u_x^i v_x^i(N) ds \tag{4.4.45}$$
$$-\int_0^t \sum_{i\in E} D_i \phi_{xx}^i(N)\phi_x^i(N) + \sum_{i\in U} D_i \phi_{xx}^i(N)\phi_x^i(N) ds.$$

Let us observe that there are no influence for terms on boundary points a_i since from boundary conditions $v^i(a_i,t)=0$ and $\phi^i_x(a_i,t)=0$, it follows that $u^i_x(a_i,t)=0$. Then we have that the above expression is always non negative under the assumptions (4.4.31), thanks to the monotonicity of the operator A on $D_\chi(A)$. In conclusion, we found that under conditions (4.4.31), the first order estimate (4.4.43) holds, and we get the proof. □

Energy estimate for $\int_0^t ||u^i||^2_{H^1}$

To prove Theorem (4.4.2) we also need to estimate the quantity

$$\int_0^t ||u^i||^2_{H^1}.$$

Let us first observe that using the Poincarè inequality we have that

$$\int_0^t ||u^i||^2_{L^2} \leq C\int_0^t ||u^i_x||^2_{L^2}, \tag{4.4.46}$$

for a positive constant C.

Now we estimate $\int_0^t ||u^i_x||^2_{L^2}$.

Lemma 4.4.5. *Let $w^i=(u^i,v^i,\phi^i)$, $i\in\mathcal{M}$, be a local solution to the problem (4.4.2), and let $u^i=u_i^+ + u_i^-$. Let us assume that the fluxes on the node N, $v^i(N,t)$, $i\in\mathcal{M}$, verify condition (4.4.18).*

Then we have

$$\int_0^t ||u^i_x||^2_{L^2}ds \leq (||u^i_0||^2_{H^1}+||v^i_0||^2_{H^1})+\lambda_i||v^i||^2_{H^1} \tag{4.4.47}$$

$$+2\lambda_i\int_0^t ||v^i||^2_{H^1}ds+\frac{\chi_0}{\lambda_i}\sup_{0\leq s\leq t}||u^i||_{H^1}\int_0^t(||u^i_x||^2_{L^2}+||\phi^i_x||^2_{L^2})ds.$$

Proof. For each $i\in\mathcal{M}$, let us consider the second equation of problem (4.4.2),

$$v^i_t+\lambda_i u^i_x = -v^i + u^i\chi(\phi^i)\phi^i_x. \tag{4.4.48}$$

Let $i\in E$ (calculations for $i\in U$ are analogous). Multiplying this equation for $\lambda_i u^i_x$ and integrating with respect to x and t, we have that

(4.4.49)

$$\lambda_i^2\int_0^t\int_{I_i}(u^i_x)^2dxds = -\lambda_i\int_0^t\int_{I_i}v^i_t u^i_x dxds - \lambda_i\int_0^t\int_{I_i}v^i u^i_x dxds+\lambda_i\int_0^t\int_{I_i}u^i u^i_x\chi(\phi^i)\phi^i_x dxds.$$

We have

$$-\int_0^t \int_{I_i} v^i u_x^i dxds \quad \leq \quad \frac{1}{2\delta}\int_0^t ||v^i||_{L^2}^2 ds + \frac{\delta}{2}\int_0^t ||u_x^i||_{L^2}^2 ds,$$

and we have

$$\int_0^t \int_{I_i} u^i u_x^i \chi(\phi^i)\phi_x^i dxds \quad \leq \quad \frac{\chi_0}{2}\sup_{0\leq s\leq t}||u^i||_{H^1}\int_0^t (||u_x^i||_{L^2}^2 + ||\phi_x^i||_{L^2}^2)ds.$$

Moreover we have to study the term

$$-\lambda_i \int_0^t \int_{I_i} v_t^i u_x^i dxds.$$

We have

$$v_t^i u_x^i = (v^i u_x^i)_t - v^i u_{xt}^i = (v^i u_x^i)_t - (v^i u_t^i)_x + (v_x^i)^2.$$

Thus

$$\begin{aligned}
-\lambda_i \int_0^t \int_{I_i} v_t^i u_x^i dxds &= -\lambda_i \int_{I_i} (v^i(t)u_x^i(t) - v_0^i u_{x,0}^i)dx - \lambda_i \int_0^t ||v_x^i||_{L^2}^2 ds \\
&\quad +\lambda_i^2 \int_0^t v^i(N,s)v_x^i(N,s)ds \\
&\leq \frac{\lambda_i}{2}(||u_0^i||_{H^1}^2 + ||v_0^i||_{H^1}^2) + \frac{\lambda_i}{2\eta}||v^i||_{L^2}^2 + \frac{\lambda_i\eta}{2}||u_x^i||_{L^2}^2 \\
&\quad +\lambda_i^2 \int_0^t v^i(N,s)v_x^i(N,s)ds.
\end{aligned}$$

Thus, substituting in (4.4.49) and summing up on $i \in \mathcal{M}$, we obtain

$$\begin{aligned}
\sum_{i\in\mathcal{M}}(\lambda_i - \frac{\eta}{2} - \frac{\delta}{2})\int_0^t ||u_x^i||_{L^2}^2 ds &\leq \sum_{i\in\mathcal{M}}(\frac{1}{2}(||u_0^i||_{H^1}^2 + ||v_0^i||_{H^1}^2) + \frac{1}{2\eta}||v^i||_{H^1}^2 \\
&\quad +\frac{1}{2\delta}\int_0^t ||v^i||_{H^1}^2 ds \\
&\quad +\frac{\chi_0}{2}\sup_{0\leq s\leq t}||u^i||_{H^1}\int_0^t (||u_x^i||_{L^2}^2 + ||\phi_x^i||_{L^2}^2)ds) \\
&\quad +\sum_{i\in E}\lambda_i^2 \int_0^t v^i(N,s)v_x^i(N,s)ds - \sum_{i\in U}\lambda_i^2 \int_0^t v^i(N,s)v_x^i(N,s)ds.
\end{aligned}$$

Thanks to condition (4.4.18) we have that the term

$$\sum_{i\in E}\lambda_i^2 v^i(N,s)v_x^i(N,s)ds-\sum_{i\in U}\lambda_i^2 v^i(N,s)v_x^i(N,s)=0.$$

So, choosing $\eta=\delta=\frac{\lambda_i}{2}$, we get:

$$\int_0^t ||u_x^i||_{L^2}^2 ds \le (||u_0^i||_{H^1}^2+||v_0^i||_{H^1}^2)+\lambda_i||v^i||_{H^1}^2$$

$$+2\lambda_i\int_0^t ||v^i||_{H^1}^2 ds+\frac{\chi_0}{\lambda_i}\sup_{0\le s\le t}||u^i||_{H^1}\int_0^t(||u_x^i||_{L^2}^2+||\phi_x^i||_{L^2}^2)ds.$$

□

Existence and uniqueness of global solutions: proof of the Theorem (4.4.2)

For each index $i\in\mathcal{M}$, let $w^i=(u^i,v^i,\phi^i)$, and let us consider the functionals

$$F_i^2(t):=\sup_{0\le s\le t}||w^i||_{H^1(I_i)}^2+\int_0^t ||w^i||_{H^1}^2 ds, \tag{4.4.50}$$

and

$$F^2(t):=\sum_{i\in\mathcal{M}}F_i^2(t). \tag{4.4.51}$$

We have that the following Lemma holds.

Lemma 4.4.6. *There exist positive constants K_0^i, K_1^i, K_2^i, K^i, K_3^i, and K_4^i depending only on initial data and coefficients of the problem such that the functional $F^2(t)$ satisfies the following estimate*

$$F^2(t)\le\sum_{i\in\mathcal{M}}(K_0^iF_i^2(0)+K_2^i(a,\lambda_i)F_i^2(t)+K_3^iF_i^3(t)+K_4F_i^4(t)). \tag{4.4.52}$$

In particular, we can choose

$$K_2^i(a,\lambda_i,\alpha,\beta)=\frac{a}{2}+2\lambda_i+D_i\alpha^2\beta$$

Proof. Collecting together all the energy estimates of the previous section we easily get the proof. □

Now we can prove Theorem (4.4.2). The proof is divided into two steps. In the first step we show that we have to choose the coefficient $\sum_{i\in\mathcal{M}}K_2^i(a,\lambda_i,\alpha,\beta)<1$ together with small initial data in order to have the global existence of the solution. In the second step we show that there exist change of variables in which the corresponding problems admit global solutions for small initial data.

Proof. **Step one:** From the previous Lemma, we have that the functional $F^2(t)$ verifies

$$F^2(t) \leq \sum_{i\in\mathcal{M}} (K_0^i F_i^2(0) + K_2^i(a,\lambda_i) F_i^2(t) + K_3^i F_i^3(t) + K_4 F_i^4(t)).$$

Using Nishida's Lemma, we have that, if the term $F_i^2(0)$ is sufficiently small and the the coefficient $\sum_{i\in\mathcal{M}} K_2^i(a,\lambda_i,\alpha,\beta)$ verifies the condition

$$\sum_{i\in\mathcal{M}} K_2^i K_2^i(a,\lambda_i,\alpha,\beta) < 1$$

, (4.4.53)

then we get a global estimate in time for the local solution (u^i, v^i, ϕ^i), $i \in \mathcal{M}$. Therefore, by Nishida's Lemma we can extend the local solution to a global one. Choosing small initial data $w_0^i = (u_0^1, v_0^i, \phi_0^i)$, $i \in \mathcal{M}$, we can make the coefficient $K_0^i F_i^2(0)$ small as we want, and choosing coefficients λ_i, a, α, and β such that condition (4.4.53) holds, then we get a global solution to problem (4.4.2).

Step two: Now we want to show that there exist change of variables such that for small initial data the corresponding problems admit global solutions.

We have that

$$K_2^i(a,b,D_i,\alpha,\beta,\lambda_i) = \frac{a}{2} + 2\lambda_i + D_i\alpha^2\beta$$

Now, for each $i \in \mathcal{M}$, let us consider the change of variables

$$y = \frac{x}{\widetilde{x}}, t = \tfrac{t}{\widetilde{t}},$$

$$U^i = \frac{u^i}{\widetilde{u^i}},\ V^i = \tfrac{v^i}{\widetilde{v^i}}, \text{ and } \Phi^i = \tfrac{\phi^i}{\widetilde{\phi^i}},$$

where $\widetilde{t}$, $\widetilde{x}$, $\widetilde{u^i}$, $\widetilde{v^i}$, $\widetilde{\phi^i}$ will be chosen in the following.

Thus the equations of our problem become in these new variables, for each $i = 1, \ldots, M$,

$$\begin{cases} U_{\widetilde{t}}^i + \frac{\lambda_i \widetilde{t}\widetilde{v^i}}{\widetilde{x}\widetilde{u^i}} V_y^i = 0, \\ V_{\widetilde{t}}^i + \frac{\lambda_i \widetilde{t}\widetilde{u^i}}{\widetilde{x}\widetilde{v^i}} U_y^i = -\widetilde{t}V^i + \frac{\widetilde{u^i}\widetilde{t}\widetilde{\phi^i}}{\lambda_i \widetilde{v^i}} U^i \chi(\Phi^i)\Phi_x^i, \\ \Phi_t^i = \frac{D_i \widetilde{t}}{\widetilde{x}^2}\Phi_{yy}^i + \frac{a\widetilde{t}\widetilde{u^i}}{\widetilde{\phi^i}} U - b\widetilde{t}\Phi^i. \end{cases} \tag{4.4.54}$$

Let us choose

$$\widetilde{u^i} = \widetilde{v^i},$$

for each index i, and

$$\widetilde{t} = \widetilde{\phi^i} = 1.$$

So the system becomes

(4.4.55)
$$\begin{cases} U^i_{\widetilde{t}} + \frac{\lambda_i}{\widetilde{x}} V^i_y = 0, \\ V^i_{\widetilde{t}} + \frac{\lambda_i}{\widetilde{x}} U^i_y = -V^i + \frac{1}{\lambda_i} U^i \chi(\Phi^i)\Phi^i_x, \\ \Phi^i_{\widetilde{t}} = \frac{D_i}{\widetilde{x}^2}\Phi^i_{yy} + a\widetilde{t}\widetilde{u^i} U - b\Phi^i. \end{cases}$$

With the previous equations we have obtained an equivalent problem, defined on new scaled intervals $\widetilde{I_i} = \widetilde{x} I_i$, and with new coefficients $\widetilde{\lambda_i}$, $\widetilde{a}$, $\widetilde{b}$, $\widetilde{D_i}$, defined as

$$\widetilde{a} = a\widetilde{u^i},\ \widetilde{b} = b,\ \widetilde{D_i} = \tfrac{D_i}{\widetilde{x}^2},\ \text{and}\ \widetilde{\lambda_i} = \tfrac{\lambda_i}{\widetilde{x}}.$$

In the new variables, the coefficient $K_2^i(a, b, D_i, \alpha, \lambda_i)$ becomes, for each $i \in \mathcal{M}$,

$$K_2^i = \frac{a\widetilde{u}^i}{2} + \frac{2\lambda_i}{\widetilde{x}} + \frac{D_i \alpha^2 \beta}{\widetilde{x}^2}.$$

Now, let us observe that choosing the parameter $\widetilde{u^i}$ such that

(4.4.56)
$$\sum_{i \in \mathcal{M}} \widetilde{u^i} < \sum_{i \in \mathcal{M}} (\frac{2}{a}(1 - \frac{2\lambda_i}{\widetilde{x}} - \frac{D_i \alpha^2 \beta}{\widetilde{x}^2})),$$

and the parameter $\widetilde{x}$ such that

(4.4.57)
$$\widetilde{x} > \min\{4\lambda_i, \sqrt{2D_i \alpha^2 \beta}\},$$

then the term $\sum_{i \in \mathcal{M}} K_2^i(a, \lambda_i, \alpha, \beta) < 1$.

Thus we can conclude the proof of the Theorem. We have that choosing the change of variables such that conditions (4.4.56) and (4.4.57), and for small initial data, then there exist constants $C > 0$ and $k > 0$ such that $F^2(t) < k$ and

$$F^2(t) \leq C(F^2(0) + F^3(t) + F^4(t)).$$

Thus, from Nishida's Lemma, we have that there exists a unique global solution (u^i, v^i, ϕ^i), $i \in \mathcal{M}$, to problem (4.4.2). $\square$

Bibliography

[1] Cazenve T. Arahaux, A. *An Introduction to Semilinear Evolution Equations*. Oxford University Press Inc., New York, 1998.

[2] Powell M. Boyce, T. Fiber density of electrospun gelatin scaffolds regulates morphogenesis of dermalâĂŞepidermal skin substitutes. *Wiley Interscience*, 2007.

[3] H. Brezis. *Functional Analysis*. Liguori Editore, 1986.

[4] Leach J. Wong J. Brown, X. Balance of chemistry, topography, and mechanics at the cellâĂŞbiomaterial interface: Issues and challenges for assessing the role of substrate mechanics on cell response. *Surface Science.*, 570:119–133, 2004.

[5] C. Cattaneo. *Sulla condizione del calore*, volume 3. 1949.

[6] Chello M. De Marco F. De Porcellinis S. Genovese J.A. Rainer A. Spadaccio C. Trombetta M. Centola, M. A g-csf functionalized plla scaïňĂold for wound repair: an in vitro preliminary study . *Conf Proc IEEE Eng Med Biol Soc.*, 2010.

[7] Crumpler L. Chaudry, H.R. Optimal patterns for suturing wounds of complex shapes to foster healing. *Journal of Biomechanics*, 34:51–58, 2001.

[8] Preziosi L. Chauviere, A. Cell mechanics from single scale-based models to multiscale modeling. 2009.

[9] Mi Q. Riviere B. Steed D. Vodovitz Y. Clermont, G. Agent-based model of inflammation and wound healing: insights into diabetic foot ulcer pathology and the role of transforming growth factor-b1 . *Wound Repair and regeneration.*, 2007.

[10] Zuazua E. Dager, R. Wave propagation, observation and control in 1d flexibility multistructure. *C.R. Math. Acad. Sci. Paris,*.

[11] Douglas S. Maini P. Sherratt J. Dallon, J. Fibroblast migration and collagen deposition during dermal wound healing: mathematical modelling and clinical implications . *Phil. Trans. R. Soc.*, 364:1385–1405, 2006.

[12] E. Di Benedetto. *Partial Differential Equations.* Birkhauser, 1995.

[13] J. DieudonnÃÍ. *ElemÃÍnts d'Analyse*, volume 2. Gauthier-Villars-Paris, 1969.

[14] Hillen T. Dolak, Y. Cattaneo models for chemosensitive movement: numerical solution and pattern formation. *J.Math. Biol.*, 46:153–170, 2003.

[15] L.C. Evans. *Partial Differential Equations.* Springer-Verlag, 1983.

[16] Piccoli B. Garavello, M. Traffic flow on networks: Conservation laws models. *AIMS Series on Applied Mathematics.*, 1, 2000.

[17] Gibb S. Hensbergen P. Ghalbzouri, A. Fibroblasts facilitate re-epithelialization in wounded human skin equivalents . *Laboratoy Investigation*, 84:102–112, 2004.

[18] Mascia C. Natalini R. Ribot M. Guarguaglini, F. Stability of constant states of qualitative behavior of solutions to a one dimensional hyperbolic model of chemotaxis. *Discrete Contin. Dyn. Syst. Ser.*, 12:39–76, 2009.

[19] Natalini R. Hanouzet, B. Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy. *Arch. Rational Mech. Anal.*, 169:89–117, 2003.

[20] Li J. Liu Y. Hong Yang, C. Influence of extracellular polymeric substances on pseudomonas aeruginosa transport and deposition profiles in porous media. *Environ. Sci. Technol.*, 41:198–205, 2007.

[21] Kedem O. Katchalsky, A. Thermodynamic analysis of the permeability of biolog- ical membrane to non-electrolytes. *Biochimica and Biophysica Acta.*, 27:229246, 1958.

[22] Segel L.A. Keller, E.F. Traveling band of chemotactic bacteria: A theoretical analysis. *J. Theor. Biol.*, 30:235–248, 1971.

[23] A. Majda. Compressible fluid flow and systems of conservation laws in several space variables. *AppliedMathematical Sciences.*, 53, 1984.

[24] Natalini R. Mascia, C. L1 non linear stability of traveling waves for a hyperbolic system with relaxation. *J. Differential Equations*, 132:275–292, 1996.

[25] V. Milisic. Stability and convergence of discrete kinetic approximations to an initial-boundary value problem for conservation laws. *PROCEEDINGS OF THE AMERICAN MATHEMATICAL SOCIETY*, 131:1727–1727, 2003.

[26] V. Milisic. Kinetic approximations of a boundary value problem for conservation laws. *Numer. Math.*, 2004.

[27] R. Mulder. ClinicianâĂŹs pocket guide to chronic wound repair. . *Wound Care Communications Network, Springhouse Corp., Springhouse.*, pages 58–67.

[28] J. D. Murray. *Mathematical biology. I*, volume 17 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, third edition, 2002. An introduction.

[29] J. D. Murray. *Mathematical biology. II*, volume 18 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, third edition, 2003. Spatial models and biomedical applications.

[30] R. Natalini. Convergence to equilibrium for the relaxation approximations of conservation laws. *Comm. Pure Appl.Math.*, 49(8):795–823, 1996.

[31] R. Natalini. A discrete kinetic approximation of entropy solutions to multidimensional scalar conservation laws. *J. Differential Equations*, 148(2):292–317, 1998.

[32] R. Natalini. Recente mathematical results on hyperbolic relaxation problems. 1998.

[33] Terracina A. Natalini, R. Convergence of relaxation approximation to a boundary value problem for conservation laws. *J. Differential Equations*, pages 35L65,35B25, 1998.

[34] Sherratt J.A. Owen, R. Pattern formation and spatiotemporal irregularity in a model formacrophage- tumour interactions. *J. Theor. Biol*, 189.

[35] C. S. Patlak. Random walk with persistence and external bias. *Bull. Math. Biophys.*, 15:311–338, 1953.

[36] S. Ramtani. Mechanical modelling of cell/ecm and cell/cell interactions during the contraction of a fibroblast-populated collagen microsphere: theory and model simulation. *Journal of Biomechanics.*, 37:1709–1718, 2004.

[37] L.A. Segel. Incorporation of receptor kinetics into amodel for bacterial chemotaxis. *J. Theor. Biol.*, 57:23–42, 1976.

[38] L.A. Segel. A theoretical study of receptormechanisms in bacterial chemotaxis. *Appl. Math.*, 32:652–665, 1977.

[39] Zuazua E. Valein, J. Stabilization of the wave equation on 1d network. *C.R. Math. Acad. Sci. Paris,*, 2008.

[40] G.B. Whitham. *Linear and non linear waves.* Wiley-Interscience, 1974.

[41] K. Yosida. *Functional Analysis.* Springer, 1980.